ON THE CORRELATION OF COMPLETELY MULTIPLICATIVE FUNCTIONS

by

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Abstract

This dissertation focuses on a conjecture of S. Chowla which asserts the equidistribution of the parity of the number of primes dividing the integers represented by a polynomial of degree d. This involves the behaviour of the classical Liouville function $\lambda(n)$ which captures the parity of the total number of prime factors of an integer n.

For any non-square polynomial $f(n)$ with integral coefficients of degree d we consider the distribution of the sequence $\{\lambda(f(n))\}_{n=1}^{\infty}$. Chowla conjectured that the partial sum average of this sequence goes to zero. In the first two chapters we study a weaker form of this conjecture for polynomials of degree 2 with integer and rational coefficients and prove that this sequence takes the values -1 and $+1$ infinitely often. In the final chapter we show that this partial sum average goes to zero when $f(n) = n(n+1)(n+2)$ and $\lambda(n)$ is replaced by the truncated Liouville function $\lambda_y(n)$.

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Contents

Preface

In this dissertation we study a conjecture of S. Chowla which asserts the equidistribution of the parity of the number of primes dividing the integers represented by a polynomial of degree d. This concerns the behaviour of the classical Liouville function $\lambda(n)$ which gives the parity of the total number of prime factors of an integer n counted with multiplicity.

We are interested in studying the distribution of the sequence $\{\lambda(f(n))\}_{n=1}^{\infty}$ where $f(n)$ is a non-square polynomial with integral coefficients of degree d. Chowla conjectured that this sequence had asymptotic mean zero. We prove this conjecture when $f(n) = n(n+1)(n+2)$ is a product of three consecutive integers and λ is replaced by a truncated Liouville function λ_y. We also study a weaker version of Chowla's conjecture for polynomials of degree 2 with integer and rational coefficients and prove that this sequence takes the values -1 and $+1$ infinitely often thereby extending some partial known results.

This dissertation is divided into five chapters: one for each of the three key results, together with an introductory and concluding chapter with some open questions for future exploration.

In Chapter 1, we give an introduction to the theory involving the Liouville function and its connection to the classical theory of the distribution of primes. This leads us to study Chowla's conjecture and we give a detailed analysis about the depth and significance of this conjecture. Finally, we mention a conjecture of Cassaigne et al. which is a weaker version of Chowla's conjecture.

In Chapter 2, we prove our first main result. We first introduce the previous results of

Cassaigne et al. conjecture known in the literature. One of our main results is that the sequence $\{\lambda(f(n))\}_{n=1}^{\infty}$ cannot be eventually constant for any quadratic integer polynomials $f(n)$ and if there is one sign change, then there are infinite many sign changes. Moreover in practice, using this result one can easily verify the Cassaigne et al. conjecture for any given integer quadratic polynomial by computer.

In Chapter 3, we establish our second result. Motivated by our earlier work in the previous chapter, we continue to study the equation $f(g(x)) = f(x)h^m(x)$ in the unknown polynomials f, g, h over any field K and we completely describe the set of solutions to this equation in the case when f is a non-constant and separable polynomial, $\deg(g) \geq 2$ and the derivative of g is nonzero. Using some explicit construction of zero sets and degree arguments we prove that the solutions include only linear and quadratic polynomials f, and in the case when f is quadratic, the polynomials g and h are given by usual Chebyshev polynomials of the first and second type, respectively. Defining the Liouville function over the rationals, one can easily prove the analogue of the main result in Chapter 2, namely, either $\lambda(f(r))$ is constant for all rational numbers r greater than the largest real root of $g(x)-x$ or it changes sign infinitely often, where $f(r)$ is a quadratic polynomial with rational coefficients.

In Chapter 4, for any arithmetic function $f(n)$ and $x > 0$, we define the triple order correlation function $C(f, x)$ by

$$C(f,x) = \sum_{n \leq x} f(n)f(n+1)f(n+2).$$

One would naturally expect from Chowla's conjecture that the sum $C(\lambda, x)$ is of order $o(x)$ when $x \to \infty$ which seems to be a hopelessly difficult task at present. This motivates us to study functions which are in some way "close" to the Liouville function but its correlation function would be easier to estimate. In particular, $\lambda_y(n)$ denotes the truncated Liouville function which equals -1 or $+1$ according to whether n has an odd or even number of prime divisors $p \leq y$ counted with multiplicity. Now in this setting, we take a modest step ahead in Chapter 4 and we succeed in obtaining an asymptotic formula for the average $\lambda_y(n(n+1)(n+2))$ up to $n \leq x$. It follows from the main theorem that the analogous Chowla's conjecture holds whenever $\log(y) = o(\log(x))$ and gives evidence in favour of Chowla's conjecture for the classical Liouville function. The crucial step here is to use the method of smooth primes which helps to transform the sum into a sieve problem and using fundamental lemma type results of Brun's sieve one obtains desired results.

The final chapter of this dissertation contains a collection of related questions and conjectures for future study.

All of the results of this thesis have been published or are in preparation to be submitted for publication. We have taken without hesitation from articles to which the author has been a major contributor ([5], [20]).

Chapter 1

Introduction

1.1 Dirichlet series and Asymptotics

One of the basic goals in analytic number theory is to obtain statistical information about a sequence $\{a_n\}$ of real or complex numbers from the analytic behaviour of an appropriate generating function, such as power series $\sum a_n z^n$ or a Dirichlet series $\sum a_n n^{-s}$.

A Dirichlet series is a series of the form $G(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ where s is a complex variable and we write $s = \sigma + it$. If we have a second Dirichlet series $H(s) = \sum_{m=1}^{\infty} b_m m^{-s}$ then

$$G(s)H(s) = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} a_k b_m (km)^{-s} = \sum_{n=1}^{\infty} \left(\sum_{km=n} a_k b_m \right) n^{-s}.$$

Thus the product is a Dirichlet series $\sum_{n=1}^{\infty} c_n n^{-s}$ whose coefficients are $c_n = \sum_{km=n} a_k b_m$. The Riemann zeta function is one of the special Dirichlet series, which for $\sigma > 1$ is defined by the absolutely convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

In order to investigate the asymptotic behaviour of the generating functions we have some special notations which are commonly used in number theory and all of mathematics. Let D be a subset of the complex numbers $\mathbb{C}$ and let $f : D \to \mathbb{C}$ be a complex valued map defined on D and let $g : D \to \mathbb{R}^+$. For example, we say '$f(x)$ is asymptotic to $g(x)$' as x

tends to infinity and write $f(x) \sim g(x)$ $(x \to \infty)$, if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

This notation is commonly used in the formulation of the Prime Number Theorem (PNT), which gives the asymptotic size of the number $\pi(x)$ of prime numbers not exceeding x; $\pi(x) = \sum_{p\leq x} 1$. Legendre conjectured about this in 1798, and it was proved independently by Hadamard [24] and de la Vallée Poussin [12] in 1896. The Prime Number Theorem asserts that

$$\pi(x) \sim \frac{x}{\log x}.$$

We can also view the above as

$$\pi(x) = (1 + o(1))\frac{x}{\log x},$$

which means that $\pi(x)$ is $x/\log(x)$ plus an error term that is in the limit negligible compared with $x/\log(x)$. In general we say, '$f(x)$ is small oh of $g(x)$' and write $f(x) = o(g(x))$, if $\lim_{x\to\infty} f(x)/g(x) = 0$. Quantitatively, the Prime Number Theorem says that

$$\pi(x) = \frac{x}{\log x} + O(\frac{x}{(\log x)^2}).$$

The above simply means that there is a constant $C > 0$ such that the inequality

$$\left|\pi(x) - \frac{x}{\log x}\right| \leq \frac{Cx}{(\log x)^2}$$

holds for all $x \geq 2$. In general, we say that '$f(x)$ is big oh of $g(x)$' and write $f(x) = O(g(x))$ if there is a constant $C > 0$ such that $|f(x)| \leq Cg(x)$ for all $x \in D$ in the domain. Note that the function f may be complex valued, but g is necessarily non-negative. Sometimes we use the notation $f(x) \ll g(x)$ instead of $f(x) = O(g(x))$ and such a situation arises when there is no main term. We say $f(x)$ is less-than-less-than $g(x)$. In this context it is worthwhile to mention that Chebyshev proved that $\pi(x) \ll x/\log(x)$. This is a much weaker statement than the Prime Number Theorem and was proved earlier in 1852. Chebyshev also showed that $\pi(x) \gg x/\log(x)$.

In general, we say that '$f(x) \gg g(x)$' if there is a positive constant c such that $|f(x)| \geq c(g(x))$ for all $x \in D$ and g is non-negative. Note that here f and g take only positive values. If both $f(x) \ll g(x)$ and $f(x) \gg g(x)$ then we say that f and g have the same order of magnitude and write $f \asymp g$. Thus Chebyshev's estimates [1] can be written as

$$\pi(x) \asymp \frac{x}{\log(x)}.$$

1.2 Multiplicative arithmetic functions

A function is called an arithmetic function if its domain is the set $\mathbb{N}$ of natural numbers or positive integers. An arithmetic function $f(n)$ is said to be multiplicative if $f(1) = 1$ and if $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. Also, an arithmetic function is called totally or completely multiplicative if $f(1) = 1$ and if $f(mn) = f(m)f(n)$ for all m and n. If f is multiplicative then the Dirichlet series $\sum f(n)n^{-s}$ factors into a product over prime powers. When the product

$$\prod_p (1 + f(p)p^{-s} + f(p^2)p^{-2s} + f(p^3)p^{-3s} + \cdots)$$

is expanded, any arbitrary term looks like

$$\frac{f(p_1^{k_1})f(p_2^{k_2})\cdots f(p_r^{k_r})}{(p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r})^s}.$$

Setting $n = p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$, since f is multiplicative, the above expression can be seen as $f(n)n^{-s}$. Now in view of the fundamental theorem of arithmetic, after rearranging the terms we get

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_p (1 + f(p)p^{-s} + f(p^2)p^{-2s} + f(p^3)p^{-3s} + \cdots).$$

If f is completely multiplicative, the product simplifies and we have

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_p \frac{1}{1 - f(p)p^{-s}}.$$

Note that in each case the product on the right-hand side is called the Euler product of the Dirichlet series. Also the above holds only under the stronger assumption of absolute convergence.

1.3 Liouville and Möbius functions

In this section we introduce two very important multiplicative arithmetic functions. Let $\omega(n)$ denote the number of distinct primes dividing n and let $\Omega(n)$ be the number of distinct prime powers dividing n. So we can write

$$\omega(n) = \sum_{p|n} 1, \quad \Omega(n) = \sum_{p^k|n} 1 = \sum_{p^k\|n} k.$$

Note that $\omega(n) \leq \Omega(n)$ for all n, with equality if and only if n is square-free. These functions are examples of additive functions because they satisfy the functional relation $f(mn) = f(m) + f(n)$ whenever $(m, n) = 1$. In other words $\Omega(n)$ is a totally additive function satisfying the above functional relation for all pairs m, n. It is important to note that an exponential of an additive function is a multiplicative function. In particular, the Liouville lambda function $\lambda(n) = (-1)^{\Omega(n)}$ is the completely multiplicative function (i.e. $\lambda(mn) = \lambda(m)\lambda(n)$ for all $m, n \in \mathbb{N}$) and is closely related to the Möbius μ function, which is defined by

$$\mu(n) = \begin{cases} (-1)^{\omega(n)} = \lambda(n) & \text{if } n \text{ is square-free,} \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, in view of fundamental theorem of arithmetic we can say that $\lambda(n)$ is the unique completely multiplicative function and takes the value -1 at every prime. So $\lambda(1) = \lambda(4) = \lambda(6) = \lambda(9) = \lambda(10) = 1$ and $\lambda(2) = \lambda(5) = \lambda(7) = \lambda(8) = -1$. Let $\zeta(s)$ denote the Riemann zeta function, defined for complex s with $\Re(s) > 1$ by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

where the product is over all prime numbers p. Thus

$$\frac{\zeta(2s)}{\zeta(s)} = \prod_p \left(1 + \frac{1}{p^s}\right)^{-1} = \prod_p \left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}. \tag{1.3.1}$$

Let $L(x)$ denote the sum of the values of $\lambda(n)$ up to x,

$$L(x) := \sum_{1 \leq n \leq x} \lambda(n)$$

so that $L(x)$ records the difference of the number of positive integers up to x with an even number of prime factors (counted with multiplicity) and those with an odd number. Pólya in 1919 showed in [48] that the Riemann Hypothesis, which says that all the non-trivial zeros of $\zeta(s)$ are on the critical line $\Re(s) = 1/2$, will follow if $L(x)$ does not change sign for sufficiently large n. There is a vast amount of literature about the study of the sign change of $L(x)$. Until 1958, Haselgrove proved that $L(x)$ changes sign infinitely often in [29]. For more discussion about this problem, we refer the reader to [4].

The summatory function of the Liouville λ-function is closely associated to some of the most fundamental problems in mathematics. In particular, the prime number theorem is equivalent to the statement that $\sum_{n\leq x}\lambda(n)=o(x)$ [37] and the Riemann hypothesis [3] is equivalent to the statement that for every $\epsilon>0$, we have $\sum_{n\leq x}\lambda(n)=O(x^{1/2+\epsilon})$.

1.4 Chowla's conjecture and the Liouville function at consecutive integers

Complementary to the prime number theorem, in 1965 Chowla [11] conjectured that

Conjecture (Chowla).

$$\sum_{n\leq x}\lambda(f(n))=o(x)$$

for any polynomial $f(x)$ with integer coefficients which is not of form $bg(x)^2$ for some $b\neq 0$ and $g(x)\in\mathbb{Z}[x]$.

The above conjecture asserts that the Liouville function assumes the values -1 and $+1$ with roughly equal frequency on polynomials which are not constant multiples of the square of a polynomial and can be said to represent the parity problem ([53], [30]) in its purest form. Some cases of Chowla's conjecture were already included in the Hardy-Littlewood [27] conjectures. The conjecture for linear functions was already proved by Landau [37] but for the degree greater than 1, the conjecture seems to be extremely hard and still remains wide open. The first non-trivial case seems to be when $f(n)=n(n+1)$ i.e. $f(n)$ is a product of consecutive integers. One might expect that this "correlation" of the Liouville function would give

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leq x}\lambda(n)\lambda(n+1)=0, \tag{1.4.1}$$

but at present even the much weaker relation

$$\liminf_{x\to\infty}\frac{1}{x}\sum_{n\leq x}\lambda(n)\lambda(n+1)<1$$

has not been proved. It is worthwhile to mention that although Halász [25] completely determined the asymptotic behaviour of the means

$$\frac{1}{x}\sum_{n\leq x}g(n)$$

for multiplicative functions g satisfying $|g| \leq 1$, unfortunately, his analytic method cannot be used to deal with (1.4.1), since the corresponding Dirichlet series do not have an Euler product representation, and it seems that the problem lies very deep.

In view of this Hildebrand [32] writes: "In other words, it is conceivable that for 'most' n we have $\lambda(n) = \lambda(n+1)$, in which case the functions $\lambda(n)$ and $\lambda(n+1)$ are far from being independent. It is possible (1.4.1) lies deep as the twin prime conjecture, for it amounts to resolving, the 'parity problem' in sieve theory, which constitutes the main obstacle to proving the twin prime conjecture by sieve methods ([19], [35])."

In a recent paper Pintz [47] establishes the following interesting connection :

Theorem (Pintz). *Suppose that with $\vartheta = \vartheta_1 > \frac{3}{4}$ and $f(n) = \lambda(n)\lambda(n+h)$, where h is any positive even integer, the following holds*

$$\sum_{q \leq N^{\vartheta-\epsilon}} \max_a \left| \sum_{\substack{n \equiv a \bmod q \\ n \leq N}} f(n) \right| \ll_{\epsilon, A} \frac{N}{\log^A N}.$$

Then $p+h$ is prime for infinitely many primes p.

The best unconditional results when $f(n) = n(n+1)$ was given by Harman, Pintz and Wolke [28] in 1985. They proved that for $x > x_0(\epsilon)$

$$-(1+o(1))\frac{1}{3} < \frac{1}{x}\sum_{n \leq x} \lambda(n(n+1)) < 1 - \frac{1}{(\log x)^{7+\epsilon}}.$$

The lower bound in the above inequality improves bounds by R. Hall (unpublished), and Graham and Hensley [22], and was generalized in Cassaigne et al. [9]. In other words, there are infinitely many sign changes, but one is very far from proving an expected result of $o(1)$. For longer patterns, later Elliott [18] in 1992 in his paper quotes that "Even more hopeless seems the conjecture that

$$x^{-1} \sum_{n \leq x} \lambda(n)\lambda(n+1)\lambda(n+2) \to 0, \quad x \to \infty.$$

It therefore comes as a surprise that

$$\limsup_{x \to \infty} \frac{1}{x} \left| \sum_{n \leq x} \lambda(n)\lambda(n+1)\lambda(n+2)) \right| \leq \frac{20}{21}$$

can be obtained." Note that this result was again improved and generalized by Cassaigne et al. [9].

Results of this type have been extended in several directions: instead of studying the Liouville λ- function at consecutive integers, one can study it on arithmetic progressions. A deep theorem of Green and Tao [23] implies that

$$\sum_{d \leq x} \sum_{n \leq x} \lambda(n)\lambda(n+d)\lambda(n+2d)\lambda(n+3d) = o(x^2).$$

This implies, averaged over d and n, that the product of the four values of the λ- function is asymptotically as often positive as it is negative.

However one can consider a weaker form of Chowla's conjecture, namely,

Conjecture (Cassaigne et al.). *If $f(x) \in \mathbb{Z}[x]$ is not of form $bg^2(x)$, then $\lambda(f(n))$ changes sign infinitely often.*

Although it is weaker, the above conjecture formulated by Cassaigne et al.[10] is still wide open for polynomials of degree > 1. One can also find similar formulation of the above conjecture in the works of Kátai [36] and Sárközy [50]. In the next two chapters we will study this weaker conjecture and prove a number of new results in this direction.

Chapter 2

Liouville function on Integer Quadratics

This chapter contains results which can be found in a collaboration with Peter Borwein and Stephen Choi (see [5] for details).

2.1 Introduction

From Chapter 1 we know that it is a consequence of the well-known prime number theorem, and also appeared in the works of Landau [37], that

$$\lim_{x\to\infty} x^{-1}\sum_{n\le x}\lambda(n) = 0.$$

Complementary to the prime number theorem, Chowla [11] made the following conjecture

Conjecture 1 (Chowla)**.** *Let $f(x)\in\mathbb{Z}[x]$ be any polynomial which is not of form $bg^2(x)$ for some $b\neq 0,\ g(x)\in\mathbb{Z}[x]$. Then*

$$\sum_{n\le x}\lambda(f(n)) = o(x). \tag{2.1.1}$$

Clearly, Chowla's conjecture is equivalent to the prime number theorem when $f(x)=x$. For polynomials of degree >1, Chowla's conjecture seems to be extremely hard and still remains wide open. One can consider a weaker form of Chowla's conjecture, namely,

Conjecture 2 (Cassaigne et al.)**.** *If $f(x)\in\mathbb{Z}[x]$ is not of form $bg^2(x)$, then $\lambda(f(n))$ changes sign infinitely often.*

Clearly, Chowla's conjecture implies Conjecture 2. In fact, suppose it is not true, i.e., there is n_0 such that $\lambda(f(n)) = \epsilon$ for all $n \geq n_0$ where ϵ is either -1 or $+1$. Then it follows that

$$\sum_{n \leq x} \lambda(f(n)) = \epsilon x + O(1)$$

which contradicts (2.1.1).

2.2 Previous Partial Results

Although it is weaker, Conjecture 2 is still wide open for polynomials of degree > 1. In [10], Conjecture 2 for special polynomials have been studied and some partial results are proved.

Theorem 2.1 (Cassaigne et al.). *Let $f(n) = (an + b_1)(an + b_2) \dots (an + b_k)$ where $a, k \in \mathbb{N}$, $b_1, \dots, b_k$ are distinct integers with $b_1 \equiv \dots \equiv b_k \pmod{a}$ then $\lambda(f(n))$ changes sign infinitely often.*

Proof. This is Corollary 2 in [10]. □

For certain quadratic polynomials, they proved

Theorem 2.2 (Cassaigne et al.). *If $f(n) = (n + a)(bn + c)$ where $a, c \in \mathbb{Z}$, $b \in \mathbb{N}, ab \neq c$ then $\lambda(f(n))$ changes sign infinitely often.*

Proof. This is Theorem 4 in [10]. □

Theorem 2.3 (Cassaigne et al.). *Let $a \in \mathbb{N}, b, c \in \mathbb{Z}$, and write $f(n) = an^2 + bn + c$, $D = b^2 - 4ac$. Assume that a, b and c satisfy the following conditions :*

(i) $2a|b$,

(ii) $D < 0$,

(iii) there is a positive integer k with

$$\lambda\left(-\frac{D}{4}k^2 + 1\right) = -1.$$

Then $\lambda(f(n))$ changes sign infinitely often.

Proof. This is Theorem 3 in [10]. □

In this chapter, we continue to study Conjecture 2 for the quadratic case. One of our main results is Theorem 2.5 below. By Theorem 2.5, in order to show that the sequence $\{\lambda(f(n))\}_{n=1}^{\infty}$ changes sign infinitely often, we only need find one pair of large integers n_1 and n_2 such that $\lambda(f(n_1)) \neq \lambda(f(n_2))$. This will make the conjecture much easier to handle. Some partial results from Theorem 2.5 are also deduced in the next section.

2.3 Main Results

Conjecture 2 for the linear polynomial is easily solved by the following result.

Theorem 2.4. *Let $P := \{n \in \mathbb{N} : \lambda(n) = +1\}$ and $N := \{n \in \mathbb{N} : \lambda(n) = -1\}$. Then both P and N cannot contain infinite arithmetic progression. In particular, $\lambda(an + b)$ changes sign infinitely often in n.*

Proof. We claim that both P and N cannot contain any infinite arithmetic progression. Suppose not and there are an n_0 and l such that

$$\lambda(n_0 + lk) = \lambda(n_0) \tag{2.3.1}$$

for $k = 0, 1, 2, \ldots$. Pick a prime p which is of the form $lm + 1$. Now put $k = mn_0$ and consider

$$\lambda(n_0 + lk) = \lambda(n_0 + lmn_0) = \lambda(n_0)\lambda(lm + 1) = \lambda(n_0)\lambda(p) = -\lambda(n_0).$$

This contradicts (2.3.1). Hence our claim is attained. $\square$

One of the main results in this paper is the following theorem.

Theorem 2.5. *Let $f(x) = ax^2 + bx + c$ with $a > 0$ and l be a positive integer such that al is not a perfect square. Then if the equation $f(n) = lm^2$ has one solution $(n_0, m_0) \in \mathbb{Z}^2$, then it has infinitely many positive solution $(n, m) \in \mathbb{N}^2$.*

Proof. Let $D = b^2 - 4ac$ be the discriminant of $f(x)$. By solving the quadratic equation

$$an^2 + bn + c = lm^2, \tag{2.3.2}$$

for n we get

$$n_0 = \frac{-b \pm \sqrt{b^2 - 4a(c - lm_0^2)}}{2a} = \frac{-b \pm \sqrt{D + 4alm_0^2}}{2a}.$$

It follows that $D + 4alm_0^2 = t_0^2$ for some integer t_0. By choosing a suitable sign of t_0, we may assume

$$t_0 \equiv b \pmod{2a}, \qquad \text{and} \qquad n_0 = \frac{-b + t_0}{2a} \in \mathbb{Z}. \tag{2.3.3}$$

This leads us to consider the diophantine equation

$$t^2 = 4alm^2 + D. \tag{2.3.4}$$

Suppose that (t_0, m_0) and (t, m) are solutions of (2.3.4). Then we have

$$t^2 = 4alm^2 + D$$

and

$$t_0^2 = 4alm_0^2 + D.$$

Subtracting the above two equations, we get

$$(t - t_0)(t + t_0) = l(m - m_0)(4am + 4am_0). \tag{2.3.5}$$

We now let s and r be

$$r(m - m_0) = 2as(t + t_0)\ ,\ 2as(4alm + 4alm_0) = r(t - t_0). \tag{2.3.6}$$

By eliminating the terms t and m respectively in (2.3.6), we get

$$(r^2 - 16a^3ls^2)m = r^2m_0 + 16a^3ls^2m_0 + 4arst_0 \tag{2.3.7}$$

and

$$(r^2 - 16a^3ls^2)t = r^2t_0 + 16a^2ls^2m_0 + 16a^3s^2lt_0. \tag{2.3.8}$$

Note that by our assumption, $16a^3l$ is not a perfect square. So the Pell equation

$$r^2 - 16a^3ls^2 = 1 \tag{2.3.9}$$

always have infinitely many solutions $(r, s) \in \mathbb{Z}^2$, and for each solution (r, s) of the Pell equation (2.3.9) gives integers m and t through (2.3.7) and (2.3.8) such that

$$m = r^2m_0 + 16a^3ls^2m_0 + 4arst_0$$

and

$$t = r^2t_0 + 16a^2lsrm_0 + 16a^3s^2lt_0.$$

One can easily verify that if $(r, s) \neq (\pm 1, 0)$ then these m and t satisfy the equations (2.3.6) and hence satisfy equation (2.3.4). Note that $r^2 \equiv 1 \pmod{2a}$ and $r(m - m_0) \equiv 0 \pmod{2a}$. Hence we have $m \equiv m_0 \pmod{2a}$ and $t \equiv t_0 \pmod{2a}$ by (2.3.6). Since there are infinitely many solutions $(r, s) \in \mathbb{Z}^2$ of the Pell equation (2.3.9) and these will give infinitely many solutions $(m, t) \in \mathbb{Z}^2$ of the equation (2.3.6). In particular, there are infinite many positive integers t such that $t \equiv t_0 \pmod{2a}$ and

$$n = \frac{-b + \sqrt{D + 4alm^2}}{2a} = \frac{-b + t}{2a}$$

to be a positive integer by (2.3.3). Therefore, there are infinitely many positive solutions $(n, m) \in \mathbb{N}^2$ of (4.4.9). This completes the proof of the theorem. $\square$

It is worth to mention that one should not expect Theorem 2.5 is true for polynomials of higher degree because they may only have finitely many integer solutions by Siegel's theorem on integral points in [54].

In view of Theorem 2.5, to determine the conjecture is true for a given quadratic polynomial $f(x)$, we only need to find one pair of positive integers n_1 and n_2 such that $\lambda(f(n_1)) \neq \lambda(f(n_2))$. This gives us the following theorem.

Theorem 2.6. *Let $f(x) = ax^2 + bx + c$ with $a \in \mathbb{N}$ and $b, c \in \mathbb{Z}$. Let*

$$A_0 = \left[\frac{|b| + (|D| + 1)/2}{2a}\right] + 1.$$

Then the binary sequence $\{\lambda(f(n))\}_{n=A_0}^{\infty}$ is either a constant sequence or it changes sign infinitely often.

Proof. Suppose $\{\lambda(f(n))\}_{n=A_0}^{\infty}$ is not a constant sequence. Then there are positive integers $n_1 \neq n_2 \geq A_0$ such that $\lambda(f(n_1)) \neq \lambda(f(n_2))$. Hence there are positive integers l_1, l_2 and m_1, m_2 such that

$$\lambda(l_1) = +1, \qquad \text{and} \qquad \lambda(l_2) = -1, \tag{2.3.10}$$

and

$$f(n_1) = l_1 m_1^2, \qquad \text{and} \qquad f(n_2) = l_2 m_2^2.$$

We claim that al_1 and al_2 are not perfect squares. If $al_j = N^2$ is a perfect square, then the diophantine equation $t^2 = D + 4al_j m^2$ has only finitely many solutions (t, m). In fact, since $(t_j - 2Nm_j)(t_j + 2Nm_j) = D$, so there is $d \neq 0$ such that $t_j + 2Nm_j = D/d$ and $t_j - 2Nm_j = d$. It follows that $2t_j = D/d + d$. Thus,

$$|t_j| \leq \frac{1}{2}\left(\frac{|D|}{|d|} + |d|\right) \leq \frac{|D|+1}{2}.$$

Since $f(n_j) = l_j m_j^2$, so

$$n_j = \left|\frac{-b \pm \sqrt{D + 4al_j m_j}}{2a}\right| \leq \frac{|b| + |t_j|}{2a} \leq \frac{|b| + (|D|+1)/2}{2a} < A_0.$$

This contradicts $n_j \geq A_0$. Therefore from Theorem 2.5, there are infinitely many n_1 and n_2 such that $\lambda(f(n_1)) \neq \lambda(f(n_2))$ and hence $\lambda(f(n))$ changes sign infinitely often. □

As we remarked above, one should not expect Theorem 2.6 to be true for polynomials of higher degree.

2.4 Special Quadratic Polynomials

We prove some partial results of special quadratic polynomials.

Theorem 2.7. *Let $f(n) = n^2 + bn + c$ with $b, c \in \mathbb{Z}$. Suppose there exists a positive integer $n_0 \geq A_0$ (with $a = 1$) such that $\lambda(f(n_0)) = -1$. Then $\lambda(f(n))$ changes sign infinitely often.*

Proof. We observe the following identity

$$f(n)f(n+1) = f(f(n) + n)$$

which can be verified directly. Hence we have

$$\lambda(f(n))\lambda(f(n+1)) = \lambda(f(f(n) + n)). \tag{2.4.1}$$

If $\{\lambda(f(n))\}_{n=1}^{\infty}$ is a constant sequence, then it follows from (2.4.1) that

$$\lambda(f(n)) = +1, \qquad \text{for all } n = 1, 2, \ldots.$$

Therefore if there is $n_0 \geq A_0$ such that $\lambda(f(n_0)) = -1$, then by Theorem 2.6, $\{\lambda(f(n))\}_{n=1}^{\infty}$ changes sign infinitely often. This proves the theorem. □

The proof of Theorem 2.5 shows that the solvability of the diophantine equation

$$X^2 - 4alY^2 = D \tag{2.4.2}$$

is critical in solving the problem. In general, there is no simple criterion to determine the solvability of the equation (2.4.2) except if we know the central norm of the continued fraction of the irrational number $\sqrt{al}$. For more discussion in this area, we refer the readers to [40]-[43]. The following theorem deals with a special case of D for which we can solve the equation (2.4.2).

Theorem 2.8. *Let $f(x) = px^2 + bx + c$ with prime number p. Suppose the discriminant $D = b^2 - 4pc$ is a non-zero perfect square. Then $\lambda(f(n))$ changes sign infinitely often.*

Proof. We first choose positive integers l_1 and l_2 such that $p\,l_1$ and $p\,l_2$ are not perfect squares and $\lambda(l_1) \neq \lambda(l_2)$. So the Pell equations

$$X^2 - 4p\,l_jY^2 = 1, \quad j = 1, 2 \tag{2.4.3}$$

have infinitely many positive solutions (X, Y). Let $D = q^2$ with $q \geq 1$. Then any positive solution (X, Y) of (2.4.3) gives a positive solution (qX, qY) of

$$X^2 - 4p\,l_jY^2 = D.$$

We can choose X large enough so that $-b + qX > 0$. On the other hand, we have $X^2 \equiv 1 \pmod{p}$ by (2.4.3) and $q^2 \equiv b^2 \pmod{p}$ because $D = b^2 - 4pc$. Therefore $(qX)^2 \equiv b^2 \pmod{p}$. Since p is a prime, so either (a) $qX \equiv b \pmod{p}$ or (b) $qX \equiv -b \pmod{p}$. We define

$$n = \frac{-b \pm qX}{2p}$$

where the sign $\pm$ is determined according to cases (a) or (b) so that n is a positive integer. Therefore (n, qX) is a positive solution of the equations $f(n) = l_jm^2$. Then our theorem follows readily from Theorem 2.5. $\square$

Chapter 3

Liouville function on Rational Quadratics

This chapter contains results which can be found in a collaboration with Jonas Jankauskas (see [20] for details).

3.1 Introduction

We observe that from the previous chapter even the much weaker conjecture of Cassaigne et al. which states

Conjecture (Cassaigne et al.) *If $f(x) \in \mathbb{Z}[x]$ and is not of the form of $bg^2(x)$ for some $b \neq 0$ and $g(x) \in \mathbb{Z}[x]$, then $\lambda(f(n))$ changes sign infinitely often.*

has not been proved unconditionally for the polynomials of degree $\deg f \geqslant 2$.

In Chapter 2, it has been proved that the sequence $\{\lambda(f(n))\}_{n=1}^{\infty}$ cannot be eventually constant for quadratic integer polynomials $f(x) = ax^2 + bx + c$, provided that at least one sign change occurs for $n > (|b| + (|D| + 1)/2)/2a$, where D is the discriminant of $f(x)$. The proof is based on the solutions of Pell-type equations. In practice, using this conditional result, one can prove the Cassaigne's conjecture for any particular integer quadratic $f(x)$, for instance, $f(x) = 3x^2 + 2x + 1$. In contrast, the only examples of degree $\deg f \geqslant 3$ for which the conjecture has been proved in [10] are $f(x) = \prod_{j=1}^{k}(ax + b_j)$, where $a, b_k \in \mathbb{N}$, b_k are all distinct, $b_1 \equiv \cdots \equiv b_k \pmod{a}$. No similar examples of irreducible integer polynomials of degree $d \geqslant 3$ are known . It appears that the problem of finding an irreducible example of degree $d = 3$ is interesting and probably difficult.

The central problem investigated in the present chapter is motivated by the following question:

Question : Do there exist integer polynomials $f(x)$, $g(x)$ and $h(x)$ of degrees $\deg f \geqslant 3$, $\deg g \geqslant 2$, $f(x)$ separable (and possibly irreducible in $\mathbb{Z}[x]$), such that $f(g(x)) = f(x)h^2(x)$?

We now explain how the composition identity in the above question could be of use to prove that $\lambda(f(n))$ or $\lambda(f(-n))$ is not eventually constant for cubic polynomials $f(x)$. Assume that the leading coefficient of $g(x)$ is positive. Since $\deg g \geqslant 2$, there exists a positive integer n_0 such that $g(n) > n$ for integers $n > n_0$. Suppose that there exist two integers $k_0, l_0 > n_0$ such that $\lambda(f(k_0)) = -\lambda(f(l_0))$. Then $\lambda(f(k_j))$ and $\lambda(f(l_j))$ also differ in sign for infinite sequences of integers k_j and l_j, defined by $k_{j+1} = g(k_j)$ and $l_{j+1} = g(l_j)$, $j \geqslant 0$, since $\lambda(f(g(n))) = \lambda(f(n))$ follows by the composition identity.

Unfortunately, the answer to our question is negative. In the next section we prove a general result which holds for polynomials with coefficients in an arbitrary field K. Our result shows that one cannot prove the conjecture for cubic polynomials $f(x)$ by using the composition identity in the question.

3.2 Main Result

The main result of this paper is the following theorem:

Theorem 3.1. *Let $m \geqslant 2$ be an integer not divisible by the characteristic of the field K. Suppose that $f(x) \in K[x]$ is non constant and separable, and the polynomial $g(x)$, $\deg g \geqslant 2$, has a non-zero derivative. Then the equation*

$$f(g(x)) = f(x)h^m(x)$$

holds if and only if:

$$I) \qquad f(x) = ax + b, \qquad a, b \in K, a \neq 0, \qquad g(x) = \left(x + \frac{b}{a}\right) h^m(x) - \frac{b}{a}$$

or

$$II) \qquad f(x) = ax^2 + bx + c, \qquad a, b, c \in K, \quad a \neq 0, \qquad m = 2,$$

with

$$g(x) = \frac{1}{2a}\left(\pm T_n\left(\frac{2ax+b}{\sqrt{D}}\right)\sqrt{D} - b\right), \qquad h(x) = \pm U_{n-1}\left(\frac{2ax+b}{\sqrt{D}}\right),$$

where $T_n(x)$, $U_n(x)$ *are Chebyshev polynomials of the first and second kind, respectively,* $D = b^2 - 4ac$ *is the discriminant of* $f(x)$.

We remark that the condition on the separability of $f(x)$ cannot be weakened in Theorem 3.1 which can be seen by taking $f(x) = g(x) = x(x-1)^m$ in $\mathbb{Q}[x]$. The requirement that $g(x)$ has a non-zero derivative for fields K of characteristic $p \neq 0$ also cannot be weaken. Indeed, consider the simple example given by $f(x) = x^d - 1$, $g(x) = x^{p^l}$ in $\mathbb{F}_p[x]$. Also, if the characteristic p divides the exponent $m \neq 0$ in the equation $f(g(x)) = f(x)h^m(x)$, then one can write $h^m(x) = h_1^{m/p}(x^p) = h_2^{m/p}(x)$, where $h_2(x)$ is a polynomial with coefficients in K.

3.3 Chebyshev Polynomials

Recall that for the field K of characteristic not equal to 2, the *Chebyshev polynomials* $T_n(x) \in K[x]$ of the *first kind* are defined by the linear recurrence of order two:

$$T_0(x) = 1, \quad T_1(x) = x, \qquad T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x). \tag{3.3.1}$$

In the similar way, the Chebyshev polynomials of the *second kind* $U_n(x) \in K[x]$ are defined by the recurrence

$$U_0(x) = 1, \quad U_1(x) = 2x, \qquad U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x). \tag{3.3.2}$$

Polynomials $T_n(x)$ and $U_n(x)$ contain only even powers of x for even n, odd powers of x for odd n. Thus, the coefficients of $g(x)$ and $h(x)$ in Theorem 3.1, (II) lie in K if n is odd and in $K(\sqrt{D})$ if n is even. Chebyshev polynomials have many other remarkable properties, see, for instance, [49]. They play a key role in the theorems of Ritt for decompositions of polynomials [51]. In addition, Chebyshev polynomials are related to permutation polynomials over finite fields called Dickson polynomials [38]. In our proof, the following property of Chebyshev polynomials will be useful:

Theorem 3.2. *Suppose that the characteristic of the field K is not equal to 2. Then all solutions of the Pell equation*

$$P^2(x) - (x^2 - 1)Q^2(x) = 1$$

in the ring $K[x]$ are given by

$$P(x) = \pm T_n(x), \qquad Q(x) = \pm U_{n-1}(x),$$

where $T_n(x)$ and $U_n(x)$ are Chebyshev polynomials of the first and second kind, respectively.

The equation wich appears in Theorem 3.2 is a special case of a general polynomial Pell equation $P(x)^2 - D(x)Q^2(x) = 1$. Solutions to general Pell equations in polynomials over complex number field $K = \mathbb{C}$ were investigated by Pastor [46]. Dubickas and Steuding [17] gave an elementary algebraic proof for arbitrary field K. The proof of Proposition 3.2 can be found in [17]. Alternative proofs (in the case $K = \mathbb{C}$) are given in [2] and [46].

3.4 Proof of Main Theorem

Proof. Set $d = \deg f$. Let $a \in K$ and $b \in K$ be the leading coefficients of polynomials $f(x)$ and $g(x)$, respectively, $ab \neq 0$. Suppose that L is the field extension of K generated by the roots of the polynomials $f(x)$, $x^m - 1$ and $x^m - b$. Then

$$f(x) = a \prod_{\alpha \in V(f)} (x - \alpha). \tag{3.4.1}$$

Here $V(f) \subset L$ denotes the set of the roots of the polynomial $f(x)$. The composition equation $f(g(x)) = f(x)h^m(x)$ factors in $L[x]$ into

$$a \prod_{\alpha \in V(f)} (g(x) - \alpha) = a \prod_{\alpha \in V(f)} (x - \alpha)h^m(x), \tag{3.4.2}$$

and one can cancel a on both sides. Observe that distinct factors $g(x) - \alpha$ on the left hand side of (3.4.2) are relatively prime in $L[x]$ since their difference is a non-zero constant. We claim that at most one factor $g(x) - \alpha$ may be relatively prime with $f(x)$ if $m \geqslant 2$ and the characteristic of K does not divide m. Indeed, suppose that $g(x) - \beta$, $\beta \in V(f)$, $\beta \neq \alpha$ is another such factor. Then both $g(x) - \alpha$ and $g(x) - \beta$ divide $h^m(x)$, so $g(x) - \alpha$ and $g(x) - \beta$ must be the m-th powers of some polynomials $u(x)$ and $v(x)$ in $L[x]$ which divide

$h(x)$, say, $g(x)-\alpha = u^m(x)$ and $g(x)-\beta = v(x)^m$. (Note that $u(x)$ and $v(x)$ belong to $L[x]$ since the field L contains all roots of $f(x)$ and the m–th roots of the leading coefficient b of the polynomial $g(x)$). Then $u(x)^m - v(x)^m = \beta - \alpha$ is a non-zero constant polynomial. On the other hand,

$$u^m(x) - v^m(x) = \prod_{j=0}^{m-1} (u(x) - \zeta^j v(x)),$$

where ζ is a primitive m–th root of unity in L and at least one of polynomials $u(x) - \zeta v(x)$ has degree greater than or equal to one which is impossible. Note that the condition that the characteristic of K does not divide m is necessary for the separability (which is important later in the degree-counting argument, in order to establish that degree of $f(x)$ must be 1 or 2), if the field has a finite characteristic $p \neq 0$.

Now, suppose that $V(f) = \{\alpha_1, \alpha_2, \ldots, \alpha_d\}$. Let V_j be the set containing all distinct common roots of the polynomial $g(x) - \alpha_j$ and the polynomial $f(x)$,

$$V_j := V(g(x) - \alpha_j) \cap V(f).$$

Then $g(x) - \alpha_j = f_j(x)u_j(x)$, where $u_j(x) \in L[x]$ and

$$f_j(x) := \prod_{\alpha \in V_j} (x - \alpha).$$

Note that $f_j(x)$ are all separable and and coprime in $L[x]$. Since $f(x)$ is also separable, the equation (3.4.2) implies

$$a \prod_{j=1}^{d} f_j(x) = f(x) \qquad \text{and consequently,} \qquad \prod_{j=1}^{d} u_j(x) = h^m(x). \tag{3.4.3}$$

The polynomials $u_j(x)$ are relatively prime, thus $u_j(x) = h_j^m(x)$, $j = 1, \ldots, d$, for some polynomials $h_j(x) \in L[x]$ whose product is equal to $h(x)$ in (3.4.3). Let $n_j := \deg f_j$, for $j = 1, \ldots, d$. Without loss of generality, assume that $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_d$. Then $n_1 \geqslant 0$. Observe that $n_2 \geqslant 1$ if $n_1 = 0$, since no two factors $g(x) - \alpha_j$ can be coprime with $f(x)$, as noted above. The first identity in (3.4.3) gives

$$n_1 + n_2 + \cdots + n_d = \deg f = d. \tag{3.4.4}$$

Since $g(x) = f_j(x)h_j(x)^m + \alpha_j$, one also has $\deg g \equiv n_j \pmod{m}$. We now consider two cases for $\deg g$ modulo m.

Case 1). Assume that $\deg g \equiv 0 \pmod{m}$. Then $n_j \geqslant m$ for $j \geqslant 2$, hence

$$d \geqslant m(d-1) \tag{3.4.5}$$

by (3.4.4). Since $m \geqslant 2$, one has $d \geqslant 2d-2$ which is possible for $d=1$ or $d=2$ only. Suppose that $d=2$. Then one also has $m \leqslant 2$ by (3.4.5).

Case 2). Assume that $\deg g \not\equiv 0 \pmod{m}$. Then $n_1 = \cdots = n_d = 1$ by (3.4.4). Let $\deg g = sm+1$, where $s := \deg h_j \geqslant 1$ for $1 \leqslant j \leqslant d$. Since $h_j^m(x) \mid g(x) - \alpha_j$, the polynomials $h_j^{m-1}(x)$ are (relatively prime) factors of the derivative $g'(x)$. By conditions of Theorem, $g'(x)$ is a non-zero polynomial, hence

$$ms \geqslant \deg g' \geqslant \deg h_1^{m-1} + \cdots + \deg h_d^{m-1} = d(m-1)s$$

and, consequently,

$$m \geqslant d(m-1). \tag{3.4.6}$$

Then $d \leqslant m/(m-1) \leqslant 2$. Suppose $d=2$. Then, in addition, (3.4.6) gives $m \leqslant 2$.

Thus it remains to consider the cases $d=1$ and $d=2$. In the first case, the polynomial $f(x)$ is linear, thus $f(x) = ax+b$ with $a, b \in K$, $a \neq 0$. The equation $f(g(x)) = f(x)h^m(x)$ is equivalent to

$$ag(x) + b = (ax+b)h^m(x),$$

so one simplification solves $g(x)$ and this completes the proof in the case $d=1$. Suppose $d=2$. Then $f(x) = ax^2+bx+c$ with $a,b,c \in K$, $a \neq 0$. Let $D = b^2-4ac$, $D \neq 0$ since $f(x)$ is separable. One also has $m=2$ by the conditions of Theorem 3.1 and the degree inequalities in the two cases above. Hence, it suffices to find the polynomials $g(x)$ and $h(x)$ in the equation $f(g(x)) = f(x)h^2(x)$. Since the characteristic of the field K is not equal to 2 by the conditions of Theorem 3.1, the linear change of variables $x \to x(t)$ defined by

$$x = \frac{t\sqrt{D}-b}{2a}$$

transforms the polynomial $f(x)$ into

$$f(x) = \frac{D}{4a}F(t),$$

where $F(t) = t^2-1$. Set

$$G(t) := \frac{1}{\sqrt{D}}\left(2ag\left(\frac{t\sqrt{D}-b}{2a}\right)+b\right), \qquad H(t) := h\left(\frac{t\sqrt{D}-b}{2a}\right).$$

By straightforward substitution, one easily checks that the map $x \to x(t)$ transforms the composition equation $f(g(x)) = f(x)h^2(x)$ into $D/4aF(G(t)) = D/4aF(t)H^2(t)$. Canceling the factor $D/4a$ on both sides, one obtains

$$F(G(t)) = F(t)H^2(t),$$

or, equivalently,

$$G^2(t) - (t^2 - 1)H^2(t) = 1.$$

By Theorem 3.2 all the solutions to this equation are given by the formulas $G(t) = \pm T_n(t)$, $H(t) = \pm U_{n-1}(t)$, where $T_n(t)$ and $U_n(t)$ are Chebyshev polynomials of the first and second kind, respectively. Application of the inverse map $t \to t(x)$ now yields the result.

□

3.5 Rational and integer examples

Let $f(x) = ax^2 + bx + c$ be a quadratic polynomial with rational coefficients. For $n = 3$ in Theorem 3.1, one has $T_3(x) = 4x^3 - 3x$ and $U_2(x) = 4x^2 - 1$. Then $f(g(x)) = f(x)h^2(x)$ holds by Theorem 3.1 for

$$\begin{aligned} g(x) &= (16a^2x^3 + 24abx^2 + (9b^2 + 12ac)x + 8bc)/D, \\ h(x) &= (16a^2x^2 + 16abx + 3b^2 + 4ac)/D. \end{aligned} \tag{3.5.1}$$

Extend the definition of λ function to the whole set of rationals $\mathbb{Q}$ by the complete multiplicativity of λ, i.e. $\lambda(\frac{m}{n}) = \frac{\lambda(m)}{\lambda(n)}$. Then, using the method outlined in Section 3.1, one can prove easily the following analogue of Theorem 2.6 in [5] for the sign changes of λ function at rational points $f(r)$, $r \in \mathbb{Q}$:

Theorem 3.3. *$\lambda(f(r))$ is either constant for all rational numbers r greater than the largest real root of $g(x) - x$ or it changes sign infinitely often.*

The question of finding all solutions of the composition equation in integer polynomials $f(x)$, $g(x)$, and $h(x)$ is closely related to the solution of the polynomial Pell equations in $\mathbb{Z}[x]$. This does not seem to be easy. We refer the reader to the references [39], [45], [55].

Chapter 4

Triple order Correlation

4.1 Introduction

Let $\lambda(n)$ denote the Liouville function : $\lambda(n) = (-1)^{\Omega(n)}$. We know from Chapter 1 that complementary to the prime number theorem, Chowla [11] made the following conjecture

Conjecture (Chowla).

$$\sum_{n \leq x} \lambda(f(n)) = o(x) \tag{4.1.1}$$

for any polynomial $f(x)$ with integer coefficients which is not of form $bg(x)^2$.

The prime number theorem is equivalent to (4.1.1) when $f(x) = x$. Chowla's conjecture is proved for linear functions, but for degree greater than 1, the conjecture seems to be extremely hard and still remains wide open.

In fact, the case when $f(n) = n(n+1)$ remains open and is commonly considered to be as hard as the Twin Prime conjecture. Now for any arithmetic function $f(n)$ and $x > 0$, one can define a second order correlation function $H(f, x)$ by

$$H(f, x) = \sum_{n \leq x} f(n) f(n+1).$$

One would expect from Chowla's conjecture that the estimation of the correlation of the Liouville function seems to be an extremely difficult task and Hildebrand [31] mentions that

" ... even the much weaker relation

$$\liminf_{x\to\infty} \frac{H(\lambda, x)}{x} < 1$$

is not known and seems to be beyond reach of the present methods". This motivates us to study functions which are in some way "close" to the Liouville function but easier to estimate the correlation function. In general, for any multiplicative function $f(n)$ and $y > 0$ let us define the multiplicative function $f_y(n)$ by

$$f_y(p^\alpha) = \begin{cases} f(p^\alpha) & \text{for } \ p \le y, \\ 1 & \text{for } \ p > y. \end{cases}$$

Alternatively one can write

$$f_y(n) = \prod_{p^\alpha \| n} f_y(p^\alpha) = \prod_{\substack{p^\alpha \| n \\ p \le y}} f(p^\alpha).$$

Let $\lambda_y(n)$ denote the completely multiplicative "truncated" Liouville function and

$$\lambda_y(p) = \begin{cases} -1 \ (= \ \lambda(p)) & \text{for } \ p \le y, \\ +1 & \text{for } \ p > y. \end{cases}$$

Thus Daboussi and Sárközy [13] studied the second order correlation of this "truncated" Liouville function and they proved

$$H(\lambda_y, x) = o(x) \quad (y \to \infty)$$

under the condition that

$$\log y = o(\log x). \tag{4.1.2}$$

Now we take a modest step ahead by replacing $(n, n+1)$ by the triplet $(n, n+1, n+2)$ and define a triple order correlation function $C(f, x)$ by

$$C(f, x) = \sum_{n \le x} f(n) f(n+1) f(n+2)$$

where $f(n)$ is completely multiplicative with $|f(n)| \le 1$. Our main result of this chapter is the following :

Theorem 4.1.1. *Let $f(n)$ be a completely multiplicative arithmetic function with*

$$| f(n) | \leq 1 \quad \forall\, n \in \mathbb{N} \tag{4.1.3}$$

$2 \leq y \leq x$, and writing

$$u = \frac{\log x}{\log y} \tag{4.1.4}$$

we have

$$C(f_y, x) = x \prod_{p \leq y} \theta(p) + O\biggl(x\Bigl((\log y)^{-9} + \exp(-\frac{u}{12})\Bigr)\biggr) \tag{4.1.5}$$

where $\theta(p)$ is defined by

$$\theta(p) \quad = \quad \frac{p + 2f(p) - 3}{p - f(p)}. \tag{4.1.6}$$

Now if we choose $f(n) = \lambda(n)$, then from (4.1.6) we have

$$\theta(5) = \frac{5 + 2\lambda(5) - 3}{5 - \lambda(5)} = 0.$$

Thus from the above theorem we have

Corollary 4.1.2. If $5 \leq y \leq x$ and u is defined by (4.1.4) then

$$C(\lambda_y, x) = O\biggl(x\Bigl((\log y)^{-9} + \exp(-\frac{u}{12})\Bigr)\biggr).$$

Hence, as $y \to \infty$ and under the assumption (4.1.2) one obtains

$$C(\lambda_y, x) = o(x).$$

4.2 Basic notations

For a fixed y, let us define the following notations

$$Q_y = \{n : \forall p \mid n \Rightarrow p \leq y\},\ P_y = \prod_{p \leq y} p,\ B_y = \sum_{p \leq y} \frac{1}{p}.$$

Define the completely multiplicative arithmetic functions $e(n), m(n)$ by

$$n = e(n)m(n), \quad e(n), m(n) \in \mathbb{N},\ e(n) \in Q_y,\ (m(n), P_y) = 1. \tag{4.2.1}$$

So $e(n)$ consists of all the prime factors of n which are less than or equal to y. Then we can write

$$f_y(n) = f(e(n)). \tag{4.2.2}$$

Let $\psi(x, y) = |\{n : n \leq x, \forall p \mid n \Rightarrow p \leq y\}|$. In other words $\psi(x, y)$ denotes the number of y-smooth integers $\leq x$. We define the multiplicative arithmetic functions $\psi(n)$ and $\eta(n)$ in the following manner:-
For any $\alpha \in \mathbb{N}$ we have

$$\psi(p^\alpha) = \begin{cases} \left(1 - \frac{1}{p}\right)\left(1 - \frac{3}{p}\right)^{-1} = \frac{p-1}{p-3} & \text{for } \; p \neq 3, \\ 1 & \text{for } \; p = 3. \end{cases}$$

We define

$$\eta(E) = \begin{cases} \frac{2}{3} & \text{for } \; 3 \mid E \\ 0 & \text{for } \; 3 \nmid E, \end{cases}$$

so that

$$\mid \psi(p^\alpha) \mid \leq 3 \quad \forall \; p \text{ and } \alpha \in \mathbb{N}, \tag{4.2.3}$$

$$\mid \eta(E) \mid \leq 1 \quad \forall \; E \in \mathbb{N} \tag{4.2.4}$$

and also we have the following

$$\begin{aligned} \mid \psi(p^\alpha) \mid &\leq \left(1 - \frac{1}{p}\right)\left(1 - \frac{3}{p}\right)^{-1} \\ &< \left(1 + \frac{1}{p}\right)\left(1 + \frac{6}{p^2}\right) \\ &\leq \left(1 + \frac{1}{p}\right)\exp\left(\frac{6}{p^2}\right). \end{aligned}$$

In other words,

$$\mid \psi(E) \mid = \prod_{p^\alpha \| E} \mid \psi(p^\alpha) \mid \leq \left(\prod_{p \mid E}\left(1 + \frac{1}{p}\right)\right)\exp\left(\sum_p \frac{6}{p^2}\right) \ll \prod_{p \mid E}\left(1 + \frac{1}{p}\right). \tag{4.2.5}$$

4.3 Preliminary Lemmas

The following lemmas play a crucial role in the estimations made in this chapter.

Lemma 4.3.1. *Let $F_1(n), \ldots, F_k(n)$ be distinct irreducible linear polynomials with integer coefficients such that*

$$F(n) = \prod_{i=1}^{k} F_i(n) = \prod_{i=1}^{k} (a_i n + b_i).$$

Let $2 \le y \le x$ and set

$$v = \frac{\log x}{\log y}, \quad B = \prod_{i=1}^{k} a_i \prod_{1 \le i < j \le k} (a_i b_j - a_j b_i)$$

and denote the number of solutions of the congruence $F(x) \equiv 0 \pmod{p}$ by $\rho(p)$ and assume that

$$\rho(p) < p \quad \text{for all primes } p.$$

Then for $x \to \infty$ we have

$$\begin{aligned} |\ \{n : n \le x, (F(n), P_y) = 1\}\ | = x \prod_{\substack{p \mid B \\ p \le y}} \left(1 - \frac{\rho(p)}{p}\right) \prod_{\substack{p \nmid B \\ p \le y}} \left(1 - \frac{k}{p}\right) \\ \times (1 + O(\exp(-v(\log v - \log\log 3v - \log k - 2)) + (\exp(-\sqrt{\log x})))). \end{aligned}$$

Proof. This is a special case of the "fundamental lemma" of Brun sieve e.g.Theorem 2.6 in [26]. □

It is worthwhile to mention that in Sieve theory fundamental lemmas have important applications in those problems where one needs specific information about the distribution of numbers which belongs to some integer sequence and have no "very" small prime factors. Such problems arise in the study of additive arithmetic functions defined over polynomial sequences.

Lemma 4.3.2. *If $2 \le y \le x$, then*

$$|\{n : n \le x, (n, P_y) = 1\}| \ll \frac{x}{\log y}.$$

Proof. This is special case of Lemma 4.3.1 above with $F(n) = n$. □

Lemma 4.3.3. *Assume that $h(n)$ is a multiplicative function such that*

$$h(p) = h(p^2) = \cdots = h(p^\alpha) = \cdots \quad \text{for all } p$$

and there are numbers $K > 0, L \geq 0$ with

$$\Big|h(p) - K\Big| \leq \frac{L}{p} \quad \text{for all } p.$$

Then for all $2 \leq y \leq z$ we have

$$\sum_{\substack{z \leq n \\ n \in Q_y}} \frac{h(n)}{n} \ll (\log y)^K \exp\left(-\frac{\log z}{\log y}\right)$$

where the constants depends on K and L only.

Proof. This is Lemma 3 in [13]. $\square$

Lemma 4.3.4. *For all $2 \leq y \leq z$ we have*

$$\sum_{\substack{z \leq n \\ n \in Q_y}} \frac{1}{n} \ll (\log y) \exp\left(-\frac{\log z}{\log y}\right).$$

Proof. This follows from Lemma 4.3.3 above with $h(n) \equiv 1, K = 1, L = 0$. $\square$

Lemma 4.3.5. *If $(\log x)^2 \leq y \leq x$ and u is defined by (4.1.4), then*

$$\psi(x, y) \ll x \exp(-u \log u).$$

Proof. This follows from de Brujin's classical estimate (e.g.Theorem 2 in Part II of [7]). $\square$

Lemma 4.3.6. *Let $e(n)$ and u be defined as in (4.2.1) and (4.1.4) respectively, then we have*

$$\sum_{\substack{n \leq x \\ e(n) > x^{1/12}}} 1 \ll x \exp\left(-\frac{u}{12}\right). \tag{4.3.1}$$

Proof. We have

$$\sum_{\substack{n \leq x \\ e(n) > x^{1/12}}} 1 = \sum_{\substack{n \leq x \\ x^{1/12} < e(n) \leq \frac{x}{y}}} 1 + \sum_{\substack{n \leq x \\ \frac{x}{y} < e(n)}} 1 = K_1 + K_2. \tag{4.3.2}$$

If n is counted in the first term, then writing $e(n) = e, m(n) = m$ and $n = em$ we have $x^{1/12} < e \leq \frac{x}{m} \leq \frac{x}{y},\ e \in Q_y, (m, P_y) = 1$ so now using Lemma 4.3.2 and then Lemma 4.3.4, we get

$$
\begin{aligned}
K_1 &\le \sum_{\substack{x^{1/12} < e \le \frac{x}{y} \\ e \in Q_y}} \sum_{\substack{m \le \frac{x}{e} \\ (m, P_y) = 1}} 1 \\
&\ll \sum_{\substack{x^{1/12} < e \le \frac{x}{y} \\ e \in Q_y}} \frac{x}{e \log y} \\
&\le \frac{x}{\log y} \sum_{\substack{x^{1/12} < e \\ e \in Q_y}} \frac{1}{e} \\
&\ll \frac{x}{\log y} (\log y) \exp\left(-\frac{\log x^{1/12}}{\log y} \right) = x \exp\left(-\frac{u}{12} \right).
\end{aligned}
$$

If n is counted in the second term in (4.3.2), then in $n = em, (m, P_y) = 1$ we have

$$
m \le \frac{x}{e(n)} < y
$$

which forces $m = 1$ and we have

$$
K_2 = \sum_{\substack{\frac{x}{y} < e \le x \\ e \in Q_y}} 1 \le \sum_{\substack{e \le x \\ e \in Q_y}} 1 = \psi(x, y).
$$

If $(\log x)^2 \le y$, then by Lemma 4.3.5 it follows that

$$
K_2 \ll (x) \exp\left(-\frac{\log(x)}{\log y} \log \frac{\log(x)}{\log y} \right) \le x \exp(-u \log u) \ll x \exp(-\frac{u}{12}).
$$

If $2 \le y < (\log x)^2$, then again by Lemma 4.3.5 we have

$$
\begin{aligned}
K_2 &\le \psi(x, y) \le \psi(x, (\log x)^2) \\
&\ll (x) \exp\left(-\frac{\log(x)}{\log(\log x)^2} \log \frac{\log(x)}{\log(\log x)^2} \right) \\
&\le x \exp\left(-\frac{\log x}{\log(\log x)^2} \log \frac{\log x}{\log(\log x)^2} \right) \\
&\ll x \exp\left(-\left(\frac{1}{2} + o(1) \right) \log x \right) = x^{1/2 + o(1)} \ll x \exp\left(-\frac{1}{12} \frac{\log x}{\log 2} \right) \\
&\le x \exp\left(-\frac{u}{12} \right)
\end{aligned}
$$

for $2 \le y < (\log x)^2$. Hence (4.3.1) follows from (4.3.2) and estimation of K_1 and K_2. $\square$

4.4 Proof of the Main Theorem

Let us fix an integer n and writing $n = {p_1}^{\alpha_1} \cdots {p_k}^{\alpha_k} p_{k+1}^{\alpha_{k+1}} \cdots p_t^{\alpha_t}$ we have

$$f(n) = f(p_1)^{\alpha_1} \cdots f(p_k)^{\alpha_k} f(p_{k+1})^{\alpha_{k+1}} \cdots f(p_t)^{\alpha_t}$$

by complete multiplicativity of f. Now let us assume that $p_k < y < p_{k+1}$, then

$$\begin{aligned} f_y(n) &= f_y({p_1}^{\alpha_1} \cdots {p_k}^{\alpha_k} p_{k+1}^{\alpha_{k+1}} \cdots p_t^{\alpha_t}) \\ &= f_y(p_1^{\alpha_1}) \cdots f_y(p_k^{\alpha_k}) f_y(p_{k+1}^{\alpha_{k+1}}) \cdots f_y(p_t^{\alpha_t}) \quad \text{(by multiplicativity of } f_y\text{)} \\ &= f(p_1^{\alpha_1}) \cdots f(p_k^{\alpha_k}) \times 1 \cdots 1 \quad \text{(by definition of } f_y\text{).} \end{aligned}$$

Our goal is to estimate the sum $C(f_y, x) = \sum_{n \leq x} f_y(n) f_y(n+1) f_y(n+2)$. First we set

$$\phi = 90 B_y = 90 \sum_{p \leq y} \frac{1}{p}.$$

Writing

$$S = \sum_{n \in N_0} f_y(n) f_y(n+1) f_y(n+2) \tag{4.4.1}$$

where we define N_0, N_1, N_2 and N_3 by

$$\begin{aligned} N_1 &= \{n : n \leqslant x, \omega(e(n)e(n+1)e(n+2)) > \phi\} \\ N_2 &= \{n : n \leqslant x, \Omega(e(n)e(n+1)e(n+2)) - \omega(e(n)e(n+1)e(n+2)) > \phi\} \\ N_3 &= \{n : n \leqslant x, e(n)e(n+1)e(n+2) > x^{\frac{1}{4}}\} \\ N_0 &= \{n : n \leqslant x\} \setminus \{N_1 \cup N_2 \cup N_3\}. \end{aligned}$$

It follows that

$$\begin{aligned} \left| \sum_{n \leq x} f_y(n) f_y(n+1) f_y(n+2) - S \right| &= \left| \sum_{n \in N_1 \cup N_2 \cup N_3} f_y(n) f_y(n+1) f_y(n+2) \right| \\ &\leq \sum_{n \in N_1 \cup N_2 \cup N_3} | f_y(n) f_y(n+1) f_y(n+2) | \\ &\leq \sum_{n \in N_1 \cup N_2 \cup N_3} 1 \leq |N_1| + |N_2| + |N_3|. \end{aligned}$$

It remains to estimate each of these sets N_1, N_2, N_3 and S.

First we consider N_1. For any integer $n \in N_1$ we have

$$(n, n+1, n+2) = 1 \Rightarrow (e(n), e(n+1), e(n+2)) = 1$$

and it follows that

$$\omega(e(n)e(n+1)e(n+2)) = \omega(e(n)) + \omega(e(n+1)) + \omega(e(n+2)) > \phi.$$

Hence

$$\max(\omega(e(n)), \omega(e(n+1)), \omega(e(n+2)) > \frac{\phi}{3}.$$

Thus either of $n, n+1$or $n+2$ is divisible by a number e such that for $e \in Q_y$ we have $\omega(e) > \frac{\phi}{3}$ which in turn gives

$$\begin{aligned} | N_1 | &\leq \sum_{\substack{e \in Q_y \\ \omega(e) > \frac{\phi}{3}}} | \{n \leq x : e \mid n\} | + | \{n \leq x : e \mid n+1\} | + | \{n \leq x : e \mid n+2\} | \\ &\leq 3 \sum_{\substack{e \in Q_y \\ \omega(e) > \frac{\phi}{3}}} \frac{x+2}{e} \leq 9x \sum_{\substack{e \in Q_y \\ \omega(e) > \frac{\phi}{3}}} \frac{1}{e} \\ &= 9x \sum_{k > \frac{\phi}{3}} \sum_{\substack{e \in Q_y \\ \omega(e) = k}} \frac{1}{e} \leq 9x \sum_{k > \frac{\phi}{3}} \frac{1}{k!} \left(\sum_{p \leq y} \sum_{\alpha=1}^{\infty} \frac{1}{p^\alpha} \right)^k = 9x \sum_{k > \frac{\phi}{3}} \frac{1}{k!} \left(\sum_{p \leq y} \frac{1}{p-1} \right)^k \\ &\leq 9x \sum_{k > \frac{\phi}{3}} \frac{1}{k!} \left(2 \sum_{p \leq y} \frac{1}{p} \right)^k. \end{aligned}$$

By Stirling's formula we have

$$\begin{aligned} |N_1| &\leq 9x \sum_{k > \frac{\phi}{3}} \left(\frac{3}{k} \right)^k (2B_y)^k < 9x \sum_{k > \frac{\phi}{3}} \left(\frac{18B_y}{\phi} \right)^k \\ &= 9x \sum_{k > \frac{\phi}{3}} 5^{-k} \ll x5^{-\phi/3} \ll x(\log y)^{-30}. \end{aligned}$$

Recall that $N_2 = \{n : n \leq x, \Omega(e(n)e(n+1)e(n+2)) - \omega(e(n)e(n+1)e(n+2)) > \phi\}$. If $n \in N_2$ then either

$$\Omega(e(n)) - \omega(e(n)) > \frac{\phi}{3} \tag{4.4.2}$$

or

$$\Omega(e(n+1)) - \omega(e(n+1)) > \frac{\phi}{3} \tag{4.4.3}$$

or

$$\Omega(e(n+2)) - \omega(e(n+2)) > \frac{\phi}{3} \tag{4.4.4}$$

has to hold. If (4.4.2) holds then writing $e(n)$ in the form

$$e(n) = r^2 s \quad \text{with } |\mu(s)| = 1,$$

we have

$$\frac{\phi}{3} < \Omega(e(n)) - \omega(e(n)) = 2\Omega(r) + \omega(s) - \omega(e(n));$$

hence

$$\Omega(r) > \frac{1}{2}\left(\frac{\phi}{3} + \omega(e(n)) - \omega(s)\right) \geq \frac{\phi}{6}.$$

Therefore we have

$$r^2 \geq (2^{\Omega(r)})^2 \geq 2^{\phi/3}.$$

Inequalities (4.4.3) and (4.4.4) can handled similarly. Now

$$\begin{aligned}
|N_2| &\leq \sum_{r^2 \geq 2^{\phi/3}} \Big(|\{n : n \leq x, r^2 \mid n\}| + |\{n : n \leq x, r^2 \mid n+1\}| + |\{n : n \leq x, r^2 \mid n+2\}| \Big) \\
&\leq \sum_{r^2 \geq 2^{\phi/3}} 3|\{n : n \leq x, r^2 \mid n\}| \\
&\ll \sum_{r^2 \geq 2^{\phi/3}} \frac{x}{r^2} \\
&= x \sum_{r^2 \geq 2^{\phi/3}} \frac{1}{r^2} \ll x 2^{-\phi/6} \ll x(\log y)^{-9}.
\end{aligned}$$

Finally let us consider $N_3 = \{n : n \leq x, e(n)e(n+1)e(n+2) > x^{1/4}\}$ we have

$$\begin{aligned}
N_3 \subset \{n : n \leq x, e(n) > x^{1/12}\} &\cup \{n : n+1 \leq x, e(n+1) > x^{1/12}\} \\
&\cup \{n : n+2 \leq x, e(n+2) > x^{1/12}\} \cup \{n : x < n+1 \leq x+1\} \cup \{n : x < n+2 \leq x+2\}
\end{aligned}$$

and by Lemma 4.3.6 it follows that

$$|N_3| \leq 3 \sum_{\substack{n \leq x \\ e(n) > x^{1/12}}} 1 + 2 \ll x \exp\left(-\frac{u}{12}\right). \tag{4.4.5}$$

We now consider the sum S in (4.4.1) where the sum is defined over N_0 which can be redefined as

$$\begin{aligned}
N_0 = \; &\{n : n \leq x, \omega(e(n)e(n+1)e(n+2)) \leq \phi, \\
&\Omega(e(n)e(n+1)e(n+2)) - \omega(e(n)e(n+1)e(n+2)) \leq \phi, e(n)e(n+1)e(n+2) \leq x^{1/4}\}
\end{aligned}$$

and from (4.2.2) it follows that

$$\begin{aligned} S &= \sum_{n \in N_0} f_y(n) f_y(n+1) f_y(n+2) \\ &= \sum_{n \in N_0} f(e(n)) f(e(n+1)) f(e(n+2)) \\ &= \sum_{(e_1,e_2,e_3)} f(e_1) f(e_2) f(e_3) \mid \{n : n \in N_0, e(n) = e_1, e(n+1) = e_2, e(n+2) = e_3\} \mid \end{aligned}$$

where we sum over all triplets (e_1, e_2, e_3) such that there is at least one n with

$$n \in N_0, \ e(n) = e_1, e(n+1) = e_2, e(n+2) = e_3. \tag{4.4.6}$$

Now if there is at least one n with these properties, then we must have

$$e_1 = e(n) \in Q_y, \quad e_2 = e(n+1) \in Q_y, \quad e_3 = e(n+2) \in Q_y \tag{4.4.7}$$

and by the definition of N_0 we have

$$\begin{aligned} \omega(e_1 e_2 e_3) &= \omega(e(n)e(n+1)e(n+2)) \leq \phi, \\ \Omega(e_1 e_2 e_3) - \omega(e_1 e_2 e_3) &= \Omega(e(n)e(n+1)e(n+2)) - \omega(e(n)e(n+1)e(n+2)) \leq \phi, \\ e_1 e_2 e_3 &= e(n)e(n+1)e(n+2) \leq x^{1/4}. \end{aligned}$$

Beside this by (4.4.6) we have $e_1 \mid n$, $e_2 \mid n+1$ and $e_3 \mid n+2$ and therefore $(e_1, e_2, e_3) = 1$. Since we are summing over all triplets (e_1, e_2, e_3) by the above properties we restrict our attention to a smaller set

$$\begin{aligned} \Delta \;=\; &\{(e_1, e_2, e_3) : e_1, e_2, e_3 \in Q_y, \omega(e_1 e_2 e_3) \leq \phi, \\ &\Omega(e_1 e_2 e_3) - \omega(e_1 e_2 e_3) \leq \phi, e_1 e_2 e_3 \leq x^{1/4}, (e_1, e_2, e_3) = 1\}. \end{aligned}$$

If $(e_1, e_2, e_3) \in \Delta$ then the sum S can be written as

$$S = \sum_{(e_1,e_2,e_3) \in \Delta} f(e_1 e_2 e_3) \mid \{n : n \leq x, e(n) = e_1, e(n+1) = e_2, e(n+2) = e_3\} \mid . \tag{4.4.8}$$

For a fixed $(e_1, e_2, e_3) \in \Delta$, n satisfies (4.4.6) if and only if the representation of $(n, n+1, n+2)$ has the following forms :
If n is odd :

$$n = e_1 m_1 \quad (m_1, P_y) = 1 \tag{4.4.9}$$

$$n+1 = e_2 m_2 \quad (m_2, P_y) = 1 \tag{4.4.10}$$

$$n+2 = e_3 m_3 \quad (m_3, P_y) = 1. \tag{4.4.11}$$

Otherwise, if n is even :

$$\begin{aligned} n &= 2\tilde{e_1}m_1 \quad (m_1, P_y) = 1,\ \tilde{e_1} = e_1/2 \\ n+1 &= \tilde{e_2}m_2 \quad (m_2, P_y) = 1,\ \tilde{e_2} = e_2 \\ n+2 &= 2\tilde{e_3}m_3 \quad (m_3, P_y) = 1,\ \tilde{e_3} = e_3/2. \end{aligned}$$

Note that in particular for each of the above case we have

$$(e_1, e_2) = (e_2, e_3) = (e_1, e_3) = 1 \tag{4.4.12}$$

$$(\tilde{e_1}, \tilde{e_2}) = (\tilde{e_2}, \tilde{e_3}) = (\tilde{e_1}, \tilde{e_3}) = 1. \tag{4.4.13}$$

Both the cases can be handled in a similar way. Here let us consider the case when n is odd. First of all from (4.4.12) it follows that there exist unique integers $\bar{e_1}, \bar{e_2}$ and $\bar{e_3}$ such that the following holds

$$e_3\bar{e_3} \equiv 1 \pmod{e_1e_2} \Rightarrow e_3\bar{e_3} \equiv 1 \pmod{e_1},\ e_3\bar{e_3} \equiv 1 \pmod{e_2} \tag{4.4.14}$$

$$e_2\bar{e_2} \equiv 1 \pmod{e_1},\ e_1\bar{e_1} \equiv 1 \pmod{e_2}. \tag{4.4.15}$$

Consequently from the above equations (4.4.14) and (4.4.15) we have

$$e_2\bar{e_2} - e_1\bar{e_1} = 1 \qquad (-e_2 < \bar{e_1} \le 0) \tag{4.4.16}$$

$$e_3\bar{e_3} + e_2\bar{e_2} = 1 \qquad (-e_3 < \bar{e_2} \le 0). \tag{4.4.17}$$

Now from (4.4.9),(4.4.10) and (4.4.11) we have

$$\begin{aligned} e_2m_2 - e_1m_1 &= 1 \\ e_3m_3 - e_2m_2 &= 1. \end{aligned}$$

Solving the above system of linear equations we get

$$\begin{aligned} m_1 &= \frac{e_3}{e_1}z - \frac{2}{e_1} \\ m_2 &= \frac{e_3}{e_2}z - \frac{1}{e_2} \\ m_3 &= z. \end{aligned}$$

Then the positive integers m_1, m_2 and m_3 satisfy

$$(n =)\ e_1m_1 = e_2m_2 - 1 = e_3m_3 - 2 \le x$$

if and only if there exist a positive integer z such that

$$e_3 z \equiv 2 \pmod{e_1}$$
$$e_3 z \equiv 1 \pmod{e_2}.$$

Now since $(e_1, e_2) = 1$ by Chinese Remainder Theorem we have

$$e_3 z \equiv 2e_2\bar{e_2} + e_1\bar{e_1} \pmod{e_1 e_2},$$
$$\text{or,}\quad z \equiv 2e_2\bar{e_2}\bar{e_3} + e_1\bar{e_1}\bar{e_3} \pmod{e_1 e_2}.$$

In other words there exist a positive integer r such that

$$m_1 = e_2 e_3 r + \frac{2(e_2\bar{e_2}e_3\bar{e_3} - 1) + e_1\bar{e_1}e_3\bar{e_3}}{e_1} \tag{4.4.18}$$

$$m_2 = e_1 e_3 r + \frac{2e_2\bar{e_2}e_3\bar{e_3} + e_1\bar{e_1}e_3\bar{e_3} - 1}{e_2} \tag{4.4.19}$$

$$m_3 = e_1 e_2 r + 2e_2\bar{e_2}\bar{e_3} + e_1\bar{e_1}\bar{e_3}. \tag{4.4.20}$$

Note that we clearly see that m_1, m_2 and m_3 are linear polynomials in r with integral coefficients and the latter fact follows from (4.4.14) and (4.4.15).
By (4.4.9)-(4.4.11),(4.4.18), (4.4.19) and (4.4.20), S can be rewritten as

$$S = \sum_{(e_1,e_2,e_3)\in\Delta} f(e_1e_2e_3)\Bigg|\Bigg\{r : r \le \frac{x + 2 - 2e_2\bar{e_2}e_3\bar{e_3} - e_1\bar{e_1}e_3\bar{e_3}}{e_1e_2e_3}, \quad (m_1m_2m_3, P_y) = 1\Bigg\}\Bigg|. \tag{4.4.21}$$

If $(e_1, e_2, e_3) \in \Delta$, then by definition of Δ we have

$$e_1e_2e_3 \le x^{1/4}. \tag{4.4.22}$$

Let us assume for simplicity that

$$g(x) = \frac{x + 2 - 2e_2\bar{e_2}e_3\bar{e_3} - e_1\bar{e_1}e_3\bar{e_3}}{e_1e_2e_3}.$$

Hence from (4.4.16) and (4.4.17) it follows that

$$\frac{x}{e_1e_2e_3} + 1 + \bar{e_3} - 2e_2\bar{e_2} \ge g(x) \ge \frac{x}{e_1e_2e_3} \ge x^{3/4}. \tag{4.4.23}$$

Now we may assume that $y < x^{3/4}$. Then by (4.4.23), Lemma 4.3.1 can be applied with $g(x)$ and $m_1m_2m_3$ in place of x and $F(n)$, respectively. We clearly see that each of these linear polynomials m_1, m_2 and m_3 are distinct and also direct calculation gives

$$B = a_1a_2a_3(a_1b_2 - b_2a_1)(a_2b_3 - a_3b_2)(a_1b_3 - a_3b_1) = 2(e_1e_2e_3)^3 \neq 0$$

and also

$$\rho(p) = 1 \quad \text{for } p \mid B.$$

By (4.4.23),

$$v = \frac{\log\big(g(x)\big)}{\log y} \geq \frac{3}{4}u.$$

Hence it follows from Lemma 4.3.1 that for $(e_1, e_2, e_3) \in \Delta$ we have

$$|\ \{r : r \leq g(x), \Big(m_1(r)m_2(r)m_3(r), P_y\Big) = 1\}\ | \tag{4.4.24}$$

$$\begin{aligned} = \quad & g(x)\prod_{p\mid B}\Big(1-\frac{1}{p}\Big)\prod_{\substack{p\nmid B\\ p\leq y}}\Big(1-\frac{3}{p}\Big) \\ & \times\Big(1+O\Big(\exp\Big(-\frac{u}{2}\log u\Big)+\exp(-\sqrt{\log x})\Big)\Big) \\ = \quad & \frac{x}{e_1e_2e_3}\prod_{p\mid 2(e_1e_2e_3)^3}\Big(1-\frac{1}{p}\Big)\prod_{\substack{p\nmid 2(e_1e_2e_3)^3\\ p\leq y}}\Big(1-\frac{3}{p}\Big) \\ & \times\Big(1+O\Big(\exp\Big(-\frac{u}{2}\log u\Big)+\exp(-\sqrt{\log x})\Big)\Big)+O(1)+O(\bar{e_3})+O(e_2\bar{e_2}). \end{aligned}$$

Note that in the above estimate the main term depends only on the value of the product $e_1e_2e_3$. Thus we introduce the notation $E = e_1e_2e_3$ and from (4.4.21) and (4.4.24) we rewrite S in the following manner

$$S = \sum_E |\ \{(e_1, e_2, e_3) : (e_1, e_2, e_3) \in \Delta, e_1e_2e_3 = E\}\ | \tag{4.4.25}$$

$$\begin{aligned} \times \quad & f(E)\Big(\frac{x}{E}\prod_{p\mid 2E^3}\Big(1-\frac{1}{p}\Big)\prod_{\substack{p\nmid 2E^3\\ p\leq y}}\Big(1-\frac{3}{p}\Big) \\ \times \quad & \Big(1+O\Big(\exp\Big(-\frac{u}{2}\log u\Big)+\exp(-\sqrt{\log x})\Big)\Big)+O(1)+O(\bar{e_3})+O(e_2\bar{e_2})\Big). \end{aligned}$$

where the sum E runs over all integers that can be expressed in the from $E = e_1e_2e_3$ with $(e_1, e_2, e_3) \in \Delta$ and we denote this set of integers by Γ. In other words

$$\Gamma = \{E : E \in Q_y, \omega(E) \leq \phi, \Omega(E) - \omega(E) \leq \phi, E \leq x^{\frac{1}{4}}\} \tag{4.4.26}$$

and we have

$$| \{(e_1, e_2, e_3) : (e_1, e_2, e_3) \in \Delta, e_1e_2e_3 = E\} | \tag{4.4.27}$$

$$\begin{aligned} &= | \{(e_1, e_2, e_3) : e_1e_2e_3 = E, (e_1, e_2) = (e_2, e_3) = (e_1, e_3) = 1\} | \\ &= 3^{\omega(E)}. \end{aligned}$$

From (4.4.26) it follows that we can drop the error terms $O(1), O(\bar{e_3})$ and $O(e_2\bar{e_2})$ since they are smaller than the other error terms. Hence from (4.4.25) and (4.4.27) it follows that

$$\begin{aligned} S &= x \sum_{E\in\Gamma} \frac{f(E)3^{\omega(E)}}{E} \prod_{p|2E}\left(1-\frac{1}{p}\right)\prod_{\substack{p\nmid 2E\\ p\leq y}}\left(1-\frac{3}{p}\right) \\ &\quad +O\left(x\exp\left(-\frac{u}{2}\log u\right)\sum_{E\in\Gamma}\frac{3^{\omega(E)}}{E}\prod_{p|2E}\left(1-\frac{1}{p}\right)\prod_{\substack{p\nmid 2E\\ p\leq y}}\left(1-\frac{3}{p}\right)\right) \\ &\quad +O\left(x\exp(-\sqrt{\log x})\sum_{E\in\Gamma}\frac{3^{\omega(E)}}{E}\prod_{p|2E}\left(1-\frac{1}{p}\right)\prod_{\substack{p\nmid 2E\\ p\leq y}}\left(1-\frac{3}{p}\right)\right). \end{aligned}$$

We rewrite the sum S by adding new variables S_1 and S_2 as

$$S = xS_1 + O\left(x\exp\left(-\frac{u}{2}\log u\right)S_2\right) + O(x\exp(-\sqrt{\log x})S_2). \tag{4.4.28}$$

Then the first sum, say S_1 in S can be handled as

$$\begin{aligned} S_1 &= \sum_{E\in\Gamma}\frac{f(E)3^{\omega(E)}}{E}\prod_{p|E}\left(1-\frac{1}{p}\right)\prod_{\substack{p\nmid E\\ p\leq y}}\left(1-\frac{3}{p}\right) \\ &= \prod_{\substack{2\leq p\leq y\\ p\neq 3}}\left(1-\frac{3}{p}\right)\sum_{E\in\Gamma}\frac{f(E)3^{\omega(E)}\eta(E)\psi(E)}{E}. \end{aligned}$$

Now as before we write

$$\begin{aligned} \Gamma_1 &= \{E : E \in Q_y, \omega(E) > \phi\} \\ \Gamma_2 &= \{E : E \in Q_y, \Omega(E) - \omega(E) > \phi\} \\ \Gamma_3 &= \{E : E \in Q_y, E > x^{\frac{1}{4}}\} \end{aligned}$$

so that

$$\Gamma = Q_y \setminus (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3).$$

From the above construction it follows that

$$\begin{aligned}
S_1 &= \prod_{\substack{2\le p\le y\\ p\neq 3}}\left(1-\frac{3}{p}\right)\Bigg(\sum_{E\in Q_y}\frac{f(E)3^{\omega(E)}\eta(E)\psi(E)}{E} \\
&\qquad + O\Bigg(\sum_{E\in\Gamma_1\cup\Gamma_2\cup\Gamma_3}\frac{\mid f(E)\mid 3^{\omega(E)}\eta(E)\psi(E)}{E}\Bigg)\Bigg) \\
&= \prod_{\substack{2\le p\le y\\ p\neq 3}}\left(1-\frac{3}{p}\right)\sum_{E\in Q_y}\frac{f(E)3^{\omega(E)}\eta(E)\psi(E)}{E} + O\Bigg((\log y)^{-3}\Bigg(\sum_{E\in\Gamma_1}\frac{3^{\omega(E)}\psi(E)}{E} + \\
&\qquad \sum_{E\in\Gamma_2}\frac{3^{\omega(E)}\psi(E)}{E} + \sum_{E\in\Gamma_3}\frac{3^{\omega(E)}\psi(E)}{E}\Bigg)\Bigg) \\
&= \prod_{\substack{2\le p\le y\\ p\neq 3}}\left(1-\frac{3}{p}\right)\sum_{E\in Q_y}\frac{f(E)3^{\omega(E)}\eta(E)\psi(E)}{E} + O\Bigg((\log y)^{-3}\Bigg(\sum_{\substack{E\in Q_y\\ \omega(E)>\phi}}\frac{3^{\omega(E)}3^{\omega(E)}}{E} + \\
&\qquad \sum_{\substack{E\in Q_y\\ \Omega(E)-\omega(E)>\phi}}\frac{3^{\omega(E)}3^{\omega(E)}}{E} + \sum_{\substack{E\in Q_y\\ E>x^{1/4}}}\frac{3^{\omega(E)}\psi(E)}{E}\Bigg)\Bigg) \\
&= M + O((\log y)^{-3}(M_1+M_2+M_3)), \quad \text{say.}
\end{aligned}$$

So now we need to estimate

$$S_1 = M + O((\log y)^{-2}(M_1+M_2+M_3)). \tag{4.4.29}$$

Note that in order to estimate S_1 we have to estimate each of the four terms M, M_1, M_2

and M_3. First we proceed to estimate the main term M with $E \in Q_y$ and we have

$$\begin{aligned}
M &= \prod_{\substack{2\le p\le y\\ p\ne 3}}\left(1-\frac{3}{p}\right)\sum_{E\in Q_y}\frac{f(E)3^{\omega(E)}\eta(E)\psi(E)}{E}\\
&= \prod_{\substack{2\le p\le y\\ p\ne 3}}\left(1-\frac{3}{p}\right)\sum_{i=1}^{\infty}\sum_{\substack{\bar{E}\in Q_y\\ 3^i\bar{E}=E,3\nmid\bar{E}}}\frac{f(3^i)f(\bar{E})3^{1+\omega(\bar{E})}\psi(\bar{E})}{3^i\bar{E}}\frac{2}{3}\\
&= \prod_{\substack{2\le p\le y\\ p\ne 3}}\left(1-\frac{3}{p}\right)2\sum_{i=1}^{\infty}\frac{f(3^i)}{3^i}\sum_{\substack{\bar{E}\in Q_y\\ 3\nmid\bar{E}}}\frac{f(\bar{E})3^{\omega(\bar{E})}\psi(\bar{E})}{\bar{E}}\\
&= 2\sum_{i=1}^{\infty}\left(\frac{f(3)}{3}\right)^i\prod_{\substack{2\le p\le y\\ p\ne 3}}\left(1-\frac{3}{p}\right)\left(1+\sum_{\beta=1}^{\infty}\frac{f(p^\beta)3^{\omega(p^\beta)}\psi(p^\beta)}{p^\beta}\right)\\
&= 2\sum_{i=1}^{\infty}\left(\frac{f(3)}{3}\right)^i\prod_{\substack{2\le p\le y\\ p\ne 3}}\left(1-\frac{3}{p}\right)\left(1+3\psi(p)\sum_{\beta=1}^{\infty}\frac{f(p^\beta)}{p^\beta}\right)\\
&= \prod_{p\le y}\theta(p)
\end{aligned}$$

where $\theta(p)$ is defined by (4.1.6). Now since $f(n)$ is completely multiplicative then we have

$$\theta(3) = 2\sum_{i=1}^{\infty}\left(\frac{f(3)}{3}\right)^i = \frac{2f(3)}{3-f(3)} = \frac{3+2f(3)-3}{3-f(3)}$$

and

$$\theta(p) = \left(1-\frac{3}{p}\right)\left(1+3\frac{p-1}{p-3}\frac{f(p)}{p-f(p)}\right) = \frac{p+2f(p)-3}{p-f(p)} \quad \text{for } p\ne 3.$$

It remains to compute the error terms M_1, M_2 and M_3 which appears in the sum S_1 which can be handled in the following way :-

$$\begin{aligned}
M_1 &= \sum_{\substack{E\in Q_y\\ \omega(E)>\phi}}\frac{3^{\omega(E)}3^{\omega(E)}}{E} = \sum_{\substack{E\in Q_y\\ \omega(E)>\phi}}\frac{9^{\omega(E)}}{E} = \sum_{k>\phi}\sum_{\substack{E\in Q_y\\ \omega(E)=k}}\frac{9^k}{E}\\
&\le \sum_{k>\phi}\frac{1}{k!}\left(\sum_{p\le y}\sum_{\alpha=1}^{\infty}\frac{9}{p^\alpha}\right)^k = \sum_{k>\phi}\frac{1}{k!}\left(9\sum_{p\le y}\frac{1}{p-1}\right)^k \le \sum_{k>\phi}\frac{1}{k!}(18B_y)^k
\end{aligned}$$

and by Stirling's formula

$$M_1 \ll \sum_{k>\phi}\left(\frac{54B_y}{\phi}\right)^k = \sum_{k>\phi}(\frac{5}{3})^{-k} \ll (\frac{5}{3})^{-\phi} = O((\log y)^{-45}).$$

So we have

$$M_1 \ll (\log y)^{-45}. \tag{4.4.30}$$

Similarly for M_2 we write each $E \in Q_y$ with $\Omega(E)-\omega(E) > \phi$ in the form of $E = r^2 s$ where s is square-free. Then we have

$$\phi < \Omega(E) - \omega(E) = 2\Omega(r) + \omega(s)) - \omega(E)$$

hence

$$\Omega(r) > \frac{1}{2}\Big(\phi + \omega(E) - \omega(s)\Big) > \frac{\phi}{2}.$$

Therefore it follows that

$$r^2 \geq (2^{\Omega(r)})^2 \geq 2^{\phi} = 2^{90B_y} > (\log y)^{54}.$$

So we have

$$\begin{aligned}
M_2 &= \sum_{\substack{E \in Q_y \\ \Omega(E)-\omega(E)>\phi}} \frac{3^{\omega(E)}3^{\omega(E)}}{E} \\
&\leq \sum_{\substack{E \in Q_y \\ \Omega(E)-\omega(E)>\phi}} \frac{9^{\omega(E)}}{E} \\
&\leq \sum_{r^2 > (\log y)^{54}} \sum_{\substack{s \in Q_y \\ |\mu(s)|=1}} \frac{9^{\omega(r^2 s)}}{r^2 s} \\
&\leq \sum_{r > (\log y)^{27}} \frac{9^{\omega(r)}}{r} \sum_{\substack{s \in Q_y \\ |\mu(s)|=1}} \frac{9^{\omega(s)}}{s} \\
&\ll (\log y)^{-26} \prod_{p \leq y}\left(1 + \frac{9}{p}\right) \\
&\leq (\log y)^{-26} \exp(9B_y).
\end{aligned}$$

Hence we have

$$M_2 \ll (\log y)^{-17}. \tag{4.4.31}$$

Now to estimate M_3 we have

$$\psi(E) = \prod_{p^\alpha \| E} \psi(p^\alpha) \ll \prod_{p|E}\left(1 + \frac{1}{p}\right).$$

Hence it follows from above and (4.2.5)

$$M_3 \leq \sum_{\substack{E\in Q_y\\ E>x^{1/4}}} \frac{3^{\omega(E)}\psi(E)}{E} \ll \sum_{\substack{E\in Q_y\\ E>x^{1/4}}} \frac{3^{\omega(E)}}{E}\prod_{p|E}\left(1+\frac{1}{p}\right).$$

The function

$$h(E) = 3^{\omega(E)}\prod_{p|E}\left(1+\frac{1}{p}\right)$$

is multiplicative and

$$h(p) = h(p^2) = \ldots = h(p^\alpha) = \ldots = 3\left(1+\frac{1}{p}\right).$$

So we can apply Lemma 4.3.3 with $K = 3$, $L = 3$ and substituting $x^{1/4}$ in place of z we have

$$M_3 \ll (\log y)^3 \exp\left(-\frac{\log x^{1/4}}{\log y}\right) = (\log y)^3 \exp\left(-\frac{u}{4}\right). \tag{4.4.32}$$

Now all we need to calculate is the sum S_2 appearing in the error term in S

$$\begin{aligned} S_2 &= \sum_{E\in\Gamma} \frac{3^{\omega(E)}}{E}\prod_{p|E}\left(1-\frac{1}{p}\right)\prod_{\substack{p\nmid E\\ p\leq y}}\left(1-\frac{3}{p}\right) \\ &= \prod_{\substack{p\leq y\\ p\neq 3}}\left(1-\frac{3}{p}\right)\sum_{E\in\Gamma}\frac{3^{\omega(E)}\eta(E)\psi(E)}{E} \\ &\ll (\log y)^{-3}\sum_{E\in Q_y}\frac{3^{\omega(E)}}{E}\prod_{p|E}\left(1+\frac{1}{p}\right). \end{aligned}$$

Now we need to simplify

$$\sum_{E\in Q_y}\frac{3^{\omega(E)}}{E}\prod_{p|E}\left(1+\frac{1}{p}\right).$$

Since the sum is absolutely convergent we can express it as an Euler product and we get

$$\begin{aligned} \sum_{E\in Q_y}\frac{3^{\omega(E)}}{E}\prod_{p|E}\left(1+\frac{1}{p}\right) &= \prod_{p\leq y}\left(1+\left(\sum_{\alpha=1}^{\infty}\frac{3^{\omega(p^\alpha)}}{p^\alpha}\right)\left(1+\frac{1}{p}\right)\right) \\ &= \prod_{p\leq y}\left(1+\left(\sum_{\alpha=1}^{\infty}\frac{3}{p^\alpha}\right)\left(1+\frac{1}{p}\right)\right) \\ &= \prod_{p\leq y}\left(1+\frac{3(p+1)}{p(p-1)}\right). \end{aligned}$$

Now a little computation shows that

$$\begin{aligned} 1+\frac{3(p+1)}{p(p-1)} &= 1+\frac{3}{p}+\frac{6}{p(p-1)} = \left(1+\frac{3}{p}\right)\left(1+\frac{6}{p^2+2p-3}\right) \\ &\leq \left(1+\frac{3}{p}\right)\exp\left(\frac{6}{p^2+2p-3}\right). \end{aligned}$$

Since $2p-3 \geq 1$ then

$$\frac{6}{p^2+2p-3} \leq \frac{6}{p^2} = O\left(\frac{1}{p^2}\right).$$

Thus substituting back in the original sum S_2 we get

$$\begin{aligned} S_2 &\ll (\log y)^{-3}\sum_{E\in Q_y}\frac{3^{\omega(E)}}{E}\prod_{p|E}\left(1+\frac{1}{p}\right) \\ &= (\log y)^{-3}\prod_{p\leq y}\left(1+\frac{3(p+1)}{p(p-1)}\right) \\ &\leq (\log y)^{-3}\prod_{p\leq y}\left(1+\frac{3}{p}\right)\exp\left(\frac{6}{p^2+2p-3}\right) \\ &\leq (\log y)^{-3}\prod_{p\leq y}\left(1+\frac{3}{p}\right)\exp\left(O\left(\frac{1}{p^2}\right)\right) \\ &\ll (\log y)^{-3}\prod_{p\leq y}\left(1+\frac{3}{p}\right). \end{aligned}$$

So we have

$$S_2 = O(1). \tag{4.4.33}$$

Now collecting the main term M and it follows from (4.4.28), (4.4.29), (4.4.30),(4.4.31), (4.4.32) and (4.4.33) that

$$\begin{aligned} S &= x\prod_{p\leq y}\theta(p) \\ &+ O\left(x(\log y)^{-3}\left((\log y)^{-45}+(\log y)^{-17}+(\log y)^{3}\exp\left(-\frac{u}{4}\right)\right)\right) \\ &+ O\left(x\left(\exp\left(-\frac{u}{2}\log u\right)+\exp(-\sqrt{\log x})\right)\right) \\ &= x\prod_{p\leq y}\theta(p)+O\left(x\left((\log y)^{-20}+\exp\left(-\frac{u}{4}\right)+\exp(-\sqrt{\log x})\right)\right). \end{aligned}$$

Now we see that

$$\exp(-\sqrt{\log x}) \ll \max\left\{(\log y)^{-20}, \exp\left(-\frac{u}{4}\right)\right\}.$$

So dropping the term $\exp(-\sqrt{\log x})$ we get

$$S = x\prod_{p\leq y}\theta(p) + O\left(x\left((\log y)^{-20} + \exp\left(-\frac{u}{4}\right)\right)\right). \tag{4.4.34}$$

Finally we get the best possible error terms after comparing with the error terms we got from N_1, N_2 and N_3 respectively.

Chapter 5

Future directions

This dissertation mainly focuses on Chowla's conjecture and its weaker version later formulated by Cassaigne et al. There are some natural questions which arise from these topics and some of these questions might be very difficult to solve. In this chapter, we have gathered some of these questions with the hope of presenting a clearer picture for future work surrounding these topics.

5.1 Consecutive integer case

The best unconditional upper and lower bound given by Harman, Pintz and Wolke [28] is,

$$-(1+o(1))\frac{1}{3} < \frac{1}{x}\sum_{n\leqslant x}\lambda(n(n+1)) < 1-\frac{1}{(\log x)^{7+\epsilon}}.$$

Can we improve the upper bound by a constant, or any function smaller than $1-\frac{1}{\log^7 x}$? Also proving (unconditionally) that

$$\liminf_{x\to\infty}\frac{1}{x}\sum_{n\leq x}\lambda(n)\lambda(n+1) < 1-\epsilon$$

for any $\epsilon > 0$ will be a very interesting result.

5.2 Cassaigne et al. conjecture

In Chapter 2 and 3 we proved the Cassaigne et al. conjecture for quadratic polynomials with integer and rational coefficients. The next step should be to prove cubic polynomials. So the first goal might be to show that $\lambda(x^3 \pm 1)$ changes signs infinitely often. It is interesting to note that $\lambda(x^3 \pm 1) = \lambda(x \pm 1)\lambda(x^2 \mp x + 1)$ and we do know that each factor $\lambda(x \pm 1)$ and $\lambda(x^2 \mp x + 1)$ changes signs infinitely often. It will also be interesting to show that for any irreducible cubic of the form $x^3 + p$ where p is any prime, $\lambda(x^3 + p)$ changes signs infinitely often.

5.3 Extending work of Kátai

In 2010 two Hungarian number theorists, Lazlo Germain and Imri Kátai [21] proved that $\sum_{n \leq x} \lambda((n)(n+1)) = o(x)$ under the assumption that there is an exceptional Siegel zero. One might try to extend there idea to solve the parity problem conditionally for some other integer polynomials of degree deg ≥ 2. More precisely, under what assumptions relating to Siegel zeroes can one prove the parity problem for a general class of polynomials $f(n)$ with integer coefficients?

5.4 k-fold correlation

One could also try to generalize the results mentioned in Chapter 4 where we compute the third order correlation of completely multiplicative functions. The next step should be to prove the general case where for any fixed integer k we consider the k-fold correlation function defined by the polynomial.

5.5 Chowla's conjecture for finite fields

One natural direction is to study Chowla's conjecture on polynomials over finite fields. Recently Carmon and Rudnick [8] proved the following finite field version of Chowla's conjecture.

Following their notations, let $\mathbb{F}_q$ be a finite field with q elements and $\mathbb{F}_q[x]$ the polynomial ring over $\mathbb{F}_q$. For any nonzero F we define $\mu(F) = (-1)^r$ if $F = cP_1 \ldots P_r$ where $c \in \mathbb{F}_q^*$, $P_1, \ldots P_r$ are distinct monic irreducible polynomials and $\mu(F) = 0$ otherwise.

Let M_n denote the set of monic polynomials of degree n over $\mathbb{F}_q$. For $r > 0$, distinct polynomials $\alpha_1, \dots \alpha_r \in \mathbb{F}_q[x]$, with $\deg \alpha_j < n$ and $\epsilon_i \in \{1, 2\}$, not all even, set

$$C(\alpha_1, \dots \alpha_r; n) := \sum_{F \in M_n} \mu^{\epsilon_1}(F + \alpha_1) \dots \mu^{\epsilon_r}(F + \alpha_r).$$

Carmon and Rudnick [8] proved that for $n > 1, r > 1$ we have

Theorem 5.5.1. *Fix $r > 1$ and assume $n > 1$, q odd. Then for any choice of distinct polynomials $\alpha_1, \dots, \alpha_r \in \mathbb{F}_q[x]$ with max $\deg \alpha_j < n, \epsilon_i \in \{1, 2\}$, not all even*

$$| \, C(\alpha_1, \dots \alpha_r; n) \, | \leq 2rnq^{n-\frac{1}{2}} + 3rn^2 q^{n-1}.$$

Thus under the assumption of the above theorem for fixed $n > 1$,

$$\lim_{q \to \infty} \frac{1}{q^n} \sum_{F \in M_n} \mu^{\epsilon_1}(F + \alpha_1) \dots \mu^{\epsilon_r}(F + \alpha_r) = 0.$$

So perhaps in a more general setting one might try to prove the following : fix a natural number d. Let $f(T)$ be a polynomial of degree d over $\mathbb{F}_q$, with f not a constant multiple of a square. Then the sum of $\lambda(f(A))$, as A ranges over the monic polynomials of degree n over $\mathbb{F}_q$, is $o(q^n)$ as $q \to \infty$, uniformly in the choice of f. Note that d is fixed and q is tending to infinity. It might be possible to obtain something if d is allowed to tend infinity much slower that q. This result is important as it can be seen as depicting the analogies between the theory of rational primes and the distribution of irreducible polynomials over finite fields. Therefore, proving this problem asserts the equidistribution of the parity of the number of irreducible factors of a polynomial over finite fields.

5.6 A related Chowla's conjecture on Liouville function

Let $CM(-1, 1)$ denote the class of completely multiplicative functions taking values in $\{-1, 1\}$. For any such $f \in CM(-1, 1)$, let

$$N(f, x, \epsilon_1, \epsilon_2, \dots, \epsilon_k) = | \, \{n \leq x : f(n + i) = \epsilon_i \in \{-1, 1\}, \; i = 1, \dots, k\} \, | \; .$$

The following conjecture of Chowla which appeared in the work of Hildebrand [34] states that

Conjecture 3 (Chowla)**.** *For each of the 2^k many choices of signs,*

$$N(\lambda, x, \epsilon_1, \ldots, \epsilon_k) \to \infty \quad \text{as } x \to \infty.$$

Hildebrand [33] proved the first nontrivial case when $k = 3$ and he showed that each of the eight sign patterns $(\pm 1, \pm 1, \pm 1)$ occurs infinitely often in the sequence $(\lambda(n), \lambda(n+1), \lambda(n+2))$, and remarked that the same is true for $f \in CM(-1, 1)$ with $f(2) = f(3) = f(5) = f(7) = f(29) = f(31) = -1$.

In the recent work of Y. Buttkewitz and C. Elsholtz [6], the authors study functions $f \in CM(-1, 1)$ at arguments with constant gap d. Assuming necessary conditions on f, it is shown that all sixteen patterns of length 4 occur infinitely often. While many of the earlier results in this area use methods from analytic number theory, their work is of a rather combinatorial nature and uses graph theory.

It is worth to mention that the above conjecture of Chowla was proved by Cassaigne et al. [9] assuming Schinzel's hypothesis H [52] but unconditionally its still open for $k \geq 4$. Heuristically one expects that

$$N(\lambda, x, \epsilon_1, \epsilon_2, \ldots, \epsilon_k) \sim \frac{x}{2^k},$$

for all of the 2^k many sign combinations. Even for $k = 2$ one is very far away from a result of this type as it goes back to establish the correlation function

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} \lambda(n)\lambda(n+1) = 0$$

which seems to be very hard at the moment.

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