COSTAS ARRAYS, GOLOMB RULERS AND
WAVELENGTH ISOLATION SEQUENCE PAIRS

by

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Abstract

This thesis studies two combinatorial objects arising from applications in digital information processing. We firstly consider “wavelength isolation sequence pairs” (WISPs), a type of binary sequence pair introduced by Golay in 1951 but largely neglected since. Two previously overlooked examples of such sequence pairs are presented. We construct all known examples of WISPs from perfect Golomb rulers, and give partial classification results. We secondly consider Costas arrays, a generalisation of Golomb rulers dating from 1965. We examine whether a Costas array can contain every possible toroidal distance vector; contrary to claims elsewhere, this is still an open question. We constrain the (non-toroidal) distance vectors in Costas arrays by introducing “mirror pairs”. Structural properties of all Costas arrays are established via the number and type of their mirror pairs, with stronger results for G-symmetric Costas arrays, Welch Costas arrays and Golomb Costas arrays.
I owe many thanks to my senior supervisor, Jonathan Jedwab, for his guidance throughout my degree. My thesis has benefitted greatly from the influence of Jonathan’s various annoying habits, and I hope I have picked some of them up along the way. I also wish to thank the other members of my committee, especially Matt DeVos, whose insightful comments often pointed me towards the “right way” to think about my mathematical problems. Thanks go to the (current and former) members of Jonathan’s research group for listening to me talk about WISPs and Costas arrays at more than my fair share of group meetings, and for always offering ideas and encouragement. Scott Rickard, who maintains www.costasarrays.org, provided me with the Costas array database, which has been instrumental in my research. I wish to thank Chris Duffy, for his genuine (and apparently insatiable) interest in my research, and Kyle Robertson, for his patient programming help. Finally, I am grateful for the wide-ranging support of my friends and family, both near and far.
A Note from the Author

The recipe below provided required sustenance and welcomed distraction throughout the research phase of this thesis. Like this thesis, it is the result of trial and error, collaboration and feedback; like research, it is a work in progress. As this document represents the current state of my research, I offer the current recipe to you, the reader, to help you digest the work contained herein.

Mastery cookies

1/2 cup butter
1/2 cup brown sugar
1/4 cup granulated sugar
1 egg
1 1/2 tsp vanilla
2/3 cup flour
1 cup oats
1/2 tsp baking soda
3/4 tsp cinnamon
pinch salt
3/4 cup dark chocolate chips
1/2 cup raisins

Cream butter, sugars, egg and vanilla. Add flour, oats, baking soda and cinnamon, and mix until combined. Fold in chocolate chips and raisins by hand. Spoon onto a cookie sheet and bake at 350° F for 8 to 10 minutes. Remove from oven and transfer to a wire rack while cookies are still soft in the middle.
## Contents

Approval ii

Abstract iii

Acknowledgments iv

A Note from the Author v

Contents vi

List of Tables viii

List of Figures ix

1 Overview 1

2 Introduction 6

2.1 Wavelength isolation sequence pairs 9

2.1.1 History and motivation 9

2.2 Costas arrays 13

2.2.1 History and definitions 13

2.2.2 Equivalence under the action of $D_4$ 23

2.2.3 Construction techniques 27

2.2.4 Enumeration and existence results 34

2.3 Golomb rulers 38
2.3.1 Perfect Golomb rulers ................................. 40

3 Wavelength Isolation Sequence Pairs .......................... 42
  3.1 Characterisation and examples of WISPs .................. 42
  3.2 Construction of WISPs from perfect Golomb rulers .......... 46
  3.3 Are there WISPs of length greater than 13? ............... 47

4 Costas Arrays ................................................. 49
  4.1 Toroidal distance vectors in Costas arrays ................. 49
  4.2 The difference parallelogram .............................. 59
  4.3 Mirror pairs in Costas arrays .............................. 65
    4.3.1 Mirror pairs in G-symmetric arrays .................. 78
    4.3.2 Mirror pairs in Welch Costas arrays ................. 83
    4.3.3 Mirror pairs in Golomb Costas arrays ............... 86

5 Questions and Conjectures ....................................... 91

Bibliography .................................................... 92
List of Tables

2.4 Nontrivial WISPs known to Golay, up to equivalence .......................... 13
2.10 The elements of $D_4$ ........................................................................ 24
2.11 The action of $D_4$ on $A$ .................................................................. 25
2.12 The effect on $T(\sigma)$ of four dihedral symmetries ......................... 27
2.16 Costas array enumeration data ......................................................... 36
2.17 Existence table for Costas arrays up to order 200 ......................... 39
3.1 All known nontrivial WISPs, up to equivalence ................................. 43
4.2 Toroidal distance vectors present in the $W_1(5,2,1)$ Costas array of Figure 4.1 ... 51
4.4 Toroidal distance vectors present in the array $W^*$ of Figure 4.3 .......... 52
4.6 Toroidal distance vectors in the augmented Golomb Costas array of Figure 4.5 ... 55
4.8 Toroidal distance vectors of width 1 in the Welch Costas arrays $W$ and $W'$ of Figure
4.7 ........................................................................................................... 57
4.17 The Costas arrays with no width 1 mirror pair ................................. 73
List of Figures

2.1 Schematic representation of Golay’s spectrometer design ...................... 10
2.2 Example of one stream of a multislit spectrometer ............................ 11
2.3 Example of a multislit spectrometer with entrance and exit slit patterns satisfying Conditions (a) and (b) ......................................................... 12
2.5 The vector between two dots in the array corresponding to a permutation \( \sigma \) ................................................................. 18
2.6 Examples of vectors in arrays .......................................................... 19
2.7 Three configurations of dots that violate the Costas condition ................. 20
2.8 Costas array corresponding to the permutation \([2, 4, 3, 1]\) ...................... 22
2.9 An equivalence class of Costas arrays ............................................... 23
2.13 The \( W_1(11, 2, 0) \) Costas array .................................................. 29
2.14 Examples of Golomb Costas arrays .................................................. 31
2.15 A Costas array with consecutive symmetry ........................................ 37
4.1 A Welch Costas array that does not contain every possible toroidal distance vector ................................................................. 50
4.3 The augmented array \( W^+ \) obtained from the Welch Costas array in Figure 4.1 ................................................................. 52
4.5 An augmented Golomb Costas array .................................................. 54
4.7 Two Welch Costas arrays inequivalent under the action of \( D_4 \) and under cyclic column permutation ................................. 57
4.9 Number of missing toroidal distance vectors in Welch and Golomb Costas arrays up to order 40 ......................................................... 60
4.10 Number of missing toroidal vectors in Costas arrays up to order 29 ........... 61
4.11 Difference parallelogram of \([1, 7, 4, 5, 3, 6, 8, 2]\) .............................. 63
4.12 Difference parallelogram of \( \sigma = [4, 3, 6, 1, 5, 2, 7] \), illustrating Proposition 53 for
\[ w = 1, \ j = 2, \ k = 2 \] and \( c = 2 \) .......................................................... 65

4.13 Examples of mirror pairs in Costas arrays .................................................. 66

4.14 Number of mirror pairs in Costas arrays up to order 29 ............................... 69

4.15 Mirror pairs under the action of \( D_4 \) ........................................................ 70

4.16 Illustration of Example 63 .......................................................... 72

4.18 Mirror pairs of width 1 in Costas arrays .................................................. 75

4.19 Costas array with no mirror pair of width 1 and no mirror pair of width 2 .... 77

4.20 Difference triangle of a graceful (non-Costas) permutation .......................... 77

4.21 Difference triangle of a permutation satisfying the conditions of Lemma 72 ... 81

4.22 Difference triangle \( T(\gamma) \) for the proof of Theorem 75 ........................... 83
Chapter 1

Overview

In 1951, M. J. E. Golay [28] considered the problem of measuring radiation of a particular wavelength in the presence of background radiation. He defined a type of binary sequence pair whose autocorrelation properties make it ideal for use in the design of multislit spectrometers, which isolate radiation of interest from background radiation. Golay presented examples of these pairs for lengths 3, 5 and 8, but, unable to find more, speculated that further examples do not exist. He then turned his attention to an alternative solution to the spectrometer design problem, for which he introduced the binary sequence pairs now known as Golay complementary sequence pairs. These pairs can be constructed for infinitely many lengths and have been widely studied in the sixty years since Golay first defined them (see [29], [44] and [25]). Meanwhile, the sequence pairs that Golay originally sought, which we call wavelength isolation sequence pairs (WISPs), have been largely neglected.

In this thesis, we show that, in fact, Golay’s speculation was incorrect: we present examples of WISPs for two new lengths (7 and 13). Further, we describe in Theorem 35 a procedure for constructing WISPs from perfect Golomb rulers. This construction explains all known examples of WISPs, but cannot be used to produce additional examples since there are only four inequivalent perfect Golomb rulers. This leads us to question whether our construction characterises WISPs, or whether there exist WISPs that do not arise from perfect Golomb rulers. We present partial results towards a classification of WISPs by proving a number of constraints on the sequences that form these pairs.
We then turn our attention to Costas arrays, a class of permutation matrices with ideal autocorrelation properties, which can be viewed as a two-dimensional generalisation of Golomb rulers. Costas arrays were introduced by J. P. Costas in 1965 as a means of improving the performance of radar and sonar systems [16]. They are typically represented as square grids of dots and blanks (instead of 1s and 0s). Early research led to two main constructions for Costas arrays, the Welch construction and the Golomb construction, which are both based on the theory of finite fields. These constructions, together with a number of secondary construction techniques that are derived from them, produce Costas arrays for infinitely many orders (sizes) but not for all orders [15]. After nearly fifty years of research, some of the most fundamental questions about Costas arrays remain unanswered, notably, Is there a Costas array of every order? The answer to this question for order 32 has been called the “Holy Grail” of Costas array research [36], as this order has been, since 1984, the smallest order for which no example is known [33].

Computer enumeration of Costas arrays (up to order 29 so far [20]) has provided some insight into the existence pattern for Costas arrays, while also prompting further questions. For example, exhaustive search has revealed that over 90% of Costas arrays up to order 29 are sporadic, meaning that they do not arise from any of the known constructions. However, the percentage of Costas arrays that are sporadic declines dramatically after a plateau between orders 13 and 20 [19]. It remains to be seen whether sporadic arrays die out at higher orders, a question that is closely linked to the fundamental existence question for Costas arrays. Meanwhile, attempts to understand the structure of sporadic Costas arrays, with the goal of finding new construction methods, have had very little success. The structure of these arrays remains poorly understood, and no reliable construction technique has resulted from the study of sporadic Costas arrays.

Much effort has been made to establish structural constraints on Costas arrays, both as a means of reducing the computational burden required in exhaustive searches for higher orders, and in the hope that such constraints might point the way towards new construction methods. Various symmetry constraints were considered in early studies of Costas arrays [7] and, in general, arrays with these symmetry properties are understood better than general Costas arrays. More recently, researchers have sought to constrain the distance vectors present in Costas arrays — that is, the relative positions of their dots. Progress has been made in establishing constraints on the number and type of common vectors between Costas arrays, both by considering the cross-correlations of two Costas arrays [26], [17] and by considering the vectors present in Costas arrays when they are
viewed as being written on the surface of a torus [18]. This periodic setting gives Costas arrays additional structure.

We consider the distance vectors present in Costas arrays, in both toroidal and non-toroidal contexts. We ask whether there exists a Costas array that contains every possible (neither horizontal nor vertical) toroidal distance vector; contrary to claims in [18], this is still an open question. This is shown in Proposition 43, where we prove that the answer to this question is no for G-symmetric Costas arrays of even order, and therefore for Welch Costas arrays (which were previously thought to contain all possible toroidal distance vectors). We define augmented Welch and Golomb Costas arrays and show that augmented Welch Costas arrays contain every possible toroidal distance vector (Theorem 38), while augmented Golomb Costas arrays contain every possible toroidal distance vector except for a small, predictable set of vectors (Theorem 40). For G-symmetric Costas arrays of even order $n$ we identify $\frac{n}{2} - 1$ specific missing toroidal distance vectors. Finally, we present data on the numbers of missing toroidal distance vectors in (non-augmented) Welch and Golomb Costas arrays up to order 40, and in general Costas arrays up to order 29, by analysing the database of Costas arrays up to order 29 [35]. These results lead us to conjecture that no Costas array contains all possible toroidal distance vectors.

We constrain the (non-toroidal) distance vectors in Costas arrays by introducing a structural feature involving pairs of vectors which we call mirror pairs. In our study of mirror pairs in Costas arrays, we draw upon previous research on the common distance vectors between Costas arrays. We consider these results and techniques from a new perspective to obtain constraints on the internal structure of individual Costas arrays. For example, in Corollary 59 we prove the existence of mirror pairs using a result from [26] on the existence of common vectors between Costas arrays (Theorem 58). We then turn to the database of Costas arrays up to order 29 for insight into the number and type of mirror pairs present in Costas arrays. Firstly, we observe that the number of mirror pairs in a Costas array of order $n$ appears to increase with $n$, suggesting that the study of mirror pairs might yield enough constraints on Costas arrays to improve search times. Secondly, we observe that every Costas array of order $n \geq 9$ in the database contains a mirror pair of width 1 and a mirror pair of height 1. In Question 64 we ask whether this holds in general for Costas arrays of order $n \geq 9$, and we present a partial answer in Theorem 66, which shows that every Costas array of order $n \geq 6$ has a mirror pair of width 1 or 2 and a mirror pair of height 1 or 2. We then focus attention on certain sub-classes of Costas arrays—G-symmetric Costas arrays of even order, which
include Welch Costas arrays, and Golomb Costas arrays—and show that the answer to Question 64 is yes for each of them.

For G-symmetric Costas arrays of even order, the answer to Question 64 follows from Proposition 68, which gives the exact number of width 1 mirror pairs in G-symmetric Costas arrays of a given order, and Theorem 75, which classifies the G-symmetric Costas arrays of even order containing no height 1 mirror pair. The answer for Welch Costas arrays then follows (Corollary 77). These results are obtained by combining symmetry constraints with arguments about the difference triangle associated with the Costas arrays. We use our results on the toroidal distance vectors present in augmented Welch and Golomb Costas arrays, combined with results from [18], to show that every sufficiently large Welch Costas array contains a mirror pair whose width and height sum to at most 3 (Theorem 81) and every sufficiently large Golomb Costas array contains both a mirror pair of width 1 and height 1, 2 or 3 and a mirror pair of height 1 and width 1, 2 or 3 (Theorem 83 and Corollary 84). The answer to Question 64 for Golomb Costas arrays then follows from this result.

The remainder of the thesis is organised as follows. In Chapter 2, we present background on wavelength isolation sequence pairs (Section 2.1), Costas arrays (Section 2.2) and Golomb rulers (Section 2.3), including history, motivation and relevant results from other researchers. In so doing, we describe the questions of interest relating to each of these objects, and identify key ideas that will be used in the thesis. The results in Chapters 3 and 4 are new unless explicitly attributed to another source. Our results on WISPs are given in Chapter 3. In Section 3.1 we provide a characterisation of WISPs, present all known examples and prove some structural constraints on the sequences that form these pairs. Our method for constructing WISPs from perfect Golomb rulers is described in Section 3.2 and partial results towards a classification of all WISPs are given in Section 3.3. Chapter 4 contains our results on Costas arrays. We begin in Section 4.1 with a discussion of whether there exists a Costas array that contains all possible toroidal distance vectors. Section 4.2 introduces a natural extension of the difference triangle to a difference parallelogram, which allows us to more easily recognise and describe relationships between entries. We then use this extension to establish some properties of the difference triangle of a (Costas) permutation. These results are used in Section 4.3, where we discuss mirror pairs in Costas arrays. We prove the existence of mirror pairs in general Costas arrays, including mirror pairs with constrained width and height, and present numerical data on the number and type of mirror pairs in Costas arrays up
to order 29. These data lead us to ask whether all Costas arrays of sufficient size contain a mirror pair of width 1 and a mirror pair of height 1. We then answer this question for G-symmetric Costas arrays of even order (Section 4.3.1), Welch Costas arrays (Section 4.3.2) and Golomb Costas arrays (Section 4.3.3), in turn. In Chapter 5, we summarise the open questions and conjectures arising from the thesis.
Chapter 2

Introduction

This chapter provides the context for our study of wavelength isolation sequence pairs, Costas arrays and Golomb rulers. We introduce each of these objects and motivate our study by describing their history and applications. We also identify the main questions of interest in these fields of research and assemble previously known results and key ideas that we will use in later chapters.

As we will see in Sections 2.1 and 2.2, both wavelength isolation sequence pairs and Costas arrays, our two main objects of interest, were initially defined as the solution to a specific engineering problem due to their favourable autocorrelation properties. Informally, the *autocorrelation function* of a sequence or array describes how closely it resembles a shifted copy of itself. Both periodic and aperiodic autocorrelation are meaningful concepts, dealing with cyclic and acyclic shifts, respectively. For the most part, however, we consider only aperiodic autocorrelation.

The definition of the aperiodic autocorrelation function for \( m \times n \) binary arrays is given below.

**Definition 1.** For an \( m \times n \) binary array \( A = [A_{i,j} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n] \), for \( i, j \in \mathbb{Z} \) let

\[
A'_{i,j} = \begin{cases} 
A_{i,j} & \text{if } 1 \leq i \leq m \text{ and } 1 \leq j \leq n \\
0 & \text{otherwise}.
\end{cases}
\]

The aperiodic autocorrelation function of \( A \) is given by

\[
C_A(u,v) = \sum_{i,j} A_{i,j} A'_{i+u,j+v} \quad \text{for } u, v \in \mathbb{Z}.
\]

6
We note that, in Definition 1, the positive vertical direction for the translation vector \((u, v)\) is downwards. Furthermore, the vertical component of the translation vector is its second component \(v\), while the vertical (row) position of an array entry \(A_{i,j}\) is given by its first index \(i\). This explains the sums \(i + v\) and \(j + u\) in the definition.

For \(\{0, 1\}\) binary arrays, the value of the autocorrelation function at shift \((u, v)\) is simply the number of 1s that coincide when two copies of the array are placed one on top of the other and one copy is translated to the right by \(u\) columns and down by \(v\) rows. Since no 1s coincide if \(|u| > n - 1\) or \(|v| > m - 1\), \(C_A(u, v) = 0\) for these values of \(u\) and \(v\). We therefore need to calculate the value of \(C_A(u, v)\) only for \(|u| \leq n - 1\) and \(|v| \leq m - 1\). It is convenient to record these values in a \((2m-1) \times (2n-1)\) array \(C_A\), called the autocorrelation array of \(A\). This is illustrated in Example 2.

**Example 2.** Consider the \(2 \times 4\) binary array

\[
A = \begin{bmatrix}
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 \\
\end{bmatrix}.
\]

For \(-3 \leq u \leq 3\) and \(-1 \leq v \leq 1\), we record the autocorrelation of \(A\) at shift \((u, v)\) in the \(3 \times 7\) array \(C_A\), with \(C_A(0, 0)\) in the central position and \(C_A(u, v)\) to the right of the central column when \(u > 0\) and below the central row when \(v > 0\), to obtain

\[
C_A = \begin{bmatrix}
1 & 0 & 1 & 2 & 0 & 1 & 1 \\
2 & 1 & 1 & 5 & 1 & 1 & 2 \\
1 & 1 & 0 & 2 & 1 & 0 & 1 \\
\end{bmatrix}.
\]

Clearly, for any \(\{0, 1\}\) binary array, the entry in the central position of the autocorrelation array \(C_A\) is the number of 1s in \(A\), since this entry corresponds to the \((0, 0)\)-shift. We also note that \(C_A\) has rotational symmetry of order 2, so it is completely determined by the set of entries corresponding to nonnegative horizontal translations (for example). This follows from Definition 1, since \(C_A(u, v) = C_A(-u, -v)\). (This can also be explained visually, by the observation that translating one copy of \(A\) by \(u\) positions to the right and \(v\) positions downward is the same as translating the other copy of \(A\) by \(u\) positions to the left and \(v\) positions upward).

As mentioned, the definition of the autocorrelation function for a binary sequence \(B\) of length \(n\) is implied by Definition 1, since \(B\) is equivalent to a \(1 \times n\) binary array. In this case, \(v = 0\), and we denote the value of the autocorrelation function of \(B\) at horizontal shift \(u\) by \(C_B(u)\). As with arrays, the autocorrelation function of \(B\) is completely determined by its value at shifts \(u\) in the range \(0 \leq u \leq n - 1\).
For example, the autocorrelation function of the sequence \( B = [1\ 0\ 1\ 1\ 0\ 0\ 1] \) is given by \( C_B = [4\ 1\ 1\ 2\ 1\ 0\ 1] \), whose entries correspond to the values of the autocorrelation function of \( B \) at nonnegative shifts. The first entry of \( C_B \) is the value of the autocorrelation function of \( B \) at shift \( u = 0 \) and therefore corresponds to the weight of \( B \), namely its number of 1s.

It is sometimes useful to consider how closely two different sequences or arrays, \( A \) and \( B \), resemble each other. This resemblance is measured by the cross-correlation function of \( A \) and \( B \), which we denote by \( C_{A,B} \).

**Definition 3.** For two \( m \times n \) binary arrays, \( A = [A_{i,j} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n] \) and \( B = [B_{i,j} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n] \), let

\[
A'_{i,j} = \begin{cases} 
A_{i,j} & \text{if } 1 \leq i \leq m \text{ and } 1 \leq j \leq n \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
B'_{i,j} = \begin{cases} 
B_{i,j} & \text{if } 1 \leq i \leq m \text{ and } 1 \leq j \leq n \\
0 & \text{otherwise}
\end{cases}
\]

The (aperiodic) cross-correlation function of \( A \) with \( B \) is given by

\[
C_{A,B}(u,v) = \sum_{i,j} A'_{i,j}B'_{i+v,j+u} \quad \text{for } u, v \in \mathbb{Z}.
\]

For \( \{0,1\} \) binary arrays \( A \) and \( B \), the value of \( C_{A,B}(u,v) \) is the number of 1s that coincide when \( A \) is placed on top of \( B \) and then translated by \( u \) positions to the right and \( v \) positions downwards. The autocorrelation function of an array \( A \) can then be viewed as the cross-correlation of \( A \) with itself. We observe the same conventions for cross-correlation as for autocorrelation. For example, the cross-correlation function of the 2 \( \times \) 3 binary arrays

\[
A = \begin{bmatrix} 0 & 1 & 0 \\
1 & 0 & 1 
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 1 
\end{bmatrix}
\]

is represented by the array

\[
C_{A,B} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\
0 & 2 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 
\end{bmatrix}.
\]

We note that the cross-correlation array \( C_{A,B} \) does not necessarily have any symmetry, and so we must calculate \( C_{A,B}(u,v) \) for every shift \((u,v)\) with \(|u| \leq m - 1 \text{ and } |v| \leq n - 1\).
We will use the cross-correlation function in Section 4.3. However, we are primarily interested in the autocorrelation function, which is central to the study of Costas arrays and closely linked to the definition of a wavelength isolation sequence pair. We are now ready to introduce these objects.

2.1 Wavelength isolation sequence pairs

In the early 1950s, M. J. E. Golay studied binary sequence pairs whose autocorrelation properties make them ideal for use in the design of multislit spectrometers. He found examples for some small lengths, but, unable to find more, he suggested that perhaps further examples do not exist, and turned his attention to an alternative solution to the design problem (involving what are now called Golay complementary sequence pairs). The original sequence pairs that Golay sought, which we call wavelength isolation sequence pairs (WISPs), appear to have been overlooked in the sixty years since. In this section we provide an introduction to WISPs, beginning with a discussion of their history and the application that motivated their discovery.

2.1.1 History and motivation

A spectrometer is a device that produces a spectrum from a source of electromagnetic radiation (see [8] for background on spectrometers). Such a device may be used in the analysis of light emitted from an unknown incandescent material in order to establish its chemical makeup, for example. When the incident radiation comprises more than one wavelength, it is often desirable to distinguish a particular wavelength of interest from background radiation.

In 1951, Golay [28] discussed a spectrometer design that isolates radiation of interest (desired radiation) from background radiation by processing incoming radiation in two “streams”, each consisting of an entrance mask, an exit mask and a detector. The entrance and exit masks are opaque surfaces with a pattern of narrow, equally spaced rectangular slits through which radiation passes on its way to identical detectors. The principle is that if radiation of a background wavelength is always passed through the two streams in equal quantities, while radiation of the desired wavelength is passed differentially by the two streams, then the difference in total energy as measured by the two detectors is wholly attributable to radiation of the desired wavelength. Figure 2.1 shows a schematic representation of such a spectrometer.

Golay’s multislit spectrometer design takes advantage of diffraction to regulate the passage of
radiation through the two streams. Diffraction causes radiation to bend as it passes through a narrow opening. A pattern of “open” and “closed” slits is inscribed in the (otherwise opaque) surface of each entrance mask; incident radiation is blocked by the closed slits but passes through the open slits and is diffracted. Since the angle of diffraction varies with wavelength, this separates the incoming radiation into a spectrum, so that each wavelength can be treated differently as it passes through the rest of the spectrometer. In particular, the exit masks are similarly inscribed with a pattern of open and closed slits, which block some radiation and pass the rest to the detectors. The amount of radiation of a given wavelength that is passed by each stream is determined by the entrance and exit slit patterns. The slit patterns must be chosen to isolate the desired wavelength reliably.

It is convenient to assume that the desired radiation does not undergo diffraction, and thus will reach the detector whenever there is an open slit in the exit pattern aligned with an open slit in the entrance pattern. Then, if background radiation of wavelength $\lambda_u$ is diffracted such that it arrives
at the exit mask \( \mu \) positions (slits) to the right or left, then radiation of wavelength \( \lambda_\mu \) will reach the detector whenever there is an open slit in the exit mask \( \mu \) positions to the right or left, respectively, of an open slit in the entrance mask. The (more realistic) case where the desired radiation does undergo diffraction can be treated by simply translating both exit masks by an appropriate amount relative to the entrance masks.

Golay represented the entrance and exit slit patterns as binary \( \{0, 1\} \) sequences, in which 0s represent closed slits and 1s represent open slits. Figure 2.2 shows radiation of background wavelength \( \lambda_1 \), which is diffracted by one position to the right, passing through the entrance and exit masks of one stream of a spectrometer, along with the binary sequences associated with the entrance and exit slit patterns. The stream pictured allows one passage of radiation of wavelength \( \lambda_1 \) to the detector.

\[
\begin{array}{c|cc|cc|cc|cc}
\text{Entrance slit pattern} & 1 & 1 & 0 & 1 & 0 \\
\text{Exit slit pattern} & 1 & 0 & 1 & 0 & 0 \\
\end{array}
\]

Figure 2.2: Example of one stream of a multislit spectrometer

Golay [28] proposed that effective isolation of the desired wavelength could be achieved by entrance slit patterns \( A \) and \( B \) and exit slit patterns \( A' \) and \( B' \) with the following properties.

(a) \( A' \) is an exact copy of \( A \), and \( B' \) is the complement of \( B \).

(b) The number of open slits in \( A \) that are followed at distance \( u > 0 \) (reading from left to right) by an open slit is equal to the number of open slits in \( B \) that are followed at distance \( u \) by a closed slit, and also equal to the number of closed slits in \( B \) followed at distance \( u \) by an open slit.

Condition (a) guarantees that all of the desired radiation passed by entrance slit pattern \( A \) reaches the detector whereas none of the desired radiation passed by entrance slit pattern \( B \) does so. Condition (b) guarantees that radiation of a background wavelength is always passed identically by the two streams, whether it is diffracted to the right (hence the open-closed condition) or the left (hence the closed-open condition).
Since the two exit slit patterns are determined by the two entrance slit patterns, the optical system described above is modelled by an ordered pair of binary \( \{0, 1\} \) sequences \( A \) and \( B \), which represent the entrance slit patterns \( A \) and \( B \), respectively. The system illustrated in Figure 2.3 corresponds to the sequence pair \( A = (11010) \), \( B = (10001) \). Figure 2.3(a) shows the differential passage of the desired wavelength through both streams, while Figure 2.3(b) shows the identical passage of background wavelength \( \lambda_1 \) through both streams.

![Passage of desired radiation through both streams of a multislit spectrometer](a)

![Passage of one wavelength of background radiation through both streams of a multislit spectrometer](b)

Figure 2.3: Example of a multislit spectrometer with entrance and exit slit patterns satisfying Conditions (a) and (b)

In 1951, Golay found examples of sequences satisfying Conditions (a) and (b) by hand for lengths 3, 5 and 8 [28]. These examples are listed in Table 2.4. Unable to find further (nontrivial) examples, he stated that “the possibility must be reckoned with, that solutions for such patterns with more than 8 slits do not exist.” He diverted his attention to an alternative solution to the problem — one that uses a two-row array of slits rather than a single row, the patterns for which
can be constructed for infinitely many lengths using what are now known as Golay complementary
sequence pairs (see, for example, [29], [44], [25] for background on these complementary pairs).
The search for sequences suitable for single row entrance slit patterns was apparently forgotten for
the next sixty years.

Table 2.4: Nontrivial WISPs known to Golay, up to equivalence

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(110)</td>
<td>(010)</td>
</tr>
<tr>
<td>2</td>
<td>(11010)</td>
<td>(10001)</td>
</tr>
<tr>
<td>3</td>
<td>(11001010)</td>
<td>(10000001)</td>
</tr>
</tbody>
</table>

In Chapter 3, we show that in fact there is a WISP of length 13 as well as a WISP of length 7
that Golay overlooked. We present a mathematical characterisation of WISPs and some structural
constraints on the sequences $A$ and $B$. Finally, we describe a construction method that produces all
of the known examples of WISPs, and prove some partial results on the classification of all WISPs.

2.2 Costas arrays

2.2.1 History and definitions

In the 1960s, while working on a project for the US Navy, J. P. Costas studied permutation ma-
trices with ideal autocorrelation properties in order to overcome the poor performance of sonar
systems [16]. In a typical radar or sonar application, it is useful to produce a sequence of distinct
frequencies in consecutive time slots. This sequence, which is known as a ping, can be repre-
sented by an $m \times n$ array $[A_{i,j}]$ of ones and zeros, with rows indexed by the frequencies $f_1, \ldots, f_m$
and columns indexed by the time intervals $t_1, \ldots, t_n$, where $A_{i,j} = 1$ if and only if frequency $f_i$
is transmitted in time interval $t_j$ [32]. When this signal is reflected off a target, the echo returns to
the source, where it is detected by a receiver. The echo is shifted in frequency (compared to the
transmitted signal) by an amount corresponding to the target’s velocity (toward or away from the
source), and the length of time between signal transmission and echo detection corresponds to the
target’s distance.

In practical applications, the signal detected by the receiver is always noisy, and it is neces-
sary to distinguish the returning echo from background noise. To this end, the detected signal is
compared with each of the \((2m - 1)(2n - 1)\) possible time-frequency shift combinations of the
transmitted signal; it is desired that the only translate of the original signal having high correlation
with the received signal be the one whose time shift corresponds to the target’s position and whose
frequency shift corresponds to the target’s velocity. It is therefore necessary that the transmission
pattern be chosen to have low correlation with itself at all nonzero time-frequency shifts [33].

Costas [12] argued that, due to physical constraints, the most suitable signal for the application
is one in which the number of frequencies equals the number of time intervals, each frequency is
transmitted exactly once, and exactly one frequency is transmitted in each time interval. He was
interested in such patterns whose autocorrelation is at most one at all nonzero shifts, and phrased
the problem in terms of permutation matrices as follows:

“Place \(n\) ones in an otherwise null \(n \times n\) matrix such that each row contains a single
one as does each column. Make the placement such that for all possible \(x-y\) shift
combinations of the resulting (permutation) matrix relative to itself, at most one pair
of ones will coincide.” [12]

These permutation matrices are now known as Costas arrays. Costas’s formulation of the problem
gives us the first of three equivalent definitions of a Costas array. We will present each of the three
definitions, followed by a proof of their equivalence.

**Definition 4** (First Costas array definition). An \(n \times n\) permutation matrix is a Costas array of order \(n\)
(has the Costas property) if its aperiodic autocorrelation function takes only values 0, 1 and \(n\).

We consider a Costas array of order \(n\) to be nontrivial if \(n \geq 3\).
**Example 5.** Consider the $6 \times 6$ matrix

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

Using Definition 1, for $-5 \leq u, v \leq 5$, we record the autocorrelation of $A$ at the shift $(u, v)$ relative to itself in the matrix $C_A$, with the $(0,0)$-shift in the central position, to obtain

\[
C_A = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

As in Example 2, $C_A$ has rotational symmetry of order 2, and so is completely determined by the values in two adjacent quadrants. Further, because $A$ is a permutation matrix, the values in the central row and central column of $C_A$ are all zero except for the entry at their intersection; this entry corresponds to the $(0,0)$-shift and thus its value is equal to the order of the permutation matrix. Since all entries in $C_A$ are 0, 1 or 6, the matrix $A$ is a Costas array.

In the literature on Costas arrays, it is customary to represent an $n \times n$ permutation matrix visually as an $n \times n$ grid with dots in place of the 1s and blanks in place of the 0s. The correspondence between permutation matrices of size $n \times n$ and permutations of the set $\{1, 2, \ldots, n\}$ allows us to regard Costas arrays as permutations in $S_n$ where convenient. In such cases, we represent
the permutation \( \alpha \in S_n \) in single row notation. The Costas array in Example 5 corresponds to the permutation \( \alpha = [3, 1, 6, 2, 5, 4] \), following the convention that \( A_{i,j} = 1 \) if and only if \( \alpha(j) = i \).

Associated with a permutation \( \sigma \) is its difference triangle, which records the differences between pairs of entries in \( \sigma \).

**Definition 6.** Let \( \sigma \) be a permutation of \( \{1, 2, \ldots, n\} \), for \( n \in \mathbb{N} \). The difference triangle \( T(\sigma) \) of \( \sigma \) is the set \( \{t_w(\sigma) : w = 1, 2, \ldots, n-1\} \), where \( t_w(\sigma) \) is the sequence \( (\sigma(w + j) - \sigma(j) : j = 1, 2, \ldots, n-w) \). We call \( t_w(\sigma) \) the \( w \)th row of the difference triangle and we denote the \( j \)th element of \( t_w(\sigma) \) by \( t_w,j(\sigma) \).

From the definition, the \( j \)th entry of row \( w \) is given by

\[
t_w,j(\sigma) = \sigma(w + j) - \sigma(j),
\]

for \( 1 \leq w \leq n-1 \) and \( 1 \leq j \leq n-w \). Letting the rows \( t_w(\sigma) \) of the difference triangle \( T(\sigma) \) define the rows of a Young tableau gives a visual representation of \( T(\sigma) \). (Since each row of a Young tableau must be at least as long as the row below it, the order of the rows is determined.) For \( 1 \leq j \leq n-1 \) we define the \( j \)th column of \( T(\sigma) \) to be the sequence \( (t_{k,j}(\sigma) : k = 1, \ldots, n-j) \), so that \( t_{w,j}(\sigma) \) is the entry in the \( w \)th row and the \( j \)th column of \( T(\sigma) \), and for \( 2 \leq j \leq n \) we define the \( j \)th antidiagonal of \( T(\sigma) \) to be the sequence \( (t_{j-k,k}(\sigma) : k = 1, \ldots, j-1) \). Column \( j \) then contains the entries \( \sigma(k) - \sigma(j) \) for \( j < k \leq n \) and antidiagonal \( j \) contains the entries \( \sigma(j) - \sigma(k) \) for \( 1 \leq k < j \) (so the first entry of antidiagonal \( j \) is in row \( j-1 \) of \( T(\sigma) \)).

**Example 7** (Continuation of Example 5). The permutation \( \alpha = [3, 1, 6, 2, 5, 4] \) corresponds to the permutation array

```
  ●   ●
●   ●   ●
●   ●   ●
●

and has difference triangle \( T(\alpha) = \{(-2, 5, -4, 3, -1), (3, 1, -1, 2), (-1, 4, -2), (2, 3), (1)\} \).```
The visual representation of $T(\alpha)$ is the Young tableau

\[
\begin{array}{cccc}
-2 & 5 & -4 & 3 \\
3 & 1 & -1 & 2 \\
-1 & 4 & -2 \\
2 & 3 \\
1
\end{array}
\]

Although our definition of the difference triangle could be applied to general sequences without modification, we will focus on difference triangles of permutations, because our goal is to study Costas arrays (or permutations). As we will see in Section 4.2, where we examine the properties of the difference triangle in detail, difference triangles of permutations have a great deal of structure which we will exploit in our study of Costas arrays. Proposition 8 describes the first of these constraints.

**Proposition 8.** [15] Let $\sigma$ be a permutation of \{1, 2, \ldots , n\}. Then for $1 \leq k \leq n - 1$, the difference triangle $T(\sigma)$ contains exactly $n - k$ entries from \{-$k$, $k$\}.

**Proof.** Among the $n$ entries of $\sigma$ there are $n - k$ pairs whose values differ by $k$, specifically the pairs \{i, $k + i$\} for $1 \leq i \leq n - k$. □

Consider again the Costas permutation $\alpha = [3, 1, 6, 2, 5, 4]$ from Example 7, with the given difference triangle. Notice that the rows of $T(\alpha)$, the sequences $(-2, 5, -4, 3, -1), (3, 1, -1, 2), (-1, 4, -2), (2, 3)$ and $(1)$, are each free of repeated entries. This property characterises Costas arrays, and so provides us with the second of the three equivalent definitions of Costas arrays; this characterisation was first noted by Costas [12].

**Definition 9** (Second Costas array definition). A permutation $\alpha$ of \{1, 2, \ldots , n\} is a Costas permutation (has the Costas property) if for each $w = 1, 2, \ldots , n - 1$, the $n - w$ entries of $t_w(\alpha)$ are all distinct. In this case we call the corresponding array a Costas array.

Costas used this definition to find Costas arrays of order $n$ (that is, Costas permutations in $S_n$) for $n \leq 12$ by hand. The difference triangle characterisation of the Costas property provides a method to check whether the property is satisfied by a given permutation that, compared with other methods, is computationally less demanding. Rather than computing $2(n - 1)^2$ autocorrelations (that is, two quadrants of the autocorrelation array), we can verify the Costas property by checking
only \( \binom{n}{2} \) differences. In fact, we will see later in this section that the Costas property can be verified with even fewer calculations.

Of course, it often requires fewer calculations to show that a given array \textit{does not} satisfy the Costas property than to verify that a Costas array is indeed Costas: the first autocorrelation value we compute might be greater than 1 or, using the difference triangle characterisation, we might find a repeated value in the first row of differences that we compute. However, neither of the characterisations presented so far allows us to verify or rule out the Costas property simply by looking at the given array. The third definition describes the Costas property in terms of the relative positions of the dots in the array. This often allows us to rule out the Costas property at a glance, and for small \( n \) allows us to verify the Costas property with a careful visual examination.

Given a permutation array of dots and blanks, every pair of dots determines a line segment with a particular length and slope, and this line segment describes the separation between the two dots. However, in keeping with the convention established by other authors, we refer to the separation between dots as a vector. We draw it as a line segment, without arrowheads, and we take its horizontal component to be positive. A Costas array of order \( n \) then has \( \binom{n}{2} \) distance vectors from a set of \( 2(n - 1)^2 \) possible vectors (since both components are nonzero). Figure 2.5 shows the vector between two dots in the array corresponding to a permutation \( \sigma \), and indicates how this vector relates to entries in the difference triangle \( T(\sigma) \).

![Figure 2.5: The vector between two dots in the array corresponding to a permutation \( \sigma \)](image)

We also follow the convention for describing vectors in the plane, with horizontal component in the first coordinate and vertical component in the second, while still following the convention for denoting matrix elements with row index in the first coordinate and column index in the second.
Consequently, the first coordinate in the distance vector between two dots is given by the difference between their second coordinates, and vice versa. These remarks are formalized in Definition 10; notice that the positive vertical direction for the vector is defined to be downwards.

**Definition 10.** Given a permutation array \( A = [A_{i,j}] \), the vector between \( A_{i,j} \) and \( A_{k,\ell} \), for \( j < \ell \), is \((\ell - j, k - i)\).

**Example 11.** The arrays in Figure 2.6 illustrate the vectors \((3,-2)\) and \((1,2)\), respectively.

![Figure 2.6: Examples of vectors in arrays](image)

Remark 12, which relates vectors in a permutation array to entries in the difference triangle of its associated permutation, follows from Figure 2.5.

**Remark 12.** Let \( \sigma \) be a permutation corresponding to the array \( A \) and let \( T(\sigma) \) be its difference triangle. Then there exists a pair of dots in \( A \) separated by the vector \((w,h)\) if and only if \( h \in t_w(\sigma) \).

**Definition 13** (Third Costas array definition). An \( n \times n \) array of dots and blanks with \( n \) dots, one in each row and one in each column, is a Costas array of order \( n \) (has the Costas property) if the \( \binom{n}{2} \) vectors formed by joining pairs of dots are all distinct.

Looking again at the Costas array in Example 7, we can verify that the array contains \( 15 = \binom{6}{2} \) distinct vectors.

The condition described in Definition 13 is equivalent to the condition that a Costas array \( A \) does not contain a (possibly degenerate) parallelogram. The permutation arrays in Figure 2.7 illustrate three such configurations of dots.

We now show that Definitions 4, 9 and 13 are equivalent.

**Proposition 14.** Let \( A = [A_{i,j}] \) be a permutation matrix of order \( n \) corresponding to the permutation \( \alpha \). The following are equivalent.
(i) $C_A(u,v) \leq 1$ for all shifts $(u,v) \in \{-n+1, \ldots, n-1\}^2 \backslash \{(0,0)\}$,

(ii) For each $w \in \{1, \ldots, n-1\}$, the entries of $t_w(\sigma)$ are all distinct,

(iii) The $\binom{n}{2}$ vectors between distinct pairs of dots in $A$ are all distinct.

Proof. By symmetry of $C_A$, we may assume without loss of generality that $u \geq 0$. Now, to see that (i) and (iii) are equivalent, notice that the autocorrelation $C_A(u,v)$ is equal to the number of pairs of dots in $A$ that are separated by the vector $(u,v)$. The equivalence of (ii) and (iii) follows from Remark 12. □

As mentioned in our earlier discussion of the three equivalent definitions of the Costas property, the difference triangle characterisation given in Definition 9 seems to provide an advantage in the number of calculations required to check whether a given permutation (or array) has the Costas property compared to the autocorrelation characterisation given in Definition 4. Corollary 16 further reduces the number of calculations required. The result was first given in terms of the autocorrelation definition of Costas arrays [9]. We follow [5] in proving Corollary 16 from Proposition 15.

**Proposition 15.** Let $\sigma$ be a permutation of $\{1, \ldots, n\}$ with difference triangle $T(\sigma)$ and let $w \in \{1, \ldots, n-2\}$. Then for $1 \leq j \leq n-w$ and $1 \leq c \leq n-w-j$,

$$t_{w,j}(\sigma) - t_{w,j+c}(\sigma) = t_{c,j}(\sigma) - t_{c,j+w}(\sigma).$$
CHAPTER 2. INTRODUCTION

Proof. By (2.1),

\[ t_{w,j}(\sigma) - t_{w,j+c}(\sigma) = \sigma(w + j) - \sigma(j) - (\sigma(w + j + c) - \sigma(j + c)) \]
\[ = \sigma(c + j) - \sigma(j) - (\sigma(c + j + w) - \sigma(j + w)) \]
\[ = t_{c,j}(\sigma) - t_{c,j+w}(\sigma). \]

Visually, Proposition 15 can be explained by the fact that a violation of the Costas property leads to a (possibly degenerate) parallelogram in the array, as shown in Figure 2.7. The difference triangle row indices \( w \) and \( c \) correspond to the widths of the vectors that form the sides of the parallelogram in the array. Figures 2.7(a) and 2.7(b) give examples with \( \{w, c\} = \{1, 2\} \) and \( \{w, c\} = \{1, 3\} \), respectively. The case \( w = c \) of Proposition 15 corresponds to the case where the array contains a degenerate parallelogram formed by only three dots, which is illustrated in Figure 2.7(c) with \( w = c = 2 \). Corollary 16 is equivalent to the statement that at least one of the vectors forming the sides of a parallelogram in a permutation array of order \( n \) must have width at most \( \frac{n-1}{2} \).

Corollary 16. Let \( \sigma \) be a non-Costas permutation of \( \{1, \ldots, n\} \) with difference triangle \( T(\sigma) \). Then \( T(\sigma) \) contains a repeated value in one of the first \( \lfloor \frac{n-1}{2} \rfloor \) rows.

Proof. Since \( \sigma \) is not a Costas permutation, \( n > 2 \) and there exists a row \( w \) of \( T(\sigma) \), with \( 1 \leq w \leq n - 2 \), such that \( t_{w,j}(\sigma) = t_{w,j+c}(\sigma) \), for some \( j \) and \( c \) such that \( j, c \geq 1 \) and

\[ 1 \leq j + c \leq n - w. \] (2.2)

Then by Proposition 15, there is a repeated entry in row \( c \) of \( T(\sigma) \). Since \( j \geq 1 \), (2.2) gives \( w + c \leq n - 1 \), so at least one of \( w \) and \( c \) is at most \( \lfloor \frac{n-1}{2} \rfloor \). \( \square \)

Corollary 16 allows us to verify the Costas property for a particular permutation of \( \{1, \ldots, n\} \) by calculating at most \( \frac{1}{8} 3n(n - 2) \) entries of the difference triangle. (This is the total number of entries in the first \( \lfloor \frac{n-1}{2} \rfloor \) rows of the difference triangle when \( n \) is even; the number is less for \( n \) odd.) In fact, the entries of the difference triangle that must be checked are further restricted in [5] to a subset of the entries in the first \( \lfloor \frac{n-1}{2} \rfloor \) rows. In Section 4.2 we will see additional internal structure of the difference triangle.
Although aperiodicity is central to the definition of Costas arrays, it is sometimes informative to consider their properties in a periodic setting. For example, in Section 4.1, we analyse the vectors present in Costas arrays when the vectors are allowed to “wrap around” in both directions (or, equivalently, when the arrays are viewed as being written on the surface of a torus). To that end, we introduce the concept of a *toroidal distance vector*.

Consider the Costas array $A$ corresponding to the permutation $[2, 4, 3, 1]$, displayed in Figure 2.8, with two highlighted dots. By our convention for labelling distance vectors in Costas arrays, the vector joining the two highlighted dots in $A$ is $(3, -1)$, where the first component is taken to be positive to account for the fact that the vector from $A_{2,1}$ to $A_{1,4}$ is the same as the vector from $A_{1,4}$ to $A_{2,1}$. When $A$ is viewed as being written on the surface of a torus, we may always write the vector from $A_{i,j}$ to $A_{k,\ell}$ with both components positive (wrapping around at the boundaries as necessary). Moreover, following this convention, there are two vectors joining $A_{i,j}$ and $A_{k,\ell}$ on the torus, namely the vector from $A_{i,j}$ to $A_{k,\ell}$ and the vector from $A_{k,\ell}$ to $A_{i,j}$. For example, the vector from $A_{2,1}$ to $A_{1,4}$ is $(3, 3)$ (wrapping around in the vertical direction) and the vector from $A_{1,4}$ to $A_{2,1}$ is $(1, 1)$ (wrapping around in the horizontal direction).

**Definition 17.** Let $A$ be an $m \times n$ array. The *toroidal distance vector* from $A_{i,j}$ to $A_{k,\ell}$ is $((\ell - j) \mod n, (k - i) \mod m)$.

Note that in the above definition (and throughout the thesis), the notation $k \mod n$, for $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, refers to the unique integer $d \in \{0, \ldots, n - 1\}$ such that $d \equiv k \mod n$. A Costas array of order $n$ has $2\binom{n}{2} = n(n - 1)$ toroidal distance vectors (two for each pair of dots) from a set of $(n - 1)^2$ possible vectors (since both components are nonzero).
2.2.2 Equivalence under the action of $D_4$

Consider a Costas array $A$, and suppose that we obtain another array by rotating or reflecting $A$. From Definition 13, it is clear that this second array will also have the Costas property, as each of these transformations preserves the relative positions of the dots in $A$. When we consider all eight symmetries of the square — that is, all eight elements of the dihedral group $D_4$ — we see that $A$ may be transformed (by the action of $D_4$) to yield a family of Costas arrays, namely the orbit of $A$ under the action of $D_4$. Such a family is illustrated in Figure 2.9. The arrays obtained from the identity, 90°, 180° and 270° counterclockwise rotations, respectively, are given in the first row, while the arrays obtained from diagonal, horizontal, antidiagonal and vertical reflections, respectively, are given in the second row.

![Figure 2.9: An equivalence class of Costas arrays](image)

Each orbit forms an equivalence class, and we say that the Costas arrays in a given equivalence class are equivalent (under the action of $D_4$). Proposition 18 shows that the equivalence class of a Costas array $A$ has four or eight members, depending on whether $A$ has diagonal or antidiagonal symmetry. In many situations, for example in conducting an exhaustive search or reporting the
results of an enumeration of Costas arrays [19], it is convenient to consider only one representative from each equivalence class; usually the representative is chosen to be the array corresponding to the lexicographically first permutation.

Since our study of Costas arrays will involve vectors, permutations and difference triangles, we will present in this section the effect of the various dihedral symmetries on each of these objects. Let $A$ be a permutation matrix corresponding to the permutation $\sigma = [\sigma(1), \ldots, \sigma(n)]$. Table 2.11, below, describes how each of the dihedral symmetries (reflections about the horizontal axis, vertical axis and two diagonal axes, as well as $0^\circ$, $90^\circ$, $180^\circ$ and $270^\circ$ rotation) transforms $A$, its entries, the vectors it contains, and the permutation $\sigma$. To this end, we note that the transpose of $A$ corresponds to the permutation $\sigma^{-1}$ and the reflection of $A$ about a vertical axis (which we will henceforth call a *vertical reflection* of $A$) corresponds to the permutation $[\sigma(n), \ldots, \sigma(1)]$, which we will denote by $\sigma_v$. We denote the transpose operation by $T$ and the vertical reflection by $v$. Because the actions of $v$ and $T$ on the permutation $\sigma$ are easily described (as above), it is convenient to represent the elements of $D_4$ using $D_4 = \langle v, T \rangle$. We compose functions from right to left.

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>as word in $v &amp; T$</th>
<th>Symmetry</th>
<th>as word in $v &amp; T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identity</td>
<td>—</td>
<td>Diagonal reflection</td>
<td>$T$</td>
</tr>
<tr>
<td>$90^\circ$ CCW rotation</td>
<td>$Tv$</td>
<td>Horizontal reflection</td>
<td>$TvT$</td>
</tr>
<tr>
<td>$180^\circ$ rotation</td>
<td>$vTvT$</td>
<td>Antidiagonal reflection</td>
<td>$vTv$</td>
</tr>
<tr>
<td>$270^\circ$ CCW rotation</td>
<td>$vT$</td>
<td>Vertical reflection</td>
<td>$v$</td>
</tr>
</tbody>
</table>

Table 2.10: The elements of $D_4$

Table 2.10 helps us to determine the effect of $D_4$ on the entries and vectors of $A$ and on the associated permutation $\sigma$. We use this to determine the entries of Table 2.11. For example, consider the effect of $90^\circ$ counterclockwise rotation on the permutation $\sigma$. Since this is the element $Tv$ of $D_4$, and since the operations $v$ and $T$ reverse and invert the permutation $\sigma$, respectively, rotation by $90^\circ$ counterclockwise produces the permutation $(\sigma_v)^{-1}$. Now, letting the $j^{th}$ entry of the
permutation \((\sigma_v)^{-1}\) be \(k\), we have

\[
(\sigma_v)^{-1}(j) = k \iff j = \sigma_v(k) \\
\iff j = \sigma(n + 1 - k) \\
\iff \sigma^{-1}(j) = n + 1 - k \\
\iff k = n + 1 - \sigma^{-1}(j).
\]

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>Entry ((i, j))</th>
<th>Vector ((w, h))</th>
<th>Permutation (\sigma = [\sigma(j)])</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identity</td>
<td>(1) (2) (3)</td>
<td>(1) (2) (3)</td>
<td></td>
</tr>
<tr>
<td>(90^\circ) CCW rotation</td>
<td>(2) (3) (1) (4)</td>
<td>(\frac{h}{</td>
<td>h</td>
</tr>
<tr>
<td>(180^\circ) rotation</td>
<td>(3) (4) (2) (1)</td>
<td>(\frac{h}{</td>
<td>h</td>
</tr>
<tr>
<td>(270^\circ) CCW rotation</td>
<td>(4) (1) (3) (2)</td>
<td>(\frac{h}{</td>
<td>h</td>
</tr>
<tr>
<td>Diagonal reflection</td>
<td>(1) (4) (2) (3)</td>
<td>(\frac{h}{</td>
<td>h</td>
</tr>
<tr>
<td>Horizontal reflection</td>
<td>(4) (3) (1) (2)</td>
<td>(\frac{h}{</td>
<td>h</td>
</tr>
<tr>
<td>Antidiagonal reflection</td>
<td>(3) (2) (4) (1)</td>
<td>(\frac{h}{</td>
<td>h</td>
</tr>
<tr>
<td>Vertical reflection</td>
<td>(2) (1) (3) (4)</td>
<td>(\frac{h}{</td>
<td>h</td>
</tr>
</tbody>
</table>

Table 2.11: The action of \(D_4\) on \(A\)

We note that \(\frac{h}{|h|}\) is simply the sign of \(h\).

**Proposition 18.** Let \(A\) be a Costas array of order \(n > 2\). The size of the equivalence class of \(A\) under the action of \(D_4\) is four if \(A\) has diagonal or antidiagonal symmetry, and eight otherwise.
Proof. The equivalence class of $A$ is the orbit $\text{orb}_{D_4}(A)$ of $A$ under the action of $D_4$. Letting $\text{stab}_{D_4}(A) = \{ \phi \in D_4 : \phi(A) = A \}$ denote the stabiliser of $A$ in $D_4$, the Orbit-Stabiliser Theorem gives $|\text{orb}_{D_4}(A)||\text{stab}_{D_4}(A)| = 8$, so $(|\text{orb}_{D_4}(A)|,|\text{stab}_{D_4}(A)|) = (8, 1), (4, 2), (2, 4)$ or $(1, 8)$. Using the representations of the elements of $D_4$ given in Table 2.10, we will show that

$$|\text{stab}_{D_4}(A)| = \begin{cases} 
1 & \text{if } A \neq T(A) \text{ and } A \neq vTv(A) \\
2 & \text{otherwise},
\end{cases}$$

from which the desired result follows.

Since $A$ is a permutation matrix, $A \neq v(A)$ and $A \neq vTv(T(A))$. We will show that $vTvT$, $vT$ and $Tv$ are not in $\text{stab}_{D_4}(A)$.

Suppose for a contradiction that $vTvT \in \text{stab}_{D_4}(A)$. By Table 2.11, the vector joining the dot in the first column to the dot in the second column of $A$ maps to itself under 180° rotation, which is a contradiction since $n > 2$. It follows that $vT \notin \text{stab}_{D_4}(A)$, since $\text{stab}_{D_4}(A)$ is a subgroup. Subsequently, $Tv \notin \text{stab}_{D_4}(A)$, since $Tv = (vT)^{-1}$.

Therefore, $|\text{stab}_{D_4}(A)| \leq 3$. So if $T \in \text{stab}_{D_4}(A)$ or $vTv \in \text{stab}_{D_4}(A)$ then $|\text{stab}_{D_4}(A)| = 2$, and otherwise, $|\text{stab}_{D_4}(A)| = 1$. □

Given an array $A$ corresponding to the permutation $\sigma$, and its image $A'$ under a dihedral symmetry, Table 2.11 characterises $\sigma'$, the permutation corresponding to $A'$, in terms of $\sigma$. We now wish to characterise the difference triangle $T(\sigma')$ of $\sigma'$ in terms of $T(\sigma)$. Since the permutation $\sigma$ can be recovered from its difference triangle, it is possible to completely determine $T(\sigma')$ from $T(\sigma)$, regardless of which dihedral symmetry was used to obtain $\sigma'$ from $\sigma$. However, for our purposes (in Section 4.3), it will suffice to have a complete characterisation of $T(\sigma')$ for four dihedral symmetries, namely the identity, horizontal and vertical reflection, and rotation through 180°. As shown in the fourth column of Table 2.11, these elements of $D_4$ preserve the magnitude of both components of vectors in $A$, while the other elements of $D_4$ swap the magnitudes of the vector components. This distinction will be used in Chapter 4. Table 2.12 characterises $t_w(\sigma')$ in terms of $t_w(\sigma)$, for $1 \leq w \leq n - 1$, when $A'$ is obtained from $A$ by one of these four symmetries. These characterisations are determined using Definition 6 and the fifth column of Table 2.11. The sequence $t_w(\sigma)_v$ is the sequence $t_w(\sigma)$ with the order of its elements reversed, so element $j$ of $t_w(\sigma)_v$ is element $n + 1 - w - j$ of $t_w(\sigma)$, since $t_w(\sigma)$ has length $n - w$.
Symmetry       \( t_w(\sigma') \)  
Identity       \( t_w(\sigma) \)  
180° rotation  \( t_w(\sigma)_v \)  
Horizontal reflection  \( -t_w(\sigma) \)  
Vertical reflection  \( -t_w(\sigma)_v \)  

Table 2.12: The effect on \( T(\sigma) \) of four dihedral symmetries

**Example 19.** Consider the Costas array \( A \) corresponding to the permutation \( \alpha = [1, 4, 5, 3, 2] \), which is illustrated in Figure 2.9 along with its images under the action of \( D_4 \). The difference triangle of \( \alpha \) is

\[
T(\alpha) = \begin{bmatrix}
3 & 1 & -2 & -1 \\
4 & -1 & -3 \\
2 & -2 \\
1
\end{bmatrix}
\]

Rotating \( A \) through 180° produces the array \( A' \) corresponding to the permutation \( \alpha' = [4, 3, 1, 2, 5] \), which has difference triangle

\[
T(\alpha') = \begin{bmatrix}
-1 & -2 & 1 & 3 \\
-3 & -1 & 4 \\
-2 & 2 \\
1
\end{bmatrix}
\]

### 2.2.3 Construction techniques

Early research on Costas arrays led to two main algebraic construction techniques, known as the Welch construction, and the Golomb construction (which generalises the earlier Lempel construction). These generate Costas arrays for infinitely many (but not all) orders, and are based on the theory of finite fields, using primitive elements of \( \mathbb{F}_q \) to generate Costas permutations. Both the Welch and Lempel constructions were initially presented without proof (by L. R. Welch and A. Lempel, respectively) and later proved by S. Golomb in [30]. In the same paper, Golomb presented his generalisation of the Lempel construction. In addition to the algebraic constructions, there are a number of secondary construction procedures which involve modifying a known Costas array in a way that preserves the Costas property, where possible, to produce an inequivalent Costas array. Many of these can be systematically applied to certain Welch or Golomb Costas arrays and
are guaranteed to produce a Costas array. In other cases there is no guarantee, and one must test whether the resulting array has the Costas property. We begin with a detailed discussion of the algebraic constructions and the arrays that they produce.

**Theorem 20** (Welch Construction $W_1(p, \phi, c)$). Let $\phi$ be a primitive element of $\mathbb{F}_p$, where $p$ is a prime, and let $c$ be a constant. Then the permutation matrix $A = [A_{i,j}]$ of order $p - 1$ with

$$A_{i,j} = 1 \text{ if and only if } \phi^{j+c-1} \equiv i \pmod{p}$$

is a Costas array.

**Proof.** [30] Since $[\phi^{j+c-1} \pmod{p} : j = 1, \ldots, p - 1]$ is a permutation of $\{1, \ldots, p - 1\}$, $A$ is a permutation matrix. Suppose for a contradiction that $A$ contains two distinct pairs of points,

$$\{(\phi^{j+c-1} \pmod{p}, j), (\phi^{k+j+c-1} \pmod{p}, k + j)\},$$

$$\{(\phi^{\ell+c-1} \pmod{p}, \ell), (\phi^{k+\ell+c-1} \pmod{p}, k + \ell)\},$$

with $1 \leq j < \ell < p - 1$ and $1 \leq k \leq p - 1 - \ell$, separated by the same vector. That is,

$$(\phi^{k+j+c-1} \pmod{p}) - (\phi^{j+c-1} \pmod{p}) = (\phi^{k+\ell+c-1} \pmod{p}) - (\phi^{\ell+c-1} \pmod{p}).$$

Then $\phi^{j+c-1}(\phi^{k} - 1) \equiv \phi^{\ell+c-1}(\phi^{k} - 1) \pmod{p}$. Now, since $\phi^{k} - 1 \not\equiv 0 \pmod{p}$ and $1 \leq j, \ell \leq p - 1$, this forces $j = \ell$, a contradiction. □

The above construction for the array $A$ is often referred to as the *exponential* Welch construction and denoted by $W_1^{exp}(p, \phi, c)$, in contrast to the *logarithmic* Welch construction, which generates the array $T(A)$. In view of this equivalence, we will refer only to the exponential Welch construction. For any property of exponential Welch Costas arrays, there is a corresponding (equivalent) property of logarithmic Welch Costas arrays, though we will not describe the latter explicitly.

The parameter $c$ represents a cyclic shift through the columns of $A$, so we may restrict $c$ to the range $0, \ldots, p - 2$. Consequently, every $W_1(p, \phi, c)$ Welch Costas array is singly periodic; if copies of the array are placed side-by-side to form a horizontal tiling, any $p - 1$ consecutive columns form a Costas array. Golomb and H. Taylor [33] conjectured that Welch Costas arrays are characterised by this periodicity property; the conjecture is still open and is discussed in [31]. For $n > 2$ there are no doubly periodic Costas arrays (that is, Costas arrays that, when used to tile the plane, produce a Costas array in every $n \times n$ square). This result is attributed to H. Taylor [37].
Example 21. The Costas permutation \([1, 2, 4, 8, 5, 10, 9, 7, 3, 6]\), corresponding to the array shown in Figure 2.13, is generated by the \(W_1(11, 2, 0)\) construction. Setting \(c = 3\), for example, cyclically shifts the permutation by three places, yielding \([8, 5, 10, 9, 7, 3, 6, 1, 2, 4]\). We note that the arrays produced by the \(W_1\) construction for a given \(p\) are not all inequivalent; the array shown in Figure 2.13 is the vertical reflection of the array produced by \(W_1(11, 6, 1)\), which corresponds to the permutation \([6, 3, 7, 9, 10, 5, 8, 4, 2, 1]\), since \(6 = 2^{-1}\) in \(\mathbb{F}_{11}\).

The Welch Costas array shown in Example 21 has the property that its left half (that is, the first five columns) is a horizontal reflection of its right half. This property is known as \(G\)-symmetry \([41]\) (an abbreviation of glide-reflection symmetry) or anti-reflective symmetry. It is extended to arrays of odd order in \([43]\) (under the name central anti-reflective symmetry, due to the necessary dot in the central position of the array).

Definition 22. Let \(G\) be an \(n \times n\) array corresponding to the permutation \(\gamma\). We say that \(G\) is \(G\)-symmetric if

1. \(\gamma(j + \frac{n}{2}) + \gamma(j) = n + 1\) for \(1 \leq j \leq \frac{n}{2}\) when \(n\) is even, or
2. \(\gamma(\frac{n+1}{2}) = \frac{n+1}{2}\) and \(\gamma(j + \frac{n+1}{2}) + \gamma(j) = n + 1\) for \(1 \leq j < \frac{n+1}{2}\) when \(n\) is odd.

In fact, all \(W_1(p, \phi, c)\) Costas arrays are \(G\)-symmetric. This is noted without proof in \([24]\); a proof is given in \([18]\).
Proposition 23. Every $W_1(p, \phi, c)$ Costas array is $G$-symmetric.

Proof. Let $\alpha$ be the permutation corresponding to a $W_1(p, \phi, c)$ Costas array. The case $p = 2$ is trivial. For $p > 2$, let $1 \leq j \leq \frac{p-1}{2}$. Then, by the Welch Construction,

$$\alpha(j + \frac{p-1}{2}) + \alpha(j) \equiv \phi^{j+c-1} + \phi^{j+c-1+\frac{p-1}{2}} \pmod{p}$$
$$\equiv \phi^{j+c-1}(1 + \phi^{\frac{p-1}{2}}) \pmod{p}$$
$$\equiv 0 \pmod{p},$$

since $\phi$ is primitive in $\mathbb{F}_p$ and so $\phi^{\frac{p-1}{2}} \equiv -1 \pmod{p}$. Now since $1 \leq \alpha(j), \alpha(j + \frac{p-1}{2}) \leq p - 1$, we have $\alpha(j) + \alpha(j + \frac{p-1}{2}) = p$. □

We will use the G-symmetry property of Welch Costas arrays in Chapter 4 to prove some additional structural constraints on these arrays.

Theorem 24 (Golomb construction $G_2(q, \phi, \rho)$). [30] Let $\phi$ and $\rho$ be primitive elements (not necessarily distinct) of $\mathbb{F}_q$, where $q$ is a power of a prime. Then the permutation matrix $A = [A_{i,j}]$ of order $q - 2$ with $A_{i,j} = 1$ if and only if

$$\phi^i + \rho^j = 1$$

is a Costas array.

Proof. For $x \in \mathbb{F}_{q}^* \setminus \{1\}$, define $\log_{\rho}(x)$ to be the integer $t$ such that $1 \leq t \leq q - 2$ and $\rho^t = x$. Since $\rho$ is primitive in $\mathbb{F}_q$, the integer $t$ is unique. For $1 \leq i, j \leq q - 2$, we may then write the condition $\phi^i + \rho^j = 1$ as $j = \log_{\rho}(1 - \phi^i)$. As $i$ runs from 1 to $q - 2$, $1 - \phi^i$ takes each value in $\mathbb{F}_{q}^* \setminus \{1\}$ exactly once, so $\log_{\rho}(1 - \phi^i)$ takes each value in $\{1, \ldots, q - 2\}$ exactly once, and the array $A$ is a permutation matrix. Suppose for a contradiction that $A$ is not a Costas array. Then it contains two distinct pairs of points,

$$\{(i, \log_{\rho}(1 - \phi^i)) , (i + k, \log_{\rho} (1 - \phi^{i+k}))\}$$
$$\{(\ell, \log_{\rho}(1 - \phi^\ell)) , (\ell + k, \log_{\rho}(1 - \phi^{\ell+k}))\},$$

with $1 \leq i < \ell \leq q - 2$ and $1 \leq k \leq q - 2 - \ell$, separated by the same vector. That is,

$$\log_{\rho}(1 - \phi^{i+k}) - \log_{\rho}(1 - \phi^i) = \log_{\rho}(1 - \phi^{\ell+k}) - \log_{\rho}(1 - \phi^\ell).$$
We then obtain the following series of equations.

\[
\log_{\rho} \left( \frac{1 - \phi^{i+k}}{1 - \phi^i} \right) = \log_{\rho} \left( \frac{1 - \phi^{\ell+k}}{1 - \phi^\ell} \right)
\]

\[
\frac{1 - \phi^{i+k}}{1 - \phi^i} = \frac{1 - \phi^{\ell+k}}{1 - \phi^\ell}
\]

\[
(1 - \phi^{i+k})(1 - \phi^i) = (1 - \phi^{\ell+k})(1 - \phi^\ell)
\]

\[
\phi^i(\phi^k - 1) = \phi^\ell(\phi^k - 1).
\]

Since \(1 \leq k \leq q - 4\), we have \(\phi^k - 1 \neq 0\), which then gives \(\phi^i = \phi^\ell\). The ranges of \(i\) and \(\ell\) then force the contradiction \(i = \ell\). □

Figure 2.14(a) shows the Golomb Costas array generated using primitive elements \(\phi = 2, \rho = 7\) of \(\mathbb{F}_{11}\).

![Golomb Costas array](image)

(a) Golomb Costas array obtained from \(G_2(11,2,7)\)

![Symmetric (non-Lempel) Golomb Costas array](image)

(b) Symmetric (non-Lempel) Golomb Costas array

Figure 2.14: Examples of Golomb Costas arrays

The special case of the Golomb construction with \(\phi = \rho\) is known as the Lempel construction and sometimes denoted by \(L_2(q,\phi)\). It is easily verified that this produces a Costas array with diagonal symmetry — that is, it has a dot at \((i, j)\) if and only if it has a dot at \((j, i)\). However, diagonal symmetry does not characterise the Lempel construction [33]; the symmetric \(G_2(9,\phi,\rho)\)
Costas array in Figure 2.14(b) is generated using $\phi = 2x$, $\rho = x + 1$, with $\mathbb{F}_9$ constructed using the primitive polynomial $x^2 + x + 2$.

The subscripts 1 and 2 in the notations $W_1(p, \phi, c)$ and $G_2(q, \phi, \rho)$ refer to the difference between the order of the array produced and the order of the finite field over which it is generated (for example, the Welch construction $W_1(p, \phi, c)$ generates an array of order $p - 1$ over the finite field of order $p$). This notation is convenient when denoting variants of the two main constructions. As mentioned previously, these variants involve manipulating a known Costas array to produce a new one. The most common manipulations involve removing (or adding) corner dots to produce a new array whose order differs by 1 from that of the original array. For example, the $W_1(11, 2, 0)$ Welch Costas array of order 10 shown in Figure 2.13 has a corner dot at position $(1, 1)$. Removing this dot (and eliminating the first row and column) produces a $W_2(11, 2, 0)$ Costas array, of order 9. Subsequently, since this new array also has a dot at $(1, 1)$, we may once again remove the first row and first column to produce a $W_3(11, 2, 0)$ Costas array, of order 8.

It is no coincidence that the $W_1(11, 2, 0)$ Costas array of Example 21 admits the two manipulations discussed above. From the definition of the Welch construction, for all $p$ and $\phi$, the $W_1(p, \phi, 0)$ Costas array has a dot at $(1, 1)$, so there is a $W_2(p, \phi, 0)$ Costas array. Furthermore, a $W_1(p, 2, 0)$ Costas array has a dot at $(1, 1)$ and at $(2, 2)$, so there is a $W_3(p, 2, 0)$ Costas array whenever 2 is a primitive element of $\mathbb{F}_p$. Further variant constructions, involving the removal of at most 3 dots or the addition of at most 2 dots, are summarised in [42] and discussed in detail in [16]. The notations for the other variants are similarly derived from the constructions on which they are based.

In 2011, Drakakis proposed a classification of all known Costas arrays [16] into four categories: generated, predictably emergent, unpredictably emergent and sporadic. (This proposed classification supersedes an earlier proposal made in 2008 [42].) The category to which a given Costas array belongs depends on how well we understand its origin; the four categories above are listed in order from most understood (for example, Costas arrays arising from constructions that are guaranteed to produce a Costas array) to least understood (for example, those that were discovered only by computer search).

Of the constructions that are guaranteed to produce Costas arrays, the very best are those that produce infinite families of Costas arrays and whose capacity to produce Costas arrays of a given
order can be determined from conditions on the order alone. Arrays arising from these constructions are classified as “generated”. For example, the Welch $W_1(p, \phi, c)$ construction is guaranteed to produce a Costas array of order $p - 1$ whenever $p$ is prime, so these Costas arrays belong to the generated category. Similarly, since every finite field $\mathbb{F}_q$ contains two (possibly equal) primitive elements $\phi, \rho$, the $G_2(q, \phi, \rho)$ Costas arrays (and the $L_2(q, \phi)$ Costas arrays) are also in this category.

Predictably emergent Costas arrays also arise from constructions (or variants) that are guaranteed to produce Costas arrays whenever they are applied. However, in this case, the applicability of the construction to a given order cannot be determined from the order alone; other factors, such as the available primitive elements in the given finite field, must also be considered. For example, the $W_3(p, 2, 0)$ construction, which involves removing dots from positions $(1, 1)$ and $(2, 2)$ of a $W_1(p, 2, 0)$ Costas array, is guaranteed to produce a Costas array of order $p - 3$ whenever it is applicable, but it is applicable only if 2 is primitive in $\mathbb{F}_p$. Drakakis [15] provides a detailed inventory of the known conditions on predictably and unpredictably emergent Costas arrays.

Sporadic Costas arrays are those whose origin and structure are not understood. These arrays are discovered by computer search (or by chance, for small orders) and cannot be explained by any known construction or variant. Unpredictably emergent Costas arrays “essentially behave like sporadic ones, except that they have the advantage to be ‘reachable’ through simple transformations of the other known Costas arrays” [19]. For example, the Golomb-Rickard construction [14] involves augmenting a $G_2$ Costas array by adding an empty row to the bottom and an empty column to the right of the array, with a dot at their intersection. The rows and columns are then cyclically permuted (independently), and the resulting array is tested for the Costas property. This is done for every combination of row and column permutations. It is, essentially, a hybrid between a construction and an exhaustive search, in which the construction defines the search space. An analogous procedure, called the Welch-Rickard construction, can be performed on $W_1$ Costas arrays [37]. Four previously unknown Costas arrays (two of order 29 and one of each order 36 and 42) have been obtained this way, but further attempts to apply this method (up to order 300) have failed [19]. This illustrates an important difference between emergent Costas arrays and generated Costas arrays, namely that emergent arrays do not necessarily belong to infinite families.

As we learn more about the structure of Costas arrays and the algebraic structures behind the known constructions, known Costas arrays could be moved from “sporadic” to “emergent”
and from “emergent” to “generated” in this classification system. For example, when Golomb published his construction $G_2(q,\phi,\rho)$ in 1984 [30], he noted that when $\phi + \rho = 1$ (in $\mathbb{F}_q$) the removal of a corner dot from position $(1,1)$ produces a Costas array of order $q - 3$, and if $q$ is a power of 2 a dot can be simultaneously removed from $(2,2)$ to produce a Costas array of order $q - 4$. (These are the $G_3(q,\phi)$ and $G_4(2^m,\phi)$ variations, respectively, denoted without a third parameter since $\rho = 1 - \phi$ is a necessary condition for their applicability.) Golomb conjectured that such a pair of primitive elements could be found for every finite field, and in the early 1990s, this conjecture was proved [10]. As a result, arrays obtained using Golomb’s construction variant may be classified as “generated”, rather than “emergent”. The proof of Golomb’s conjecture also answered the existence question for infinitely many more orders, without the need to exhibit a pair of suitable primitive elements.

There is no known infinite family of Costas arrays whose construction is independent of finite fields. In particular, recursive construction methods do not seem to yield Costas arrays. As Drakakis, Rickard and Gow note [23],

“Virtually every novice in the field spends several hours playing around with this idea of a recursive construction for Costas arrays involving “interlacing”), until (s)he gets eventually disappointed as no array produced seems to have the Costas property.”

These authors then prove that this particular recursive construction cannot be used to generate nontrivial Costas arrays. Little else has been published on non-algebraic constructions, leaving (for the optimist, at least) the possibility that a recursive construction for Costas arrays will one day be discovered.

### 2.2.4 Enumeration and existence results

Despite the extensive work on constructions for infinite families of Costas arrays discussed in the previous section, researchers still have not answered one of the most fundamental existence questions for Costas arrays, namely,

*Does there exist a Costas array of every order?*

In 1984, Golomb and H. Taylor [33] conjectured that the answer is yes. The smallest open cases were — and, twenty-eight years later, still are — orders 32 and 33. As a result, the answer to the existence question for order 32 has been called the “Holy Grail” of Costas array research [36].
While new algebraic constructions for Costas arrays have proved elusive, significant computational resources from multiple machines have become widely available, allowing researchers to enumerate all Costas arrays of a given order by exhaustive search. The most recent enumerations are for order 27 (2008) [22], order 28 (2010) [19], and order 29 (2011) [20]. Using current algorithms and equipment, the enumeration of Costas arrays of order 32 is estimated to take 45000 years of CPU time [19]. This was described as “definitely a large number but not prohibitively so”, indicating that this “Holy Grail” of exhaustive searches is finally (just) within reach.

Some early computer searches restricted the search space by imposing symmetry constraints, as a possible shortcut to finding a Costas array of order 32. This strategy was first discussed in [7], where three types of symmetry were considered. Recall from Section 2.2.3 that every $W_1(p, \phi, c)$ Costas array has G-symmetry and every $L_2(q, \phi)$ array has diagonal symmetry. However, there exist G-symmetric Costas arrays that do not arise from the Welch construction (notice that Table 2.16, which gives enumeration data for various types of Costas arrays, lists G-symmetric Costas arrays for orders that are not one less than a prime) and diagonally symmetric Costas arrays that do not arise from the Lempel construction (see Figure 2.14(b)). The third type of symmetry is defined as follows.

**Definition 25.** [7] Let $A$ be a permutation matrix of even order $n$ corresponding to the permutation $\sigma$. We say that $A$ has consecutive symmetry (is consecutive) if $|\sigma(j) - \sigma(n + 1 - j)| = 1$ for $j = 1, \ldots, n$.

For example, the Costas array in Figure 2.15 has consecutive symmetry.

A restricted search in 2009 [43] extended the enumeration of Costas arrays with diagonal symmetry, G-symmetry and consecutive symmetry, respectively, to larger orders, and introduced some variations on these types of symmetry. However, all searches for order 32 Costas arrays using symmetry restrictions have failed.

Table 2.16 presents the known data on the total number of Costas arrays and equivalence classes, as well as the numbers of inequivalent diagonal, G-symmetric, consecutive, generated and sporadic Costas arrays for each order up to 36. A blank cell indicates that no data exist, while a - indicates that the property is not defined for the given order. Results shown in italics are lower bounds.

Alongside their conjecture on the fundamental existence question, Golomb and H. Taylor [33] presented a number of other conjectures on the enumeration and density of Costas arrays (some of
### Table 2.16: Costas array enumeration data

<table>
<thead>
<tr>
<th>Order</th>
<th>Total</th>
<th>Equivalence Classes</th>
<th>Diagonal Symmetry</th>
<th>G-symmetry</th>
<th>Consecutive Symmetry</th>
<th>Generated</th>
<th>Sporadic</th>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-</td>
<td>1</td>
<td>0</td>
</tr>
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<td>1</td>
<td>1</td>
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Among them was the conjecture, based on the available enumeration results at the time (which extended to order 13), that the total number $C(n)$ of Costas arrays of order $n$ increases monotonically with $n$. As shown in Table 2.16, this conjecture is false; in particular, $C(17) < C(16)$. To date, exhaustive search has been completed for all orders up to 29 [20].

For orders 1 to 29, over 90% of the Costas arrays are sporadic. However, there is a notable absence of sporadic Costas arrays for orders 28 and 29 (two of the twenty-three inequivalent Costas arrays of order 29 are unpredictably emergent Welch-Rickard Costas arrays [20]), and, after a plateau between orders 13 and 20, the proportion of sporadic arrays among all Costas arrays declines dramatically. The most recent new Costas arrays to be discovered by exhaustive search are three sporadic arrays of order 26 [38] and one sporadic array of order 27 (found by exhaustive search in 2008 by two groups independently [22]). It is conjectured in [19] that sporadic Costas arrays eventually die out, and that there is an order beyond which all Costas arrays are either generated or predictably emergent.

Table 2.17 displays current existence results for orders up to 200, and results for orders up to 359 are given in [41]. There are no known nonexistence results for Costas arrays of a given order; a blank cell in the table indicates that it is unknown whether there is a Costas array of the
In Chapter 4, we present new results on Costas arrays. We discuss the toroidal distance vectors present in Costas arrays, and show that Welch Costas arrays, Golomb Costas arrays and G-symmetric Costas arrays cannot contain every possible toroidal distance vector. We identify a structural feature of Costas arrays called “mirror pairs”, which was first observed numerically, and prove existence results for mirror pairs with various properties. This provides constraints on Costas arrays.

2.3 Golomb rulers

Golomb rulers were studied by Babcock [4] in 1953 for use in eliminating third-order interference between radio communications channels. Since then, they have been studied by different researchers under a variety of names, including distinct difference sets [3]. They are named for S. Golomb, who conducted a systematic study of their properties (see [13] for details). In addition to Babcock’s application in radio communications, Golomb rulers can be used in X-ray crystallography to distinguish crystal lattice structures whose diffraction patterns are identical; in coding theory to produce self-orthogonal codes; and in radio astronomy, both in locating distant radio sources and in determining the best layout of linear antenna arrays [34].

Golomb rulers are equivalent to Sidon sets, as defined by Sidon [39] in 1932 in connection with a problem in combinatorial number theory; the two objects were studied independently for many years before the connection was made [13].

**Definition 26.** A **Golomb ruler** is a set \( R \) of integers such that every pair of distinct 2-subsets \( \{m_1, m_2\}, \{m_3, m_4\} \) of \( R \) satisfies \(|m_2 - m_1| \neq |m_4 - m_3|\).

The set \( R \) can be viewed as a set of marks at integer positions along a ruler, with no two distinct pairs of marks the same distance apart. It is easy to see, both from the definition and from the visual representation of a Golomb ruler, that a Costas array can be viewed as a two-dimensional Golomb ruler. The number \( r \) of marks (that is, \(|R|\)) is called the order of the ruler, and the largest distance \( \ell \) between any two marks (that is, the difference between the largest and smallest element of \( R \)) is called the length of the ruler.

By convention, we write the set of marks \( \{m_0, m_1, \ldots, m_{r-1}\} \) of a Golomb ruler of length \( \ell \) and order \( r \) in increasing order, taking the smallest mark to be 0 so that the largest is \( \ell \).
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Table 2.17: Existence table for Costas arrays up to order 200
Example 27. The ruler below corresponds to the set \( R = \{0, 4, 6\} \).

![Ruler diagram](image)

The distances between pairs of marks are 2, 4 and 6, so this is a Golomb ruler. Its order is 3 and its length is 6.

Several optimisation problems have been considered for Golomb rulers. For example, a shortest Golomb ruler of a given order \( r \) is called an optimal Golomb ruler; the problem of determining the length of an optimal Golomb ruler of order \( r \) has been studied extensively but no general solution is known [13]. In the next section, we discuss Golomb rulers that satisfy a different optimisation condition. Unlike optimal Golomb rulers, these rulers (known as perfect Golomb rulers), have been completely characterised. In Section 3.2, we prove that wavelength isolation sequence pairs can be constructed from perfect Golomb rulers.

### 2.3.1 Perfect Golomb rulers

**Definition 28.** A Golomb ruler \( R \) of length \( \ell \) and order \( r \) is perfect if for every integer \( d \) satisfying \( 1 \leq d \leq \ell \), there is exactly one pair of marks \( m_1, m_2 \in R \) such that \( m_2 - m_1 = d \).

By the pigeonhole principle, a Golomb ruler of length \( \ell \) and order \( r \) satisfies \( \ell \geq \binom{r}{2} \); if the ruler is perfect then \( \ell = \binom{r}{2} \). A perfect Golomb ruler can be obtained from the Golomb ruler in Example 27 by adding a mark at position 1.

Proposition 29 states that there are only four perfect Golomb rulers, the longest of which has length 6 and order 4.

**Proposition 29.** (Golomb, see [13]) Up to reversal and translation, the only perfect Golomb rulers are

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<tr>
<td>4</td>
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Proof. It is straightforward to verify that, up to reversal and translation, the only perfect Golomb rulers of order \( r \leq 4 \) are those listed in the statement of the proposition. Suppose, for a contradiction, that \( R \) is a perfect Golomb ruler of order \( r \geq 5 \). Then \( 0, \ell \in R \), where \( \ell = \binom{r}{2} \geq 10 \). Since \( R \) has a pair of marks separated by distance \( \ell - 1 \), we may assume, by symmetry, that \( 1 \in R \). Then \( 2, \ell - 1 \notin R \), since distance 1 is covered by the pair of marks 0, 1. In order to achieve distance \( \ell - 2 \), we must have \( \ell - 2 \in R \). This also achieves distance 2 and implies that \( 3, \ell - 3, \ell - 4 \notin R \), as these marks would duplicate distances 2, 1 and 2, respectively. Distance \( \ell - 3 \) is achieved by the pair of marks 1, \( \ell - 2 \), and in order to realize distance \( \ell - 4 \) we must have \( 4 \in R \). This also achieves distance 3 and implies that \( 5, \ell - 5 \notin R \) (with \( \ell - 5 > 4 \) and \( 5 < \ell - 4 \) since \( \ell \geq 10 \)) as these would duplicate distances 1 and 3, respectively. However, this leaves no pair of marks at distance \( \ell - 5 \), giving the desired contradiction. \( \square \)
Chapter 3

Wavelength Isolation Sequence Pairs

In Section 2.1 we discussed the history of wavelength isolation sequence pairs and described the application in multislit spectrometry that led Golay to search for WISPs in the 1950s. In this chapter, we present new results on WISPs. In Section 3.1 we give a characterisation of WISPs and prove some structural constraints on the sequences $A$ and $B$. We present a complete list of currently known WISPs, including two new examples. In Section 3.2 we describe a construction method that explains all of the known examples, by making a connection to perfect Golomb rulers. Finally, in Section 3.3, we provide partial results on the classification of all WISPs.

3.1 Characterisation and examples of WISPs

Before we present a mathematical characterisation of WISPs in Definition 30, we introduce some notation that we will use throughout this chapter. Let $A = (a_0, \ldots, a_{n-1})$ be a binary $\{0, 1\}$ sequence of length $n$ and let $x, y \in \{0, 1\}$. For $u \geq 0$, we define

$$S_A(x, y, u) = |\{(j, j+u) : (a_j, a_{j+u}) = (x, y) \text{ and } 0 \leq j < n-u\}|$$

to be the number of positions in $A$ containing an $x$ followed at distance $u$ by a $y$. For example, if $A = (10100100)$ then $S_A(1, 1, 3) = 1$ and $S_A(1, 0, 4) = 2$. We note that $S_A(1, 1, u)$ is the aperiodic autocorrelation of $A$ at shift $u$, after relabelling the entries of $A$ from $1$ to $n$. We write $w(A)$ for the weight of $A$. We now formally define a WISP.
Definition 30. Let $A = (a_0, \ldots, a_{n-1})$ and $B = (b_0, \ldots, b_{n-1})$ be binary sequences of length $n$. We say that $(A, B)$ is a wavelength isolation sequence pair (WISP) if

$$w(A) \geq 1 \quad \text{and} \quad S_A(1,1,u) = S_B(1,0,u) = S_B(0,1,u) \quad \text{for} \quad 1 \leq u < n.$$ (3.1)

It is easily verified by reference to Property (b) in Section 2.1.1 that if $A$ and $B$ form a WISP then they will be suitable for use as the entrance slit patterns of a multislit spectrometer (Condition (3.1) ensures that some radiation is passed). Without loss of generality, we can take $a_0 = 1$ (by left-shifting the elements of $A$ and padding with zeroes on the right). We can also form an equivalent WISP by reversing the subsequence of $A$ from its initial ‘1’ element $a_0$ to its final ‘1’ element. Further, if $(A, B)$ is a WISP then so is $(A, \bar{B})$, where $\bar{B}$ is the complement of $B$, since $S_B(1,0,u) = S_B(0,1,u)$. Thus we may take $w(B) \leq \frac{n}{2}$. There is a WISP of every length, namely $A = (10 \ldots 0)$ and $B = (0 \ldots 0)$, whose corresponding multislit spectrometer is trivial. We consider a WISP to be nontrivial if $w(A) > 1$.

In addition to the three nontrivial examples of WISPs found by Golay [28], which were presented in Table 2.4, we present in Table 3.1 two new examples of WISPs, having lengths 7 and 13. Up to equivalence, these five examples constitute the complete list of currently known nontrivial WISPs.

Table 3.1: All known nontrivial WISPs, up to equivalence

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<td>$(0000001000000)$</td>
</tr>
<tr>
<td>$(11001010)$</td>
<td>$(10000001)$</td>
</tr>
</tbody>
</table>

We see that, for each WISP presented in Table 3.1, $B$ is symmetric. Proposition 31 states that in fact this is a necessary condition for the binary sequence pair $(A, B)$ to be a WISP.

Proposition 31. If $A$ and $B$ form a WISP then $B$ is symmetric.
Proof. Suppose that \( A = (1, a_1 \ldots a_{n-1}) \) and \( B = (b_0 b_1 \ldots b_{n-1}) \) form a WISP of length \( n > 1 \). Then by (3.2) with \( u = n - 1 \), we obtain

\[
b_0 = b_{n-1}.
\]

We may therefore take \( n > 3 \). We now prove by induction on \( i \) that \( b_i = b_{n-1-i} \) for \( 0 \leq 2i < n - 1 \), so that \( B \) is symmetric. The base case \( i = 0 \) is given by (3.3). Assume that cases up to \( i - 1 \) hold, where \( 2 \leq 2i < n - 1 \), so that \( B \) has the form

\[
B = (b_0 \ b_1 \ \ldots \ b_{i-1} \ | \ b_i \ \ldots \ b_{n-1-i} \ | \ b_{i-1} \ \ldots \ b_1 \ b_0).
\]

We wish to prove that \( b_i = b_{n-1-i} \).

By (3.2) with \( u = n - 1 - i \), we have

\[
S_B(1, 0, n-1-i) = S_B(0, 1, n-1-i).
\]

But by the inductive hypothesis, \((b_j, b_{j+n-1-i}) = (b_{n-1-j}, b_{i-j})\) for \( 1 \leq j \leq i - 1 \), so that the contributions to \( S_B(1, 0, n-1-i) \) arising from index pairs \((j, j+n-1-i)\) with \( 1 \leq j \leq i - 1 \) are exactly balanced by the contributions to \( S_B(0, 1, n-1-i) \) arising from index pairs \((i-j, n-1-j)\) with \( 1 \leq j \leq i - 1 \). Accounting for the remaining contributions to \( S_B(1, 0, n-1-i) \) and \( S_B(0, 1, n-1-i) \) from index pairs \((0, n-1-i)\) and \((i, n-1)\), and using (3.4), then gives

\[
(b_0, b_{n-1-i}) = (1, 0) \iff (b_i, b_{n-1}) = (0, 1)
\]

and

\[
(b_0, b_{n-1-i}) = (0, 1) \iff (b_i, b_{n-1}) = (1, 0).
\]

Using (3.3), we obtain \( b_i = b_{n-1-i} \) as required, thus completing the induction. \( \square \)

In light of the symmetry of \( B \), the conditions on a WISP may be rephrased to give an alternative definition.

Alternative Definition 32. Let \( A = (a_0, \ldots, a_{n-1}) \) and \( B = (b_0, \ldots, b_{n-1}) \) be binary sequences of length \( n \). We say that \( A \) and \( B \) form a wavelength isolation sequence pair (WISP) if

- \( B \) is symmetric,
- \( w(A) \geq 1 \) and
- \( S_A(1, 1, u) = S_B(1, 0, u) \) for \( 1 \leq u < n \).
We will present a second structural constraint in Proposition 34 concerning the weights of members of WISPs. In preparation, we will prove Lemma 33. For \( u \geq 0 \), we define
\[
P_A(x, y, u) = \left| \{ (j, j + u) : (a_j, a_{(j+u) \mod n}) = (x, y) \text{ and } 0 \leq j < n \} \right|,
\]
a periodic analogue of \( S_A(x, y, u) \).

**Lemma 33.** For every binary \( \{0, 1\} \) sequence \( C \) of length \( n \),
\[
\sum_{u=1}^{n-1} P_C(1, 1, u) = w(C)^2 - w(C). \tag{3.7}
\]
Furthermore, if \( A \) and \( B \) form a WISP of length \( n \), then
\[
P_A(1, 1, u) + P_B(1, 1, u) = w(B) \quad \text{for } 1 \leq u < n. \tag{3.8}
\]

**Proof.** For (3.7), we note that
\[
\sum_{u=1}^{n-1} P_C(1, 1, u) = w(C)(w(C) - 1), \tag{3.9}
\]
since each ordered pair of distinct ‘1’ entries in \( C \) contributes exactly 1 to the sum. It is easily verified that
\[
S_C(x, y, u) + S_C(x, y, n - u) = P_C(x, y, u) \quad \text{for } 1 \leq u < n \tag{3.10}
\]
(which is a restatement of a well-known relation between the periodic and aperiodic autocorrelations of a binary sequence). Let \( 1 \leq u < n \). Applying (3.10) with \((C, x, y) = (A, 1, 1)\) and \((B, 1, 0)\) gives
\[
P_A(1, 1, u) = P_B(1, 0, u), \tag{3.11}
\]
by (3.6). There are \( w(B) \) 1s in \( B \), of which \( P_B(1, 1, u) \) are followed by a 1 at (periodic) distance \( u \) and \( P_B(1, 0, u) \) are followed by a 0. Therefore
\[
P_B(1, 0, u) + P_B(1, 1, u) = w(B),
\]
which combines with (3.11) to give (3.8). \( \square \)

Proposition 34 now follows easily from Lemma 33.
CHAPTER 3. WAVELENGTH ISOLATION SEQUENCE PAIRS

Proposition 34. Suppose that $A = (a_0, \ldots, a_{n-1})$ and $B = (b_0, \ldots, b_{n-1})$ form a WISP of length $n$. Then

$$w(A)^2 + w(B)^2 = w(B) n + w(A).$$

Proof. Summing (3.8) over $u = 1, \ldots, n - 1$ gives

$$\sum_{u=1}^{n-1} P_A(1, 1, u) + \sum_{u=1}^{n-1} P_B(1, 1, u) = (n - 1) w(B).$$

Substitution from (3.7) gives the result. □

3.2 Construction of WISPs from perfect Golomb rulers

Theorem 35 describes two construction procedures, each of which produces a WISP from a perfect Golomb ruler of length $\ell$. The constructed WISPs are inequivalent for $\ell \neq 1$.

Theorem 35. Let $R$ be a perfect Golomb ruler of order $r \geq 1$ and length $\ell = (\ell^2)$. For $0 \leq j \leq \ell$, let

$$c_j = \begin{cases} 1 & \text{for } j \in R \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{cases} A = (1 \ c_1 \ldots \ c_\ell \ 0) \\ B = (1 \ 0 \ldots \ 0 \ 1) \end{cases}$$

(3.12)

is a WISP of length $\ell + 2$ and

$$\begin{cases} A = (c_0 \ldots c_{\ell-1} \ 1 \ 0 \ldots \ 0) \\ B = (0 \ldots \ 0 \ 1 \ 0 \ldots \ 0) \end{cases}$$

(3.13)

is a WISP of length $2\ell + 1$.

Proof. We will show that $A$ and $B$ satisfy the conditions of Alternative Definition 32 under both Constructions (3.12) and (3.13). Clearly, Condition (3.5) is satisfied and $B$ is symmetric in both cases, so we need only show that Condition (3.6) is satisfied in both cases (with $n = \ell + 2$ for the pair (3.12) and $n = 2\ell + 1$ for the pair (3.13)). By construction, the positions of the 1s in $A$ are the marks of the perfect Golomb ruler $R$, and

$$S_A(1, 1, u) = S_B(1, 0, u) = \begin{cases} 1 & \text{for } 1 \leq u \leq \ell \\ 0 & \text{for } u > \ell. \end{cases}$$

□
Each of the known WISPs (or one that is equivalent), presented in Table 3.1, can be constructed by (3.12) or (3.13) from one of the perfect Golomb rulers listed in Proposition 29. Trivial WISP lengths 2 and 1 arise from the perfect Golomb ruler of length 0, WISP lengths 3 and 3 from length 1, WISP lengths 5 and 7 from length 3, and WISP lengths 8 and 13 from length 6.

3.3 Are there WISPs of length greater than 13?

By Proposition 29, there are no more perfect Golomb rulers to use in Theorem 35, and computer search rules out the existence of additional WISPs for lengths less than 33. We were unable to determine whether there are any more WISPs. However, Propositions 36 and 37 give partial results on the classification of all WISPs.

**Proposition 36.** Up to equivalence, the only nontrivial WISPs \((A, B)\) with \(w(B) = 1\) are those listed in Table 3.1.

**Proof.** Let \(A\) and \(B\) form a nontrivial WISP of length \(n\) with \(w(B) = 1\). Then, since \(B\) is symmetric, \(n\) is odd and

\[
b_i = \begin{cases} 
1 & \text{for } i = \frac{n+1}{2} \\
0 & \text{otherwise.}
\end{cases}
\]

Thus

\[
S_B(1, 0, u) = \begin{cases} 
1 & \text{for } 1 \leq u \leq \frac{n+1}{2} \\
0 & \text{for } \frac{n+1}{2} < u < n,
\end{cases}
\]

which, since \(A\) and \(B\) form a WISP, forces

\[
S_A(1, 1, u) = \begin{cases} 
1 & \text{for } 1 \leq u \leq \frac{n-1}{2} \\
0 & \text{for } \frac{n+1}{2} < u < n.
\end{cases}
\]

Then the subsequence of \(A\) from its initial to final ‘1’ element (this sequence having \(\frac{n+1}{2} + 1\) elements) is a perfect Golomb ruler of length \(\frac{n-1}{2}\). By Proposition 29, \(\frac{n-1}{2} = 0, 1, 3\) or 6 and \(A\) is determined up to reversal and translation. □

A similar argument shows that WISPs with \(B = (10\ldots01)\) are characterised by Construction (3.12) of Theorem 35. Proposition 37 rules out another case in which \(w(B) = 2\), where the 1s are in the central positions of \(B\).
Proposition 37. Suppose that $A$ and $B$ form a WISP of length $n > 2$. Then $B \neq (0 \ldots 0110 \ldots 0)$.

Proof. Suppose for a contradiction that $B = (0 \ldots 0110 \ldots 0)$. Then $n$ is even, and

$$S_A(1, 1, u) = S_B(1, 0, u) = \begin{cases} 
1 & \text{for } u = 1, \frac{n}{2} \\
2 & \text{for } 2 \leq u \leq \frac{n}{2} - 1 \\
0 & \text{for } \frac{n}{2} < u < n.
\end{cases} \quad (3.14)$$

Without loss of generality, applying (3.14) $\frac{n}{2}$ times with $u = n-1, t-2, \ldots, \frac{n}{2}$, respectively, gives $A = (1a_1 \ldots a_{\frac{n}{2}-1}10 \ldots 0)$. Then in the case $n = 4$ we derive a contradiction from $S_A(1, 1, 1) = 1$, and in the case $n > 4$ the condition $S_A(1, 1, \frac{n}{2} - 1) = 2$ forces $A = (11a_2 \ldots a_{\frac{n}{2}-2}110 \ldots 0)$, contradicting $S_A(1, 1, 1) = 1$. \hfill $\square$
Chapter 4

Costas Arrays

In Section 2.2 we provided an overview of the study of Costas arrays since they were first defined in the 1960s. We discussed the radar/sonar application that motivated this study, and described important results and considerations in Costas array research, including algebraic constructions and infinite families, symmetry and equivalence considerations, and existence and enumeration results. In this chapter, we draw upon the background presented in Section 2.2 and upon numerical data to identify and prove new structural constraints on Costas arrays. In Section 4.1 we consider toroidal distance vectors in Costas arrays. In particular, we show that, contrary to claims elsewhere, the question of whether a Costas array can contain every possible toroidal distance vector is still open. We answer this question for G-symmetric Costas arrays, which include Welch Costas arrays, and make a conjecture for general Costas arrays. In Section 4.3, we identify a new structural feature of Costas arrays, which we call mirror pairs, and prove existence results for mirror pairs with various properties. We discuss the application of these constraints to reducing the computational burden required for exhaustive searches for Costas arrays. In proving constraints on mirror pairs, we rely heavily on the properties of the difference triangle of a permutation. We discuss these properties in Section 4.2.

4.1 Toroidal distance vectors in Costas arrays

It is often useful to consider the distance vectors present in Costas arrays when we view the array periodically. This is equivalent to viewing the array as being written on the surface of a torus.
However, unless otherwise stated, we treat all arrays as being written in the plane while allowing vectors to “wrap around” at the array boundaries; we prefer this viewpoint because it avoids confusion arising from the convention that the rows and columns of Costas arrays are numbered from 1 rather than from 0. Vectors that are considered in this context are called toroidal distance vectors, as defined in Section 2.2.1. Consideration of the toroidal distance vectors in Costas arrays can be an intermediate step in proving results about the (non-toroidal) vectors present in Costas arrays. We use this technique, from [18], in Sections 4.3.2 and 4.3.3 to prove that certain pairs of vectors must appear in Welch and Golomb Costas arrays.

As a caution, we distinguish the topic of toroidal distance vectors in Costas arrays from that of doubly periodic Costas arrays, which was mentioned in Section 2.2.3. When considering toroidal distance vectors, all vectors wrap around and all cyclic row/column permutations are equivalent. By contrast, a doubly periodic Costas array is defined as a Costas array for which every cyclic row/column permutation produces another Costas array (and vectors do not cross the boundaries of the array).

Theorem 5 of [18] states that “All possible [toroidal] distance vectors \([(w, h)]\) are contained within a Welch Costas array (assuming the array wraps around at the boundaries).” The proof of the statement involves a system of congruences which has a unique solution for each \((w, h)\), leading to the conclusion that each possible toroidal distance vector is contained exactly once in a Welch Costas array. However, as we saw in Section 2.2.1, there are \(n(n-1)\) toroidal distance vectors in a Costas array of order \(n\) and only \((n-1)^2\) possible values for the vectors, so the claim of uniqueness in the proof of Theorem 5 cannot hold. In fact, the statement of the theorem is also incorrect. For a counterexample, consider the \(W_1(5, 2, 1)\) Welch Costas array \(W\), which is shown in Figure 4.1. For convenience, the dots are replaced with letters so that we can easily list the toroidal vectors present in \(W\); these are given in Table 4.2. The toroidal vector \((2, 2)\) is absent from \(W\), while the vectors

![Figure 4.1: A Welch Costas array that does not contain every possible toroidal distance vector](image-url)
Table 4.2: Toroidal distance vectors present in the $W_1(5,2,1)$ Costas array of Figure 4.1

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$AB$</td>
<td>$(1,2)$</td>
<td>$BA$</td>
<td>$(3,2)$</td>
<td></td>
</tr>
<tr>
<td>$AC$</td>
<td>$(2,1)$</td>
<td>$CA$</td>
<td>$(2,3)$</td>
<td></td>
</tr>
<tr>
<td>$AD$</td>
<td>$(3,3)$</td>
<td>$DA$</td>
<td>$(1,1)$</td>
<td></td>
</tr>
<tr>
<td>$BC$</td>
<td>$(1,3)$</td>
<td>$CB$</td>
<td>$(3,1)$</td>
<td></td>
</tr>
<tr>
<td>$BD$</td>
<td>$(2,1)$</td>
<td>$DB$</td>
<td>$(2,3)$</td>
<td></td>
</tr>
<tr>
<td>$CD$</td>
<td>$(1,2)$</td>
<td>$DC$</td>
<td>$(3,2)$</td>
<td></td>
</tr>
</tbody>
</table>

(1, 2), (2, 1), (2, 3) and (3, 2) each appear twice.

The ideas in [18] are useful for proving two related results, Theorem 38 and Theorem 40, which we will use in Section 4.3.

**Theorem 38.** Let $W$ be a $W_1(p, \phi, c)$ Costas array and let $W^+$ be the $p \times (p - 1)$ array obtained by adding an empty row on the bottom of $W$. Then $W^+$ contains every toroidal distance vector $(w, h) \in \{1, \ldots, p - 2\} \times \{1, \ldots, p - 1\}$ exactly once.

**Proof.** Let $(w, h) \in \{1, \ldots, p - 2\} \times \{1, \ldots, p - 1\}$. The toroidal distance vector $(w, h)$ appears in $W^+$ whenever there exist $i, j \in \{1, \ldots, p - 1\}$ such that

\[
i \equiv \phi^{i+c-1} \pmod{p} \quad \text{and} \quad \text{(4.1)}
\]

\[
i + h \equiv \phi^{i+c+w-1} \pmod{p}. \quad \text{(4.2)}
\]

(Since $\phi$ is primitive, $i + h \not\equiv 0 \pmod{p}$, and so these equations do not admit solutions in which a dot appears in the empty row of $W^+$. Thus all solutions arise from vectors $(w, h)$ joining dots in $W$.) Multiplying the first congruence by $\phi^w$ and subtracting the second congruence gives $i(\phi^w - 1) \equiv h \pmod{p}$, so, since $\phi^w \not\equiv 1$,

\[
i \equiv h(\phi^w - 1)^{-1} \pmod{p}. \quad \text{(4.3)}
\]

Then, by (4.1),

\[
\phi^{i+c-1} \equiv h(\phi^w - 1)^{-1} \pmod{p}. \quad \text{(4.4)}
\]

Since there exist unique $i, j \in \{1, \ldots, p - 1\}$ satisfying (4.3) and (4.4), the toroidal distance vector $(w, h)$ appears exactly once in $W^+$. $\square$

We call the array $W^+$ of Theorem 38 the augmented array associated with $W$. In general, the toroidal distance vector $(w, h)$ in $W^+$ corresponds to the toroidal distance vector $(w, h)$ in $W$ if
j + h < p (where \( j \) satisfies (4.4)), and to the toroidal distance vector \((w, h - 1)\) in \(W\) otherwise. This explains the oversight in [18]. For example, adding an empty row to the Welch Costas array shown in Figure 4.1 yields the \(5 \times 4\) array \(W^+\) shown in Figure 4.3. The toroidal distance vectors of \(W^+\) are given in Table 4.4, with each vector \((w, h) \in \{1, 2, 3\} \times \{1, 2, 3, 4\}\) appearing exactly once.

![Figure 4.3: The augmented array \(W^+\) obtained from the Welch Costas array in Figure 4.1](image)

<table>
<thead>
<tr>
<th></th>
<th>(A)</th>
<th>(B)</th>
<th>(C)</th>
<th>(D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(AB)</td>
<td>(1, 2)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(AC)</td>
<td>(2, 1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(AD)</td>
<td>(3, 4)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(BC)</td>
<td>(1, 4)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(BD)</td>
<td>(2, 2)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(CD)</td>
<td>(1, 3)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.4: Toroidal distance vectors present in the array \(W^+\) of Figure 4.3

The vectors that are different from their counterparts in \(W\) (see Table 4.2) are written in boldface; these are exactly the vectors that wrap around in the vertical direction. Removing the empty row from \(W^+\) (“collapsing” \(W^+\) back to \(W\)) reduces the vertical component of each of these vectors by 1, eliminating the vector \((2, 2)\) and causing repeats of other vectors. Algebraically (as suggested in the proof of Theorem 38), the introduction of the empty row in \(W^+\) accounts for the fact that (4.2) has no solution when \(i + h \equiv 0 \pmod{p}\). No such adjustment is needed in the horizontal direction because the column indices occur only as exponents of \(\phi\) and are therefore automatically reduced modulo \(p - 1\) (which is the order of \(W\)).

Before proving a similar result to Theorem 38 for Golomb Costas arrays, we note in passing that the set of positions of the dots in an augmented Welch Costas array is equivalent to a direct product difference set, in which differences are taken periodically [21].
Definition 39. [27] Let $H$ and $N$ be groups of order $m$ and $n$, respectively, and let $G = H \times N$. A $k$-subset $R$ of $G$ is an $(m, n, k, \mu)$ direct product difference set in $G$ relative to $H$ and $N$ if the multiset
\[ \{ r_1r_2^{-1} : r_1, r_2 \in R \text{ and } r_1 \neq r_2 \} \]
comprises each non-identity element of $G \setminus (H \cup N)$ exactly $\mu$ times.

Given a $W_1(p, \phi, c)$ Costas array $W$, relabel the rows of $W^+$ from 0 to $p - 1$ and the columns from 0 to $p - 2$. Then the set $R$ of positions in which (the relabelled array) $W^+$ has a dot is a $(p, p - 1, p - 1, 1)$ direct product difference set in $\mathbb{Z}_p \times \mathbb{Z}_{p-1}$.

We now prove a result similar to Theorem 38 for Golomb Costas arrays.

Theorem 40. Let $G$ be a $G_2(q, \phi, \rho)$ Costas array and let $G^+_\pm$ be the $(q - 1) \times (q - 1)$ array obtained by adding an empty row on the bottom of $G$ and then an empty column on the right. Let $(w, h) \in \{1, 2, \ldots, q - 2\}^2$. Then $G^+_\pm$ contains the toroidal distance vector $(w, h)$ exactly once if
\[ \rho^w \neq \phi^h \text{ and otherwise never. Furthermore, if the toroidal distance vector } (w, h) \text{ appears in } G^+_\pm \text{ starting from a dot at } G_{i, j} \text{ then} \]
\[ \rho^j = 1 - \phi^j = (\phi^h - \rho^w)^{-1}(\phi^h - 1). \] (4.5)

Proof. Let $(w, h) \in \{1, 2, \ldots, q - 2\}^2$. The toroidal distance vector $(w, h)$ occurs in $G^+_\pm$ whenever there exist $i, j \in \{1, \ldots, q - 2\}$ such that
\begin{align*}
\phi^i + \rho^j &= 1 \quad \text{and} \quad (4.6) \\
\phi^{i + h} + \rho^{j + w} &= 1. \quad (4.7)
\end{align*}
(These equations do not admit solutions in which a dot appears in the blank row or the blank column of $G^+_\pm$, since $\phi$ and $\rho$ are primitive and so $i + h \not\equiv 0 \pmod{q - 1}$ and $j + w \not\equiv 0 \pmod{q - 1}$. Therefore all solutions arise from vectors $(w, h)$ joining dots in $G$.)

Multiplying (4.6) by $\phi^h$ and subtracting (4.7) gives $\rho^j(\phi^h - \rho^w) = \phi^h - 1$. If $\phi^h = \rho^w$ then this has no solution. Otherwise,
\[ \rho^j = (\phi^h - \rho^w)^{-1}(\phi^h - 1) \] (4.8)
and then by (4.6),
\[ \phi^j = 1 - \rho^j. \] (4.9)
Equation (4.8) has a unique solution for $j \in \{1, \ldots, q-2\}$, since $\phi^h \neq 1$ and $\rho^w \neq 1$. Therefore, (4.9) has a unique solution for $i \in \{1, \ldots, q-2\}$, since $\rho^j \notin \{0, 1\}$. Equation (4.5) rewrites (4.8) and (4.9).

\[\square\]

**Remark 41.** By Theorem 40, there are exactly $q-2$ toroidal distance vectors missing from $G^+_+$ (out of $(q-2)^2$ possible vectors), namely the solutions $(w, h) \in \{1, \ldots, q-2\}^2$ to $\rho^w = \phi^h$ in $\mathbb{F}_q$. Since this equation has exactly one solution for each $w$ and one solution for each $h$, each $w$ occurs exactly once and each $h$ occurs exactly once in the set of missing vectors.

For example, let $G$ be the $G_2(8, x + 1, x)$ Costas array, with $\mathbb{F}_8$ constructed using the primitive polynomial $x^3 + x + 1$. The augmented array $G^+_+$ is shown in Figure 4.5 (with letters instead of dots) and its toroidal distance vectors and missing toroidal distance vectors are listed in Table 4.6.

![Figure 4.5: An augmented Golomb Costas array](image)

The idea of adding an empty row to a Welch or Golomb Costas array of order $n$ is mentioned in [37], where these arrays are noted to be 1-gap periodic, meaning that if they are tiled vertically with an empty row between adjacent arrays then any selection of $n + 1$ consecutive rows contains no repeated vectors. Viewing the augmented array $G^+_+$ of Lemma 40 as being written on the surface of a torus provides an alternative interpretation of the Golomb-Rickard construction mentioned in Section 2.2.3 (and discussed in [37]). Indeed, we may view the Golomb-Rickard construction as starting with the toroidal $G^+_+$ array, adding a dot at the intersection of its blank row and column, and then trying to “cut” the torus along vertical and horizontal boundaries in such a way that the resulting $(q-1) \times (q-1)$ array (in the plane) has the Costas property. The list of toroidal distance vectors present in $G^+_+$ is known by Theorem 40; we wish to cut the torus so that at least one of
each pair of repeated toroidal distance vectors, arising from the introduction of the extra dot, is eliminated.

We have seen a class of \{0, 1\} matrices, namely augmented Welch Costas arrays, whose multi-set of toroidal distance vectors satisfies two conditions: it is free of repeated vectors and it contains every possible toroidal distance vector. We have already shown (by a counting argument) that the first condition cannot be satisfied by a (non-augmented) Costas array. These results prompt the following natural question.

**Question 42.** Are there (non-augmented) Costas arrays of order \(n > 2\) containing every possible toroidal distance vector \((w, h) \in \{1, \ldots, n - 1\}^2\)?

As mentioned, Theorem 5 of [18] claims that Welch Costas arrays contain all possible toroidal distance vectors, which would provide a positive answer to Question 42. Our next result shows that, in fact, the answer to Question 42 is no for G-symmetric Costas arrays of even order (which include Welch Costas arrays). In fact, we identify \(\frac{n^2}{2} - 1\) specific toroidal distance vectors from the set \(\{1, \ldots, n - 1\}^2\) that are missing from such a Costas array of order \(n\).
Proposition 43. Let $G$ be a $G$-symmetric Costas array of even order $n$. Then $G$ contains no toroidal distance vector $(\frac{n}{2}, h)$ with $h$ even.

Proof. Suppose that the toroidal distance vector $(\frac{n}{2}, h)$ appears in $G$ from position $(i, j)$ to position $(i + h - an, j + \frac{n}{2} - \beta n)$, where $\alpha, \beta \in \{0, 1\}$. Then, by $G$-symmetry, $i + (i + h - an) = n + 1$. Reducing this equation modulo 2 then gives $h \equiv 1 \pmod{2}$. □

Since all Welch Costas arrays are $G$-symmetric, Proposition 43 implies that no nontrivial Welch Costas array contains every possible toroidal distance vector. (This does not contradict Theorem 38, which deals with the augmented array $W^\ast$.) In fact, there is a great deal of regularity in the sets of missing toroidal distance vectors for Welch Costas arrays (and, to a lesser extent, for $G$-symmetric Costas arrays). Clearly, since the parameter $c$ of a $W_1(p, \phi, c)$ Costas array represents a cyclic permutation of the columns of a $W_1(p, \phi, 0)$ Costas array, varying $c$ (for given $p$ and $\phi$) does not change the toroidal distance vectors present in the array. Further, the arrays $W_1(p, \phi, 0)$ and $W_1(p, \phi^{-1}, 0)$ are vertical reflections of each other. As we show in Proposition 44, this, together with the $G$-symmetry property of Welch Costas arrays, implies that these two arrays contain the same toroidal distance vectors.

Proposition 44. Let $G$ be a $G$-symmetric permutation matrix of even order $n$ and let $G'$ be the image of $G$ under rotation by $180^\circ$, horizontal reflection or vertical reflection. Then $G$ and $G'$ contain the same multiset of toroidal distance vectors.

Proof. By Table 2.11, rotating $G$ by $180^\circ$ does not change the (non-toroidal) vectors present, so it does not change the relative position of each pair of dots. Therefore, $180^\circ$ rotation does not change the two toroidal distance vectors between these dots. Since $G$ is $G$-symmetric and $n$ is even, reflecting $G$ horizontally is equivalent to cyclically permuting the columns of $G$ by $\frac{n}{2}$ positions, which preserves toroidal distance vectors. Finally, by Table 2.10, vertical reflection is equivalent to horizontal reflection followed by $180^\circ$ rotation, and so preserves toroidal distance vectors. □

In general, two Welch Costas arrays that are inequivalent both under the action of $D_4$ and under cyclic column permutation do not contain the same multiset of toroidal distance vectors. For example, Figure 4.7 shows two inequivalent Welch Costas arrays (with letters instead of dots), and Table 4.8 lists their toroidal distance vectors of width 1, illustrating that the two multisets of toroidal distance vectors are not equal. However, as we show in Corollary 48, all Welch Costas
Figure 4.7: Two Welch Costas arrays inequivalent under the action of $D_4$ and under cyclic column permutation

<table>
<thead>
<tr>
<th></th>
<th>Vector in $W$</th>
<th>Vector in $W'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AB$</td>
<td>(1, 1)</td>
<td>(1, 7)</td>
</tr>
<tr>
<td>$BC$</td>
<td>(1, 2)</td>
<td>(1, 1)</td>
</tr>
<tr>
<td>$CD$</td>
<td>(1, 4)</td>
<td>(1, 7)</td>
</tr>
<tr>
<td>$DE$</td>
<td>(1, 7)</td>
<td>(1, 8)</td>
</tr>
<tr>
<td>$EF$</td>
<td>(1, 5)</td>
<td>(1, 6)</td>
</tr>
<tr>
<td>$FG$</td>
<td>(1, 9)</td>
<td>(1, 3)</td>
</tr>
<tr>
<td>$GH$</td>
<td>(1, 8)</td>
<td>(1, 9)</td>
</tr>
<tr>
<td>$HI$</td>
<td>(1, 6)</td>
<td>(1, 3)</td>
</tr>
<tr>
<td>$IJ$</td>
<td>(1, 3)</td>
<td>(1, 2)</td>
</tr>
<tr>
<td>$JA$</td>
<td>(1, 5)</td>
<td>(1, 4)</td>
</tr>
</tbody>
</table>

Table 4.8: Toroidal distance vectors of width 1 in the Welch Costas arrays $W$ and $W'$ of Figure 4.7
arrays of the same order contain the same number of distinct toroidal distance vectors (and so are missing the same number of toroidal distance vectors). In fact, for $p$ prime and $\phi$ a primitive element of $\mathbb{F}_p$, we establish a one to one correspondence between the toroidal distance vectors in an arbitrary Welch Costas array $W$ of order $p - 1$ and the toroidal distance vectors in the $W_1(p, \phi, 1)$ Welch Costas array. In order to prove Proposition 47, from which Corollary 48 follows, we make the following definition.

**Definition 45.** Let $A = (A_{i,j})$ be an $n \times n$ array and let $k \in \mathbb{Z}$ satisfy $\gcd(k, n) = 1$. The $k$-decimation of $A$ with respect to columns is the $n \times n$ array $(A_{i,(jk-1) \mod n + 1})$.

The index $(jk-1) \mod n + 1$ in Definition 45 is the unique integer in $\{1, \ldots, n\}$ that is congruent to $jk$ modulo $n$. This “adjustment” accounts for the fact that columns are numbered from 1 to $n$ rather than from 0 to $n - 1$. (Specifically, the adjustment is necessary for the case where $jk \mod n = 0$.) A more intuitive description of $k$-decimation is given by the observation that the $k$-decimation of $A$ with respect to columns is the array whose columns (in order) are column $k$ of $A$ followed by every $k$th column of $A$, wrapping around as necessary. This leads to the following remark.

**Remark 46.** The toroidal distance vector $(w, h)$ is contained in the $k$-decimation of $A$ with respect to columns exactly when the toroidal distance vector $((wk) \mod n, h)$ is contained in $A$. (We note that no adjustment is necessary in this context, since $0 < w < n$ and $\gcd(k, n) = 1$, so $(wk) \mod n \neq 0$.)

Corollary 48, which establishes the correspondence between the multisets of toroidal distance vectors for Welch Costas arrays, follows from the next result, concerning $k$-decimation of Welch Costas arrays.

**Proposition 47.** Let $p$ be prime and let $\phi$ and $\rho$ be primitive in $\mathbb{F}_p$. The $W_1(p, \rho, 1)$ Welch Costas array is the $k$-decimation with respect to columns of the $W_1(p, \phi, 1)$ Welch Costas array, where $\rho = \phi^k$.

**Proof.** Since $\phi$ and $\rho$ are both primitive in $\mathbb{F}_p$, $\gcd(k, p - 1) = 1$. By the Welch construction, for $1 \leq i, j \leq p - 1$, there is a dot at position $(i, j)$ in $W_1(p, \rho, 1)$ exactly when $i \equiv \rho^j \pmod{p}$, or equivalently, when $i \equiv \phi^{jk} \pmod{p}$. This occurs exactly when there is a dot at position $(i, \ell)$ in $W_1(p, \phi, 1)$, where $1 \leq \ell \leq p - 1$ and $\ell \equiv jk \pmod{p - 1}$. These conditions force

$$\ell = (jk - 1) \mod (p - 1) + 1.$$
The result then follows from Definition 45.

\[\square\]

**Corollary 48.** For \( p \) prime, let \( \phi \) be primitive in \( \mathbb{F}_p \) and let \( W \) be a Welch Costas array of order \( p-1 \). The multiset of toroidal distance vectors in \( W \) is in one to one correspondence with the multiset of toroidal distance vectors in the \( W_1(p,\phi,1) \) Welch Costas array.

**Proof.** Let \( W \) be the \( W_1(p,\rho,c) \) Costas array, where \( \rho \) is primitive in \( \mathbb{F}_p \) and \( 0 \leq c \leq p-2 \). Since cyclic column permutation does not affect toroidal distance vectors, \( W \) contains the toroidal distance vector \((w,h)\) exactly when \( W_1(p,\rho,1) \) does so. By Proposition 47, \( W_1(p,\rho,1) \) is the \( k \)-decimation with respect to columns of \( W_1(p,\phi,1) \), for \( k \) satisfying \( \rho = \phi^k \). Then by Remark 46, \( W_1(p,\rho,1) \) contains the toroidal distance vector \((w,h)\) if and only if \( W_1(p,\phi,1) \) contains the toroidal distance vector \(((wk) \mod (p-1), h)\). \(\square\)

We note that, in particular, Corollary 48 implies that all Welch Costas arrays of order \( p-1 \) contain the same number of distinct toroidal distance vectors. Moreover, all of their multisets of toroidal distance vectors have the same set of multiplicities.

Golomb Costas arrays do not exhibit the same regularity in the number of distinct toroidal distance vectors that Welch Costas arrays do. For primes \( p \) and prime powers \( q \) up to 40, Figure 4.9(a) shows the number of missing toroidal distance vectors in Welch Costas arrays of order \( p-1 \) and Figure 4.9(b) shows the minimum, mean and maximum number of missing toroidal distance vectors in Golomb Costas arrays of order \( q-2 \).

Analysis of the Costas array database reveals that the answer to Question 42 is \textit{no} for \( n \leq 29 \). In fact, the number of missing toroidal distance vectors appears to increase with \( n \). Figure 4.10 shows the minimum, mean and maximum number of missing toroidal distance vectors in Costas arrays of order \( n \) for \( 2 \leq n \leq 29 \).

These data on the numbers of missing toroidal distance vectors lead us to conjecture that the answer to Question 42 is \textit{no}.

**Conjecture 49.** No nontrivial Costas array contains every possible toroidal distance vector.

### 4.2 The difference parallelogram

While the Costas property can be expressed in many ways (see Section 2.2.1), the difference triangle is our primary tool for investigating the structure of Costas arrays, especially those that are
CHAPTER 4. COSTAS ARRAYS

(a) Number of missing toroidal distance vectors in Welch Costas arrays

(b) Number of missing toroidal distance vectors in Golomb Costas arrays

Figure 4.9: Number of missing toroidal distance vectors in Welch and Golomb Costas arrays up to order 40
Figure 4.10: Number of missing toroidal vectors in Costas arrays up to order 29
CHAPTER 4. COSTAS ARRAYS

not algebraically constructed. In this section, we present some results on the internal structure of the difference triangle for general permutations, describing relationships between the entries of the difference triangle. We will apply these results to our study of Costas arrays in Section 4.3.

However, our main result (Proposition 53) can be stated more naturally if we extend the difference triangle to include the differences \( \sigma(w + j) - \sigma(j) \) where \(-n + 1 \leq w \leq 0\). Although we will only require special cases of Proposition 53 derivable from the (unextended) difference triangle, we give the result in its more general form because this allows for a much cleaner statement and proof. To record these extra differences, we append extra rows to the difference triangle to form a difference parallelogram, whose name comes from its natural visual representation as a skew Young tableau.

**Definition 50.** Let \( \sigma \) be a permutation of \( \{1, \ldots, n\} \), for \( n \in \mathbb{N} \). The difference parallelogram \( P(\sigma) \) of \( \sigma \) is the set \( \{p_w(\sigma) : w = -n+1, -n+2, \ldots, n-1\} \), where \( p_w(\sigma) \) is the sequence \( \sigma(w + j) - \sigma(j) : j = \max(1, 1 - w), \ldots, \min(n, n - w) \). We call \( p_w(\sigma) \) the \( w \)th row of the difference parallelogram and we denote the \( j \)th element of \( p_w(\sigma) \) by \( p_{w,j}(\sigma) \).

It is easy to see that \( \{p_w(\sigma) : w = 1, \ldots, n - 1\} = T(\sigma) \), and that \( p_{-w}(\sigma) = -p_w(\sigma) \). Note that the entries of negative-numbered rows are not indexed starting at 1. For example, for \( \sigma = [1, 7, 4, 5, 3, 6, 8, 2] \), row \(-2\) of \( P(\sigma) \) is the sequence \((-3, 2, 1, -1, -5, 4)\), which comprises the differences \( \sigma(-2 + j) - \sigma(j) \) for \( j = 3, \ldots, n \). This convention is convenient when considering these entries in the context of the whole difference parallelogram, especially in its visual representation. As mentioned, we obtain this visual representation by letting the sequences \( p_w(\sigma) \) define the rows of a skew Young tableau, with \( p_0(\sigma) \) in the middle (containing the differences \( \sigma(j) - \sigma(j) \)), \( p_w(\sigma) \) for \( w > 0 \) below it (left justified) and \( p_w(\sigma) \) for \( w < 0 \) above it (right justified). The difference parallelogram of the Costas permutation \( \alpha = [1, 7, 4, 5, 3, 6, 8, 2] \) is shown in Figure 4.11.

Given this visual representation of \( P(\sigma) \) (and our convention for numbering the elements of negative rows of \( P(\sigma) \)), it is natural to define the \( j \)th column of \( P(\sigma) \) to be the sequence \( p_{w,j}(\sigma) : w = 1 - j, \ldots, n - j \) and the \( j \)th antidiagonal of \( P(\sigma) \) to be the sequence \( p_{j-k,k}(\sigma) : k = 1, \ldots, n \), for \( j = 1, \ldots, n \). Column \( j \) then contains the entries \( \sigma(k) - \sigma(j) \) and antidiagonal \( j \) contains the entries \( \sigma(j) - \sigma(k) \), for \( k = 1, \ldots, n \), and the first entry of antidiagonal \( j \) lies in row \( j - 1 \). For example, referring to Figure 4.11, column 2 of \( P(\sigma) \) is the sequence \((-6, 0, -3, -2, -4, -1, 1, -5)\), which comprises all of the differences \( \sigma(k) - 7 \) for \( k = 1, \ldots, 8 \), since \( \sigma(2) = 7 \). Antidiagonal 3 of \( P(\sigma) \) is the sequence \((3, -3, 0, -1, 1, -2, -4, 2)\), which comprises all of differences \( 4 - \sigma(k) \) for
Our first result on the difference parallelogram of a permutation $\sigma$ is a direct consequence of the permutation property. Its restriction to the difference triangle (Corollary 52) is instrumental in proving many of the results in Section 4.3. These results generalise a lemma presented in [23].

**Lemma 51.** Let $\sigma$ be a permutation of $\{1, \ldots, n\}$ and let $P(\sigma)$ be its difference triangle. Then the entries of every column of $P(\sigma)$ are distinct and the entries of every antidiagonal of $P(\sigma)$ are distinct.

**Proof.** Let $j$ satisfy $1 \leq j \leq n$. For $k = 1, \ldots, n$, column $j$ of $P(\sigma)$ contains the entries $\sigma(k) - \sigma(j)$ and antidiagonal $j$ contains the entries $\sigma(j) - \sigma(k)$. Since $\sigma$ is a permutation, the values $\sigma(k)$ are all distinct. □

**Corollary 52.** Let $\sigma$ be a permutation of $\{1, \ldots, n\}$ and let $T(\sigma)$ be its difference triangle. Then the entries of every column of $T(\sigma)$ are distinct and the entries of every antidiagonal of $T(\sigma)$ are distinct.

\begin{figure}[h]
\centering
\begin{tabular}{cccccccc}
-1 & -7 & 5 & & & & & \\
-5 & -1 & 2 & -2 & 1 & -4 & 3 & \\
-4 & 4 & -2 & -3 & 1 & & & \\
-3 & 2 & 1 & -1 & -5 & 4 & & \\
-6 & 3 & -1 & 2 & -3 & -2 & 6 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & -3 & 1 & -2 & 3 & 2 & -6 & \\
3 & -2 & 1 & 5 & -1 & 4 & -3 & \\
4 & -4 & 2 & 3 & 1 & & & \\
2 & -1 & 4 & & & & & \\
5 & 1 & -2 & & & & & \\
7 & -5 & & & & & & \\
1 & & & & & & & \\
\end{tabular}
\caption{Difference parallelogram of $[1, 7, 4, 5, 3, 6, 8, 2]$}
\end{figure}
Proof. For the given range for triangle. Then \( t \) use in Section 4.3. Corollary 54 deals with the case Proposition 53 for \( w \) as \( x) \) and \( z = w \). By Definition 50, Proof. Let \( n \) Corollary 55. Let \( \sigma \) be a permutation of \( \{1, \ldots, n\} \) and let \( x = p_{w,j}(\sigma) \) and \( y = p_{k,j+c}(\sigma) \), with \( w, k, c > 0 \). Define \( v = p_{w-c,j+c}(\sigma) \) (so that \( v \) is in the same column as \( y \) and the same antidiagonal as \( x \)) and \( z = p_{k+c,j}(\sigma) \) (so that \( z \) is in the same column as \( x \) and the same antidiagonal as \( y \)). Then \( x + y = v + z \).

Proof. By Definition 50, \( p_{w,j}(\sigma) = \sigma(w+j) - \sigma(j) \), so \( x + y = \sigma(w+j) + \sigma(k+j+c) - \sigma(j) - \sigma(j+c) = v + z \).

For example, the difference parallelogram of the (non-Costas) permutation \( \sigma = [4, 3, 6, 1, 5, 2, 7] \) is shown in Figure 4.12 with entries \( x = 3 \), \( y = 1 \), \( v = 5 \) and \( z = -1 \) highlighted. This illustrates Proposition 53 for \( w = 1 \), \( j = 2 \), \( k = 2 \) and \( c = 2 \).

For convenience, we restate two special cases of Proposition 53, both with \( w = k \), which we will use in Section 4.3. Corollary 54 deals with the case \( w = k = n - 2 \) and \( j = c = 1 \), and Corollary 55 deals with the case where \( v \) falls in \( p_{0}(\sigma) \). Both corollaries are stated as results about the difference triangle, since they will be used in arguments where negative rows of the difference parallelogram are not considered.

Corollary 54. Let \( \sigma \) be a permutation of \( \{1, \ldots, n\} \), for \( n \geq 4 \), and let \( T(\sigma) \) be its difference triangle. Then \( t_{n-3,2}(\sigma) = t_{n-2,1}(\sigma) + t_{n-2,2}(\sigma) - t_{n-1,1}(\sigma) \).

Corollary 55. Let \( n \geq 3 \) and let \( \sigma \) be a permutation of \( \{1, \ldots, n\} \) with difference triangle \( T(\sigma) \). Then, for \( w \in \{1, \ldots, n-2\} \), \( j \in \{1, \ldots, n-1-w\} \) and \( c \in \{1, \ldots, n-w-j\} \),

\[
 t_{w,j}(\sigma) + t_{w,j+c}(\sigma) = t_{w+c,j}(\sigma) \iff c = w.
\]

Proof. For the given range for \( w \), \( j \) and \( c \), we have \( t_{w,j}(\sigma) = p_{w,j}(\sigma) \), \( t_{w,j+c}(\sigma) = p_{w,j+c}(\sigma) \) and \( t_{w+c,j}(\sigma) = p_{w+c,j}(\sigma) \). Then by Proposition 53 with \( w = k \),

\[
 t_{w,j}(\sigma) + t_{w,j+c}(\sigma) = p_{w-c,j+c}(\sigma) + t_{w+c,j}(\sigma),
\]
and $p_{w-c,j+c}(\sigma) = 0$ if and only if $w - c = 0$. □

In Section 4.3.1 we present additional constraints on the difference triangle $T(\sigma)$ when $\sigma$ is G-symmetric.

## 4.3 Mirror pairs in Costas arrays

In this section we describe a structural feature of Costas arrays, which we call a **mirror pair**. We provide constraints on Costas arrays according to the number and type of their mirror pairs, with stronger constraints for G-symmetric Costas arrays (4.3.1), Welch Costas arrays (4.3.2), and Golomb Costas arrays (4.3.3). We discuss the application of mirror pair constraints to reducing the computational burden required for an exhaustive search for Costas arrays.

Consider the Costas arrays in Figure 4.13. In each Costas array, a pair of vectors of the form $((w, h), (w, -h))$ is highlighted. (For example, in Figure 4.13(b) the vectors $(3, 1)$ and $(3, -1)$ are shown). We call such a pair of vectors a **mirror pair**, as defined below.
CHAPTER 4. COSTAS ARRAYS

Definition 56. Let $A$ be a Costas array. A pair of vectors $(w, h), (w, -h)$ in $A$, where $w, h > 0$, is called a mirror pair of width $w$ and height $h$, or a $(w, h)$-mirror pair.

By Remark 12, the Costas array $A$ corresponding to the permutation $\alpha$ has a $(w, h)$-mirror pair if and only if $-h, h \in t_w(\alpha)$. We also note that a $(w, h)$-mirror pair in a Costas array $A$ must involve four distinct dots; the vectors $(w, h)$ and $(w, -h)$ cannot share a dot, as that would violate the permutation property of $A$. Consequently, we consider mirror pairs only for arrays of order $n \geq 4$.

Example 57. The difference triangle for the Costas array shown in Figure 4.13(b) is

\[
\begin{array}{cccccc}
3 & 2 & -6 & 5 & -3 & 1 \\
5 & -4 & -1 & 2 & -2 & \\
-1 & 1 & -4 & 3 & \\
4 & -2 & -3 & \\
1 & -1 & \\
2 & 
\end{array}
\]

The highlighted $(3, 1)$-mirror pair corresponds to the pair $-1, 1$ in the third row. This Costas array also has a $(1, 3)$-mirror pair, a $(2, 2)$-mirror pair and a $(5, 1)$-mirror pair.

Analysis of the database of Costas arrays up to order 29 reveals that for $4 \leq n \leq 29$, every Costas array of order $n$ contains a mirror pair. We will show that in fact every Costas array of order...
CHAPTER 4. COSTAS ARRAYS

$n \geq 4$ contains a mirror pair, using the following result of Freedman and Levanon [26]. The result was originally stated in terms of ambiguity sidelobes of Costas arrays, that is, the values of the cross-correlation function for nonzero shifts $(u,v)$. We state it here in terms of the vectors present in Costas arrays; by Proposition 14, the Costas property is equivalently characterised by conditions on its vectors and on its autocorrelation values. The proof we present is adapted from the proof presented in [23].

**Theorem 58.** Every pair of Costas arrays $A$, $B$ of order $n \geq 4$ has a vector in common.

**Proof.** Let $A$ and $B$ be Costas arrays of order $n \geq 4$, corresponding to permutations $\alpha$ and $\beta$ having difference triangles $T(\alpha)$ and $T(\beta)$, respectively. For $w = 1, \ldots, n - 1$, let $t_w(\alpha,\beta)$ be the multiset $\{t_{w,j}(\alpha), t_{w,j}(\beta) : j = 1, \ldots, n - w\}$ and let $T(\alpha,\beta) = \{t_w(\alpha,\beta) : w = 1, \ldots, n - 1\}$ (where $t_w(\alpha,\beta)$ is called the $w^{th}$ row of $T(\alpha,\beta)$). Assume, for a contradiction, that $A$ and $B$ do not have a vector in common. Then by Remark 12, for $w = 1, \ldots, n - 1$, the entries of $t_w(\alpha,\beta)$ are all distinct (that is, the multiset $t_w(\alpha,\beta)$ is a set). For $k = 1, \ldots, n - 1$, we prove by induction that

$$\{−k, k\} \subseteq t_w(\alpha,\beta)$$

(4.10) for $w = 1, \ldots, n - k$.

For the base case $k = 1$, Proposition 8 implies that $T(\alpha,\beta)$ contains exactly $2(n - 1)$ entries from $\{-1, 1\}$, distributed over its $n - 1$ rows. By assumption, no three such entries are contained in the same row of $T(\alpha,\beta)$, so, for $w = 1, \ldots, n - 1$, the row $t_w(\alpha,\beta)$ contains exactly two entries from $\{-1, 1\}$. In particular, by the Costas property, $\{-1, 1\} \subseteq t_w(\alpha,\beta)$ for $w = 1, \ldots, n - 1$. Assume that (4.10) holds for cases up to $k - 1$. By Proposition 8, $T(\alpha,\beta)$ contains exactly $2(n - k)$ entries from $\{-k, k\}$. By the inductive hypothesis, the $2i$ entries of $t_{n-i}(\alpha,\beta)$ are $\{-i, \ldots, -1, 1, \ldots, i\}$ for $1 \leq i \leq k - 1$, so entries from $\{-k, k\}$ occur only in the first $n - k$ rows of $T(\alpha,\beta)$. By assumption, no three such entries are contained in the same row of $T(\alpha,\beta)$, so, for $w = 1, \ldots, n - k$, row $t_w(\alpha,\beta)$ contains exactly two entries from $\{-k, k\}$. In particular, by the Costas property, $\{-k, k\} \subseteq t_w(\alpha,\beta)$, for $w = 1, \ldots, n - k$. This completes the induction.

It follows from (4.10) that $t_{n-k}(\alpha,\beta) = \{-k, \ldots, -1, 1, \ldots k\}$ for $k = 1, \ldots, n - 1$ and, in particular, that

$$t_{n-1}(\alpha,\beta) = \{-1, 1\} \text{ and } t_{n-2}(\alpha,\beta) = \{-2, -1, 1, 2\} \text{ and } t_{n-3}(\alpha,\beta) = \{-3, -2, -1, 1, 2, 3\}.$$  (4.11)

We may assume, without loss of generality, that $t_{n-1}(\alpha) = (1)$ and so $t_{n-1}(\beta) = (-1)$. By Corollary 52, $t_{n-2}(\alpha)$ does not contain the entry 1 and $t_{n-2}(\beta)$ does not contain the entry −1. Then the
entries of \( t_{n-2}(\alpha) \) are not \( \{-2,2\} \), as this would imply \(-1 \in t_{n-2}(\beta)\), and by Corollary 55, the entries of \( t_{n-2}(\alpha) \) are not \( \{-1,2\} \) since \( 1 \neq n - 2 \). Thus the entries of \( t_{n-2}(\alpha) \) are \( \{-2,-1\} \). Then by Corollary 54, \( t_{n-3}(\alpha) \) contains the entry \(-1 + (-2) - 1 = -4\), contradicting (4.11).  

Theorem 58 was used in [23] to rule out a proposed recursive construction technique for Costas arrays that involved “interlacing” two Costas arrays of order \( n \) together to produce an array of order \( 2n \). Our next result, which is the first existence result for mirror pairs, provides a new perspective on Theorem 58. Rather than discussing the vectors in two different Costas arrays, we use the result to prove constraints on the vectors within a single Costas array. In this way, we obtain information about the internal structure of Costas arrays.

**Corollary 59.** Every Costas array of order \( n \geq 4 \) has a mirror pair.

**Proof.** Let \( A \) be a Costas array of order \( n \geq 4 \). By Theorem 58, \( A \) and \( v(A) \), the vertical reflection of \( A \), have a common vector, \((w,h)\). Then, by Table 2.11, \( A = v(v(A)) \) contains the vector \((w,-h)\).

The existence of mirror pairs in Costas arrays appears to be a new observation. Moreover, data obtained from an analysis of the database of Costas arrays up to order 29, as presented in Figure 4.14, suggest that the number of mirror pairs in a Costas array of order \( n \) increases with \( n \). These observations represent a new avenue in the pursuit of structural constraints on Costas arrays, which has the potential to improve search times by reducing the search space. It is also possible that further results on mirror pairs will provide a deeper understanding of the structure of Costas arrays. This, in turn, could lead to a new construction method, or perhaps to the first nonexistence result for Costas arrays.

The number of mirror pairs in a Costas array \( A \) of order \( n \) is naively bounded above by \( \frac{1}{2} \binom{n}{2} \), which is obtained by assuming that every vector in \( A \) is involved in a mirror pair. (This bound can be improved slightly by considering the difference triangle of the permutation \( \alpha \) associated with \( A \), since row \( t_w(\alpha) \) contains at most \( \lfloor \frac{n-w}{2} \rfloor \) pairs \(-h,h\)).

While our focus here is on mirror pairs (vectors that mirror each other by vertical or horizontal reflection), we note in Proposition 60 that Theorem 58 can be similarly applied to arrays \( A \) and \( A' \), where \( A' \) is obtained from \( A \) by some other dihedral symmetry (for example, diagonal reflection or rotation by 90° counterclockwise), to ensure the existence of other kinds of vector pairs. These provide additional constraints on Costas arrays.
Proposition 60. Every Costas array of order $n \geq 4$ has a pair of vectors $(w_1, h_1)$, $\frac{h_1}{|h_1|}(h_1, -w_1)$ and a pair of vectors $(w_2, h_2)$, $\frac{h_2}{|h_2|}(h_2, w_2)$.

Proof. Let $A$ be a Costas array of order $n \geq 4$. Referring to Table 2.11, we see that the vector pair $(w_1, h_1)$, $\frac{h_1}{|h_1|}(h_1, -w_1)$ is obtained by applying Theorem 58 to the arrays $A$ and $A'$, where $A'$ is obtained from $A$ by $90^\circ$ or $270^\circ$ counterclockwise rotation, and the vector pair $(w_2, h_2)$, $\frac{h_2}{|h_2|}(h_2, w_2)$ is obtained by applying Theorem 58 to the arrays $A$ and $A''$, where $A''$ is obtained from $A$ by diagonal or antidiagonal reflection. \hfill $\square$

Returning our attention to (vertical/horizontal reflection) mirror pairs, we make the following remark, which can be verified using column 4 of Table 2.11.

Remark 61. Suppose that $A$ has a $(w, h)$-mirror pair. Then so does its image under vertical reflection, horizontal reflection and rotation by $180^\circ$. Its image under diagonal reflection, antidiagonal reflection and rotation by $90^\circ$ and $270^\circ$ has an $(h, w)$-mirror pair.
For example, the Costas array $A$ in Figure 4.15(a) has a $(3, 1)$-mirror pair, while its transpose $T(A)$ in Figure 4.15(b) has a $(1, 3)$-mirror pair.

![Figure 4.15: Mirror pairs under the action of $D_4$](image)

Table 2.11 also allows us to make a connection between mirror pairs in a Costas array $A$ and the cross-correlation of $A$ with its vertical reflection $v(A)$ (see Definition 3). The cross-correlation properties of Costas arrays and their images under horizontal and vertical reflections are discussed in [17], where bounds are given on the maximum value of $C_{A, v(A)}(u, v)$ over all shifts $(u, v)$ and on the value of $C_{A, v(A)}(0, 0)$.

**Proposition 62.** Let $A$ be a Costas array, let $V = v(A)$ be the image of $A$ under vertical reflection, and let $S = \{-n+1, -n+2, \ldots, n-1\}^2$. The number of mirror pairs in $A$ is

$$\frac{1}{2} \sum_{(u, v) \in S} \left( \frac{C_{A, v}(u, v)}{2} \right).$$

**Proof.** Consider a $(w, h)$-mirror pair in $A$ formed by the vector $(w, h)$ between $A_{i, j}$ and $A_{i+h, j+w}$ and the vector $(w, -h)$ between $A_{k, \ell}$ and $A_{k-h, \ell+w}$. By Table 2.11, the vector $(w, h)$ is contained in $V$ between $V_{k, n+1-\ell}$ and $V_{k-h, n+1-\ell-w}$, and the vector $(w, -h)$ is contained in $V$ between $V_{i, n+1-j}$ and $V_{i+h, n+1-j-w}$. Exactly one shift (namely $(u_1, v_1) = (k-h-i, n+1-j-\ell-w)$) maps the vector $(w, h)$ of $A$ onto the same vector in $V$, so the pair of dots forming the vector $(w, h)$ in $A$ contributes exactly 1 to $\left( \frac{\sum_{(u, v) \in S} C_{A, v}(u, v)}{2} \right)$ (namely when $(u, v) = (u_1, v_1)$). Further, exactly one shift (namely $(u_2, v_2) = (i+h-k, n+1-j-\ell-w)$) maps the vector $(w, -h)$ of $A$ onto the same
vector in \( V \), so the pair of dots forming the vector \((w, -h)\) contributes exactly 1 to \( \sum_{(u,v) \in S} \frac{C_{A,v}(u,v)}{2} \) (namely when \((u,v) = (u_2, v_2)\)). Therefore, the \((w, h)\)-mirror pair in \( A \) contributes exactly 2 to the sum \( \sum_{(u,v) \in S} \frac{C_{A,v}(u,v)}{2} \).

\[\square\]

**Example 63.** The Costas array \( A \) corresponding to the permutation \( \alpha = [1, 5, 3, 8, 7, 4, 6, 2] \) is shown with its difference triangle in Figure 4.16(a). The fourteen shaded entries of \( T(\alpha) \) reveal that \( A \) has exactly seven mirror pairs, with \((w, h) = (1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (4, 6) \) and \((5, 1)\). The cross-correlation array \( C_{A,v(A)} \) of \( A \) and its vertical reflection \( v(A) \) is shown in Figure 4.16(b), and we see that the entries of \( C_{A,v(A)} \) that contribute to the sum \( \sum_{(u,v) \in S} \frac{C_{A,v(A)}(u,v)}{2} \) are 2, 2, 4 and 4. Therefore

\[
\frac{1}{2} \sum_{(u,v) \in S} \left( \frac{C_{A,v(A)}(u,v)}{2} \right) = \frac{1}{2} (2(1) + 2(6)) = 7,
\]

in agreement with Proposition 62.

Figure 4.16(c) shows \( A \) and \( v(A) \), each with four dots highlighted. These sets of dots coincide at shift \((u, v) = (0, -1)\) (where \( A \) is shifted up by one row) to give \( C_{A,v(A)}(0, -1) = 4 \), which appears above the central zero in the cross-correlation array. Each of the \( \binom{6}{2} \) vectors between pairs of highlighted dots in \( A \) forms a mirror pair with the vector joining the pre-image (under the mapping \( v \)) of the corresponding dots in \( v(A) \).

The bound given in [17] on the maximum cross-correlation of a Costas array \( A \) and its vertical reflection \( v(A) \) is

\[
\max_{u,v} C_{A,v(A)}(u,v) \leq \left\lfloor \frac{n}{2} \right\rfloor, \tag{4.12}
\]

for \((u,v) \in \{-n+1, -n+2, \ldots, n-1\}^2\). Direct application of this bound to the result in Proposition 62 gives a much weaker bound on the number of mirror pairs in \( A \) than the naïve bound of \( \frac{1}{2} \binom{n}{2} \) discussed previously.

Further analysis of the database of Costas arrays up to order 29 shows that for \( 4 \leq n \leq 29 \), every Costas array of order \( n \) has a mirror pair of height 1 or a mirror pair of width 1, except for a single Costas array of order 6 (corresponding to the permutation \([2, 4, 5, 1, 6, 3]\)) which has neither. In fact, up to equivalence, the database contains only 8 Costas arrays of order greater than 3 having no mirror pair of width 1; these are displayed in Table 4.17. By Remark 61, a width 1 mirror pair in \( A \) implies a height 1 mirror pair in \( T(A) \). Figure 4.18(a) shows the minimum, mean and maximum number of width 1 mirror pairs in Costas arrays of order \( n \) for \( 4 \leq n \leq 29 \), and, for each
(a) A Costas array $A$ and its associated difference triangle (mirror pairs shaded)

(b) The cross-correlation array $C_{A,A} \circ (A)$ of $A$ and $\nu(A)$

(c) The dots of $A$ and $\nu(A)$ that coincide at shift $(0, -1)$

Figure 4.16: Illustration of Example 63
Table 4.17: The Costas arrays with no width 1 mirror pair
of these orders, Figure 4.18(b) shows the fraction of arrays attaining the minimum number of width 1 mirror pairs. In particular, for \(9 \leq n \leq 29\), every Costas array of order \(n\) has a width 1 mirror pair. The number of width 1 mirror pairs in a Costas array of order \(n\) is bounded above by \(\left\lfloor \frac{n-1}{2} \right\rfloor\); we note that the data displayed in Figure 4.18(a) suggest that a slightly lower bound of \(\frac{n^3}{2}\) may exist for arrays of odd order. These data were obtained by analysing all Costas arrays of each order (rather than inequivalent Costas arrays). Therefore, the minimum and maximum numbers of width 1 mirror pairs for a given order are also the minimum and maximum numbers of height 1 mirror pairs (by Remark 61), and we see that for \(9 \leq n \leq 29\) every Costas array of order \(n\) also contains a height 1 mirror pair.

These observations prompt the following question.

Question 64. Does every Costas array of order \(n \geq 9\) have a mirror pair of width 1 and a mirror pair of height 1?

Analysis of the Costas array database also shows that for \(7 \leq n \leq 29\) every Costas array of order \(n\) has a mirror pair of width 2 or a mirror pair of height 2, and for \(14 \leq n \leq 29\) every Costas array of order \(n\) has a mirror pair of width 2 and therefore, by Remark 61, a mirror pair of height 2. This leads to the following question.

Question 65. Does every Costas array of order \(n \geq 14\) have a mirror pair of width 2 and a mirror pair of height 2?

A partial answer to Questions 64 and 65 is given in Theorem 66. In Sections 4.3.1, 4.3.2 and 4.3.3 we will answer Question 64 for three sub-classes of Costas arrays.

Theorem 66. Every Costas array of order \(n \geq 6\) has a mirror pair of width 1 or 2 and a mirror pair of height 1 or 2.

Proof. We claim that every Costas array of order \(n \geq 6\) contains a height 1 mirror pair or a height 2 mirror pair. It then follows, for a Costas array \(A\) of order \(n \geq 6\), that both \(A\) and \(T(A)\) contain a mirror pair of height 1 or 2. This implies that every such Costas array \(A\) has a mirror pair of height 1 or 2 and a mirror pair of width 1 or 2.

To prove the claim, let \(A\) be a Costas array of order \(n \geq 6\), corresponding to permutation \(\alpha\), and suppose for a contradiction that \(A\) contains neither a mirror pair of height 1 nor a mirror pair of height 2. Then no row of \(T(\alpha)\) contains more than one entry from \([-1, 1]\) and no row of \(T(\alpha)\)
(a) Number of width 1 mirror pairs in Costas arrays up to order 29

(b) Fraction of Costas arrays attaining the minimum number of width 1 mirror pairs

Figure 4.18: Mirror pairs of width 1 in Costas arrays
contains more than one entry from \{-2, 2\}. Then, since \( T(\alpha) \) contains exactly \( n - 1 \) entries from \{-1, 1\} (by Proposition 8), it must contain exactly one such entry in each of the \( n - 1 \) rows. Now, since \( t_{n-1}(\alpha) \) contains only one entry (namely, \( \pm 1 \)) and \( T(\alpha) \) contains exactly \( n - 2 \) entries from \{-2, 2\}, there is exactly one entry from \{-2, 2\} in each of the first \( n - 2 \) rows. By Table 2.12, rotating \( A \) through \( 180^\circ \) reflects the rows of \( T(\alpha) \), and reflecting \( A \) in a horizontal axis negates the entries of \( T(\alpha) \). Further, by Remark 61, neither of these transformations changes the width or height of the mirror pairs present in \( A \). Thus, we may assume without loss of generality that \( t_{n-1}(\alpha) = (1) \) and \( t_{n-2}(\alpha) = (x, y) \), where \( x \in \{-1, 1\} \) and \( y \in \{-2, 2\} \). Applying Corollary 52 and Corollary 55 with \( c = 1 \), we see that in fact \( x = -1 \) and \( y = -2 \). Then by Lemma 54, \( t_{n-3}(\alpha) = (u, -4, v) \), where \( \{|u|, |v|\} = \{1, 2\} \). Now, by Corollary 52, \( u \notin \{-1, 1\} \) and \( v \neq 1 \), so \( (u, v) = (2, -1) \) or \((-2, -1) \). However, since \( n \geq 6 \), we have \( u + v + 1 \) by Corollary 55 with \( c = 2 \). This forces \( (u, v) = (-2, -1) \), so the last three rows of \( T(\alpha) \) are as shown below.

\[
\begin{array}{ccc}
-2 & -4 & -1 \\
-1 & -2 \\
1 &
\end{array}
\]

From these entries of \( T(\alpha) \), using (2.1), we obtain \( \alpha = [m, m + 3, m + 2, \ldots, m - 2, m - 1, m + 1] \), for some \( m \in \{3, \ldots, n - 3\} \). Then \( t_1(\alpha) \) contains the elements \((m + 2) - (m + 3) = -1 \) and \((m - 1) - (m - 2) = 1 \), yielding the desired contradiction. Thus \( A \) contains a mirror pair of height 1 or 2.

In fact, by examining all Costas arrays of order 4 and 5, we can obtain a slightly stronger result than Theorem 66, classifying all Costas arrays that do not have both a mirror pair of width 1 or 2 and a mirror pair of height 1 or 2. Up to equivalence, and for order at least 4, there is only one such Costas array. One member of this equivalence class, corresponding to the permutation \([2, 3, 5, 1, 4]\), is shown in Figure 4.19. The array shown has a height 1 mirror pair but no width 1 mirror pair and no width 2 mirror pair, as do three other members of its equivalence class. The remaining four members of the equivalence class have a width 1 mirror pair but no height 1 mirror pair and no height 2 mirror pair.

As noted in Remark 12, a \((1, h)\)-mirror pair in a Costas array \( A \) corresponds to an occurrence of the pair \(-h, h\) in \( t_1(\alpha) \), where \( \alpha \) is the permutation associated with \( A \). Thus, \( A \) contains no width 1 mirror pair if and only if the entries of \( t_1(\alpha) \) are all distinct in absolute value. Permutations with this property are called graceful permutations \([1]\), as defined below.
Definition 67. A permutation $\sigma$ of the set $\{1, 2, \ldots, n\}$ is a graceful $n$-permutation if

$$\{ |\sigma(2) - \sigma(1)|, |\sigma(3) - \sigma(2)|, \ldots, |\sigma(n) - \sigma(n - 1)| \} = \{1, 2, \ldots, n - 1\}.$$ 

We note that the difference triangle of a graceful permutation $\sigma$ does not necessarily have distinct entries in rows 2 through $n - 1$, so the graceful property does not imply the Costas property. For example, the permutation $[1, 7, 2, 6, 3, 5, 4]$, whose difference triangle is given in Figure 4.20, is a graceful permutation but not a Costas permutation.

The number of graceful $n$-permutations is denoted by $G(n)$; the sequence $G(n)$ is not well known, even asymptotically [1]. The first 29 terms of the sequence are given in the On-Line Encyclopedia of Integer Sequences [40], where $G(n)$ is sequence number A006967. It is known to grow exponentially, with an asymptotic lower bound of $\left(\frac{5}{3}\right)^n$ [2]. (This lower bound has been improved to $2.37^n$ [1].)

Since a Costas array corresponding to permutation $\alpha$ has no mirror pair of width 1 if and only if $\alpha$ is a graceful permutation, an affirmative answer to Question 64 is equivalent to the statement
that the set of Costas permutations and the set of graceful permutations are disjoint for \( n \geq 9 \). In that case, the size of the search space for Costas arrays of order \( n \) is bounded above by \( n! - G(n) \). Similarly (if the answer to Question 64 is yes), discoveries about the number of Costas arrays of order \( n \) could shed light on the sequence \( G(n) \).

In Sections 4.3.1, 4.3.2 and 4.3.3, we prove some stronger mirror pair results, including answering Question 64 in the affirmative, for G-symmetric Costas arrays of even order (with a partial answer for G-symmetric Costas arrays of odd order) and for Welch and Golomb Costas arrays. We partially answer Question 65 for G-symmetric Costas arrays (and therefore for Welch Costas arrays).

### 4.3.1 Mirror pairs in G-symmetric arrays

In this section, we answer Question 64 for G-symmetric Costas arrays of even order in two steps. We firstly show that every G-symmetric Costas array of order \( n \geq 4 \) has a mirror pair of width 1 (and that every G-symmetric Costas array of order \( n \geq 6 \) has a mirror pair of width 2, a partial answer to Question 65). Secondly, by arguing about the difference triangle of the associated permutation, we show that every G-symmetric Costas array of even order \( n > 6 \) has a mirror pair of height 1.

For G-symmetric Costas arrays, the existence of width 1 and 2 mirror pairs in Costas arrays of sufficient size follows easily from the definition of G-symmetry. Proposition 68 establishes a stronger result, giving the exact number of width 1 mirror pairs in a G-symmetric Costas array.

**Proposition 68.** Every G-symmetric Costas array of order \( n \geq 4 \) has exactly \( \frac{n}{2} - 1 \) mirror pairs of width 1.

**Proof.** Let \( G \) be a G-symmetric Costas array of order \( n \geq 4 \) corresponding to the permutation \( \gamma \). By G-symmetry, for \( 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \), the vector between the dots in columns \( j \) and \( j + 1 \) forms a mirror pair with the vector between the dots in columns \( \left\lfloor \frac{n}{2} \right\rfloor + j \) and \( \left\lfloor \frac{n}{2} \right\rfloor + j + 1 \), yielding \( \left\lfloor \frac{n}{2} \right\rfloor - 1 \) mirror pairs of width 1. To see that these are the only mirror pairs of width 1 in \( G \), we note that \( t_1(\gamma) \) contains exactly \( n - 1 \) entries, so \( G \) has at most \( \left\lfloor \frac{n-1}{2} \right\rfloor \) mirror pairs of width 1. This completes the proof for \( n \) even. For \( n \) odd, the above argument guarantees \( \frac{n-3}{2} \) mirror pairs of width 1 (involving the first \( \frac{n-3}{2} \) and last \( \frac{n-3}{2} \) entries of \( t_1(\gamma) \)). Since the two central entries of \( t_1(\gamma) \) (that is, entries \( t_{1,\frac{n-1}{2}}(\gamma) \) and \( t_{1,\frac{n+1}{2}}(\gamma) \)) arise from only three dots in the array, they cannot form a mirror pair as this would violate the permutation condition. \( \square \)
Similarly, Proposition 69 specifies (to within one) the number of mirror pairs of width 2 in a G-symmetric Costas array. This partially answers Question 65 for G-symmetric Costas arrays, guaranteeing a width 2 mirror pair but not necessarily a height 2 mirror pair.

**Proposition 69.** Every G-symmetric Costas array of order \( n \geq 6 \) has either \( \lfloor \frac{n-4}{2} \rfloor \) or \( \lfloor \frac{n-2}{2} \rfloor \) mirror pairs of width 2.

**Proof.** Let \( G \) be a G-symmetric Costas array of order \( n \geq 6 \) corresponding to the permutation \( \gamma \). By G-symmetry, for \( 1 \leq j \leq \lfloor \frac{n}{2} \rfloor - 2 \), the vector between the dots in columns \( j \) and \( j + 2 \) forms a mirror pair of width 2 with the vector between the dots in columns \( \lfloor \frac{n}{2} \rfloor + j \) and \( \lfloor \frac{n}{2} \rfloor + j + 2 \), so \( G \) has at least \( \lfloor \frac{n-4}{2} \rfloor \) mirror pairs of width 2. Since \( t_2(\gamma) \) contains exactly \( n - 2 \) entries, \( G \) has at most \( \lfloor \frac{n-2}{2} \rfloor \) mirror pairs. \(\square\)

As we saw in our discussion of mirror pairs in general Costas arrays (earlier in Section 4.3), the existence of a width 1 mirror pair in every Costas array of a given order \( n \) implies the existence of a height 1 mirror pair in every Costas array of order \( n \). However, we cannot use the same argument (involving \( T(A) \)) together with Proposition 68 to show that every G-symmetric Costas array \( G \) of order \( n \geq 4 \) also has a height 1 mirror pair, since, in general, \( T(G) \) is not G-symmetric. Nonetheless, in Theorem 75 we prove that every G-symmetric Costas array of even order \( n > 6 \) has a mirror pair of height 1. In preparation to prove this result (by contradiction), we establish several constraints on G-symmetric Costas arrays that do not have mirror pairs of height 1 in Lemmas 70, 72, and 74.

**Lemma 70.** Let \( G \) be a G-symmetric Costas array of even order \( n \) corresponding to the permutation \( \gamma \), and suppose that \( G \) has no mirror pair of height 1. Then the entries \( \{ \gamma(j) : j = 1, \ldots, \frac{n}{2} \} \) all have the same parity.

**Proof.** By G-symmetry, there is no vector of height 1 completely contained in either half of \( G \), as any such vector would be mirrored in the other half of \( G \) to form a mirror pair of height 1. Therefore, no two dots in the left half of \( G \) are in consecutive rows and no two dots in the right half of \( G \) are in consecutive rows. Then the dot in row \( i \) of \( G \) must occur in the opposite half from the dot in row \( i + 1 \) of \( G \). The result follows. \(\square\)

Lemma 70 implies that a permutation \( \gamma \) corresponding to a G-symmetric Costas array of even order \( n \) with no mirror pair of height 1 must have all even values from \( \{1, \ldots, n\} \) in its first half and all odd values in its second half, or vice versa.
Lemmas 72 and 74 provide constraints on the difference triangle of the permutation corresponding to a G-symmetric Costas array of even order. It is convenient to define four regions of the difference triangle that will be of particular interest.

**Definition 71.** Let \( \alpha \) be a permutation corresponding to a Costas array \( A \) of even order \( n \). We define \( T_1, T_2, T_3 \) and \( T_4 \) to be the regions of \( T(\alpha) \) as illustrated below.

![Diagram of regions](image)

**Lemma 72.** Let \( G \) be a G-symmetric permutation matrix of even order \( n \) corresponding to the permutation \( \gamma \), and suppose that the first \( \frac{n}{2} \) entries of \( \gamma \) have the same parity. Then

(i) The entries in \( T_1 \) are even and \( T_2 = -T_1 \),

(ii) The entries in \( T_3 \) are odd and row \( \frac{n}{2} - w \) of \( T_4 \) is row \( w \) of \( T_3 \), and

(iii) The entries in \( t_\frac{n}{2}(\gamma) \) are \( \{n - 3, n - 7, \ldots, 1 - n\} \) if \( \gamma(1) \) is even and \( \{n - 1, n - 5, \ldots, 3 - n\} \) if \( \gamma(1) \) is odd.

**Proof.** For (i), let \( 1 \leq w \leq \frac{n}{2} - 1 \). By (2.1), row \( w \) of \( T_1 \) is the sequence \( (\gamma(w + j) - \gamma(j) : j = 1, \ldots, \frac{n}{2} - w) \). Since \( \gamma(w + j) \) and \( \gamma(j) \) have the same parity by assumption, the entries of \( T_1 \) are all even. The \( w \)th row of \( T_2 \) is the sequence

\[
(\gamma(w + j) - \gamma(j) : j = \frac{n}{2} + 1, \ldots, n - w) = (\gamma(\frac{n}{2} + w + j) - \gamma(\frac{n}{2} + j) : j = 1, \ldots, \frac{n}{2} - w)
\]

\[
= (n + 1 - \gamma(w + j) - (n + 1 - \gamma(j)) : j = 1, \ldots, \frac{n}{2} - w),
\]

where the last step follows from the definition of G-symmetry. Therefore \( T_2 = -T_1 \).

For (ii), let \( 1 \leq w \leq \frac{n}{2} - 1 \). By (2.1), row \( w \) of \( T_3 \) is the sequence

\[
(\gamma(w + j) - \gamma(j) : j = \frac{n}{2} + 1 - w, \ldots, \frac{n}{2}).
\]
Since $\gamma(w + j)$ and $\gamma(j)$ have opposite parity for each $j$ by assumption, the entries in row $w$ of $T_3$ are odd. Row $\frac{n}{2} - w$ of $T_4$ is the sequence

$$
(\gamma(n - w + j) - \gamma(j) : j = 1, \ldots, w) = (\gamma(j + \frac{n}{2}) - \gamma(w + j - \frac{n}{2}) : j = \frac{n}{2} + 1 - w, \ldots, \frac{n}{2})
$$

$$
= (n + 1 - \gamma(j) - (n + 1 - \gamma(w + j)) : j = \frac{n}{2} + 1 - w, \ldots, \frac{n}{2}),
$$

where the last step follows from the definition of G-symmetry. Finally, for (iii), we note that

$$
t_{\frac{n}{2}}(\gamma) = (\gamma(j + \frac{n}{2}) - \gamma(j) : j = 1, \ldots, \frac{n}{2})
$$

$$
= (n + 1 - 2\gamma(j) : j = 1, \ldots, \frac{n}{2}),
$$

by G-symmetry. □

**Example 73.** Consider the (non-Costas) permutation $\gamma = [2, 6, 4, 8, 10, 9, 5, 7, 3, 1]$, which satisfies the conditions of Lemma 72 (that is, $\gamma$ corresponds to a G-symmetric permutation matrix and its first five entries are all even). The difference triangle of $\gamma$ is given in Figure 4.21.

![](image)

**Figure 4.21: Difference triangle of a permutation satisfying the conditions of Lemma 72**

**Lemma 74.** Let $G$ be a G-symmetric Costas array of order $n$ corresponding to the permutation $\gamma$, and suppose that the first $\frac{n}{2}$ entries of $\gamma$ all have the same parity. Then for $w = 1, \ldots, \frac{n}{2} - 1$,

$$
\{ |k| : k \text{ is in row } w \text{ of } T_1 \} = \{2, 4, \ldots, n - 2w\}. 
$$
CHAPTER 4. COSTAS ARRAYS

Proof. By Proposition 8, for \( k = 1, \ldots, \frac{n}{2} - 1 \), the difference triangle \( T(\gamma) \) contains exactly \( n - 2k \) entries of magnitude \( 2k \). Further, by Lemma 72, these even entries of \( T(\gamma) \) are exactly the entries of \( T_1 \) and \( T_2 \). Then, since \( T_2 = -T_1 \), \( T_1 \) contains exactly \( \frac{n}{2} - k \) entries of magnitude \( 2k \) distributed over its \( \frac{n}{2} - 1 \) rows, with no two such entries in the same row. The result now follows by a pigeonhole argument.

We are now ready to classify the G-symmetric Costas arrays of even order \( n \geq 4 \) that have a mirror pair of height 1.

**Theorem 75.** The only G-symmetric Costas arrays of even order \( n \geq 4 \) with no mirror pair of height 1 are the arrays corresponding to the permutations \([3, 1, 2, 4]\) and \([2, 6, 4, 5, 1, 3]\) and their images under vertical reflection, horizontal reflection and 180° rotation.

**Proof.** Let \( G \) be a G-symmetric Costas array of even order \( n \geq 4 \), corresponding to the permutation \( \gamma \), and suppose that \( G \) has no mirror pair of height 1. By Proposition 8, \( T(\gamma) \) has exactly \( n - 1 \) entries from \( \{-1, 1\} \), and by assumption no two are in the same row. Thus,

\[
eachrowT\gamma\textrm{containsexactlyoneentryfrom}\{-1,1\}. \quad (4.13)
\]

Since vertical and horizontal reflection and 180° rotation preserve G-symmetry, by Table 2.12, we may assume without loss of generality that \( t_{n-1,1}(\gamma) = 1 \) and subsequently, using Table 2.12 and Corollary 52, that \( t_{n-2,1}(\gamma) = -1 \). Letting \( \gamma(1) = m \), we then have \( \gamma(n-1) = m-1 \) and \( \gamma(n) = m+1 \). This, in turn, gives \( t_{1,n-1}(\gamma) = 2 \) and, by Lemma 72(i), \( t_{1,\frac{n}{2}-1}(\gamma) = -2 \).

In the case \( n = 4 \), this last conclusion implies that \( t_{1,1}(\gamma) = -2 \) and so \( \gamma = [m, m-2, m-1, m+1] \), forcing \( \gamma = [3, 1, 2, 4] \).

For \( n > 4 \), Lemmas 74 and 52 together give \( t_{\frac{n}{2}-1,1}(\gamma) = 2 \), and subsequently, by Lemma 72(i), \( t_{\frac{n}{2}-1,\frac{n}{2}+1}(\gamma) = -2 \). For \( n = 6 \), this implies that \( \gamma = [m, m+4, m+2, m+3, m-1, m+1] \), which forces \( \gamma = [2, 6, 4, 5, 1, 3] \).

It is easily verified that in both cases \( n = 4 \) and \( n = 6 \), \( G \) is a G-symmetric Costas array but its transpose is not, giving the eight listed examples.

For \( n \geq 8 \), we let \( x = t_{1,1}(\gamma) \) and \( y = t_{n-3,2}(\gamma) \) (see Figure 4.22).

By the Costas property, \( x \neq -2 \), so \( \gamma(2) \neq m-2 \). In turn, \( y = \gamma(n-1) - \gamma(2) \neq 1 \), which forces \( t_{n-3,3}(\gamma) = -1 \) by (4.13) and Corollary 52. This implies that \( \gamma(3) = m+2 \), which then gives \( t_{2,1}(\gamma) = 2 \), a contradiction to Corollary 52 since \( 2 \neq \frac{n}{2} - 1 \). □
The eight exceptional Costas arrays of Theorem 75 belong to equivalence classes of size 8. The other four Costas arrays in each equivalence class (obtained from the arrays corresponding to $[3, 1, 2, 4]$ and $[2, 6, 4, 5, 1, 3]$ by $90^\circ$ or $270^\circ$ rotation or diagonal or antidiagonal reflection) are not G-symmetric, as noted in the proof of Theorem 75, but are guaranteed by Proposition 68 to have mirror pairs of height 1. By Remark 61, these arrays lack width 1 mirror pairs; they are represented by the arrays corresponding to permutations $[1, 4, 2, 3]$ and $[2, 4, 3, 6, 1, 5]$ in Table 4.17.

We now give an affirmative answer to Question 64 for G-symmetric Costas arrays of even order. This follows from Proposition 68 and Theorem 75.

**Remark 76.** Every G-symmetric Costas array of even order $n \geq 8$ has a mirror pair of width 1 and a mirror pair of height 1. (The answer to Question 64 is yes for G-symmetric Costas arrays of even order.)

### 4.3.2 Mirror pairs in Welch Costas arrays

Recall from Section 2.2.3 that every Welch Costas array is G-symmetric. In this section, we restate the results from Section 4.3.1 for Welch Costas arrays. Further, we use some results from Section 4.1 to prove a stronger constraint on the mirror pairs in Welch Costas arrays, namely that every Welch Costas array of sufficient size has a $(w, h)$-mirror pair with $w + h \leq 3$. 
Our first result on mirror pairs in Welch Costas arrays classifies the Welch Costas arrays of order \( n \geq 4 \) that have both a mirror pair of width 1 and a mirror pair of height 1. This result follows from Proposition 68 and Theorem 75.

**Corollary 77.** For \( p \geq 5 \), every \( W_1(p, \phi, c) \) Costas array has exactly \( \frac{p-3}{2} \) mirror pairs of width 1 and, for \( (p, \phi, c) \notin \{ (5, 2, 3), (5, 2, 1), (5, 3, 2), (5, 3, 0), (7, 3, 2), (7, 5, 5), (7, 5, 0), (7, 5, 2) \} \), at least one mirror pair of height 1.

**Proof.** Since every \( W_1(p, \phi, c) \) array is a G-symmetric Costas array of order \( p-1 \), the first assertion follows from Proposition 68. Let \( G \) and \( G' \) be the Costas arrays corresponding to the permutations \([3, 1, 2, 4]\) and \([2, 6, 4, 5, 1, 3]\), respectively. By Theorem 75, a Welch Costas array of order at least 4 having no mirror pair of height 1 must be the image of \( G \) or \( G' \) under the identity element of \( D_4 \), horizontal or vertical reflection, or 180° rotation. It can be verified that the four sets of parameters \((5, 2, 3), (5, 2, 1), (5, 3, 2) \) and \((5, 3, 0) \) yield the four images of \( G \), respectively, while the four sets of parameters \((7, 3, 2), (7, 3, 5), (7, 5, 5) \) and \((7, 5, 2) \) yield the four images of \( G' \), respectively. \( \square \)

The answer to Question 64 for Welch Costas arrays now follows from Corollary 77.

**Remark 78.** For \( p > 7 \) every \( W_1(p, \phi, c) \) Welch Costas array has a mirror pair of width 1 and a mirror pair of height 1. (The answer to Question 64 is yes for Welch Costas arrays.)

The G-symmetry property of Welch Costas arrays leads to a partial answer to Question 65 for Welch Costas arrays.

**Corollary 79.** For \( p \geq 5 \) every \( W_1(p, \phi, c) \) Costas array has either \( \frac{p-5}{2} \) or \( \frac{p-3}{2} \) mirror pairs of width 2.

**Proof.** The result follows from Proposition 69. \( \square \)

As mentioned, constraints on the number, width and height of mirror pairs present in Costas arrays may be useful in reducing the computational burden in exhaustive searches for Costas arrays. We have proved several existence results for mirror pairs of small width or height, which provide corresponding constraints on the entries of the difference triangle. For example, a mirror pair of height 1 corresponds to a pair of entries \(-1, 1\) in the same row of the difference triangle. A stronger constraint can be obtained by restricting the row(s) of the difference triangle in which these \(-1, 1\)
entries appear. This is equivalent to obtaining simultaneous constraints on the width and height of mirror pairs present in Costas arrays.

In Theorem 81, we show that every Welch Costas array has a mirror pair of small overall length (that is, a \((w, h)\)-mirror pair with \(w + h \leq 3\)). The proof of this result relies on Theorem 38 (from Section 4.1) and Lemma 80 below. In Section 4.3.3, we will prove analogous results for Golomb Costas arrays.

Theorem 38 shows that each of the toroidal distance vectors \((1, 1)\), \((1, p-1)\), \((2, 1)\) and \((2, p-1)\) occurs in an augmented \(W_1(p, \phi, c)\) Welch Costas array, provided \(p \geq 5\). We will use this result to prove (in Lemma 80) that certain pairs of (non-toroidal) vectors from \{\((1, 1), (1, -1), (2, 1), (2, -1)\}\} are present in the array. We will then show (in Proposition 81) that every sufficiently large Welch Costas array has a mirror pair of small width and height. The proof techniques for these results are based on ideas found in [18].

**Lemma 80.** For \(p \geq 5\), every \(W_1(p, \phi, c)\) Welch Costas array contains a distance vector from \{\((1, 1), (2, 1)\}\} and a distance vector from \{\((1, -1), (2, -1)\}\}.

**Proof.** Let \(W\) be a \(W_1(p, \phi, c)\) Welch Costas array, with \(p \geq 5\), and let \(W^+\) be the array obtained by adding an empty row to the bottom of \(W\). We firstly show that at least one of the toroidal distance vectors \((1, 1)\) and \((2, 1)\) occurs in \(W\). Suppose for a contradiction that \(W\) contains neither of these vectors. By Theorem 38, each of the toroidal distance vectors \((1, 1)\) and \((2, 1)\) is present in \(W^+\). Then the toroidal distance vectors \((1, 1)\) and \((2, 1)\) must both cross a boundary in \(W^+\). Since the horizontal boundary of \(W^+\) is adjacent to an empty row, no toroidal distance vector of height 1 can cross this boundary. Thus the toroidal vector \((1, 1)\) must occur between columns \(n\) and 1 of \(W^+\) and \((2, 1)\) must occur either between columns \(n-1\) and 1 or between columns \(n\) and 2. In either case, the permutation property of \(W\) dictates that the toroidal distance vectors \((1, 1)\) and \((2, 1)\) must share a dot. However, since they both have height 1, this means that the two unshared endpoints of these vectors lie in the same row of \(W^+\) (in adjacent columns), which violates the permutation condition of \(W\) and yields the desired contradiction. It remains to show that at least one of \((1, -1)\) and \((2, -1)\) occurs in \(W\). Since these vectors correspond to the toroidal distance vectors \((1, p-1)\) and \((2, p-1)\) in \(W^+\), the result follows from Theorem 38 and symmetry. \(\square\)

**Theorem 81.** For \(p \geq 5\), every \(W_1(p, \phi, c)\) Welch Costas array has a \((1, 1)\) mirror pair, a \((1, 2)\) mirror pair, or a \((2, 1)\) mirror pair.
Proof. Let $p \geq 5$ and let $W$ be a $W_1(p, \phi, c)$ Welch Costas array. By Lemma 80, $W$ contains a distance vector from $\{(1, 1), (2, 1)\}$ and a distance vector from $\{(1, -1), (2, -1)\}$. If it contains $(1, 1)$ and $(1, -1)$, or $(2, 1)$ and $(2, -1)$, we are done. Assume, then, that $W$ contains $(1, 1)$ and $(2, -1)$ but not $(1, -1)$ or $(2, 1)$. By G-symmetry, the vectors $(1, 1)$ and $(2, -1)$ both straddle the centre of the array, with dots in positions $(i, \frac{p-1}{2})$, $(i + 1, \frac{p-1}{2} + 1)$ and either $(i - 1, \frac{p-1}{2} + 2)$ (in which case the dot at $(i, \frac{p-1}{2})$ is common to both vectors) or $(i + 2, \frac{p-1}{2} - 1)$ (in which case the dot at $(i + 1, \frac{p-1}{2} + 1)$ is common to both vectors). In both cases, the two unshared endpoints of these vectors lie both in the right half or both in the left half of the array and are separated by the vector $(1, 2)$. By G-symmetry, this yields a $(1, 2)$-mirror pair. By symmetry, the result also holds for the case where $W$ contains the distance vectors $(1, -1)$ and $(2, 1)$ but not $(1, 1)$ or $(2, -1)$. □

4.3.3 Mirror pairs in Golomb Costas arrays

As we did for Welch Costas arrays in Theorem 81, we use results from Section 4.1 to show that every sufficiently large Golomb Costas array contains a mirror pair with constrained width and height. To wit, we show that every sufficiently large Golomb Costas array has a $(w, h)$-mirror pair with $(w, h) \in \{(1, 1), (1, 2), (1, 3)\}$. We use this stronger result to prove that every sufficiently large Golomb Costas array has a mirror pair of width 1 and a mirror pair of height 1, answering Question 64 for Golomb Costas arrays.

Recall from Section 2.2.3 that a $G_2(q, \phi, \rho)$ Golomb Costas array $G$ of order $q - 2$ has a dot at $(i, j)$ if and only if $\phi^i + \rho^j = 1$ in $\mathbb{F}_q$, where $q$ is a prime power and $\phi$ and $\rho$ are primitive elements of $\mathbb{F}_q$. Lemma 82 establishes some necessary conditions for the absence of certain vectors from $G$. This result, in turn, will be used to show that every Golomb Costas array of sufficient size has a $(w, h)$-mirror pair with $(w, h) \in \{(1, 1), (1, 2), (1, 3)\}$. The proof of Lemma 82 relies on Theorem 40, which establishes the conditions under which a given toroidal distance vector $(w, h)$ appears in an augmented Golomb Costas array $G^+_a$ and determines the position of the vector if it does appear. For certain vectors $(w, h)$, we now describe the conditions that $\phi$ and $\rho$ must satisfy if the (non-toroidal) vector $(w, h)$ is absent from the non-augmented array $G$. To do this, we consider the various ways in which $(w, h)$ may be absent from $G$, namely if the toroidal vector $(w, h)$ is absent from $G^+_a$ or if it crosses a boundary of $G^+_a$.

Lemma 82. [18] Let $G$ be a $G_2(q, \phi, \rho)$ Costas array.

(i) For $q \geq 4$, if the vector $(1, 1)$ does not appear in $G$ then $\rho = \phi$. 
(ii) For $q \geq 4$, if the vector $(1, -1)$ does not appear in $G$ then $\rho = \phi^{-1}$.

(iii) For $q \geq 5$, if the vector $(1, 2)$ does not appear in $G$ then $\rho = \phi^2$ or $\phi + \rho = 0$.

(iv) For $q \geq 5$, if the vector $(1, -2)$ does not appear in $G$ then $\rho = \phi^{-2}$ or $\phi + \rho^{-1} = 0$.

(v) For $q \geq 7$, if the vector $(1, 3)$ does not appear in $G$ then $\rho = \phi^3$ or $\phi^2 + \phi + \rho = 0$ or $\phi^2 + \phi \rho + \rho = 0$.

(vi) For $q \geq 7$, if the vector $(1, -3)$ does not appear in $G$ then $\rho = \phi^{-3}$ or $\phi^2 + \phi + \rho^{-1} = 0$ or $\phi^2 + \phi \rho^{-1} + \rho^{-1} = 0$.

Proof. We prove parts (i), (iii) and (v). Parts (ii), (iv) and (vi) then follow by replacing $\rho$ with $\rho^{-1}$, since the array $G_2(q, \phi, \rho^{-1})$ is the vertical reflection of the array $G_2(q, \phi, \rho)$, and $v(G)$ contains the vector $(w, h)$ if and only if $G$ contains the vector $(w, -h)$.

To prove parts (i), (iii) and (v), suppose that for some $h \in \{1, 2, 3\}$ and $q - 2 \geq h + 1$, the distance vector $(1, h)$ does not appear in $G$. We distinguish two cases.

**Case 1:** The toroidal distance vector $(1, h)$ does not appear in $G^+$. Then $\phi = \rho^h$ by Theorem 40, which is one of the possibilities listed as a conclusion in (i), (iii) and (v).

**Case 2:** The toroidal distance vector $(1, h)$ appears in $G^+$. Since the distance vector $(1, h)$ does not appear in $G$, the toroidal distance vector $(1, h)$ crosses a boundary in $G^+$. Suppose this toroidal distance vector starts at $G_{i,j}$. Then by (4.5),

$$(1 - \phi) (\phi^h - \rho) = \phi^h - 1,$$

or equivalently, multiplying by $\phi^{-i}$,

$$\phi^h - \phi^{-i} + \phi^{-i} \rho - \rho = 0. \quad (4.14)$$

Now, since the vector $(1, h)$ is absent from $G$, it must cross a boundary of $G^+$. Since the vertical boundary is adjacent to an empty column, no vector of width 1 can cross this boundary. Thus the absence of the vector $(1, h)$ from $G$ implies that the solution $i$ to (4.14) satisfies

$$i < q - 1 < i + h. \quad (4.15)$$

For (i), take $h = 1$ in (4.15) to give a contradiction, as required.
For (iii), take $h = 2$ in (4.15) to give $i = q - 2$. Then (4.14) gives

$$\phi^2 - \phi + \phi\rho - \rho = 0$$

$$\iff (\phi + \rho)(\phi - 1) = 0,$$

which implies that $\phi + \rho = 0$, as required.

For (v), take $h = 3$ in (4.15) to give $i = q - 3$ or $i = q - 2$. In the case that $i = q - 3$, (4.14) gives

$$\phi^3 - \phi^2 + \phi^2\rho - \rho = 0$$

$$\iff (\phi - 1)(\phi^2 + \phi\rho + \rho) = 0,$$

which implies that $\phi^2 + \phi\rho + \rho = 0$. Otherwise, $i = q - 2$ and then (4.14) gives

$$\phi^3 - \phi + \phi\rho - \rho = 0$$

$$\iff (\phi - 1)(\phi^2 + \phi + \rho) = 0,$$

which implies that $\phi^2 + \phi + \rho = 0$. □

An extended version of Lemma 82 was used in [18] to prove that any two sufficiently large Golomb Costas arrays have a common vector from a specific set of eight distance vectors. Rather than comparing vectors present in two different Costas arrays, we will use the result, and other ideas from [18], to constrain the mirror pairs present in a single Golomb Costas array. In particular, we show that every sufficiently large Golomb Costas array has a $(w, h)$-mirror pair with $(w, h) \in \{(1, 1), (1, 2), (1, 3)\}$. We note that $G$ has a $(1, h)$-mirror pair unless at least one of $(1, h)$ or $(1, -h)$ is absent. For each $h \in \{-3, -2, -1, 1, 2, 3\}$, Lemma 82 describes exactly when the vector $(1, h)$ is absent from a $G_2(q, \phi, \rho)$ Costas array $G$ of sufficient size by providing a set of conditions on $\phi$ and $\rho$. In Theorem 83, we combine conditions from Lemma 82 to determine exactly when $G$ has no $(1, h)$-mirror pair for all $h \in \{1, 2, 3\}$, thereby classifying the Golomb Costas arrays that have such a mirror pair. (We restrict to $q \geq 7$ since Costas arrays of order $n \leq 3$ are trivially lacking mirror pairs.)

**Theorem 83.** Let $G$ be a $G_2(q, \phi, \rho)$ Golomb Costas array, $q \geq 7$, and suppose that $G$ has no $(1, h)$-mirror pair for all $h \in \{1, 2, 3\}$. Then $G$ corresponds to the permutation $[5, 3, 2, 6, 1, 4]$, $[3, 6, 1, 5, 4, 2]$, $[4, 1, 6, 2, 3, 5]$ or $[2, 4, 5, 1, 6, 3]$. 
Proof. We are given that $G$ has no $(1, h)$-mirror pair with $h \in \{1, 2, 3\}$. Then by parts (i) and (ii) of Lemma 82,

$$\rho = \phi \quad \text{or} \quad \rho = \phi^{-1},$$

(4.16)

and by parts (iii) and (iv) of Lemma 82,

$$\rho = \phi^2 \quad \text{or} \quad \phi + \rho = 0,$$

or

$$\rho = \phi^{-2} \quad \text{or} \quad \phi + \rho^{-1} = 0,$$

(4.17)

and by parts (v) and (vi) of Lemma 82,

$$\rho = \phi^3 \quad \text{or} \quad \phi^2 + \phi + \rho = 0 \quad \text{or} \quad \phi^2 + \phi + \rho + \rho = 0$$

or

$$\rho = \phi^{-3} \quad \text{or} \quad \phi^2 + \phi + \rho^{-1} = 0 \quad \text{or} \quad \phi^2 + \phi + \rho^{-1} + \rho^{-1} = 0.$$

(4.18)

We consider all possible cases. The two main cases, given by (4.16), are $\rho = \phi$ and $\rho = \phi^{-1}$.

We first consider the case $\rho = \phi$. By (4.17), there are four subcases.

a. $\rho = \phi^2$. Since $\rho = \phi$, this implies that $\phi = \phi^2$, so $q = 2$, a contradiction (since $\phi$ is primitive).

b. $\rho = \phi^{-2}$. This implies that $\phi^3 = 1$, so $q = 2$ or $4$, a contradiction.

c. $\phi + \rho^{-1} = 0$. This gives $\phi = -\phi^{-1}$, so $\phi^2 = -1$, implying that $q = 5$, a contradiction.

d. $\phi + \rho = 0$. This implies that $2\phi = 0$, so $q$ is a power of $2$. By (4.18), there are six further subcases.

i. $\rho = \phi^3$. This implies that $\phi = \phi^3$, so $\phi^2 = 1$ and thus $q = 3$, a contradiction.

ii. $\rho = \phi^{-3}$. This gives $\phi = -\phi^{-3}$, so $\phi^4 = 1$. Since $q$ is a power of $2$, this forces $q = 2$, a contradiction.

iii. $\phi^2 + \phi + \rho = 0$. This gives $\phi^2 = 0$, a contradiction.

iv. $\phi^2 + \phi + \rho = 0$. This gives $\phi = 0$, a contradiction.

v. $\phi^2 + \phi + \rho^{-1} = 0$. This gives $\phi^3 + \phi^2 + 1 = 0$. Since the polynomial $x^3 + x^2 + 1$ is irreducible over $\mathbb{F}_2$, it is the minimal polynomial of $\phi$, so $q = 8$. Then $G$ is the $G_2(8, \phi, \phi)$ Costas array where $\phi^3 + \phi^2 + 1 = 0$. This generates the Costas permutation $[5, 3, 2, 6, 1, 4]$.

vi. $\phi^2 + \phi + \rho^{-1} + \rho^{-1} = 0$. This gives $\phi^3 + \phi + 1 = 0$. Since the polynomial $x^3 + x + 1$ is irreducible over $\mathbb{F}_2$, it is the minimal polynomial of $\phi$, so $q = 8$. Then $G$ is the $G_2(8, \phi, \phi)$ Costas array where $\phi^3 + \phi + 1 = 0$. This generates the Costas permutation $[3, 6, 1, 5, 4, 2]$. 
CHAPTER 4. COSTAS ARRAYS

This concludes the case $\rho = \phi$. For the case $\rho = \phi^{-1}$, we note that the array $G_2(q, \phi, \rho^{-1})$ is the vertical reflection of the array $G_2(q, \phi, \rho)$. Thus this case yields the vertical reflection of each of the arrays obtained in the previous case. These arrays correspond to the Costas permutations $[4, 1, 6, 2, 3, 5]$ and $[2, 4, 5, 1, 6, 3]$.

The set of Costas arrays corresponding to the permutations listed in Theorem 83 is closed under the transpose operation. Therefore, unlike Welch Costas arrays (or G-symmetric Costas arrays of even order), for which two separate arguments were required to classify the arrays containing a width 1 mirror pair and those containing a height 1 mirror pair, the following classification of Golomb Costas arrays containing a $(w, 1)$-mirror pair with $w \in \{1, 2, 3\}$ is a direct consequence of Theorem 83.

**Corollary 84.** Let $G$ be a $G_2(q, \phi, \rho)$ Golomb Costas array, $q \geq 7$, and suppose that $G$ has no $(w, 1)$-mirror pair with $w \in \{1, 2, 3\}$. Then $G$ corresponds to the permutation $[5, 3, 2, 6, 1, 4]$, $[3, 6, 1, 5, 4, 2]$, $[4, 1, 6, 2, 3, 5]$ or $[2, 4, 5, 1, 6, 3]$.

The answer to Question 64 for Golomb Costas arrays now follows from Theorem 83 and Corollary 84.

**Remark 85.** For $q \geq 9$ every Golomb Costas array has a mirror pair of width 1 and a mirror pair of height 1. (The answer to Question 64 is yes for Golomb Costas arrays.)

In fact, Theorem 83 and Corollary 84 imply a much stronger result for Golomb Costas arrays, by guaranteeing the existence of width 1 and height 1 mirror pairs formed by vectors whose other component is constrained.
Chapter 5

Questions and Conjectures

In this chapter we summarise questions and conjectures arising from the thesis.

- (Section 3.3.) Are there WISPs of length greater than 13?

- Question 42. Are there (non-augmented) Costas arrays of order \( n > 2 \) containing every possible toroidal distance vector \( (w, h) \in \{1, \ldots, n-1\}^2 \)? (Conjecture 49: \( \text{No.} \))

- (Section 4.3.) What additional constraints on Costas arrays can be obtained from knowledge of the existence of vector pairs of the form \( (w_1, h_1), \frac{h_1}{|h_1|}(h_1, -w_1) \) and \( (w_2, h_2), \frac{h_2}{|h_2|}(h_2, w_2) \), as guaranteed by Proposition 60?

- (Section 4.3.) Is the number of mirror pairs of width 1 in a Costas array of odd order \( n \) bounded above by \( \frac{n-3}{2} \)?

- Question 64. Does every Costas array of order \( n \geq 9 \) have a mirror pair of width 1 and a mirror pair of height 1?

- Question 65. Does every Costas array of order \( n \geq 14 \) have a mirror pair of width 2 and a mirror pair of height 2?
Bibliography


