SOME COLORING PROBLEMS IN RAMSEY THEORY

by

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of
Mathematics

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**APPROVAL**

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Abstract

In this thesis, we present new results which are concerned with the following four coloring problems in Ramsey Theory.

i. The finite form of Brown’s Lemma.

ii. Some 2-color Rado numbers.

iii. Ramsey results involving Fibonacci sequences.

iv. Coloring the odd-distance plane graph.

A few of the results in this thesis are cited from earlier works of other authors. Several results are cited from papers co-authored by the present author. All other results are new.
To Tom Brown
İlim ilim bilmektir
İlim kendin bilmektir
Sen kendin bilmez isen
Ya nice okumaktır.

YUNUS EMRE
Acknowledgments

First and foremost I would like to express my sincerest thankfulness and gratitude to my senior supervisor Thomas C. Brown for his patience, support and belief in me through the years. He was, and will be, more than a supervisor for me. Without him, this thesis would have never been completed.

There are so many people I feel obliged to thank that I cannot finish writing them here. Among them, my supervisor Ladislav Stacho and my colleague Veselin Jungić have their special places. It is a privilege to know them and work with them.

I also thank both my parents for supporting me all through my education both financially and emotionally. I especially thank my mother, Fatma Ardal, who was my first math teacher, and all time mentor and counselor.

Lastly, I thank my wife, Yeşim, above all the other things, for her love and encouragement. She made my studies a lot easier than it would be without her. I would also like to thank her for bearing with me 10000 km away from home.
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Chapter 1

Introduction

Ramsey Theory is the study of the preservation of properties under set partitions. In other words, given a particular set $S$ that has a property $P$, is it true that whenever $S$ is partitioned into finitely many subsets, one of the subsets must also have property $P$ \[29\]

Ramsey Theory is named after Frank P. Ramsey and his famous theorem, which is now called Ramsey’s Theorem. His original theorem states that

**Theorem 1.1.** (Ramsey’s Theorem [37]) For any positive integers $k_1$ and $k_2$ there is a (least) positive integer $n$ such that if the edges of the complete graph on $n$ vertices is colored with two colors then there is either a monochromatic complete graph on $k_1$ vertices of the first color or a monochromatic complete graph on $k_2$ vertices of the second color.

The infinite Ramsey’s Theorem states that

**Theorem 1.2.** For any positive integer $r$ and any $r$-coloring of the edges of an infinite complete graph there is an infinite monochromatic complete graph.

In 1916, I. Schur [39] proved one of the first results in Ramsey Theory, 12 years before Ramsey proved his original theorem. Schur’s Theorem states that

**Theorem 1.3.** (Schur’s Theorem [39]) For every positive integer $r$, there is a least positive integer $s(r)$ such that for every $r$-coloring of the set $\{1, 2, \ldots, s(r)\}$ there is a monochromatic solution to $x + y = z$.

In 1927, van der Waerden [43] proved that
**Theorem 1.4.** (van der Waerden’s Theorem [43]) For all positive integers \( k \) and \( r \), there is a least positive integer \( w(k; r) \) such that for every \( r \)-coloring of \( \{1, 2, \ldots, w(k; r)\} \) there is a monochromatic \( k \)-term arithmetic progression.

Another classical result is Rado’s Theorem. R. Rado, a student of Schur, in his Ph.D. Thesis [36], proved a theorem that generalized both Schur’s Theorem and van Der Waerden’s Theorem. To state Rado’s theorem we need the following definition.

**Definition 1.5.** (Columns condition) Let \( C = (\overrightarrow{c}_1, \overrightarrow{c}_2, \ldots, \overrightarrow{c}_n) \) be a \( k \times n \) matrix, where \( \overrightarrow{c}_i \in \mathbb{Z}^k \) for \( 1 \leq i \leq n \). We say that \( C \) satisfies the columns condition if we can order the columns \( \overrightarrow{c}_i \) in such a way that there exist indices \( i_0 = 1 < i_1 < i_2 < \cdots < i_t = n \) such that the following two conditions hold for \( \overrightarrow{s}_1 = \sum_{i_1}^{i_t} \overrightarrow{c}_i \) and \( \overrightarrow{s}_j = \sum_{i_{j-1}+1}^{i_j} \overrightarrow{c}_i, 2 \leq j \leq t \):

i. \( \overrightarrow{s}_1 = \overrightarrow{0} \).

ii. \( \overrightarrow{s}_j \) can be written as a rational linear combination of \( \overrightarrow{c}_1, \overrightarrow{c}_2, \ldots, \overrightarrow{c}_{i_{j-1}} \) for \( 2 \leq j \leq t \).

**Theorem 1.6.** (Rado’s Theorem [36]) Let \( S \) be a system of linear homogeneous equations. Write \( S \) as \( A\overrightarrow{x} = \overrightarrow{0} \). Then \( S \) is regular if and only if \( A \) satisfies the columns condition. Furthermore, \( S \) has a monochromatic solution in distinct positive integers if and only if \( S \) is regular and there exist distinct (not necessarily monochromatic) integers that satisfy \( S \).

For a positive integer \( r \), an \( r \)-coloring of a set \( S \) is any function \( \chi : S \to \{1, 2, \ldots, r\} \), where \( \{1, 2, \ldots, r\} \) is a set with \( r \) elements. If \( \chi \) is a coloring of a set \( S \), and \( A \) is a subset of \( S \), we say that \( A \) is monochromatic under \( \chi \) if \( \chi \) is constant on \( A \). For any \( c \) in \( \{1, 2, \ldots, r\} \), \( \chi^{-1}(c) \) is called a color class of \( \chi \).

Let \( r \) be a positive integer. We say that a family of sequences \( \mathcal{A} \) is \( r \)-regular if for every \( r \)-coloring of \( \mathbb{N} \) there is a monochromatic member of \( \mathcal{A} \). If \( \mathcal{A} \) is \( r \)-regular for all \( r \) then \( \mathcal{A} \) is called regular. If \( \mathcal{A} \) is not regular then the largest \( r \) for which \( \mathcal{A} \) is \( r \)-regular is called the degree of regularity of \( \mathcal{A} \). Therefore, in this terminology, van der Waerden’s Theorem (1.4) states that \( \mathcal{A} \), the family of all \( k \)-term arithmetic progressions in \( \mathbb{N} \), is regular for each \( k \in \mathbb{N} \). A system of linear equations \( S \) is called \((r-)regular\) if the set of solutions of \( S \) is \((r-)regular\).
Chapter 2

Finite Form of Brown’s Lemma

2.1 Introduction

Brown’s Lemma is named after T.C. Brown, after it was introduced by him in [7]. Before stating the theorem, we define, for a finite set \( A = \{a_1 < a_2 < \cdots < a_n\} \subseteq \mathbb{N} \), the gap size of \( A \), denoted by \( gs(A) \), to be

\[
\max_{1 \leq i \leq n-1} (a_{i+1} - a_i).
\]

If \( |A| = 1 \), we define \( gs(A) = 1 \).

**Theorem 2.1 (Brown’s Lemma [7]).** For any finite coloring of \( \mathbb{N} \), there exist a fixed positive integer \( d \) and arbitrarily large monochromatic sets \( A \) such that

\[
gs(A) \leq d.
\]

For a proof, see [29]. Some applications appeared in [5], [23], [25], [29], [30], and [41]. A generalization of it can be found in [9].

The finite version of Brown’s Lemma states;

**Theorem 2.2 (Finite form of Brown’s Lemma).** For every positive integer \( r \), and every function \( f : \mathbb{N} \to \mathbb{N} \), there exists a (smallest) positive integer \( B(f,r) \) such that for every \( r \)-coloring of the interval \([1, B(f,r)]\), there exists a monochromatic set \( A \) such that

\[
|A| > f(gs(A)).
\]

This theorem has applications to the theory of locally finite semigroups, in particular to Burnside’s problem for semigroups of matrices(see [41]).
In [25], it is shown that there is a 2-coloring \( \chi \) of \( \mathbb{N} \) and a function \( f : \mathbb{N} \to \mathbb{N} \) such that if \( A = \{ a, a + d, a + 2d, \ldots \} \) is any monochromatic arithmetic progressions then \( |A| < f(d) \). Hence, one cannot require the monochromatic set \( A \) in Theorem 2.2 to be an arithmetic progression.

We will give a proof of Theorem 2.2 here (from [29]). But before that we will prove the following.

**Theorem 2.3.** Theorem 2.1 and Theorem 2.2 are equivalent.

**Proof.** First, assume Theorem 2.1.

Assume without loss of generality that \( f : \mathbb{N} \to \mathbb{N} \) is a nondecreasing function and let \( r \in \mathbb{N} \) be arbitrary. Assume for a contradiction that \( B(f, r) \) does not exist, i.e., for every \( m \in \mathbb{N} \), there exists an \( r \)-coloring \( \chi_m \) of \([1, m]\) such that for every monochromatic set \( A \subseteq [1, m], |A| \leq f(gs(A)) \). We will define an \( r \)-coloring \( \chi \) of \( \mathbb{N} \) in the following way. Let \( I_j^{(1)} = \{ m : \chi_m(1) = j \} \). Then, as \( I_1^{(1)} \cup I_2^{(1)} \cup \cdots \cup I_k^{(1)} \) is infinite, for some \( j = j_1 \), \( I_j^{(1)} \) is infinite. Then define \( \chi(1) = j_1 \). Now, let \( I_{j_1}^{(2)} = \{ m \geq 2 : m \in I_{j_1}^{(1)} \} \) and \( \chi_m(2) = j_1 \). Then \( I_1^{(2)} \cup I_2^{(2)} \cup \cdots \cup I_k^{(2)} = I_{j_1}^{(1)} \cap [2, \infty) \). Hence, for some \( j = j_2 \), \( I_{j_2}^{(2)} \) is infinite. Then define \( \chi(2) = j_2 \). We continue in this way to find, for each \( i \geq 1 \), some color \( j_i \) such that \( I_{j_i}^{(i)} = \{ m \geq i : m \in I_{j_{i-1}}^{(i-1)} \} \) is infinite and define \( \chi(i) = j_i \). The coloring \( \chi \) has the property that for any \( n \in \mathbb{N} \), \( \chi(i) = \chi_m(i) \) for any \( i \in [1, n] \) and \( m \in I_j^{(n)} \).

Therefore, by assumption, there exists \( d \geq 1 \) and arbitrarily large monochromatic sets with gap size at most \( d \). Let \( A \) be one such with \( |A| > f(d) \). Let \( n = \max A \) and \( m \in I_{j_n}^{(n)} \) be arbitrary. Then \( m \geq n \) and \( \chi_m(i) = \chi(i) \) for all \( i, 1 \leq i \leq n \). But then \( A \subseteq [1, n] \subseteq [1, m] \) is monochromatic under \( \chi_m \) such that \( |A| > f(d) \geq f(gs(A)) \), since \( f \) is increasing. But this contradicts to our assumption.

Conversely, assume Theorem 2.2. Assume also, for a contradiction, that Theorem 2.1 is false, i.e., there exists \( r \geq 1 \) and an \( r \)-coloring \( \chi \) of \( \mathbb{N} \) with the property that for any \( d \in \mathbb{N} \), there exists \( n = n_d \in \mathbb{N} \) such that for any monochromatic set \( A \),

\[
\text{If } |A| \geq n \text{ then } gs(A) > d. \tag{2.1}
\]

Define a function \( f : \mathbb{N} \to \mathbb{N} \) as \( f(d) = n_d \). Then by Theorem 2.2, there exists \( m \) and \( A \subseteq [1, m] \) monochromatic under \( \chi \) such that \( |A| > f(gs(A)) \). Let \( d = gs(A) \). Then \( |A| > f(gs(A)) = f(d) = n_d \). But then we have monochromatic set \( A \) such that \( |A| > n_d \) and \( gs(A) = d \), contradicting to (2.1).

\( \square \)
CHAPTER 2. FINITE FORM OF BROWN’S LEMMA

Proof of Theorem 2.2. Let \( f : \mathbb{N} \to \mathbb{N} \) be a nondecreasing function. We use induction on \( r \).

Clearly, \( B(f, 1) = f(1) + 1 \), since if the interval \([1, f(1) + 1]\) is colored with one color, the interval itself constitutes a monochromatic set \( A \) with \( gs(A) = 1 \), so that \( |A| > f(gs(A)) = f(1) \).

Now, let \( r \geq 2 \) and assume that \( B(f, r - 1) \) exists. Let \( m = rf(B(f, r - 1)) + 1 \).

We will show that \( B(f, r) \leq m \). Assume for a contradiction that there is an \( r \)-coloring \( \chi \) of \([1, m]\) such that for every monochromatic set \( A \), we have \( |A| \leq f(gs(A)) \). Let \( C_i = \{ j : \chi(j) = i \}, 1 \leq i \leq r \). Then \( |C_i| \leq f(gs(C_i)) \).

Also, \( gs(C_i) \leq B(f, r - 1) \) for each \( i \in [1, r] \); otherwise, for some \( a \geq 1 \), the set \( \{ a + 1, a + 2, \ldots, a + B(f, r - 1) \} \subseteq [1, m] \) would be colored only with \( r - 1 \) colors. Hence, by the inductive hypothesis, this would give a monochromatic set \( T \) with \( |T| > f(gs(T)) \), contradicting our assumption.

Since \( f \) is nondecreasing, \( f(gs(C_i)) \leq f(B(f, r - 1)) \). Hence, \( |C_i| \leq f(gs(C_i)) \leq f(B(f, r - 1)) \) for all \( i \in [1, r] \). Since \([1, m] = C_1 \cup C_2 \cup \cdots \cup C_r \), we get \( m \leq rf(B(r - 1)) \), contradicting to (2.2). Therefore, any \( r \)-coloring of \([1, m]\) satisfies the conditions of the theorem. This proves the existence of \( B(f, r) \), since \( B(f, r) \leq m = rf(B(f, r - 1)) + 1 \).

If \( f : \mathbb{N} \to \mathbb{N} \) is an arbitrary function then define a new function \( g : \mathbb{N} \to \mathbb{N} \) by \( g(n) = \max \{ f(m) : 1 \leq m \leq n \} \). Then \( g \) is an increasing function and for any set \( n, g(n) \geq f(n) \). Then it is easy to see that \( B(f, r) \leq B(g, r) \) for any \( r \geq 1 \).

Our main concern in this chapter is to find upper and lower bounds for the function \( B(id, r) \) where \( id \) denotes the identity function on \( \mathbb{N} \). For simplicity of notation, we will denote \( B(id, r) \) by \( B(r) \).

The proof of Theorem 2.2 gives the following upper bound:

\[
B(r) < \lfloor r! \cdot e \rfloor.
\] (2.3)

This is the only previously known bound for any \( B(f, r) \), and is mentioned in [8]. There is a big gap between this upper bound and the known values/lower bounds of \( B(r) \). In Table 2.1, we give all the known values or the best lower bounds (known to date) for \( B(r) \).
In Section 2.2, we will improve this upper bound and in Section 2.3, we will find a lower bound for $B(r)$. In Section 2.4, we will find an upper bound for $B(m, x, r)$ where $m \in \mathbb{N}$ using the techniques of Section 2.2. We need some definitions and lemmas to start.

**Definition 2.4.** Let $A$ be a finite subset of $\mathbb{N}$. We say that $A$ has Property $P$ if $|B| \leq gs(B)$ for any subset $B$ of $A$.

**Theorem 2.5.** Let $A = \{a_1 < a_2 < \cdots < a_n\}$ be a subset of $\mathbb{N}$. Then the following are equivalent.

(i) $A$ has Property $P$.

(ii) For each $1 \leq i < j \leq n$

$$|[a_i, a_j]| \leq gs([a_i, a_j])$$

where $[a_i, a_j] = \{a_i, a_{i+1}, \cdots, a_j\}$.

**Proof.** The forward implication is true by definition.

We will prove the contrapositive of the backward implication. Assume that $A$ does not have Property $P$, so that there exists a subset $B$ of $A$ such that $|B| > gs(B)$.

Let $i = \min \{k : a_k \in B\}$ and $j = \max \{k : a_k \in B\}$. Then $B \subseteq [a_i, a_j]$. Since $a_i, a_j \in B$ and $B \subseteq [a_i, a_j]$, $gs([a_i, a_j]) \leq gs(B)$. Hence

$$gs([a_i, a_j]) \leq gs(B) < |B| \leq |[a_i, a_j]|,$$

therefore (ii) does not hold. \[ \Box \]

Note that a finite set of $A$ of positive integers has Property $P$ if and only if for any $a \in \mathbb{Z}$ with $a > \min A$, $A - a$ has Property $P$. This fact suggests the following definitions.

<table>
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<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>13</td>
</tr>
<tr>
<td>4</td>
<td>35</td>
</tr>
<tr>
<td>5</td>
<td>$\geq 74$</td>
</tr>
<tr>
<td>6</td>
<td>$\geq 143$</td>
</tr>
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Table 2.1: All Known Values/Lower Bounds of $B(r)$. 
Definition 2.6. Let \( A = \{a_1 < a_2 < \cdots < a_n\} \) be a subset of \( \mathbb{N} \), for some \( n \in \mathbb{N} \). Then we define the difference sequence \( d(A) \) of \( A \) as \( d(A) = () \), the length 0 vector, if \( n = 1 \) and
\[
d(A) = (a_2 - a_1, a_3 - a_2, \ldots, a_n - a_{n-1}).
\]
if \( n \geq 2 \).

Definition 2.7. Let \( d = (d_1, d_2, \ldots, d_n) \in \mathbb{N}^n \) for some \( n \geq 1 \). Then we say that \( d \) has Property \( P' \) if
\[
\max_{a \leq i \leq b} d_i \geq b - a + 2
\]
for all \( 1 \leq a \leq b \leq n \) i.e., any \( l \) consecutive numbers in \( d \) have maximum bigger than or equal to \( l + 1 \).

The following theorem gives the correspondence between Property \( P \) and Property \( P' \).

Theorem 2.8. A finite subset \( A \) of \( \mathbb{N} \) has Property \( P \) if and only if \( d(A) \) has Property \( P' \).

Proof. Let \( A = \{a_1 < a_2 < \cdots < a_n\} \) be a subset of \( \mathbb{N} \) and let \( d(A) = (d_1, d_2, \ldots, d_{n-1}) \) be the difference sequence of \( A \) where \( d_i = a_{i+1} - a_i \) for \( 1 \leq i \leq n - 1 \). Then
\[
A \text{ has Property } P \quad \iff \quad |[a_i, a_j]| \leq gs([a_i, a_j]) \text{ for all } 1 \leq i < j \leq n
\]
\[
\quad \iff \quad j - i + 1 \leq \max_{i \leq t \leq j-1} a_{t+1} - a_t \text{ for all } 1 \leq i < j \leq n
\]
\[
\quad \iff \quad t - i + 2 \leq \max_{t \leq i \leq t} d_i \text{ for all } 1 \leq i \leq t \leq n - 1 \ (t = j - 1)
\]
\[
\quad \iff \quad d(A) \text{ has Property } P'.
\]

2.2 Upper Bound

Definition 2.9. For a positive integer \( n \), define
\[
D_n = \{d = (d_1, d_2, \ldots, d_n) \in \mathbb{N}^n : d \text{ has Property } P'\}.
\]

Lemma 2.10. Let \( d = (d_1, d_2, \ldots, d_n) \) and \( d' = (d'_1, d'_2, \ldots, d'_m) \) for some positive integers \( n, m \in \mathbb{N} \), and \( t \in \mathbb{N} \), \( t > n + m + 1 \) be arbitrary. For \( d'' = (d_1, d_2, \ldots, d_n, t, d'_1, d'_2, \ldots, d'_m) \)
\[
d'' \text{ has Property } P' \text{ if and only if both } d \text{ and } d' \text{ have Property } P'.
\]
Proof. The forward implication follows directly from the definition.

Now, assume both \( d \) and \( d' \) have Property \( P' \) and let \( 1 \leq a \leq b \leq n + m + 1 \) be arbitrary. Then

Case 1. If \( b \leq n \) then
\[
\max_{a \leq i \leq b} d''_i = \max_{a \leq i \leq b} d_i \geq b - a + 2,
\]
since \( d \in D_n \).

Case 2. If \( a \leq n + 1 \leq b \) then
\[
\max_{a \leq i \leq b} d''_i \geq t \geq n + m + 2 \geq b - a + 2,
\]
since \( d''_{n+1} = t \).

Case 3. If \( a \geq n + 2 \) then
\[
\max_{a \leq i \leq b} d''_i = \max_{a \leq i \leq b} d'_i \geq b - a + 2,
\]
since \( d' \in D_n \).

Therefore, \( d'' \) has Property \( P' \).

\[ \Box \]

Corollary 2.11. Let \( d^{(i)}(n_i), 1 \leq i \leq m \) for some \( m \geq 1 \) and \( n_1, n_2, \ldots, n_m \in \mathbb{N} \). Let
\[
n = m - 1 + \sum_{i=1}^{m} n_i
\]
Then \( d = (d^{(1)}, t_1, d^{(2)}, t_2, \ldots, t_{m-1}, d^{(m)}) \in D_n \) for any \( t_i > n, 1 \leq i \leq m - 1 \).

Corollary 2.12. Let \( d \in D_n \) and \( m \geq 2 \) be arbitrary. Then
\[
d' = (d, t, d, t, \ldots, t, d) \in D_{m \cdot n + m - 1}
\]
for any \( t > m \cdot n + m - 1 \), where in \( d' \), \( d \) is repeated \( m \) times.

Now, we define a function \( \sigma \) on \( \bigcup_{n \geq 1} D_n \) in the following way. If \( d = (d_1, d_2, \ldots, d_n) \in \mathbb{N}^n \), define
\[
\sigma(d) = \sum_{i=1}^{n} d_i.
\]
Note that, if \( A = \{a_1 < a_2 < \ldots < a_n\} \subset \mathbb{N} \) for some \( n \geq 2 \), then \( \sigma(d(A)) = a_n - a_1 \).

Now, define another function \( F : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\} \) by \( F(0) = 0 \) and
\[
F(n) = \min_{d \in D_n} \sigma(d),
\]
for \( n \geq 1 \), i.e.,

\[
F(n) = \min \{ \max A - \min A : A \subset \mathbb{N} \text{ with } |A| = n, \text{ has Property } P \}.
\]

Note that \( F \) is a strictly increasing function and, \( F(1) = 2 \), \( F(2) = 5 \) and \( F(3) = 8 \).

We will use \( F \) to find an upper bound for \( B(r) \). For that, we need more information about \( F \).

The following two lemmas give a recursive definition for \( F(n) \).

**Lemma 2.13.** Let \( n \in \mathbb{N} \). Then

\[
F(n) = n + 1 + F(n - m) + F(m - 1)
\]

for some \( m \) in \([1, n]\).

**Proof.** Let \( d = (d_1, d_2, ..., d_n) \) in \( D_n \) be such that

\[
F(n) = \sigma(d) = \sum_{i=1}^{n} d_i.
\]

By the definition of Property \( P' \), \( \max_{1 \leq i \leq n} d_i \geq n + 1 \). And, by the minimality of \( F(n) \), \( \max_{1 \leq i \leq n} d_i \leq n + 1 \), since otherwise we could replace any \( d_i \) greater than \( n + 1 \) by \( n + 1 \) and obtain another sequence with smaller sum. Therefore,

\[
\max_{1 \leq i \leq n} d_i = n + 1
\]

Let \( m \in [1, n] \) be such that \( d_m = n + 1 \). Then

\[
d = (d_1, d_2, ..., d_{m-1}, n + 1, d_{m+1}, d_{m+2}, ..., d_n)\).
\]

Again, by the minimality of \( F(n) \) and Lemma 2.10,

\[
\sum_{i=1}^{m-1} d_i = F(m - 1) \text{ and } \sum_{i=m+1}^{n} d_i = F(n - m).
\]

Therefore,

\[
F(n) = n + 1 + F(m - 1) + F(n - m)
\]

for some \( m \) in \([1, n]\).
Lemma 2.14. For $n \geq 2$

\[
F(n) = F(n - 1) + \lfloor \log_2 n \rfloor + 2,
\]

\[
F(n) = n + 1 + F \left( \left\lfloor \frac{n - 1}{2} \right\rfloor \right) + F \left( \left\lceil \frac{n - 1}{2} \right\rceil \right).
\]

Proof. We’ll prove both equalities by induction on $n$, at the same time.

Clearly, both equalities are true for $n = 2$ and $n = 3$.

Now assume that they are true for all $m < n$ for some $n > 3$. Hence, $F(m) = F(m - 1) + \lfloor \log_2 m \rfloor + 2$ for all $m < n$. This implies that $F(m) - F(m - 1) = \lfloor \log_2 m \rfloor + 2$ for all $m < n$. Therefore,

\[
F(m) - F(m - 1) \geq F(m - 1) - F(m - 2)
\]

for all $m \in [2, n)$. Hence, for $1 \leq l < m < n$,

\[
F(m) - F(m - 1) \geq F(l + 1) - F(l)
\]

which implies

\[
F(m) + F(l) \geq F(m - 1) + F(l + 1)
\]

(2.4)

Therefore, if $m < n$,

\[
\min_{0 \leq l \leq m} (F(l) + F(m - l)) = F \left( \left\lfloor \frac{m}{2} \right\rfloor \right) + F \left( \left\lceil \frac{m}{2} \right\rceil \right).
\]

(2.5)

From Lemma 2.10, it follows that

\[
F(n) \leq n + 1 + F(m - 1) + F(n - m)
\]

(2.6)

for all $m$ in $[1, n]$. And by Lemma 2.13,

\[
F(n) = n + 1 + F(m - 1) + F(n - m)
\]

(2.7)

for some $m$ in $[1, n]$.

Hence by the minimality of $F(n)$, (2.6) and (2.7)

\[
F(n) = n + 1 + \min_{1 \leq m \leq n} (F(m - 1) + F(n - m))
\]

\[
= n + 1 + F \left( \left\lfloor \frac{n - 1}{2} \right\rfloor \right) + F \left( \left\lceil \frac{n - 1}{2} \right\rceil \right), \text{ by (2.5)}
\]

(2.8)

Now we’ll show that

\[
F(n) = F(n - 1) + \lfloor \log_2 n \rfloor + 2
\]
Case 1. Suppose that \( n = 2t \) for some \( t \geq 2 \). Then
\[
F(2t) = 2t + 1 + F(t - 1) + F(t), \quad \text{by (2.8)}
\]
\[
F(2t - 1) = 2t + F(t - 1) + F(t - 1), \quad \text{by induction hypothesis.}
\]
Hence,
\[
F(2t) - F(2t - 1) = 1 + F(t) - F(t - 1)
= 1 + \lfloor \log_2 t \rfloor + 2, \quad \text{by induction hypothesis}
= \lfloor \log_2 2t \rfloor + 2,
\]
i.e.,
\[
F(n) = F(n - 1) + \lfloor \log_2 n \rfloor + 2.
\]

Case 2. Suppose now that \( n = 2t + 1 \) for some \( t \geq 2 \). Then
\[
F(2t + 1) = 2t + 2 + F(t) + F(t), \quad \text{by (2.8)}
\]
\[
F(2t) = 2t + 1 + F(t) + F(t - 1), \quad \text{by induction hypothesis.}
\]
Hence
\[
F(2t + 1) - F(2t) = 1 + F(t) - F(t - 1)
= 1 + \lfloor \log_2 t \rfloor + 2, \quad \text{by induction hypothesis}
= \lfloor \log_2 2t \rfloor + 2
= \lfloor \log_2 (2t + 1) \rfloor + 2, \quad \text{since } 2t + 1 \text{ is odd},
\]
i.e.,
\[
F(n) = F(n - 1) + \lfloor \log_2 n \rfloor + 2.
\]

\[\Box\]

**Theorem 2.15.** For any \( k \) in \( \mathbb{N} \), \( F(2^k - 1) = k \cdot 2^k \).

**Proof.** The equality is clear for \( k = 1 \).

Assume that the assumption is true for \( k - 1 \), for some \( k \geq 2 \). Then
\[
F\left(2^k - 1\right) = 2 \cdot F\left(2^{k-1} - 1\right) + 2^k, \quad \text{from Lemma 2.14}
= 2 \cdot \left((k - 1) \cdot 2^{k-1}\right) + 2^k, \quad \text{by induction hypothesis}
= k \cdot 2^k
\]
\[\Box\]
We need two more theorems to obtain the new upper bound of $B(r)$.

**Theorem 2.16.** For any $k$ in $\mathbb{N}$, $F(2^k - k) = k(2^k - k) + 1$.

**Proof.** Let $k \in \mathbb{N}$ be given. Then
\[
F(2^k - 1) = F(2^k - k) + \sum_{i=1}^{k-1} \left( \left\lfloor \log_2(2^k - i) \right\rfloor + 2 \right), \text{ by Lemma 2.14}
\]
\[
= F(2^k - k) + (k - 1)((k - 1) + 2)
\]
\[
= F(2^k - k) + (k^2 - 1).
\]
Hence,
\[
F(2^k - k) = F(2^k - 1) - (k^2 - 1)
\]
\[
= k \cdot 2^k - (k^2 - 1)
\]
\[
= k \cdot (2^k - k) + 1
\]
\[\square\]

**Theorem 2.17.** Let $r \in \mathbb{N}$ be given. Then if there exists $N \in \mathbb{N}$ such that $B(r) > rN + 1$ then $F(N) \leq rN$.

**Proof.** Assume that $B(r) > rN + 1$ for some $N \in \mathbb{N}$. Then there exists an $r$-coloring of $[1, rN + 1]$ such that each color class has Property $P$. By the pigeon hole principle, at least one of the color classes has at least $N + 1$ elements, say $C$. Then
\[
d(C) = (d_1, d_2, ..., d_{|C|-1}) \in D_{|C|-1}
\]
by Theorem 2.8.

So,
\[
F(N) \leq F(|C| - 1) \leq \sigma(d(C)).
\]
But since $C \subset [1, rN + 1]$,
\[
\sigma(d(C)) \leq rN.
\]
Therefore, $F(N) \leq rN$. \[\square\]

**Theorem 2.18.** Let $r \geq 1$ be given. Then
\[
B(r) \leq r(2^r - r) + 1.
\]

**Proof.** The proof follows from Theorems 2.16 and 2.17, upon setting $N = 2^r - r$. \[\square\]
2.3 Lower Bound

In this section, we will obtain a lower bound for $B(r)$ by recursively constructing a $2^s$-coloring of the interval $[1, n_s]$, where $n_s = 2^s \cdot \prod_{i=0}^{s-1} (2^i + 1)$, in such a way that all color classes will have Property $P$ and therefore we will conclude that $B(2^s) \geq n_s$. We will represent this coloring by a matrix $M_s$ with $2^s$ rows and $\prod_{i=0}^{s-1} (2^i + 1)$ columns where the rows of $M_s$ are the color classes of the coloring.

Let $J_s$ denote the $2^s \times \prod_{i=0}^{s-1} (2^i + 1)$ matrix of all 1’s. Let $d(M_s)$ denote the difference sequence of the first row of $M_s$.

For $s = 0$,

$$n_0 = 1,$$

$$M_0 = [1].$$

For $s = 1$,

$$n_1 = 2^1 \cdot 2 = 4,$$

$$M_1 = \begin{bmatrix}
M_0 & M_0 + 2n_0 J_0 \\
M_0 + n_0 J_0 & M_0 + 3n_0 J_0
\end{bmatrix} = \begin{bmatrix}
M_0 & M_0 + 2J_0 \\
M_0 + J_0 & M_0 + 3J_0
\end{bmatrix} = \begin{bmatrix}
1 & 3 \\
2 & 4
\end{bmatrix}.$$
Note that all the rows of $M_1$ and $M_2$ are obtained by shifting the first row of the corresponding matrix. And this will turn out to be true for each $M_s$ so that each row of $M_s$ has the same difference sequence as the first row of $M_s$, namely $d(M_s)$.

Note also that $d(M_2) = (2, 6, 2, 6, 2)$, so by Theorem 2.8 and Definition 2.7, each row of $M_2$ has Property $P$.

Now, assume that $M_s$ is constructed in such a way that all the color classes (i.e., rows of $M_s$) have Property $P$. Then construct $M_{s+1}$ as follows.

$$M_{s+1} = \begin{bmatrix} M_s & M_s + 2n_sJ_s & M_s + 4n_sJ_s & \cdots & M_s + 2^{s+1}n_sJ_s \\
M_s + n_sJ_s & M_s + 3n_sJ_s & M_s + 5n_sJ_s & \cdots & M_s + (2^{s+1} + 1)n_sJ_s \end{bmatrix}.$$  

Since each row of $M_s$ is obtained by shifting the first row of $M_s$, the same is true for $M_{s+1}$. Hence,

$$d(M_{s+1}) = (d(M_s), t_s, d(M_s), \cdots, t_s, d(M_s))$$

where $d(M_s)$ is repeated $2^s$ times and $t_s$ is the gap between the smallest number in the first row of $M_s + 2n_sJ_s$, which is clearly $2n_s + 1$ and the largest number in the first row of $M_s$. Since the length of the sequence $d(M_{s+1})$ is $n_{s+1}/2^s - 1 = (\prod_{i=0}^{s} (2^i + 1)) - 1$ and from Lemma 2.19 below $t_s = \prod_{i=0}^{s} (2^i + 1)$, by Corollary 2.12, $(M_s)_1$ has Property $P$ and therefore all the color classes have Property $P$.

**Lemma 2.19.** For any $s \geq 0$, $t_s = \prod_{i=0}^{s} (2^i + 1)$.

*Proof.* It follows from the recursive construction that $t_s = 1 + n_1 + n_2 + \cdots + n_s = t_{s-1} + n_s$.

We will prove the Lemma by induction on $s$.

For $s = 1$, $t_1 = 6 = (2^0 + 1)(2^1 + 1)$.

Assuming that the assumption is true for some $s \geq 1$, we get

$$t_{s+1} = t_s + n_{s+1} = \prod_{i=0}^{s} (2^i + 1) + 2^{s+1} \cdot \prod_{i=0}^{s} (2^i + 1)$$

$$= \prod_{i=0}^{s+1} (2^i + 1).$$

Therefore, our claim is true for all $s \geq 1$. \[\square\]

Hence, $B(2^s) \geq n_s$. But

$$n_s = 2^s \prod_{i=0}^{s-1} (2^i + 1) \geq 2^s \cdot 2^{2^s-s}$$

$$= (2^{s+1})^{\frac{1}{2}}.$$
Therefore, \( B(2^s) \geq \left( 2^{s+1} \right)^{\frac{c}{2}} \).

Now, let \( r \geq 1 \) be given. Then there exists \( s \geq 1 \) such that \( 2^s \leq r < 2^{s+1} \). Hence,

\[
B(2^s) \geq \left( 2^{s+1} \right)^{\frac{c}{2}} > r^{c \log r}
\]

for some \( c > 0 \).

Therefore, \( B(r) \geq r^{c \log r} \).

**Remark 2.20.** A slight modification of the above construction gives better lower bounds of \( B(r) \) for individual \( r \)'s, but it does not improve the asymptotic lower bound.

### 2.4 Upper Bound for \( B(mx, r) \)

This section is dedicated to finding an upper bound for \( B(f, r) \) where \( f(x) = mx \) for some \( m \in \mathbb{N} \). We will give the theorems and lemmas without proof, as the proof of each is analogous to the proof of the corresponding theorem/lemma in Section 2.2.

Let \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) be a strictly increasing continuous function such that \( f(n) \in \mathbb{N} \) for all \( n \in \mathbb{N} \).

**Definition 2.21.** Let \( A \) be a finite subset of \( \mathbb{N} \). We say that \( A \) has Property \( P_f \) if \( |B| \leq f \left( gs(B) \right) \) for any subset \( B \) of \( A \).

**Theorem 2.22.** Let \( A = \{a_1 < a_2 < \cdots < a_n\} \) be a subset of \( \mathbb{N} \). Then the following are equivalent.

(i) \( A \) has Property \( P_f \).

(ii) For each \( 1 \leq i < j \leq n \)

\[
|[a_i, a_j]| \leq f \left( gs([a_i, a_j]) \right).
\]

**Definition 2.23.** Let \( d = (d_1, d_2, \ldots, d_n) \in \mathbb{N}^n \). Then we say that \( d \) has Property \( P'_f \) if

\[
\max_{a \leq i \leq b} d_i \geq f^{-1}(b - a + 2)
\]

for all \( 1 \leq a \leq b \leq n \), i.e., any \( l \) consecutive numbers in \( d \) have maximum bigger than or equal to \( f^{-1}(l + 1) \).
The following theorem gives the correspondence between Property $P_f$ and $P'_f$.

**Theorem 2.24.** A finite subset $A$ of $\mathbb{N}$ has Property $P_f$ if and only if $d(A)$ has Property $P'_f$.

**Definition 2.25.** For a positive integer $n$, define

$$D_{n,f} = \{ \mathbf{d} = (d_1, d_2, \ldots, d_n) \in \mathbb{N}^n : \mathbf{d} \text{ has Property } P'_f \}. $$

Now, define the function $F_f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ by $F_f(0) = 0$ and

$$F_f(n) = \min_{\mathbf{d} \in D_{n,f}} \sigma(\mathbf{d}),$$

for $n \geq 1$, i.e.,

$$F_f(n) = \min \{ \max A - \min A : A \subset \mathbb{N}, \text{ with } |A| = n, \text{ has Property } P_f \}. $$

**Theorem 2.26.** For any positive integer $n$,

$$D_{n,f} = \{ ([f^{-1}(d_1)], [f^{-1}(d_2)], \ldots, [f^{-1}(d_n)] ) : (d_1, d_2, \ldots, d_n) \in D_n \}$$

**Proof.** Let $(d_1, d_2, \ldots, d_n) \in D_n$ be arbitrary. Then by the definition of $D_n$,

$$\max_{a \leq i \leq b} d_i \geq b - a + 2$$

for all $1 \leq a \leq b \leq n$. Hence,

$$\max_{a \leq i \leq b} [f^{-1}(d_i)] \geq \max_{a \leq i \leq b} f^{-1}(d_i) \geq f^{-1}(b - a + 2)$$

for all $1 \leq a \leq b \leq n$, so that

$$([f^{-1}(d_1)], [f^{-1}(d_2)], \ldots, [f^{-1}(d_n)]) \in D_{n,f}.$$

Now, let $(d_1, d_2, \ldots, d_n) \in D_{n,f}$ be arbitrary. Then

$$\max_{a \leq i \leq b} d_i \geq f^{-1}(b - a + 2)$$

for all $1 \leq a \leq b \leq n$. Hence,

$$\max_{a \leq i \leq b} f(d_i) = f \left( \max_{a \leq i \leq b} d_i \right) \geq b - a + 2$$

for all $1 \leq a \leq b \leq n$. Hence

$$(f(d_1), f(d_2), \ldots, f(d_n)) \in D_n.$$ 

The result follows since $d_i = f^{-1}(f(d_i)) = \lceil f^{-1}(f(d_i)) \rceil$. 

$\square$
Theorem 2.27. Let \( r \in \mathbb{N} \) be given. Then if there exists an \( N \in \mathbb{N} \) such that \( F_f(N) > rN \) then \( B(f, r) \leq rN + 1 \).

Proof. Analogous to the proof of Theorem 2.17.

In the rest of this section, we will only consider linear functions on \( \mathbb{N} \). For ease of notation, we will write \( F_m(n) \), \( B_m(n) \) and \( D_{n,m} \) for \( F_f(n) \), \( B(n,f) \) and \( D_{n,f} \), respectively, if \( f(x) = mx \) for some \( m \in \mathbb{N} \).

Lemma 2.28. Let \( m \) and \( n \) be positive integers. Then

\[
F_m(n) \geq \frac{1}{m} F(n).
\]

Proof. 

\[
F_m(n) = \min_{d \in D_{n,m}} \sigma(d) = \min_{d \in D_n} \sum_{i=1}^n \left\lceil \frac{d_i}{m} \right\rceil, \text{ from Theorem 2.26}
\]

\[
\geq \frac{1}{m} \min_{d \in D_n} \sum_{i=1}^n d_i
\]

\[
= \frac{1}{m} F(n).
\]

Lemma 2.29. For any positive integers \( m \) and \( k \),

\[
F_m \left( 2^{mk} - mk \right) \geq k \left( 2^{mk} - mk \right) + 1.
\]

Theorem 2.30. Let \( r \) and \( m \) be positive integers. Then

\[
B_m(r) \leq r(2^{mr} - mr) + 1.
\]

Remark 2.31. The method used in Section 2.3 to obtain a lower bound for \( B(2^s, id) \) can be extended, in the obvious way, to obtain the following lower bound for \( B(2^s, mx) \) for any positive integers \( m \) and \( s \).

\[
B(2^s, mx) \geq n_s = m 2^s \prod_{i=0}^{s-1} (2^{2i} + 1).
\]

Therefore, for any positive integer \( k \),

\[
B(k, mx) \geq (mk)^{c \log k}
\]

for some \( c > 0 \).
2.5 Remarks and Questions

In Sections 2.2 and 2.3, we showed

\[ r^{c \log r} \leq B(r) \leq r(2^r - r) + 1 \]

where \( c = \frac{\log_2 e}{2} \).

There is a very big gap between the established bounds. The values in Table 2.1 suggest that the upper bound is a better estimate than the lower bound. We wonder if the exact value is "closer" to the upper bound than the lower bound.

In Section 2.4, we have only considered linear functions. What possible upper bounds can we get for a more general class of functions?
Chapter 3

Ramsey results involving Fibonacci sequences

In this chapter we will consider two coloring problems which are related to the Fibonacci sequence.

In the first section, we are going to define the chromatic number of a (infinite) word and present an upper bound for that of the Fibonacci word.

In the second section, we'll look at some Ramsey properties of the set of all Fibonacci numbers, denoted $F$. Namely, we will investigate the degree of regularity and the degree of accessibility of the set $F$. We will first give what is known and then present a new upper bound for the latter.

We'll use the following definition for the Fibonacci numbers.

For $n \geq 1$, let $F_n$ denote the $n$th Fibonacci number where $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. The first few Fibonacci numbers are given in Table 3.1.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_n$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
</tr>
</tbody>
</table>

Table 3.1: Some Fibonacci numbers.
3.1 The Fibonacci Word

3.1.1 Definitions

Unless otherwise stated, the definitions and theorems in this section have been taken from Lothaire [31] and [32].

An (infinite) word $w$ is an infinite sequence of symbols drawn from a finite alphabet. If $w$ is finite, it is called a finite word. An element of the alphabet is called a letter. Any finite contiguous subsequence of $w$ is called a factor of $w$. A factor which starts from the beginning of $w$ is called a prefix of $w$. If $v$ is a finite word, the length of $v$ is defined to be the number of letters in $v$ counted with multiplicity and denoted by $|v|$. For $n \in \mathbb{N}$, $w_n$ is the $n$th symbol in $w$.

If $u$ and $v$ are two finite words then the product of $u$ and $v$, $uv$, is the finite word obtained by concatenation of $u$ and $v$.

A word $w$ is a Sturmian word if for all $n \in \mathbb{N}$, $w$ has exactly $n+1$ distinct factors of length $n$. Therefore, $w$ uses exactly two letters from the alphabet. Without loss of generality, we can assume that these letters are 0 and 1.

The following well-known result gives a characterization of Sturmian words over the alphabet $\{0,1\}$.

Theorem 3.1. [32] A word $w = (w_n)_{n \in \mathbb{N}}$ over $\{0,1\}$ is a Sturmian word if and only if there exists $\alpha, \rho \in \mathbb{R}$ with $0 < \alpha < 1$ irrational such that

$$w_n = \lfloor \alpha (n+1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor$$

for all $n \in \mathbb{N}$.

The Sturmian word corresponding to $\rho = 0$ is also called the characteristic sequence of the irrational number $\alpha$ and denoted by $c_\alpha$ (See Brown [6]). $c_\alpha$ can also be constructed recursively in the following way;

Let $[0; a_1, a_2, \ldots, a_n, \ldots]$ be the continued fraction expansion of $\alpha$. Define $s_1 = 1$, $s_2 = 0$, and $s_{n+2} = s_{n+1} + s_n$ for $n \geq 1$. Every finite word in the sequence, except the first one, is a prefix of the next ones, so that the sequence converges to an (infinite) word, which is $c_\alpha$.

The Fibonacci word is the characteristic sequence of $\theta = 2 - \phi = 1/\phi^2 = \frac{3-\sqrt{5}}{2}$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. We will denote the Fibonacci word by $f$, and the prefixes in the above construction by $f^{(n)}$. These are called the finite Fibonacci words.
Note that the continued fraction expansion of $\theta$ is $[0; 2, 1, 1, \ldots]$. Hence, following the above construction, we get

- $f^{(1)} = 1$,
- $f^{(2)} = 0$,
- $f^{(n)} = f^{(n-1)} f^{(n-2)}$, for $n \geq 3$.

Hence, for any $n \in \mathbb{N}$, $|f^{(n)}| = F_n$.

The following Lemma is an immediate result of the construction of the Fibonacci word given above.

**Lemma 3.2.** For any $n \geq 2$,

$$f = f^{(n)} f^{(n-1)} f^{(n)} f^{(n+1)} \ldots \quad (3.1)$$

Let $\mathcal{F}$ denote the set of all finite Fibonacci words and let $\mathcal{F}^* = \mathcal{F} \setminus \{1\}$. Denote by $\emptyset$ the set of all prefixes of $f$. Note that $\mathcal{F}^* = \mathcal{F} \cup \emptyset$, by Lemma 3.2 i.e., $f^{(n)} \in \emptyset$ for all $n \geq 2$.

### 3.1.2 Preliminaries

In the next two lemmas, we will prove that distinct finite Fibonacci words do not commute i.e., for any two positive integers $m$ and $n$, $f^{(m)} f^{(n)} \neq f^{(n)} f^{(m)}$.

**Lemma 3.3.** For any $n \geq 1$, $f^{(n+1)} f^{(n)} \neq f^{(n)} f^{(n+1)}$. In fact, they only differ at the very last two positions.

**Proof.** We will prove by induction on $n$ that $f^{(n+1)} f^{(n)}$ and $f^{(n)} f^{(n+1)}$ differ only at their last two digits.

For $n = 1$, $f^{(2)} f^{(1)} = 01$ and $f^{(1)} f^{(2)} = 10$.

Assume that for some $n \geq 1$, $f^{(n+1)} f^{(n)}$ and $f^{(n)} f^{(n+1)}$ differ only at their last two digits. Then

$$f^{(n+2)} f^{(n+1)} = f^{(n+1)} f^{(n)} f^{(n+1)}$$

and

$$f^{(n+1)} f^{(n+2)} = f^{(n+1)} f^{(n+1)} f^{(n)}.$$

Hence, by the induction hypothesis, $f^{(n+2)} f^{(n+1)}$ and $f^{(n+1)} f^{(n+2)}$ differ only at their last two digits. \qed
Lemma 3.4. Let $k$ and $n$ be positive integers. Then

$$f^{(n+k)} f^{(n)} \neq f^{(n)} f^{(n+k)}.$$  

Proof. Since $f^{(1)} = 1$ and the first letter of $f^{(n)}$ is 0 for all $n \geq 2$, the Lemma is true for $n = 1$. The case $k = 1$ follows from the Lemma 3.3.

Now, let $k, n \geq 2$ be arbitrary. Then since $f^{(n+k)} = f^{(n)} f^{(n-1)} f^{(n+1)} ... f^{(n+k-2)}$

$$f^{(n+k)} f^{(n)} = f^{(n)} f^{(n-1)} f^{(n+1)} ... f^{(n+k-2)} f^{(n)}$$

and

$$f^{(n)} f^{(n+k)} = f^{(n)} f^{(n-1)} f^{(n)} f^{(n+1)} ... f^{(n+k-2)}.$$  

Therefore, $f^{(n+k)} f^{(n)} \neq f^{(n)} f^{(n+k)}$ since the underlined parts are not equal by Lemma 3.3. \qed

Lemma 3.5. For any positive integers $n$ and $k$, $f^{(n+k)} f^{(n)} \in \wp$.

Proof. Let $k \geq 1$. Then since $n + k - 1 \geq n$, $f^{(n)}$ is a prefix of $f^{(n+k-1)}$. Hence, $f^{(n+k)} f^{(n)}$ is a prefix of $f^{(n+k+1)} = f^{(n+k)} f^{(n+k-1)}$ which itself is a prefix. Therefore, $f^{(n+k)} f^{(n)} \in \wp$. \qed

Corollary 3.6. For any positive integers $k$ and $n$, $f^{(n)} f^{(n+k)} \notin \wp$.

Lemma 3.7. Let $n \geq 4$. Then $f^{(n)} = P_1 P_2$ for some prefixes $P_1$ and $P_2 \in \wp$ if and only if

$P_2 = f^{(i)}$ for some $i \in \{n - 2k : 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \}$.

Proof. The backward implication follows easily by induction as, for any positive integer $n \geq 4$, $f^{(n)} = f^{(n-1)} f^{(n-2)}$ and any prefix of $f^{(n)}$ is a prefix of $f$.

We will prove the forward implication by induction on $n$.

For $n = 4$, $f^{(4)} = 010$ and therefore the only possibility is $P_1 = 01$ and $P_2 = 0 = f^{(2)}$.

Assume that the assumption holds for all $m \leq n$, for some $n \geq 4$. Then if $f^{(n+1)} = P_1 P_2$ for some $P_1, P_2 \in \wp$, $f^{(n)} f^{(n-1)} = P_1 P_2$.

Now, if $|P_2| > \left| f^{(n-1)} \right|$ then $P_2 = P_2' f^{(n-1)}$ for some $P_2' \in \wp$. Then $f^{(n)} f^{(n-1)} = P_1 P_2 = P_1 P_2' f^{(n-1)}$. Hence $f^{(n)} = P_1 P_2'$ for some $P_1, P_2' \in \wp$. Then, by induction hypothesis, $P_2' = f^{(n-2k)}$ for some $k \in \left[1, \left\lfloor \frac{n}{2} \right\rfloor \right]$. Therefore, $P_2 = P_2' f^{(n-1)} = f^{(n-2k)} f^{(n-1)}$. But, by Corollary 3.6, $P_2$ is not a prefix since $2k > 1$. Therefore, $|P_2| \leq |f^{(n-1)}|$.  

So, we must have $f^{(n-1)} = P_1' P_2$ for some $P_1' \in \wp$ (possibly empty). Hence, by induction hypothesis, $P_2 = F_i$ for some $i \in \{n - 1 - 2k : 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \}$. Since this set is a subset of $\{n + 1 - 2k : 1 \leq k \leq \left\lfloor \frac{n+1}{2} \right\rfloor \}$, the proof is complete. \qed
For each $n \geq 2$, there are infinitely many copies of $f^{(n)}$ in $f$. Table 3.2 shows the first three copies of $f^{(6)}$ in $f$. As it can be seen from the table, the first copy is at the beginning, the second copy starts right after the first one and the third one starts after $0100101001001010 = f^{(6)} f^{(5)}$ which also follows from Lemma 3.2. We will prove in the following lemma that this observation is true for all $n \geq 4$.

**Lemma 3.8.** For $n \geq 4$, the first three appearances of $f^{(n)}$ in $f$ are

i. $f = f^{(n)} \ldots$, at the beginning.

ii. $f = f^{(n)} f^{(n)} \ldots$, right after its first copy.

iii. $f = f^{(n)} f^{(n-1)} f^{(n)} \ldots = f^{(n+1)} f^{(n)} \ldots$, right after $f^{(n+1)}$.

**Proof.** Let $n \geq 4$ be a positive integer. The first appearance of $f^{(n)}$ is clearly at the beginning as $f^{(n)}$ is a prefix of $f$.

Now, if the second appearance $f^{(n)}$ is before the one in (ii) then we must have have

$$f^{(n)} = P_1 P_2 = P_2 P_3$$

for some $P_1, P_2, P_3 \in \wp$. Then by Lemma 3.7, $P_2 = f^{(n-2i)}$ and $P_3 = f^{(n-2j)}$ for some $i, j \in \{n - 2k : 1 \leq k \leq \lfloor \frac{n}{2} \rfloor \}$. But this implies that

$$|f^{(n-1)}| + |f^{(n-2)}| = |f^{(n)}| = |P_2| + |P_3| \leq |f^{(n-2)}| + |f^{(n-2)}| .$$

This is possibly only when $n = 3$. Since $n \geq 4$, we must have the second copy of $f^{(n)}$ right after the first copy.

The third copy of $f^{(n)}$ gives a factorization of the second copy of $f^{(n)}$ into two prefixes. But by Lemma 3.7, the second prefix has to be $f^{(n-2k)}$ for some $k \in \lfloor \frac{n}{2} \rfloor$. Therefore, the third copy cannot appear before the one in (iii).
**Lemma 3.9.** For any \( n \in \mathbb{N} \), there exist unique positive integers \( i_1, i_2, \ldots, i_k \geq 2 \) for some \( k > 0 \) such that

\[
  i_j > i_{j+1} + 1 \quad \text{for all } j \in [1, k - 1] \quad \text{and} \\
  n = F_{i_1} + F_{i_2} + \cdots + F_{i_k}.
\]

For \( n \in \mathbb{N} \), let \( C(n) = i_k \), where \( i_k \) is the smallest index in Lemma 3.9. For example, since \( 10 = 8 + 2 = F_6 + F_3 \), \( C(10) = 3 \).

The following theorem easily follows from the definition of the function \( C \).

**Theorem 3.10.** Let \( n \) be a positive integer and assume \( C(n) = m \) for some positive integer \( m \geq 2 \). Then

\[
  C(n + t) = C(t) \quad \text{for all } t \in [1, F_{m-1} - 1] \\
  C(n + F_{m-1}) > m.
\]

Let \( P^{(n)} \) denote the prefix of \( f \) of length \( n \). Define a new sequence \( l \) as follows. For \( n \in \mathbb{N} \), let \( l_n \) be the length of the longest prefix \( P \) such that there is a copy of \( P \) in \( f \) which starts at the \((n+1)\)st position i.e.,

\[
  l_n = \max\{m \in \mathbb{N} : P^{(n)}P^{(m)} \text{ is a prefix of } f\}.
\]

Now, we will show some properties of the function \( l \).

**Lemma 3.11.** For any integer \( m \geq 2 \),

\[
  l_{F_m} = F_{m+1} - 2.
\]

**Proof.** Let \( m \geq 2 \) be given. Then by Lemma 3.2, \( f = f^{(m)}f^{(m-1)}f^{(m)}f^{(m+1)}\ldots \). Since \( |f^{(m)}| = F_m \) and \( f^{(m+1)} = f^{(m)}f^{(m-1)} \) differs from \( f^{(m-1)}f^{m} \) only at the very last two positions, by Lemma 3.3, \( l_{F_m} = F_{m+1} - 2 \).

**Lemma 3.12.** Let \( m \geq 2 \). Then

\[
  l_n < F_m - 2 \quad \text{for all } n < F_m \quad \text{and} \\
  l_n = l_{n-F_m} \quad \text{for all } F_m < n < F_{m+1}.
\]
Proof. The proof will go by induction on $m$.

The assumptions are clearly true for $m = 2$ as $F_2 = 1$ and $F_3 = 2$.

Suppose that the assumptions are true for some $m \geq 2$ and let $F_{m+1} < n < F_{m+2}$ be arbitrary.

Since $n - F_{m+1} < F_{m+2} - F_{m+1} = F_m$, by induction hypothesis

$$l_{n-F_{m+1}} < F_m - 2$$

i.e., the length of the longest prefix starting from the position $n - F_{m+1} + 1$ is less than $F_m - 2$.

Since $l_{F_{m+1}} = F_{m+2} - 2$ and $n + l_{n-F_{m+1}} < F_{m+2} + F_m - 2 \leq F_{m+1} + F_{m+2} - 2 = F_{m+1} + l_{F_{m+1}}$, any prefix which starts at the position $n + 1$ should live inside the prefix of length $l_{F_{m+1}} = F_{m+2} - 2$ starting at the position $F_{m+1} + 1$. Hence, $l_n = l_{n-F_{m+1}}$.

Since $l_n < F_m - 2 \leq F_{m+1} - 2$ for all $n < F_m$, and $l_n = l_{n-F_m} < F_{m-1} - 2 < F_{m+1} - 2$ for all $F_m < n < F_{m+1}$, and $l_{F_m} = F_{m+1} - 2$, $l_n < F_{m+1} - 2$ for all $n < F_{m+1}$.

Therefore, the assumptions are true for all $m \geq 2$. \hfill \Box

Corollary 3.13. For any positive integer $n$,

$$l_n = F_{C(n)+1} - 2.$$

Theorem 3.14. Let $n \in \mathbb{N}$ be given. Then for all $i$ in \{1, 2, ..., $l_n$\} \{F_{C(n)-1}\}, $l_{n+i} < l_n$ and $l_{n+F_{C(n)-1}} > l_n$.

Proof. The second claim follows from the Corollary 3.13 since $C(n + F_{C(n)-1}) > C(n)$ by Theorem 3.10.

Let $i \in \{1, 2, ..., l_n\} \{F_{C(n)-1}\} = \{1, 2, ..., F_{C(n)-1} - 1\} \cup \{F_{C(n)-1} + 1, ..., F_{C(n)+1} - 2\}$ be arbitrary.

If $i \in \{1, 2, ..., F_{C(n)-1} - 1\}$ then $C(n+i) = C(i) < C(n) - 1$ by Theorem 3.10. So

$$C(n+i) = C(i) < C(n) - 1 < C(n) + 1.$$  

If $i \in \{F_{C(n)-1} + 1, ..., F_{C(n)+1} - 2\}$ then $i = F_{C(n)-1} + j$ for some $j \in \{1, 2, ..., F_{C(n)} - 2\}$.

So

$$C(n+i) = C(n + F_{C(n)-1} + j)$$

$$= C(j)$$ since $C(n + F_{C(n)-1}) > C(n)$ by Theorem 3.10

$$< C(n)$$

$$< C(n) + 1.$$
Hence $C(n + i) < C(n) + 1$ for all $i \in \{1, 2, \ldots, l_n\} \setminus \{F_{C(n) - 1}\}$. Therefore,

\[
l_{n+i} = F_{C(n)+1} - 2 < F_{C(n)+1} - 2 < l_n
\]

for all $i \in \{1, 2, \ldots, l_n\} \setminus \{F_{C(n) - 1}\}$.

\[\square\]

### 3.1.3 Chromatic Number

**Theorem 3.15.** Let $w = w_1w_2\ldots$ be an infinite word and let $r \in \mathbb{N}$ be arbitrary. Then for any $r$-coloring of the set of factors of $w$, there exists $n \geq 1$ such that $w_nw_{n+1}\ldots$ can be written as a product of (distinct) monochromatic factors.

**Proof.** Let $\psi$ be an $r$-coloring of the set of all factors of $w$ and let $G$ denote the complete graph on $\mathbb{N}$. Define an $r$-coloring $\gamma$ of the edges of $G$ as

\[
\gamma(i, j) = \psi(w_i \ldots w_{j-1})
\]

for $i < j \in \mathbb{N}$.

Then by Ramsey’s Theorem 1.2, there exists $n_1 < n_2 < \cdots$ such that $\gamma(n_i, n_j)$ is constant for all $i < j \in \mathbb{N}$. The result follows by requiring $n_{i+1} - n_i$, $i \geq 1$ to be a strictly increasing sequence.

Note that the number $n$ in Theorem 3.15 depends on the number of colors, $r$, and can be very large. This suggest the following definition.

**Definition 3.16.** Let $w$ be an infinite word. Define the chromatic number of $w$, denoted $\chi(w)$, to be the largest integer $r$, if it exists, such that for any $r$-coloring of the set of all factors of $w$, there is a monochromatic set of distinct factors $\{u_1, u_2, u_3, \ldots\}$ satisfying $w = u_1u_2u_3\ldots$.

We will prove that the chromatic number of the Fibonacci word is at most 2. The result will follow easily from the following two lemmas.

**Lemma 3.17.** The Fibonacci word can not be written as a product of distinct elements of $\mathcal{F}$ i.e., if $f = f^{(n_1)}f^{(n_2)}f^{(n_3)}\ldots$ then $n_i = n_j$ for some $i \neq j$. 
Proof. Assume on the contrary that \( f = f^{(a)} f^{(b)} f^{(c)} \ldots \) for some \( a, b, \ldots \) with \( a \neq b \) whenever \( i \neq j \). Let \( j = \min \{ i : n_i \geq n_i + 2 \} \). Hence \( n_j \geq n_1 + 2 \).

We will show that \( n_j = n_1 + 2 \).

Note that the first place that \( f^{(j)} \) can occur in the factorization is starting from the position \( |f^{(j)}| + 1 \), by Lemma 3.8. Therefore, the first \( |f^{(j)}| \) letters should be written as a product of distinct elements of \( \{ f^{(i)} : -1 \leq i \leq n_1 + 1 \} \). But since

\[
\sum_{i=-1}^{n_1+1} |f^{(i)}| = |f^{(n_1+3)}| - 1,
\]

we must have \( n_j \leq n_1 + 2 \). Therefore, \( n_j = n_1 + 2 \).

Hence, we have

\[
f = f^{(a)} A f^{(n_1+2)} \ldots
\]

where \( A \) is a product of distinct elements of \( \{ f^{(i)} : 1 \leq i \leq n_1 - 1 \} \cup \{ f^{(n_1+1)} \} \).

The smallest possible length of \( A \) occurs if \( f^{(n_1+2)} \) starts from the position \( |f^{(n_1+2)}| + 1 \) (corresponds to the second case of Lemma 3.8). In this case,

\[
|A| = |f^{(n_1+2)}| - |f^{(n_1)}| = |f^{(n_1+1)}|.
\]

The second smallest possible length of \( A \) occurs if \( f^{(n_1+2)} \) starts from the position \( |f^{(n_1+3)}| + 1 \) (corresponds to the third case of Lemma 3.8) which gives

\[
|A| = |f^{(n_1+3)}| - |f^{(n_1)}| = 2 |f^{(n_1+1)}|.
\]

But, since \( A \) is a product of distinct elements of \( \{ f^{(i)} : 1 \leq i \leq n_1 - 1 \} \cup \{ f^{(n_1+1)} \} \) and

\[
\sum_{i=1}^{n_1-1} |f^{(i)}| + |f^{(n_1+1)}| = \left( |f^{(n_1+1)}| - 1 \right) + |f^{(n_1+1)}| = 2 |f^{(n_1+1)}| - 1 < 2 |f^{(n_1+1)}|,
\]

the only possibility is \( |A| = |f^{(n_1+1)}| \). And since

\[
\sum_{i=1}^{n_1-1} |f^{(i)}| = |f^{(n_1+1)}| - 1,
\]
$f^{(n_1+1)}$ must be used in this factorization of $A$. Therefore, $A = f^{(n_1+1)}$. But then we must have

$$f = f^{(n_1)} f^{(n_1+1)} f^{(n_1+2)} \ldots$$

which is not true by Corollary 3.6.

Therefore, $f$ can not be written as a product of distinct elements of $\mathcal{F}$. \qed

**Remark 3.18.** It follows from Lemma 3.2 that if we delete the first 0 from $f$ then the resulting word, denote it by $f^*$, can be written as

$$f^* = f^{(1)} f^{(2)} f^{(3)} \ldots$$

**Lemma 3.19.** $f$ can not be written as a product of, not necessarily distinct, elements of $\varnothing - \mathcal{F}^*$ i.e., if $f = P^{(n_1)} P^{(n_2)} P^{(n_3)} \ldots$ for some $n_1, n_2, \ldots \in \mathbb{N}$ then $n_i \in F$ for some $i \in \mathbb{N}$.

**Proof.** Assume by contradiction that $f = P^{(n_1)} P^{(n_2)} P^{(n_3)} \ldots$ for some $n_1, n_2, \ldots \in \mathbb{N}$ where none of $n_i$’s is a Fibonacci number.

For $i \geq 2$, let $m_i = \sum_{j=1}^{i-1} n_j$. Then $n_i = |P^{(n_i)}| \leq l_{m_i}$ for all $i \geq 2$. Hence, for all $i \geq 2$, $l_{m_{i+1}} = l_{m_i+1} < l_{m_i}$ by Theorem 3.14 since $n_i \in \{1, 2, \ldots, l_{m_i}\}$ and it is not a Fibonacci number.

Therefore, $n_i \leq m_{i} \leq m_{i+1} - i + 2$ for all $i \geq 2$.

Therefore, such a factorization is impossible. \qed

**Theorem 3.20.** The chromatic number of the Fibonacci word is at most 2.

**Proof.** We will give a 3-coloring of the factors of $f$ such that $f$ cannot be written as a product of distinct factors with the same color.

Color all factors in $\varnothing - \mathcal{F}^*$ by 0, all factors in $\mathcal{F}$ by 1 and everything else by 2. Then the result follows from Lemmas 3.17, 3.19 and the fact that first factor of the product has to be a prefix. \qed

### 3.2 Families of sequences defined in terms of $F$

Let $D$ be a set of positive integers. Denote by $AP_D$ the collection of all arithmetic progressions whose gaps belong to $D$. If $D = \mathbb{N}$ then $AP_D = AP$, the set of all arithmetic progressions in $\mathbb{N}$. An element of $AP_D$ is called a $D$-a.p. Also, for a positive integer $d$,
by a $d$-a.p. we mean an arithmetic progression whose gap is $d$. Since $AP_D$ is a subset of $AP$, van der Waerden theorem does not guarantee regularity of $AP_D$. (For the definition of regularity refer to Chapter 1.)

We call a set $D$ $r$-large if $AP_D$ is $r$-regular and large if $AP_D$ is regular.

In [10], T. Brown, et al. have investigated some properties of $r$-large sets (especially 2-large) and large sets. Some of their results will be given below without proof.

**Theorem 3.21.** If $D$ is a 2-large set then for each integer $m$, $D$ contains a multiple of $m$.

Therefore, $D$ contains an infinite number of multiples of $m$, for each integer $m$.

**Theorem 3.22.** Let $D = \{a_k\}_{k=1}^{\infty}$ be a sequence of positive integers where either

$$a_k \geq 3a_{k-1} \text{ for } k \geq 2$$

or

$$a_1 = 1, \ a_2 = 2 \text{ and } a_k \geq 3a_{k-1} \text{ when } k \geq 3.$$

Then $D$ is not 2-large.

**Theorem 3.23.** If $D = \{a_k\}_{k=1}^{\infty}$ is an increasing sequence of positive integers where $a_k$ divides $a_{k+1}$ for all $k \geq 1$, then $D$ is not 2-large.

**Theorem 3.24.** If $C$ is not $r$ large and $D$ is not $s$-large then $C \cup D$ is not $rs$-large.

**Theorem 3.25.** Let $c > 1$ be a fixed real number. Let $D = \{a_k\}_{k=1}^{\infty}$ be a sequence of positive integers such that $a_i > ca_{i-1}$ for all but a finite number of $i \geq 2$. Then $D$ is not large.

In [28], Landman and Robertson investigated the Ramsey properties of a particular superset of $AP_D$, namely, $D$-diffsequences.

**Definition 3.26.** Let $D$ be a set of positive integers. A sequence $x_1 < x_2 < \cdots < x_k$ of positive integers is called a $k$-term $D$-diffsequence if $x_{i+1} - x_i \in D$ for $i = 1, 2, \ldots, k - 1$.

**Definition 3.27.** Let $r \geq 1$. A set of positive integers $D$ is called $r$-accessible if for every $r$-coloring of $\mathbb{N}$ and for every $k \geq 1$, there is a monochromatic $k$-term $D$-diffsequence. If $D$ is $r$-accessible for all $r$, we say $D$ is accessible.

If $D$ is not an accessible set then the maximum $r$ for which $D$ is $r$-accessible is called the degree of accessibility of $D$, and is denoted by $doa(D)$. 
Note that if $D$ is $r$-large for some $r$ then it is also $r$-accessible as every $D$-a.p. is also a $D$-diffsequence. Hence if a set of positive integers is large then it is also accessible. Brown [29] has conjectured that the contrapositive of this statement is also true. Namely, he conjectured that a set of positive integers is accessible if and only if it is large. In 2005, Jungić [24] has settled this conjecture in the opposite.

As an immediate corollary of Theorem 3.25 (from [10]), $AP_F$ is not regular, since the Fibonacci numbers have the asymptotic ratio $F_i/F_{i-1} \sim (1 + \sqrt{5})/2$. By directly applying the method of the proof of Theorem 3.25, as presented in [10], to the set $F$, one finds that $dor(AP_F) \leq 7$. One can obtain a better upper bound (Proposition 3.28) by employing a different line of reasoning. The proof uses the Theorems 3.21, 3.22 and 3.24.

**Proposition 3.28.** [1] The degree of regularity of $AP_F$ is at most 3.

It is obvious from the definitions that for all $D$, $doa(D) \geq dor(AP_D)$. A lower bound $doa(F)$ has been noted in [28].

**Theorem 3.29.** [28] The degree of accessibility of $F$ is at least 2.

But an upper bound on $doa(F)$ was not known. In the following theorem, we prove that $doa(F)$ is at most 5.

**Theorem 3.30.** The degree of accessibility of $F$ is at most 5.

The proof makes use of the following three lemmas and the Corollary 3.34. Lemma 3.31 is well-known and Lemma 3.32 is due to T.C. Brown[6].

Let $\alpha = \frac{\sqrt{5} - 1}{2}$ and let $g$ denote the function defined on $\mathbb{N}$ by $g(m) = 4m + 2\lfloor m\alpha \rfloor$. The function $g$ arises naturally from the proof of Theorem 3.30.

**Lemma 3.31.** For any $n \geq 0$, $2 \cdot \sum_{i=1}^{n} F_{3i+1} = F_{3n+3}$.

**Lemma 3.32.** [6] For any $N \geq 2$, if $N - 1 = F_{i_1} + F_{i_2} + \cdots + F_{i_k}$ for some $i_1, i_2, \ldots, i_k$ where $i_{j+1} \geq i_j + 2$ for all $1 \leq j \leq k - 1$ and $i_1 \geq 2$ then

$$\lfloor N\alpha \rfloor = F_{i_{k-1}} + F_{i_{k-2}} + \cdots + F_{i_1} + 1.$$

**Lemma 3.33.** For any $n \in \mathbb{N}$,

$$g \left( \sum_{i=1}^{n} F_{3i-1} \right) = F_{3n+3} - 4 \text{ and } g \left( 1 + \sum_{i=1}^{n} F_{3i-1} \right) = F_{3n+3} + 2.$$
Proof. Since \( g(1) = 4 \) and \( g(2) = 10 \), the claim is obviously true for \( n = 1 \). Now by Lemma 3.32, for \( n \geq 2 \)

\[
g \left( \sum_{i=1}^{n} F_{3i-1} \right) = 4 \cdot \sum_{i=1}^{n} F_{3i-1} + 2 \cdot \left\lfloor \alpha \cdot \sum_{i=1}^{n} F_{3i-1} \right\rfloor
\]

This, along with Lemma 3.31 and the fact that for all positive integers \( m \), \( F_{3m-2} + 2F_{3m-1} = F_{3m+1} \), implies that

\[
g \left( \sum_{i=1}^{n} F_{3i-1} \right) = 4 + 2 \cdot \sum_{i=1}^{n} F_{3i-1} + 2 \cdot \sum_{i=2}^{n} F_{3i-2}. \tag{3.2}
\]

Also, for \( n \geq 2 \), using 3.2 we have

\[
g \left( 1 + \sum_{i=1}^{n} F_{3i-1} \right) = 4 + 4 \cdot \sum_{i=1}^{n} F_{3i-1} + 2 \cdot \left\lfloor \alpha \cdot \left( 1 + \sum_{i=1}^{n} F_{3i-1} \right) \right\rfloor
\]

which completes the proof. \( \square \)

The following is an immediate corollary.

**Corollary 3.34.** For any \( m \in \mathbb{N} \), \( \{g(m), g(m) + 2\} \cap F = \emptyset \).

Proof. If the claim were false then for some positive integer \( m \), \( g(m) = F_{3n} \) or \( g(m) + 2 = F_{3n} \) for some \( n \geq 2 \) since \( g(m) \geq 4 \) is even. From Lemma 3.33 and the fact that \( g \) is an increasing function, it would then follow that

\[
\sum_{i=1}^{n-1} F_{3i-1} < m < 1 + \sum_{i=1}^{n-1} F_{3i-1},
\]

which is impossible. \( \square \)
Now we have all tools to prove Theorem 3.30.

Proof of Theorem 3.30. To prove the theorem, we give a 6-coloring of $\mathbb{N}$ that avoids 2-term monochromatic $F$-diffsequences.

Let $w$ be the sequence obtained from the Fibonacci word $f$ by replacing 0 by 6 and 1 by 4 i.e.,

$$w = 6466464664\ldots$$

Therefore, for any $n \in \mathbb{N}$,

$$w_n = 6 - 2 (\lfloor \theta (n + 1) \rfloor - \lfloor \theta n \rfloor) = 4 + 2 (\lfloor \alpha (n + 1) \rfloor - \lfloor \alpha n \rfloor)$$

where $\theta = \frac{3 - \sqrt{5}}{2} = 1 - \alpha$, from Section 3.1.

For $n \in \mathbb{N}$, let $t_n = 1 + \sum_{i=1}^{n-1} w_i$.

Let $C$ be the 6-coloring on $\mathbb{N}$ determined by the partition consisting of the sets $C_1, C_2, \ldots, C_6$ defined as follows. Let $C_1 = \{t_n : n \in \mathbb{N}\}$ and for $2 \leq i \leq 6$, let $C_i = (i - 1 + C_1) - \bigcup_{j=1}^{i-1} C_j$. Clearly,

$$k \neq l \Rightarrow C_k \cap C_l = \emptyset.$$

Since gaps between any two consecutive elements of $C_1$ are 4 or 6, for any positive integer $n$, there are $k \in \mathbb{N}$ and $i \in [1, 6]$ so that $n = t_k + i - 1$. Thus, $\mathbb{N} = \bigcup_{i=1}^{6} C_i$.

Next, note that to prove the theorem, it is enough to show that there is no 2-term $F$-diffsequence contained in $C_1$. Moreover, since all elements of $C_1$ are odd, it is enough to prove that for any positive integers $m$ and $n$, $n < m$,

$$t_m - t_n \notin \{F_{3i} : i \in \mathbb{N}\}.$$

Let $n$ and $m$ be positive integers such that $n < m$. Let $N = m - n$. Since for any two real numbers $x$ and $y$,

$$|x+y| - (|x| + |y|) \in \{0, 1\},$$

we have

$$t_m - t_n = t_{n+N} - t_n = \sum_{i=n}^{n+N-1} w_i = 4N + 2 (\lfloor (n+N)\alpha \rfloor - \lfloor n\alpha \rfloor) \in \{g(N), g(N) + 2\}.$$

Hence, by Corollary 3.34, $t_m - t_n \notin F$. Therefore, $C_1$ does not contain any 2-term $F$-diffsequence, which completes the proof. \qed
3.3 Remarks and Questions

In Section 3.1, we showed that the chromatic number of the Fibonacci word, $\chi(f)$, is at most 2. Hence $\chi(f) = 1$ or 2. The actual value of $\chi(f)$ is yet to find. We have considered only the Fibonacci word. We wonder if the ideas of this section can be generalized to obtain upper bounds on the chromatic number of some other Sturmian word.

The set of Fibonacci numbers $F$ can be obtained as the set of the denominators of the convergents of the simple continued fraction for $\alpha = \frac{\sqrt{5} - 1}{2}$. In Section 3.2, we showed $doa(F) \leq 5$. This bound is the best possible with the algorithm used. Can a similar algorithm be used to find upper bounds for the degree of accessibility of some other sets which are obtained similar to $F$?

See also Section 5.3 for an application of degree of accessibility.
Chapter 4

Some 2-color Rado numbers

4.1 Introduction

In 1936, Rado [36] proved the following theorem.

Theorem 4.1. Let $a, b$ and $c$ be nonzero integers, not all positive or all negative. Then for any 2-coloring of $\mathbb{N}$, there exists $x, y,$ and $z$ monochromatic such that $ax + by + cz = 0$, i.e., $ax + by + cz = 0$ is 2-regular.

Note that in the above theorem, the assumption on $a, b$ and $c$ is to guarantee that $ax + by + cz = 0$ has solution in $\mathbb{N}$.

Fox and Radoićić [16] showed that $x + 2y - 4z = 0$ is not 3-regular. Therefore, 2 colors in Theorem 4.1 is best possible.

Harborth and Maasberg [21], [22] have considered a special case of Theorem 4.1, namely the equation $a(x + y) = bz$ for $a, b > 0$. They have calculated the corresponding 2-color Rado numbers for any $a, b \in \mathbb{N}$.

In [20] they considered the equation $ax + by + cz = 0$ as a homogeneous second order linear recurrence and studied the Ramsey properties of the corresponding collection of sequences which satisfy the recurrence. Namely, they considered the following problem.

Problem 4.2. [20] Let $a, b, c, s$ be nonzero integers with $s \geq 3$. Find the largest $r \in \mathbb{N}$, if it exists, such that every $r$-coloring of $\mathbb{N}$ yields a monochromatic $s$-term sequence $x_1, x_2, \ldots, x_s$ that satisfies the homogeneous second order recurrence $ax_i + bx_{i+1} + cx_{i+2} = 0$.

If such an $r$ exists, it is called the degree of partition regularity of the given recurrence.
for \( s \)-term sequences and denoted by \( k_0(s; a, b, c) \). If there is no \( r \) for which the statement is true, we write \( k_0(s; a, b, c) = 0 \), and \( k_0(s; a, b, c) = \infty \) if it is true for all \( r \).

**Remark 4.3.** Let \( a, b, c \in \mathbb{Z} \). Then for any \( s \geq 3 \), the sequence \( x_1, x_2, \ldots, x_s \) satisfies the recurrence \( ax_i + bx_{i+1} + cx_{i+2} = 0 \) if and only if the sequence \( y_1, y_2, \ldots, y_s \), where \( y_i = x_{s-i+1} \), satisfies the recurrence \( cy_i + by_{i+1} + ay_{i+2} = 0 \). Therefore, \( k_0(s; a, b, c) = k_0(s; c, b, a) \). for all \( a, b, c \in \mathbb{Z} \) and \( s \geq 3 \).

**Remark 4.4.** Let \( a, b, c \in \mathbb{Z} \). Then

i. \( k_0(s; a, b, c) = k_0(s; na, nb, nc) \), for any nonzero integer \( n \).

ii. For any \( s \geq 3 \), \( k_0(s + 1; a, b, c) \leq k_0(s; a, b, c) \).

One can find a detailed study on \( k_0(s; a, b, c) \) in [20] and [21]. We list some of their results here, without proof.

**Theorem 4.5.** [20] \( k_0(s; a, b, c) = \infty \) if and only if one of the following is true.

i. \( s = 3 \) and one of \( a + b + c, a + b, a + c, b + c \) is equal to zero.

ii. \( s = 4 \) and \( a + b + c = 0 \) or \( a = b = -c \) or \( a = -b = c \).

iii. \( s \geq 5 \) and \( a + b + c = 0 \)

The proof only uses the columns condition (definition 1.5).

For \( s = 3 \) terms, the following results are obtained.

**Theorem 4.6.**

(a) If \( k_0(3; a, b, c) \neq \infty \) then
\[
k_0(3; a, b, c) \leq p - 2
\]
where \( a, b > 0 \) and \( c < 0 \) and \( p \) is a prime number greater than \( \max\{a + b, -c\} \).

(b)
\[
k_0(3; a, a, c) \leq \begin{cases} 3 & \text{if } -2 < a/c < -1/4, a/c \neq -1 \\ & \text{and } a/c \neq -1/2 \\ 2 & \text{if } a/c \leq -2 \text{ or } -1/4 \leq a/c \leq 0 \end{cases}
\]
where \( a > 0 \) and \( c < 0 \).
For $s = 4$ terms, the following results are obtained.

**Theorem 4.7.**

(a) \[ k_0(4, 1, 1, -1) = \infty, \]

i.e., for any finite coloring of $\mathbb{N}$, there is a 4-term monochromatic sequence $x_1, x_2, x_3, x_4$ that satisfies the recurrence $x_i + x_{i+1} = x_{i+2}$. (It is known that any 2-coloring of $[1, 17]$ produces such a monochromatic sequence. [21])

(b) If $\gcd(a, b, c) = 1$ and if there is a prime $p$ that divides exactly two of the coefficients to the same power then $k_0(4; a, b, c) \leq 2$.

(c) $k_0(4; 1, -1, c) \leq 3$ if $c < 0, c \neq 1 - 2^\gamma$ for $\gamma > 1$.

(d) $k_0(4; 1, b, -1) \leq 3$ if $|b| \geq 2$ and there is a prime $q$ such that $q | (b^2 + 1)$ but $q \nmid (b + 1)$.

And for $s = 5$ terms, the following results are obtained.

**Theorem 4.8.** $k_0(5; 1, -1, -1) = k_0(5; -1, 1, 1) = 1$.

We introduce a new Ramsey type function $S_r(a, b, c)$ which is defined as the maximum $s \geq 0$ such that for any $r$-coloring of $\mathbb{N}$ there is a monochromatic sequence $x_1, x_2, \ldots, x_s$ satisfying the recurrence $ax_i + bx_{i+1} + cx_{i+2} = 0, 1 \leq i \leq s - 2$. If the statement is true for all $s$, we write $S_r(a, b, c) = \infty$. Alternatively, we can define $S_r(a, b, c)$ as

\[ S_r(a, b, c) = \max \{ s \geq 0 : k_0(s; a, b, c) \geq r \} \]

We will investigate $S_2(a, b, c)$.

Before proceeding, we mention some notations that we will use. For $x, t \in \mathbb{N}$ if $x = t^u(tv + w)$ for some integers $u, v, w \in \mathbb{Z}$ with $u, v \geq 0$ and $1 \leq w \leq n - 1$ then we will write $x = (u, v, w)_t$. For a prime $p$, if $l \in \mathbb{Z}$ is such that $p \nmid l$, $o_p(l)$ denotes the order of $l$ in the multiplicative group $\mathbb{Z}_p^*$. For $n, x, y \in \mathbb{Z}$, by $x \equiv y \pmod{n}$, we mean $x \equiv y \pmod{n}$. And lastly, for $n \in \mathbb{Z}$, $(n)_2$ is the remainder when $n$ is divided by 2.
4.2 The function $S_2(a, b, c)$

As an immediate corollary of Remarks 4.3 and 4.4, we get the following.

**Corollary 4.9.** Let $a, b, c \in \mathbb{Z}$. Then

i. $S_2(a, b, c) = S_2(c, b, a)$.

ii. $S_2(a, b, c) = S_2(na, nb, nc)$, for any nonzero integer $n$.

Therefore, we will only consider those $(a, b, c)$ where $\text{gcd}(a, b, c) = 1$.

Theorem 4.1 implies the following corollary.

**Corollary 4.10.** Let $a, b$ and $c$ be nonzero integers, not all positive or all negative. Then

i. $k_0(3; a, b, c) \geq 2$.

ii. $S_2(a, b, c) \geq 3$.

**Corollary 4.11.** (Rado [36]) Let $a, b, c \in \mathbb{Z}$ such that $a + b + c = 0$. Then $S_2(a, b, c) = \infty$.

**Theorem 4.12.** Let $a, b, c \in \mathbb{N}$ with $c \leq b$. Then $S_2(a, b, -c) \leq 4$.

**Proof.** Let $\alpha = (a + b)/c$. For each positive integer $i$, let $B_i = [\alpha^i, \alpha^{i+1}) \cap \mathbb{N}$. Let $\chi$ be the 2-coloring of $\mathbb{N}$ defined by $\chi(x) = (i)_2$ if $x \in B_i$ for some $i \geq 1$. We will show that under $\chi$ there is no 5-term monochromatic sequence satisfying the recurrence $ax_n + bx_{n+1} = cx_{n+2}$.

Assume for a contradiction that $\{x_1, x_2, x_3, x_4, x_5\}$ is monochromatic and satisfies the recurrence $ax_n + bx_{n+1} = cx_{n+2}$. Since $b \geq c$ and $a > 0$, we have $x_2 < x_3 < x_4 < x_5$. If $x_2, x_3 \in B_i$ for some $i$, then

$$\alpha^{i+1} \leq x_4 = \frac{a}{c}x_2 + \frac{b}{c}x_3 < \alpha^{i+2}$$

which is impossible since this would imply $\chi(x_4) \neq \chi(x_3)$. Similarly, there is no $i$ such that $x_3, x_4 \in B_i$. Since $\chi(x_2) = \chi(x_3)$ and $x_2 < x_3$, there exists $i < j$, $j - i > 0$ even, such that $x_2 \in B_i$ and $x_3 \in B_j$. But then, we have

$$\alpha^{j} \leq x_3 < x_4 < \alpha x_3 < \alpha^{j+2},$$

i.e., $x_4 \in B_j \cup B_{j+1}$. Hence $x_4$ has to be in $B_j$, which gives the desired contradiction. \qed
Corollary 4.13. \((\text{Theorem 4.5 (ii)}\) \(S_2(1,1,-1) = 4\) \((\text{In fact, } S_r(1,1,-1) = 4 \text{ for all } r \geq 1)\).\\

Theorem 4.14. \(S_2(1,-b,1) = 3\) for all \(b \geq 3\).

Proof. Since \(b\) is positive, \(S_2(1,-b,1) \geq 3\)

First, assume that \(b\) is odd and define the 2-coloring \(\chi\) as

\[
\chi(x) = \begin{cases} 
0 & \text{if } x \equiv_b 1, 2, \ldots, \frac{b-1}{2}, \\
1 & \text{if } x \equiv_b \frac{b+1}{2}, \ldots, b-1, \\
\chi(x/b) & \text{if } x \equiv_b 0.
\end{cases}
\]

Assume that \(\{x_1, x_2, x_3, x_4\}\) is monochromatic and satisfies the recurrence \(x_i - bx_{i+1} + x_{i+2} = 0\) with \(x_4\) minimal. Then \(x_1 + x_3 \equiv_b 0\) and \(x_2 + x_4 \equiv_b 0\). This is possible only if \(x_1 \equiv_b x_2 \equiv_b x_3 \equiv_b x_4 \equiv_b 0\). Let \(y_i = x_i/b < x_i, 1 \leq i \leq 4\). Then \(\chi(y_i) = \chi(x_i)\) for all \(i \in \{1, 4\}\) and \(y_i - by_{i+1} + y_{i+2} = 0, i \in \{1, 2\}\). This is a contradiction since \(y_4 < x_4\).

Now assume that \(b\) is even. Let \(b = 2b'\) for some \(b' \geq 2\). Define the 2-coloring \(\chi\) as

\[
\chi(x) = \begin{cases} 
0 & \text{if } x \equiv_b 1, 2, \ldots, b'-1, \\
1 & \text{if } x \equiv_b b' + 1, \ldots, b-1, \\
\chi(x/b') & \text{if } x \equiv_{b'} 0.
\end{cases}
\]

The proof is similar to the proof of the first case. \(\square\)

Theorem 4.15. Let \(a, b, c\) be integers such that \(\gcd(a, b, c) = 1\). If there is a prime \(p\) which divides exactly two of the coefficients to the same power, then

\[S_2(a, b, c) \leq 3.\]

Proof. Suppose that \(p\) is a prime which divides exactly two elements of the set \(\{a, b, c\}\) to the same power and let \(k \in \mathbb{N}\) be that power.

Define the 2-coloring \(\chi\) as \(\chi(x) = \left(\left\lfloor \frac{x}{k} \right\rfloor \right)_2\) where \(x = (u, v, w)_p\).

Assume that \(p \nmid c\) and \(a = Ap^k\) and \(b = Bp^k\) with \(p \nmid A\) and \(p \nmid B\).

Assume \(\{x_1, x_2, x_3, x_4\}\) is monochromatic and satisfies \(ax_i + bx_{i+1} + cx_{i+2} = 0\). Let \(x_i = (u_i, v_i, w_i)_p\) for some \(u_i, v_i, w_i, 1 \leq i \leq 4\). Then

\[
Ap^{u_1+k}(pv_1 + w_1) + Bp^{u_2+k}(pv_2 + w_2) = -cp^{u_3}(pv_3 + w_3) \quad (4.1)
\]
\[
Ap^{u_2+k}(pv_2 + w_2) + Bp^{u_3+k}(pv_3 + w_3) = -cp^{u_4}(pv_4 + w_4) \quad (4.2)
\]
If \( u_1 < u_2 \) then \( u_1 + k = u_3 \) and hence,
\[
\left\lfloor \frac{u_3}{k} \right\rfloor = 1 + \left\lfloor \frac{u_1}{k} \right\rfloor.
\]
But this is not possible since \( \chi(x_1) = \chi(x_3) \). Similarly, \( u_2 < u_1 \) is not possible. Hence, we must have \( u_1 = u_2 \). In the same way, we must have \( u_2 = u_3 \). Then, since \( k \geq 1 \), \( p^{u_3+1} \) divides the left-hand side of (4.1) but not the right-hand side, a contradiction.

The proof works the same way if we assume \( p \nmid a \) or \( p \nmid b \).

\[\blacksquare\]

**Theorem 4.16.** Let \( p \) be a prime and let \( a, c \) and \( l \) be arbitrary integers not divisible by \( p \). Let \( C \equiv_p -c/a \). Then

(i) If \( p \) is odd and \( o_p(C) \) is even then \( S_2(a, -p^kl, c) \leq 3 \).

(ii) If \( p \) is odd and \( o_p(C) > 1 \) is odd then \( S_2(a, -p^kl, c) \leq 5 \).

(iii) If \( p = 2, k \geq 2 \) and \( a \equiv_4 c \) then \( S_2(a, -2^kl, c) \leq 3 \).

**Proof.** We will first define a 2-coloring \( \psi \) of \( \mathbb{Z}_p^* \) that we will use in the proof of (i) and (ii).

Let \( l \in \mathbb{Z} \) be such that \( p \nmid l \). Let \( d = o_p(l) \) be the order of \( l \) in the multiplicative group \( \mathbb{Z}_p^* \).

Let \( H = \langle l \rangle \) be the cyclic subgroup generated by \( l \) and let \( \{a_1, a_2, \ldots, a_t\} \) be a complete set of representatives in \( \mathbb{Z}_p^*/H \).

Define a 2-coloring \( \psi_{(p,l)} \) of \( \mathbb{Z}_p^* \) as \( \psi_{(p,l)}(x) = (i)_2 \) if \( x = aj^i \) for some \( 1 \leq i \leq d-1 \) and \( 1 \leq j \leq t \). Then
\[
\psi_{(p,l)}(x) = \psi_{(p,l)}(lx) \text{ if and only if } d \text{ is odd and } x = aj^d \text{ for some } j.
\]  

(i) Define the 2-coloring \( \chi \) as \( \chi(x) = \psi_{(p,C)}(w) \) where \( x = (u, v, w)_p \). Assume that \( \{x_1, x_2, x_3, x_4\} \) is monochromatic satisfying the recurrence \( ax_i - (p^k) x_{i+1} + cx_{i+2} = 0 \).

Let \( x_i = (u_i, v_i, w_i)_p \) for some \( u_i, v_i \) and \( w_i, 1 \leq i \leq 4 \). Then \( \chi(x_i) = \psi_{(p,C)}(w_i) \), i.e., \( \{w_1, w_2, w_3, w_4\} \) is monochromatic under \( \psi_{(p,C)} \), and
\[
\begin{align*}
ap^{u_1}(pv_1 + w_1) + cp^{u_3}(pv_3 + w_3) &= p^{u_2+k}(pv_2 + w_2) \quad (4.4) \\
ap^{u_2}(pv_2 + w_2) + cp^{u_4}(pv_4 + w_4) &= p^{u_3+k}(pv_3 + w_3) \quad (4.5)
\end{align*}
\]

If \( u_1 < u_3 \) then \( u_1 = u_2 + k \) by (4.4). But then, by (4.5) \( u_2 = u_4 \) and hence
\[
p^{u_2} (p(av_2 + cw_4) + aw_2 + cw_4) = p^{u_3+k}(pv_3 + w_3).
\]
Since \( u_2 < u_3 + k \), this is possible only if \( aw_2 + cw_4 \equiv_p 0 \). Hence \( w_2 \equiv_p Cw_4 \). But since \( \alpha_p(C) \) is even and \( \psi_{(p,C)}(w_2) = \psi_{(p,C)}(w_4) \), this is not possible by (4.3).

If \( u_3 < u_1 \) then \( u_3 = u_2 + k > u_2 \), from (4.4). As above, we must then have \( u_2 = u_4 \) and \( w_2 \equiv_p Cw_4 \), which is a contradiction as above.

So assume \( u_1 = u_3 \). Since \( \psi_{(p,C)}(w_1) = \psi_{(p,C)}(w_3) \), by (4.3) \( aw_1 + cw_3 \not\equiv_p 0 \). Hence \( u_1 = u_3 = u_2 + k \), which is again contradiction.

Hence, in the case of \( p \) is odd and \( \alpha_p(C) \) is even, \( S_2(a, -p^k l, c) \leq 3 \).

(ii) Define \( \chi \) as in (i). Assume that \( \{x_1, x_2, x_3, x_4, x_5, x_6\} \) is monochromatic satisfying the recurrence \( ax_i - (p^k l) x_{i+1} + cx_{i+2} = 0 \). Let \( x_i = (u_i, v_i, w_i)_p \) for some \( u_i, v_i \) and \( w_i \), \( 1 \leq i \leq 6 \). Then \( \{w_1, w_2, w_3, w_4, w_5, w_6\} \) is monochromatic under \( \psi_{(p,C)} \), as in (i).

Now we have, equations (4.4), (4.5) and

\[
\begin{align*}
ap^{u_3}(pv_3 + w_3) + cp^{v_3}(pv_5 + w_5) &= p^{u_4+k}(pv_4 + w_4) \quad (4.6) \\
ap^{u_4}(pv_4 + w_4) + cp^{v_4}(pv_6 + w_6) &= p^{u_5+k}(pv_5 + w_5) \quad (4.7)
\end{align*}
\]

If \( u_1 < u_3 \) then \( u_1 = u_2 + k, u_2 = u_4 \) and \( w_2 \equiv_p Cw_4 \), same as in (i). Since \( u_2 + k = u_1 < u_3, u_4 + k < u_3 \). Hence, from (4.6), we get \( u_5 < u_3 \). If we now apply the preceding to the equations (4.6) and (4.7), we get \( w_4 \equiv_p Cw_6 \). Therefore,

\[
\begin{align*}
w_2 &\equiv_p Cw_4 \equiv_p C^2w_6 \\
\psi_{(p,C)}(w_2) &= \psi_{(p,C)}(w_4) = \psi_{(p,C)}(w_6)
\end{align*}
\]

which contradicts (4.3).

If \( u_3 < u_1 \) then \( u_3 = u_2 + k, u_2 = u_4 \) and \( w_2 \equiv_p Cw_4 \), same as in (i). So \( u_4 + k = u_3 \). From (4.6), we get \( u_5 \geq u_4 + k \) and hence, from (4.7), \( w_4 \equiv_p Cw_6 \). This is a contradiction as above.

So assume \( u_1 = u_3 \). Assume also that \( w_1 \not\equiv_p Cw_3 \). Then \( u_3 = u_2 + k \), from (4.4). Hence we get the same contradiction,

\[
\begin{align*}
w_2 &\equiv_p Cw_4 \equiv_p C^2w_6 \\
\psi_{(p,C)}(w_2) &= \psi_{(p,C)}(w_4) = \psi_{(p,C)}(w_6)
\end{align*}
\]
Hence, \( w_1 \equiv_p C w_3 \). From (4.4), we get \( u_1 = u_3 < u_2 + k \). Therefore, from (4.5), we get \( u_4 + k > u_3 \). But this implies, from (4.5), \( u_3 = u_5 \) and \( w_3 \equiv_p C w_5 \). Hence

\[
w_1 \equiv_p C w_3 \equiv_p C^2 w_5
\]

\[
\psi_{(p,C)}(w_1) = \psi_{(p,C)}(w_3) = \psi_{(p,C)}(w_5).
\]

contradicting to (4.3).

Hence, in the case of \( p \) and \( o_p(C) \) are both odd, \( S_2(a, -p^k l, c) \leq 5 \).

(iii) Define \( \chi \) as \( \chi(x) = (v)_2 \) where \( x = (u, v, 1)_2 = 2^u(2v + 1) \). Assume \( \{x_1, x_2, x_3, x_4\} \) is monochromatic and satisfies \( ax_i - (2^k l)x_{i+1} + cx_{i+2} = 0 \). Let \( x_i = (u_i, v_i, 1) \) for some \( u_i, v_i \geq 0, 1 \leq i \leq 4 \). Then \( v_1, v_2, v_3 \) and \( v_4 \) are all of the same parity. Then

\[
2^{u_1}a(2v_1 + 1) + 2^{u_3}c(2v_3 + 1) = 2^{u_2+k}l(2v_2 + 1) \tag{4.8}
\]

\[
2^{u_2}a(2v_2 + 1) + 2^{u_4}c(2v_4 + 1) = 2^{u_3+k}l(2v_3 + 1) \tag{4.9}
\]

If \( u_1 < u_3 \) then from (4.8), \( u_1 = u_2 + k \) and from (4.9) \( u_2 = u_4 \). Hence

\[
2^{u_2}(2(av_2 + bv_4) + a + c) = 2^{u_3+k}l(2v_3 + 1)
\]

Since \( a \) and \( c \) are both odd and \( a \equiv 4 \), \( a + c \equiv 2 \). Hence,

\[
2^{u_2+1}\left((av_2 + bv_4) + \frac{a + c}{2}\right) = 2^{u_3+k}l(2v_3 + 1)
\]

Since \( a \) and \( b \) are both odd, and \( v_2 \) and \( v_4 \) have the same parity, \( av_2 + bv_4 \) is even. Therefore, \( u_2 + 1 = u_3 + k \), since \( \frac{a + c}{2} \) is odd. But we also have \( u_2 + k < u_3 \). Since \( k \geq 2 \), this is not possible.

Similarly, if \( u_3 < u_1 \) then we get \( u_3 = u_2 + k \), \( u_2 = u_4 \) and \( u_2 + 1 = u_3 + k \), which is again not possible.

So, assume \( u_1 = u_3 \). Then

\[
2^{u_3+1}\left((av_1 + bv_3) + \frac{a + c}{2}\right) = 2^{u_2+k}l(2v_2 + 1)
\]

Hence, \( u_3 + 1 = u_2 + k \). Since \( k \geq 2 \), \( u_3 + k > u_2 + 1 \). But from (4.9), we have \( u_2 + 1 \geq u_3 + k \), a contradiction.

Hence, in the case of \( k \geq 2 \) and \( a \equiv 4 \), \( S_2(a, -2^k l, c) \leq 3 \).
The following discussion is necessary before stating the next theorem. The numbered equations will be used frequently later on.

Let $p$ be a prime number and let $a, b, l \in \mathbb{Z}$ be such that $p \nmid a, b, l$. Let $k$ be a positive integer.

Consider the recurrence equation,

$$ax_i + bx_{i+1} = p^k l \cdot x_{i+2} \quad (4.10)$$

Let $x_1, x_2, \ldots$ satisfy 4.10 and let $x_i = (u_i, v_i, w_i)_p$ for some $u_i, v_i \geq 0$ and $1 \leq w_i \leq p - 1$.

Then

$$ap^{u_i}(pv_i + w_i) + bp^{u_{i+1}}(pv_{i+1} + w_{i+1}) = p^{u_{i+2}+k}(pv_{i+2} + w_{i+2}).$$

If $u_1 < u_2$ then

$$p^{u_1}(apv_1 + aw_1 + bp^{u_2-u_1}(pv_2 + w_2)) = p^{u_3+k}(pv_3 + w_3).$$

Hence,

$$u_1 = u_3 + k \text{ and } aw_1 \equiv_p lw_3$$

$$u_i = u_{i+1} + k \text{ and } bw_i \equiv_p lw_{i+1} \quad \text{for all } i \geq 3. \quad (4.11)$$

Similarly, if $u_2 < u_1$ then

$$u_i = u_{i+1} + k \text{ and } bw_i \equiv lw_{i+1} \quad \text{for all } i \geq 2. \quad (4.12)$$

If $u_1 = u_2$ then

$$p^{u_2}(p(av_1 + bv_2) + aw_1 + bw_2) = p^{u_3+k}(pv_3 + w_3).$$

If we also assume that $aw_1 + bw_2 \not\equiv_p 0$, we get

$$u_1 = u_2 = u_3 + k$$

$$u_i = u_{i+1} + k \quad \text{for all } i \geq 2$$

$$bw_i \equiv_p lw_{i+1} \quad \text{for all } i \geq 3. \quad (4.13)$$

**Theorem 4.17.** Let $p$ be an odd prime, $k \geq 1$ and $l \in \mathbb{Z}$ such that $p \nmid l$. Let $a, b \in \mathbb{Z}$ be such that $a \equiv_p 1$ and $\gcd(p, b) = 1$. Let $B \equiv_p -b$, $L \equiv_p l/b$ (the division is done in $\mathbb{Z}_p^*$), $s = o_p(B)$, $d = o_p(L)$ and $t = \gcd(s, d)$. Then, if $s$ is even,

$$S_2(a, b, -p^k l) \leq \begin{cases} 3 & \text{if } s/t \text{ is even} \\ 3 & \text{if } s/t \text{ and } d/t \text{ are both odd} \\ 4 & \text{if } s/t \text{ is odd and } d/t \text{ is even} \end{cases}$$
Proof. Let $H = \{1, L, L^2, \ldots, L^{d-1}\}$ and $K = \{1, B, B^2, \ldots, B^{s-1}\}$ be subgroups of $\mathbb{Z}_p^*$. Let $G = HK = \{B^jL^i : 0 \leq i \leq s - 1 \text{ and } 0 \leq j \leq d - 1\}$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be a complete set of representatives of classes in $\mathbb{Z}_p^*/G$.

Note that, $B^{s/t} = L^{u(d/t)}$ for some integer $n$ such that $\gcd(n, t) = 1$ and if $B^i \equiv_p B^j$ for some $i, j \in \mathbb{Z}$ then $(s/t) | i$ and $(d/t) | j$.

First, assume that $s/t$ is even.

We 2-color $G$ by $f \left( B^iL^j \right) = (i)_2$ for $i, j \in \mathbb{Z}$. Now, if $B^{i_1}L^{j_1} \equiv_p B^{i_2}L^{j_2}$ for some $i_1, i_2, j_1, j_2$, then $B^{i_1-i_2} = L^{j_2-j_1}$. But then we have, $(s/t) | (i_1 - i_2)$. Since $s/t$ is even, this implies $i_1 \equiv_2 i_2$. Therefore, $f \left( B^{i_1}L^{j_1} \right) = f \left( B^{i_2}L^{j_2} \right)$. Hence $f$ is well-defined. Now, we extend this coloring to a 2-coloring of $\mathbb{Z}_p^*$ as $F(x) = f \left( x\alpha_j^{-1} \right)$ if $x \in G\alpha_j$. Note that, for any $x \in \mathbb{Z}_p^*$,

$$F(Bx) \neq F(x) \tag{4.14}$$

$$F(Lx) = F(x) \tag{4.15}$$

Finally, define $\chi : \mathbb{N} \to \{0, 1\}$ as

$$\chi(x) = \left( \left\lfloor \frac{u}{k} \right\rfloor + F(w) \right)_2$$

where $x = (u, v, w)_p$.

Suppose that $\{x_1, x_2, x_3, x_4\}$ is monochromatic and satisfies $ax_i + bx_{i+1} = (p^k)l_{i+2}$. Let $x_i = (u_i, v_i, w_i)_p$. Then

Case 1. If $u_1 \neq u_2$ then from (4.11) and (4.12) $u_3 = u_4 + k$ and $w_3 \equiv_p Lw_4$. Hence,

$$\left\lfloor \frac{u_3}{k} \right\rfloor + F(w_3) = 1 + \left\lfloor \frac{u_4}{k} \right\rfloor + F(Lw_4) = 1 + \chi(x_4).$$

But this is not possible since $\chi(x_3) = \chi(x_4)$.

Case 2. Assume that $u_1 = u_2$. Then $F(w_1) = F(w_2)$ and from (4.14), it follows $w_1 \not\equiv_p Bw_2$, i.e., $aw_1 + bw_2 \not\equiv_p 0$. Hence, from (4.13), $u_3 = u_4 + k$ and $w_3 \equiv_p Lw_4$, which is a contradiction as in Case 1.

Therefore, in the case of $s/t$ even, $S_2(a, b, -p^k) \leq 3$.

Secondly, assume that $s/t$ and $d/t$ are both odd. Since $s$ is even, $t$, and hence $d$, must also be even.

Define a 2-coloring on $G$ by $f \left( B^iL^j \right) = (i+j)_2$ for $i, j \in \mathbb{Z}$. Now, if $B^{i_1}L^{j_1} \equiv_p B^{i_2}L^{j_2}$ for some $i_1, i_2, j_1, j_2$, then $B^{i_1-i_2} = L^{j_2-j_1}$. But then we have, $(s/t) | (i_1 - i_2)$ and $(d/t) | (j_2 - j_1)$.
Since $s/t$ and $d/t$ are both odd, this implies $i_1 - i_2 \equiv j_2 - j_1$. Therefore, $f(B^{i_1}L^{j_1}) = f(B^{i_2}L^{j_2})$. Hence $f$ is well-defined.

We extend this coloring to a 2-coloring of $\mathbb{Z}_p^*$ the same as in Case 1. Note that, for any $x \in \mathbb{Z}_p^*$, \[ F(Bx) \neq F(x) \] (4.16) \[ F(Lx) \neq F(x) \] (4.17)

Define $\chi : \mathbb{N} \to \{0, 1\}$ as \[ \chi(x) = F(w) \] where $x = (u, v, w)_p$.

Suppose that $\{x_1, x_2, x_3, x_4\}$ is monochromatic and satisfies $ax_i + bx_{i+1} = p^k l x_{i+2}$. Let $x_i = (u_i, v_i, w_i)_p$. Then

**Case 1.** If $u_1 \neq u_2$ then from (4.11) and (4.12) $w_3 \equiv_p Lw_4$. Then
\[ \chi(x_3) = F(w_3) = F(Lw_4) \neq \chi(x_4) \]
a contradiction.

**Case 2.** Assume that $u_1 = u_2$. Then $F(w_1) = F(w_2)$. Hence from (4.16), $w_1 \not\equiv_p Bw_2$, i.e., $aw_1 + bw_2 \not\equiv_p 0$. Hence, from (4.13), $w_3 \equiv_p Lw_4$, which is again a contradiction.

Therefore, in the case of both $s/t$ and $d/t$ even, $S_2(a, b, -p^k l) \leq 3$.

Lastly, assume that $s/t$ is odd and $d/t$ is even. As before, $t$ and $d$ must be even.

We 2-color $G$ by $f(B^iL^j) = \left(i + \left\lfloor \frac{j}{d} \right\rfloor \right)_2$ for $i, j \in \mathbb{Z}$. Now, if $B^{i_1}L^{j_1} \equiv_p B^{i_2}L^{j_2}$ for some $i_1, i_2, j_1, j_2$, then $B^{i_1-i_2} = L^{j_2-j_1}$. But then we have, $(s/t) | (i_1 - i_2)$ and $(d/t) | (j_1 - j_2)$. Let $i_1 - i_2 = (s/t)m_1$ and $j_1 - j_2 = (d/t)m_2$ for some $m_1, m_2 \in \mathbb{Z}$. Then
\[ L^{j_2-j_1} = B^{i_2-i_1} = \left(B^{s/t}\right)^{m_1} = \left(L^{n(d/t)}\right)^{m_1} = L^{m_1n(d/t)}. \]

Hence, $m_1n(d/t) + j_1 - j_2 = (d/t)(m_1n + m_2)$ is a multiple of $d$. Hence, $m_1n + m_2$ is divisible by $t$. Hence $m_1n + m_2 \equiv 0$, since $t$ is even. Since $t$ is even and $\gcd(n, t) = 1$, $n$ must be
odd. And since $d/t$ divides $j_1 - j_2$,

$$\frac{j_1 - j_2}{d/t} = \left\lfloor \frac{j_1}{d/t} \right\rfloor - \left\lfloor \frac{j_2}{d/t} \right\rfloor.$$

But

$$m_1 n + m_2 = \frac{i_1 - i_2}{s/t} n + \frac{j_1 - j_2}{d/t},$$

and

$$\frac{i_1 - i_2}{s/t} n + \left\lfloor \frac{j_1}{d/t} \right\rfloor - \left\lfloor \frac{j_2}{d/t} \right\rfloor \equiv_2 (i_1 - i_2) + \left\lfloor \frac{j_1}{d/t} \right\rfloor - \left\lfloor \frac{j_2}{d/t} \right\rfloor$$

since $s/t$ and $n$ are both odd.

Hence,

$$(i_1 - i_2) + \left\lfloor \frac{j_1}{d/t} \right\rfloor - \left\lfloor \frac{j_2}{d/t} \right\rfloor \equiv_2 0.$$

which implies

$$i_1 + \left\lfloor \frac{j_1}{d/t} \right\rfloor \equiv_2 i_2 + \left\lfloor \frac{j_2}{d/t} \right\rfloor$$

Therefore, $f(B^{i_1} L^{j_1}) = f(B^{i_2} L^{j_2})$. Hence $f$ is well-defined. Now, extend this coloring in the usual way. For any $x \in \mathbb{Z}_p$, we have

$$F(Bx) \neq F(x)$$

and

$$F(Lx) \neq F(x) \Rightarrow F(L^2 x) = F(Lx)$$

since $d/t > 0$ is even.

Define $\chi : \mathbb{N} \to \{0, 1\}$ as

$$\chi(x) = \left( \left\lfloor \frac{u}{k} \right\rfloor + F(w) \right)_2$$

where $x = (u, v, w)_p$.

Suppose that $\{x_1, x_2, x_3, x_4, x_5\}$ is monochromatic and satisfies $ax_i + bx_{i+1} = (p^k l)x_{i+2}$.

Let $x_i = (u_i, v_i, w_i)_p$. Then
CHAPTER 4. SOME 2-COLOR RADO NUMBERS

Case 1. If $u_1 \neq u_2$ then from (4.11) and (4.12) $u_3 = u_4 + k$, $u_4 = u_5 + k$, $w_3 \equiv_p Lw_4$ and $w_4 \equiv_p Lw_5$. Hence,

$$
\chi(x_3) \equiv 2 \left\lfloor \frac{u_3}{k} \right\rfloor + F(w_3) \\
\equiv 2 \left\lfloor \frac{u_4}{k} \right\rfloor + F(Lw_4) \\
\equiv 2 + \chi(x_4) + F(Lw_4) + F(w_4).
$$

But since $\chi(x_3) = \chi(x_4)$, we must have $F(Lw_4) \neq F(w_4)$. In the same way, we must have $F(Lw_5) \neq F(w_5)$. This contradicts to (4.2), since $w_4 \equiv_p Lw_5$.

Case 2. Assume that $u_1 = u_2$. Then $F(w_1) = F(w_2)$. Hence from (4.2), $w_1 \not\equiv_p Bw_2$, i.e., $aw_1 + bw_2 \not\equiv_p 0$. Hence, from (4.13), $u_3 = u_4 + k$, $u_4 = u_5 + k$, $w_3 \equiv_p Lw_4$ and $w_4 \equiv_p Lw_5$, which is a contradiction as in Case 1.

Therefore, in the case of $s/t$ odd and $d/t$ even, $S_2(a, b, -p^k l) \leq 4$.

Corollary 4.18. Let $p$ be an odd prime, $k \geq 1$ and $l \in \mathbb{Z}$ such that $p \nmid l$. Let $a, b \in \mathbb{Z}$ be such that $p \nmid a, b$. Let $B \equiv_p -b/a$, $L \equiv_p l/b$, $s = o_p(B)$, $d = o_p(L)$ and $t = \gcd(s, d)$. Then, if $s$ is even

$$
S_2(a, b, -p^k l) \leq \begin{cases} 
3 & \text{if } s/t \text{ is even} \\
3 & \text{if } s/t \text{ and } d/t \text{ are both odd} \\
4 & \text{if } s/t \text{ is odd and } d/t \text{ is even}
\end{cases}
$$

Proof. Let $A \in \mathbb{Z}$ be such that $aA \equiv_p 1$, and let $a' = aA$, $b' = bA$ and $l' = lA$. Then

$$
S_2(a, b, -p^k l) = S_2(aA, bA, -p^k lA) = S_2(a', b', -p^k l')
$$

Since $-b/a = -b'/a' \equiv_p -b'$ and $l/b = l'/b'$, $B \equiv_p -b'$ and $L \equiv_p l'/b'$. Therefore, the result follows by Theorem 4.17.

Theorem 4.19. Let $a, b$ and $l$ be odd integers such that $a \equiv_4 b$. Then

(i) If $k = 2$ then $S_2(a, b, -2^k l) \leq 4$.

(ii) If $k \geq 3$ then $S_2(a, b, -2^k l) \leq 3$.

Proof. Let $L = l - b$. 

(i) We will prove this in two cases, \( L \equiv_4 0 \) or 2.

Case 1. Assume \( L \equiv_4 0 \) and define the 2-coloring \( \chi \) as \( \chi(x) = \left( \left\lfloor \frac{x}{2} \right\rfloor + v \right)_2 \) where \( x = (u, v, 1)_2 \).

Suppose \( \{x_1, x_2, x_3, x_4, x_5\} \) is monochromatic and satisfies \( ax_i + bx_{i+1} = (2^k l)x_{i+2} \).
Let \( x_i = (u_i, v_i, 1)_2 \).

If \( u_1 \neq u_2 \) then from (4.11) and (4.12), \( u_2 \geq u_3 + 2 \) and \( u_3 = u_4 + 2 \). Hence \( 2^{u_2 - u_3}a(2v_2 + 1) + b(2v_3 + 1) = l(2v_4 + 1) \). But this implies \( 2^{u_2 - u_3 - 1}a(2v_2 + 1) + bv_3 = lv_4 + L/2 \). Since \( L \equiv_4 0 \) and \( u_2 - u_3 - 1 \geq 1 \), this implies \( v_3 \equiv_2 v_4 \). But then,

\[
\chi(x_3) \equiv_2 \left\lfloor \frac{u_3}{2} \right\rfloor + v_3 \\
\equiv_2 1 + \left\lfloor \frac{u_4}{2} \right\rfloor + v_4 \\
\equiv_2 1 + \chi(x_4),
\]

a contradiction.

If \( u_1 = u_2 \), since \( \chi(x_1) = \chi(x_2) \), we have \( v_1 \equiv_2 v_2 \). Then

\[
2^{u_2}(2(au_1 + bv_2) + a + b) = 2^{u_3 + 2}l(2v_3 + 1)
\]

Since \( a \equiv_4 b, a+b \equiv_4 2 \). Hence, \( u_2 = u_3 + 1 > u_3 \). Then from (4.12), we get \( u_4 = u_5 + 2 \) and \( v_4 \equiv_2 v_5 \). But this implies \( \chi(x_4) \neq \chi(x_5) \).

Case 2. Assume \( L \equiv_4 2 \) and define the 2-coloring \( \chi \) as \( \chi(x) = (v)_2 \) where \( x = (u, v, 1)_2 \).

Suppose \( \{x_1, x_2, x_3, x_4, x_5\} \) is monochromatic and satisfies \( ax_i + bx_{i+1} = (2^k l)x_{i+2} \).
Let \( x_i = (u_i, v_i, 1)_2 \). Then \( v_i \)'s are either all even or all odd.

If \( u_1 \neq u_2 \) then as in Case 1, we get \( u_2 - u_3 - 1 \geq 1 \) and \( 2^{u_2 - u_3 - 1}a(2v_2 + 1) + bv_3 = lv_4 + L/2 \). Since \( L \equiv_4 2 \), this implies \( v_3 \equiv_2 1 + v_4 \), a contradiction.

If \( u_1 = u_2 \), we get \( v_4 \equiv_2 1 + v_5 \), a contradiction again.

(ii) Similar to (i), we will have 2 cases, namely \( L \equiv_4 0 \) and \( L \equiv_4 2 \).

Let \( k \geq 3 \).

Case 1. Assume \( L \equiv_4 0 \) and define the 2-coloring \( \chi \) as \( \chi(x) = \left( \left\lfloor \frac{x}{2} \right\rfloor + v \right)_2 \) where \( x = (u, v, 1)_2 \).

Suppose \( \{x_1, x_2, x_3, x_4\} \) is monochromatic and satisfies \( ax_i + bx_{i+1} = (2^k l)x_{i+2} \). Let \( x_i = (u_i, v_i, 1)_2 \).
If \( u_1 \neq u_2 \) then from (4.11) and (4.12), \( u_2 \geq u_3 + k \) and \( u_3 = u_4 + k \). Hence 
\( 2^{u_2-u_3}a(2v_2+1) + b(2v_3+1) = l(2v_4+1) \). But this implies \( 2^{u_2-u_3-1}a(2v_2+1) + bv_3 = lv_4 + L/2 \). Since \( L \equiv_4 0 \) and \( u_2 - u_3 - 1 \geq 2 \), this implies \( v_3 \equiv_2 v_4 \). But then,
\[
\chi(x_3) \equiv_2 \left[ \frac{u_3}{k} \right] + v_3 \\
\equiv_2 1 + \left[ \frac{u_4}{k} \right] + v_4 \\
\equiv_2 1 + \chi(x_4),
\]
a contradiction.

If \( u_1 = u_2 \), since \( \chi(x_1) = \chi(x_2) \), we have \( v_1 \equiv_2 v_2 \). Then
\[
2^{u_2} (2(av_1 + bv_2) + a + b) = 2^{u_3+k}l(2v_3 + 1)
\]
Since \( a \equiv_4 b \) are odd, \( a + b \equiv_2 2 \). Hence, \( u_2 = u_3 + k - 1 > u_3 + 1 \). Then from (4.12), we get \( u_3 = u_4 + k \) and \( v_3 \equiv_2 v_4 \). But this implies \( \chi(x_3) \neq \chi(x_4) \).

**Case 2.** Assume \( L \equiv_4 2 \) and define the 2-coloring \( \chi \) as \( \chi(x) = (v)_2 \) where \( x = (u, v, 1)_2 \).

Suppose \( \{x_1, x_2, x_3, x_4\} \) is monochromatic and satisfies \( ax_i + bx_{i+1} = (2^k l)x_{i+2} \). Let \( x_4 = (u_i, v_i, 1)_2 \). Then \( v_i \)'s are either all even or all odd.

If \( u_1 \neq u_2 \) then as in Case 1, we get, \( u_2 - u_3 \geq k \geq 3 \) and 
\[
2^{u_2-u_3-1}a(2v_2+1) + bv_3 = lv_4 + L/2.
\]
Since \( L \equiv_4 2 \), this implies \( v_3 \equiv_2 1 + v_4 \), a contradiction.

If \( u_1 = u_2 \), as above, we get \( v_3 \equiv_2 1 + v_4 \), a contradiction again.

\[
\square
\]

**Corollary 4.20.** Let \( c \in \mathbb{N} \). Then
\[
S_2(1, 1, -c) = \begin{cases} 
4 & \text{if } c = 1 \text{ or } c = 4, \\
\infty & \text{if } c = 2, \\
3 & \text{if } c \equiv_8 0, \\
3 & \text{if } c = p^k l \text{ for some odd prime } p, k \geq 1 \text{ and } l \in \mathbb{N} \text{ such that } p \nmid l \text{ and } \text{ord}_p(l) \not\equiv_4 0.
\end{cases}
\]
In all the other cases, \( 3 \leq S_2(1, 1, -c) \leq 4 \).

**Remark 4.21.** Let \( p \equiv_4 3 \) be a prime and let \( l \in \mathbb{N} \) be such that \( p \nmid l \). Then either \( \text{ord}_p(l) \) is odd or \( \text{ord}_p(l) \equiv_4 2 \). Therefore, if \( c \in \mathbb{N} \) is divisible by \( p \) then
\[
S_2(1, 1, -c) = 3
\]
4.3 More Values: A New Technique

There are still more cases which are not covered in Section 4.2. For example, consider the recurrence $8x_i - 6x_{i+1} + x_{i+2} = 0$. It is not covered by any of the preceding theorems. In this section we introduce a different technique which gives upper bounds for $S_2(a, b, c)$ for some of the cases not covered in Section 4.2 as well as some which are already covered. We will explain the technique through an example.

**Theorem 4.22.** $S_2(8, -6, 1) \leq 5$.

**Proof.** Let $\pi$ be a permutation on $\mathbb{Z}_{11}^2$ defined by

$$\pi(a, b) = (b, 6b - 8a).$$

We consider the recurrence $-8x_i + 6x_{i+1} = x_{i+2}$ modulo 11. Excluding the trivial cycle $(0, 0)$, we represent the cycles of this permutation by Table 4.1:

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<th>10</th>
<th>1</th>
<th>3</th>
<th>10</th>
<th>3</th>
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<tr>
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<td>6</td>
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Table 4.1: The cycles of the permutation $\pi$ of $\mathbb{Z}_{11}^2$.

(In the first row, 0 1 6 6 . . . means $(0,1) \xrightarrow{\pi} (1,6) \xrightarrow{\pi} (6,6) \xrightarrow{\pi} . . . $)

Let $f$ be a 2-coloring of $\mathbb{Z}_{11}$ such that

$$f(m) = \begin{cases} 0 & \text{if } m \in \{1, 2, 3, 5, 7\} \\ 1 & \text{if } m \in \{4, 6, 8, 9, 10\} \end{cases}.$$

and assume that 0 is colored by both colors.
CHAPTER 4. SOME 2-COLOR RADO NUMBERS

We observe that no 6 consecutive elements in any of the cycles have the same color, but that there is a cycle with five consecutive elements colored by the same color; 6 9 6 8 0 or 7 5 7 2 0, for example. Also, we note that a single 0 is among any five consecutive elements of the same color, in any of the cycles.

Let \( \chi : \mathbb{N} \rightarrow \{0, 1\} \) be defined as \( \chi(x) = f(w) \) if \( x = (u, v, w)_{11} \) for some \( u, v \geq 0 \) and \( 1 \leq w \leq 10 \). Then it is easy to see that, under this coloring there is no monochromatic 6-term sequence \( x_1, x_2, x_3, x_4, x_5, x_6 \) satisfying the recurrence \( x_{n+2} = 6x_{n+1} - 8x_n \) \( \square \)

Note that if we take \( x_1 = 1, x_2 = 11 = 1 \cdot 11, x_3 = 6 \cdot 11 - 8 \cdot 1 = 58 = 5 \cdot 11 + 3, x_4 = 6 \cdot 58 - 8 \cdot 11 = 260 = 23 \cdot 11 + 7, \) and \( x_5 = 6 \cdot 260 - 8 \cdot 58 = 1096 = 99 \cdot 11 + 7 \) we get that \( \chi(x_1) = f(1) = 1, \chi(x_2) = f(1) = 1, \chi(x_3) = f(3) = 1, \chi(x_4) = f(7) = 1, \) and \( \chi(x_5) = \chi(7) = 1 \). Therefore, for the coloring \( \chi \), there is a monochromatic 5-term sequence that satisfies the given recurrence.

Note also that the above proof also implies that if \( a, b, c \in \mathbb{Z} \) is such that \( a \equiv 11 \pmod{8}, b \equiv -6 \pmod{11} \) and \( c \equiv 1 \pmod{11} \) then \( S_2(a, b, c) \leq 5 \).

The method of the above theorem can be summarized as follows.

Given the recurrence relation \( ax_i + bx_{i+1} = cx_{i+2} \), we pick a prime number \( p \) such that \( p \not| c \) and consider the recurrence as a permutation on \( \mathbb{Z}_p^2 \) defined by \( \pi(x, y) = (y, aCx + bCy) \) where \( C \in \mathbb{Z} \) is such that \( cC \equiv 1 \pmod{p} \). Then we find a 2-coloring of \( \mathbb{Z}_p \) in a way that we have the minimum number of monochromatic consecutive elements in the cycles of the permutation \( \pi \), assuming that 0 is colored by both colors. Note that, no cycle should be monochromatic. We can do this for several primes and pick the best one among them.

Some computer generated results of this method is summarized in Table 4.2. The computer program used is due to Z. Dvorak.

### 4.4 Remarks and Questions

It is a very interesting fact that for all the cases that we have considered, \( S_2(a, b, c) \leq 6 \) except when \( a + b + c = 0 \), and in that case \( S_2(a, b, c) = \infty \). We wonder if this is the case for any triple \( (a, b, c) \) with \( a + b + c \neq 0 \).

Considering the case \( a = b = 1 \), we see from Corollary 4.10 that when \( S_2(1, 1, -c) \neq 3 \) or \( \infty \), the exact value is known only when \( c = 1 \) and \( c = 4 \). \( c = 1 \) is easy to check and \( c = 4 \) is done in [21] with the help of a computer by showing that any 2-coloring of the interval
### Table 4.2: Some more bounds for $S_2(a, b, c)$

[1,71] contains a monochromatic 4-term sequence satisfying the recurrence. We wonder if there are other values of $c$ for which $S_2(1, 1, -c) = 4$. For example, is $S_2(1, 1, -10) = 4$? (10 is the smallest value of $c$ for which the exact value of $S_2(1, 1, -c)$ is unknown. The next number to consider is 26.)
Chapter 5

Coloring the odd-distance plane graph

5.1 Introduction

This chapter is dedicated to finding a lower bound for the chromatic number of the odd-distance plane graph. (For definition see Section 5.2). This problem arises as a generalization of the famous unit-distance problem.

In 1950 Edward Nelson posed the following problem which is now known as the unit-distance problem:

**Problem 5.1.** What is the minimum number of colors needed to color the points of the plane so that any two points which are unit distance apart receive different colors.

Nelson himself showed that at least four colors are needed. Shortly after, John Isbell found a proper 7-coloring of the plane. These bounds are still unchanged.

Several variations of the problem has been studied through the years. One such variation, for example, is to require the color classes to be Lebesque measurable. K.J. Falconer [13] had shown that, under this requirement, the chromatic number is at least five, increasing the lower bound by one. For other variations of the problem see [4].
5.2 The odd-distance graph

Let $D \subset \mathbb{N}$ be a nonempty set of real numbers and let $(X, d)$ be a metric space. We denote by $G^D(X)$ the graph whose vertices are the points in $X$ and whose edges are between points in $X$ whose distance is in $D$. Denote by $\chi^D(X)$ the chromatic number of the graph $G^D(X)$: the minimum number of colors needed to color the points in $X$ in such a way that any two points whose distance is in $D$ receive different colors.

Note that in this terminology, the unit-distance problem is to find $\chi^{\{1\}}(\mathbb{R}^2)$.

Denote by $\text{Odd}$ the set of all odd natural numbers. The odd-distance graph is the graph $G^{\text{Odd}}(\mathbb{R}^2)$.

We’ll find a lower bound for the chromatic number of $G^{\text{Odd}}(\mathbb{R}^2)$. One obvious lower bound comes from the fact that the unit-distance graph is a subgraph of the odd-distance graph. Hence we have $\chi^{\text{Odd}}(\mathbb{R}^2) \geq 4$. One might obtain a lower bound by considering the "clique number" of $G^{\text{Odd}}(\mathbb{R}^2)$, which is defined to be the maximum $n$ such that $K_n$, the complete graph on $n$ vertices, is a subgraph of $G^{\text{Odd}}(\mathbb{R}^2)$. Since four points in the plane cannot have pairwise odd distances (see [17], [38]), $K_4$ is not a subgraph of $G^{\text{Odd}}(\mathbb{R}^2)$. Hence the clique number of $G^{\text{Odd}}(\mathbb{R}^2)$ is three as, clearly, $K_3$ is a subgraph of it. But this only gives a lower bound of three which does not improve the lower bound.

Next, we are going to consider another subgraph of $G^{\text{Odd}}(\mathbb{R}^2)$, the triangular grid. We will show that the chromatic number of the grid itself is four but this will enable us to show that $\chi^{\text{Odd}}(\mathbb{R}^2) \geq 5$, using Lemma 5.4.

5.2.1 The triangular grid

Definition 5.2. The triangular grid is the metric space on the set

$$T = \left\{ \left( n + \frac{m}{2}, \frac{m}{2} \sqrt{3} \right) : n, m \in \mathbb{Z} \right\} \subset \mathbb{R}^2$$

with the Euclidean distance.

Note that there are points in $T$, other than the obvious ones on the same grid line, which are odd distance apart. Consider, for example, the part of the grid in Figure 5.1. As $d(C, D) = 3$, $d(B, C) = 5$ and $\angle BCD = 120^0$, 

**Lemma 5.4.**
\[ d(B, D)^2 = d(B, C)^2 + d(C, D)^2 - 2 \cos(120^\circ)d(B, C)d(C, D) \]
\[ = 5^2 + 3^2 + 3 \cdot 5 \]
\[ = 49. \]

Hence \( d(B, D) = 7 \). Similarly, \( d(A, E) = d(A, H) = d(B, G) = 7 \).

Figure 5.1: A part of the triangular grid in which each side of each small equilateral triangle is 1 and hence, the distance of \( A \) and \( B \) is 8.

**Theorem 5.3.** The chromatic number of \( G^{Odd}(T) \) is four.

**Proof.** We’ll first show that \( \chi^{Odd}(T) > 3 \). Since \( T \) contains an equilateral triangle of side three (e.g., \( \triangle ACD \) in Figure 5.1), \( \chi^{Odd}(T) \geq 3 \). If \( \chi^{Odd}(T) = 3 \) then there is a proper 3-coloring, say \( c \) of \( G^{Odd}(T) \).

Let \( c_1, c_2 \) and \( c_3 \) denote the three colors. Consider the points in Figure 5.1. Assume, without loss of generality, that \( c(A) = c_1 \). Since \( \triangle ACD \) is an equilateral triangle with side three, \( A, C \) and \( D \) must get distinct colors. Assume \( c(C) = c_2 \) and \( c(D) = c_3 \). Then \( c(B) = c_1 \) as \( d(B, C) = 5 \) and \( d(B, D) = 7 \). Since \( d(D, E) = 5 \) and \( d(B, E) = 3 \), \( c(E) = c_2 \). Similarly, we get \( c(F) = c(H) = c_3 \) and \( c(G) = c_2 \). Now, since \( d(I_1, E) = d(I_1, F) = 3 \), and \( c(E) = c_2 \) and \( c(F) = c_3 \), \( c(I_1) = c_1 \). Similarly, \( c(I_2) = c(I_3) = c_1 \). This is a contradiction as \( \triangle I_1I_2I_3 \) is an equilateral triangle with side one.
Therefore, $\chi^{Odd}(T) > 3$.

Now, define the 4-coloring $c$ of $T$ as follows:

Let $X$ be a point in $T$. Then there exist two integers $m$ and $n$ such that $X = (n + \frac{m}{2}, \frac{m}{2} \sqrt{3})$.

Then let

\[
c(X) = \begin{cases} 
  c_1 & \text{if } n \text{ and } m \text{ are both even,} \\
  c_2 & \text{if } n \text{ is odd and } m \text{ is even,} \\
  c_3 & \text{if } n \text{ is even and } m \text{ is odd,} \\
  c_4 & \text{if } n \text{ and } m \text{ are both odd.}
\end{cases}
\]

Each color class of this coloring can be obtained by a suitable translation of the first color class. Hence, it is enough to show that any two points colored with color $c_1$, do not have an odd distance between them.

Let $X, Y \in T$ be such that $c(X) = c(Y) = c_1$. Then there are four even integers $m_1, m_2, n_1, n_2$ such that $X = (n_1 + \frac{m_1}{2}, \frac{m_1}{2} \sqrt{3})$ and $Y = (n_2 + \frac{m_2}{2}, \frac{m_2}{2} \sqrt{3})$.

\[
d(X, Y)^2 = \left( n_1 - n_2 + \frac{m_1 - m_2}{2} \right)^2 + \left( \frac{m_1 - m_2}{2} \sqrt{3} \right)^2 \\
= (n_1 - n_2)^2 + (n_1 - n_2)(m_1 - m_2) + (m_1 - m_2)^2.
\]

Since all four numbers are even integers, $d(X, Y)^2$ is also an even integer. Therefore, $d(X, Y)$ cannot be an odd integer. Hence, $c$ is a proper 4-coloring of $T$ and $\chi^{Odd}(T) = 4$. \hfill \Box

Now we know that $G^{Odd}(T)$ can be properly 4-colored. We prove in the following lemma that any proper 4-coloring of $G^{Odd}(T)$ is "highly structured". This observation is due to a computer program written by J. Mańuch.

**Lemma 5.4.** Let $c$ be a proper 4-coloring of $G^{Odd}(T)$. Then for any two points $A$ and $B$ on the grid line of $T$ with $d(A, B) = 8$, $c(A) = c(B)$.

**Proof.** Let $A$ and $B$ be any two points on the grid line of $T$ with $d(A, B) = 8$. Then we have a piece of the grid as depicted in Figure 5.1. Let $c_1, c_2, c_3, c_4$ denote the four colors. Assume for a contradiction that $c(A) \neq c(B)$. We may assume that $c(A) = c_1$ and $c(B) = c_2$. Then a similar argument as in the beginning of the proof of Theorem 5.3 yields $c(I_1), c(I_2), c(I_3) \in \{c_1, c_2\}$ which gives the desired contradiction. Therefore, $c(A) = c(B)$. \hfill \Box
Note that for any two points \(X, Y \in \mathbb{R}^2\) with \(d(X, Y) = 8\), there is a copy of \(T\) in \(\mathbb{R}^2\) in which \(X\) corresponds to \(A\) and \(Y\) corresponds to \(B\). Therefore, we get the following obvious corollary.

**Corollary 5.5.** If there is a proper 4-coloring \(c\) of \(G^{Odd}(\mathbb{R}^2)\) then for any two points \(X, Y \in \mathbb{R}^2\) with \(d(X, Y) = 8\), \(c(X) = c(Y)\).

**Corollary 5.6.** \(\chi(G^{Odd}(\mathbb{R}^2)) \geq 5\).

**Proof.** The proof follows from Corollary 5.5 by considering an isosceles triangle with sides 8, 8 and 1. \(\square\)

### 5.2.2 The chromatic number of \(G^{Odd}(\mathbb{Q}^2)\)

Unlike the unit-distance graph, the chromatic number of one of its dense subgraph \(G^{(1)}(\mathbb{Q}^2)\) is easy to calculate. It is shown that \(\chi^{(1)}(\mathbb{Q}^2)\) is two [19, 42]. They used the following fact from graph theory:

The chromatic number of a graph is two if and only if it contains no odd cycle.

We will do the same thing for \(G^{Odd}(\mathbb{Q}^2)\). Let’s start with some easy observations that we will need to prove our claim.

**Lemma 5.7.** Let \(x, y\) and \(z\) be integers such that \(z = x^2 + y^2\). If \(z \equiv 0\) (mod 4) then \(x \equiv y \equiv 0\) (mod 2).

**Corollary 5.8.** Let \(x, y\) and \(z\) be integers such that \(z = x^2 + y^2\). If \(2^{2k}\) divides \(z\) for some \(k \in \mathbb{N}\) then \(2^k\) divides both \(x\) and \(y\).

**Lemma 5.9.** Let \(X = (x, y) = (2^am, 2^bn)\) for some integers \(a, b, m\) and \(n\). If \(x^2 + y^2\) is an odd integer then both \(x\) and \(y\) are integers with different parity modulo 2 i.e., \(x + y\) is odd. (Note that \(m\) or \(n\) need not be odd.)

**Proof.** Let

\[
x^2 + y^2 = (2^am)^2 + (2^bn)^2 = t
\]

for some odd integer \(t\). Assume without loss of generality that \(a \leq b\). Since \(x\) and \(y\) are clearly integers when \(a \geq 0\) we can assume \(a < 0\). Then the equation 5.1 becomes

\[
m^2 + (2^b-a)n)^2 = 2^{-2a}t.
\]
Since $m$ and $2^{b-a}n$ are both integers, by Corollary 5.8, $2^{-a}$ divides both $m$ and $2^{b-a}n$. Therefore, $x = 2^am$ and $y = 2^bn$ are both integers.

Since $x$ and $y$ are integers and $(x+y)^2 = x^2 + y^2 + 2xy$, $x+y$ has to be an odd integer. □

**Corollary 5.10.** Let $X = (x, y) = (2^km, 2^ln)$ and $U = (u, v) = (2^pr, 2^qs)$ for some integers $k, l, m, n, p, q, r, s$. If $d(X, U)$ is an odd integer, then $x - u$ and $y - v$ are both integers with different parity modulo 2.

**Proof.** Let $a = \min(k, p)$ and $b = \min(l, q)$. Then $x - u = 2^a(m - r)$, $y - v = 2^b(n - s)$. Since $d(X, U)$ is odd and $(x - u)^2 + (y - v)^2 = d(X, U)^2$, the result follows by Lemma 5.9. □

Now we are going to prove that the chromatic number of $G^{Odd}(\mathbb{Q}^2)$ is two by showing that it contains no odd cycle.

**Theorem 5.11.** The chromatic number of $G^{Odd}(\mathbb{Q}^2)$ is two.

**Proof.** We’ll show that $G^{Odd}(\mathbb{Q}^2)$ contains no odd cycle. Assume that it contains an odd cycle, say $(x_1, y_1), (x_2, y_2), \ldots, (x_{2t+1}, y_{2t+1})$ for some $t \in \mathbb{N}$. Then $x_i = \frac{2^ki}{m}$ and $y_i = \frac{2^li}{m}$ for some $k_i, l_i, n_i, p_i \in \mathbb{Z}$ for $1 \leq i \leq 2t + 1$ and $m \in \mathbb{Z}$ odd. Therefore, $(2^kn_1, 2^lp_1)$, $(2^kn_2, 2^lp_2)$, $\ldots$, $(2^kn_{2t+1}, 2^lp_{2t+1})$ is also an odd cycle in $G^{Odd}(\mathbb{Q}^2)$. By Lemma 5.10, $(2^kn_i - 2^kn_{i+1}) + (2^lp_i - 2^lp_{i+1})$ is odd, for all $1 \leq i \leq 2t + 1$ where indices are taken modulo $2t + 1$. Thus we have

$$\sum_{i=1}^{2t+1} \left(2^kn_i - 2^kn_{i+1}\right) + \left(2^lp_i - 2^lp_{i+1}\right) \equiv 1 \mod 2,$$

since there are $2t + 1$ terms on the left hand side and each of them is odd. On the other hand,

$$\sum_{i=1}^{2t+1} \left(2^kn_i - 2^kn_{i+1}\right) + \left(2^lp_i - 2^lp_{i+1}\right)$$

$$= \sum_{i=1}^{2t+1} 2^kn_i - \sum_{i=1}^{2t+1} 2^kn_{i+1} + \sum_{i=1}^{2t+1} 2^lp_i - \sum_{i=1}^{2t+1} 2^lp_{i+1} = 0,$$

a contradiction. Therefore, the chromatic number of $G^{Odd}(\mathbb{Q}^2)$ is two. □
5.3 An Observation

Let $D$ be a set of positive integers. In Section 3.2 we defined the degree of accessibility of $D$, $\text{doa}(D)$, to be the maximum positive integer $r$, if it exists, so that for any $r$-coloring of $\mathbb{N}$ and for any $k \in \mathbb{N}$, there is a monochromatic sequence $\{a_1 < a_2 < \ldots < a_k\}$ such that $a_{i+1} - a_i \in D$ for all $1 \leq i \leq n - 1$ (called a $D$-diff sequence). Hence $\text{doa}(D) < r$ if and only if there exist $k > 1$ and an $r$-coloring of $\mathbb{N}$ such that there is no monochromatic $k$-term $D$-diff sequence. Considering the distance graph $G_D^{(\mathbb{R})}$, if $\chi^D(\mathbb{R})$ is finite, there is such a coloring for $k = 2$ and $r = \chi^D(\mathbb{R})$. Therefore, $\text{doa}(D) < \chi^D(\mathbb{R})$.

**Theorem 5.12.** Let $D$ be a set of positive integer. If $\text{doa}(D)$ exists then $G_D^{(\mathbb{R})}$ is finitely colorable i.e., $\chi^D(\mathbb{R})$ is also finite.

**Proof.** Let $r = \text{doa}(D) + 1$. Then there exist $k > 1$ and an $r$-coloring $\psi$ of $\mathbb{N}$ such that there is no monochromatic $k$-term $D$-diff sequence in $\mathbb{N}$. Extend the domain of $\psi$ to $\mathbb{Z}$ and let $\psi(-n + 1) = \psi(n)$ for all $n \geq 1$. Clearly, under this coloring there is no monochromatic $D$-diff sequence in $\mathbb{Z}$ with $2k - 1$ terms. Define another coloring $\psi^*$ of $\mathbb{Z}$ by

$$\psi^*(n) = (\psi(n), l_n)$$

where $l_n$ denotes the length of the longest $D$-diff sequence, with $n$ as its smallest element, in $\mathbb{Z}$ which is monochromatic under $\psi$. Since, under $\psi$, there is no $(2k - 1)$-term monochromatic $D$-diff sequence in $\mathbb{Z}$, $1 \leq l_n \leq 2k - 2$ for all $n \in \mathbb{Z}$. Hence $\psi^*$ is an $r(2k - 2)$ coloring of $\mathbb{Z}$.

Note that if $\psi(m) = \psi(n)$ and $|m - n| \in D$ then $l_m \neq l_n$. Hence, if $|m - n| \in D$ then $\psi^*(m) \neq \psi^*(n)$. Therefore, under $\psi^*$, there is no 2 term monochromatic $D$-diff sequence in $\mathbb{Z}$.

We now extend $\psi^*$ to $\mathbb{R}$, in the obvious way, by letting $\psi^*(x) = \psi^*(\lfloor x \rfloor)$. Clearly, $\psi^*$ is a proper coloring of $G_D^{(\mathbb{R})}$.

\[\square\]

**Corollary 5.13.** A set of positive integers $D$ is accessible if and only if $\chi^D(\mathbb{R}) = \aleph_0$.

Therefore, if $\text{doa}(D)$ is finite then $\text{doa}(D) < \chi^D(\mathbb{R}) < \aleph_0$.

In [1], it is noted that for any 5-coloring of $\{1, 2, \ldots, 13\}$, there exists a 2 term monochromatic $F$-diff sequence where $F$ is the set of Fibonacci numbers. This implies that $\chi^F(\mathbb{R}) > 5$. On the other hand the 6-coloring that we used in the proof of Theorem 3.30 can be used to obtain a proper 6-coloring of $G^F(\mathbb{R})$, giving $\chi^F(\mathbb{R}) \leq 6$. Therefore, $\chi^F(\mathbb{R}) = 6$. 
5.4 Remarks and Questions

We could not find any finite upper bound for the chromatic number of the odd-distance graph. The only known upper bound is the trivial bound $\aleph_0$, which believe to be the real value of the chromatic number. It has been shown that if the color classes are measurable then the chromatic number is actually $\aleph_0$ (see [14]).

In Section 5.2.2 we have shown that $\chi^{Odd}(\mathbb{Q}^2)$ is two. By the same method one can show that $\chi^{Odd}(\mathbb{Q}^3)$ is also two. What about for higher dimensions?

If Section 5.3 we have discussed the relation between the degree of accessibility of a set of positive integers and the chromatic number of the corresponding graph on $\mathbb{R}$ and we showed that $\chi^F(\mathbb{R}) = 6$. This makes us wonder if $\chi^F(\mathbb{R}^2)$ can be found. No bounds are known for it other than the trivial ones: $6 \leq \chi^F(\mathbb{R}^2) \leq \aleph_0$. 
Bibliography


