A COMPARATIVE STUDY OF SPECTRAL EMBEDDING METHODS FOR TRIANGLE MESHES

by

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Abstract

In order to have a better understanding and use of the strengths and weaknesses of different spectral approaches in the field of geometry processing and analysis, this preliminary comparative study investigates the behaviors of four spectral embedding methods: Global Point Signatures Embedding (GPSE), Heat Kernel Signature Embedding (HKSE), Multi-Dimensional Scaling Embedding (MDSE), and Spectral Embedding using Gaussian-filtered affinity matrices (SEG), by applying them onto three applications. Therefore experimental observations are obtained based on the limited test cases. For the mesh segmentation, HKSE and GPSE tend to outperform MDSE and SEG with better segments and boundaries. For the symmetry detection, all the embedding methods produce similar results. For the shape correspondence, MDSE and SEG show better performance and more stability than GPSE and HKSE. All in all, there is no single one embedding method that clearly outperforms the others across all three applications.

Keywords: spectral embedding, geometry processing, 3D shape, mesh segmentation, symmetry detection, shape correspondence
To my family
“Being deeply loved by someone gives you strength, while loving someone deeply gives you courage.”

— Lao Tzu, 6th century BC
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Chapter 1

Introduction

1.1 Background

A spectral method is defined as solving “a problem by examining or manipulating the eigenvalues, eigenvectors, eigenspace projections, or a combination of these quantities, derived from an appropriately defined linear operator” in [ZvKD10]. Since Taubin [Tau95] initially applied spectral analysis based on a mesh Laplacian operator to mesh smoothing in 1995, spectral methods have been developed into multiple variations using different linear operators, different eigenstructures and been employed to solve a diversity of problems in the field of geometry processing, such as shape classification, graph partitioning, clustering, mesh parameterization, mesh segmentation, shape correspondence, and symmetry detection. Meanwhile from a broader view spectral methods also play a role in the fields of graph theory, computer vision, machine learning, visualization, graph drawing, high performance drawing and computer graphics.

Over the past two decades, various spectral methods have been proposed in the literature, which based on different mesh operators, use different eigenstructures depending on various purposes. A comprehensive survey [ZvKD10] focusing on spectral methods in geometry processing has provided a concrete base with theoretical background, useful references, and thoughtful insights, for researchers to have a thorough understanding of up-to-date methodology and applications. Also several short survey papers discuss particular operators, like [Lev06] focusing on Laplace-Beltrami operator, and particular applications, like [Got03] talking about spectral partitioning.

Among all spectral methods, eigenvectors derived from appropriate linear operators are
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Figure 1.1: Framework of spectral embedding methods in geometry processing. (a) Input mesh, (b) Linear operator is constructed from the input mesh, for example, geodesic distances as operator elements are shown, (c) Leading m eigenvectors are in order consisting a matrix, and each vertex of the input mesh is represented by the corresponding row of this matrix in the m-dimensional embedding, shown in the blue rectangle [ZvKD10].

often used to transform input data from its original domain, such as points in 2D space or mesh vertices in 3D space, into a spectral domain for further operation. We call the transformation process the spectral embedding method and it can be classified according to the number of eigenvectors used to construct the spectral domain. For example, one-dimensional embeddings have been applied to segment images by recursively partitioning using only one eigenvector at a time [SM97]. Two-dimensional embeddings have been used for surface flattening [ZKK02], mesh parameterization [ZSGS04] and three-dimensional embeddings for mesh segmentation [KLT05]. Furthermore, even higher-dimensional embeddings have been discovered to solve geometric problems like mesh symmetry [OSG08], which utilize 15 and 30 eigenvectors, and shape correspondence [DK10], which choose 5 to 10 eigenvectors. Figure 1.1 illustrates the framework of spectral embedding methods in geometry processing.

1.2 Motivation

Although a variety of spectral methods have been discovered and employed by considerable amount of applications, it is still not clear what the advantages and disadvantages, reasons and consequences of applying one particular spectral method over another are. And no one has done any in depth analysis or comparison of these methods. Spectral methods
are not straightforward and are difficult to visualize due to their nature, therefore a better understanding of the strengths and weaknesses of spectral methods will help researchers to better use the properties of particular spectral methods so that optimal results can be achieved for the applications.

Considering the diversity of the spectral methods used in the field of geometry processing, this thesis will focus our investigation on spectral embedding methods. By exploring the behaviours of four spectral embedding methods: Global Point Signatures Embedding (GPSE), Heat Kernel Signature Embedding (HKSE), Multi-Dimensional Scaling Embedding (MDSE), and Spectral Embedding using Gaussian-filtered affinity matrices (SEG), working on three applications: mesh segmentation, symmetry detection and shape correspondence, attention will be drawn to the differences of the methods and informative insights will be extracted. It will be beneficial for application researchers to make a better use of the right spectral method and for theoretical researchers to explore the potential of various spectral methods.

Our current investigation focuses on triangle meshes that are complete, relatively noise-free, and possess a nice tessellation. Spectral embedding is often expected not to work robustly on partial shapes as it is a global method. Meshes still are the dominant shape representation and are the most typical input for the studied applications. Pre-processing to remove noise and highly non-uniform tessellations can be performed. The experiments performed in this thesis research have been based on limited data, hence the conclusions drawn should not be considered as entirely general. However, the choice of the dataset were unbiased and it is our hope that this preliminary study reveals sufficient insight and potential to motivate a larger-scale study.

1.3 Four spectral embedding methods

With the help of the framework of spectral embedding methods illustrated above, we can see that variants of spectral embedding methods may differ in the properties used to form linear operators, the number of eigenvectors selected and the way being employed according to the specific application. Due to a diversity of spectral embedding methods, the definitions of the four embedding methods being researched in this paper are clarified and summarized below in this section. Later a review of the literature will provide more thorough information in Chapter 2 and implementation of the four methods will demonstrate the results so as to
compare the strength and weakness in the following chapters.

1.3.1 Global Point Signatures Embedding (GPSE)

Rustamov defined a "deformation invariant shape representation" [Rus07] called Global Point Signatures (GPS), to be used in shape processing based on the discovery in [Lev06] that eigenfunctions of the Laplace-Beltrami operator contain global information of the surface. Similarly to the geodesic distance, the Laplace-Beltrami operator is also invariant to natural deformations which is a basic property wanted for further mesh analyses such as symmetry, correspondence, and retrieval. Furthermore, the Laplace-Beltrami operator overcomes the geodesic distance disadvantage of being sensitive to topological changes. With these properties GPS can be seen as a transformation of the surface into infinite dimensional space. This mapping process will be referred as Global Point Signatures Embedding (GPSE) in this thesis. In summary, GPSE is composed of two fundamental elements: (1) the discrete Laplace-Beltrami operator as the embedding operator, and (2) embedding using the eigenvectors corresponding to the smallest eigenvalues of the operator. GPSE cab be represented as:

$$GPSE(p) = \left( \frac{\phi_1(p)}{\sqrt{\lambda_1}}, \frac{\phi_2(p)}{\sqrt{\lambda_2}}, \frac{\phi_3(p)}{\sqrt{\lambda_3}}, ... \right)$$  

(1.1)

At first, GPSE has been applied to a straightforward segmentation algorithm, and the results show that consistent segmentation can be obtained across meshes with different poses. Moreover shape classification using GPSE is able to group meshes with small topological changes and meshes with non-rigid transformations together [Rus07]. In a more complex way, unique properties of GPSE were discovered and applied to solve the symmetry detection problem [OSG08]. This finding theoretically simplified the search process after intrinsic symmetry was transformed into extrinsic symmetry in the embedding space. Note that while extrinsic symmetry focuses on invariance under rigid transformations and scaling, intrinsic symmetry covers non-rigid transformations of non-rigid shapes. For example, a human body with extrinsic symmetry may be also intrinsically symmetric, but a human body with hands in different poses is only intrinsically not extrinsically symmetric. Similarly, GPSE is used to provide surface descriptors for correspondence detection among shapes with non-rigid transformations [DK10].
1.3.2 Heat Kernel Signature Embedding (HKSE)

Sun et al. proposed a “transformation invariant shape signature” [SOG09] called Heat Kernel Signature (HKS) based on the heat diffusion function and heat kernel. By inheriting useful properties from the heat kernel and restricting to the temporal domain, the HKS of a point is defined as a scale value instead of a vector like GPS. But the nature of HKS is to transform the original shape representations into another space for effective and efficient comparison and other applications. Therefore, alternatively with the usage of scaled eigenfunctions, a spectral embedding method, called canonical diffusion embedding in [SH10], can be obtained and the norm of a point in the embedding space is the HKS of the point. In this thesis we will refer to this embedding method as the Heat Kernel Signature Embedding (HKSE). In conclusion, HKSE is composed of two fundamental elements: (1) the discrete Laplace-Beltrami operator as the embedding operator, and (2) embedding using the eigenvectors corresponding to the smallest eigenvalues of the operator and the time parameter.

HKSE can be represented as:

\[
HKSE(p) = \left( \sqrt{e^{-\lambda_1 t}} \phi_1(p), \sqrt{e^{-\lambda_2 t}} \phi_2(p), \sqrt{e^{-\lambda_3 t}} \phi_3(p), ... \right)
\] (1.2)

At first, HKS has been applied to find multi-scale shape matching and detect repeated structures within the same shape and a collection of shapes by setting the time parameter differently [SOG09]. Then, to achieve better performance on shape retrieval, Scale-invariant HKS (SIHKS) [BK10] and Volumetric HKS [RBBK10] have been designed to overcome the disadvantages of the HKS. Moreover, Wave Kernel Signature (WKS) [ASC11], which uses the Schrödinger equation instead of the heat equation used by HKS, has been derived to resolve limitations of HKS. And in a related but different way, [XHW10] used heat kernels by embedding every nodes in a graph into a vector space. The direct embedding method used in this study weights the eigenvectors the same way as HKSE.

1.3.3 Multi-Dimensional Scaling Embedding (MDSE)

Elad and Kimmel defined a “bending invariant signature” [EK03] which is an embedding method that uses the Multi-Dimensional Scaling (MDS) process to map 3D shapes into a small dimensional Euclidean space. The idea is to make use of the nature of MDS which is able to map similarity or dissimilarity information into Euclidean distances in the projection space. Although in the area of mesh processing, geodesic distances between pairs
CHAPTER 1. INTRODUCTION

of vertices are usually used to construct the dissimilarity matrix for MDS so that the resulting embedding is invariant to rigid and non-rigid deformations, we consider the MDS process based on any kind of dissimilarity matrix as Multi-Dimensional Scaling Embedding (MDSE). In summary, MDSE is composed of two fundamental elements: (1) dissimilarity matrix as embedding operator, and (2) embedding using the eigenvectors corresponding to largest eigenvalues of the operator. MDSE can be represented as:

\[ \text{MDSE}(p) = \left( \sqrt{\lambda_1} \phi_1(p), \sqrt{\lambda_2} \phi_2(p), \sqrt{\lambda_3} \phi_3(p), \ldots \right) \]  

(1.3)

MDS methods have been widely used in the field of geometry processing. Narrowing down to MDSE, classical MDS is originally used to transform high-dimensional data into low-dimensional, usually 2D, to help visualize the relation among data points. It has been used to embed a 2D contour extracted from other embeddings into a plane for segmentation analysis [LZ07] and to embed 3D shapes into a plane for texture mapping [ZKK02]. Besides 2D embeddings, MDSE is also used for 3D and higher-dimensional embeddings for feature point extraction and pose-invariant normalizations [KLT05, ATCO+10].

1.3.4 Spectral Embedding using Gaussian-filtered affinity matrices (SEG)

The affinity matrix is widely used as an operator of spectral methods in the fields of computer vision and machine learning. Each entry of the affinity matrix is a measure between a pair of feature points. The Gaussian kernel is often employed onto the pairwise relation to obtain an affinity matrix which would be used to compute the eigendecomposition and then to embed the relation into the eigenspace. We call this kind of spectral embedding method, using the Gaussian filtered affinity matrix as the operator, as Spectral Embedding with Gaussian kernel (SEG) in this thesis. Our research shows that different studies used various measures, such as Euclidean distance for clustering, well-known geodesic distances to deal with shape bending, and combined distances to serve a specific purpose. In summary, SEG embedding is composed of two fundamental elements: (1) the Gaussian filtered affinity matrix as embedding operator, and (2) embedding using the eigenvectors corresponding to the largest eigenvalues of the operator. SEG can be represented as:

\[ \text{SEG}(p) = \left( \sqrt{\lambda_2} \phi_2(p), \sqrt{\lambda_3} \phi_3(p), \sqrt{\lambda_4} \phi_4(p), \ldots \right) \]  

(1.4)

At first, SEG is adopted by researchers in the field of computer vision to partition 2D images with the help of color, brightness and texture information [SM97]. It is also used
to solve clustering problems in 2D space by using Euclidean distances [NJW01]. Later on, SEG embedding is migrated into the field of computer graphics and used for processing of 3D shapes, such as segmenting meshes based on spectral clustering [LZ04], finding vertex-to-vertex correspondence between two similar meshes [JZvK07], and achieving robust shape retrieval in a database of articulated 3D shapes [JZ07].

1.4 Summary of contributions

Three applications: mesh segmentation, symmetry detection and shape correspondence, serve the purpose as test scenarios to show the performance of the chosen spectral embedding methods in order for us to visualize and discover the reason and usages. We hope by visualizing the results it will help mathematicians to discover more interesting questions and application researchers to make a better use of these methods.

By applying these four embedding methods onto three applications, the major observations of this comparative study are presented with the following aspects:

- GPSE and HKSE perform very similarly due to the Laplace-Beltrami operator, and MDSE and SEG perform very similarly due to the geodesic distance matrix.
- HKSE tends to perform better when the time parameter is set to small values for all three applications.
- For the first application - mesh segmentation, GPSE and HKSE tend to outperform SEG and MDSE with better segments and boundaries.
- For the second application - symmetry detection, all the embeddings produce similarly satisfying results, but they are very sensitive to details of meshes, which have higher probability on causing asymmetry in connectivity, such as fingers or with intrinsically symmetric poses.
- For the third application - shape correspondence, SEG and MDSE give consistently satisfying results on all the input meshes, while GPSE and HKSE have local inconsistency and even fail on some meshes.
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1.5 Thesis organization

The remaining parts of this thesis are organized as follows. In chapter 2, we summarize various spectral embedding methods including their operators and applications, focusing on the four methods we are going to experiment on, and also offer an overview of related research. Chapter 3, 4 and 5 present methods, results and discussions of our experiments by applying the four types of embedding methods onto three applications: mesh segmentation, symmetry detection and shape correspondence, respectively. And finally in chapter 6 we conclude this preliminary comparative study and propose ideas for possible future work.
Chapter 2

Related Work

In this chapter we will give a brief survey of the previous research, including the introduction of the four spectral embedding methods: Global Point Signatures Embedding (GPSE), Heat Kernel Signature Embedding (HKSE), Multi-Dimensional Scaling Embedding (MDSE), and Spectral Embedding using Gaussian-filtered affinity matrices (SEG), in the field of mesh processing, and the applications of these methods with their specific properties. We are also interested in the survey of the literature which will allow us to observe the work done in mesh processing in more details.

A comprehensive survey done by [ZvKD10] focusing on spectral methods in geometry processing has provided a concrete base with theoretical background, useful references, and thoughtful insights, for interested researchers to have a thorough understanding of up-to-date methodology and applications. It covered a wide range of existing works on different parts of spectral methods, such as historical background, operators, applications and various ways of employing eigenstructures, which included spectral embeddings in 1D, 2D, 3D and higher-dimension. Comparing with the survey concentrating on the width of the research, our paper is going to go deeper into the 3D and higher-dimensional embeddings and investigate the effects of four typical spectral embeddings.

There are also some shorter survey papers discussing particular operators, like [Lev06] which focus on eigenvectors of the Laplace-Beltrami operator and its potential usage in geometry processing, and particular applications, like [Got03] that focuses on spectral partitioning and [Lux07] that studies spectral clustering with emphasis and discussion on un-normalized and normalized graph Laplacian. The most recent study from Mitra et al.
[MPWC12] discusses the numerous methods to detect partial and global, extrinsic and intrinsic symmetries appeared in Eurographics 2012.

### 2.1 Global Point Signatures Embedding (GPSE)

Global Point Signature (GPS) is first defined by Rustamov [Rus07] to characterize any point on a surface without using any external coordinate system. Based on the nature of the Laplace-Beltrami operator, GPS makes use of the resulting eigenvectors and can be further seen as transformation of the surface into an infinite-dimensional space. This kind of the transformation will be referred as Global Point Signatures Embedding (GPSE) of the surface into spectral domain. Some beneficial properties of GPSE, such as “GPS embedding is an isometry invariant”, are quickly used by researchers to solve mesh processing problems. Some results to demonstrate the properties of GPSE are shown in Figure 2.1.

Intrinsic symmetry detection of general shapes is always a difficult problem. Even after transforming the intrinsic symmetry into extrinsic symmetry in a high-dimensional signature space, the search space is still huge, considering that the symmetry axis can be anywhere,
more than one symmetry can occur on a shape and moreover rotational symmetry may
appear accompanied with reflectional symmetry. Ovsjanikov et al. [OSG08] is trying to find
the intrinsic symmetry of a shape by employing the unique property of GPSE and restricting
the search space to reflectional symmetry by only considering non-repeated eigenvalues. The
unique property of GPSE is that the symmetry axis will always be one of the base axes of
the high-dimensional signature space, which will not be true for the MDSE and SEG. This
finding and generalization decreases the search space of extrinsic symmetry enormously.
Figure 2.2 shows the symmetry results with this algorithm on different shapes.

Following the same path Dubrovina and Kimmel [DK10] used the GPSE method as
surface descriptors to find correspondence among articulations of 3D shapes. Figure 2.3
visualizes the correspondence information carried by eigenvectors of the Laplace-Beltrami
operator.

More recently, Bronstein and Bronstein [BB11] compared GPS with diffusion distance on
Figure 2.3: Coloring first 3 eigenvectors (from left to right) of the Laplace-Beltrami operators on two articulations of a human body shows the correspondence [DK10].
shape processing and analyzed the hidden relation between them. As a result, they proposed a new shape similarity measure which employed the advantages of both approaches.

2.2 Heat Kernel Signature Embedding (HKSE)

With the similar concept of Global Point Signatures (GPS), another kind of point signature has been proposed based on the widely used diffusion function and heat kernel, called Heat Kernel Signature (HKS) [SOG09]. By inheriting useful properties from the heat kernel and restricting it to the temporal domain, the HKS of a point is defined as a scale value instead of a vector like GPS. But the nature of HKS is to transform the original shape representations into another space for effective and efficient comparison and other applications. Therefore, with the usage of eigenvectors, a spectral embedding method, called canonical diffusion embedding [SH10], can be obtained and the norm of a point in the embedding space is the HKS of the point. In this thesis we will refer to it as the Heat Kernel Signature Embedding (HKSE).

HKS and HKSE are also highly related to the concepts of diffusion maps and diffusion distances [NLCK05]. We can see that in the spectral space HKSE constructed, the Euclidean distance of embeddings of two points is actually the diffusion distance.

Even though both point signatures, GPS and HKS, are obtained by computing the smallest eigenvalues and their corresponding eigenvectors of the Laplace-Beltrami operator, GPSE weights the eigenvectors by the root of the inverse of the eigenvalues while HKSE weights by the root of the exponential of the eigenvalues and the time parameter. GPSE and HKSE are all able to represent the intrinsic geometry of shapes and to stay stable under small perturbations of shapes. However with the addition of the time parameter into HKSE, it allows for the capturing of the information about the shape in a multi-scale way, like partial shape matching.

In [SOG09], multi-scale shape matching is performed by setting the time parameter of the HKS differently. Figure 2.4 shows some results of shape matching with different time parameters. Based on the original HKS, two variations have been developed. One is the Scale-invariant HKS (SIHKS) [BK10] designed to overcome one disadvantage of HKS, which is its dependence on the global scale of the shape, so that better performance can be achieved for shape retrieval. Figure 2.5 shows the influence of global scale on HKS and SIHKS. In the top left image, the solid green line represents HKS of the green point on the original
shape and the dotted green line represents HKS of the corresponding green point on the globally scaled shape. So the similarity of the same colored solid and dotted lines shows the influence of global scale. We can see that the solid and dotted green lines are very close to each other in the top right image which are the SIHKS results, while the solid and dotted green lines are farther apart in the top left image which are the HKS results. Therefore the lines help to visualize the stability of SIHKS over HKS on scaled shapes in this figure.

The other variation is Volumetric HKS [RBBK10] which makes use of volume instead of boundary of the shape to build isometry-invariant descriptors.

Another related research is [OMMG10] which defines the Heat Kernel Map and uses it to find intrinsic symmetries of shapes. Figure 2.6 shows some symmetry detection results using this algorithm.

More recently, Wave Kernel Signature (WKS) [ASC11], which uses the Schrödinger equation instead of the heat equation used by HKS, is derived to resolve limitations of HKS. [XHW10] used heat kernels differently by embedding every node in a graph into a vector space. The research explored two types of embedding methods, which are referred to as direct and indirect embedding individually. The direct embedding method is based on the Young-Householder decomposition of the heat kernel and weights the eigenvectors the same way as HKSE.

2.3 Multi-Dimensional Scaling Embedding (MDSE)

Using MDS to embed 3D shapes dates back to 1989 when Schwartz et al. [SSW89] tried to visualize the curved surface of cortical areas in a planar mode. More recently Zigelman et al. [ZKK02] also presented a technique to flatten surfaces for texture mapping based on MDS. Both these papers only transformed arbitrary surfaces into a 2D plane, until Elad and Kimmel [EK03] proposed “bending invariant signature” which is an embedding method using the MDS process to map 3D shapes into higher dimensional space other than a plane. As a result of this mapping, the geodesic distances of the original shapes are preserved in the Euclidean distances of the embedded finite high-dimensional space and the embedded signature can be seen as an isometric representation of the 3D mesh. Figure 2.7 shows the transformation of two hand meshes with different poses into a very similar 3D embedding while preserving distances.

Zhou et al. [ZSGS04] employed MDS in a more complex way to achieve chartification and
Figure 2.4: The points coloured in blue are the found matchings of the point marked by the red dot, based on small time parameter (first row), and all time parameters (second row), of HKS [SOG09].
Figure 2.5: Example results to compare HKS (left column) and SIHKS (right column) when shapes differ by global (top row) and local (bottom row) transformations. The RGB colored lines corresponds to the three points on the foot (red), hand (green) and head (blue). That means, the similarity of the same colored solid and dotted lines show the stability of HKS and SIHKS [BK10].
parameterization of meshes based on the matrix of geodesic distance. Spectral analysis of
the eigendecomposition also shows its ability to decrease stretch on mesh parameterization.

Katz et al. [KLT05] applied this MDS method to embed mesh vertices into a pose-
invvariant space. The idea behind this is that MDS is able to uncover the geometric structure
from the dissimilarity information and using geodesic distance to construct the dissimilarity
matrix ensures the embedding to be invariant to poses. Compared with previous work,
three eigenvectors are used to obtain a 3D embedding. Figure 2.8 gives an example of
this 3D embedding. Then, feature points and core components are extracted based on this
normalized space, which will lead to the hierarchical segmentation algorithm that is able to
partition the mesh along the natural seams of the models. Figure 2.9 provides an overview
of the whole segmentation process.

Au et al. [ATCO+10] presented a method to find correspondence between two seman-
tically similar shapes by examining the underlying skeleton information. These shapes are
usually geometrically dissimilar, including different poses, surface details and similarity on
part of the shapes. To avoid the mismatching of similar components, like the front leg of
a dog and the back leg of an elephant, MDS is used to normalize the shapes with pose
variance. Instead of applying MDS to the shapes directly, 3D skeletons derived from the
shapes are normalized with MDSE to extract the intrinsic spatial relationship. Figure 2.10
Figure 2.7: Transformation process of two hand meshes with different poses into a very similar 3D embedding with the preserving distances using MDSE [EK03].
CHAPTER 2. RELATED WORK

Figure 2.8: An example of 3D embedding using MDSE: (a) - input mesh (monkey), (b) - transformation after MDS embedding [KLT05].

Figure 2.9: Overview of segmentation algorithm: (a) - MDS embedding and its convex hull, (b) - extracted feature points shown in red dots on the original mesh, (c) - core extraction of the mesh, (d) - segmentation result [KLT05].
Figure 2.10: 3D skeletons derived from the shapes are normalized with MDSE to extract the intrinsic spatial relationship. Left column – the input meshes with their skeleton, and right column – their pose-invariant embedding results [ATCO+10].

illustrates the results of this process.

2.4 Spectral Embedding using Gaussian-filtered affinity matrices (SEG)

As early as 1997, Shi and Malik [SM97] applied the graph partitioning concept to solve grouping problems in the field of computer vision. In order to extract global information, eigenvectors of a Gaussian filtered matrix are computed and utilized for the segmentation of brightness, color and texture images. Figure 2.11 shows an example of the segmentation on an image. Depending on the application, the matrix may employ Euclidean distance, color intensity or a mixture of feature similarity terms and spatial proximity terms. The matrix is later defined as the “normalized affinity matrix” by Weiss [Wei99] when compared of with the other two eigenvector segmentation methods. Although all three methods made use of eigenvectors, the normalized matrix in [SM97] showed advantages for real images. On the other hand the analysis also suggested using more than two eigenvectors to improve the algorithm based on the experimental results.
Ng et al. [NJW01] presented a straightforward spectral method to solve clustering problems by embedding input points using eigenvectors of an affinity matrix. Here the normalized affinity matrix is constructed using the Euclidean distance and obtained using the Gaussian kernel. Then k-means clustering is applied to the entries of the un-scaled eigenvectors corresponding to the largest eigenvalues which are projected onto the unit k-sphere. And this research also provided proof and conditions for the positive behaviour of this algorithm based on matrix perturbation theory. Figure 2.12 compared some clustering results obtained using this algorithm with other previous methods.

Similarly Liu and Zhang [LZ04] applied the spectral clustering algorithm to segment 3D meshes. The normalized affinity matrix from the Gaussian kernel is still used, but with the geodesic distance and the angle distance on meshes instead of the Euclidean distance among the points. A newly added step to improve the performance is the pre-processing of cluster centers before directly applying k-means clustering onto entries of un-scaled eigenvectors. Figure 2.13 shows the results of using this method.

Compared with symmetry existing in one shape which may involve non-rigid transformations, correspondence is a meaningful interpretation between two similar shapes which may be different in shape, pose, scale, besides rigid and non-rigid transformation. Jain et al.
Figure 2.12: Spectral clustering results with, (a) - (c) from Ng et al. algorithm, (d) from Kannan et al. algorithm, (e) from Meila and Shi algorithm and (f) from k-means [NJW01].
[JZvK07] presented an algorithm to find this vertex-to-vertex correspondence between two 3D shapes by embedding them into the spectral domain. The spectral embedding method chosen is SEG as the use of geodesic distance enables the algorithm to deal with shape bending. Moreover, together with the Gaussian kernel, this embedding ensures the mapping process invariant to uniform scaling and rigid body transformation. Also, this research pointed out some insightful observations about spectral method properties and limitations. Figure 2.14 shows the correspondence results of different pairs of shapes obtained with this algorithm.

Besides employing the SEG embedding to solve correspondence problems, Jain and Zhang [JZ07] also applied this embedding method for robust shape retrieval in a database of articulated 3D shapes. All the shapes are transformed into the spectral domain using SEG embedding in order to against bending, rigid-body transformation and uniform scaling. Then shape retrieval is performed on these embeddings. Figure 2.15 provides the visualization of the similarity of the embeddings of shapes with non-rigid transformation.
CHAPTER 2. RELATED WORK

Figure 2.14: Color plots to show the correspondence results of the semantic related pairs of 3D shapes [JZvK07].

Figure 2.15: Top row is a list of 3D shapes with intrinsic similarity. Bottom row is their spectral embeddings after normalization [JZ07].
2.5 Other spectral embedding methods

Considering the wide range of spectral embedding methods, there are also other papers that use other or a mix of spectral embeddings to achieve optimal results besides the four types we focused on in this thesis.

A paper using three different embedding methods to segment 3D meshes in their research is from Liu and Zhang [LZ07]. It firstly got two 2D embeddings of an input mesh using different operators, one is the classic graph Laplacian, the other is a newly defined curvature Laplacian. With these two embeddings, both structural and geometric segmentability information is preserved and extracted. Then, taking the contour of the resulting 2D embedding as input, the MDSE with inner distance matrix as the operator is performed to find concavity. Figure 2.16 shows an example of the two 2D embeddings obtained from the same 3D mesh.

Qiu and Hancock [QH07] explored the properties of commute time distance and used commute time embedding on two applications, image segmentation and multibody motion tracking.

Different variants of spectral methods may differ in the construction of operators which may use different distances according to the applications, may also differ in the way of the normalization of the operators, or the way of using eigenvectors and eigenvalues. With so many possibilities, besides the widely used geodesic distance and Laplace-Beltrami operator, some new concepts have also contributed to the development of spectral methods. Lipman et al. [LRF10] proposed a new surface distance measure called biharmonic distance,
while Rustamov [Rus11] introduced multiscale biharmonic kernels on meshes, and Au at al. [AZC+12] designed a concavity-sensitive Laplacian. [RBB+11] altered the metric in the Laplace-Beltrami operator to construct affine-invariant local and global diffusion geometric structures and applied the framework on shape retrieval, correspondence, and symmetry detection applications. Furthermore, modal analysis [HWAG09] [HSvTP12] is borrowed from traditional engineering field for use in shape analysis due to its connection with Laplace operators. Energy-based operators derived from modal analysis are designed to have a better representation of surfaces, such as shape deformation and sharp bends.
Chapter 3

Application 1 - Mesh Segmentation

In this chapter we will show how we applied the four spectral embedding methods: Global Point Signatures Embedding (GPSE), Heat Kernel Signature Embedding (HKSE), Multi-Dimensional Scaling Embedding (MDSE) and Spectral Embedding using Gaussian-filtered affinity matrices (SEG), into the first application — mesh segmentation and compared the resulting performances of different embedding methods. From the similarities and differences, we will propose some insights and analyses.

3.1 Mesh segmentation algorithm

Mesh segmentation is one of the most basic problems in the field of geometry processing, and moreover it is usually the first step or a part of more complex problems. A 3D shape can be segmented in many ways, but a meaningful segmentation should align with what a human performs, that is mostly based on the structure and semantics of the shape. With the ongoing cognitive studies on shape segmentation and diversity in assorted shapes, mesh segmentation becomes a fundamental but non-trivial undertaking. Therefore, due to its popularity and complication we choose this problem as the first application to test our embedding methods. In order to obtain a relatively fair comparison of the embedding methods so that their nature and applications can be discovered, our goal is to implement a straightforward segmentation algorithm which will not favour any of these methods.

The segmentation algorithm we implemented makes use of the classical k-means clustering method to segment 3D meshes. We take a 3D mesh with vertices and faces information
as input, and apply each of the four embeddings to transform every vertex into an \( m \)-dimensional vector. Then the k-means clustering algorithm is applied to the \( m \)-dimensional data. The corresponding vertices belonging to the same cluster will be in one segment. To help visualize the results, original meshes are color plotted according to the clustering of vertices. The color for each segment in each result is chosen randomly, therefore there are no mapping relations among different results of the same mesh.

### 3.2 Dataset and parameters

A variety of seven shapes were chosen as the input dataset with four from the Princeton Segmentation Benchmark [CGF09]: “Human” — a most commonly used human model in walking pose, “Dancing children” — one model with four dancing children on a shared base, two animal models — “Horse” and “Skeleton dinosaur”, two manmade objects with sharp features — “Lamp” and “Chair”, and “Neptune” — a human holding a trident on a shared base, which has a high-genius number. These models represent varied levels of complexity in geometry and topology shown in Figure 3.1.

In this experiment of 3D shape segmentation, there are two parameters we need to set carefully. First is the number of segments for each shape, that is, the choice of \( k \) for the k-means clustering method; and second is the dimension of the embedding space, that is,
the number \( m \) of eigenvalues used for the embedding methods.

Moreover, for HKSE, different settings of the time parameter will lead to different results. In general, smaller time tends to reveal the local structure of the model and larger time tends to reveal the global structure of the model. Therefore, we choose one small time and one big time based on the eigenvalues of each mesh to discover the performance of HKSE.

### 3.3 Results

In this section, we are going to compare the similarities and differences of results induced by different parameters and embedding methods, to explore the possible causes and to discover the usage of embedding methods based on their advantages and disadvantages.

#### 3.3.1 Number of segments

We need to choose the number of segments for each shape according to their different geometry. To choose the number of clusters \( k \) for the k-means clustering method, we tried \( k = 1 \) to 20 to search for the best one. And we ran 10 times for each value of \( k \) in order to get the average criteria value for each \( k \) on each mesh. Then, plotting the k-means criteria of each \( k \) will provide a sense of the converging point. When testing with different embedding methods, the resulting plots of \( k \) converge similarly. Figure 3.2 shows two plots of \( k \) based on the same mesh using the GPSE and MDSE methods individually. Combining with the common sense of human beings, we are now able to choose \( k \) for every model. In this way, a reasonable number of clusters will be found and carried on to the segmentation experiments.

The resulting number of segments \( k \) we choose for each of the seven meshes are: “Human”, “Horse” and “Skeleton dinosaur” — \( k = 6 \), “Dancing children” — \( k = 5 \), “Lamp” — \( k = 4 \), “Chair” — \( k = 6 \), “Neptune” — \( k = 3 \).

#### 3.3.2 Number of eigenvalues and eigenvectors

In our implementation of achieving mesh segmentation via spectral clustering, we need to consider how many eigenvectors from the eigendecomposition to use. When choosing the dimension of the embedding space, we tried to use one small number and one large number on each mesh to see the effect. Considering our computing power based on the number of vertices of meshes, and using existing spectral embedding algorithms as references, 3 and
3.3 Comparisons

All four kinds of embedding methods produced similar and reasonable results on three of the meshes: “Human”, “Skeleton dinosaur” and “Lamp” shown in Figure 3.6, Figure 3.7, Figure 3.8 separately.

For another mesh “Neptune”, which has a high-genius number, all the results were less...
Figure 3.3: GPSE and HKSE methods show subtle differences in segmentation results when using 3 eigenvectors (first row) and 15 eigenvectors (second row) on two meshes. One is the mesh “Chair”, GPSE and HKSE perform a bit better when \( m = 15 \) than \( m = 3 \). The other one is the mesh “Dancing children”, GPSE produces a better result when \( m = 3 \) while HKSE produces a better result when \( m = 15 \).
Figure 3.4: Segmentation results of the mesh “Chair” from four embedding methods, with (a) - (e) using 3 eigenvectors, and (f) - (j) using 15 eigenvectors.

Figure 3.5: Segmentation results of the mesh “Dancing children” from four embedding methods, with (a) - (e) using 3 eigenvectors, and (f) - (j) using 15 eigenvectors.
Figure 3.6: All four embedding methods produce similar segmentation results on the mesh “Human”, with (a) - (e) using 3 eigenvectors, and (f) - (j) using 15 eigenvectors.
CHAPTER 3. APPLICATION 1 - MESH SEGMENTATION

Figure 3.7: All four embedding methods produce similar segmentation results on the mesh “Skeleton dinosaur”, with (a) - (e) using 3 eigenvectors, and (f) - (j) using 15 eigenvectors.

Figure 3.8: All four embedding methods produce similar segmentation results on the mesh “Lamp”, with (a) - (e) using 3 eigenvectors, and (f) - (j) using 15 eigenvectors.
accurate, as shown in Figure 3.10. This mesh is naturally described in three parts: a human body, a trident and a base, which is the ground truth we use to evaluate the results. However we can see that no embedding methods can detect it easily. Only HKSE is able to divide the human body and the base into one segment when the time parameter is large.

For the other three meshes, GPSE and HKSE tend to outperform MDSE and SEG with more moderate segments and higher quality cut boundaries, as shown in Figure 3.11. From results of the mesh “Chair” in the first row we can see that segments generated by GPS and HKS embeddings have more clear and clean boundaries than MDSE and SEG. From the results of the mesh “Horse” in the second row we can see that segment parts produced by different embedding methods are all semantically similar and reasonable, but boundaries of segments by GPSE and HKSE embeddings tend to agree with the convexity changes which fit more with human perception. More results of the mesh “Horse” are shown in Figure 3.9. Moreover from the results of the mesh “Dancing children” in the the last row, GPSE and HKSE detect the five parts — four children and a base, which it is divided naturally — better than MDSE and SEG.

In all the discussions above, HKSE produces better results when the time parameter is small rather than large. We should be aware that the choice of time parameter changes results dramatically for HKSE. Even though we know based on the principles of HKSE that larger time leads to more global structure and smaller time leads to more local structure, the best result does not always appear at large time. From the results shown in Figure 3.12
small time actually performs better than large time on most of the meshes. This finding shows that choosing the right time for HKSE is a difficult and important problem, but with the flexibility HKSE is also able to reveal more scales of details than the others.

### 3.4 Conclusion

Regarding the input meshes, all the embedding methods are not able to deal with high-genius number mesh accurately, but are able to deal with mesh with sharp features.

The number of eigenvectors used hardly makes any difference on MDSE and SEG, and a small difference on GPSE and HKSE. But there is no guarantee that more eigenvectors always produce better results.
Figure 3.11: Segmentation results of three meshes “Chair”, “Horse” and “Dancing children” showing GPSE and HKSE outperform MDSE and SEG embedding methods with more moderate segments and higher quality cut boundaries.
Figure 3.12: Segmentation results showing the difference of HKSE method with time parameter being small and big, and small time actually performs better than big time on most of the meshes.
All in all, HKSE and GPSE tend to outperform MDSE and SEG on this segmentation application with better segments and boundaries. However the ability of setting the extra time parameter for HKSE adds difficulty for finding the best result but provides the potential for discovering more levels of information.
Chapter 4

Application 2 - Symmetry Detection

In this chapter we will show how we applied the four spectral embedding methods: Global Point Signatures Embedding (GPSE), Heat Kernel Signature Embedding (HKSE), Multi-Dimensional Scaling Embedding (MDSE), and Spectral Embedding using Gaussian-filtered affinity matrices (SEG), into the second application — symmetry detection and compare the resulting performances of different embedding methods. From the similarities and differences, we will propose some insights and analyses.

4.1 Symmetry Detection algorithm

Symmetry can be found everywhere in the world among both natural and man-made objects, from the human body and animals to airplanes and architectures. Due to its wide existence, symmetry information is useful for various applications, such as surface remeshing, mesh segmentation, face detection and shape matching. In order to extract this high-level structural information, numerous methods have been proposed in the area of computer graphics and computer vision to find global and partial symmetries, extrinsic and intrinsic symmetries.

To investigate the performance of different embedding methods on the symmetry problem, we implemented a well-known global intrinsic symmetry detection algorithm designed by [OSG08]. Even though the paper [OSG08] is making use of the properties of GPS, we
still substituted GPS with the other three embedding methods. Following the same sym-
metry detection procedure, we would like to observe the similarity and differences of the
results in order to gain a better understanding of the spectral embedding methods and the
corresponding operators.

4.2 Dataset and parameters
The meshes used as input data for symmetry detection are from the Surface Correspon-
dence Benchmark [KLF11] which is composed with SCAPE [ASK+05], TOSCA [BBK08]
and Watertight datasets from SHREC, the shape retrieval contest. Considering the compu-
tational complexity of geodesic distance matrices used by MDSE and SEG methods, all the
meshes are remeshed to a coarser level, around 1000 vertices, and used as input for all the
embeddings for a fair comparison.

In our experiments of symmetry detection, we used 15 (m = 15) non-repeated eigenvalues
and their corresponding eigenvectors same as the paper [OSG08], that means, a mesh is
always embedded into a 15-dimensional signature space after each embedding method and
then followed by the same symmetry detection procedure.

4.3 Results
While substituting the GPS embedding in the original algorithm with the other three em-
bedding methods, we are able to observe the differences in the overall performance of these
embedding methods.

4.3.1 Negative eigenfunction detection
In the paper [OSG08], it is proved that the intrinsic symmetries of a shape would be trans-
formed into extrinsic symmetries of its GPS embedding. However, the search space of
extrinsic symmetries is still large. Here an important property of eigenfunctions associated
with non-repeated eigenvalues is employed, which is that these kind of eigenfunctions are
either positive or negative. The definition of positive and negative is based on the fact that
each eigenfunction contains a value for each vertex of a shape. For an intrinsic symmetry,
a vertex and its symmetric vertex either have the same value in an eigenfunction, which
is called positive, or have the same absolute value with opposite signs in an eigenfunction,
which is called negative. In other words, by only considering eigenfunctions associated with non-repeated eigenvalues in the signature space, the search space of extrinsic symmetries is reduced to reflection symmetries around the principle axes. Every intrinsic symmetry can be represented as a sequence of positive and negative eigenfunctions. Moreover, it is claimed that in practice around half of the eigenfunctions are always positive, the problem of finding intrinsic symmetries is transformed into finding negative eigenfunctions.

In the symmetry detection procedure of the algorithm, negative eigenfunctions need to be found according to the errors computed based on each eigenfunction. Theoretically, negative eigenfunctions should have errors equal to zero, but practically we are supposed to consider those with small errors as negative. From our experiments we noticed that it is very difficult to decide how small the errors should be for negative eigenfunctions, since the values of errors are quite different for different meshes. That means, the error threshold highly depends on the particular mesh itself, we need to set the threshold for each mesh and there is no guideline on how to set it.

To achieve the goal, alternative ways are examined to find the thresholds. Firstly, by sorting eigenfunctions according to their errors, we tried to decide the boundary of negative and positive eigenfunctions when the error increment is suddenly much bigger than previous ones. Unfortunately, a good cut on the error plot will not guarantee a satisfying symmetry result. Continuing to try different thresholds turns out to be time-consuming, and moreover the outcomes have very low accuracy.

Secondly, instead of relying on the error values, we also made use of the visualization of each eigenfunction on the mesh. Since for every eigenfunction there is one value corresponding to each vertex of the mesh, coloring the mesh based on the values will help easily distinguish the symmetric color plots which represent the positive eigenfunctions. Considering the errors that human eyes can make on the color plots, we combined them with the sorted error values of eigenfunctions to detect negative eigenfunctions more accurately.

Here an example is shown to help explain the details of the negative eigenfunction detection process. These results are based on GPS embedding and used a human mesh as input. Figure 4.1 shows 15 eigenfunctions sorted by their error values and Figure 4.2 are color plots of the 15 eigenfunctions onto the human mesh. Thus we can easily find 9 positive eigenfunctions with symmetric color plots, which also turn out to be the last 9 eigenfunctions in the sorted list which has the largest errors.
4.3.2 Comparisons

With the help of our alternative negative eigenfunction detection method, we tried to substitute GPS embedding used in the paper [OSG08] with HKSE, MDSE and SEG embeddings. The results show that all embeddings are able to produce similar reasonable results. And the first several eigenfunctions with the largest errors are always set to positive, which fits the “negative eigenfunctions have the smallest errors” rule in the paper [OSG08]. But the number of positive eigenfunctions for the same mesh are different for the four embeddings. MDSE and SEG embeddings have similar eigenfunctions and usually set the same number of eigenfunctions to positive or negative. Figure 4.3 and Figure 4.4 list some results of human meshes and animal meshes using different embedding methods.

4.3.3 Limitations

With the results showing above, we observe that all four embeddings perform similarly and are able to give satisfying results. However for some meshes the results are not as accurate as others. Figure 4.5 shows some examples, with the first two rows showing the results of
Figure 4.2: (a)–(o) are color plots of 15 eigenfunctions ordered by their errors with last 9 symmetric color plots being positive and (p) is the symmetry result showing with red lines.
Figure 4.3: Similar satisfying symmetry results of human meshes using different embedding methods.
Figure 4.4: More symmetry results show similarity when using different embedding methods.
two human meshes with fingers and the last row listing results of a cat mesh with different poses. We can see that the results in the first two rows are not as good as those of the human meshes without fingers shown in Figure 4.3, and the results in the last row are not as good as the cat mesh shown in Figure 4.4. The reason causing this difference becomes an issue we should be aware of when using this symmetry detection algorithm.

Firstly, we suspected the remeshing of the original meshes into coarser meshes played an important role. By taking the original fine meshes with around 5000 vertices, the algorithm with GPS embedding gives results, shown in Figure 4.6, which are very similar to the coarser meshes. Although this investigation is not able to tell us why some meshes produce worse results than others, it demonstrates the stability of eigenfunctions to preserve the structure of fine to coarse meshes.

Secondly, the contrasting results between the human meshes with fingers and those without fingers lead to thinking about the details of the meshes. Compared with the results in last row of Figure 4.4, which are also obtained from a human mesh with fingers, it is better than those in the first two rows of Figure 4.5. Plus the varied results of the cat meshes with different poses make us take a closer look at the connectivity of the meshes. It seems that meshes with better symmetry in connectivity tends to have better symmetry detection result. In Figure 4.7 we can see that the cat mesh at left is extrinsically symmetric, which means both sides of the cat have the same pose. On the other hand, the cat mesh at right is the same cat but with intrinsic symmetry, which means two sides of the cat have different poses. The differences in two sides cause less symmetric triangulation, such as the upper part of the front left leg and that of front right leg, which are supposed to be symmetric opponents. The symmetry results of the two cats in Figure 4.4 and Figure 4.5 suggest that the cat mesh with intrinsic symmetry has lower level of symmetry in connectivity and therefore has worse symmetry results.

Thirdly, even with the satisfying results, accurate point-to-point mapping is still a problem. Like the results in the last row of Figure 4.4, the general symmetry is found on the human mesh with fingers, but with a closer look we can see that finger-to-finger mapping is not obtained by any of the embedding methods.

All in all, during the exploration, it is shown that eigenfunctions are consistent of preserving mesh structure from fine to coarse meshes. Also the symmetry method using spectral methods indicates itself are sensitive to the details of the meshes, such as level of symmetry in connectivity.
Figure 4.5: Less satisfying symmetry results using different embedding methods.
Figure 4.6: Symmetry results of finer meshes (around 5000 vertices) using GPS embedding method.
Figure 4.7: Connectivity symmetry of meshes.
4.4 Conclusion

When using the symmetry detection algorithm, all the embedding methods are able to produce similar acceptable results. It is difficult to say which method is superior. But one advantage of GPS embedding is its ability to deal with larger meshes under the same computing power because of its sparse operator.

However, the step of deciding on negative eigenfunctions is the vital part of the whole process. Inaccurate choices of negative eigenfunctions will affect the quality of the symmetry results.

In addition, the input meshes are also an important factor. Fine and coarse meshes do not make a huge difference, but meshes with details like fingers or with intrinsically symmetric poses, which have a higher possibility of causing asymmetry in connectivity, tend to produce less satisfying results.
Chapter 5

Application 3 - Shape Correspondence

In this chapter we will show how we applied the four spectral embedding methods: Global Point Signatures Embedding (GPSE), Heat Kernel Signature Embedding (HKSE), Multi-Dimensional Scaling Embedding (MDSE), and Spectral Embedding using Gaussian-filtered affinity matrices (SEG), into the third application — shape correspondence and compared the resulting performances of different embedding methods. From the similarities and differences, we will propose some insights and analyses.

5.1 Correspondence algorithm

Shape correspondence is trying to find a mapping between two structurally and semantically similar meshes, which is also a fundamental way of human beings to understand the world. The two meshes can be from the same object with rigid and non-rigid transformations such as a human in different poses, or different but structurally similar objects such as a cat and a dog. The effort to automate the mapping process has been made in the field of computer vision on 2D space for a long time, for 3D meshes it is also a principle problem in the field of computer graphics for applications such as mesh parameterization, object recognition and shape registration. More recently, with the establishing of a correspondence measure, shape retrieval can be achieved and searching models in a huge database of 3D shapes will be helpful both for research and practical use.
To investigate different embedding performance on shape correspondence, we implemented a spectral correspondence algorithm designed by [JZvK07] which finds a vertex-to-vertex mapping between 3D shapes. Even though the paper [JZvK07] chooses to use SEG embedding, we still substituted SEG embedding with other embeddings to discover the advantages and disadvantages. Following the same correspondence detection procedure, we would like to see the similarity and differences of the results to gain a better understanding of the spectral embedding methods and the corresponding operators.

5.2 Dataset and parameters

Meshes used as input data for correspondence detection are from the Surface Correspondence Benchmark [KLF11] which is composed of SCAPE [ASK+05], TOSCA [BBK08] and Watertight datasets from SHREC, the shape retrieval contest. Considering the computational complexity of the geodesic distance matrix used by MDSE and SEG methods, all the meshes are remeshed to a coarser level, around 1000 vertices, and used as input for all the embeddings for fair comparisons.

We chose three categories of meshes, with meshes in each category being structurally similar: human – different human models with different poses, animals — like cat, wolf and horse with different poses, and different kinds of birds.

In our experiments of correspondence detection, we use 5 (m = 5) eigenvalues and their corresponding eigenvectors as in the paper [JZvK07], that means, a mesh is always embedded into a 5-dimensional signature space after each embedding method and then followed by the same correspondence detection procedure.

Moreover, for HKSE different settings of the time parameter will lead to different results. In general smaller time tends to reveal local structure of the model and bigger time tends to reveal global structure of the model. Therefore we choose one small time and one big time based on the eigenvalues of each pair of meshes to discover the performance of HKSE.

5.3 Results

While substituting the SEG embedding in the original application algorithm with other embeddings, we are able to observe the differences in the overall performance of these embedding methods.
5.3.1 Symmetrically flipped correspondence

As discussed in the paper [JZvK07], for shapes with intrinsic symmetry, it is possible that the correspondence happens to be symmetrically flipped. As shown in Fig 5.1, the pair of shapes Cat:Wolf are both intrinsically symmetric from left to right side. We can see the result of SEG embedding in (e) that the left side of the cat is mapped into the right side of the wolf, and vice versa for the other side. That means the left front leg and the left back leg of the cat corresponds to the right front leg and the right back leg of the wolf separately. This phenomenon is caused by constructing the affinity matrix from geodesic distance, with this measure symmetric points can not be distinguished. However the mapping is still consistent across the shape and it happens randomly on different meshes when using different embedding methods, we consider the results with symmetrically flipped correspondence acceptable.

Another similar but not acceptable flipped correspondence is shown in Fig 5.1(a) — the result of GPSE. Here we can see that the left front leg of the cat corresponds to the left front leg of the wolf while the left back leg of the cat to the right back leg of the wolf. This flipping is not symmetrical anymore and also the mapping is not consistent across the shape, therefore this kind of non-symmetrically flipped correspondence is not seen as an acceptable result.

5.3.2 Comparisons

For the human meshes, all the embedding methods are able to match the human with different poses and different human models correctly. Fig 5.2 and Fig 5.3 show the results of two pair of meshes using different kinds of embedding methods. Only GPSE and HKSE show symmetrically flipped correspondence.

For the animal and bird meshes, while SEG and MDSE still produce satisfying results, GPSE and HKSE give less satisfying results and even fail to find the right mapping on some meshes, as shown in Fig 5.4, Fig 5.5, Fig 5.6 and Fig 5.7. We can see that results from GPSE tend to be inconsistent on some local areas, such as at the back of the cat in Fig 5.5(a) and on the wings of the bird in Fig 5.7(a). HKSE appears to have the same situation in Fig 5.5(b) and Fig 5.6(b), and HKSE fails to find the mapping in Fig 5.4 and Fig 5.7.

From our experiments, we can see that SEG and MDSE perform very similarly and both produce satisfying results over all three kinds of meshes. GPSE are able to give satisfying
Figure 5.1: Results with symmetrically flipped correspondence.
Figure 5.2: All the embedding methods are able to produce satisfying correspondence results on human meshes with different poses.

results on human meshes, but have local inconsistencies on animal and birds meshes. HKSE works similarly to GPSE with some local inconsistency on some meshes, but failed to find the right mapping on some meshes. And HKSE tends to perform better at small time.

5.3.3 Limitations

With the satisfying performance of SEG and MDSE on meshes from the same category, we tested the limit of the correspondence algorithm with meshes which are structurally similar but not from the same category, such as Centaur:Horse in Fig 5.8 and Human:Gorilla in Fig 5.9. We can see that for the pair of meshes Centaur:Horse, MDSE and SEG produce great mapping results in Fig 5.8(d) and (e), while GPSE and HKSE fail to find the right correspondence. Even the mesh Centaur has two extra arms than the Horse, the meaningful
Figure 5.3: All the embedding methods are able to produce satisfying correspondence results on this pair of male-female human meshes.
Figure 5.4: SEG and MDSE outperform GPSE and HKSE with more satisfying correspondence results on animal meshes.
Figure 5.5: SEG and MDSE outperform GPSE and HKSE with more satisfying correspondence results on animal meshes.
Figure 5.6: SEG and MDSE outperform GPSE and HKSE with more satisfying correspondence results on bird meshes.
Figure 5.7: SEG and MDSE outperform GPSE and HKSE with more satisfying correspondence results on bird meshes
mapping is still found.

And for the pair of meshes Human:Gorilla, we can see that all the embedding methods are able to give the same results, with mapping the arms of the Human to the legs of the Gorilla, and vice versa. Even with this upside-down mapping, the consistency still exists, meaning that the same side of the arm and the leg of the Human get mapped to the same side of the leg and the arm of the Gorilla. We suspect the reason is that even though the Human and the Gorilla are structurally similar, the difference of proportion is much bigger than the different human models.

Here we can see that embedding methods are able to preserve the general structure of meshes and are robust to small differences among meshes, such as arms of the Centaur.

5.4 Conclusion

From our experiments, we can see that SEG and MDSE perform very similarly and both produce satisfying results over all three kinds of meshes. GPSE is able to give satisfying results on human meshes, but has local inconsistencies on animal and bird meshes. HKSE works similarly to GPSE with some local inconsistencies on some meshes, but failed to find the right mapping on some meshes. HKSE tends to perform better at small time.
Figure 5.8: GPSE and HKSE failed to find correspondence on centaur horse meshes while SEG and MDSE give satisfying results.
Figure 5.9: All the embedding methods are NOT able to produce satisfying correspondence results on the pair of Human:Gorilla meshes.
Chapter 6

Conclusion and Future Work

6.1 Conclusion

In this thesis we have conducted a preliminary comparative study by first selecting four popular spectral embedding methods: Global Point Signatures Embedding (GPSE), Heat Kernel Signature Embedding (HKSE), Multi-Dimensional Scaling Embedding (MDSE), and Spectral Embedding using Gaussian-filtered affinity matrices (SEG). Then they were applied onto three applications in the field of geometry processing and analysis: mesh segmentation, symmetry detection and shape correspondence. Therefore we are able to obtain experimental observations based on the limited test cases so that the differences and similarities of the behaviours of different spectral embedding methods are explored.

In the first application, mesh segmentation, HKSE and GPSE tend to outperform MDSE and SEG with better segments and boundaries. In the second application, symmetry detection, all the embedding methods produce similar results based on the sensitivity of the symmetry detection algorithm itself. In the third application, shape correspondence, MDSE and SEG show better performance and more stability than GPSE and HKSE. All in all, there is no single one embedding method to overpower the others across all three applications. It highly depends on the application algorithm, the input mesh, the computing power, the level of information and so on.
6.2 Future Work

In the future, this study can be extended by adopting a larger mesh dataset as input for each application so that the conclusions can be more convincing with the statistics of quantitative data. The more important and challenging problem is to discover reasons for the different behaviours and to extract insights based on all the observations from our experiments, so that researchers can have a deeper understanding and better use of various spectral methods to achieve optimal outcome.
Appendix A

Summary of embedding methods

Here we list fundamental equations of the four spectral embedding methods we mainly focused on in this survey.

With the eigendecomposition of the linear operators, eigenvalues $\lambda_i$ and the corresponding eigenvectors $\phi_i$ are used to embed input data from its original domain into a spectral domain. For each point $p$ on the manifold, it will be transformed into a $m$-dimensional vector as shown below for each embedding method, with $m$ being the number of eigenvalues used and $\phi_i(p)$ being the value of the eigenvector $\phi_i$ at the point $p$.

A.1 Global Point Signatures Embedding (GPSE)

With the use of the Laplace-Beltrami operator and the eigenvectors corresponding to the smallest eigenvalues, GPSE is defined as:

$$GPSE(p) = \left( \frac{\phi_1(p)}{\sqrt{\lambda_1}}, \frac{\phi_2(p)}{\sqrt{\lambda_2}}, \frac{\phi_3(p)}{\sqrt{\lambda_3}}, ... \right)$$  \hspace{1cm} (A.1)

A.2 Heat Kernel Signature Embedding (HKSE)

With the use of the Laplace-Beltrami operator, the eigenvectors corresponding to the smallest eigenvalues and the time parameter $t$, HKSE is defined as:

$$HKSE(p) = \left( \sqrt{e^{-\lambda_1 t}}\phi_1(p), \sqrt{e^{-\lambda_2 t}}\phi_2(p), \sqrt{e^{-\lambda_3 t}}\phi_3(p), ... \right)$$ \hspace{1cm} (A.2)
A.3 Multi-Dimensional Scaling Embedding (MDSE)

With the use of the dissimilarity matrix as embedding operator and the eigenvectors corresponding to the largest eigenvalues, MDSE is defined as:

\[
MDSE(p) = \left( \sqrt{\lambda_1} \phi_1(p), \sqrt{\lambda_2} \phi_2(p), \sqrt{\lambda_3} \phi_3(p), \ldots \right) \tag{A.3}
\]

A.4 Spectral Embedding using Gaussian-filtered affinity matrices (SEG)

With the use of the Gaussian-filtered affinity matrix as embedding operator and the eigenvectors corresponding to the largest eigenvalues, SEG is defined as:

\[
SEG(p) = \left( \sqrt{\lambda_2} \phi_2(p), \sqrt{\lambda_3} \phi_3(p), \sqrt{\lambda_4} \phi_4(p), \ldots \right) \tag{A.4}
\]
Bibliography


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