EQUILIBRIA OF A NONLOCAL MODEL FOR BIOLOGICAL AGGREGATIONS: LINEAR STABILITY AND BIFURCATION STUDIES

by

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Abstract

In this thesis, we study a nonlocal hyperbolic model for biological aggregations in one spatial dimension. In particular, we investigate the linear stability of the spatially homogeneous steady states and perform bifurcation studies. Two cases are considered; the first with constant velocity and the second with density-dependent velocity. We derive the dispersion relation and illustrate some examples for both cases. Numerical simulations of the model confirm the results obtained through linear stability analysis. We also provide the stability regions for some of the steady states by changing the magnitudes of attraction and repulsion and show that the instability region tends to increase in the presence of nonconstant velocities.
I would like to dedicate this body of work to my family members, who have supported me throughout my life.
“A person who never made a mistake never tried anything new.”

ALBERT EINSTEIN
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Chapter 1

Introduction

Flocks of birds, swarms of insects, schooling behavior of fish, herds of quadrupeds, bacterial swarms, and many other biological aggregations have attracted the attention of scientists for a long time.

Animals aggregate for different reasons. Examples include protecting against predators, avoiding harsh environmental conditions, increasing the chance of finding mates, or to take advantage of foraging efficiency. The migration of monarch butterflies is a good example. In Fall, the entire eastern North American population migrates approximately 5000 km to the southern US and central parts of Mexico to avoid severe winter conditions [5]. At the end of their migration 20 million monarch butterflies can be seen to settle in fewer than 1000 trees. Forests act for them as an umbrella to keep them dry and protected during winter storms. Their aggregation also helps protect them against their predators. See [5] for more details.

Aggregation also increases foraging efficiency. For example, Pitcher et al. [34] found out that the foraging time decreases for shoals of minnows and goldfish as the size of the group increases. Aggregation behavior provides spawning benefits in some species. For example, coral reef fish aggregate for the purpose of spawning [26, 31].

Animals aggregate based on internal or external factors. As an example of external factors, we mention fish aggregation around aggregating devices [30, 36]. Some hypotheses attempt to explain aggregations around floating objects. For example, the meeting point hypothesis suggests that some types of fish such as tuna use these floating objects to form larger groups and therefore use the benefits of being in a group. Soria et al. [36] tested the meeting point hypothesis for a type of pelagic fish called the bigeye scad. They verified that isolated individuals or small groups spend more time around the floating object than around any other arbitrary point. They start aggregating around the floating object and leave it in a larger group [36]. In this example, floating objects (like logs) are considered to be external factors that cause aggregations [31].

Alternately, internal factors focus on the social interactions between individuals, such as attraction, repulsion and alignment. Attraction is the tendency between distant individuals to get closer to each other. Repulsion is the social force that causes individuals in close proximity to repel from each other. Alignment represents the tendency to align with neighbours.

Huth and Wisselb [20] introduced a three dimensional model that explains the behavior of fish schools based on internal interactions between individuals. Specifically, they considered attraction, repulsion, and parallel orientation. They averaged the influences of at least four neighbours on these social interactions. They showed that the group mobility in fish schools can be explained by internal interactions between members of the group without
any external stimuli [20]. This thesis focuses on the aggregations caused by internal factors.

The study and understanding of animal aggregations can have valuable application in industry. For example, increasing knowledge of how, where and when fish aggregations form, or understanding the cause of schooling behaviour can help fishermen to catch fish more efficiently [30, 35]. Recently, engineers have shown interest in swarms, too. Understanding the way individuals communicate with each other and make coordinated decisions can help engineers to develop automated systems such as remote-controlled vehicles and autonomous multi-robot teams [1, 13, 15, 21].

1.1 Mathematical models

Mathematical models help us gain a better understanding and a clearer explanation for the observed group behaviours in nature. There exist two different categories for mathematical models of biological aggregations: Lagrangian and Eulerian models.

1.1.1 Lagrangian models

Lagrangian models are individual-based models that study the collective behavior of the animal groups. Most of these models consider the three different interactions among group members, that is, attraction towards distant individuals, repulsion towards the close ones and a tendency to align with neighbors [6, 17, 18, 20].

As an example, Couzin et al. [6] introduced a Lagrangian model in three dimensional space and observed four types of group structures: swarm, tori, dynamic parallel groups and highly parallel groups. The model by Couzin et al. also shows the transition between these group structures by changing the size of the interaction zones; see [6] for more details.

Most of the Lagrangian models assume that the individuals do not slow down or speed up as the result of social interactions with neighbours [6, 18, 20, 41]. There exist a few Lagrangian models in the literature that consider nonconstant velocities [4, 17, 16]. For instance, Buhl et al. [4] showed that by increasing the density of the individuals, a fast transition from disordered movement to extremely aligned motions can be obtained.

Analytical results are hard to obtain in Lagrangian models. Since the implementation of these models is difficult for large number of individuals, Eulerian models have been introduced in the case of dense groups.
1.1.2 Eulerian models

Eulerian models use partial differential equations and track the density of the individuals. The majority of these models are in two spatial dimensions. Eulerian models use hyperbolic or parabolic equations which describe the evolution of the density function. Most of the previous Eulerian models use attraction and repulsion [27, 28, 39] or only alignment [25, 32] as their social interactions. There are only a few Eulerian models in two spatial dimensions that consider all three interactions simultaneously [3, 11].

Eulerian models can be local or nonlocal. In local Eulerian models, nearby individuals play an important role. In nonlocal ones, distant individuals interact with each other. In the following, we briefly review a few of the Eulerian models in the literature:

- Lutscher et al. [25] propose a one dimensional hyperbolic model to describe the coordinated movements of cells in the presence of local interactions. Their model supports the rippling behaviour observed in Myxobacteria, in which right and left traveling ridges of high cell densities pass through each other.

- Mogilner et al. [28] introduce a nonlocal parabolic model and obtain traveling band solutions. They only consider attraction and repulsion as the social interactions between individuals. Mogilner et al. [28] also assume that the velocity of the individuals can vary as a result of interaction with neighbors. Their model is in one spatial dimension.

- Topaz et al. [39] construct a nonlocal Eulerian model and investigated the stationary solutions. In particular, their model shows in one spatial dimension that each group or clump is highly localized, and the density outside the group is zero. Topaz et al. [39] assume that the velocity is a function of the density. It should be noted that stationary solutions refer to distinct groups of individuals with sharp boundaries.

Here, we summarize the spatial patterns obtained through Eulerian models existent in the literature. Local parabolic models in one spatial dimension can give rise to ripple patterns [29], while nonlocal parabolic models can only support stationary pulses [39] or traveling pulses [28]. Hyperbolic models, on the other hand, can support ripples [25], aggregations [25], or traveling pulses [24]. See [9] and the references therein for more details.

In this research, we focus on a nonlocal hyperbolic model introduced originally by Pfistner [32], and improved by Eftimie, de Vries, Lewis, and Lutscher [10]. Eftimie et al. [10]
consider different communication mechanisms between individuals. In particular, they studied five submodels that correspond to five various types of receiving signals from neighbors. In addition to all the spatial patterns mentioned above, their model gives rise to many more complex spatial and spatiotemporal patterns.

1.2 Thesis outline and main accomplishments

In this thesis, we study equilibria of the nonlocal hyperbolic model introduced by Eftimie et al. [10]. We investigate the model for both constant and density dependent velocities.

In Section 2.1, we introduce the nonlocal hyperbolic model proposed by Eftimie et al. [10]. In particular, we talk about the way individuals receive information from neighbors and the way they respond to these perceived signals. In Section 2.2, we derive the constant velocity model via the correlated random walk. In the last section of Chapter two, we present the nonconstant velocity case proposed by Fetecau and Eftimie [12].

In Section 3.1, we obtain all the spatially homogeneous steady states. In Section 3.2, we perform a linear stability analysis of these steady states for the constant velocity and density dependent velocity cases. In Section 3.3, we plot the dispersion relation for some of the steady states. We solve the equations numerically by considering initial conditions that are small perturbations of the steady states. Finally, we confirm the analytical results obtained via linear stability using the numerical simulations of the model. In Section 3.4, we present the bifurcation diagrams for the constant and nonconstant velocity cases.

In Section 4.1, we introduce a new visualization tool, called ‘Paraglide’. In section 4.2, we obtain the stability regions for some of the steady states in the constant and nonconstant velocity cases by changing two parameters, the magnitude of attraction and repulsion. In this section, we also study and compare the stability regions for the two cases. We conclude from this comparison that the instability region tends to increase when the velocity is density dependent compared to the constant velocity case.

Here, we list the main accomplishments of this thesis:

- We derive the dispersion relation for the constant and nonconstant velocity cases;
- We confirm the analytical results obtained through the linear stability of the steady state with the numerical simulations of the model for specific cases;
- We plot and explore the bifurcation diagram;
• We have been the first user of the ‘Paraglide’ tool and assist in the improvements of the tool by giving feedbacks;

• We compare the stability regions for the constant and nonconstant velocity cases.
Chapter 2

Model Description

In this chapter, we present a one dimensional hyperbolic model for constant and density dependent velocities. In particular, in Section 2.1, we describe the model for the constant velocity case. In Section 2.2, we derive the model using a correlated random walk approach. Finally, in Section 2.3, we present the model with density dependent speed.

2.1 A nonlocal hyperbolic model with constant velocity

Eftimie et al. [8] introduced the following nonlocal hyperbolic model to describe biological aggregations in one spatial dimension:

\[ \begin{align*}
\partial_t u^+(x, t) + \gamma \partial_x (u^+(x, t)) &= -\lambda^+ [u^+, u^-] u^+(x, t) + \lambda^- [u^+, u^-] u^-(x, t) \\
\partial_t u^-(x, t) - \gamma \partial_x (u^-(x, t)) &= \lambda^+ [u^+, u^-] u^+(x, t) - \lambda^- [u^+, u^-] u^-(x, t)
\end{align*} \]

\[ u^\pm(x, 0) = u^\pm_0, \quad x \in \mathbb{R}. \]  \hspace{1cm} (2.1)

Here, \( u^+(x, t) \) and \( u^-(x, t) \) represent, respectively, the densities of the right and left moving individuals at position \( x \) and time \( t \). The velocity \( \gamma \) is constant. Some studies [12, 9] consider the density dependent velocity. We describe the nonconstant velocity case in more detail in Section 2.3. The parameter \( \lambda^+ (\lambda^-) \) is the turning rate for initially right (left) moving individuals that turn left (right). The density of right moving individuals decreases when they turn left. Therefore, the \( \lambda^+ u^+ \) term enters in the equation (2.1) with a negative sign. A similar explanation holds for the positive sign of \( \lambda^- u^- \).

Turning rates in system (2.1) model signals received by the individuals. Eftimie et al. [10] defined turning rates to be a bounded, positive and monotone function of the signals
perceived from the neighbors. They assumed $\lambda^\pm$ to be

$$
\lambda^+ = \lambda_0 + 0.5\lambda_2 \tanh(y^+[u^+, u^-] - y_0)) \\
\lambda^- = \lambda_0 + 0.5\lambda_2 \tanh(y^-[u^+, u^-] - y_0)),
$$

(2.2)

where $\lambda_0$ and $\lambda_2$ are constant turning rates.

Eftimie et al. [10] assume the total signal received by a right or left moving individual from all three interaction zones to be

$$
y^\pm[u^+, u^-] = y^+_{r}[u^+, u^-] - y^+_{a}[u^+, u^-] + y^\pm_{al}[u^+, u^-].
$$

(2.3)

The term $y^\pm_{i}[u^+, u^-]$, for $i \in \{a, r\}$, is defined to be the sum of all the signals that a right (+) or a left (-) moving individual receives from the neighbours in front of or behind it, located in the attraction ($a$) or in the repulsion zone ($r$). The term $y^\pm_{al}[u^+, u^-]$ corresponds to the summation of all the received signals from the neighbors that are located in the alignment zone and move toward the considered individual. It should be noted that the attractive and repulsive terms in (2.3) have opposite signs. The term $y^\pm_{i}$ for $i \in \{a, r, al\}$ can be described by the following mathematical expressions:

$$
y^+_{a,r}[u^+, u^-] = q_{a,r} \int_0^\infty K_{a,r}(s)(u(x+s,t) - u(x-s,t)) \, ds \\
y^-_{a,r}[u^+, u^-] = q_{a,r} \int_0^\infty K_{a,r}(s)(u(x-s,t) - u(x+s,t)) \, ds,
$$

(2.4)

and

$$
y^+_{al}[u^+, u^-] = q_{al} \int_0^\infty K_{al}(s)(u^-(x+s,t) - u^+(x-s,t)) \, ds \\
y^-_{al}[u^+, u^-] = q_{al} \int_0^\infty K_{al}(s)(u^+(x-s,t) - u^-(x+s,t)) \, ds.
$$

(2.5)

In equation (2.4),

$$
u(x \pm s):= u^+(x \pm s) + u^-(x \pm s)
$$

(2.6)

represents the total density. This model considers different spatial ranges for attraction, repulsion and alignment interactions as shown in Figure 2.1. It further assumes that any individual located at $(x, t)$ turns to repel the individual located in the repulsion zone, to approach the individuals located in the attraction zone or to align with the individuals located in the alignment zone.
CHAPTER 2. MODEL DESCRIPTION

Figure 2.1: Attraction, repulsion and alignment interaction zones. It is biologically realistic to assume that $s_r$ is less than $s_a$ or $s_{al}$.

The attraction, repulsion, and alignment kernels denoted by $K_a, K_r,$ and $K_{al}$ respectively weigh the information received from different neighbours. In this study, we use odd kernels for attraction and repulsion to ensure that distinct individuals exert equal forces with opposite signs on each other. The attraction and repulsion kernels are as follows

$$K_i(s) = \frac{s}{2s_i^2} \exp(-s^2/(2s_i^2)), i = r, a, s \in (-\infty, \infty). \quad (2.7)$$

We use a translated Gaussian kernel for alignment as,

$$K_{al}(s) = \frac{1}{\sqrt{2\pi m_{al}^2}} \exp(-(s - s_{al})^2/(2m_{al}^2)), s \in [0, \infty). \quad (2.8)$$

We also assume that more than 98% of the mass of the alignment kernel is in the interval $(0, \infty)$, i.e,

$$\int_0^\infty K_{al}(s) \, ds \approx 1. \quad (2.9)$$

Figure 2.2(a) illustrates the graph of attraction and repulsion kernels given by formula (2.7). Figure 2.2(b) shows the translated Gaussian alignment kernel as given by equation (2.8).

The interaction terms $y_{a,r}^\pm$ in (2.4) can be simplified to

$$y_{a,r}^\pm[u^+, u^-] = q_{a,r} \int_{-\infty}^{\infty} K_{a,r}(s)u(x \pm s, t) \, ds, \quad (2.10)$$

due to odd properties of attraction and repulsion kernels given in (2.7). Therefore, equation (2.3) can be rewritten as

$$y^\pm[u^+, u^-] = + q_r \int_{-\infty}^{\infty} K_r(s)u(x \pm s) \, ds$$

$$- q_a \int_{-\infty}^{\infty} K_a(s)u(x \pm s) \, ds$$

$$+ q_{al} \int_0^\infty K_{al}(s)(u^\mp(x \pm s) - u^\pm(x \mp s)) \, ds, \quad (2.11)$$

by considering (2.5) and (2.10).
CHAPTER 2. MODEL DESCRIPTION

2.2 Constant velocity model derivation

In this section, we present the derivation of the model introduced by Eftimie et al. [10] using the correlated random walk. Assume $p^\pm(x, t)$ is the probability of a randomly chosen right (left) moving individual to be at location $x$ at time $t$. Also, assume that $\lambda^\pm$ represents the probability that a right (left) moving individual turns left (right). The space and time are divided into intervals of size $\Delta x$ and $\Delta t$ respectively. We are interested in finding the probability of a randomly chosen right (left) moving individual to be at location $x$ at time step $t + \Delta t$.

An arbitrary selected right (left) moving individual will end up to be at location $x$ at time $t + \Delta t$ if at the previous time step $t$ it was located at $x - \Delta x$ and was moving to the right or it was moving to the left at $x + \Delta x$ and turned at the end of the time step. Figure 2.3 illustrates these information. Therefore, the probability of a randomly chosen
space time \[ x-\Delta x \quad x \quad x+\Delta x \] 

time \[ t \quad t+\Delta t \]

Figure 2.3: Model derivation via correlated random walk approach

right moving individual to be at position \( x \) at time \( t + \Delta t \) is

\[
p^+(x, t + \Delta t) = p^+(x - \Delta x, t)(1 - \lambda^+ \Delta t) + p^-(x + \Delta x, t)\lambda^-\Delta t \tag{2.12}
\]

Similarly,

\[
p^-(x, t + \Delta t) = p^-(x + \Delta x, t)(1 - \lambda^- \Delta t) + p^+(x - \Delta x, t)\lambda^+\Delta t. \tag{2.13}
\]

Taylor approximation around \((x, t)\) gives us

\[
p^+(x, t) + p^+_t(x, t)\Delta t = [p^+(x, t) - p^+_x(x, t)\Delta x](1 - \lambda^+ \Delta t) + [p^-(x, t) + p^-_x(x, t)\Delta x]\lambda^-\Delta t, \tag{2.14}
\]

and

\[
p^-(x, t) + p^-_t(x, t)\Delta t = [p^-(x, t) + p^-_x(x, t)\Delta x](1 - \lambda^- \Delta t) + [p^+(x, t) - p^+_x(x, t)\Delta x]\lambda^+\Delta t. \tag{2.15}
\]

Simplify equations (2.14) and (2.15) further to get

\[
p^+_t(x, t) + \frac{\Delta x}{\Delta t}p^+_x(x, t) = -\lambda^+ p^+(x, t) + \lambda^- p^-(x, t) + \Delta x\lambda^+ p^+_x(x, t) + \Delta x\lambda^- p^-_x(x, t),
\]

\[
p^-_t(x, t) - \frac{\Delta x}{\Delta t}p^-_x(x, t) = \lambda^+ p^+(x, t) - \lambda^- p^-(x, t) - \Delta x\lambda^+ p^+_x(x, t) - \Delta x\lambda^- p^-_x(x, t). \tag{2.16}
\]
By letting $\Delta x \to 0$ and $\Delta t \to 0$ such that $\frac{\Delta x \Delta t}{\Delta t} \to \gamma$, we obtain
\[
p_t^+(x, t) + \gamma p_x^+(x, t) = -\lambda^+ p^+(x, t) + \lambda^- p^-(x, t),
\]
\[
p_t^-(x, t) - \gamma p_x^-(x, t) = \lambda^+ p^+(x, t) - \lambda^- p^-(x, t).
\] (2.17)

Note that $p^\pm(x, t)$ represents the probability density function. The probability of a randomly chosen right or left-moving individual to be in the interval $[x - \frac{\Delta x}{2}, x + \frac{\Delta x}{2}]$ at time $t$ for a population of size $N$ is
\[
p^\pm(x, t) = \frac{1}{N} \int_{x - \frac{\Delta x}{2}}^{x + \frac{\Delta x}{2}} u^\pm(s, t) \, ds \to \frac{\Delta x u^\pm(x, t)}{N} \quad \text{when } \Delta x \to 0. \tag{2.18}
\]

By substituting $p^\pm(x, t)$ from (2.18) into (2.17), we obtain the system given in (2.1).

### 2.3 A nonlocal hyperbolic model with nonconstant velocity

In this section, we present a more general version of the model given in (2.1). To generalize the constant velocity model, we let the individuals to speed up or slow down as a response to their social interactions with neighbors [28, 14]. There exist some parabolic [6, 13, 7, 8] and hyperbolic models [19, 24] in the mathematical literature that study the motion of the animal aggregations with nonconstant velocities.

In this work, we assume that the speed depends only on attractive and repulsive interactions. Furthermore, we focus on the velocity function given by Fetecau and Eftimie [12]. Therefore, we consider the following nonlocal hyperbolic model with the density dependent velocity [12].

\[
\frac{\partial}{\partial t} u^+(x, t) + \frac{\partial}{\partial x} \left( \Gamma^+[u^+, u^-] u^+(x, t) \right) = -\lambda^+ u^+(x, t) + \lambda^- u^-(x, t)
\]
\[
\frac{\partial}{\partial t} u^-(x, t) - \frac{\partial}{\partial x} \left( \Gamma^-[u^+, u^-] u^-(x, t) \right) = \lambda^+ u^+(x, t) - \lambda^- u^-(x, t)
\]
\[
u^\pm(x, 0) = u^\pm_0, \quad x \in \mathbb{R}, \tag{2.19}
\]

where
\[
\Gamma^+[u^+, u^-] = \gamma(1 + \tanh(K(x) * u(x, t)))
\]
\[
\Gamma^-[u^+, u^-] = \gamma(1 - \tanh(K(x) * u(x, t))), \tag{2.20}
\]

and
\[
K(x) = -q_a K_a(x) + q_r K_r(x). \tag{2.21}
\]
The symbol ‘∗’ in (2.20), denotes the convolution. As we mentioned in Section 2.1, the total density \( u(x,t) \) is given by (2.6). We also use the same attraction, repulsion and alignment kernels defined by equations (2.7) and (2.8).

Using the ‘tanh’ function in the velocity equation (2.20) is an arbitrary choice. There exist some studies in the literature that consider the identity function instead of ‘tanh’ [28, 39, 22]. For instance, Leverents et al. in [22] investigate the asymptotic dynamics of a specific class of swarming models. Their one dimensional model consists of a conservation equation that describes the evolution of the population density. Leverents et al. in [22] consider the velocity to be the convolution of the density function with a kernel that represents the social interactions.
Chapter 3

Linear Stability Analysis and Bifurcation

In Section 3.1, we obtain the spatially homogeneous steady states. In Section 3.2, we study the linear stability analysis and derive the dispersion relation in the case of constant and density dependent velocities. In Section 3.3, we illustrate examples of the dispersion relation and confirm the analytical results obtained via linear stability. In the last section, we study the bifurcations of the spatially homogeneous steady states.

3.1 Steady States

In this section, we study the spatially homogeneous steady states of the system (2.1) denoted by $u^+(x,t) = u^*$ and $u^-(x,t) = u^{**}$. We attain the steady state equation by setting the time and space derivative terms in (2.1) to zero,

$$-\lambda^+[u^*, u^{**}]u^* + \lambda^-[u^*, u^{**}]u^{**} = 0,$$

(3.1)

where

$$\lambda^\pm[u^*, u^{**}] = \lambda_0 + 0.5\lambda_2 \tanh(y^\pm[u^*, u^{**}] - y_0)$$

(3.2)
and
\begin{align*}
y^\pm[u^*, u^{**}] &= + q_r \int_{-\infty}^{\infty} K_r(s)(u^* + u^{**}) \, ds \\
&\quad - q_a \int_{-\infty}^{\infty} K_a(s)(u^* + u^{**}) \, ds \\
&\quad + q_{al} \int_{0}^{\infty} K_{al}(s)(\pm u^{**} \mp u^*) \, ds.
\end{align*}
(3.3)

Note that \( A = u^* + u^{**} \) is a constant number and represents the total density. Equation (3.3) can be simplified to
\begin{align*}
y^+ [u^*, u^{**}] &= q_{al}(A - 2u^*) \\
y^- [u^*, u^{**}] &= q_{al}(2u^* - A),
\end{align*}
(3.4)
since \( K_a \) and \( K_r \) are odd kernels and the assumption (2.9) holds for the alignment kernel. Substitute (3.4) into (3.1) and (3.2) to get the steady state equation as
\[ h(u^*, q_{al}, \lambda, A) = 0, \] (3.5)
where
\[ h(u, q_{al}, \lambda, A) := - u[1 + \lambda \tanh(Aq_{al} - 2uq_{al} - y_0)] \\
+ (A - u)[1 + \lambda \tanh(-Aq_{al} + 2uq_{al} - y_0)], \]
(3.6)
and \( \lambda \) is defined as
\[ \lambda = \frac{0.5\lambda_2}{0.5\lambda_2 + \lambda_1}. \]
(3.7)
Figure 3.1 shows the graph of \( h(u, q_{al}, \lambda, A) \) in terms of \( u \) for three different values of \( q_{al} \) by fixing these parameters: \( A = 2, y_0 = 2, \lambda_1 = 0.2 \) and \( \lambda_2 = 0.9 \). As we infer from Figure 3.1, the steady state equation (3.5) can have one, three or five solutions based on the values of \( q_{al} \).

By investigating the slope of \( h(u, q_{al}, \lambda, A) \) with respect to \( u \), critical values \( Q^* \) and \( Q^{**} \) can be found [10]. These critical values determine the number of steady states. \( Q^* \) is given implicitly by
\[ h(u^*, q_{al}, \lambda, A) = 0, \quad \frac{\partial h(u, Q^*, \lambda, A)}{\partial u}|_{u = u^*} = 0, \quad \text{where } u^* \neq \frac{A}{2}. \]
(3.8)
The critical value \( Q^{**} \) can be obtained explicitly by
\[ \frac{\partial h(u, Q^{**}, \lambda, A)}{\partial u}|_{u = A/2} = 0. \]
(3.9)
Figure 3.1: The graph of the steady state equation for different values of $q_{al}$. The values of the fixed parameters are: $\lambda_1 = 0.2, \lambda_2 = 0.9, A = 2$. (a) The steady state equation has one zero where $q_{al} = 1.5$. (b) Five steady states can be obtained for $q_{al} = 2.2$. (c) The three steady states are observed for $q_{al} = 4$.

As you see from (3.8) and (3.9), the critical values, $Q^*$ and $Q^{**}$, depend on the turning rate, $\lambda$. A threshold value for $\lambda$ can be obtained when we set the critical values equal each other. When the value of the turning rate parameter is less than the threshold value, we can have one, three, or five steady states. As the value of $\lambda$ gets larger than the threshold value, only one or three steady states can be observed. See [10, 9] for more details.

Here, we derive a formula for $Q^{**}$ using equation (3.9). The first step is to compute the derivative of $h(u, q_{al}, \lambda, A)$ with respect to $u$:

$$
\frac{\partial h}{\partial u} = -(1 + \lambda \tanh(Aq_{al} - 2uq_{al} - y_0) + 2q_{al}\lambda u(1 - \tanh^2(Aq_{al} - 2uq_{al} - y_0))

- (1 + \lambda \tanh(-Aq_{al} + 2uq_{al} - y_0) + 2q_{al}\lambda(A - u)

\times (1 - \tanh^2(-Aq_{al} + 2uq_{al} - y_0)).

\tag{3.10}
$$

Substitute $u = \frac{A}{2}$ into (3.10), and set it equal to zero:

$$
\lambda Aq_{al}(1 - \tanh^2 y_0) = (1 - \lambda \tanh(y_0)).
$$

\tag{3.11}
We then solve for $q_{al}$ in (3.11) and define it as $Q^{**}$. Therefore,

$$Q^{**} = \frac{1 - \lambda \tanh(y_0)}{\lambda A(1 - \tanh^2 y_0)}.$$  

(3.12)

The spatially homogeneous steady states are illustrated in Figure 3.2. As shown in this Figure, there is only one steady state $u_3^* = (u_3^*, u_3^*)$ where $q_{al}$ is less than $Q^*$. As $q_{al}$ approaches $Q^*$, four steady states are created: $u_1^* = (u_1^*, u_1^*)$, $u_2^* = (u_2^*, u_2^*)$, $u_4^* = (u_4^*, u_4^*)$ and $u_5^* = (u_5^*, u_5^*)$. When $q_{al}$ passes through $Q^{**}$, two of the steady states $(u_2^*, u_4^*)$ and $(u_4^*, u_2^*)$ cease to exist.

![Figure 3.2: The spatially homogeneous steady states based on different values of $q_{al}$.](image)

Figure 3.2: The spatially homogeneous steady states based on different values of $q_{al}$. $u_3^*$ is the only steady state when $q_{al} < Q^*$. Five steady states $u_1^*$, $u_2^*$, $u_3^*$, $u_4^*$, and $u_5^*$ exist when $Q^* \leq q_{al} \leq Q^{**}$. Two steady states $u_2^*$, and $u_4^*$ disappear when $q_{al} > Q^{**}$.

3.2 Linear Stability Analysis

3.2.1 Linear stability analysis: constant velocity case

In this Subsection, we obtain the dispersion relation for model (2.1). We consider this model on a domain of length $L$ for the sake of numerical simulation. To study the linear stability
analysis for this system, we perturb the steady state \((u^*, u^{**})\) by spatially nonhomogeneous terms, i.e., we substitute

\[
\begin{align*}
    u^+(x, t) &= u^* + u_p(x, t) \\
    u^-(x, t) &= u^{**} + u_m(x, t),
\end{align*}
\]

(3.13)

into system (2.1). We take

\[
\begin{align*}
    u_p(x, t) &= c_p e^{\sigma t + i k x} \\
    u_m(x, t) &= c_m e^{\sigma t + i k x}, \quad c_{p,m} \in \mathbb{C},
\end{align*}
\]

(3.14)

where \(\sigma\) denotes the growth rate and \(k\) represents the wave number, \(k_n = \frac{2n\pi}{L}\), for \(n \in \mathbb{N}\).

The first step in obtaining the dispersion relation is to substitute the perturbations into (2.11) and to compute \(y^\pm\). The next step is to find \(\tanh(y^\pm - y_0)\) by using the Taylor expansion and ignoring higher order terms. We obtain the dispersion relation by simplifying the right hand side and substituting the perturbations into the left hand side. The equations below implement these steps.

The term \(y^+ [u^+, u^-]\) after adding the perturbations is

\[
y^+ = q_r \int_{-\infty}^{\infty} K_r(s)(u^* + u_p(x + s, t) + u^{**} + u_m(x + s, t)) \, ds
- q_a \int_{-\infty}^{\infty} K_a(s)(u^* + u_p(x + s, t) + u^{**} + u_m(x + s, t)) \, ds
+ q_{al} \int_{0}^{\infty} K_{al}(s)(u^{**} + u_m(x + s)) \, ds - q_{al} \int_{0}^{\infty} K_{al}(s)(u^* + u_p(x - s)) \, ds.
\]

(3.15)

Use the property of \(K_{al}\) given in (2.9), and consider the choice of odd attraction and repulsion kernels to get

\[
y^+ = q_r \int_{-\infty}^{\infty} K_r(s)u_p(x + s, t) \, ds + q_r \int_{-\infty}^{\infty} K_r(s)u_m(x + s, t) \, ds
- q_a \int_{-\infty}^{\infty} K_a(s)u_p(x + s, t) \, ds - q_a \int_{-\infty}^{\infty} K_a(s)u_m(x + s, t) \, ds
+ q_{al}(u^{**} - u^*) + q_{al} \int_{0}^{\infty} K_{al}(s)u_m(x + s, t) \, ds
- q_{al} \int_{0}^{\infty} K_{al}(s)u_p(x - s, t) \, ds.
\]

(3.15)
Substitute $u_p(x,t)$ and $u_m(x,t)$ from (3.14) into (3.15):

\[
y^+ = q_r c_p e^{\sigma t + ikx} \int_{-\infty}^\infty K_r(s)e^{iks} ds + q_r c_m e^{\sigma t + ikx} \int_{-\infty}^\infty K_r(s)e^{iks} ds
\]

\[
- q_a c_p e^{\sigma t + ikx} \int_{-\infty}^\infty K_a(s)e^{iks} ds - q_a c_m e^{\sigma t + ikx} \int_{-\infty}^\infty K_a(s)e^{iks} ds
\]

\[
+ q_al(u^{**} - u^*) + q_al c_m e^{\sigma t + ikx} \int_{0}^\infty K_al(s)e^{iks} ds
\]

\[
- q_al c_p e^{\sigma t + ikx} \int_{0}^\infty K_al(s)e^{-iks} ds.
\]

(3.16)

Notice that

\[
\int_{-\infty}^\infty K_i(s)e^{\pm iks} ds = \hat{K}_i(k), \quad i \in \{a, r, al\},
\]

(3.17)

where $\hat{K}$ represents the Fourier transform of $K$. By using (2.9) and (3.17):

\[
\int_{0}^\infty K_al(s)e^{iks} ds \approx \int_{-\infty}^\infty K_al(s)e^{iks} ds = \hat{K}_al(k).
\]

(3.18)

We deduce

\[
y^+ = q_r \hat{K}_r^+ u_p + q_r \hat{K}_r^+ u_m - q_a \hat{K}_a^+ u_p - q_a \hat{K}_a^+ u_m
\]

\[
+ q_al(u^{**} - u^*) + q_al \hat{K}_al^+ u_m - q_al \hat{K}_al^+ u_p.
\]

(3.19)

Since $u_p$ and $u_m$ are small perturbations, we can use the Taylor expansion of the tanh function around $q_al(u^{**} - u^*) - y_0$ to compute $\tanh(y^+ - y_0)$.

Introduce

\[
M_1 := q_al(u^{**} - u^*),
\]

(3.20)

to obtain

\[
\tanh(y^+ - y_0) = \tanh(M_1 - y_0) + (1 - \tanh^2(M_1 - y_0))(q_r \hat{K}_r^+ u_p + q_r \hat{K}_r^+ u_m
\]

\[
- q_a \hat{K}_a^+ u_p - q_a \hat{K}_a^+ u_m + q_al \hat{K}_al^+ u_m - q_al \hat{K}_al^+ u_p).
\]

Define

\[
L_1 := \lambda_0 + 0.5\lambda_2 \tanh(M_1 - y_0)
\]

\[
P_1 := 0.5\lambda_2(1 - \tanh^2(M_1 - y_0)),
\]

(3.21)
and ignore the second and higher order terms to get

\[-(\lambda_0 + 0.5\lambda_2 \tanh(y^+ - y_0))(u^* + u_p) = -L_1 u^* - L_1 u_p - P_1 u^* (q_r \hat{K}_r^- u_p + q_r \hat{K}_r^+ u_m - q_a \hat{K}_a^- u_p - q_a \hat{K}_a^+ u_m) + q_a \hat{K}_a^- u_p - q_a \hat{K}_a^+ u_m + q_a \hat{K}_a^- u_p - q_a \hat{K}_a^+ u_m).\]

(3.22)

By a similar strategy, we derive

\[y^- = q_r \hat{K}_r^- u_p + q_r \hat{K}_r^- u_m - q_a \hat{K}_a^- u_p - q_a \hat{K}_a^- u_m + q_a(u^* - u^{**}) + q_a \hat{K}_a u_p - q_a \hat{K}_a^+ u_m.\]

(3.23)

Use the Taylor expansion around \(q_a(u^* - u^{**}) - y_0:\)

\[\tanh(y^- - y_0) = \tanh(-M_1 - y_0) + (1 - \tanh^2(-M_1 - y_0))(q_r \hat{K}_r^- u_p + q_r \hat{K}_r^- u_m - q_a \hat{K}_a^- u_p - q_a \hat{K}_a^- u_m + q_a \hat{K}_a^- u_p - q_a \hat{K}_a^+ u_m).\]

Define

\[L_2 := \lambda_0 + 0.5\lambda_2 \tanh(-M_1 - y_0)\]

\[P_2 := 0.5\lambda_2(1 - \tanh^2(-M_1 - y_0)),\]

and ignore the second and higher order terms to find

\[(\lambda_0 + 0.5\lambda_2 \tanh(y^- - y_0))(u^{**} + u_m) = L_2 u_m + L_2 u^{**} + P_2 u^{**} (q_r \hat{K}_r^- u_p + q_r \hat{K}_r^- u_m - q_a \hat{K}_a^- u_p - q_a \hat{K}_a^- u_m + q_a \hat{K}_a^- u_p - q_a \hat{K}_a^+ u_m).\]

(3.24)

Therefore, the right hand side of the first equation in system (2.1) after adding the perturbations, is

\[-\lambda^+(u^* + u_p) + \lambda^-(u^{**} + u_m) = I_c \times u_p + II_c \times u_m,\]

(3.25)

where

\[I_c = -L_1 - q_r M_3 \hat{K}_r^+ + q_a M_5 \hat{K}_a^- + q_a M_3 \hat{K}_a^-\]

\[II_c = L_2 - q_r M_3 \hat{K}_r^+ + q_a M_5 \hat{K}_a^- - q_a M_5 \hat{K}_a^+ ,\]

(3.26)

and

\[M_5 := P_1 u^* + P_2 u^{**}.\]

(3.27)
The subscript in the coefficients $I_c$ and $\Pi_c$ indicates the constant velocity case. Note that the right hand side of the second equation in system (2.1) is of opposite sign to the first equation. The left hand sides of this system after adding the perturbations are

$$
\begin{align*}
\partial_t (u^* + u_p) + \partial_x (\gamma u^* + \gamma u_p) &= (\sigma_c + ik\gamma) u_p, \\
\partial_t (u^{**} + u_m) - \partial_x (\gamma u^{**} + \gamma u_m) &= (\sigma_c - ik\gamma) u_m.
\end{align*}
$$

(3.28)

Therefore, system (2.1), after substituting the perturbations, can be written as

$$
\sigma_c \begin{bmatrix} u_p \\ u_m \end{bmatrix} = \begin{bmatrix} I_c - ik\gamma & \Pi_c \\ -I_c & -\Pi_c + ik\gamma \end{bmatrix} \begin{bmatrix} u_p \\ u_m \end{bmatrix},
$$

(3.29)

where $I_c$ and $\Pi_c$ can be obtained from (3.26). As we see from (3.29), $\sigma$ is the eigenvalue of the $2 \times 2$ matrix

$$
J_c = \begin{bmatrix} I_c - ik\gamma & \Pi_c \\ -I_c & -\Pi_c + ik\gamma \end{bmatrix}.
$$

(3.30)

Consequently, $\sigma$ can be obtained from the dispersion relation

$$
\sigma^2_c - \text{tr}(J_c)\sigma_c + \det(J_c) = 0,
$$

(3.31)

where

$$
\text{tr}(J_c) = -L_1 - L_2 + M_5q_a(\hat{K}_{al}^- + \hat{K}_{al}^+),
$$

(3.32)

and

$$
\det(J_c) = \gamma^2 k^2 + ik\gamma (L_2 - L_1 + M_5q_a(\hat{K}_{al}^- - \hat{K}_{al}^+)) \\
- 2M_5ik\gamma (-q_a\hat{K}_{a}^+ + q_r\hat{K}_{r}^+).
$$

(3.33)

For convenience, we collect below all parameters introduced in the derivation of the dispersion relation:

$$
\begin{align*}
L_1 &= \lambda_1 + 0.5\lambda_2 + 0.5\lambda_2 \tanh(M_1 - y_0) \\
L_2 &= \lambda_1 + 0.5\lambda_2 + 0.5\lambda_2 \tanh(-M_1 - y_0) \\
P_1 &= 0.5\lambda_2 (1 - \tanh^2(M_1 - y_0)) \\
P_2 &= 0.5\lambda_2 (1 - \tanh^2(-M_1 - y_0)) \\
M_1 &= q_a(u^{**} - u^*) \\
M_5 &= P_1 u^* + P_2 u^{**}.
\end{align*}
$$

(3.34)
3.2.2 Linear stability analysis: nonconstant velocity case

In this section, we obtain the dispersion relation for the nonconstant velocity model given in (2.19), (2.20) and (2.21). As we see from the systems given in (2.1) and (2.19), the constant and nonconstant velocity models have the same right-hand sides. Since the left-hand sides for both systems include time and space derivatives, we can conclude that the spatially homogeneous steady states are the same. As mentioned earlier, the spatially homogeneous steady states are the solutions to the equations (3.5) and (3.6). The difference between constant and nonconstant velocity models in the dispersion relation comes from the left-hand sides.

Here, we derive the dispersion relation for the nonconstant velocity model. The same strategy as in Subsection 3.2.1 is followed. We substitute the perturbations given in (3.13) and (3.14) into the convolution term in (2.20), to get

$$(K * u)(x) = \int_{-\infty}^{\infty} K(s)(u^+(x-s) + u^-(x-s)) ds$$

$$= \int_{-\infty}^{\infty} K(s)(u^* + u_p(x-s) + u^{**} + u_m(x-s)) ds$$

$$= \int_{-\infty}^{\infty} c_p K(s)e^{\sigma t+ik(x-s)} ds + \int_{-\infty}^{\infty} c_m K(s)e^{\sigma t+ik(x-s)} ds$$

$$= u_p \hat{K}^- - u_m \hat{K}^+. \tag{3.35}$$

Equation (3.35) is derived for odd attraction and repulsion kernels. Therefore, the term $\partial_x(\Gamma^+u^+)$ can be simplified to

$$\partial_x(\Gamma^+u^+) = \partial_x[(\gamma + \gamma \tanh(u_p\hat{K}^- - u_m\hat{K}^+))(u^* + u_p)]$$

$$= ik\gamma u_p + ik\gamma(u^* + u_p)(1 - \tanh^2(u_p\hat{K}^- - u_m\hat{K}^+))(u_p\hat{K}^- - u_m\hat{K}^+).$$

Use the Taylor expansion of tanh function around the origin, and ignore the second, and higher order terms to get

$$\partial_x(\Gamma^+u^+) = ik\gamma(1 + u^*\hat{K}^-)u_p - ik\gamma u^*\hat{K}^+u_m. \tag{3.36}$$

By a similar calculation,

$$\partial_x(\Gamma^-u^-) = -ik\gamma u^{**}\hat{K}^-u_p + ik\gamma(1 + u^{**}\hat{K}^+)u_m. \tag{3.37}$$
Using (3.36), and (3.37) the left-hand sides of system (2.19) become

\[
\begin{align*}
\partial_t u^+ + \partial_x (\Gamma^+ u^+(x,t)) &= [\sigma_{nc} + ik\gamma + ik\gamma u^*(-q_a \hat{K}_a^- + q_r \hat{K}_r^-)] u_p \\
&\quad\quad + [-ik\gamma u^*(-q_a \hat{K}_a^+ + q_r \hat{K}_r^+)] u_m \\
\partial_t u^- - \partial_x (\Gamma^- u^-(x,t)) &= [ik\gamma u^{**}(-q_a \hat{K}_a^- + q_r \hat{K}_r^-)] u_p \\
&\quad\quad + [\sigma_{nc} - ik\gamma - ik\gamma u^{**}(-q_a \hat{K}_a^+ + q_r \hat{K}_r^+)] u_m.
\end{align*}
\]

(3.38)

The subscript in \(\sigma_{nc}\) indicates the nonconstant velocity case. System (2.19), after adding the perturbations, can be written as

\[
\sigma_{nc} \begin{bmatrix} u_p \\ u_m \end{bmatrix} = \begin{bmatrix} I_{nc} & II_{nc} \\ III_{nc} & IV_{nc} \end{bmatrix} \begin{bmatrix} u_p \\ u_m \end{bmatrix},
\]

(3.39)

where

\[
\begin{align*}
I_{nc} &= I_c - ik\gamma - ik\gamma u^*(-q_a \hat{K}_a^- + q_r \hat{K}_r^-) \\
II_{nc} &= II_c + ik\gamma u^*(-q_a \hat{K}_a^+ + q_r \hat{K}_r^+) \\
III_{nc} &= -I_c - ik\gamma u^{**}(-q_a \hat{K}_a^- + q_r \hat{K}_r^-) \\
IV_{nc} &= -II_c + ik\gamma + ik\gamma u^{**}(-q_a \hat{K}_a^+ + q_r \hat{K}_r^+),
\end{align*}
\]

(3.40)

and \(I_c\) and \(II_c\) are given by (3.26). As we see from equation (3.39), \(\sigma_{nc}\) is the eigenvalue of a 2 \(\times\) 2 matrix which we denote by \(J_{nc}\). Therefore, the dispersion relation for the system (2.19) is given by

\[
\sigma_{nc}^2 - \text{tr}(J_{nc})\sigma + \text{det}(J_{nc}) = 0,
\]

(3.41)

where

\[
\text{tr}(J_{nc}) = \text{tr}(J_c) + ik\gamma A(-q_a \hat{K}_a^+ + q_r \hat{K}_r^+),
\]

\[
\text{det}(J_{nc}) = \text{det}(J_c) + ik\gamma A(-q_a \hat{K}_a^- + q_r \hat{K}_r^-)(L_1 + L_2)
\]

\[+ ik\gamma q_a M_5(-q_a \hat{K}_a^+ + q_r \hat{K}_r^+)(\hat{K}_a^+ + \hat{K}_a^-) + k^2\gamma^2(u^* - u^{**})(-q_a \hat{K}_a^- + q_r \hat{K}_r^-).
\]

(3.42)

Here, \(\text{tr}(J_c)\) and \(\text{det}(J_c)\) can be obtained from (3.32), and (3.33) respectively.
3.3 Linear Stability: examples

3.3.1 Examples of the dispersion relation: constant velocity case

In this section, we present examples of the dispersion relation for some of the spatially homogeneous steady states, as the value of the bifurcation parameter varies.

Figure 3.3 illustrates the real and imaginary parts of the eigenvalue $\sigma_c$ for the steady state $u_3^s$. Note that $\sigma_c$ is the solution of the equation (3.31). In Figure 3.3(a), $\sigma_c$ has a negative real part for all the values of $k$. Therefore, the steady state $u_3^s$ is stable where $q_{al} = 0.5$.

As we increase $q_{al}$, the real part of $\sigma_c$ becomes zero at a critical mode $k_{15}$, where $q_{al} = 1.3$; see Figure 3.3(b). By increasing $q_{al}$ further, we get a band of unstable modes for $q_{al} > 1.3$.

Figure 3.3(c) shows that the steady state $u_3^s$ is unstable where $q_{al} = 2$. A similar transition from stable to unstable for the steady state $u_3^s$ can be observed in Figure 3.3(d-f), by varying $q_a$. The difference is that the critical mode at which the real part of $\sigma_c$ is equal to zero is at $k_1$, as compared to part (c) in which the critical mode was at $k_{15}$.

Figure 3.4 shows the transition from stable to unstable for the steady state $u_1^s$. This transition can be detected by decreasing $q_{al}$ in Figure 3.4(a-c), or by increasing $q_a$, in Figure 3.4(d-f). As we see from Fig 3.4, the steady state $u_1^s$ is stable for $q_{al} = 2.5$ and $q_a = 0$ in parts (a) and (d). This steady state becomes unstable for $q_{al} = 2$, and $q_a = 7$ in parts (c), and (f). The real part of $\sigma_c$ for the steady state $u_1^s$ equals zero at $k_{16}$, for $q_{al} = 2.2$, and at $k_1$, for $q_a = 3.05$. 


Figure 3.3: Some examples of the dispersion relation for the steady state $u^*_3$. The solid line represents the real part and the dashed line represents the imaginary part of $\sigma_c$. We fixed the following parameters: $L = 10$, $y_0 = 2$, $A = 2$, $\gamma = 0.1$, $q_r = 0$, $s_a = 1$, $s_r = 0.25$. In (a), (b), and (c) $\lambda_1 = 1.33$, $\lambda_2 = 6$, $q_a = 0$, and $s_{al} = 0.5$. In (d), (e), and (f) $\lambda_1 = 0.4$, $\lambda_2 = 1.8$, $q_{al} = 0.5$, and $s_{al} = 1.25$. In parts (a), (b), and (c) $q_{al}$ is the bifurcation parameter and takes the values 0.5, 1.3, and 2 respectively; while in parts (d), (e), and (f) $q_a$ is the bifurcation parameter, and takes the values 0, 1.16, and 5 respectively. A transition from stability to instability is observed for the steady state $u^*_3$, as we increase $q_{al}$ in parts (a), (b), and (c) or as we increase $q_a$ in parts (d), (e), and (f). The steady state $u^*_3$ is stable for $q_{al} = 0.5$, and $q_a = 0$, in parts (a) and (d). This steady state becomes unstable for $q_{al} = 2$, and $q_a = 5$ in parts (c) and (f). The real part of $\sigma_c$ for the steady state $u^*_3$ equals to zero at $k_{15}$ for $q_{al} = 1.3$, and at $k_1$ for $q_a = 1.16$. 
Figure 3.4: Some examples of the dispersion relation for the steady state $u_1^*$. The solid line represents the real part and the dashed line represents the imaginary part of $\sigma_c$. We fixed the following parameters: $L = 10, y_0 = 2, A = 2, \gamma = 0.1, s_a = 1, s_r = 0.25$. In (a), (b), and (c) $\lambda_1 = 0.4, \lambda_2 = 1.8, q_a = 0, q_r = 0$, and $s_al = 0.5$. In (d), (e), and (f) $\lambda_1 = 0.2, \lambda_2 = 0.9, q_al = 2, q_r = 1$, and $s_al = 1.25$. In parts (a), (b), and (c) $q_al$ is the bifurcation parameter and takes the values 2.5, 2.2, and 2 respectively; while in parts (d), (e), and (f) $q_a$ is the bifurcation parameter, and takes the values 0, 3.05, and 7 respectively. A transition from stability to instability is observed for the steady state $u_1^*$, as we decrease $q_al$ in parts (a), (b), and (c) or as we increase $q_a$ in parts (d), (e), and (f). The steady state $u_1^*$ is stable for $q_al = 2.5$, and $q_a = 0$, in parts (a) and (d), respectively. This steady state becomes unstable for $q_al = 2$, and $q_a = 7$ in parts (c) and (f). The real part of $\sigma_c$ for the steady state $u_1^*$ equals to zero at $k_{16}$ for $q_al = 2.2$, and at $k_1$ for $q_a = 3.05$. 
3.3.2 Examples of the dispersion relation: nonconstant velocity case

Here, we present the dispersion relation for the spatially homogeneous steady state $u_5^*$ in the nonconstant velocity model (2.19).

Figure 3.5 illustrates the transition from stable to unstable for the steady state $u_5^*$, as we decrease the value of $q_{al}$ in the nonconstant velocity model. As shown in Figure 3.5(a), the steady state $u_5^*$ is stable for $q_{al} = 3.3$. The real part of $\sigma_{nc}$ becomes zero where $q_{al} = 1.98$, at the critical mode $k_{16}$. The steady state is unstable where $q_{al} = 1.9$. Note that $\sigma_{nc}$ is the solution to the equation (3.41). As the real part of $\sigma_{nc}$ passes through zero, the imaginary
part stays nonzero. This confirms the occurrence of Hopf bifurcation shown at $q_{al} = q_h$, in Figure 3.14(b).

3.3.3 Illustration of the linear instability

Linear stability analysis gives us the stable and unstable wave numbers of the perturbations. To verify the results obtained through linear stability analysis, we set the initial condition to be random perturbations of the spatially homogeneous steady state, and simulate the model numerically. We use a Fourier pseudospectral method to discretize the space, and a fourth order Runge-Kutta method to advance the solution in time. We consider the periodic boundary conditions. We also choose the time step $\Delta t$ and the space step $\Delta x$ such that the Courant-Friedrichs-Lewy (CFL) condition holds, i.e., $\frac{\gamma \Delta t}{\Delta x} > 1$ where $\gamma$ represents the constant velocity.

The simulations show spatial patterns for unstable wave numbers since the perturbations grow. On the other hand, no patterns can be observed for the stable wave numbers since the perturbations damp out. Moreover, the number of groups in the spatial pattern agrees with the first wave number that becomes unstable, i.e., critical wave number.

Figure 3.6 shows the spatial pattern for the constant velocity model (2.1). All the parameter values correspond to Figure 3.3(b). By setting the initial condition to be the random perturbations of the spatially homogeneous steady state $u_3^*$, and solving the constant velocity model numerically, we obtain Figure 3.6. The cross section of this figure at time $= 190$ is given in Figure 3.7. As we see from these figures, 15 groups have been formed which agrees with the first wave number that becomes unstable, i.e. critical wave number $k_{15}$, obtained in Figure 3.3(b). This verifies the linear stability analysis in this specific case.

Figure 3.8 presents the numerical solution of the constant velocity model where the initial condition is the random perturbations of $u_3^*$. The cross section of this figure at time $= 76$ is given in Figure 3.9. The two groups in the spatial pattern matches with the critical wave number $k_2$ obtained in Figure 3.3(f).

All parameter values in Figure 3.10 and Figure 3.4(b) are the same. Figure 3.10 shows the density profile over time for the constant velocity model when the initial condition is the random perturbation of the spatially homogeneous steady state $u_1^*$. The cross section of this figure at time $= 95$ is given in Figure 3.11. The sixteen bumps in the cross section figure agree with the critical wave number $k_{16}$ obtained from Figure 3.4(b). See these figures for more details.
Figure 3.12 corresponds to the numerical simulation of the nonconstant velocity model. The initial condition is a spatial perturbation of the steady state $u^*_5$. The cross section of this spatial pattern at time = 57 is given in Figure 3.13. The parameter values are the same as the ones used in Figure 3.5. In the simulation, we fix the total density $A = 2$. As shown in the spatial pattern, individuals form 16 groups. This number matches with the critical wave number $k_{16}$ illustrated in Figure 3.5(b).

Figure 3.6: Spatial pattern obtained through solving the constant velocity model numerically. All the parameter values agree with Figure 3.3(b). The initial condition is a random perturbation of the steady state $u^*_3$. As we see from this figure, the individuals form 15 groups which agrees with the critical wave number $k_{15}$ in Figure 3.3(b).

Figure 3.7: The cross section of the spatial pattern given in Figure 3.6 at time = 190. The horizontal axis represents space and the vertical one represents the density $u$. 
Figure 3.8: Spatial pattern obtained through solving the constant velocity model numerically. The horizontal and vertical axes represents space and time, respectively. The color shows the density. All the parameter values correspond to Figure 3.3(f). The initial condition is a random perturbation of the steady state $u^*_3$. As we see from this figure, the individuals form two groups which agrees with the critical wave number $k_2$ in Figure 3.3(f).

Figure 3.9: The cross section of the spatial pattern given in Figure 3.8 at time = 76. The horizontal axis represents space and the vertical one represents the density $u$. 
Figure 3.10: Spatial pattern obtained through solving the constant velocity model numerically. The horizontal and vertical axes represents space and time, respectively. The color shows the density. All the parameter values correspond to Figure 3.4(b). The initial condition is a random perturbation of the steady state $u^*_1$. As we see from this figure, the individuals form 16 groups which agrees with the critical wave number $k_{16}$ in Figure 3.4(b).

Figure 3.11: The cross section of the spatial pattern given in Figure 3.10 at time = 95. The horizontal axis represents space and the vertical one represents the density $u$. 
Figure 3.12: Spatial pattern obtained through solving the nonconstant velocity model numerically. The horizontal and vertical axes represent space and time, respectively. The color shows the density. All the parameter values correspond to Figure 3.5. The initial condition is a random perturbation of the steady state $u^*_5$. As we see from this figure, the individuals form 16 groups which agrees with the critical wave number $k_{16}$ in Figure 3.5(b).

Figure 3.13: The cross section of the spatial pattern given in Figure 3.12 at time = 57. The horizontal axis represents space and the vertical one represents the density $u$. 
3.4 Bifurcation

Figure 3.14 illustrates the bifurcation diagrams for all the spatially homogeneous steady states in the case of constant and density dependent velocities with $q_{al}$ as the bifurcation parameter. The filled circles denote the stable spatially homogeneous steady states and the unfilled ones denote the unstable steady states. The fixed parameter values in Figure 3.14(a) correspond to Figure 3.4(a-c), while the parameter values in Figure 3.14(b) correspond to Figure 3.5.

This figure is obtained via solving the steady state equation (3.5), and (3.6) using Newton’s method for each value of $q_{al}$. The points at which the stability of the steady state changes can be obtained using the dispersion relation equations given in (3.31), and (3.41).

The only steady state is $u^*_3$, when the value of $q_{al}$ is less than $Q^*_\sim 1.9$. This steady state loses its stability at the critical value $Q^{**} \sim 3.4$, in the case of constant and density dependent velocities.

**Saddle-node bifurcation at $q_{al} = Q^*$:** A saddle-node bifurcation is related to the creation and destruction of the steady states as the value of the bifurcation parameter varies. At $q_{al} = Q^*$, four steady states $u^*_1$, $u^*_2$, $u^*_4$, and $u^*_5$ are created. Therefore, these fixed points undergo a saddle-node bifurcation at $q_{al} = Q^*$ for both constant and density dependent velocities. As we increase $q_{al}$, the two steady states $u^*_2$, and $u^*_4$ approach each other and collide at $q_{al} = Q^{**}$. These two steady states cease to exist as soon as $q_{al} > Q^{**}$.

**Subcritical pitchfork bifurcation at $q_{al} = Q^{**}$:** It is common for the fixed points of a spatially symmetrical system to go through a pitchfork bifurcation. In this type of bifurcation, the steady states tend to be created and destroyed in symmetrical pairs [37]. The two fixed points $u^*_2$, and $u^*_4$ are unstable before the bifurcation, where $q_{al} < Q^{**}$ and disappear after the bifurcation has occurred. Therefore, a subcritical pitchfork bifurcation has occurred at $q_{al} = Q^{**}$ as demonstrated in Figure 3.14(a-b).

**Hopf bifurcation at $q_h$, and $q_{h1}$:** As depicted in Figure 3.14(a), the fixed points $u^*_1$, and $u^*_5$ lose their stability at $q_h \simeq 2.2$. By investigating the graph of the dispersion relation, we conclude that as the real part of one or both of the eigenvalues passes through zero, the imaginary part stays nonzero. This confirms the occurrence of Hopf bifurcation. The graph of the dispersion relation for the steady state $u^*_1$ is given in Figure 3.4(a-c) for three different values of $q_{al}$. The same explanation holds for Hopf bifurcation that is taken place for the steady state $u^*_1$, and $u^*_5$ in the nonconstant velocity case at $q_{h1} \simeq 2$. The occurrence
of Hopf bifurcation at $q_{h_1}$ for the steady state $u_5^*$ is illustrated in Figure 3.5.

Figure 3.14: Bifurcation diagrams for the cases of (a) constant and (b) density dependent velocities. The filled (unfilled) circles represent the stable (unstable) spatially homogeneous steady states. In both cases, $q_{al}$ is the bifurcation parameter. At $Q^* \simeq 1.9$, four fixed points $u_1^*, u_2^*, u_4^*$, and $u_5^*$ are created through a saddle-node bifurcation. As we approach $Q^{**} \simeq 3.4$ from below, two of the steady states $u_2^*$ and $u_4^*$ disappear through a subcritical pitchfork bifurcation. A Hopf bifurcation has occurred at $q_h \simeq 2.2$ for the steady states $u_1^*$ and $u_5^*$ when the velocity is constant. In the case of density dependent velocity, the steady states $u_1^*$ and $u_5^*$ undergo Hopf bifurcations at $q_{h_1} \simeq 2$. The values of the fixed parameters in case (a) correspond to Figure 3.4(a-c), and in case (b) correspond to Figure 3.5.
Chapter 4

Parameter Space Visualization: 
Stability Region

The constant and density dependent velocity models given in (2.1) and (2.19) include fourteen parameters. We are interested in determining the effects of attraction and repulsion on the stability of the spatially homogeneous steady states. For this purpose, we use a visualization tool for multidimensional data called Paraglide. In Section 4.1, we give an overview of this tool. In Section 4.2, we demonstrate the stability regions for the steady states \((u_1^*, u_5^*)\) and \((u_3^*, u_3^*)\) by only varying attraction and repulsion parameters. In Section 4.3, we present some applications of Paraglide.

4.1 Paraglide tool

Paraglide is a visualization tool for multidimensional data introduced by Bergner et al. [2] in 2011. Here, we briefly describe the steps involved in running projects in Paraglide.

In the first step, Paraglide connects to our simulation code in MATLAB. This connection can be established when the user fills out the form shown in Figure 4.1. In this Figure, Set_par_both written in front of the run command, is a MATLAB program that determines the stability of the steady state in both constant and density dependent velocity models. Paraglide performs heavy or offline computations in the run command. The pick parameters command involve the parameters that vary with their initial values. The show command can be used for real-time simulations. The save and load commands allow
one to store computed results in mat files.

It is also possible to extract dependent variables. For instance, \texttt{stab} and \texttt{stab\_nc} are the two dependent variables presented in Figure 4.1. These two variables display the stability type for the steady state of the constant and density dependent velocity models. Paraglide obtains the values of these dependent variables from defined parameters \texttt{stab\_type}, and \texttt{stab\_type\_nc} in the \texttt{Set\_par\_both} function.

The second step is to choose a range for parameters that vary and determine the number of sample points and the desired sampling method. Paraglide allows for the linear, random or lattice methods. Lattice sampling is used for all the results in this chapter.

Each sample point generated by Paraglide corresponds to a specific set of parameters. All these sample points appear in the scatter plot matrix (SPloM) as shown in Figure 4.2. The SPloM consists of several scatter plots. Each scatter plot investigates the relation between two or at most three parameters. The third parameter is visualized via color mapping. The SPloM is a compact way of displaying the relation between more than three parameters. Figure 4.2 illustrates ten scatter plots in the SPloM. Each row and each column of the SPloM corresponds to a particular parameter.

Note that the parameters that show up in the SPloM are the ones we chose in the \texttt{pick parameters} command. As shown in Figure 4.2, each small circle in every scatter plot represents a sample point that includes a specific set of parameters. For each sample point, Paraglide connects to the MATLAB code, runs the simulation and shows the results for that particular set of parameters. For example, in Figure 4.2, Paraglide reports the spatial pattern, the bifurcation, and the stability type for the highlighted sample.
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Figure 4.2: Overview of a project in Paraglide. The SPloM shows all the possible combinations of \( q_a \), \( q_r \), \( q_{al} \), \( \lambda_1 \), and \( \lambda_2 \). In this project, we only vary \( q_a \) and \( q_r \). The magnitudes of attraction and repulsion are chosen between 0 and 4. Small circles shown in the scatter plots represent the sample points. Each sample point contains a full set of parameters. By clicking on each circle, the bifurcation and the spatial pattern corresponding to that specific sample point appear. The shown figures are related to the highlighted sample point where \( q_a = 1.1771 \), and \( q_r = 3.3982 \). The values of the fixed parameters for all the sample points are: \( \lambda_1 = 1.33 \), \( \lambda_2 = 6 \), and \( q_{al} = 2.5 \).

Using Paraglide has some advantages. For instance, one can observe all the sample points simultaneously and get an overview of the data sets. This helps analyze the data more efficiently. Furthermore, this tool allows the user to perform a denser exploration of the parameter space. Coarse to dense refinement of the domain of interest is one of the features in Paraglide. We have used this feature to obtain the stability region for the steady state \( u^*_1 \) in Section 4.2.

4.2 Comparison: stability region

In this section, we obtain the stability regions for the spatially homogeneous steady state \( u^*_3 \) for the constant and density dependent velocities. We also consider the stability region
for the steady state $u_1^*$ when the velocity is constant. The following parameters are fixed: $\lambda_1 = 1.33$, $\lambda_2 = 6$, $s_a = 1$, $s_r = 0.25$, and $s_{al} = 0.5$.

### 4.2.1 Steady State $u_3^*(A/2, A/2)$

Figure 4.1 presents the stability regions for the steady state $u_3^* = (A/2, A/2)$ for the constant and density dependent velocity cases. These figures are obtained through running Paraglide. The number of samples is 1051 and the values of attraction and repulsion are chosen to be in the intervals $0 \leq q_a \leq 4.5$, and $0 \leq q_r \leq 8$. The magnitude of alignment is $q_{al} = 0.5$.

The filled (unfilled) circles correspond to the specific values of attraction and repulsion at which the steady state is stable (unstable). The steady state $u_3^*$ is unstable for about 63% of the total samples in the constant velocity case. This percentage increases to about 71% when the velocity is not constant. Therefore, the instability region increases in the case of density dependent velocity.

![Stability region for the steady state $u_3^*$](image)

Table 4.1: Stability region for the steady state $u_3^*$. The right figure corresponds to the density dependent speed, while in the left one, the velocity is constant. Filled (unfilled) circles show that the steady state $u_3^*$ is stable (unstable) for the corresponding values of $q_a$ and $q_r$. The horizontal axis denotes the magnitude of repulsion, $q_r$, and the vertical axis denotes the magnitude of attraction, $q_a$.

By fixing $q_r$ and increasing $q_a$, $u_3^*$ becomes unstable. Large enough values of $q_a$ in the presence of small repulsion force the steady state to become unstable. In this case, the social interactions are large enough to produce the group formation. On the other hand, large enough $q_r$ and small values of $q_a$ make the steady state stable. Actually, large repulsion forces cause the individuals to repel each other strongly and settle in a state of rest, that
is, a stable state. The steady state $u_3^*$ is stable for small values of $q_a$ and $q_r$. In this case, social interactions are not strong enough to make the individuals aggregate.

As we see from both figures, increasing $q_a$ increases the possibility of the instability. As $q_a$ gets large, a greater value for $q_r$ is needed to neutralize the instability effect of $q_a$. By comparing the two figures, we realize that a larger value of $q_r$ is needed to make the steady state stable in the nonconstant velocity case. For instance, for a fixed value of $q_a \simeq 1$, the steady state $u_3^*$ is stable where $q_r \simeq 0.3$ in the constant velocity case compared to $q_r \simeq 1.6$ for the nonconstant speed.

It should be noted that the steady state $u_3^*$ does not exist biologically for large values of $q_r$. The existence of this steady state is an artifact of the numerical simulation. For the infinite domain, the individuals spread when repulsion is the dominant parameter, while in the case of a finite domain with periodic boundaries, the individuals cannot leave the domain.

It is worth mentioning that the steady state $u_3^*$ undergoes a bifurcation as the magnitude of attraction, or repulsion varies across the boundaries in Table 4.1. In particular, the imaginary part of the growth rate $\sigma_c$ stays zero while the steady state, passes through the bifurcation. This type of bifurcations is illustrated in Figures 3.3 and 3.4 in the previous chapter.

### 4.2.2 Steady State $u_1^* = (u_1^*, u_5^*)$

In this section, we are interested in running a project with Paraglide to obtain the stability region for the steady state $u_1^*$ in the case of constant velocity. First, we start a project with a small number of samples. Figure 4.3 illustrates the scatter plot matrix for our project. The filled (unfilled) circles show that the steady state is stable (unstable) for the corresponding values of attraction and repulsion parameters.

This coarse sampling gives us a rough idea of the stability region. We need to generate more samples to have a better understanding of the stability region. As shown in Figure 4.3, reasonable ranges for attraction and repulsion for producing more samples are: $0 \leq q_a \leq 6.5$, and $0 \leq q_r \leq 12$. We generated 400 samples in our domains of interest. The stability regions for the refined and coarse sampling in the case of constant velocity are given in Table 4.2.

Further investigations of the stability regions show that the imaginary part of the growth rate $\sigma_c$, stays nonzero when the real part of $\sigma_c$ passes through zero. We conclude that the steady state $u_1^*$ undergoes a Hopf bifurcation as it crosses the boundaries of the stability
Figure 4.3: Scatter plot matrix for the steady state $u_1^*$ in the case of constant velocity. Filled (unfilled) circles show that the steady state $u_1^*$ is stable(unstable) for the corresponding values of $q_a$ and $q_r$.

4.3 Other applications of Paraglide

In this section, we present interesting directions that can be taken using Paraglide for a deeper understanding the cohesive behavior of individuals.

Mapping the parameter space could be very interesting. We initiated a project in Paraglide to map the various spatial patterns observed in [12] into the parameter plane. The axes in the parameter plane represent the biased and random turning rates, when the magnitude of repulsion is greater than the magnitude of attraction. A similar work has been performed by Couzin et al. [6]. They constructed a Lagrangian model in three spatial dimensions and considered all three social interactions: attraction, repulsion, and alignment. Their model reveals the following collective behaviors: swarm, torus, dynamic parallel group, and highly parallel group. They mapped these patterns into a plane in which the axes represent the widths of the attraction and repulsion zones. They performed this mapping based on the values of the defined polarization variable.

Vabø et al. [40] developed a two dimensional Lagrangian model to describe the schooling behavior of fish. Their model exhibits the antipredatory behaviors such as splitting and
Table 4.2: Stability region for the steady state $u^*_1$. The left figure corresponds to the coarse sampling, while the right one shows the refined one. Filled (unfilled) circles show that the steady state $u^*_1$ is stable (unstable) for the corresponding values of $q_a$ and $q_r$. The horizontal axis denotes the magnitude of repulsion, $q_r$, and the vertical axis denotes the magnitude of attraction, $q_a$.

merging. The model introduced by Eftimie et al. in [10] is the first Eulerian model that supports the succession of activities observed in nature. In particular, the authors consider the following loop of activities: forage, rest, travel. To demonstrate this succession of activities, they started with a perturbation of the steady state $u^*_1$, and simulated the model for a specific set of parameters. Then, they changed the parameter values and defined the initial condition to be the density obtained in the previous step in order to capture new behaviors. Paraglide can be beneficial for further investigation of this aspect.
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Conclusion

In this thesis, we obtained the spatially homogeneous steady states for a nonlocal hyperbolic system that models biological aggregations. We considered two separate cases. In the first, the velocity is assumed to be constant. In the second, the individuals are allowed to speed up or slow down as a result of interactions with neighbors.

We derived the dispersion relation for both constant and density dependent speeds and confirmed the analytical results using the numerical simulation of the model. Furthermore, we investigated various types of bifurcations that can occur for the spatially homogeneous steady states. In particular, we concluded that these steady states can undergo subcritical pitchfork, Hopf, and saddle-node bifurcations.

We also illustrated the stability region for two of the spatially homogeneous steady states for specific sets of parameters. The instability region tends to increase in the case of density dependent speed. Paraglide, a visualization tool for multidimensional data, is used to produce these stability regions. In the following, we present some potentially interesting extensions of this work.

- In Chapter 4, we investigated the effects of attraction and repulsion on the stability region for two of the spatially homogeneous steady states. It would be interesting to analyze the impact of the other parameters such as the magnitude of alignment and the turning rate on the stability region. Obtaining and comparing the stability region for the other steady states can be an interesting path to take as well.

- Further investigation can be performed on the stability regions illustrated in Chapter 4. For instance, identifying and classifying the spatial patterns for the unstable region of the steady state can be an intriguing topic. Both constant and density dependent speeds can be considered.

- We detected the occurrence of a Hopf bifurcation for one of the spatially homogeneous steady states in Chapters 3 and 4. Determining the type of Hopf bifurcation: supercritical, subcritical, or degenerate, can be the next step.

- The extension of this model to two spatial dimensions has been accomplished very recently [11]. Obtaining steady states, performing stability analysis, and studying bifurcations in the two dimensional model could be very worthwhile.
Bibliography


