Investigating Students’ Understanding of Square Numbers and Square Roots

by

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Abstract

The ideas related to square numbers and square roots, despite the apparent simplicity, appear problematic to high school students. I inquired into students' understanding of square numbers and square roots, focusing on obstacles that students face in considering these concepts and solution approaches that they prefer. The study followed a modified analytic induction methodology. A written questionnaire was administered to 51 pre-calculus 11 students and followed up with clinical interviews with 9 students. The study revealed significant obstacles in three categories: confusion of concepts, distribution of square numbers within the natural numbers and opaque representation of square numbers. Confusion of concepts included confusion both within and between the concepts of square numbers and square roots. Students used five different solution approaches; brute force, guess and check, rule application, pattern spotting and attention to structure. Students switched between solution approaches often, but showed personal preference toward particular solution approaches.

Keywords: mathematics education; square numbers; square roots; definitions; distribution; representation
To my loving husband, Mark,

my children, James and Rebecca and my family;
you have all put up with a lot from me for the last two years but have never been anything but supportive.

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1. Introduction

In my experience as a high school mathematics teacher, I have found that often teachers feel that some mathematical concepts are so simple and basic that there should be no difficulty with understanding them. These concepts are taught as if to understand them only requires being informed of their pertinent properties and from then on no confusion should be possible. Square numbers are such a concept. What could be simpler than arranging dots into a square shape? And yet students do have a variety of ways of comprehending square numbers, and square numbers are not quite as simple as might appear at first glance. As such, there remains the possibility of differences in understanding or incorrect inferences about square numbers by students. When students move forward towards much more complex ideas like general exponents, square roots and exponential growth, it is imperative that they link their new understanding to a firm foundation of square numbers. Square roots are complementary to square numbers and while they are usually given more attention from teachers, square roots are often treated as being almost distinct from square numbers. With this thesis I examine how students understand both square numbers and square roots.

This chapter discusses square numbers and square roots in general terms, my personal interest in this topic and describes the layout of the thesis as a whole.

1.1. Square Numbers and Square Roots

A number \( n \), is called a square number if \( m \cdot m = n \) for integer \( m \). The first ten non-zero square numbers are: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100. The origin of the term ‘square number’ comes from the ability to arrange a square number of dots into a square array. Figure 1 shows the first four non-zero square numbers arranged into squares.
Square numbers are also called perfect squares. Square numbers have some basic properties of interest to this study:

- The product of two square numbers is itself a square number.
- Any whole number power of a square number is itself a square number.
- Square are the only numbers that have an odd number of factors.
- The list of prime factors of a square number will have even multiplicities of factors.
- The difference between two consecutive square numbers increases as the numbers get larger.

The square root of a number is the value that when multiplied by itself yields the original number. Every real positive number is not a square number, but every real positive number does have a square root; square numbers are those whose square root is an integer, rather than an irrational number.

1.2. Personal Interest

My interest in students’ understanding and ideas about square numbers and square roots came after having spent an hour reviewing square roots with a class of grade 9 students while working as a Teacher-on-Call. These students had worked with square numbers and square roots for two previous lessons, and we were reviewing the previous days’ homework together as a class. The homework involved estimating the square root of an integer that was not a square number by using a number line with whole numbers written above the lines and square roots of perfect squares written below. One question asked students to estimate the square root of 48, with the appropriate number line provided (Figure 2).
Estimate $\sqrt{48}$

48 is not a square number but it is very close to the square number 49, whose square root is 7. An appropriate estimate for the square root of 48 should therefore be just a little smaller than 7, perhaps 6.9. A large proportion of the class, perhaps $1/3^{rd}$ of the students, was having tremendous difficulty with this problem. The remaining students found the problem overly simple and could not understand what difficulty the others were having. After much probing and questioning it came to light that some of the students experiencing difficulty were unable to estimate the square root as just under 7 because 48 is equal to 6 times 8, so “the square root could be either 6 or 8”. These students had an understanding of square numbers that was tied to the factorization of the number. For perfect squares, these students had practiced listing the factors of the number in ascending order and circling the middle one to find the square root. This method is valid for perfect squares but is clearly not applicable to estimating the square root of a non-square whole number. These students knew that finding the square root required the middle factor of the number, but did not know any alternative method to use when there was no middle factor, only a pair of middle factors. These students had learned one process for finding the square root of a square number but did not have the conceptual understanding of the number either being a square or being representable as a square.

Previously, I had never considered square numbers or square roots to be particularly difficult concepts because like many mathematics teachers, I had considerable experience with students having difficulty with “harder” related topics, such as higher order exponents, rational exponents, or negative exponents. To me, square numbers and square roots seemed like simple concepts that only required explaining on my part, and required very little effort on the students’ part to understand.
Since this experience I have become interested in how students understand square numbers and square roots, but beyond mere calculation. In particular I wished to know:

1) What obstacles do students encounter when attempting to solve problems with square numbers and square roots?
2) What strategies do students use to solve problems relating to square numbers and square roots?

These are the research questions that I aim to address with this work. My focus is on students’ conceptual understanding of square numbers and square roots and how they can use their knowledge to solve problems. This focus has shifted slightly towards more abstract understanding and away from mere problems of estimation or calculation.

1.3. Thesis Organization

This thesis is organized in 5 chapters. Chapter 1, the introduction, briefly describes square numbers and square roots, gives an overview of my personal interest in this topic and describes the outline of the thesis as a whole.

Chapter 2 is the literature review in which I examine the literature as it relates to this work, in the following areas: the usual definitions of square numbers in both rigorous mathematics texts and in high school texts, students’ understanding of square numbers, representation of numbers and solution approaches used by students. There are several complimentary definitions of square numbers commonly in use, however student texts often supply a definition that is not fully rigorous. There is little research relating to understanding of square numbers and square roots, perhaps because this topic is such a “simple” one. The vast majority of research in this area focuses on exponential growth, general exponents or exponential functions. I review the topic of representation of numbers, and how these representations may help or hinder students’ understanding. I also report the common solution approaches used by students while solving problems of various types.

In Chapter 3, I discuss my methodology in detail. This includes a description of the participants, the instruments, and examples of the tasks given on each of these
instruments. The rationale behind all the selections is also given. The methodology also includes a brief overview of the data analysis procedures and rationale including a description of the theoretical constructs used to focus the analysis.

The data analysis is described in detail in Chapter 4. This description includes the detailed account of how the analysis was conducted for each instrument, the rationale behind the analysis, an in depth analysis of responses from each item, and the findings that the analysis brought to light.

In Chapter 5, I offer pedagogical considerations and suggestions for improved teaching and learning of square numbers and square roots. I summarize my work and discuss the contributions that I have made. I also discuss the limitations of this work and include considerations for further research.
2. Literature Review

In this chapter, I describe the relevant literature in three parts: square numbers and square roots, representation, and solution approaches. First, square numbers and square roots are described in two different aspects: the mathematical definitions of the square numbers and square roots and students’ ability to learn and understand square numbers and square roots. While focussing on the mathematical definitions, the common definitions in use in both rigorous mathematics texts and in high school mathematics texts are described. While focussing on students’ ability to learn and understand square numbers and square roots, the role that a student’s concept image and concept definition may play in their understanding is described as well as any literature relating to understanding square numbers and square roots in particular. I describe the relevant research into students’ understanding of square numbers and square roots, including the role that definitions play and the confusion that may occur between two terms that have similar names and similar definitions but are distinct entities.

Secondly, I relate the previous research into the role that representation plays when students are attempting to solve problems in a wide variety of mathematical subject areas. How a number or expression is represented in a problem can affect the ability of students to work with it.

Finally, I report the research into solution approaches chosen by students when working on mathematics problems. Common approaches across mathematical topics include guess and check and pattern spotting. Each of these parts is connected to the research questions described in Chapter 1.
2.1. Square Numbers and Square Roots

I begin by describing the common definitions in use for square numbers. The difference between the definitions given in mathematics texts designed for mathematics professionals and those given in texts designed for high school students is small but significant. While the former are simple and rigorous, often the latter are “fuzzy” and require students to make assumptions that may not be clear or obvious to them.

2.1.1. Definitions of Square Numbers and Square Roots

The mathematical definition creates the mathematical meaning. (Pólya, 1945, p.46)

Mathematical terms may be defined in different equally valid ways that highlight different aspects of the concept, and square numbers are no exception. However, a clear definition is still required to avoid confusion and incorrect assumptions. Without a clear and simple definition to work from, mathematical terms can exist in a hazy indeterminate state within the mind of the student. Of course, this may still be true when a clear definition is given; however, a definition allows an external standard against which to measure understanding.

Definitions in Rigorous Texts

In mathematics texts designed for a more advanced reader, the importance of clear and concise definitions is made clear. Even simple or basic terms are defined in order to avoid confusion or unjustified assumptions. Definitions for square numbers can be found in mathematics texts pertaining particularly to number theory, or in mathematical encyclopaedias.

Perhaps the most common definition is that given by Shapiro (2008): “an integer is a square if it equals $m \cdot m = m^2$” (p. 6) which can be paraphrased as ‘the product of an integer and itself is called a square number’. This definition is what most mathematics teachers and students have in mind when thinking of square numbers, and can be traced back to Euclid in his Elements. Definition 18 of Book 1 states “[a] square number is equal multiplied by equal, or a number which is contained by two equal numbers.”
Previously in Definition 2 of Book 1, Euclid had stated that “a number is a multitude composed of units,” which together yield Shapiro’s definition for $m = 2, 3, 4$ ...

Square numbers are also defined as the main diagonal of the multiplication times table beginning with 1 (Conway & Guy, 1996, p. 30). But perhaps less commonly, square numbers are defined as a special subset of the figurate numbers. The equation for a general group of r-gonal numbers is $f_r(X) = (r - 2) \times \frac{X(X - 1)}{2} + X$, for $X = 1, 2, 3, 4 ...$, which becomes the equation $f_4(X) = X^2 = 2 \times \frac{X(X - 1)}{2} + X$, for $X = 1, 2, 3, 4 ...$ for square numbers (Ribenboim, 1995, p. 11).

In the online mathematical encyclopaedia, Wolfram MathWorld, a ubiquitous mathematical reference guide, Goodman and Weisstein combine the definitions given by Shapiro and Ribenboim: “A square number, also called a perfect square, is a figurate number of the form $S_n = n^2$, where $n$ is an integer. The square numbers for $n = 0, 1, 2 ...$ are $0, 1, 4, 9, 16, 25, 36, 49, ...$” This definition also gives the alternative term for square number as perfect square.

Here each definition serves a different purpose that highlights different connections between square numbers and other mathematical concepts. Ribenboim’s definition shows square numbers as a smaller part of a larger group of two-dimensional figurate numbers, while Shapiro goes on to link square numbers to cubes, and other exponential numbers. These differing definitions are equivalent and, to a mathematics professional, are not ambiguous or contradictory. It is worth noting, however, that these definitions do appear to differ on whether 0 or 1 would be considered square numbers. Euclid’s definition excludes both 0 and 1, as 1 is not a number but the unit, both Conway and Guy, and Ribenboim include 1 but exclude 0, while Shapiro and Wolfram include both 0 and 1.

Clear and precise definitions of the sort found in rigorous mathematics texts are unambiguous and allow advanced concepts to be built up from a strong and stable foundation. However, overly complex definitions may not be clear or beneficial for students who lack experience working with definitions.
Definitions in High School Texts

Square numbers and square roots are both first introduced to students in British Columbia in grade 8 mathematics. At this level, students are expected to “demonstrate an understanding of perfect square and square root, concretely, pictorially, and symbolically (limited to whole numbers)” and “determine the approximate square root of numbers that are not perfect squares (limited to whole numbers)” (Ministry of Education, Province of British Columbia, 2008, p. 62). However, by grade 9, mathematics square numbers, referred to in the curriculum documents as perfect squares, have been relegated to a special case of exponents and are no longer offered special treatment apart from dealing with square roots of rational numbers. Students are required to “determine the square root of positive rational numbers that are perfect squares” and “determine an approximate square root of positive rational numbers that are non-perfect squares” (Ministry of Education, Province of British Columbia, 2008, p. 77). It is worth noting that although no definition of square numbers is given in the curriculum documents, it is clear that the Ministry does not require square numbers to be integers. At further grade levels, students are expected to work with square numbers and square roots, but they are rarely treated independently. So how do the texts for these grade levels define square numbers for students?

In high school mathematics texts, square numbers are “defined” in three ways:

1) No Definition
2) Description
3) Fuzzy Definitions

No Definition

Some of the high school mathematics textbooks recently in use do not offer a definition or description for square number either in the main text, or in the glossary if one exists. Perhaps square numbers are generally assumed to be too simple to require a definition. These are texts that discuss exponents and powers, and many include definitions of square roots (Connelly, et al., 1987; Knill, et al., 1998; McAskill, et al., 2011). If definitions are given for the square root, it may not reference square numbers
at all. Knill et al. (1998) give the definition of a square root as “One of two identical factors of a number” (p. 650).

**Description**

Square numbers are fairly simple, so it is not so surprising that often they are not defined rather they are explained through examples only, or described quite generally. The most common description uses a square array of dots or pebbles (Bye et al., 1984a; Ebos, Tuck, Schofield, & Hamaguchi, 1987; Knill et al., 1996b). Figure 1 given in the introduction shows the most common format of these arrays. These arrays may come with a description similar to “Perfect square numbers can be represented by squares of pebbles” (Knill et al., 1996b, p. 6), or none at all. Note here the combination of terms “perfect square” and “square number” into the new term “perfect square number”. The visual representation of square numbers as a square of pebbles is sometimes paired with an algebraic expression, but not in all cases (Bye et al., 1984a; Knill et al., 1996b). When describing the ability of square numbers to be displayed in a physical square there is an implicit assumption that the square number must be a whole number whose square representation will have a whole number of pebbles per side, but this assumption is never stated explicitly.

The second common type of explanation in textbooks is “$8 \times 8 = 64$ so 64 is called a square number and 8 is the square root of that number” (Knill et al., 1996a, p. 29). In this case, once again, there is an assumption that all the “numbers” should be whole numbers, but the assumption is somehow weaker than in the first case. Imagining this phrase with decimal numbers is not difficult and, as far as students are concerned, may still be valid: “$2.5 \times 2.5 = 6.25$ so 6.25 is called a square number and 2.5 is the square root of that number”. The fact that the second half of the statement is correct may further lead students to the belief that 6.25 may be a square number. Since every positive real number does have a square root, students may believe that every positive real number could be a square number.

Descriptive definitions offer the learner valuable visual representations or examples to support the construction of their understanding of square numbers and square roots. However with just a description, learners are may create their own flawed or incomplete definitions for square numbers.
Fuzzy Definitions

If a definition of square number is given, it is often similar to the form above and is not fully rigorous. When definitions lack rigour and clarity they leave some matters to be inferred by learners. The grade 8 mathematics textbook currently in use at many schools in British Columbia gives the definition of a square number as

A square number is the product of the same two numbers $3 \times 3 = 9$ so 9 is a square number. A square number is also known as a perfect square. A number that is not a perfect square is called a non-perfect square. (McAskill et al., 2008, p. 80)

This definition does not make explicit that the ‘same two numbers’ must be integers and may leave the reader with the question of what a non-perfect square could be. Other definitions are also vaguely circular in their terms, with statements such as “The perfect squares (sometimes called squares) are a set of numbers that have special properties. These are the numbers: 1, 4, 9, 16, ... they are formed by squaring the natural numbers, 1, 2, 3, 4...” (Alexander, 1998) Although this definition specifies that the domain is limited to the natural numbers, it does not make clear what ‘squaring’ means.

These various definitions and descriptions of square numbers all have some common elements. They are not perfectly clear, and each definition has some underlying assumptions that may not be shared by students. Usually this information is that “number” really means “natural number” or “integer”. These omissions appear to be attempts to make the definition clear and easy to understand – but leaving students to wonder at the meaning of a definition is not beneficial to them. All the fuzzy definitions use small numbers as examples in any visual description – never a number larger than 16 as the square, and 4 is the only non-prime to be used as the root of the square.

These descriptions also all avoid referring to the geometrical background of squaring; that is forming a square with two equal extensions. Perhaps the authors wish to avoid the confusion that may result if students believe that those equal extensions could be any length rather than integer lengths. The texts that show the square numbers as a square array of pebbles also often clearly use the word ‘representation’ to describe the array (Knill et al., 1996b).
Lack of clarity and preciseness, use of only small numbers, and a lack of multifaceted approaches, especially visual ones, may lead to a shallow understanding of square numbers, and by extension square roots.

2.1.2. Students’ Understanding of Square Numbers and Square Roots

Students’ understanding of square numbers and square roots can be affected by both their own personal understanding of the concepts as well as the manner in which square numbers and square roots are introduced. I describe the literature that discusses the role that concept image and concept definition may play as well as how the manner of instruction of the topics at hand may influence a student’s learning and understanding of square numbers and square roots.

Concept Image and Concept Definition

Tall and Vinner’s article “Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity” (1981) is the earliest work describing concept image and concept definition using those terms. The authors describe concept image as “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures” (p. 152). They also describe concept definition as “a form of words used to specify that concept. It may be learnt by an individual in a rote fashion or more meaningfully learnt and related to a greater or lesser degree to the concept as a whole” (p. 152). A concept image is usually more personal than the corresponding concept definition, which is often learnt from somewhere else. As an individual’s concept image is built up of many parts and is developed through experience, some portions of the concept image may be incorrect or incomplete.

The role that concept image and concept definition play in student understanding has been studied with regards to a wide range of mathematical topics from those as ‘simple’ as division (Tirosh & Graeber, 1990) to those as ‘complex’ as Tall and Vinner’s own work on limits and continuity (1981). With respect to prime numbers, Zazkis and
Liljedahl (2004) found that students have an incomplete concept image: “if a number is composite, it must be divisible by a small prime” (p.175).

Concept image has also been found to be resistant to change. Tirosh and Graeber found that students continue to hold on to their belief that division makes things smaller even after they completed an example of division by a decimal number less than one. “Although all of the subjects included in the study correctly completed the algorithm for division by a decimal less than one, some of them apparently believed that "division makes smaller" more firmly than they believed in the result of their calculation” (1990, p.106).

There is little literature dealing with concept image and concept definition with regards to square numbers and square roots in particular. However, Bingolbali and Monaghan (2008) give as an example of incorrect concept image: “squaring makes things larger”.

**Learning and Understanding Square Numbers and Square Roots**

There are two overlapping styles of articles written regarding square numbers and square roots with respect to mathematics education. One style consists of articles published in professional journals written primarily for teachers, that give suggestions or instruction on how to teach students square roots or exponents more effectively (Boomer, 1969; Edmonds, 1970; Goodman & Bernard, 1979; Kamins, 1969; Thompson, 1992). These articles often focus on an intuitive approach to exponents and square roots or give tips and tricks for teaching. The majority of these articles emphasise square roots rather than square numbers, or focus on exponents in general, often with the specific purpose of extensions for explaining rational exponents or negative exponents. Some articles discuss only the zero exponent, possibly the most confusing aspect of exponents for students (Astin, 1984; Bernard, 1982). There also exists a selection of articles from a time when scientific calculators were not widely available or used, that give various algorithms for calculating the square root (Chow & Lin, 1981; Edge, 1979). From these articles, it is clear that the nature of exponential growth and the relationship between whole number exponents and rational exponents are the aspects of exponents that teachers are most concerned with when teaching.
The second style of work comprises scholarly papers that research students’ learning and understanding. However, the vast majority of these articles focus on much higher-level mathematical topics than square numbers or square roots that cause more obvious problems for students. These topics include general exponents involving negative, rational or the zero exponent (Pitta-Pantazi, Christou & Zachariades, 2007; Sastre & Mullet, 1998; Vinner, 1977), exponential growth (Brown, 2005; Ebersbach & Wilkening, 2007), and exponential functions (Confrey & Smith, 1994; Confrey & Smith, 1995). Of particular interest is Sastre and Mullet’s (1998) work that shows that when estimating the magnitude of an exponential expression, students used mathematical models that included both the base of the expression and the exponent.

There are very few articles or papers from either area that specifically address students’ understanding of square numbers or simple square roots (i.e., not complex roots or special cases). However, Gough (2007) discusses difficulties teaching square roots and argues that the vocabulary and terms used when teaching square numbers and square roots can be confusing and detrimental to student understanding. Square number and square root are similar sounding phrases that evoke images from our everyday English language use of those words; while ‘square number’ may yield a useful image, ‘square root’ does not convey much meaning in and of itself.

Instead of a simple definition, or descriptive fact-statement, we have a not immediately obvious backwards process, which, importantly, reverses a process of “squaring” which is assumed to have been taught and learned beforehand. When students first encounter “square root,” they are expected to pull together geometric ideas (recognizing a square shape) and numerical ideas (a number multiplied together with itself), and then mentally re-work the process of squaring so it becomes a new, but related, reverse process of “treat it like a square and find the side”.

(Gough, 2007, p.55)

The difficulty here is both the backwards nature of the square root as an inverse operation but also the confusion generated between the English words for square number and square root. Similarly, in a paper discussing the derivative function and the derivative at a point, Park (2013) found that the confusion between the two concepts was exacerbated by the similarity of the terms. She points out that “in Korean and
Japanese, the terms for the concepts do not share the same word, and thus the terms are not confusing but do not show the relation between the concepts” (p. 624).

Although Gough lamented the lack of a simple ‘forwards’ concept and definition for the square root, the lack a simple precise definition given to students for square numbers may also be lamented. Pólya points out “Technical terms in mathematics are of two kinds. Some are accepted as primitive terms and are not defined” (1945, p. 85). Tall and Vinner (1981) also suggest that a lack of formal definition is acceptable and natural,

Many concepts which we use happily are not formally defined at all, we learn to recognize them by experience and usage in appropriate contexts. Later these concepts may be refined in their meaning and interpreted with increasing subtlety with or without the luxury of a precise definition. (p. 151)

This seems to often be the case with square numbers. As discussed in the introduction, the definition for square numbers may be given as the numbers on the diagonal of the multiplication table (Conway & Guy, 1996), as a special case of figurate numbers (Ribenboim, 1995) or as the product of an integer and itself (Shapiro, 2008). Zazkis and Leikin (2008) point out that “that mathematical concepts rather often have several equivalent definitions and the choice of a definition useful for a particular task is determined by the context” (p. 135). However, in high school, it is often pragmatic to give examples of square numbers in lieu of a precise definition, and so many students may be left without a clear definition for square numbers.

Although Pólya and Tall and Vinner believe that some mathematical terms do not require a definition, argument for precise definitions has also been made. Under the heading of communication standards, the National Council of Teachers of Mathematics states “Beginning in the middle grades, students should understand the role of mathematical definitions and should use them in mathematical work. Doing so should become pervasive in high school” (NCTM, 2000, p. 63). Levenson (2012) studies teachers’ understanding of definitions relating to exponents in general and the zero exponent in particular and found that two of the three teachers interviewed believed that definitions could be proven rather than being arbitrary agreed-upon conventions. She argued that these teachers are in danger of confusing their students about the role of
definitions and theorems and of depicting mathematics as a group of disjoint rules with no internal consistency.

The role of definitions has not been studied with particular respect to square numbers or square roots, but it is clear that the similarity of the terms square number and square root may be an obstacle for students. I suggest that the lack of clear and concise definitions of square numbers and square roots given to students, will also be an obstacle for students to overcome when attempting to solve square number problems.

2.2. Representation

Zazkis and Gadowsky (2001) describe that many definitions of sets of numbers rely on the representation of that number. For example, a rational number is a number that can be represented as \( a/b \), for integer values of \( a \) and \( b \). In a similar manner a square number is a number that can be represented as \( a^2 \) for some integer \( a \). However, like all numbers, square numbers may be represented in numerous ways some of which make the ‘squareness’ more or less apparent. Representations may be referred to as opaque or transparent (Lesh, Behr & Post, 1999). Representations that highlight a desired property of a number may be said to be transparent to that property, while representations that obscure a property are said to be opaque to that property. Zazkis and Gadowsky (2001) argue that all representations are opaque; some properties are always highlighted while others are obscured. The example used by Zazkis and Gadowsky is the number 46,656, which is both a square number and a cubic number. It may be represented as \( 216^2 \); this representation is transparent with respect to the square, but opaque with respect to the cube. Written as \( 36^3 \) the same number is opaque to the fact that it is indeed a square number, but now transparent to the fact that is a cubic number.

The role of representation dealing with numbers, rather than algebraic or geometric or other mathematical representations, has been well documented with respect to prime numbers (Zazkis & Liljedahl, 2004; Zazkis, 2005), irrational numbers, (Zazkis & Sirotic, 2004; Zazkis, 2005), divisibility and prime factorization (Zazkis & Campbell, 1996; Zazkis, 2008).
The lack of transparent representation has been found to be hindrance to students when solving problems or attempting to generate examples. Zazkis and Liljedahl (2004) found “the lack of transparent representation for prime numbers creates an obstacle for acting on them, and thus, creates an additional difficulty for constructing a mental object” (p. 180).

However, if students do not have a clear understanding of the structure of a problem or expression, a transparent representation will not guarantee success. Zazkis and Sirotic (2004) found that only 60% of respondents gave the correct response to the question of whether $53/83$ was rational or irrational after performing the division on a calculator. Although the representation of the number was transparent to rationality a large proportion of respondents did not attend to this representational feature. These students were not attending to the rational expression, but were focused instead on the partial decimal representation shown by their calculators. While studying divisibility, Zazkis and Campbell (1996) found that when asked if $M = 3^3 \times 5^2 \times 7$ was divisible by 7, some participants solved the problem by first performing the multiplication, and then dividing by 7. Again the representation of the number $M$ is transparent with respect to divisibility by 7, but students must have sufficient understanding of the multiplicative structure in order to attend to the transparent features.

Zazkis and Gadowsky (2001) claim that as students are very familiar with computational tasks, they do not have the “inclination to ‘think’ before turning to calculations” (p. 47). Opaque representation of power expressions as squares is another obstacle that students may need to overcome when attempting to solve problems involving square numbers or square roots.

2.3. Solution Approaches

Solution approaches adopted by students vary from problem to problem and student to student. Students are likely to use more than one approach while attempting to solve a problem and are likely to switch between solution approaches often. Switching approaches occurs when one approach has failed to yield the desired results or if the approach appears too difficult (Huntley & Davis, 2008; Lannin, 2005; Orton, 1999; Stacy
Common solution approaches across problem types include guess and check and looking for patterns, both of which are described below.

2.3.1. **Guess and Check**

Guess and check, also known as guess and adjust, guess and test or guess and try, has been shown to be a very common solution strategy for students of all ages and ability levels (Capraro, An, Ma, Rangel-Chavez & Harbaugh, 2012; Johanning 2004; Lannin, 2005).

Lannin (2005) describes guess and check as “guessing a rule without regard as to why this rule might work”; episodes of students using guess and check in this manner are described as *wild* guessing (p. 234). In contrast Johanning (2007) defines *systematic* guess and check as “a form of model-based reasoning where the problem solver works with the situational context and applies relational reasoning to solve the problem” (p. 123). Capraro et al. (2012) distinguish between guess and check as a “systematic problem solving strategy” and random guess and try as “guessing without organized rational thinking” (p. 112). Guess and check solution approaches lie along a continuum that ranges from wild guessing at random to strategic guessing coupled with rational reasoning.

With systematic guess and check, students use information from previous guesses to adjust their next guess and move systematically towards the correct solution. Guerrero (2010) argues that, if used intelligently in a strategic manner, guess and check can help students develop theories and narrow down the possible solutions until they are able to solve their problem. Systematic guess and check may also allow students to find solutions to problems when they have not had instruction in more formal methods. Johanning (2007) shows that elementary students working on word problems of an algebraic nature without prior formal instruction in algebra predominantly used systematic guess and check to arrive at correct solutions. Similarly, Chow and Lin (1981) describe a method to use guess and check and successive approximations to find a square root using division when other methods are not available. Capraro et al. (2012) show that when given a semi open-ended problem, preservice teachers’ primary solution strategy was guess and check, but the nature of their guess and check strategies
affected their outcomes drastically. Used strategically “guess and check is as much about developing mathematical reasoning and problem solving as it is about finding a correct solution” (Guerrero, 2010, p. 394).

However, despite the promise of systematic guess and check as a learning strategy, many have pointed out the limitations to its use. Johanning (2007) points out that in order to use this strategy successfully students need to understand the relationships between the quantities in the problem context. Although often maligned as a less valuable solution strategy, guess and check, even wild guess and check can play a useful role when used appropriately. When little data is provided or students are unsure of the solution approach to pursue guess and check can be a valuable strategy to help students move forwards. “In many cases guess and check is important strategy to find a breakthrough points when limited clues are provided” (Capraro et al., 2012, p. 114). However, in this case students must analyze their attempts in order to come up with a method or suggestion or path. “Many a guess has turned out to be wrong but nevertheless useful in leading to a better one” (Pólya, 1945, p. 99).

Systematic or strategic guess and check may have some limitations but wild guess and check has repeatedly been shown to be a poor solution strategy with little lasting benefit to the student. When students used guess and check in a manner that was not strategic and did not analyze their solutions, guess and check became simply a method to stumble upon the correct answer without learning about the problem or solution. Lannin (2005) determines that [wild] “guess-and-check strategies often moved student attention away from the problem situation, discouraging them from establishing a geometric scheme” (p. 253). Similarly Capraro et al. (2012) claim:

another weakness of random guessing or guessing without systematic thinking may influence PTs [preservice teachers’] ability to see the whole scope of the problem. The PTs who tried to solve the problem using a random guess did not understand the entire scope of the problem. The process of repeatedly experimenting with different combinations of numbers provided them no evidence of the possibility of multiple solutions unless they unexpectedly obtained multiple answers (p. 113)

A crucial finding with guess and check used as a strategy is the importance of the thinking behind the guess and check. Wild guessing is often an unproductive
strategy that results in wasted time, wasted effort, and little knowledge gained about the problem, or useful strategies if the correct solution is stumbled upon. Guess and check used systematically, purposefully and with logical analysis of guesses can be a useful strategy that can lead to algebraic thinking, further understanding and intuition about the structure of the problem.

Now it is not so much that this student guessed, it is the thinking behind the guess that matters. In order to develop a plan to systematically guess and check with, a student has to understand the underlying structure of the problem and articulate this into a formal plan.


I will use Johanning’s term systematic guess and check when referring to strategic guessing and information gathering and Lannin’s term wild guess and check when referring to the other end of the guess and check spectrum. Here wild guess and check will be modified to include examples of students attempting to find only a single solution rather than a ‘rule’.

2.3.2.  Pattern Spotting

Finding and understanding patterns is a major goal in mathematics education. The NCTM stresses patterns in its Principles and Standards for School Mathematics in both its Reasoning and Proof standard, and its Algebra standard for all grade levels (NCTM, 2000). They suggest in particular “People who reason and think analytically tend to note patterns, structure, or regularities in both real-world situations and symbolic objects; they ask if those patterns are accidental or if they occur for a reason; and they conjecture and prove” (NCTM, 2000, p.56). Similarly, the BC mathematics curriculum documents also stress patterns:

Mathematics is about recognizing, describing and working with numerical and non-numerical patterns. Patterns exist in all of the mathematical topics, and it is through the study of patterns that students can make strong connections between concepts in the same and different topics. Working with patterns also enables students to make connections beyond mathematics. The ability to analyze patterns contributes to how students understand their environment.
Students need to learn to recognize, extend, create and apply mathematical patterns. This understanding of patterns allows students to make predictions and justify their reasoning when solving problems. (Ministry of Education, Province of British Columbia, 2008)

Looking for patterns, and learning to extend them, is a popular path to understanding algebra. Both the NCTM and the BC Ministry of Education push this belief: “Students in the middle grades should learn algebra both as a set of concepts and competencies tied to the representation of quantitative relationships and as a style of mathematical thinking for formalizing patterns, functions, and generalizations” (NCTM, 2000, Algebra Standard for Grades 6-8) and “Learning to work with patterns helps develop students’ algebraic thinking, which is foundational for working with more abstract mathematics” (Ministry of Education, Province of British Columbia, 2008).

However using patterns to lead to learning algebra can be problematic. Orton and Orton (1999) found that when working on a pattern of numbers, students could continue the pattern through addition but could not note the overall structure of the pattern and noted that “It is perhaps not unreasonable to conclude that this approach of asking pupils to find more terms in the sequences was not productive way of attempting to elicit the general term” (p. 116). While in a study looking at pictorial patterns Orton, Orton and Roper found that “success on the fifth, ten or even fiftieth term did not automatically lead to an acceptable expression for the general term” (p. 126). Liljedahl (2004) described the difference between repeating patterns such as ‘a, b, a, b, a, b’, and number patterns that rely on a mathematical structure such as ‘1, 5, 9…’. He found that students were likely to treat number patterns as if they were repeating patterns; these students could extend the pattern but not necessarily use the structure to their advantage.

The phrase ‘train spotting’ comes from Hewitt (1992), who describes the practice of looking for trains and recording their numbers; at the end of the day you have numbers but no trains. In the same manner he describes reducing mathematical tasks to a table of values in order to spot patterns within the numbers. He criticizes this method as removing all the mathematics from the problem, and leaving only train spotting, which may allow students to spot the patterns, but not allow reference to the original problem. I will use the complimentary term of ‘pattern spotting’ rather than ‘train spotting’.
Whatever the initial mathematical situation once the numbers are collected into a table a separate activity begins to find patterns in the numbers. Their attention is with the numbers and if that’s taken away from the original situation. After a period of time, some children have difficulty reminding themselves where all the numbers came from. I suggest that for many children, what they find out about the numbers remains exactly that; it does not mean it like anything about the original mathematical situation only about sets of numbers in a table. (Hewitt, 1992, p.7)

Pattern spotting is an easy trap to fall prey to and false pattern spotting is very common, Hewitt (1992) notes "Children can find many patterns in their table, even if they have made some errors in the entries. They may find all sorts of rules, none of which apply to the original situation" (p. 2). Lee and Frieman (2006) also describe the tendency of students to look for patterns that do not exist because they are so often given work with patterns. They note that “Although seeing a pattern is an important mathematical skill noting a pattern where there is none can lead to mathematical disasters or quite simply be a huge waste of time months in the case of practicing mathematicians” (p. 430).

Pattern spotting also refers to describing a number pattern without enough emphasis on explaining the origin of these patterns or to relating the pattern to the original problem. Hewitt (1992) explains, “The trouble is that the general statements are statements about the results rather than the mathematical situation from which they came” (p. 7). Zazkis and Liljedahl (2002) describe different levels of understanding patterns: “Being able to continue a pattern can be taken as an understanding of a repeating pattern. Being able to describe a ‘general’ element can be seen as a solution of a linear pattern” (p. 387). Pattern spotting covers both the situation of continuing the pattern, and possibly of describing the general element of a pattern if students cannot relate the general element of the pattern to the general element of the original problem.

Using either guess and check or pattern spotting as a solution strategy is only successful in the long term if enough personal analysis is applied to the problem to gain insight and understanding. Using either approach only as a means to the solution of a particular problem is unreliable.
Mathematics should make sense to students; they should see it as reasoned and reasonable. Their experience in school should help them recognize that seeking and finding explanations for the patterns they observe and the procedures they use help them develop deeper understandings of mathematics.

(NCTM, 2000, Reasoning and Proof Standard for Grades 9-12)

These solution approaches are strategies that students may use to solve problems relating to square numbers and square roots among others. The importance is in both the approach chosen, but also the intent behind the choice and the manner in which students pursue their approach.
3. Methodology

This study followed the methodology of modified analytic induction, as laid out by Bogdan and Biklen (1998). This methodology designates a way to collect and analyze data as well as test a theory. Modified analytic induction requires a phenomenon of interest and a working hypothesis or theory. One develops a loose descriptive theory, collects data and then rewrites and modifies the theory to fit the new data. This process is recursive and continues until no new phenomena are encountered. Modified analytic induction uses purposeful sampling in order to choose subjects that will facilitate the expansion of the developing theory. Bogdan and Biklen also describe how the research questions may be modified during the study:

Not only is the theory modified during the research process to fit all new facts that arise, but the research question also can be redefined (narrowed) to exclude the cases that defy explanation by it. By choosing what categories to include or exclude, you also control the breadth of the work by limiting the theory's scope. (p. 65)

In this study, the phenomenon of interest was students’ understanding of square numbers and square roots in general, and the research questions listed in the introduction in particular. These research questions were:

1) What obstacles do students encounter when attempting to solve problems with square numbers and square roots?
2) What strategies do students use to solve problems relating to square numbers and square roots?

The working theory that addresses these questions began as an assumption that the representation of the expression would play a large role in students’ abilities to solve square number problems. In particular, opaque representation of square numbers was expected be an obstacle for students in solving square number and square root problems.
As modified analytic induction dictates, the theories, possible explanations and even phenomena of interest were modified throughout the study in order to formulate a relationship between the definition and explanation of the particular phenomenon (Bogdan & Biklen, 1998, p. 65). The methodology described here reflects the changes that occurred throughout the study.

3.1. The Participants

The participants in this study were 51 grade 11 pre-calculus students, from a high school in the lower mainland area of British Columbia. The students attended a high school designated “inner-city” in a culturally diverse, low socio-economic neighbourhood. The students in the school generally perform near the mean of schools in same geographic area in mathematics on provincial exams and other standardized tests. Pre-calculus 11 is designed to lead students into calculus at the university level and is the most theoretically rigorous grade 11 mathematics course available at this school. The students in this course range greatly in ability; they include some of the most mathematically talented students in the school as well as students who are much less capable. Pre-calculus students were chosen for this study in order to capture students with both a great deal of familiarity with square numbers and a wide range of knowledge and ability. The students were in their first month of pre-calculus 11 at the time this study was implemented; they had no recent practice with exponent laws, prime factorization or any other topics particularly germane to square numbers or square roots. However, these students have a great deal of previous experience with these topics beginning in grade 8. Students were told ahead of time that the questionnaire would be focusing on square numbers and square roots, but they were not asked to study as the purpose of this investigation was to determine the students’ current understanding of square numbers and square roots, rather than their memory or their ability to carry out calculations.
3.2. The Tasks

Two instruments were used; a written questionnaire, found in Appendix A, completed by all participants and a follow-up semi-structured clinical interview designed to gain more insight into student responses from the questionnaire, completed by nine participants. The nine students who participated in the clinical interviews all had a grade of ‘A’ or ‘B’ in both their current mathematics course as well as their previous mathematics course. These students were selected through purposeful sampling by choosing students for the interview based on their willingness to explain their reasoning on the questionnaire responses and their willingness to attempt the problems in good faith.

3.2.1. The Questionnaire

The complete questionnaire can be found in Appendix A, while a brief overview of the tasks is given here. The written questionnaire had nine distinct questions, some of which consisted of a series of related parts grouped together to attempt to pinpoint a specific area of students’ understanding. The nine questions were arranged in such a way as to lead the students through a series of related problems to discover differences in their responses to structurally similar questions whose representation became more opaque. The questionnaire was administered during the participants’ regular mathematics class. They had just over one hour to work on the tasks, however the average length of time taken was approximately 40 minutes. Students were not permitted to use a calculator while completing the questionnaire; this was to encourage students to attempt to answer the questions conceptually rather than by brute force calculation. Although calculators may assist strategic thinking in some cases, in my experience, I have found that students are more likely to use a calculator in place of conceptual thinking.

The items on the questionnaire originated from various sources, including but not limited to, discussions with colleagues, interesting exam questions from previous courses and topics from the literature.
The questionnaire began by giving examples of square numbers in the following manner: “Note: \(2^2 = 4, 3^2 = 9, 4^2 = 16, 10^2 = 100\) numbers like these are known as square numbers or perfect squares.” Rather than a rigorous definition, I chose to only include examples of square roots because I wished simply to remind students of their previous knowledge of the terms square number and perfect square rather than impose upon them my definition of the terms.

The questionnaire tasks can be broken down into the following categories based on the structure of the problem: expressions with only one variable, expressions with two variables, and expressions with only numerals. However, these structural categories are not to be confused with the intent of the problems, that is to discover student thinking surrounding certain conceptual themes. These conceptual themes were designed to investigate students’ understanding or use of: distribution of square numbers throughout the natural numbers, irrational numbers, prime factorization and the representation of the square numbers in the problem. These themes represent the main obstacles that were hypothesized to hamper students attempting to solve problems with square numbers. The questionnaire tasks in each category are summarized in Table 1. Further discussion of the conceptual themes follows.

**Conceptual Themes**

*Distribution of Square Numbers*

Square numbers are not regularly spaced along the number line; the difference between two subsequent square numbers gets larger as the square numbers themselves get larger. Students often have small square numbers memorized, and may not be attentive to the underlying organisation of the distribution throughout the natural numbers. Unfortunately, due to time constraints limiting the length of the questionnaire, only one question dealing with distribution was included.
Table 1. Questionnaire Tasks in Each Structural Category and Conceptual Theme

<table>
<thead>
<tr>
<th>Structural Categories</th>
<th>Conceptual Themes</th>
<th>Distribution</th>
<th>Irrational Numbers</th>
<th>Prime Factorization</th>
<th>Representation</th>
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<tr>
<td>Numerals</td>
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</table>

* Questionnaire tasks in parentheses were analyzed but have been excluded from the analysis in Chapter 4.

Item 8: How many perfect squares are there between 100 and 10,000?

Item 8 was chosen to illuminate students’ ideas regarding the distribution of square numbers within the natural numbers, based on their choice of solution strategy. A possible incorrect strategy that shows a misconception about the distribution of square numbers is based on counting the square numbers in a smaller interval and multiplying that number to scale to the larger interval. A possible solution may state: ‘if there are 10 perfect squares from 0 – 100, there must be $10 \cdot 100 = 1000$ from 0 – 10,000.’ This is an example of a whole-object method that is often chosen by students to incorrectly scale up a linear problem (Stacey, 1989; Lannin, 2005). A correct strategy could use the square roots of the end-points of the interval in question, as the numbers are not distributed equally throughout the interval. For example, ‘since $\sqrt{100} = 10, \sqrt{10000} = 100, 100 – 10 = 90$. There are 90 square numbers between 100 and 10,000.’
**Irrational Numbers**

I was interested in whether study participants would consider the possibility of irrational numbers when asked to solve a problem.

Both Items 1 and 2 on the questionnaire involved explicitly including or excluding irrational numbers from the domain for the fully generalized solution. Irrational numbers are not mentioned specifically in the questionnaire, but neither did the questionnaire limit students to whole numbers. The use of word “value” in the questionnaire rather than the word “number” was an intentional choice to limit provoking any preconceived ideas about “number”.

**Item 1:** For what values of $a$, if any, can the following be perfect squares? For each answer explain briefly.

1) $a^2$
2) $2 \cdot a^2$

Correct solutions to Item 1 should specifically attend to the possibility of irrational numbers. The correct solution to Item 1.1 is “$a$ can be any integer” while the correct solution to Item 1.6 is “$a = x/\sqrt{2}$”. In the first case, specifying integers excludes the possibility of $a$ being an irrational number, while the second case requires $a$ to be an irrational number. Note that neither correct response requires use of the term irrational number.

**Item 5:** Given a number $a$ such that $a^2$ is an integer. Does this mean that $a$ must also be an integer? Circle your answer Yes/No. Explain / Exemplify your decision.

Item 5 was the only question chosen exclusively to investigate if students would use irrational numbers when looking for examples or a counterexample to support their claim. A very simple example can be chosen to support the suggestion that if $a^2$ is an integer $a$ does not have to be an integer; if $a^2 = 2$, an integer, then $a = \sqrt{2}$, which is not an integer. Once again the term irrational number need not be used.

**Prime Factorization**

Factorization of large numbers by prime decomposition is a skill taught in each mathematics course from grade 8 until grade 10 in BC, therefore the pre-calculus students have had frequent instruction in this topic. I wished to determine if participants
would access and make use of this information without specific prompting or if they would use this knowledge in a problem that was not specifically asking for a calculation.

**Item 9:** The number 576 is a perfect square. Find \( \sqrt{576} \).

Item 9 asked participants to calculate the square root of a number, without the use of a calculator. When calculators are not available, an efficient solution to this problem uses factorization. However, students may not attempt to factor but employ a guess and check method of solution or another alternative method.

**Item 4:** Which of the following are perfect squares? Explain briefly.

1) \( 13^2 \cdot 17^2 \cdot 23^2 \)
2) \( 13^2 \cdot 17^2 \cdot 23 \)
3) \( 13^2 \cdot 17^2 \cdot 9 \)

Item 4 attempted to discover if students would make use of factorization when not explicitly asked for a calculation. Item 4.1) and 4.3) are perfect squares because they are products of perfect squares. Item 4.2) is not a perfect square because 23 is not a square. Prime factorization shows easily that products of square numbers are also square numbers. Students who are unaware of, or inattentive to this feature of square numbers may attempt to use an alternative method for solving this problem; such as brute force calculation or looking for clues within the factors themselves.

**Representation**

Four questionnaire items, 1, 2, 3, and 4, were created to illuminate students’ thinking surrounding the representation and structure of square numbers. These items may be completed either by using the exponent laws or by using knowledge of the fact that any product of two square numbers will itself be a square number and the product of a square number and a number that is not square cannot be a square number. It was expected, however, that these participants would primarily make use of exponent laws rather than the product of squares as they have more experience with the exponent laws. The students in this study have had a minimum of three years’ of mathematics instruction dealing with exponent laws and have a great deal of experience with simplifying expressions such as \((x^2)^3 = (x^3)^2 = x^6\). I was interested to see if students
would access this information and use it in a problem when not specifically prompted to do so.

Item 3: Consider $36^2, 36^3, 36^4, 36^5, 36^6, 36^7$. Circle the perfect squares. Explain briefly.

Item 3 is an example of a question whose representation of the square numbers is opaque, and was selected to investigate students’ attention to the representation. All six expressions should be circled in the correct solution because 36 is itself a square number. As discussed in Chapter 2, the representation of the square number is opaque in this problem; the square-ness is “hidden” in the number 36. A change of representation using the exponent laws serves to make the representation of the square transparent: $36^x = (6^2)^x = (6^x)^2$. The aim of this problem was to see whether students would attend to the base, the number 36, where the square is evident, or only attend to the exponent, where square-ness is not always shown transparently.

Item 2: For what values of $x$ and $y$, if any, can the following be perfect squares? For each answer explain briefly.

1) $x \cdot y$
2) $x^2 \cdot y$
3) $x^2 + y^2$

Some of the items on the questionnaire are expressed in an unfamiliar manner to what students are accustomed in their mathematics experience. In an unfamiliar format, the representation of the problem may be opaque to students even if it is transparent to someone with more mathematics experience. For example, students have extensive experience with the Pythagorean theorem, but almost exclusively in the format $a^2 + b^2 = c^2$; often students are even instructed to chant the expression “a squared plus b squared equals c squared”. Item 2.3 was designed to see if students would access their knowledge about the Pythagorean theorem when the equation was presented in a slightly unfamiliar format, using $x$ and $y$ rather than $a$ and $b$. The appropriate solution to this problem is to give the smaller two numbers of any Pythagorean triple, for example 3 and 4 of the Pythagorean triple $x = 3, y = 4, x^2 + y^2 = 5^2$. There are two probable incorrect solutions. Students may state that this expression can never yield a perfect square because there is no exponent law that allows the simplification of addition of
squares. Other students are also likely to suggest that this expression will always yield a perfect square due to an incorrect application or understanding of an exponent law.

### 3.2.2. The Clinical Interview

The clinical interviews were designed to gather additional information of a different nature than that gained from the questionnaire. While the questionnaire was designed to indicate *which* questions students had difficulty answering, the clinical interviews were designed to explore *why* students had difficulty answering a particular question.

The correctness or incorrectness of a solution given on the questionnaire could not provide any information that the students had not chosen to share. When a student supplied a correct solution, the questionnaire could not yield the method that led the student to a successful solution. Similarly, if a student did not answer a question, the lack of response could not explain the nature of the difficulty; the student ran out of time, the student didn’t know how to perform the mathematical task, or the student didn’t understand the question, etc.

The clinical interview was chosen as an instrument to augment the information gained from the questionnaire. The interviews were to determine how a student came to a particular solution and for determining the thought process behind their decisions. In particular, the clinical interviews were designed to gain information relating to: what struggles students had when performing a task, what prompted students to make a particular decision about their work and what methods students used when attempting to solve a problem.

The clinical interviews were one-on-one semi-structured interviews with participants that lasted between 25 minutes and one hour. The average length for an interview was 50 minutes. All participants were supplied with scratch paper to use during the interviews, and any non-trivial questions were written on the scratch paper as well as asked aloud. The interviews were audio recorded and transcribed within two weeks of having been recorded. The written transcripts together with the participants’ scratch paper were used for analysis rather than the recordings. As mentioned previously,
participants were chosen for the interviews based on their responses to the questionnaire tasks, most importantly their willingness to explain their reasoning and their effort to answer the questions in good faith. Other practical concerns also contributed to the choice of participants including their teacher's opinion of who would be the least anxious and most interested in participating. All nine participants in the clinical interview were considered strong mathematics students who had earned a grade of either ‘A’ or ‘B’ in both their current mathematics class and their previous mathematics class.

The semi-structured clinical interviews were designed to be very flexible in order to pursue intriguing areas of student understanding. As Posner and Gertzog maintain “The method is highly flexible, allowing a skillful researcher both to probe the areas of the knowledge domain of particular interest and to let the subject speak freely, while constantly checking his or her spontaneous remarks for those that will prove genuinely revealing” (p. 197). The clinical interviews were also designed to allow further questions to be added to include more questions regarding certain topics that were not fully investigated through the questionnaire, or to add follow up questions for topics without sufficient information gathered through the questionnaire.

All nine clinical interviews began with the same three questions and thereafter followed the flow of discussion from participants’ responses. As suggested by Ginsburg (1997), these three questions were designed to act as warm-up exercises to give the participants some success and to allow them to become more comfortable with the setting of the interview. The interviews began with the following questions; all expressions in square brackets were written down for the participants:

1. Do you remember how to simplify these exponent laws? 
   \[ (a^3 \cdot a^4) \text{ and } [(a^3)^4] \text{?} \]
2. Can you give me an example of a square number? (And another…) 
3. I have this number \( M \); \( M \) is 8 squared times 10 squared. Can you tell me the square root of the number \( M \)? \[ [M = 8^2 \cdot 10^2, \sqrt{M} = ?] \]

The first item was chosen to assess student’s memory of exponent laws shown in the traditional manner. My working assumption was that if students had difficulty with the “simple” application of the exponent laws, they were likely to struggle more with the
subsequent items. The second item is what Zazkis and Hazzan (1998) describe as a “give an example task” (p. 433). In this case, students were prompted to give another example, and another, and then prompted to give an example between two large numbers at which point the task becomes an example of a “construction task” (p. 433). Follow up questions pursued each of these lines of questioning based on the individual participant’s responses, answers to previous questions and their previous questionnaire responses. Further questions were chosen from a pool of possible questions with a wide range of task types and structure to give the participants a change of pace if they became tired of a particular question or overwhelmed by a series of questions.

3.3. The Analysis

The analysis sections describes in brief the theoretical constructs that the research was analyzed with respect to, summarizes the method of the analysis of the questionnaire and summarizes the analysis of the clinical interviews.

3.3.1. Theoretical Constructs

The analysis was viewed through the lens of three theoretical constructs. The three theoretical constructs are: concept image and concept definition, opaque and transparent representation and pattern spotting. Each construct was described in general in Chapter 2 as part of the literature review, but will be discussed with particular emphasis on square numbers and square roots. Each construct is described in brief and particular examples relevant to square numbers and square roots are given.

Concept Image and Concept Definition

I have described concept image and concept definition in general terms in Section 2.1.2; here I describe the particular attention I will pay to concept image and concept definition with respect to square numbers and square roots. The questionnaire and the clinical interviews will both be analyze through the lens of concept image and concept definition. I will look for examples of students demonstrating a robust and multi-dimensional concept image and examples of students demonstrating a weak or shallow
concept image. Examples of conflicting concept image to concept definition will also be sought.

Examples of incomplete concept images of square numbers would include statements that ‘anything squared’ is a square number or that ‘perfect cubes cannot be perfect squares’. These statements suggest an incomplete concept image of a square number; a more complete concept image would correctly limit the domain to integers as ‘an integer times itself’.

Opaque and Transparent Representation

Zazkis and Gadowsky (2001) assert that all representations are opaque to some features and transparent to others. With respect to square numbers and square roots, the representation of an expression may be more or less opaque to certain features. One representation of a given expression may be more opaque or more transparent to the “squareness” of the expression. An example of an expression that is transparent with respect to “squareness” is $a^2$. Here the exponent ‘2’ clearly shows that this number is a square number based on the definition and common description of a square. The expression $a^4$ is more opaque and less transparent than the first example, but the squareness is still somewhat evident if only attending to the exponent in the expression. $36^3$ is now very opaque as the exponent shows no sign of the expression being a square number and the square number must be found by attending to the base of the expression to discover the square.

However opaque and transparent representations are also dependent on the knowledge and experience of the viewer. A student who is just learning square numbers may not find $a^4$ to be a transparent representation, if they have not yet had experience with other powers. Similarly a more experienced person may nevertheless find $36^3$ to be a transparent representation of a square number he or she immediately recognizes that 36 is a square number and that it is a salient feature of the expression.

Pattern Spotting

I have described pattern spotting in general in Section 2.3.2. With particular emphasis on square numbers and square roots pattern spotting may hinder student understanding. Some examples of how pattern spotting may come in to play include
students who may expect the distribution of square numbers to follow an easily visible pattern, or students who use pattern spotting to attempt to solve problems without dealing with the underlying structure of the square number.

### 3.3.2. Questionnaire Analysis

The analysis of the questionnaire tasks unfolded organically in stages as is customary with the modified analytic induction protocol (Bogdan & Biklen, 1998, p. 65). Further rounds of analysis are based upon the results obtained from previous rounds. In this manner, the analysis of the questionnaire tasks unfolded in three rounds that were determined by shortcomings of the previous round.

Round I consisted of coding responses of each question via a 6-point scale that I designed to rate the suitability of the response. Each question was coded based on whether or not the response was correct or incorrect and whether or not the response included correct or incorrect justification or reasoning. The codes ranged from 1: correct answer with correct reasoning to 4: incorrect answer with incorrect reasoning, with two additional codes for ‘left blank’ and ‘other’. These six codes varied slightly between different questionnaire tasks depending on the type of the task.

**Item 2:** For what values of $x$ and $y$, if any, can the following be perfect squares? For each answer explain briefly.
1) $x \cdot y$
2) $x^2 \cdot y$
3) $x^2 + y^2$

For example, responses to Item 2 of the questionnaire yielded only examples, rather than any explanation or rationale, so the six codes were modified to reflect the possible solutions given. An example of how the codes were modified is discussed in detail in Section 4.1.1.

Subsequent to coding round I, each question was returned to and classified more finely with codes particular to each question, which enabled each response type to be grouped together. The full range of round I and II codes for each item can be found in Appendix B. This round allowed a very detailed view of student response but little
general information about the types of obstacles that students were encountering while completing the questionnaire.

To alleviate this shortfall, the round III coding assigned a thematic code to every response type from round II. This yielded information about common stumbling blocks or views across all students and all problems. These codes dealt with confusion of concepts, distribution and representation. These themes were chosen based on both the original conceptual themes that the questionnaire was designed to investigate as well as themes that arose organically from the analysis. The full analysis is examined in detail in Chapter 4.

3.3.3. Clinical Interview Analysis

The clinical interview analysis focused on finding instances of responses that fit the themes found from round III of questionnaire coding. This was done by thoroughly and repeatedly reading through the interview transcripts and coding all responses (or sections of responses) with a thematic code. In this manner many new themes were found apart from the original themes of interest determined a priori during the questionnaire development stage and from the questionnaire analysis stage. The most common themes were then grouped together while underlying explanations were sought. Some themes were not pursued further because they lay outside the scope of this work, while others were combined under categories since they were very interconnected. The final refined list of themes of interest contained three items: confusion of concepts, distribution of square numbers and representation of square numbers and square roots. The confusion of concepts theme contained two distinct groups: concept definition confusion and inconsistent concept image confusion.

Additionally, while coding and analyzing the clinical interview transcripts, the solution approaches chosen by the students became too striking to be ignored. Therefore the solution approaches of the students were also analysed, coded and grouped into the following categories: brute force, guess and check, rule application pattern spotting, and attention to structure. The full analysis is found in Chapter 4.
4. Data and Results

The data analysis was designed to unfold naturally with subsequent rounds of analysis predicated on the results from the previous round, as suggested by modified analytic induction and described briefly in Chapter 3. I began round I with the themes that determined the design of the questionnaire and were discussed above in Section 3.2.1. These themes are: distribution of square numbers, irrational numbers, prime factorization and representation; they were the foundation from which the analysis began. Throughout the data analysis process, additional themes were identified and added as warranted, and two of the original themes were discarded as areas of interest. These themes were discarded due to a lack of evocative results or results falling outside of the scope of this work.

This chapter emulates the chronological stages of the analysis. I begin with the questionnaire data, which was partially analyzed before the clinical interviews took place and helped mould the interview questions to align more carefully with the themes of increasing interest. Section 4.1 describes in detail the rounds of questionnaire data coding and gives very general analysis of each round before considering the relevant questionnaire items on an item-by-item basis for a detailed analysis.

The analysis of the interview data follows the analysis of the questionnaire data and is discussed in Section 4.2.1. I begin with a description of how the analysis was conducted, the themes I chose to focus on and those I chose to exclude. The analysis of the clinical interviews is organized according to theme rather than item-by-item due to the nature of clinical interviews as a unique experience and product for each participant.

Lastly in Section 4.2.2 I examine the solution approaches employed by the students while working through the problems given during the clinical interviews. These solution approaches range from a brute force approach that generally requires more effort and less skill, to an approach that attends to the structure of the mathematics that
generally requires less effort but more skill to solve the problem. In this part of the chapter I also discuss examples of students dropping one strategy in favour of another and what prompts their decision to switch approaches.

4.1. Questionnaire Analysis

The questionnaire data was coded in three consecutive rounds that each focused on different aspects of student response. Round I of coding focused on the appropriateness of the solutions given by the students. It coded the responses according the whether or not the solution was correct or incorrect and if it included correct or incorrect reasoning. Round II focused on quantifying and grouping the common responses and for identifying common difficulties that the students were experiencing. Round III was devoted to coding the grouped responses according to theme across questionnaire items; these themes were those mentioned in Section 3.2.1 and one additional theme identified from the common difficulties found in round II of coding. Each round after round I was created to delve deeper into the data and to gain insight into the participants’ solutions, based on a deficiency of the previous round. The questionnaire items were analyzed with a special view to the theoretical construct of representation, as responses that related to this construct were easily identifiable on the written questionnaire. The remaining two theoretical constructs, concept image and concept definition, and pattern spotting, were not as easy to apply to written responses, but there were examples of responses relating to each of these constructs on the questionnaire.

All of the questionnaire items were analyzed in this manner using all three rounds of data analysis; however, I have chosen to focus only on questionnaire items 1, 2, 3 and 8 in this report. The reasons for omitting the other five items vary considerably but include: the majority of student responses were trivial; a great deal of students misunderstood the question; a majority of students left the question blank; or the information gained from their analysis of that item ultimately did not fall within the scope of this research.
4.1.1. Description of Coding

Round I Coding

In round I of coding each questionnaire item was considered individually and assigned a code from a 6-point scale that I designed to be a preliminary categorisation of the suitability of the student response. Codes 1 through 4 on this scale are designed to rank the responses according to appropriateness, from 1: “correct answer with correct reasoning”, through code 4: “incorrect answer with incorrect reasoning”. Code 5 was given to responses left blank, and code 6 was added as an “other” code. Examples of responses that received code 6 include: completely illegible responses, responses that were incoherent, responses that were self-contradictory and students having misunderstood the question. The last reason was the most common cause for a response to earn code 6. Many students misunderstood one or more questionnaire items; they often answered a related question instead. Many of these responses gave correct answers to the related question so assigning a code 4 for “incorrect answer with incorrect reasoning” would yield a distorted view of the responses to that question. The general form of all round I codes can be found in Table 2.

Table 2. Round I Codes for Questionnaire Analysis

<table>
<thead>
<tr>
<th>Code</th>
<th>Explanation of Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Correct answer and correct reasoning</td>
</tr>
<tr>
<td>2</td>
<td>Partially correct answer with partially correct reasoning or no reasoning</td>
</tr>
<tr>
<td>3</td>
<td>Incorrect answer with partially correct reasoning</td>
</tr>
<tr>
<td>4</td>
<td>Incorrect answer with incorrect reasoning or no reasoning</td>
</tr>
<tr>
<td>5</td>
<td>Left blank</td>
</tr>
<tr>
<td>6</td>
<td>Other (illegible response, misunderstanding of question, etc.)</td>
</tr>
</tbody>
</table>
Examples of student responses that fall within each code are given in Table 3 for Item 8 of the questionnaire.

**Table 3. Round I Questionnaire Codes - Sample Student Responses for Item 8**

<table>
<thead>
<tr>
<th>Code</th>
<th>Sample Student Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(100^2 = 10,000)</td>
</tr>
<tr>
<td></td>
<td>(10^2 = 100)</td>
</tr>
<tr>
<td></td>
<td>(100 - 10 = 90)</td>
</tr>
<tr>
<td></td>
<td>(correct)</td>
</tr>
<tr>
<td>2</td>
<td>(100^2 = 10,000)</td>
</tr>
<tr>
<td></td>
<td>(10^2 = 100)</td>
</tr>
<tr>
<td></td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>(student did not subtract)</td>
</tr>
<tr>
<td>3</td>
<td>100, 121, 144, ...</td>
</tr>
<tr>
<td></td>
<td>(student begins to list them, but gives up)</td>
</tr>
<tr>
<td>4</td>
<td>They can all be perfect squares</td>
</tr>
<tr>
<td></td>
<td>(square number / square root confusion)</td>
</tr>
</tbody>
</table>

The correct answer to Item 8 was 90 (or 88 depending on the definition of “between”) and could be arrived at by subtracting the square root of 100 from the square root of 10,000. An example of a partially correct answer was the solution that showed both square roots but did not show a subtraction. The student who began listing the perfect squares but eventually gave up and could not complete the problem exemplified an incorrect answer with partially correct reasoning. Lastly an example of an incorrect answer and incorrect reasoning for this problem was the solution that “they can all be perfect squares”. However as discussed in both Section 4.1.2 and Section 4.2.1, this could have been a result of unconventional interpretation of the term perfect squares. Codes 5 and 6 have not been included in this table because there is clearly no need to show a sample of “left blank” and there were no questionnaire responses given a code 6 for Item 8.
Item 2: For what values of $x$ and $y$, if any, can the following be perfect squares? For each answer explain briefly.

1) $x \cdot y$
2) $x^2 \cdot y$
3) $x^2 + y^2$

The round I codes have been given in their general form, however based on the style of the particular questionnaire item and the student responses for that item, some of the codes required minor modification. For example, no students supplied a generalized solution for Item 2.3. However many students supplied examples of cases where the expression did yield a perfect square. In this case code 1, “correct answer with correct reasoning”, was modified to include “gives at least two non-trivial examples”. Codes were modified in this manner on a question-by-question basis as needed. Often not all codes were used. For some questions, no participant gave the correct answer with the correct reasoning. Codes were therefore not completely consistent between questionnaire items, however the absolute uniformity of the codes was not the objective; rather the relevance of the codes to the questionnaire item was paramount.

The round I coding and analysis was preliminary and was used as a means to suggest particular avenues to pursue within the themes of interest during the clinical interviews and to design the next round of coding. There were limited concrete results from this stage of coding but those few results were used to begin to refine the themes of interest. It became clear that participants had difficulty in fully generalizing their solutions and that they were often using a naïve view of number that generally consisted of only natural numbers. Some round I codes contained a wide range of student responses within them that were dissimilar to each other and showed significant differences in the nature of their response. It was apparent at this stage that a further round of coding that separated the student responses into smaller categories was required.

Round II Coding

After the round I coding was complete I desired a more refined coding scheme that would allow a more nuanced look at the student responses. I examined each questionnaire item individually and then grouped them according to their round I code. For each code, I compiled the questionnaires into groups of similar responses. I
classified the responses with the letters A-F to distinguish the round II codes from the round I numerical codes. The codes A-F were based on the particular responses given for that question and the round I code. In this manner the round II codes are all subordinate to the round I code; these subordinate round II codes are linked to their round I code. The codes A-F were chosen to represent a slight hierarchy of responses, if applicable; within a round I code category, the round II codes range from A, the most “correct”, and move towards less correct responses. However, when responses were about equally correct or equally incorrect the round II codes were assigned in order of frequency of response with code A representing the most popular response and moving towards less frequent responses. For some questions and round I code combinations no further refinement was required so all responses within that category were given the round II code A. For other combinations, students demonstrated a wide range of possible responses and codes A through F were used. All round II codes were generated directly from the student responses, without any reference to or bias from the themes of interest. For each questionnaire item the codes were generated again from scratch, but to aid in the coding process and to reduce confusion I attempted to keep them similar between different questionnaire items. Code 5 “left blank” is the only round I code with no round II subordinate code as it was never required for any questionnaire item. An example of the round I and round II codes is given in Table 4 for the responses to the questionnaire Item 8.

Questionnaire responses given a round I code “correct answer and correct reasoning” were separated into two groups: one group consisted of responses with the correct solution 90 (100 − 10), while another group consisted of solutions that included a subtraction error and answered 99. The response was still coded as correct, as the error is only mechanical and not an error of conceptual understanding or of the student’s chosen procedure. For this item there was no need to further divide the code 2 or code 3 responses. Code 4 was separated into five different groups, as there were a variety of incorrect answers with incorrect reasoning.
Table 4. Round II Questionnaire Codes - Sample Student Responses for Item 8

<table>
<thead>
<tr>
<th>Round I Code</th>
<th>Round II Code</th>
<th>Explanation of Code / Sample Student Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
<td>90 ((100 - 10)) (correct)</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>99 (error in subtraction)</td>
</tr>
<tr>
<td>2</td>
<td>A</td>
<td>100 (did not subtract)</td>
</tr>
<tr>
<td>3</td>
<td>A</td>
<td>100, 121, 144 ... (incomplete list)</td>
</tr>
<tr>
<td>4</td>
<td>A</td>
<td>All of them</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>An infinite amount</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>A random guess</td>
</tr>
<tr>
<td></td>
<td>D</td>
<td>9900 ((10,000 - 100))</td>
</tr>
<tr>
<td></td>
<td>E</td>
<td>4900 (\frac{10,000}{2} - \frac{100}{2})</td>
</tr>
<tr>
<td>5</td>
<td>No Code</td>
<td>Left blank</td>
</tr>
<tr>
<td>6</td>
<td>A</td>
<td>None given for this item</td>
</tr>
</tbody>
</table>

The round II coding was used to refine the responses, to gain insight into the frequency of a particular type of student response and to further identify themes that may be of interest. In order to determine the frequency of responses all the round II data was tallied and tabulated. The round II analysis showed that often a partially correct solution with partially correct reasoning could be very different conceptually from another solution that met the same round I criteria. This round also showed that for some items the types of difficulties were fewer but many students were affected by that difficulty, while other items displayed a large range of responses. This round allowed a much more detailed view of the types of responses for each questionnaire item but I desired the data to be combined and categorized to give a better indication of how the themes related to the students responses.

A table that includes full round I and II codes per Item is found in Appendix B, while a table of full coding results for each student questionnaire is found in Appendix C. Appendix D gives the frequency of the responses by item.
Round III Coding

Subsequent to the round II coding no further response refinement was warranted, rather I sought unification across questionnaire items according to different themes. I generated a comprehensive list of themes that became round III codes, each of which corresponded to either a theme discussed in Chapter 3, e.g. representation, or a newly identified theme from the previous round of coding. Each response group from round II, e.g. the group of all the responses coded 2A, was given a round III code if applicable. However the round III coding process was different in many respects from that of rounds I and II. In round III, some response groups were assigned two codes, as two areas of difficulty may be experienced and expressed simultaneously, while others were not assigned a round III code at all. For example a solution that is correct with the correct reasoning cannot yield much information regarding what thematic areas students may have difficulty with, and at the other extreme, solutions that are incorrect with incorrect reasoning, often are so confused that it becomes difficult for an outside observer to pinpoint the nature of the difficulty. Unlike the round II codes, these codes were not solely generated based on student response, but also based on the themes of interest, and responses were assessed for conformity to those themes. The round III codes are also not subordinate; viewing the round III code for a particular response without the round I or II codes still provides information about the nature of the difficulty while the round I and round II codes now provide the degree of the difficulty and the specifics of the response. Lastly, this round of coding was the most subjective. These codes targeted at the perceived areas of difficulty for students are given based only on the written work that the students have supplied in their solution, from this I can only make inferences about the nature of the difficulties encountered.

From the original comprehensive list of round III codes, I have narrowed the scope of interest to three. Two codes are from the list used to design the questionnaire while one new code was incorporated from the information gained while coding rounds I and II. The original list described in Section 3.2.1, contained distribution of square numbers, irrational numbers, prime factorization and representation. Representation remained the most significant theme of interest, as the questionnaire demonstrated a broad range of responses when dealing with opaque representation; it was also the most
commonly used round III code. Distribution of square numbers has been included as a round three code despite the fact that it is only apparent in the particular questionnaire item designed to investigate it. I designed additional questions for the clinical interviews that continue the investigation into distribution. Irrational numbers and prime factorization were both excluded as themes of interest at this stage because in both cases the results yielded an overwhelming lack of evidence of use. From all of the questionnaire responses there was only one instance of a student specifically using an irrational number. Some students did mention decimal numbers but this was still very uncommon. When specifically asked to do a calculation without the use of their calculator, some students used general factorization or prime factorization, but there was no evidence of students using these concepts in any other situation. For both of these themes designing enough clinical interview questions to gain sufficient insight into the topics as they relate to square numbers and square roots seemed likely to draw the investigation off course. The paucity of data led me to conclude that the topics of irrational numbers and prime factorization would be better investigated in another work.

The new theme, identified from the round I and round II coding analysis, became particularly significant and applicable to this investigation. This theme was denoted as confusion of concepts and appears in several varieties. In some cases solutions show errors that confuse the square number and the square root; the provided solution may have been correct had the question asked for the square root rather than the square number or vice versa. In other cases, students have confused square numbers with even numbers, or are more generally confused by square numbers and their properties. This theme was confined to responses that showed an attempt to explain and understand the confused reasoning; confusion was not a catch-all theme for all incorrect responses. This theme was apparent in the questionnaire data but also became prevalent during the clinical interviews; it is analyzed in more detail in Section 4.2.2. Table 5 shows the thematic themes that are of interest and coded for in round III.
Table 5. Round III Questionnaire Thematic Codes

<table>
<thead>
<tr>
<th>Round III Code</th>
<th>Name of Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>Con</td>
<td>Confusion</td>
</tr>
<tr>
<td>Dis</td>
<td>Distribution</td>
</tr>
<tr>
<td>Rep</td>
<td>Representation</td>
</tr>
</tbody>
</table>

These themes relate to the theoretical constructs, concept image and concept definition, representation and pattern spotting, used in the analysis to varying degrees. These themes describe some of the obstacles that students encounter while solving problems while the theoretical constructs are the lens through which these difficulties are viewed. The theme of representation and the construct of representation are closely linked and are equivalent for the purposes of this study. The theme of confusion relates closely to the theoretical construct of concept image and concept definition, as often the confusion that a student is experiencing can be explained through the construct of concept image. However, the distribution theme is not closely related to the theoretical constructs at all.

The questionnaire items were designed to investigate students’ understanding and ability to work with particular concepts, and as such most questionnaire items did not include responses that yielded all three round III codes. As I have excluded the codes that do not fall within the scope of this investigation many response categories do not have a round III thematic code. Table 6 shows an example of the round III coding for Item 8 of the questionnaire.

Here the themes square number confusion and the distribution of square numbers have come into play. Three groups of student response earned the code “con” for square number confusion. For example, student 17 responded: “they could all be perfect squares, because you can multiply them by themselves”. This solution showed confusion between square numbers and square roots as it would have been correct if the question had asked for the number of square roots in the interval. Another example, from student 21, stated: “perfect squares come in even numbers. \( \frac{100}{2} = 50 \) even numbers, \( \frac{10000}{2} = 5000 \) even numbers, \( 5000 - 50 = 4950 \).” This example clearly showed
both an error in understanding what square numbers are, but also how they are arranged throughout the interval. The student stated that all of the even numbers are square numbers and that they are uniformly distributed throughout the interval. This question is looked at in more detail in Section 4.1.2.

Table 6. Round III Questionnaire Codes - Sample Student Responses for Item 8

<table>
<thead>
<tr>
<th>Item 8: How many perfect squares are there between 100 and 10,000?</th>
<th>Round I &amp; II Codes</th>
<th>Explanation of Code / Sample Student Response</th>
<th>Round III / Thematic Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A 90, 100-10 (correct)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1B 99</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2A 100</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3A 100, 121, 144, ...</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4A All of them</td>
<td></td>
<td>Con</td>
<td></td>
</tr>
<tr>
<td>4B Random guess</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4C 9900 (10,000 − 100)</td>
<td></td>
<td>Con, Dis</td>
<td></td>
</tr>
<tr>
<td>4D 4900 ( \left( \frac{10,000}{2} \right) - \frac{100}{2} )</td>
<td></td>
<td>Con, Dis</td>
<td></td>
</tr>
<tr>
<td>5 Left blank</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6A Incomprehensible answer</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It is clear from this example how the thematic codes were more subjective and ambiguous than the codes from the previous two rounds, particularly when only one or two students supplied a particular response. The in-depth analysis of the relevant questionnaire items 1, 2, 3 and 8 is found in Section 4.1.2.

After this round of coding, I was satisfied and analyzed each questionnaire item in detail using all three rounds of coding.

These themes allowed me to adjust my research questions to specifically consider the obstacles surrounding; confusion of concepts, distribution and representation.
The research questions can be restated as:

1) What obstacles do students encounter when attempting to solve problems with square numbers and square roots?
   In particular to what degree does: confusion of concepts, distribution of square numbers and representation of square numbers and square roots, hamper students attempting to solve problems?

2) What strategies do students use to solve problems relating to square numbers and square roots?

4.1.2. Questionnaire Item Analysis

All nine of the questionnaire items were analyzed using the three rounds of coding discussed above, however only four of the questionnaire items are discussed here. I examine Items 1, 2, 3, and 8, while five questionnaire items, Item 4, 5, 6, 7, and 9 have been excluded from the analysis.

Item 4: Which of the following are perfect squares? Explain briefly.
   1) $13^2 \cdot 17^2 \cdot 23^2$
   2) $13^2 \cdot 17^2 \cdot 23$
   3) $13^2 \cdot 17^2 \cdot 9$

Item 4 has been omitted because a large number of students misinterpreted the question and answered a different related question instead. However, this item was still of great interest, so I included a similar questions in the clinical interviews. Items 7 and 9 have both been omitted because they concern concrete calculations that did not offer a great deal in the way of information about the current themes of interest developed in the round III coding. Item 5 was omitted because students overwhelmingly left it blank, or gave an overly simple response that did yield much information. Item 6 was omitted because most students did not explain their rationale.

For each questionnaire item, the analysis begins with a brief description of the question, its purpose and the correct solution. The detailed analysis then examines all 51 results and the respective round I, round II and round III codes. A table is given which shows all response types, the associated codes and the frequency of those responses as a percentage of the total responses. Following the table there is analysis of the relevant features of the responses and then any particularly notable responses
along with sample student responses of that type are given. This analysis focuses on the solutions that can provide insight into the thematic areas of interest developed in the round III coding. Lastly any links to other items are noted.

This analysis clearly shows the limitations of questionnaire data and the need for clinical interviews or other data collection methods. For each questionnaire item, most of the responses yield little information as noted above. A correct response shows that a student can supply the correct solution, but does not always give insight into their thought process or if the student experienced difficulties while pursuing the solution. A very incorrect answer shows that the student did not supply the correct solution, but usually cannot indicate the source of the difficulty. An incorrect solution may be due to: misunderstanding the question, misreading the question, having a wrong conceptual idea about the topic, having no knowledge of the topic or other reasons. In this case, the responses given on the questionnaire data that yield the richest information are usually those given a round I code of 2 or 3. These are the responses that show that students were thinking about their solutions and were close to arriving at the correct solution, but made an error somewhere. Usually these errors are more easily identified than the errors made when students give a completely incorrect answer.

**Item 1: For what values of $a$, if any, can the following be perfect squares?**

Item 1, shown in full below, had six parts all of which dealt with increasingly opaque representation. The different values of the exponent obscured the possibility that the expression could be a square number for some values of $a$; the representation moves from transparent to opaque. All the items in this series featured only one variable.
Item 1: For what values of \( \alpha \), if any, can the following be perfect squares?

1) \( \alpha^2 \)
2) \( \alpha^3 \)
3) \( \alpha^6 \)
4) \( \alpha^{100} \)
5) \( \alpha^{200} \)
6) \( 2 \cdot \alpha^2 \)

1.1. \( \alpha^2 \)

This was the first item on the questionnaire. The most general correct solution is “\( \alpha \) can be any integer”. This solution would show that students were attending to the types of numbers that could be the chosen as the value of \( \alpha \), and those that must be excluded. As discussed in Section 3.2.1, the word “value” was used intentionally rather than the word “number” to avoid biasing students in favour of thinking about only whole numbers. All student responses for this item are shown in Table 7 along with all three rounds of coding.

**Table 7. Coding Data for Questionnaire Item 1.1**

<table>
<thead>
<tr>
<th>Item 1.1: For what values of ( \alpha ), if any, can the following be perfect squares?</th>
<th>( \alpha^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Round I &amp; II Code</td>
<td>Explanation of Code / Sample Student Response</td>
</tr>
<tr>
<td>1A</td>
<td>Any integer (correct)</td>
</tr>
<tr>
<td>2A</td>
<td>Any whole number</td>
</tr>
<tr>
<td>2B</td>
<td>Any number</td>
</tr>
<tr>
<td>2C</td>
<td>Gives example only</td>
</tr>
<tr>
<td>2D</td>
<td>“Yes”, no reason</td>
</tr>
<tr>
<td>3A</td>
<td>Gives the square numbers</td>
</tr>
<tr>
<td>4A</td>
<td>“Can be square if it produces a natural number”</td>
</tr>
<tr>
<td>5</td>
<td>Left blank</td>
</tr>
<tr>
<td>6A</td>
<td>Incomprehensible answer</td>
</tr>
</tbody>
</table>

As mentioned above most of the student responses do not contribute to the themes of interest. For this item it is noteworthy that no students successfully restricted
the domain to the integers, however 29 students of the 51 were aware that the domain must be limited. The majority of students did well on this item, even if they left out the possibility of negative integers or only gave an example. The round III code of note for Item 1.1 is “con”; confusion of concepts of square numbers affected two students. The solution “this can be a perfect square if it produces a natural number” (Student 40) showed confusion because the given solution $a^2 = \text{natural number}$, only makes sense if we think of the square root being a natural number. However in this case there is a great deal of ambiguity in the response given that it was supplied by only one person and it could easily be attributed to a language problem, or a problem in their ability to explain themselves. A similar explanation can be given for the student who listed the square numbers.

1.2. $a^3$

In this problem, the representation of the possibility of a square number is opaque. It may not be difficult for students to find the square root of $a^3$ in a concrete mechanical problem if they are given a value for $a$, but this problem asks them to indicate when and if it is possible for $a^3$ to be a square number. The correctly generalized solution to this problem is “if $a$ is a square number”. Table 8 lists all of the responses to this questionnaire item and the accompanying codes.

It is abundantly clear that the opaque representation of $a^3$ is a problem for students. Not one student gave the correctly generalized solution of “If $a$ is a square number”. However eight students were able to give a correct example that showed when this could be the case. Of those eight students one gave the example $a = 1$, while the other seven students all gave the example $a = 4$. None of those eight students were able to supply more than one example, and there is evidence that some of those students used a brute force calculation method to run through the natural numbers beginning with $a = 2$, until they were successful with $a = 4$. 
Table 8. Coding Data for Questionnaire Item 1.2

<table>
<thead>
<tr>
<th>Round I &amp; II Code</th>
<th>Explanation of Code / Sample Student Response</th>
<th>Frequency n = 51 %</th>
<th>Round III / Thematic Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A</td>
<td>α is a square number (correct)</td>
<td>0 0</td>
<td>—</td>
</tr>
<tr>
<td>2A</td>
<td>Gives example only</td>
<td>8 16</td>
<td>—</td>
</tr>
<tr>
<td>3A</td>
<td>“No”, “it is a cube” or “it is a volume”</td>
<td>10 20</td>
<td>Rep</td>
</tr>
<tr>
<td>3B</td>
<td>“No”, odd exponent</td>
<td>5 10</td>
<td>Rep</td>
</tr>
<tr>
<td>3C</td>
<td>“No”, “not the formula for a square”</td>
<td>4 8</td>
<td>Rep</td>
</tr>
<tr>
<td>4A</td>
<td>Any number</td>
<td>3 6</td>
<td>—</td>
</tr>
<tr>
<td>4B</td>
<td>“No”, no reason</td>
<td>2 4</td>
<td>—</td>
</tr>
<tr>
<td>4C</td>
<td>Gives incorrect example</td>
<td>4 8</td>
<td>—</td>
</tr>
<tr>
<td>4D</td>
<td>Gives example to disprove</td>
<td>4 8</td>
<td>—</td>
</tr>
<tr>
<td>4E</td>
<td>Other</td>
<td>4 6</td>
<td>—</td>
</tr>
<tr>
<td>5</td>
<td>Left blank</td>
<td>5 10</td>
<td>—</td>
</tr>
<tr>
<td>6A</td>
<td>Incomprehensible answer</td>
<td>2 4</td>
<td>—</td>
</tr>
</tbody>
</table>

19 students believed that \( a^3 \) could not ever be a square number. Their reasoning can be divided into three categories that are worded differently but all demonstrate that the representation of the number is the issue. Ten students said that \( a^3 \) could not be a square number because it is a cubic number or because it is a “volume”. It is interesting to note the either / or nature of this description; a number can be either a square number or it can be a cubic number but it cannot be both. Zazkis and Gadowsky (2001) report the same phenomenon, that is, students claiming that \( 36^2 \) could not be a perfect square because it is a perfect cube. Here, the concept image that these students harboured limited numbers to being either perfect squares or perfect cubes but not both. This belief may be reinforced by the common examples that often use either very small numbers or prime numbers as the base when working with square numbers or other exponents; these examples would rarely apply to both situations. Five students attended to the exponent in the expression and looked for evenness. Four of these five students used this criterion, evenness of exponent, throughout the entire series of Item 1 questions. A
further four students claimed that it could not be a square number because it was not the formula for a square. Clearly all of these responses point to students being unable to see other options than the given representation of the problem.

1.3. $a^6$

This item in the series was designed to investigate whether or not students would attend to the evenness of the exponent. 6 was chosen as the exponent rather than 4 to avoid confusion around the fact that 4 is itself a square number. The correctly generalized solution to this problem was accepted as: “if $a$ is an integer”. A more comprehensive generalization would be “if $a$ is the cube root of an integer” but for this level of student, this advanced generalization was not required to constitute a correct answer. Table 9 shows the fully coded responses to this question.

### Table 9. Coding Data for Questionnaire Item 1.3

<table>
<thead>
<tr>
<th>Round I &amp; II Code</th>
<th>Explanation of Code / Sample Student Response</th>
<th>Frequency</th>
<th>%</th>
<th>Round III / Thematic Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A</td>
<td>Any integer (correct)</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2A</td>
<td>Any whole number</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2B</td>
<td>Any number, even exponent</td>
<td>11</td>
<td>22</td>
<td></td>
</tr>
<tr>
<td>2C</td>
<td>Gives example only</td>
<td>15</td>
<td>29</td>
<td></td>
</tr>
<tr>
<td>2D</td>
<td>“Yes”, no reason</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3A</td>
<td>“No”, wrong exponent</td>
<td>5</td>
<td>10</td>
<td>Rep</td>
</tr>
<tr>
<td>4A</td>
<td>“No”, no reason</td>
<td>3</td>
<td>6</td>
<td>Rep</td>
</tr>
<tr>
<td>4B</td>
<td>Gives incorrect example</td>
<td>2</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>4C</td>
<td>Other</td>
<td>3</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Left blank</td>
<td>6</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>6A</td>
<td>Incomprehensible answer</td>
<td>4</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

Again, no participants completely generalized their solution to $a$ being an integer, however since they did not do so for Item 1.1 it is not surprising that they did not do so
Here. Like the previous item numerous participants attended specifically to the evenness of the exponent, in this case 11 students. 15 participants gave an example; the examples ranged from $a = 1$ to $a = 6$; $a = 2$ was the most common example given. Five students claimed that $a^6$ could not be a square number because the exponent was not “2”, while three students responded with “no”, but did not explain why they believed that this expression could not be a square number. These three participants all responded to Item 1.1 in a correct or partially correct manner which leads to the conclusion that the opaque representation of the expression as a square allowed these students to claim that $a^6$ cannot be a square number.

As mentioned in Chapter 3, these students have extensive experience with the exponent laws and are very comfortable with simplifying expression with exponents such as $(a^2)^3 = a^6$ and $(a^3)^2 = a^6$. This item showed, however, that these students did not make use of this knowledge when solving problems without being explicitly prompted to do so.

As the problems in this series became more difficult, a greater number of students left the items blank, gave an example rather than a generalized solution, or gave explanations that were incoherent.

1.4. $a^{100}$

This item has a large even exponent that is itself a square number. This item was chosen in particular to contrast with the subsequent item, $a^{200}$, which has a large even exponent that is not a square number. The accepted generalized answer for this problem, and students at this grade level, is “if $a$ is an integer”. Table 10 gives all the solutions and codes for this item.

This item clearly showed a much wider range of student responses. 12 of the 51 participants left this question blank. 7 students attended to the evenness of the exponent in this problem. One participant responded that any whole number was eligible but did not give her reasoning, while another responded with “yes” again with no reasoning. Five students responded with a primitive example, two of those students used the number $a = 2$ and three students used $a = 1$. 
Table 10. Coding Data for Questionnaire Item 1.4

<p>| Item 1.4: For what values of $a$, if any, can the following be perfect squares? $a^{100}$ |
|------------------------------------------|-----------------|-----------------|-----------------|</p>
<table>
<thead>
<tr>
<th>Round I &amp; II Code</th>
<th>Explanation of Code / Sample Student Response</th>
<th>Frequency n = 51</th>
<th>%</th>
<th>Round III / Thematic Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A</td>
<td>Any integer (correct)</td>
<td>0</td>
<td>0</td>
<td>—</td>
</tr>
<tr>
<td>2A</td>
<td>Any whole number</td>
<td>1</td>
<td>2</td>
<td>—</td>
</tr>
<tr>
<td>2B</td>
<td>Any number, even exponent</td>
<td>7</td>
<td>14</td>
<td>—</td>
</tr>
<tr>
<td>2C</td>
<td>Gives example only</td>
<td>5</td>
<td>10</td>
<td>—</td>
</tr>
<tr>
<td>2D</td>
<td>“Yes”, no reason</td>
<td>1</td>
<td>2</td>
<td>—</td>
</tr>
<tr>
<td>2E</td>
<td>“$10\times10 = 100$”</td>
<td>3</td>
<td>6</td>
<td>Con, Rep</td>
</tr>
<tr>
<td>2F</td>
<td>Guess</td>
<td>5</td>
<td>10</td>
<td>—</td>
</tr>
<tr>
<td>3A</td>
<td>“No”, wrong exponent</td>
<td>4</td>
<td>8</td>
<td>Rep</td>
</tr>
<tr>
<td>4A</td>
<td>“No”, no reason</td>
<td>7</td>
<td>14</td>
<td>Rep</td>
</tr>
<tr>
<td>4B</td>
<td>Gives incorrect example</td>
<td>0</td>
<td>0</td>
<td>—</td>
</tr>
<tr>
<td>4C</td>
<td>Other</td>
<td>2</td>
<td>4</td>
<td>—</td>
</tr>
<tr>
<td>5</td>
<td>Left blank</td>
<td>12</td>
<td>24</td>
<td>—</td>
</tr>
<tr>
<td>6A</td>
<td>Incomprehensible answer</td>
<td>3</td>
<td>6</td>
<td>—</td>
</tr>
<tr>
<td>6B</td>
<td>Contradicts self in answer</td>
<td>1</td>
<td>2</td>
<td>—</td>
</tr>
</tbody>
</table>

Especially noteworthy here are the responses that showed how the representation of $a^{100}$ had obscured the square number for students. Four students replied that it could not be a square number because the “exponent was wrong”, while seven students replied “no” without giving any reasoning. In total 11 students did not believe that the expression could be a square number based on its representation. Interestingly three students replied that it could be a square number because $10 \cdot 10 = 100$ or because $100$ itself was a square number. This type of reasoning also showed confusion about square numbers and square roots, since these students were taking the square root of the exponent.

1.5. $a^{200}$

This item is meant to contrast with the previous item; it features a large even exponent that is itself not a square number. The correctly generalized solution that
expected from this population is “$\alpha$ is any integer”. A more comprehensive generalization was not expected at this level. Table 11 gives all the solutions and codes for this item.

Table 11. Coding Data for Questionnaire Item 1.5

<table>
<thead>
<tr>
<th>Round I &amp; II Code</th>
<th>Explanation of Code / Sample Student Response</th>
<th>Frequency n = 51</th>
<th>%</th>
<th>Round III / Thematic Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A</td>
<td>Any integer</td>
<td>0</td>
<td>0</td>
<td>—</td>
</tr>
<tr>
<td>2A</td>
<td>Any whole number</td>
<td>1</td>
<td>2</td>
<td>—</td>
</tr>
<tr>
<td>2B</td>
<td>Any number, even exponent</td>
<td>6</td>
<td>12</td>
<td>—</td>
</tr>
<tr>
<td>2C</td>
<td>Gives example only</td>
<td>5</td>
<td>10</td>
<td>—</td>
</tr>
<tr>
<td>2D</td>
<td>“Yes”, no reason</td>
<td>4</td>
<td>8</td>
<td>—</td>
</tr>
<tr>
<td>2E</td>
<td>“200 is like 2”</td>
<td>3</td>
<td>6</td>
<td>Con, Rep</td>
</tr>
<tr>
<td>2F</td>
<td>Guess</td>
<td>4</td>
<td>8</td>
<td>—</td>
</tr>
<tr>
<td>3A</td>
<td>“No”, wrong exponent</td>
<td>4</td>
<td>8</td>
<td>Rep</td>
</tr>
<tr>
<td>4A</td>
<td>“No”, no reason</td>
<td>4</td>
<td>8</td>
<td>Rep</td>
</tr>
<tr>
<td>4B</td>
<td>Gives incorrect example</td>
<td>0</td>
<td>0</td>
<td>—</td>
</tr>
<tr>
<td>4C</td>
<td>Other</td>
<td>3</td>
<td>6</td>
<td>—</td>
</tr>
<tr>
<td>5</td>
<td>Left blank</td>
<td>13</td>
<td>25</td>
<td>—</td>
</tr>
<tr>
<td>6A</td>
<td>Incomprehensible answer</td>
<td>3</td>
<td>6</td>
<td>—</td>
</tr>
<tr>
<td>6B</td>
<td>Contradicts self in answer</td>
<td>1</td>
<td>2</td>
<td>—</td>
</tr>
</tbody>
</table>

This item is very similar the previous in terms of student response. 13 students left the question blank. Six participants explicitly attended to the evenness of the exponent to supply the correct solution. One student replied with “any whole number” and four students replied with “yes” with no explanation. As these five students do not explain their rationale, I cannot conclude that they have fully understood the problem, only that they supplied the correct solution, and may have guessed. Four students replied with “no” and gave no explanation while four other students gave no as their answer because the “exponent is wrong”. These eight students are not attending to possible alternate representations of this problem. However, as in the previous problem,
three students claim that the expression can give a square number, but give an odd reason why. In this case three students suggest that \( a^{200} \) can be a square number because “200 is like 2”. The representation in this case is somewhat less opaque than some of the previous examples and leads students to the correct answer, but it is clear that students are not specifically attending to the evenness of the exponent; if “200 is like 2” how might these students respond to \( a^{300} \)? Interestingly none of the three students who answered this problem with “200 is like 2” are the same as any of the three students who replied to the previous problem with “\( 10 \cdot 10 = 100 \)”. 

1.6. \( 2 \cdot a^2 \)

This was the last item of this series of questions. All the questions in this series have featured only one variable, but this is the only question to feature a constant as well. This problem was designed to investigate how students attend to the constant and if they would work with the constant to generalize the solution. It was expected that students at this level would generalize the problem to “it is not possible”, rather than give the fully generalized expression for \( a, a = \frac{\text{integer}}{\sqrt{2}} \). However this solution remained as the correct code 1 solution in order to capture if any students would use an irrational number as part of their solution. Table 12 shows all of the student responses to this problem along with the three rounds of coding.

In the response to this item no participants ever used the term irrational number, however 8 participants argued that \( 2 \cdot a^2 \) could never be a square number because of the 2 in front, e.g. “it is not a perfect square, \( \sqrt{a} = a\sqrt{2} \)” (Student 6) and “No, it will not because the number won’t be equal to a perfect square since 2 does not have an exponent” (Student 24). The representation in this problem is fairly transparent, but it is crucial to recognize that the exponent does not apply to the constant. Five students gave the response “any number”; they were clearly not attending to the constant in the expression and only regarding the exponent as significant.
Table 12. Coding Data for Questionnaire Item 1.6

<table>
<thead>
<tr>
<th>Item 1.6: For what values of ( \alpha ), if any, can the following be perfect squares? ( 2 \cdot \alpha^2 )</th>
<th>Explanation of Code / Sample Student Response</th>
<th>Frequency n = 51</th>
<th>%</th>
<th>Round III / Thematic Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>Round I &amp; II Code</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1A</td>
<td>If ( \alpha = \frac{\text{integer}}{\sqrt{2}} ) (correct)</td>
<td>0</td>
<td>0</td>
<td>—</td>
</tr>
<tr>
<td>2A</td>
<td>“No”, “because of the 2”</td>
<td>8</td>
<td>16</td>
<td>—</td>
</tr>
<tr>
<td>2B</td>
<td>“No”, with example</td>
<td>3</td>
<td>6</td>
<td>—</td>
</tr>
<tr>
<td>2C</td>
<td>“No”, no reason</td>
<td>13</td>
<td>25</td>
<td>—</td>
</tr>
<tr>
<td>3A</td>
<td>Not applicable</td>
<td>0</td>
<td>0</td>
<td>—</td>
</tr>
<tr>
<td>4A</td>
<td>Any number, divide first</td>
<td>2</td>
<td>4</td>
<td>—</td>
</tr>
<tr>
<td>4B</td>
<td>Any number</td>
<td>1</td>
<td>2</td>
<td>Rep</td>
</tr>
<tr>
<td>4C</td>
<td>Any number, 2 is irrelevant</td>
<td>4</td>
<td>8</td>
<td>Rep</td>
</tr>
<tr>
<td>4D</td>
<td>Gives incorrect example</td>
<td>4</td>
<td>8</td>
<td>—</td>
</tr>
<tr>
<td>5</td>
<td>Left blank</td>
<td>9</td>
<td>19</td>
<td>—</td>
</tr>
<tr>
<td>6A</td>
<td>Incomprehensible answer</td>
<td>3</td>
<td>6</td>
<td>—</td>
</tr>
<tr>
<td>6B</td>
<td>Gives an example, not sure if proving or disproving</td>
<td>4</td>
<td>8</td>
<td>—</td>
</tr>
</tbody>
</table>

Item 2: For what values of \( x \) and \( y \), if any, can the following be perfect squares?

Item 2 comprises three questions that were designed as a series. All three questions feature two variables. This item was designed to investigate whether students would attend to the prime factorization and use the idea that factors of the whole expression may be found within either term \( x \) or \( y \). Once again the term “values” was chosen rather than the term “number” to avoid biasing the participants towards whole numbers. I have excluded the prime factorization as theme, but still analyzed these items and included them here, because of insight gained into student difficulties with both confusion of concepts and representation.
Item 2: For what values of $x$ and $y$, if any, can the following be perfect squares?
1) $x \cdot y$
2) $x \cdot y^2$
3) $x^2 + y^2$

2.1. $x \cdot y$

This is the simplest expression of two variables in this series. The correct generalized solution would allow for any division of factors between $x$ and $y$, and may be stated as the following “if $x$ and $y$ are each factored and the combined expression yields only pairs of factors”. However, while such generalization does not rely on advanced knowledge, I expected only a few students would give this solution and therefore the solutions “if $x$ and $y$ are the same” or “if $x$ and $y$ are both perfect squares” were also accepted as correct and were coded 1B and 1C respectively. Table 13 gives all the responses for this item and the three rounds of coding.

Table 13. Coding Data for Questionnaire Item 2.1

<table>
<thead>
<tr>
<th>Round I &amp; II Code</th>
<th>Explanation of Code / Sample Student Response</th>
<th>Frequency</th>
<th>%</th>
<th>Round III / Thematic Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A</td>
<td>All factors paired in expression (correct)</td>
<td>0</td>
<td>0</td>
<td>—</td>
</tr>
<tr>
<td>1B</td>
<td>$x$ and $y$ are the same</td>
<td>13</td>
<td>25</td>
<td>—</td>
</tr>
<tr>
<td>1C</td>
<td>$x$ and $y$ are both square numbers</td>
<td>2</td>
<td>4</td>
<td>—</td>
</tr>
<tr>
<td>2A</td>
<td>Gives example, $x$ and $y$ are different</td>
<td>1</td>
<td>2</td>
<td>—</td>
</tr>
<tr>
<td>2B</td>
<td>Gives example, $x$ and $y$ are the same</td>
<td>11</td>
<td>22</td>
<td>—</td>
</tr>
<tr>
<td>3A</td>
<td>“Yes”, no reason</td>
<td>2</td>
<td>4</td>
<td>—</td>
</tr>
<tr>
<td>4A</td>
<td>“No”, “it makes a rectangle”</td>
<td>2</td>
<td>4</td>
<td>Rep</td>
</tr>
<tr>
<td>4B</td>
<td>“No”, “$x$ cannot equal $y$”</td>
<td>4</td>
<td>8</td>
<td>Rep</td>
</tr>
<tr>
<td>4C</td>
<td>Gives incorrect example</td>
<td>1</td>
<td>2</td>
<td>—</td>
</tr>
<tr>
<td>4D</td>
<td>“No”, other incorrect reasons</td>
<td>8</td>
<td>16</td>
<td>—</td>
</tr>
<tr>
<td>5</td>
<td>Left blank</td>
<td>7</td>
<td>14</td>
<td>—</td>
</tr>
<tr>
<td>6A</td>
<td>Incomprehensible answer</td>
<td>0</td>
<td>0</td>
<td>—</td>
</tr>
</tbody>
</table>
For this problem students did not fully generalize to the case of factors of the final square number existing in both variables. Instead students who gave general solutions, rather than examples, generalized to specific cases of the problem. 13 students generalized the problem for the case when $x$ is equal to $y$, while two students generalized to the case where $x$ and $y$ are each square numbers. Of the students who gave examples for their solution to this problem 11 students gave examples where $x$ was equal to $y$, while only one student gave an example where $x$ was not equal to $y$; “$x = 2, y = 8, x \cdot y = 16 = \text{perfect square}$” (Student 41). Eight students replied with “no”, but gave a variety of incorrect reasons including “No, because neither are perfect squares” (Student 42), and “Multiplying two numbers together just sum up a number, not a number that is a perfect square” (Student 24). Four students responded with the curious solution that “no it can’t because $x$ cannot equal $y$”, while two students replied “no because it makes a rectangle”. These last two responses show a belief that the two variables could not represent the same value. The representation of this problem allows any values for $x$ and $y$, and particular values may be chosen. These students seem confused by acceptability of choosing values that allow the expression to satisfy certain conditions.

2.2. $x \cdot y^2$

This question was the second in the series of problems with two variables. The question is still attempting to see if students will attend to factorization and the fact that perfect squares contain pairs of factors. The solution can be worded again “If $x$ and $y$ are fully factored, and the combined expression of $x \cdot y^2$ contains only pairs of factors”. If a participant is attending to prime factorization, this problem is not much different than the previous in terms of generalization. However if a participant is not attending to factorization, many examples can easily be found of special cases, but attempting to generalize from those examples is quite difficult. Based on their lack of use of factorization, it was unlikely that any student would be able to supply the fully general solution to this problem, so all the cases were treated separately. Table 14 gives all the student responses to this item along with all rounds of coding.
Table 14. Coding Data for Questionnaire Item 2.2

<table>
<thead>
<tr>
<th>Item 2.2:</th>
<th>For what values of $x$ and $y$, if any, can the following be perfect squares? $x \cdot y^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Round I &amp; II Code</td>
<td>Explanation of Code / Sample Student Response</td>
</tr>
<tr>
<td>1A</td>
<td>All factors paired in expression (correct)</td>
</tr>
<tr>
<td>1B</td>
<td>$x$ is a square number, $y$ any integer</td>
</tr>
<tr>
<td>1C</td>
<td>$x = 1$</td>
</tr>
<tr>
<td>1D</td>
<td>$x = y^2$</td>
</tr>
<tr>
<td>2A</td>
<td>Gives example, $x = 1$</td>
</tr>
<tr>
<td>2B</td>
<td>Gives example, $x = y^2$</td>
</tr>
<tr>
<td>2C</td>
<td>Gives example, $x$ is a square number</td>
</tr>
<tr>
<td>3A</td>
<td>Not applicable</td>
</tr>
<tr>
<td>4A</td>
<td>“No”, “wrong exponent”</td>
</tr>
<tr>
<td>4B</td>
<td>“No”, no reason</td>
</tr>
<tr>
<td>4C</td>
<td>Gives incorrect example</td>
</tr>
<tr>
<td>4D</td>
<td>“No”, other incorrect reasons</td>
</tr>
<tr>
<td>5</td>
<td>Left blank</td>
</tr>
<tr>
<td>6A</td>
<td>Incomprehensible answer</td>
</tr>
</tbody>
</table>

There was great difficulty in generalizing the solution to this problem, but many students either generalized a special case of the problem, or gave an example that fit into a special case. Three students generalized the solution to “$x$ must be a square number, and $y$ a whole number”, and one student gave an example that fit into this category. Four students simplified the problem by setting $x$ equal to 1, in which case the problem reduces to that of Item 1.1 and these students allowed $y$ to be any number. Three students gave an example of this style, where $x$ is equal to 1. Three more students gave the solution that $x$ must be equal to $y^2$, and again four other students offered examples that fit into this pattern. All of the examples offered make use of very small numbers, overwhelmingly 2 and 4 with one example using a 5 and one example a 7. There is good reasoning shown here and only nine students left this problem blank.
The opaque representation of the expression is a factor for the two students that claim the expression cannot be a perfect square because the exponent is wrong. These two students are referring to the exponent on the $x$. Interestingly one of these students (Student 20) correctly generalized the previous problem with this solution “any natural # that are the same (i.e. $2 \cdot 2 = 4, 50 \cdot 50 = 2500$)” but gave this as the solution for this problem “None that I can think of because $y^2$ = perfect but multiplying $x$ would mess it up.” I am not sure why this participant did not use the same logic on the second problem as on the first.

2.3. $x^2 + y^2$

This problem is the last of the series of items with two variables, but it is not directly attempting to investigate factoring. This problem is attempting to investigate a style of representation problem that concerns familiarity. I wished to investigate whether students would recognize this as the Pythagorean theorem in an alternative format. Students have a great deal of practice with the Pythagorean theorem, but it is almost exclusively represented as: $a^2 + b^2 = c^2$ rather than $x^2 + y^2 = z^2$, or any other representation. The solution to this problem is “when $x$ and $y$ are the two smaller values of any Pythagorean triple”, however, I did not expect students to use the term Pythagorean triple, and so a solution was considered correct if it gave more than one non-trivial example. Table 15 gives the solutions to this question along with all the rounds of coding.

No student gave the generalized the solution to this problem. Five students were able to supply an example that would satisfy the problem; the example in all cases was $x = 3$ and $y = 4$. This is a simple example that is frequently used during trigonometry and learning the Pythagorean theorem so perhaps the students remembered if from there. However none of these students mentioned the theorem at all, or even the fact that is may be possible with other values. In this case it is very difficult to tell how much these students are aware of other options. 14 students left the question blank. 11 students gave examples that were incorrect; they either made an error in the multiplication or the addition of the problem. Three students made the claim that the expression can’t be a square number “because you can’t add powers” these students are basing their argument on the fact that there is no power rule to allow you to always
simplify adding powers. The representation of the problem as an addition problem that cannot be simplified has obscured the possibility of the solution being a square number for particular values of \( x \) and \( y \). Mason and Pimm (1984) describe how students are often confused about the rules that govern general examples or theorems when they attempt to apply those rules to a particular example. A rule cannot be proven using just one example, but even if there is no rule that applies to all examples, a rule for a particular example may still exist. In this case the students understood that you can’t simplify powers when adding in a general sense, but were not attending the possibility of adding particular numbers.

Table 15. Coding Data for Questionnaire Item 2.3

<table>
<thead>
<tr>
<th>Round I &amp; II Code</th>
<th>Explanation of Code / Sample Student Response</th>
<th>Frequency n = 51</th>
<th>%</th>
<th>Round III / Thematic Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A</td>
<td>Pythagorean pairs or Gives more than one non-trivial example (correct)</td>
<td>0</td>
<td>0</td>
<td>—</td>
</tr>
<tr>
<td>2A</td>
<td>Gives example only</td>
<td>5</td>
<td>10</td>
<td>—</td>
</tr>
<tr>
<td>2B</td>
<td>“No”, “you can’t add powers”</td>
<td>3</td>
<td>6</td>
<td>Rep</td>
</tr>
<tr>
<td>2C</td>
<td>( x = 0 ) or ( y = 0 )</td>
<td>1</td>
<td>2</td>
<td>—</td>
</tr>
<tr>
<td>3A</td>
<td>Not applicable</td>
<td>0</td>
<td>0</td>
<td>—</td>
</tr>
<tr>
<td>4A</td>
<td>“No”, “it is a rectangle”</td>
<td>1</td>
<td>2</td>
<td>—</td>
</tr>
<tr>
<td>4B</td>
<td>“Yes”, any number</td>
<td>7</td>
<td>14</td>
<td>Rep</td>
</tr>
<tr>
<td>4C</td>
<td>Gives incorrect example</td>
<td>11</td>
<td>22</td>
<td>—</td>
</tr>
<tr>
<td>4D</td>
<td>“Both ( x^2 ) and ( y^2 ) are already”</td>
<td>4</td>
<td>8</td>
<td>Rep</td>
</tr>
<tr>
<td>4E</td>
<td>Other</td>
<td>5</td>
<td>10</td>
<td>—</td>
</tr>
<tr>
<td>5</td>
<td>Left blank</td>
<td>14</td>
<td>27</td>
<td>—</td>
</tr>
<tr>
<td>6A</td>
<td>Incomprehensible answer</td>
<td>0</td>
<td>0</td>
<td>—</td>
</tr>
</tbody>
</table>

The representation of this problem also led students to the opposite conclusion; Student 2 responded with “Yes, this can be a perfect square as long as the exponent is 2 it is a perfect square.” This participant was not attending to the addition in the problem.
at all, so in a sense had the opposite problem as the three students above. Here the representation showing an exponent 2 may have been blinding the students to other information given in the problem. Another four students also pointed out that $x^2$ and $y^2$ are squares already, but it is unclear what claim they were making, if any, about the sum of two square numbers.

**Item 3: Consider $36^2$, $36^3$, $36^4$, $36^5$, $36^6$, $36^7$. Circle the perfect squares.**

This problem was borrowed from Zazkis and Gadowsky (2001) and was chosen to investigate the representation of square numbers by examining whether students would attend to the base of this series of numbers, or only attend to the exponents. In this case 36 is both large enough that students would have trouble working through the problem mechanically, but small enough that most students know that it is a square number. The solution to this problem is to circle all of the expressions because 36 is a square number. Table 16 gives the solutions to this problem along with all the three rounds of coding.

**Table 16. Coding Data for Questionnaire Item 3**

<table>
<thead>
<tr>
<th>Round I &amp; II Code</th>
<th>Explanation of Code / Sample Student Response</th>
<th>Frequency $n = 51$</th>
<th>%</th>
<th>Round III / Thematic Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A</td>
<td>All circled with correct reasoning (correct)</td>
<td>1</td>
<td>2</td>
<td>—</td>
</tr>
<tr>
<td>2A</td>
<td>All circled with incorrect reasoning or no reasoning</td>
<td>1</td>
<td>2</td>
<td>—</td>
</tr>
<tr>
<td>3A</td>
<td>Power 2 circled</td>
<td>19</td>
<td>37</td>
<td>Rep</td>
</tr>
<tr>
<td>3B</td>
<td>Power 2, 4, 6 circled</td>
<td>17</td>
<td>33</td>
<td>Rep</td>
</tr>
<tr>
<td>3C</td>
<td>Power 2, 4 circled</td>
<td>3</td>
<td>6</td>
<td>Con, Rep</td>
</tr>
<tr>
<td>3D</td>
<td>Other, with reasoning</td>
<td>1</td>
<td>2</td>
<td>—</td>
</tr>
<tr>
<td>4A</td>
<td>Other (no reason)</td>
<td>4</td>
<td>8</td>
<td>—</td>
</tr>
<tr>
<td>5</td>
<td>Left blank</td>
<td>4</td>
<td>8</td>
<td>—</td>
</tr>
<tr>
<td>6A</td>
<td>Incomprehensible answer</td>
<td>1</td>
<td>2</td>
<td>—</td>
</tr>
</tbody>
</table>
This problem showed very clearly how opaque representation can influence students’ conclusions. Only two participants from the group of 51 circled every expression in the list. 19 students circled only the expression \(36^2\). Another 17 students circled the expressions with even exponents, and 3 students circled the expressions whose exponents were themselves powers of 2. Only 4 participants left this question blank. Together 39 students of the 51 participants, 76% of the participants attended to the exponent in this problem without attending to the base. Of all the responses to this problem only one response, coded 1A above, mentioned the base of the expression in his response (Student 50). These numbers were very surprising, and worth further investigation during the clinical interviews.

The three participants who circled the expressions with powers of 2 as the exponents, also show a degree of square number confusion. These participants left out the expression with the power of 6. Student 43 gave the explanation “Because those numbers can be \(\sqrt{\cdot}\) to a whole number” [indicating the exponents]. One of these three students also gave incorrect answer to Item 1.4 \((a^6)\) and gave the response “no, because it is not a multiple of four”. Another of these three students left Item 1.4 blank but the third answered Item 1.4 correctly. There is clearly some confusion surrounding square roots and exponents of this problem.

**Item 8: How many perfect squares are there between 100 and 10,000?**

This question was the only one on the questionnaire to deal explicitly with the distribution of square numbers. The problem also investigates whether the students would link the square numbers to their square roots. The problem was designed to be easy to calculate without the use of a calculator, but difficult to solve in another manner. The interval is too long to list the square numbers, and the distribution of square numbers prohibits calculating the number in a smaller interval and then multiplying. The appropriate solution to this question is 90 obtained by subtracting the square root of 100 from the square root of 10,000. Table 17 gives the students responses to this problem along with the three rounds of coding.

This was one of the few concrete calculation problems on the questionnaire. However it is apparent that it was difficult for some students. 21 students did not attempt this problem. Another 12 offered guesses as their solutions; in this case students
explicitly using the word “guess” in their solution or including a question mark on their paper indicated guesses. Together that is 65% of respondents who had major difficulty with the problem.

Table 17. Coding Data for Questionnaire Item 8

<table>
<thead>
<tr>
<th>Round I &amp; II Code</th>
<th>Explanation of Code / Sample Student Response</th>
<th>Frequency n = 51</th>
<th>%</th>
<th>Round III / Thematic Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A</td>
<td>90, 100-10 (correct)</td>
<td>5</td>
<td>10</td>
<td>—</td>
</tr>
<tr>
<td>1B</td>
<td>99</td>
<td>1</td>
<td>2</td>
<td>—</td>
</tr>
<tr>
<td>2A</td>
<td>100</td>
<td>4</td>
<td>8</td>
<td>—</td>
</tr>
<tr>
<td>3A</td>
<td>100, 121, 144, ...</td>
<td>2</td>
<td>4</td>
<td>—</td>
</tr>
<tr>
<td>4A</td>
<td>All of them</td>
<td>1</td>
<td>2</td>
<td>Con</td>
</tr>
<tr>
<td>4B</td>
<td>Guess</td>
<td>12</td>
<td>24</td>
<td>—</td>
</tr>
<tr>
<td>4C</td>
<td>9900 ((10,000 - 100))</td>
<td>2</td>
<td>4</td>
<td>Con, Dis</td>
</tr>
<tr>
<td>4D</td>
<td>(4900 \left(\frac{10,000}{2} - \frac{100}{2}\right))</td>
<td>3</td>
<td>6</td>
<td>Con, Dis</td>
</tr>
<tr>
<td>5</td>
<td>Left blank</td>
<td>21</td>
<td>41</td>
<td>—</td>
</tr>
<tr>
<td>6A</td>
<td>Incomprehensible answer</td>
<td>0</td>
<td>0</td>
<td>—</td>
</tr>
</tbody>
</table>

However five students did supply the correct answer, and another student simply made a subtraction error. Four students had a portion of the work correct but did not subtract at all. It is probable that these students simply forgot, but it is not possible to know for certain. Another two students began listing the perfect squares but realized that they could not get the solution in this manner; they were somewhat aware of the of the nature of the problem.

As mentioned in Section 4.1.1, two groups of student response demonstrated square number confusion. Student 17 responded, “they could all be perfect squares, because you can multiply them by themselves”; this solution would have been correct if the question had asked for the number of square roots of perfect squares in the interval. The same argument could be made for the group of responses that gave 9900 as the number of perfect squares. This response points to both confusion related to square
numbers and an interesting concept image from these groups of students. Their concept image of a square number may have been “any number that can be multiplied by itself” which is an incomplete image that doesn’t appropriately limit the domain of numbers to the integers. Did these students therefore consider all real numbers to be square numbers, or did they harbour inconsistent concept images of square numbers?

There was also some difficulty with the distribution of square numbers, as mentioned briefly in Chapter 3, Student 21 gave the following response: “perfect squares come in even numbers. \( \frac{100}{2} = 50 \) even numbers, \( \frac{10000}{2} = 5000 \) even numbers, \( 5000 - 50 = 4950 \).” As mentioned previously this example clearly showed both an error in understanding what square numbers are, and also how they are arranged throughout the interval. This student confused perfect squares with even numbers. Perhaps the English language is partially at fault here. When studying students’ understanding of even and odd numbers, Zazkis (1998) described a source of confusion with odd perfect squares. “‘Even’, outside of the mathematical context, means ‘smooth’, ‘balanced’, ‘equal’, ‘exact’ or ‘precise’. ‘Odd’ means ‘strange’, ‘exceptional’, ‘not regular, expected or planned’” (p. 86). This may be a possible source of difficulty for an English-speaking student to label a number that is a perfect square as an odd number.

Based on one questionnaire item only, the theme of distribution was not adequately examined; additional questions were added to the clinical interviews to gain further insight. Section 4.2.1 describes and analyzes these additional interview questions.

The three rounds of coding the questionnaire data classified all solutions and coded them according to theme. The coding began with round I focused on the appropriateness of the solutions, round II focused on quantifying and grouping the common responses and culminated with round III devoted to coding the grouped responses according to theme. These themes: confusion of concepts, distribution of square numbers, and representation of square numbers and square roots, are further discussed and expanded below in Section 4.2, with particular respect to the clinical interview data.
4.2. Clinical Interview Analysis

The clinical interview data were coded using an entirely different scheme than that used for the questionnaire due to differences in both the nature of responses given during an interview, and the nature of the information sought. There are no clear divisions between different questions and although all of the interviews began in the same manner, they each followed a different path. The clinical interview analysis was also conducted with all three theoretical constructs, concept image and concept definition, representation, and pattern spotting, in mind.

As discussed in Chapter 3, the first three questions remained the same for each interview:

1. Do you remember how to simplify these exponent laws? 
   \[ (a^3 \cdot a^4) \text{ and } (a^3)^4 \]?
2. Can you give me an example of a square number? (And another…) 
3. I have this number \( M \); \( M \) is 8 squared times 10 squared. Can you tell me the square root of the number \( M \)? \([ M = 8^2 \cdot 10^2, \sqrt{M} = ? ]\)

Evidently, as students responded differently to these initial questions, each interview took a different path and I explored intriguing responses as they arose. Therefore, the interviews were analyzed by theme rather than by question as was done for the questionnaire. As stated in Chapter 3, nine students participated in the clinical interviews; these nine students were all high achievers in the class with grades of either ‘A’ or ‘B’ in both their current mathematics course and in their previous course. These students were given pseudonyms for the clinical interview and will no longer be referred to by their student number. Table 18 lists the pseudonyms and the previously used student numbers for each clinical interview participant.
### Table 18. Pseudonyms and Student Numbers for Clinical Interview Participants

<table>
<thead>
<tr>
<th>Pseudonym</th>
<th>Previously Used Student Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adam</td>
<td>6</td>
</tr>
<tr>
<td>Claire</td>
<td>11</td>
</tr>
<tr>
<td>Kennedy</td>
<td>18</td>
</tr>
<tr>
<td>Maya</td>
<td>20</td>
</tr>
<tr>
<td>Piper</td>
<td>25</td>
</tr>
<tr>
<td>Brianna</td>
<td>42</td>
</tr>
<tr>
<td>Rachel</td>
<td>49</td>
</tr>
<tr>
<td>Jack</td>
<td>50</td>
</tr>
<tr>
<td>Sarah</td>
<td>51</td>
</tr>
</tbody>
</table>

Examples from responses that showed evidence of the identified themes discussed in Section 4.1.1 were sought first. While coding for the themes of interest, the nature of the approaches used by students while working the tasks became striking; these approaches are also discussed in Section 4.2.2.

#### 4.2.1. Themes

**Confusion of Concepts**

Confusion regarding square numbers was apparent throughout the clinical interviews, in many different settings and situations. I distinguished between some of the major types of confusion of terms demonstrated by the students; and I elected to focus on the two most common and most distinct types of confusion in this study: confusion between concept definitions of square numbers and square roots and confusion due to inconsistent evoking of concept images of square numbers and square roots.

**Concept Definition Confusion**

While coding the interviews I noticed that students used many terms, some interchangeably, and did not seem to have a strong sense of their definitions. During the clinical interviews I used the terms square number, perfect square and square root; the students used these terms as well as additional terms such as non-perfect square,
perfect number and others. The assumed meanings or working definitions of each term were not standard or consistent between the students or even between tasks during the interview with one student. The meanings given by each student to the terms in use were sometimes unclear or incorrect. These meanings were often only implicit or assumed, from the context, as participants rarely gave a definition for their terms. The second question of the interview for all participants, immediately following the question about exponent laws, asked for examples of perfect squares. The students did not seem to have any difficulty with the definition when asked this question, yet the context and simplicity of the question may be the reason for this. The concept definitions that students had were not universal and did not seem to be well defined.

I have included here only examples that show clear confusion between the terms rather than any instances of students using one term when it was clear from the context that they meant another, as this may have been simply a slip of the tongue. These instances often involved a student using both terms while speaking, but consistently choosing the correct term for the situation if I asked for clarification, or asked them to repeat their explanation.

A prime example of a clear confusion between concept definitions can be found in my interview with Jack in Excerpt 1 below. Jack was an exceptionally able student who answered the majority of questions on the questionnaire and during the interviews correctly and swiftly, but he often needed to ask for clarification if I wanted a perfect square or the square root if I used the term square number. At the end of the interview I asked explicitly for his definition of a square number, as I had noticed him using this term in an unusual manner previously in the interview.

**Excerpt 1: Jack**

1.1. Interviewer: What definition would you give for a square number?

1.2. Jack: Square number? Hmm. [pause] Umm, A whole number that is...that is being multiplied by itself to make a perfect square?

[...]

1.3. I: Hmm. Ok. And do you consider square number and perfect squares to be the same thing or to be different things?

1.4. J: Uh, different I think.
1.5.  **I:**  Ok, so how are they different?

1.6.  **J:**  Uh, the square number is like, the number that is being multiplied to get the perfect square and the perfect square is the result.

Jack was using the term square number to indicate the square root of a perfect square [1.2; 1.6]. He did not confuse perfect squares with square roots as some other students did, but he was not sure about the term square number.

Another facet of the confusion with concept definitions came from Maya, who had been asked which of the following series are square numbers: $36^2, 36^3, 36^4, 36^5, 36^6, 36^7, 36^8$. In Excerpt 2, she used her calculator and discovered that $36^3$ is a square number.

**Excerpt 2: Maya**

2.1.  **Maya:** I'm so confused! Um, well I just punched them all in on my calculator and you square root it and it comes out to a whole number but, …yeah. Wow, I did not know that.

2.2.  **Interviewer:** Ok, and you say you are confused, why are you confused?

2.3.  **M:**  Well because, what I understood of squares or perfect squares was that, well this would be a cube… wait that makes no sense cause it’s a square still…but for a square what I thought they meant was like a 2D form...

Here Maya’s definition of a square number was linked to her image of a square. This may be her concept definition image as described by Tall and Vinner (1981). In this case, her concept image of a square number was related to a geometric square, ‘a 2D form’ as she said. While this is a powerful idea, she was limited by this image in her mind and she had difficulty connecting it to the idea that a cube could also be a square number. Her concept image of a square number is narrower than it could be. I have still used this example as an example of concept definition confusion, because Maya’s confusion was not between the concepts of square numbers and square root, but within the concept definition of square number only. Note also, the representation of $36^3$ is transparent as to the number being a cube, but opaque to the number being a square. It is clear that Maya’s definition of a square number did not rely on general factors or prime factors and may be quite different than that of her peers.
Item 8: How many perfect squares are there between 100 and 10,000? Explain Briefly.

Another example of a non-conventional definition being used by a student can be found on the questionnaire. As a response to Item 8, Brianna gave the solution: “There is an infinite amount of perfect squares, any number can be squared, whether its 9 or 1.000218…. (If its whole numbers it would only have 90)”. It is clear here that Brianna’s definition was ‘a perfect square is the result of squaring any number’. However she was able to supply the correct answer to the task by limiting her definition to whole numbers only.

It is apparent from these examples that the definitions in use by the students are not always clear or consistent with mathematical conventions. Their definitions are also not locally consistent; these students do not share any definitions that are particular to their group. This may be due to the apparent lack of rigorous definitions supplied to students; students must therefore create their own concept definitions. As noted previously in Chapter 2, most textbooks recently in use in British Columbia never give a definition of perfect squares or square numbers, but rely on examples instead. The grade 8 mathematics textbook currently in use at the school these students attend, *MathLinks 8*, does give the definition of a square number: “A square number is the product of the same two numbers $3 \times 3 = 9$ so 9 is a square number. A square number is also known as a perfect square. A number that is not a perfect square is called a non-perfect square” (McAskill et al., 2008). However there are no definitions given for square numbers or square roots in any of the subsequent textbooks at use in this school. However, even if students all have the same concept definition, their will each have their own concept image that still may not conform to their definition.

*Inconsistent Concept Image*

Confusion between the concept image of square numbers and square roots or due to inconsistent evoking of images was distinguished from that of confusion of concept definitions because it is a larger conceptual problem than one of just unclear definitions. This confusion was noted in some of the questionnaire responses but was seen more clearly during the interviews because it was evident when a student simply used the wrong word when speaking.
Kennedy was asked to find a perfect square larger than 500. After finding a square larger than the target number, she became confused when the square root was smaller than the target number.

**Excerpt 3: Kennedy**

3.1. **Interviewer:** Ok, so what if I asked you for a perfect square that was larger than 500?

3.2. **Kennedy:** Um,...[pause] I guess the perfect square of 1000, would be 100,000? Like the square root of 100,000, maybe? Am I allowed to use a calculator?

3.3. **I:** You can use a calculator.

3.4. **K:** [pause] Ok, so I’m doing the square root of 10,000. Which is 100, so wait, that’s not bigger than 500.

This may have been an example of simply forgetting what the original question was, but this type of confusion was not a unique event as shown by Rachel in Excerpt 4, when asked for a perfect square that has three digits.

**Excerpt 4: Rachel**

4.1. **Interviewer:** Can you give me an example of a perfect square that has three digits?

[...]

4.2. **Rachel:** [Pause] umm, 1,000,000?

4.3. **I:** Ok, and how did you get that?

4.4. **R:** I thought that well if you cube it... oh wait its perfect squares [laughs] sorry, uh 10,000? Because if you square it, it would also be like a number. [paper shows $100^2 = 10,000$]

Rachel gave 10,000 as her final answer for a square number with three digits, because $100^2 = 10,000$. Rachel had correctly answered the previous question as well as the following question about perfect squares, but here responded with the square root. Throughout her interview Rachel repeatedly exchanged the square number for the square root in an inconsistent manner.

Both Rachel and Kennedy were having difficulty with holding the concept images as they were working through the problem. Both students had access to paper and were
making notes as they worked, but they are becoming confused as they work through the problem.

Note also Rachel’s use of the term “perfect number” in this exchange [4.4]. Due to her confusion between square numbers and square roots, it is not clear which term her “perfect number” represents.

Another particularly clear case of this confusion came once again from Maya. Maya became confused between the square number and its square root during the task that asked her to find the number of square numbers between 0 and 100.

**Excerpt 5: Maya**

5.1. Interviewer: Ok. So, how many perfect squares are there from zero to one hundred?

5.2. Maya: Um, 2, no wait... the last one would be 10. Yeah, 2, 4, 9, ...4?

5.3. I: Ok so how did you come up with 4?

5.4. M: Umm, well the last perfect square is 100, so the square root of a hundred is 10.

5.5. I: Ok,

5.6. M: Or, I don’t know why I wrote that, but yeah its 10. Then you go back down to the next number, which is 9 that would be 81,...

5.7. I: Ok,

5.8. M: But then 8 doesn’t have a square root, nor does 7, nor does 6, nor does 5, but 4 and 2 do have one. No wait, 2 does not have one. Or does it have one? No 2 does not have one. I’ll go 3. [laughs]

5.9. I: Ok

5.10. M: Or 1 wait. Is 1 a square root? I’m pretty sure 1 is a square root as well right?...I’m confused, hold on a second. Uh, yeah.

5.11. I: Ok, so 1 counts?


Maya was attempting to count the square numbers between 0 and 100; she gave her final answer on her paper as “4 – 1, 4, 9 and 100”. She began counting at 10, the square root of 100. Her confusion began just after counting 9, the square root of 81,
because “8 doesn’t have a square root” even though she had been working on the *square* of 9 not the *square root* of 9. [5.6 – 5.8]

Here there is an issue of losing sight of the problem due to the labels Maya internally assigned to numbers. Maya had 10 and then 9 in her head as square roots, but the fact that 9 is also a square number seems to have confused her. When she moved down her list to 8, she should have squared it but she became confused because she knew that 8 is not a perfect square as 9 is. Maya evoked her own concept image of 9 as a square number when she should have been evoking the image of 9 as a square root. It is not so much Maya had a confused image of either square numbers or square roots, but that she became confused during the problem about what she was trying to accomplish. Maya and others demonstrated a difficulty coordinating concept images consistently with their work.

This confusion between square numbers and square roots was also apparent in responses to the questionnaire items, where students gave answers that would have been correct if the question had asked for the alternate term. There were also numerous examples from the clinical interviews where students needed to be corrected or reminded to work on the correct term.

**Distribution**

Students’ awareness of the distribution of the square numbers was investigated through the questionnaire; however, as the information gained was not sufficient, further clinical interview questions were devised to explore in more depth. Students were asked a series of questions that began with a variation of “How many perfect squares are there from 0 to 100?” and culminated with questions comparing the number of perfect squares between two intervals of the same size. In general, students found these tasks to be very challenging. As seen in Excerpt 5 above and Excerpt 12 in Section 4.2.2, two students of the nine who participated in the clinical interviews did not supply the correct answer to the first question. Other students were able to calculate the correct answer but as Excerpt 6 with Piper shows, often a students’ desire for a “nice” pattern was not able to help her solve the problem.
Excerpt 6: Piper

6.1. Interviewer: Ok, so what if I wanted to know how many there was, up to – instead of up to 100, I want to go up to 1000. How many is there going to be up to 1000?

6.2. Piper: 100.

6.3. I: So how did you get 100?

6.4. P: I guessed. No, actually used the logic of it. So if it’s like 100 for the first question and there is 10 then - and then the answer is 10, then for 1000, you just times it by itself and its going to be 100.

 [...] 

6.5. P: Is it? 100 times 100...ah no wait, no it’s not, it’s 10,000, isn’t it?

6.6. I: So then if I just want 1000, so up to 100 you said there is 10,

6.7. P: Yeah.

6.8. I: So up to 10,000 there would be 100. So what if I just want up to 1000?

 [...] 

6.9. P: Well it can’t be 50...can it be 50? No it can’t be 50. [pause] Ah 20!

6.10. I: There is 20?

6.11. P: Yeah I think.

 [...] (She works out that 20 is too few)

6.12. P: Uh, I’ll keep going. There is going to be a lot. Wait. I think there is going to be 50.

 [...] 

6.13. I: Why do you think 50?

6.14. P: Why do I think 50? Well, hmm, half of 100 is 50?

 [...] (She works out that 50 is too many)

6.15. I: So how many do you think are going to be from 1 to 1000?


Piper did come to the correct answer that there are 31 perfect squares in 1000, but her first impulse was that there should be a round even number of perfect squares in the interval. It is clear that she believed the distribution of the square numbers should either be constant or a least uniformly decreasing in a linear manner. Piper argued that there should be 100 perfect squares in 1000, but she discovered her own mistake [6.1 –
She believed that there cannot be 50, so she guessed 20 [6.6], but when she worked out that 20 is too few she jumped back to the assumption that there must be 50 [6.12]. She did eventually work out that there will be 31 square numbers between 0 and 1000, by multiplying integers using a guess and check approach on her calculator, but the solution did not seem satisfying to her. Pólya points out that “The feeling that harmonious simple order cannot be deceitful guides the discoverer both in the mathematical and in the other sciences” (1945, p.45). In this case 31 did not “seem right” because in Pipers’ understanding it did not appear harmonious or simple. This feeling is echoed by Sinclair who suggests that the aesthetic appeal of a solution or solution approach may affect students’ choices and their beliefs about the suitability of their solutions (Sinclair, 2004).

In subsequent problems of this type, Piper reverted to listing each perfect square and no longer relied on her belief that there should be a nice pattern to the distribution of the square numbers.

Brianna successfully solved every concrete problem of this type that was asked of her but she did so slowly through a guess and check approach to find the square numbers at the ends of the intervals.

**Excerpt 7: Brianna**

7.1. *Interviewer:* Ok, so what if I asked you, up to – how many up to 500?

7.2. *Brianna:* Uh, [pause], there is 20 – 22?

7.3. *I:* Ok, so 22, what did you try?

7.4. *B:* I tried 23 because 6 uh, 25 is 625, and then 24 I assumed it would be a little too high, and then I tried 23 thinking it would just be under, but it wasn’t so then I knew 22 would be so...

Brianna found the number of perfect squares within each interval given to her with a systematic guess and check approach and was not hampered by assumptions about what the solutions should be. She never attempted to find the square root of the endpoints of the intervals to help determine the number of perfect squares, and she appeared to be working hard enough at the calculations that she did not attend to the spacing of the square numbers.
Sarah was also able to correctly answer all the concrete questions relating to distribution, but did not attend to the decreasing distribution of square numbers. Excerpt 8 shows her competency at the concrete calculations.

**Excerpt 8: Sarah**

8.1. **Interviewer:** Ok. Ok, so if I have 2000 to 3000 and 3000 to 4000 so I have two separate intervals, does one of those intervals have more square numbers in it than the other one?

8.2. **Sarah:** Um, [pause] So when I tried to find the numbers for it – the square numbers, I found each of them and then I subtracted them – subtracted it – no I added it together for each interval, and so from 2000 to 3000 I have 9, and 3000 to 4000 I have 8, so 2000 to 3000 would be larger or have more.

8.3. **I:** Ok, and if we continued on that, and I gave you the interval 3000 to 4000, and 4000 to 5000, does one of those intervals have more square numbers than the other?

8.4. **S:** So that one...[pause] this would be 70 and this would be 64, so this one from the previous question, it would be 8 and this one would be 6, so then 3000 to 4000 would be 8.

8.5. **I:** Ok, and what if I gave you, let’s see, from 10,000 to 20,000 and 20,000 to 30,000 but this time I don’t want you to use your calculator do you think you can tell me if one interval has more square numbers than the other interval?

8.6. **S:** Um, that would be very hard. The numbers are very big. [laughs] Let’s see. [pause] So...[attempting to work it out by hand]

For any given interval, Sarah was able to calculate the number of perfect squares that exist, but she did not show that she understood that the square numbers are spread further apart as they get larger; when confronted with a very large pair of intervals she continued to attempt to her previous method to calculate the largest and smallest square number within each interval in order to find how many there were. [8.6]

Some students were able to see that “the numbers are going up faster now” when working on a systematic guess and check approach to finding square numbers, but no students linked that knowledge to the distribution of square numbers within an interval. Some students had difficulty with the simpler task of finding how many were in an interval, while other students could mechanically solve the problems but could not generalize to a conceptual idea of distribution of square numbers spreading.
Representation

As in the questionnaire, the opaque representation of square numbers was often an obstacle for students to overcome in the clinical interviews. The most common issue with representation was when students only attended to exponents in expressions when looking for square numbers. They did not attend to the base when determining if an expression with an exponent was a square number, and in compound expressions that contained an exponent, any number without an exponent was usually treated as not a square number.

During the clinical interviews, each student was asked the following two questions: “Can $k^3$ ever be a square number?” and “Which of these numbers, $36^2, 36^3, 36^4, 36^5, 36^6, 36^7, 36^8$, are perfect squares?” However these questions were not always asked in this order. All participants who were asked if $k^3$ could be a square number first, before the series of numbers, claimed that it could not be. Excerpt 9 with Adam demonstrates how the representation of $k^3$ as a possible square number is opaque.

Excerpt 9: Adam

9.1. Interviewer: Ok, so, new question. Can $k$ cubed ever be a square number?
9.3. I: Why do you say that?
9.4. A: It cannot be a perfect square because it would be $k$ square root $k$.
9.5. I: Ok, so it can’t be a square number ever?
9.7. I: Ok, I have an example and I claim that for that value of $k$ it’s a square number.
9.8. A: What’s the value?
9.9. I: I claim that 16 cubed will give you a square number.
9.10. A: [checks with calculator] Oh, it can be.
9.11. I: So why do you think that is?
9.12. A: [pause] Um, like, not for all numbers just for some.
[...]
9.13. A: If you square root like the first number then cube root.
Adam was attending to the exponent in the expression $k^3$ and to the rules used to simplify exponents. He claimed that $k^3$ cannot be a square number because the expression would reduce to $k\sqrt{k}$ after attempting to find its square root [9.1 – 9.4]. This is not only true, but shows that if $\sqrt{k}$ is an integer, then $k^3$ must be a square number. In the simplified or reduced expression Adam did not attend to $\sqrt{k}$. However, after being shown that the example $16^3$ is a square number, Adam was able to see why and to offer other examples of the same type, $25^2$ and $81^2$. In this case the algebraic impossibility of $k^3$ always being a square number was obscuring the possibility of it being a square number for a particular $k$.

Note also Adam’s use of the terms square root and cube root at the end of the excerpt. On his scratch paper he gave examples that showed he was squaring the first term then finding the cube root in order to generate examples. Here his use of ‘square root’ rather than ‘square’ may point to confusion between the terms but also could simply be a spoken error [9.13].

During the clinical interviews, students that were asked questions about the series $36^2, 36^3, 36^4, 36^5, 36^6, 36^7, 36^8$ first, before the question about $k^3$, usually manually checked all the expressions and were surprised that $36^3$ was a perfect square. However, they were subsequently able to correctly answer that $k^3$ could be a perfect square for certain $k$ when confronted with that problem later in the interview.

While working through the a similar series with base 12 rather than base 36, Jack demonstrated another instance of how the representation of the expression can be misleading for students.

**Excerpt 10: Jack**

10.1. Interviewer: Ok, so I’m going to give you a series of numbers there’s 12 squared, 12 to the power of 3, 12 to the power of 4, 12 to the power of 5, 12 to the power of 6, 12 to the power of 7 and 12 to the power of 8. Can you circle the ones that are perfect squares? [12²,12³,12⁴,12⁵,12⁶,12⁷,12⁸]

10.2. Jack: Well, 12 squared, I don’t think that one would be a perfect square...I think 12 to the power of 4...[pause] ... and the other ones I’m not sure about.
10.3. I: So why did you pick 12 squared, 12 to the power of 4 and 12 to the power of 8?

10.4. J: Well this one I figured, well this one is 12 squared and that's already a perfect square pretty much, and this one I figured it was like 12 squared times 12 squared and that's a perfect square times a perfect square.

10.5. I: Hmm,

10.6. J: And then 6 I wasn’t sure about, I think it might be because that would just be 12 squared times 12 squared times 12 squared. But I’m not sure about that one. But then this one is 12 squared times 12 squared times 12 squared times 12 squared, but I thought that’s like just a bigger version of 12 to the power of 4.

As all the participants, Jack could see that $12^2$ must be a square number. He could also see easily that $12^4 = 12^2 \cdot 12^2$ and that $12^8 = 12^2 \cdot 12^2 \cdot 12^2 \cdot 12^2$ but he was not sure about $12^6$ because he saw it as $12^2 \cdot 12^2 \cdot 12^2$ [10.4; 10.6]. Subsequently, using his calculator he discovered that $12^6$ is a perfect square and then was able to work out why it must be so, but at first he was unable to see that possibility through the representation. Given the series $36^2, 36^3, 36^4, 36^5, 36^6, 36^7, 36^8$ Jack was able to see immediately that they all must be square numbers because the base is a square number. Other students had more difficulty with both series of numbers; they did not attend to the bases of the expressions and had trouble both predicting and explaining the square numbers.

As previously stated, opaque representation became an obstacle most often for students who only attended to the exponents in expressions and not to the bases. In Excerpt 11 Claire was able to generalize and simplify a procedure for multiplying two squares and then finding the square root, but when the representation was changed to obscure the second square number, the change in representation hampered her success. Claire was asked to find $\sqrt{M}$ if $M = 9^2 \cdot 10^2$.

**Excerpt 11: Claire**

11.1. Claire: [pause] Oh, well it was just 9 times 10 and that was 90. You just multiply the thing, both of them, and then you are going to get the square root of that. That’s what I was thinking earlier. Like for example, for this one, I multiplied, you multiply 5 and 3 and then you get 15. And then if you
multiply 8 and 10 it would be 80. And then 10 and 9 is 90 so...

11.2.  Interviewer: So...
11.3.  C:  You just multiply both of them.

[...]

11.4.  I:  So why do you think that works?
11.5.  C:  Why do I think that works?
11.6.  I:  Yeah.
11.7.  C:  Umm...[pause] Well, I guess since they’re both – they both have the exponent of 2 and then if you multiply you just get the same thing.

11.8.  I:  Ok, I’m going to give you a question that is similar,
11.9.  C:  Ok
11.10. I:  So now, M is 8 squared times 9. And I’d like the square root of that.
11.11. C:  8 squared time 9. [pause] Am I allowed to use my calculator?
11.13. C:  [pause] It would be 24. Umm, yeah.
11.14. I:  Can you think of...
11.15. C:  Any other way I can answer it?
11.16. I:  Yeah.
11.17. C:  [pause] [She shakes her head]

After several similar examples Claire was able to see that if \( M = 9^2 \cdot 10^2 \) then \( \sqrt{M} = 9 \cdot 10 \) [11.3]. However when she was asked to solve a similar problem for \( M = 8^2 \cdot 9 \) she could no longer use her shortcut; the “squareness” of 9 is opaque. She was able to solve the problem by using a brute force approach and mechanically multiplying and then taking the square root [11.15 – 11.17]. Later in the interview, when asked if she could create an example of the same type she attempted to use a wild guess and check method with small numbers. Claire did find one example but did find any more because she was not attending to the squareness of the second number.

When confronted with problems that forced students to attend to the base of a power expression, some students were more likely to attend to the bases in subsequent problems. However this was only common with problems that were very similar such as
$k^3$ and $36^3$. In unfamiliar problems most students continued to only attend to the exponents in the expressions.

**4.2.2. Classification of Approaches**

During the analysis of the interview transcripts, as well as during the interviews themselves, I was struck by the approaches to the problems that the students used. This was particularly apparent during the interviews as I could observe the students attempting the problem, see what they were doing on their calculators and ask for explanations or rationale even if they had not yet written anything down or given an answer. The approaches used were not differentiated by subject or topic (graphical or algebraic for example) but differentiated more by the effort involved and the conceptual understanding required. The common types of approaches that I identified were brute force (checking every possibility or performing unnecessary calculations), guess and check (both wild and systematic), rule application, pattern spotting and attention to structure. All of these approaches were successful for students a part of the time, but they require a different amount of effort to employ and a different amount of conceptual understanding. These approaches form a hierarchy that moves from most effort and least conceptual understanding (brute force methods) through to least effort and most conceptual understanding (attending to the structure of the problem). However, students did show a willingness to abandon one approach in favour of another if it seemed to be failing, and use of a “lower” approach did not indicate that a student was not able to use a “higher” approach. These approaches were easily observable during the clinical interviews and often very difficult to infer from the questionnaire responses, but in some cases solution approaches are discernable in the questionnaire responses as well. Examples of each type of solution approach are examined below as well as examples of students abandoning one solution approach in favour of another.

**Types of Approaches**

I have identified five approaches: brute force, guess and check, rule application, pattern spotting and attention to structure.
**Brute Force**

The brute force approach was characterized as a willingness or attempt to simply try every possibility, or repeatedly perform unnecessary calculations rather than attend to the structure of the problem. Given enough time and a problem that is numerical rather than algebraic, this approach is often successful. The obvious drawback is that problems may be too long, complicated, or may not be suitable for calculation. However, although the brute force method may require a great deal of effort to work through every possibility, it does not require a great deal of conceptual understanding, past what is required to perform the calculations.

Rachel provided an example of the brute force method being used in a particularly slow manner. Rachel had been asked to find how many perfect squares there are between 0 and 100.

**Excerpt 12: Rachel**

12.1. *Interviewer: So how many square numbers are there up to 100? Perfect squares.*
12.2. *Rachel: [pause] Like how many from 1 to 100? [
12.3. *R: So like, 4 and 9... [
12.4. *R: [pause] [Writes 25 on paper]
12.5. *I: So, can you explain how you found 25?*
12.6. *R: Umm, I kept going up by 1, 2, 3, 4, and square rooting each number and if they came out to a perfect number then I wrote down which numbers were perfect square numbers, so I have found 5 so far...[1,4,9,16 and 25]*
12.7. *I: So, if I asked you to find the next one, we only have a couple of minutes so if I asked you to keep going, how would you find the next one?*
12.8. *R: I think that I would continue the same way, unless, I’m not sure, I think that there would be an easier way to find out, but, I’m not sure how to figure it out that way, so I think that I would continue going up with the numbers and square rooting them and...*
12.9. *I: Ok, and you know that that way you can get them*
12.10. *R: Be sure, yeah...work*
To find the square numbers, Rachel attempted to work through the interval by finding the square root of each whole number on her calculator, if it gave her a whole number for the square root, then she wrote it down as a square number [12.6]. If she had had more time, she would have been able to successfully find all the square numbers in the given interval, however it is clear that this approach was incredibly time consuming as she would have needed to check all 100 whole numbers.

Also note that once again Rachel uses the term “perfect number” [12.6]. From the context in this case it seems like she is using perfect number to mean a whole number, but it is somewhat unclear.

While addressing the same problem, but in the interval 0 to 500, Kennedy also began with using the brute force approach but in a more strategic way. Rather than checking each number to see if it was a perfect square, she calculated all the perfect squares by squaring each integer in turn, and then counted them.

**Excerpt 13: Kennedy**

13.1. *Interviewer: So how many are there up to 500?*
13.2. *Kennedy: I guess, 50? Maybe, maybe not. Um...wait, no. [pause] Yeah, it’s not going to be 50.*
13.3. *I: So you have been listing them out...*
13.4. *K: Yeah, um I’m just listing them out and then finding whatever this number is 12, 13, 14, umm, listing it out and then doing that number to the power of 2, just to see what it’s perfect square would be.*
13.5. *I: Ok,*
13.6. *K: [pause] It probably isn't this method...[laughs] [pause] Yeah, so 12 and then another 10, so 22.*
13.7. *I: Ok, so 22 to 500.*
13.8. *K: Yeah, there are 22 perfect squares in 500.*

In this case rather than having to test 500 numbers, she only needed to work through 22 numbers. Later, shown in Excerpt 24, as the interval got larger Kennedy switched from a brute force approach to one that attended to the structure of the problem.
This approach is also evident on Item 8 of the questionnaire: “How many perfect squares are there between 100 and 10,000? Explain Briefly.” Student 29 attempted to answer the problem by listing the perfect squares but gave up and did not answer the problem.

There were many other examples during the interviews of students using the brute force approach by “checking each one just to make sure” or performing unnecessary calculations repetitively when there was a simpler method available. The brute force approach was very popular and apparent in many different problem solutions. Eight of the nine students who participated in the clinical interviews used a brute force approach at least once during their interview, and six students used it more than twice.

**Guess and Check**

The guess and check approach method is slightly more advanced than the brute force method. To successfully use the guess and check method requires some conceptual understanding of how to adjust your choice. The systematic guess and check approach requires less effort than the brute force method because there is no longer the requirement to check all of the options, but does require more conceptual understanding. Some students used this method as ‘wild guess and check’ by choosing arbitrary small numbers with very few adjustments while other students were very efficient in their adjustments and can be said to have used ‘systematic guess and check’.

Maya’s solution to the problem of finding a square number between 1365 and 1543 was a good example of the systematic guess and check method.

**Excerpt 14: Maya**

14.1. Interviewer: *Ok, so what if I asked you for one [a square number] that was bigger than – I want one that is larger than 1365 but I want it to be smaller than 1543.*

14.2. Maya: *Ok [pause], Oh wait no, [pause] A little bit smaller than 40, so 35 squared? Nope, a little bit higher, 37 squared. Does that work?*

14.3. I: *Yes, 37 works. So how did you find that number*?
Maya had a strategy for how to effectively use her guess and check method to try to narrow down the interval of interest. She began with $40 \cdot 40 = 1600$ which was too big [14.4], so she jumped to $30 \cdot 30 = 900$ [14.6] which was too small and worked back and forth until she settled on 37 as her solution [14.3].

Similarly Piper used systematic guess and check with a strategy of working with steps of 5. She used this strategy often, but she only adjusted her steps as she approached very close to the solution.

**Excerpt 15: Piper**

15.1. *Interviewer:* Ok, so, last one, what about one that was bigger than 500 but less than 700?

[...]

15.2. *Piper:* Uh, 25 squared. Is it? Yeah, no wait. It is bigger than 500 and less than 700.

15.3. *I:* Hmm. Um and how did you,… so you tried 20 first...

15.4. *P:* Yeah...

15.5. *I:* And that was too small… so then how did you know to try 25?

15.6. *P:* Oh, ‘cause its – I always do it 5 numbers bigger or smaller than the other number that I tried first so...

[...]

15.7. *I:* Just to give you so you don’t have to...

15.8. *P:* Yeah, do them one by one.

However sometimes the systematic guess and check method can be detrimental if the strategy is not used effectively. In Excerpt 16 Sarah used guess and check but skipped over the solution that she was looking for. Sarah was attempting to find another
number that would satisfy $k^3$ is a square number. She was given $4^3 = 64$ as an example.

**Excerpt 16: Sarah**

16.1. **Interviewer:** Ok, and what numbers have you tried so far?

16.2. **Sarah:** I've tried 5 and 6 but neither of them has worked because they came out as decimals.

16.3. **I:** Ok, so if I asked you to keep going, what is the next number that you would try?

16.4. **S:** Probably...8 because 8 is similar to these two numbers so it is possible that they would come out the same.

16.5. **I:** Ok...

16.6. **S:** Or not [laughs], so it came out as a decimal again, so I was wrong,

16.7. **I:** That's ok. So you're just looking for examples, so what's the next number that you would try?

16.8. **S:** Um, probably 10. To see if it works.

Sarah tried 5 and 6 as values of $k$, but when those did not yield a square number for $k^3$, she said that next she would try 8 and then 10 clearly skipping over $k = 9$ which would satisfy the condition [16.4, 16.8].

Guess and check, both wild and systematic, was a very common approach for students during the clinical interviews. It was also the most common intended approach stated when asked questions of the type “How would you solve this problem?” or “What would you do if...”. While working with this approach some students were able to refine their adjustments very effectively to arrive quickly at the solution to the problem, while other students would stick to their strategy and use a combination of systematic guess and check with brute force, e.g. checking every fifth number or every even number.

**Rule Application**

Rule application was much less common than both brute force or guess and check. It requires that students know rules of operations or structure about the topic they are working with. Rule application is a different phenomenon than attending to the structure of the problem, because of the reasoning involved. Rationale is given as “you can’t...”, “you are not allowed...” or “the rules says...” without any explanation that refers
to the structure of the expression. In rule application, students treat these rules as arbitrary or as coming from an external source; often the students cannot provide justification.

All nine students who participated in the clinical interviews were able to correctly answer the first question that asked them to simplify expressions using exponent rules. However, subsequent uses of rule application as an approach repeatedly led to an incorrect solution. This incorrect solution could be from incorrect application of a rule, or incorrect recollection of a rule, as Excerpt 17 with Kennedy shows. The question was to find $\sqrt{M}$ if $M = 9^2 \cdot 10^2$.

**Excerpt 17: Kennedy**

17.1. Interviewer: Ok, so I have this number $M$ and $M$ is 9 squared times 10 squared, and I want you to give me the square root of $M$.

17.2. Kennedy: Ok, I know you add those, and I guess you multiply that so 90 to the power of 4. Do you want me to give you the actual value of that number?

17.3. I: Sure, yeah.

17.4. K: So 90 to the power of 4 is 6 - [pause] 6,561,000

17.5. I: And that’s the square root?

17.6. K: Oh, no that’s not the square root. [pause] It’s not the square root – the square root of that is 8100.

17.7. I: Ok, um, can you think of another way to do that question?

17.8. K: You could just square root that number 90 to the power of 4.

17.9. I: Ok, without multiplying it out in between?

17.10. K: Yeah.

Although Kennedy correctly answered the first question of the clinical interview that asks to simplify using exponent rules, here she has conflated the two rules that allow simplification of exponents; $a^x \cdot b^x = (a \cdot b)^x$ and $x^a \cdot x^b = x^{a+b}$. Kennedy multiplied the bases, but also added the exponents [17.2]. She was very confident in her solution; her confidence was rooted in her confidence of the rule. When prompted to solve the problem in a different way, she used the rules incorrectly [17.8].
Later in the interview Kennedy displayed another way that rule application can lead to incorrect solutions. This is using rules for algebraic problems when dealing with arithmetic problems. The question is whether or not \( x^2 + y^2 \) can ever yield a square number.

**Excerpt 18: Kennedy**

18.1. **Interviewer:** Can \( x \) squared plus \( y \) squared ever give you a square number?

18.2. **Kennedy:** Um, I don’t think so, because I know you can’t add the values of \( x \) and \( y \), they stay separate when you – when you’re cross – or not cross multiply – when you are um, whatever it’s called when you multiply onto the equation, right? And you can’t add these either or multiply them because these are not the same – the bases are different – the value of \( x \) and \( y \) are different.

18.3. **I:** Ok, so you do think that, um, so you can’t put them together as a rule, but do you think there might be some value of \( x \) and some value of \( y \) where if you added these together you could get a square number?

18.4. **K:** Um, probably...um, I guess if you found, one of these numbers, a perfect square number and added them together, actually I’m not sure if it could give you a perfect square or...yeah, no, I don’t think so.

18.5. **I:** Ok

Kennedy is correct that as a rule you cannot simplify algebraic expressions that contain exponents, but she is confusing this with the ability to add numbers that have exponents. This belief is echoed in the questionnaire responses to Item 2.3 discussed in Section 4.1.2, where three students explicitly stated no because “you can’t add powers”. For this rule there is confusion between not being able to always perform an operation and never being able to perform the operation.

Pólya (1945) distinguishes between *mastery* “To apply a rule with natural ease with judgment noticing cases where it fits and without ever letting the words of the rule of scare the purpose of the action or the opportunities of the situation” and *pedantry* “To apply a rule to the letter rigidly unquestioningly, in cases where it fits and in cases where it does not fit” (p. 148). Here, rule application fits with pedantry. It is applying a memorized rule incorrectly due to a misunderstanding of the rule, or applying it when it is not applicable.
Kennedy was not an isolated case of rule application; other students also mentioned that “you can’t” perform a certain operation, most often concerning $x^2 + y^2$, but overall this approach was much less popular than brute force or guess and check.

**Pattern Spotting**

As mentioned in Chapter 2, the term ‘pattern spotting’ comes from Hewitt, to describe occasions when students are able to continue a pattern after seeing some examples and rely on the pattern without the conceptual understanding of the structure of the problem or the mathematical entities they are using. If the example is changed only very slightly, the pattern that they have seen or created may no longer be applicable. Pattern spotting is the most intellectually challenging of the strategies encountered so far, as it requires the creation of conjectures as opposed to simply reproducing work. It is linked to the rule application approach in that with pattern spotting students often make their own rules for application. However pattern spotting can be problematic for students if they find a false pattern in a few examples, i.e. a pattern that appears but does not hold true for all examples, or if they cannot generalize from a true pattern to the reason for the pattern. As Hewitt has stated, “Children can find many patterns in their table, even if they have made some errors in the entries. They may find all sorts of rules, none of which apply to the original situation but then some children have long ago turned their attention away from that” (1992, p. 6). Pattern spotting can be an ideal way, however, to get students to see that there is a structure within, but students must be pushed to try to uncover the reason for the pattern and to focus on the original problem.

Piper’s responses showed a number of different examples of trying to use pattern spotting to solve a problem. Piper was asked a series of questions of the type, “find $\sqrt{M}$ if $M = 8^2 \cdot 100^2$”.

**Excerpt 19: Piper**

19.1. Interviewer: Ok, ok, so I have this number $M$, and $M$ is 9 squared times 10 squared and I want to know what the square root of $M$ is.

19.2. Piper: So you want to know what the square root of $M$ is.
19.3. I: Yeah, and you can use your calculator for this as well if you want.

19.4. P: Ok. So 9 squared times 10 squared equals to 8100.

19.5. I: You can write on here, too,

19.6. P: Oh, ok. 8100 and you square root and it 90.

19.7. I: Ok.

19.8. P: So the square root of M is 90.

19.9. I: Ok, and now, I’m going to give you M is 8 squared time 10 squared and I want the square root of M again.


19.11. I: So what if M is 8 squared times 100 squared?

19.12. P: It would be 800.

19.13. I: So how did you get that?

19.14. P: The pattern of it, you just have to times both of them.

After solving several examples through a brute force approach Piper was able to see the pattern and use it to solve another example without performing all intermediate calculations [19.12 – 19.14]. However when the problem was changed slightly to $M = 3^2 \cdot 4$, which obscured the second square, she could no longer use the pattern she had found.

19.15. Interviewer: Ok, so, what if I gave you instead, so now M is going to be 3 squared times 4, and I don’t have a square there, and I still want the square root of M.

[...]


19.17. I: Ok, and if I gave you another example like that, let’s say, ah, 5 squared times 9 and I want the square root of M.

19.18. P: 5 squared times 9 – 225 ...15 is the square root of M.

[...]

19.19. I: Ok, so let’s come up – let’s make another example. So we know that this one, we know 3 squared times 4 this square root is a whole number,


19.21. I: And we know that 5 squared times 9 the square root is a whole number, and we know 5 squared times 2 is not.
19.23. I: So give me a number.
19.24. P: Um, ok, um, 4 squared.
19.25. I: Ok, 4 squared times...
19.26. P: Uh, 3?
19.27. I: 3. Ok, and does this give you a whole number?
19.28. P: ...no still not.

[...]

19.29. P: Ok, ah,... [pause] Not really, [pause], um, [pause] no, I can't. I don't know why they actually work. Hmm, [pause] Oh, all I notice is the one you have to multiply them by, is always bigger than the base, compared to mine which is smaller than the base. Yeah.

Piper had been working with the examples $M = 3^2 \cdot 4$ and $M = 5^2 \cdot 9$ [19.6; 19.18] that gave her square numbers, and the examples $M = 5^2 \cdot 2$ and $M = 4^2 \cdot 3$ [19.21; 19.28] that did not yield square numbers. She found a pattern and conjectured that $M = a^2 \cdot b$ will give a square number if $a < b$ [19.29]. This did hold for the four examples that she had been working with, but she soon found that the pattern did not hold for any $a$ and $b$.

Maya used pattern spotting in conjunction with brute force (checking “just to be sure”) while working through the series of expressions $12^2, 12^3, 12^4, 12^5, 12^6, 12^7, 12^8$ to see which were square numbers, but she did not attend to why the pattern exists. Because Maya did not move past pattern spotting into attention to structure, she made an error when her check did not work.

**Excerpt 20: Maya**

20.1. Interviewer: Ok, so if I continue this and I got to 12 to the power of 99 and 12 to the power of 100 and 12 to the power of 101 and 12 to the power of 102, and I asked you to circle the perfect squares...

20.2. Maya: Ok, Um, [pause], Ok, so obviously it doesn’t work [laughs]...

20.3. I: So you circled 12 to the power of 100?
20.5. I: What about 12 to the power of 102?
M: Well, because then I went with the pattern but then I realized that after 100 it doesn’t work anymore.

Maya originally believed that both $12^{100}$ and $12^{102}$ would be square numbers because of the pattern of even exponents, but when her calculator could not solve $\sqrt{12^{102}}$ she changed her mind and declared that the pattern did not work anymore.

Similarly Rachel provided an example where the found pattern did not work, as she attempted to extrapolate from only one instance.

**Excerpt 21: Rachel**

Interviewer: So, ok, so what if M is 2 squared times 9? And I want the square root of M?

Rachel: [pause] Um 6? [laughs] Um so I did the exponent first and multiply it by 9 which equals the root of 36 and you square root that which equals 6...

Interviewer: Could you do that another way?

Rachel: [pause] Um, if you times the two bases together, and then times it by two, the exponent which also equals 36, and then if you square root 36, again it equals 6

Interviewer: Umm, so try this one, three squared times 9.

Rachel: [pause] Hmm, it doesn’t work the same way.

Rachel saw that in the expression $2^2 \cdot 9$ the solution can be obtained by multiplying the bases by the exponent ($2 \cdot 9 \cdot 2$) but she discovered that the rule that she created did not work for the follow up example ($3^2 \cdot 9 \neq 3 \cdot 9 \cdot 2$) [21.4 – 21.6].

Pattern spotting can be very useful; if students are able to find patterns in the expressions it may help them to find solutions and be curious as to why the pattern exists. But pattern spotting is the approach that was the most likely to cause errors, as students often did not look for why the pattern existed or used too few examples to convince themselves that a pattern existed.

**Attention to Structure**

Finally, attention to structure requires the most conceptual understanding, but often the least computational effort. When attending to the structure of an expression, many problems can be solved very quickly. Attention to structure is different than rule
application, as there is understanding of how the structure can generate the rule, rather than a rule being arbitrary or externally imposed. Many students did not attend to the structure of the problem and used guess and check or brute force approaches instead. Jack demonstrated attention to structure while answering the question that asked him to find a perfect square between 2100 and 2500.

**Excerpt 22: Jack**

22.1. Interviewer: Ok, what if I wanted one – Ok, I want one that is bigger than 2100 but smaller than 2500.

[...]


22.3. I: Ok, and how did you get that number?

22.4. J: Uh, took the square root of 2500 which was 50, and I took the square root of 2100 which was 45 and some decimal, so anything between... I rounded up to 46, so anything between 50 and 46.

For similar questions most of the participants did not attend to structure by taking the square root of the endpoints of the interval but rather used a guess and check approach by squaring numbers to see if they fit within the interval. Jack showed clearly that he understood the structure of the problem not only in his approach of finding the square roots, but also in his assertion that the solution could be “anything between 50 and 46”.

Adam also exemplified attending to the structure of the problem in Excerpt 23. He answered a series of problems of the type “Find $\sqrt{M}$ if $M = a^2 \cdot b^2$.

**Excerpt 23: Adam**

23.1. Interviewer: Ok. So now, I have this number $M$ and $M$ is 9 squared times 10 squared and I want you to give me the square root of $M$

23.2. Adam: Uh, its 90, because you do like radical 9 squared is 9 and radical 10 squared is 10 and its gonna be 9 times 10 – it’s 90.

23.3. I: Ok, so and if I gave you a longer one, so 13 squared times 17 squared times 23 squared?

23.4. A: Its gonna be like 13 times 17 times 32.
23.5.  I: Ok. And if $M$ was $x$ squared times $y$ squared?

23.6.  A: It’s $xy$.

23.7.  I: Ok, so what if $M$ was, um, 9 squared times 16?

23.8.  A: Uh radical 16 is 4 so 9 times 4. That is 39...is 34. It’s 34.

23.9.  I: Ok, so what if I had 3 to the power of 4 times 2 to the power of 4?

23.10. A: It’s gonna be 3 to the power of 2 times 2 to the power of 2; 9 times 4. 36.

It is clear that Adam had a good grasp of the structure of square numbers as they relate to this problem. It is also apparent how much less mechanical effort is required by Adam to solve these problems than if he were using a brute force approach.

The attention to structure as a solution approach was much less common than using brute force, guess and check or pattern spotting; only Jack and Adam routinely attended to structure. It requires the most conceptual understanding but the least amount of time and effort. Although attention to structure is the approach that that we hope students will use, most students were more comfortable using other approaches during the clinical interviews.

Movement Between Approaches

The amount of consistency to the choice of approach varied between participants. Like Kennedy in Excerpt 13, most students during the clinical interviews began a problem using brute force as their method of choice, but as the problem got more difficult, some students would stick with brute force method while others would transfer to a new approach. Transferring to a new approach in this way is efficient, using each when it seems most appropriate. If it were possible to solve a problem using a conceptually easier approach in a moderate amount of time the simpler approach would be preferred. It was not until problems got very long, that some students would abandon the brute force approach. Also noteworthy is that most students had a preferred approach, e.g. Piper often attempted to use pattern spotting, while Kennedy was the most likely to use rule application. However, students were still quite likely to change their approach if they felt it was not working, or to move back and forth between approaches.
In the following excerpt, Kennedy used many different approaches within one problem. When one approach seemed to be failing she would abandon it in favour of another approach. Kennedy was asked how many square numbers there are up to 100, and up to 500.

**Excerpt 24: Kennedy**

24.1. Interviewer: Ok, so how many perfect squares are there up to 100?

24.2. Kennedy: Um,...well I guess 1 would count, 4, [pause], 16, ...36, ...oh, yeah 10.

24.3. I: So how many are there up to 500?

24.4. K: I guess, 50? Maybe, maybe not. Um...wait, no. [pause] Yeah, it's not going to be 50.

24.5. I: So you have been listing them out...

24.6. K: Yeah, um I'm just listing them out and then finding whatever this number is 12, 13, 14, umm, listing it out and then doing that number to the power of 2, just to see what it's perfect square would be.

24.7. I: Ok,

24.8. K: [pause] It probably isn’t this method...[laughs] [pause] Yeah, so 12 and then another 10, so 22.

24.9. I: Ok, so 22 to 500.

24.10. K: Yeah, there are 22 perfect squares in 500.

24.11. I: So what if I asked you a bigger number? Um, would you – how would you go about it?

24.12. K: There is probably some sort of pattern, that I’m not seeing. Umm, [pause] Ok, so I know that after 10 numbers it equals 100, and then – um after a total of 20 numbers it’s 400, ...but I just can’t find the relationship between them.

24.13. I: Ok, so if you had to guess, if I asked you up to 1000, how many would you guess would there be up to 1000?


24.15. I: Ok, and why do you say 30?

24.16. K: Um, because I know it goes up a lot more with each number anyways, can I check it?


24.19. I: Ok, so there is 31 up to 1000.

24.21.  I:  Ok, so what if I asked you to find how many are between 2000 and 3000?

24.22.  K:  Probably not a lot. If 35 to the power of 2 is... 45...[pause] Ok, yeah, so 45 to the power of 2 gives ...

24.23.  I:  Ok,

24.24.  K:  45 that fits in there, um, 46, 47, 48, 49, 50, 51, it seems to be going up by 100 now, 52, 53, yeah 54. 54 to the power of 2 is 2916. So there is 9.

Kennedy began with brute force when she described “counting” the perfect squares [24.2], but quickly jumped to attention to structure when she did not need to count after finding 36 as a square number. She used brute force again while listing the square numbers [24.6], and then she jumped to systematic guess and check after the numbers became large enough to force her to abandon brute force [24.13 – 24.19]. Finally, she combined guess and check with brute force [24.22]. She used systematic guess and check to find the first perfect square of the interval but then brute force to list every one until she found the last one of the interval.

This excerpt illustrates the most common movement between approaches during the clinical interviews. Students would often begin with brute force methods and if pressed by further questions with big numbers, or larger intervals, students would sometimes make the jump to another approach. Most often the next approach of choice was systematic guess and check, but it might also have been pattern spotting or attention to structure. However as this excerpts shows, when the problem became simpler or shorter again, many students would revert to their previous method. Students were very likely to move back and forth between approaches but would often revert to their favoured approaches when given the opportunity.

The analysis of the interview data followed from the analysis of the questionnaire data discussed in Section 4.1.2. The themes of focus during the clinical interview analysis were confusion of concepts, distribution of square numbers and representation of square numbers and square roots. Students did not share the simple and rigorous definitions commonly accepted by the wider mathematical community, and were therefore hampered by an ambiguity in their working definitions. Students also showed confusion between the terms square number and square root; they often exchanged one term for the other while speaking, but some students also confused the two while
attempting to solve a problem. Students did not seem aware of the distribution of square numbers within the natural numbers. Some students may have been aware of the distribution, but did not attend to the information in a manner that would allow them to solve problems more easily. Opaque representation of square numbers and square root was also a large obstacle for students to overcome when solving problems. These findings are further discussed in Chapter 5.

Student solution approaches were also described in Section 4.2.2. These solution approaches create a hierarchy that ranges through brute force, guess and check, rule application, pattern spotting to attention to structure. Brute force and guess and check were the overwhelmingly most popular solution approaches, but pattern spotting was also quite common. Rule application and attention to structure were the two least common solution approaches used by students. Most students seemed to have a preferred solution approach but where still very likely to switch frequently between solution approaches. These findings are further discussed in Chapter 5.
5. Discussion

Chapter 5 gives a summary of the study, including links to the literature, pedagogical implications, limitations of the study and suggestions for further research. The findings respond to the research questions posed in the introduction. I describe how this study shows similar occurrences to previous research, but adds to the domain of viewed phenomena. I also discuss new findings and those areas that suggest the need for further research. The pedagogical implications include suggestions for simple modifications to assignments, problems and styles of teaching. The limitations of the study and suggestions for further research are both expressed as a personal response to the questions I felt were left unanswered by this study.

5.1. Findings

The research questions that prompted this study were originally produced before the study began; the description of the original research questions is found in Section 1.2 and Chapter 3. The questions of interest were revised over the course of the study as is warranted when using a modified analytic induction approach. The revised research questions and the rationale leading to the revisions are described in Section 4.1.1. In their final form, the research questions were:

1) What obstacles do students encounter when attempting to solve problems with square numbers and square roots?
   In particular to what degree does: confusion of concepts, distribution of square numbers and representation of square numbers and square roots, hamper students attempting to solve problems?

2) What strategies do students use to solve problems relating to square numbers and square roots?

The findings related to these questions are discussed below.
5.1.1. What obstacles do students encounter when attempting to solve problems with square numbers and square roots?

This study has found that students do experience difficulty when working with square numbers and square roots, and although the topics may seem simple, there is a wide variety of ways in which to think about and work with square numbers.

In particular, the students in this study did not share the agreed-upon definitions for the terms involved with the wider mathematical community. The students in the study also did not share locally-consistent definitions with each other. Some students were unsure if square numbers must be squares of integers, or if they could be any number that could be ‘square rooted’, while others believed that ‘square number’ meant ‘square root’ as opposed to perfect square.

For some students there existed some confusion not just between the terms ‘square number’ and ‘square root’ but also between the concepts. These students had concept images of square numbers and square roots that were not distinct from one another, and the students became easily confused between the two concepts. Many students became confused while working through a problem, as to which type of number they were dealing with and which properties those numbers had.

Students also had very little notion about the distribution of square numbers within the natural numbers. Their concept image of square numbers was not complete in the area of distribution; many students seemed to have very few ideas about distribution at all.

Any opaque representation of the exponential expression as a square number also caused confusion. Students did not change between representations without being prompted to do so. However, even when the representation was transparent, many students did not attend to the transparent features of the problem.

These obstacles are discussed in more detail below.

Confusion of Concepts

Gough (2007) notes several difficulties when teaching square roots, including the backwards nature of a square root, but also the similarity between the English words in
the term “square root”. He suggested that “root” is not evocative of any portion of the square root. There is another issue with the English words in the terms square number and square root, as this study has shown there is considerable confusion between the two terms. This confusion manifested both as a confusion between the concept definitions, and as a confusion due to inconsistent evoking of the concept images held by the students. The confusion between the words and the definitions was demonstrated when students often used one term instead of the other, or were performing the actions for the alternate term. This shows a difficulty with the definitions of the two terms, and being clear as to which term is which and when to use each one. This confusion is also common outside of the mathematics environment and is spread culturally as seen in the movie *The Wizard of Oz*. The Scarecrow states the Pythagorean theorem as: “The sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the remaining side” (LeRoy & Fleming, 1939). Note as well that the scarecrow uses an isosceles triangle rather than a right triangle.

However, a unique finding is the confusion demonstrated between the concepts of square number and square root. This concept confusion was demonstrated when students became confused in the middle of a problem, as Maya did in Excerpt 5 in Section 4.2.1. Other instances of confusion between the concepts of square number and square root can be found in Excerpts 3 and 4 of Section 4.2.1. Kennedy and Rachel both showed uncertainty over which features and properties applied to which concept, or at least with which concept they were working, the square number or the square root. To what extent this confusion is prevalent and to what extent this confusion stems from the nature of the similarity of the terms square number and square root remains to be determined.

**Distribution**

When dealing with the distribution of square numbers within the natural numbers, most students did not display an understanding that the square numbers “stretch out” but are still distributed with an underlying order. When asked to find if an interval contained a square number, or to find how many an interval contained, students would most likely resort to a guess and check method or brute force approach. Students who thoughtfully tried to work out this problem often believed that the distribution should be
nice and even and display some sort or pattern; this led them to believe that the number of square numbers within a given interval should be a round number, or have some linear relationship to the numbers in a different related interval. However, when these students found that the patterns they were expecting were not there, they often resorted once again to a brute force approach to find the solution. There is little research into square numbers or square roots in general, and less with regards to distribution of square numbers, but these findings do agree with Zazkis and Gadowsky’s (2001) assertion that students will immediately turn to calculations when confronted with problems.

**Representation**

This study adds to the body of knowledge on the role of the representation of numbers, in this case square numbers and square roots. Opaque representation remains a large obstacle for students when attempting to solve problems with square numbers or square roots. When the representation was not transparent with respect to square numbers, students often claimed that the expression could not be a square number, without attempting to verify the statement. Zazkis and Sirotic (2004) found that generally students do not attend to all of the information provided in a problem; in this study students overwhelmingly did not attend to the base in exponential expressions. When not prompted to do so, students did not freely switch between different representations, by using exponent laws or other means, but if explicitly asked to simplify an expression they were able to do so.

If students did not have a strong conceptual understanding of the structure of square numbers and their properties, they were very unlikely to attend to transparent features of the representation. This study shows once again, that even when representation is transparent to experts, those transparent features may not be clear to students. Given the problem to find the square root of $M$ if $M = 8^2 \cdot 10^2$, students often did not attend to the squares as transparent features, but resorted to performing the calculations to determine the square root. This agrees with what Zazkis and Sirotic (2004) found with irrational numbers, and Zazkis and Campbell (1996) found with divisibility; that transparent features may not be clear as such to students.
5.1.2. **What strategies do students use to solve problems relating to square numbers and square roots?**

This study has described five different solution strategies that students use when solving problems relating to square numbers and square roots. The five approaches are brute force, guess and check, rule application, pattern spotting and attention to structure. Of these, brute force and guess and check were the most popular methods. However, as problems became longer or more complicated, students abandoned their simple-to-use, but time consuming, approaches in favour of ones that were generally more efficient.

When comparing the two most popular approaches, brute force seemed to yield the correct solution more often, provided students were given enough time to tediously work through all the options, compared to guess and check, which was more unreliable.

Students in this study frequently used guess and check strategies as their preferred approach. This supports the many other studies that have also shown that guess and check is a very common solution approach for students of all ages and ability levels (Capraro, An, Ma, Rangel-Chavez & Harbaugh, 2012; Johanning 2004; Lannin, 2005). If students used guess and check with good strategy as Guerrero (2010) suggests, they were likely to find the correct solution and gain insight into the structure and process of the problem. However, just as Johanning (2007) suggests, if students did not understand the problem structure enough to begin with, a strategic guess and check approach was not possible. In this case, students resorted to wild guess and check, which was not likely to yield positive results. Wild guess and check may have yielded the correct solution but did not aid in understanding, just as Lannin (2005) finds with elementary aged students.

Rule application describes applying a rule indiscriminately with little or no understanding of how, why, or when the rule should be applied. However, students who used this approach clearly believed they were applying a rule properly. Rule application was a less popular method than both brute force or guess and check. Used with little attention to the structure of the problem, rule application often led students to the incorrect answer due to an inappropriate application of the “rule”.
Pattern spotting was very apparent in this study and like guess and check, pattern spotting was also very unreliable for students who were not carefully analyzing their work. Hewitt (1992) notes at least two problems with pattern spotting, that students would focus on the pattern in the numbers rather than the mathematical structure of the problem and that students would find false patterns in the numbers. Both of these concerns were realized in this study. Students found false ‘patterns’ that were based on only one example, and other false patterns based on several examples but were not generalizable across all examples. When students did find generalizable patterns, they were unlikely to analyze the problem to look for the underlying structure that would explain the pattern. Rather, when students found generalizable patterns, if they could extend and use those patterns without analysis, they were likely to do so. Orton, Orton and Roper discuss using patterns and problem solving as a means to lead to algebra through generalizing and discuss the “hope or expectation that a rule will be found” (1999, p. 134). Students in this study did sometimes find “rules” but rarely did they attempt to find the reason for the rule. Some students were able to find legitimate patterns and through careful analysis use those patterns to gain some insight into the nature of the problem, or structure of the example, but this was less common.

Attention to the structure of the problem or expression was the most ‘advanced’ solution approach, as it requires the most conceptual understanding to apply. It was the least used approach but was also the most effective at allowing students to correctly solve the problem. Although it requires the most conceptual understanding from students, once this understanding is in place, it is the also the easiest and most efficient approach.

The study found that students had their own preferred solution approaches that they persisted with until they were forced to switch approaches due to the nature of the problem. Despite these preferred approaches, the students were still very willing to switch back and forth between approaches whenever it appeared to be necessary. Switching between approaches did not always appear to be a conscious decision. This finding agrees with similar work, that found students very willing to change solution approaches when one seemed to be failing, but they did not always switch to an appropriate approach (Capraro, An, Ma, Rangel-Chavez & Harbaugh, 2012; Lannin, 2005; Stacy, 1989).
In Excerpt 24, (Section 4.2.2), Kennedy displays an exemplary case of approach switching. She began the problem with small numbers working with a brute force approach, and as the numbers got larger she jumped to attention to structure. However, although she demonstrated the correct thinking with relatively small numbers, as the numbers became larger, she jumped to a whole-object approach described by Stacy (1989). This approach involves multiplying results from a smaller domain to obtain results for a larger domain. However this approach is not valid for any relationship that is not directly proportional. Stacy (1989) finds that “Nearly 1 in seven of those who used a linear method for the near generalization swapped to whole object shortcut method for the far generalization” (p. 155). This whole-object far generalization approach was also noted from other participants of this study.

5.2. Pedagogical Considerations

Here I suggest some pedagogical considerations to help remove or reduce the obstacles that students face while working with square numbers and square roots. In particular, I address the previously discussed obstacles of definitions, distribution and representation. I also make some suggestions that may help students use more advanced solution approaches when solving problems. Some suggestions for improving the teaching of square numbers and square roots are to: focus on definitions, provide problems that require attention to the base in exponential expressions, encourage a geometrical approach to square numbers, include ‘big number’ problems that require attention to structure, include non-routine problems and encourage a wide variety of solution approaches.

5.2.1. Attention to Definitions

The participants in this study did not share a common definition for square numbers or square roots, and the definitions that they carried were not consistent with those of the wider mathematical community. Square numbers and square roots may seem too simple to require definitions, and yet, the differences of assigned meaning by students caused problems for them during the clinical interviews. It is likely that these differences caused more severe problems during the questionnaire, as students were
unable or unwilling to ask for clarification. I suggest being more explicit when it comes to giving definitions to the terms in use in mathematics class. I believe that mathematical textbooks should include a glossary of terms that include simple rigorous definitions. Barring that unlikely eventuality, I suggest that teachers discuss the definitions of terms explicitly during each topic of discussion. A clear consistent concept definition between students cannot solve all problems, but it seems like a clear starting point. A common understanding of what applies and what does not apply to a term, situation or theorem is essential. Perhaps a focus on definitions would also help change the attitude of students away from simple calculation towards a more rigorous analysis of mathematics. A classroom focus on definitions may also help students see that there may be more than one definition of a concept that is useful in different contexts, or that definitions are arbitrary and as long as there is agreement, definitions can be created differently. Attention to definitions will ideally help students have consistent concept definitions.

5.2.2. Attention to the Base

This study has shown repeatedly that students did not attend to the base of a power expression when given problems that deal with square numbers and square roots. I believe that this is due to the fact that students are not usually given problems that require them to do so. In their mathematics texts problems commonly deal with small numbers that are also often prime, and therefore, students have little experience with problems that require attention to the base. These examples artificially limit the concept image that students have of square numbers, to either perfect squares with no exponent, or numbers with transparent even exponents. I encourage teachers to give students a wider range of problems including those that force attention to the base. By providing students with problems and examples that require students attend to another part of the expression, in the future, they may be more likely to look for salient features without being prompted to so.

Examples of problems that require attention to the base from this study are: “For what values of $a$ can $a^3$ be a perfect square?”, “Consider $36^2$, $36^3$, $36^4$, $36^5$, $36^6$, $36^7$. Circle the perfect squares.” and “For what values of $x$ and $y$ can $x \cdot y^2$ be a perfect square?” There are also very basic calculation problems that can be used both for
practice and for forcing attention to the base: “Without the use of a calculator, find the square root of 9³.”

5.2.3. **Geometrical Approach**

This research has primarily focussed on square numbers and square roots in an algebraic sense, which is not surprising based on the focus that the current curriculum places on algebra. However, square numbers and square roots are derived originally from the actions of drawing physical squares and measuring the areas and side lengths of those squares. Perhaps a more geometrical approach to teaching square numbers and square roots would allow students to create more robust concept images of these terms. Some students in this study already demonstrated the willingness to think about square numbers as squares. In Excerpt 2 of Section 4.2.1. we can see Maya using a geometrical image “Well because, what I understood of squares or perfect squares was that, well this would be a cube… wait that makes no sense cause it’s a square still…but for a square what I thought they meant was like a 2D form…”. A greater attention to the geometrical underpinnings of square numbers may aid students in their understanding and visualization of both square numbers and square roots.

5.2.4. **Big Numbers**

Problems that use big numbers, too big for brute force calculation and possibly too big for calculators, may be offered in order to force students to attend to the structure of the expression rather than simply calculating the answer. Big numbers are still concrete quantities, so students will behave in a different manner than if they are just given algebraic expressions with variables. Zazkis (2008) gives an example of a series of problems that ask students to find an example that has certain properties. This series increased the size of numbers involved, and gradually forced students to abandon guess and check approaches in favour of approaches that attended to the structure of the problem. The final problem in the series asked students to find a number that gave a remainder of 123 in division by 247. Zazkis found that students who were competent at the calculations would continue to use guess and check strategy for small numbers, and it was not until the numbers became large enough that calculations were tedious to perform that students began to attend to the structure of the problem.
Examples of problems with big numbers that apply particularly to square numbers are: “Is $3^{150}$ a square number?”, “Find the square root of $a^{168}$” and “Find the square root of $19^{280} \cdot 13^{176} \cdot 4^{555}$”. These problems still offer concrete numbers to manipulate, but move students away from calculations.

5.2.5.  **Non-Routine Problems**

Together with problems that attend to the base of a power expression and problems that include big numbers, non-routine problems should be offered to students. Often the problems assigned to students rely heavily on calculations and procedure. While knowing the correct procedures and being able to correctly perform calculations are both vital, with the current technology available to all students, this focus on the solution is short-sighted: “Routine problems even many routine problems maybe necessary in teaching mathematics but to make the students do no other kind is inexcusable” (Pólya, 1945, p.172). Problem solving and non-routine problems ensure that students have experience using their knowledge when not specifically prompted to do so. These problems will give students opportunities to understand the mathematical structure that underlies the procedures and calculations, and allow them to stretch and expand their concept images of square numbers.

Examples of non-routine problems include the problem from this study “Consider $36^2$, $36^3$, $36^4$, $36^5$, $36^6$, $36^7$. Circle the perfect squares.” During the clinical interviews this problem prompted more thought and unsolicited analysis than any other question. Students were surprised by the outcome of their calculations and were interested in the series. The companion series “Consider $12^2$, $12^3$, $12^4$, $12^5$, $12^6$, $12^7$. Circle the perfect squares.” was a formidable follow up problem, as some students believed that the evenness of the base was the reason all were perfect squares in the first series. Other suggestions include asking students to construct examples that fit certain criteria as suggested by Zazkis and Leikin (2007). Asking students to create examples also forces them to attend to the structure of the mathematical objects that they are dealing with. Other non-routine problems include, but are not limited to, problems that begin with “How many…”, “When can…”, “Find an example…” and “For what values…” All of these types of questions move students away from simple rote calculations and towards thinking about the structure of the problems.
5.2.6. **Solution Approaches**

Students should be encouraged to use more efficient solution approaches that require attention to the structure of the problem. In traditional mathematics classes, teachers often require students to “Show all your work!” and inadvertently may be keeping students from progressing to more advanced solution approaches. This requirement to show all the steps of the problem, usually a regurgitation of the steps provided by the teacher, keeps students in the solution approach of brute force, and does not allow for mathematical thinking. In this manner, some very conscientious students may continue with brute force approaches because they have learned that that is the “right way” to do mathematics.

Teachers should encourage students in their use of novel approaches or any demonstration of logical thinking. Teachers can also push students towards more advanced solution approaches by supplying problems like those listed above, that make brute force or wild guess and check too tedious to use.

5.3. **Limitations of Study**

This study exhibits several limitations in three main categories: the participants, the scope of the study and the experience of the interviewer.

With nine clinical interviews there is little that this study can say about the frequency of the phenomena described here. There may also be phenomena that were not captured at all during the interviews. How widespread the ideas about the definitions or square numbers, distribution and representation are is impossible to say. The participants were also all relatively high achieving students; their knowledge and use of square numbers may be very different from those of other students. In particular, the solution approaches chosen by these students, may be very different from those chosen by a different type of student.

There are also some limitations that involve the scope of the study. Many aspects of student understanding of square numbers and square roots were eliminated from the study, including students’ use of prime factorization and irrational numbers.
Other topics that were included in the study still require further consideration, for example, I wished to delve more deeply into students’ concept images of the distribution of square numbers and their ability to work with them. This study also offers less information that I would like relating to students’ definitions of both square numbers and square roots.

Lastly, I had little experience as an interviewer; there were avenues of questioning to explore that I did not notice until reviewing the clinical interview transcripts after the interviews were complete. A more experienced interviewer may have been able to extract more pertinent information from the participants.

5.4. Areas of Further Study

Further study is warranted in several areas: the type of learner, big numbers as a route to advanced solution approaches and mathematical language considerations.

Investigation into whether the ideas and phenomena discussed in this study are representative of a wide range of learner is called for. Younger students who have more recently been taught square numbers, as its own topic, rather than simply treated as a special case of exponents in general, may have a much different concept image of square numbers. More mathematically competent students may have a much more robust concept image of square numbers and square roots that includes connections to a wider range of phenomena. From personal anecdotal evidence I know that both the concept images and the concept definitions of square numbers among mathematics teachers vary widely. How might these differing concept images affect students’ abilities to solve problems?

Further study into ‘big numbers’ is warranted. What role do ‘big numbers’ play in ‘forcing’ students to attend to the structure of the problems that they are attempting to solve? Can big numbers discourage students from simply mechanically performing calculations and encourage them to attend the structure of the problem at hand? Similarly, what are the best practices teachers can use to encourage students to use more ‘advanced’ solution approaches?
The role of representation as it relates to familiarity is also intriguing. In this study very few students believed that $x^2 + y^2$ could be a square number, and yet these same students had a great deal of experience with the Pythagorean theorem. How many other mathematics topics do teachers constantly show in the same format? Are students able to separate the concept from the image of that format?

Finally the role that language may play in mathematics concept development is intriguing. Mathematics is often perceived as having little to do with language, besides discussing algebraic syntax or decoding word problems. However I have shown that there is confusion between the terms square number and square root. Is the confusion between square numbers and square roots unique, or do other pairs or similar sounding terms create confusion also? If this type of confusion is widespread, how can teachers address the confusion and minimize the problem?

While conducting this study I have found that I have developed a greater insight into the difference between student understanding and ability to “get the answer”. I have been used to marking assessments based on the solution provided to me, but as I have learned while conducting this study, some students are able to arrive at a correct solution through guess and check or laborious brute force methods, without as much understanding as someone else who may arrive at the incorrect solution after making an error. I have found that my original 6-point scale used in round 1 of coding the questionnaire to be more useful than the simple right / wrong marking I did in the past. This study has changed how I will teach in the future, as I discussed in the pedagogical considerations section, but also how I will mark and assess my students’ work.
References


Appendix A.

Square Numbers Questionnaire

Name:
Date:

Square Numbers Questionnaire

This questionnaire is designed to be completed with no calculator.

Note: $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $10^2 = 100$ numbers like these are known as square numbers or perfect squares.

1. For what values of $a$, if any, can the following be perfect squares? For each answer explain briefly.

1) $a^2$

2) $a^3$

3) $a^4$

4) $a^{100}$

5) $a^{200}$

6) $2 \cdot a^2$
2. For what values of $x$ and $y$, if any, can the following be perfect squares? For each answer explain briefly.

1) $x \cdot y$

2) $x \cdot y^2$

3) $x^2 + y^2$


4. Which of the following are perfect squares? Explain briefly.

1) $13^2 \cdot 17^2 \cdot 23^2$

2) $13^2 \cdot 17^2 \cdot 23$

3) $13^2 \cdot 17^2 \cdot 9$
5. Given a number \( a \) such that \( a^2 \) is an integer. Does this mean that \( a \) must also be an integer? Circle your answer. Yes/No. Explain / exemplify your decision.

6. Would you call \( 2.5^2 \) a square number? Explain your reasoning.

7. Estimate \( \sqrt{20} \).

8. How many perfect squares are there between 100 and 10,000? Explain briefly.

9. The number \( 576 \) is a perfect square. Find \( \sqrt{576} \).
## Appendix B.

### Round I & II Questionnaire Codes by Item

<table>
<thead>
<tr>
<th>Questionnaire Item Number</th>
<th>Correct Answer</th>
<th>Correct Reasoning</th>
<th>Partially Correct Answer and/or Partially Correct Reasoning</th>
<th>Incorrect Answer</th>
<th>Partially Correct Reasoning</th>
<th>Incorrect Answer</th>
<th>Incorrect Reasoning</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>A - any integer</td>
<td></td>
<td>A - any whole number</td>
<td>A - gives the values of a squared [lists the square numbers]</td>
<td>A - &quot;can be square if it produces a natural number&quot;</td>
<td>A - I do not understand what they are trying to say</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>A - if &quot;a&quot; is a square number</td>
<td>A - examples given</td>
<td>A - none, cube or volume</td>
<td>A - any number (no reason) or Yes</td>
<td>B - none (no explanation)</td>
<td>C - incorrect examples</td>
<td>D - examples to disprove</td>
<td>E - other</td>
</tr>
<tr>
<td>1.3</td>
<td>A - any integer</td>
<td></td>
<td>A - any whole number, even exponent</td>
<td>A - wrong exponent</td>
<td>A - no, none, (no reason)</td>
<td>B - incorrect use of example</td>
<td>C - other</td>
<td>A - I do not understand what they are trying to say</td>
</tr>
<tr>
<td>1.4</td>
<td>A - any integer</td>
<td></td>
<td>A - any whole number, even exponent</td>
<td>A - wrong exponent</td>
<td>A - no, none, (no reason)</td>
<td>B - incorrect use of example</td>
<td>C - other</td>
<td>A - I do not understand what they are trying to say</td>
</tr>
<tr>
<td>1.5</td>
<td>A - any integer</td>
<td></td>
<td>A - any whole number, even exponent</td>
<td>A - wrong exponent</td>
<td>A - no, none, (no reason)</td>
<td>B - incorrect use of example</td>
<td>C - other</td>
<td>A - I do not understand what they are trying to say</td>
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<tr>
<td>1.6</td>
<td>A - if a is a integer divided by sqrt 2</td>
<td>A - no because the 2</td>
<td>A - any number, divide first</td>
<td>A - I do not understand what they are trying to say</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.1</td>
<td>A - x and y are same</td>
<td>B - each a square already</td>
<td>A - given an example where they are different</td>
<td>A - I do not understand what they are trying to say</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.2</td>
<td>A - x is a square, y anything</td>
<td>B - if x is 1</td>
<td>A - given an example where x = 1</td>
<td>A - I do not understand what they are trying to say</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.3</td>
<td>A - all circled correct reasoning</td>
<td>A - did not subtract</td>
<td>A - all circled incorrect reasoning</td>
<td>A - all of them</td>
<td>B - infinite amount</td>
<td>C - random guess</td>
<td>D - 900, just subtraction</td>
<td>E - 400, etc, based on evenness</td>
</tr>
</tbody>
</table>

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## Appendix C.

### Round I & II Questionnaire Data by Student

<table>
<thead>
<tr>
<th>Student Number</th>
<th>Questionnaire Item Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2C 2A 2C S S S 2B 4C 2A 3A 2A</td>
</tr>
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Appendix D.

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