A DIXMIER-MOEGLIN EQUIVALENCE FOR SKEW LAURENT POLYNOMIAL RINGS

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Abstract

The work of Dixmier in 1977 and Moeglin in 1980 show us that for a prime ideal P in the universal enveloping algebra of a complex finite-dimensional Lie algebra the properties of being primitive, rational and locally closed in the Zariski topology are all equivalent. This equivalence is referred to as the Dixmier-Moeglin equivalence. In this thesis we will study skew Laurent polynomial rings of the form $\mathbb{C}[x_1, \ldots, x_d][z, z^{-1}; \sigma]$ where σ is a $\mathbb{C}$-algebra automorphism of $\mathbb{C}[x_1, \ldots, x_d]$. In the case that σ restricts to a linear automorphism of the vector space $\mathbb{C} + \mathbb{C}x_1 + \cdots + \mathbb{C}x_d$, we show that the Dixmier-Moeglin equivalence holds for the prime ideal (0).

To my former roommate but perennial friend Jennifer.

Valiantly I await your riposte.

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Chapter 1

Introduction

The motivation for this thesis has its origins in the study of algebraic geometry. Let k be an algebraically closed field, let $X \subseteq \mathbb{A}^n$ be an affine algebraic variety, let

$$J = \{f \in k[x_1, \ldots, x_n] \text{ such that } f \text{ vanishes on } X\}$$

and let $\mathcal{O}(X) = k[x_1, \ldots, x_n]/J$ be the coordinate ring of X. Then there is a bijective correspondence between the points of X and the maximal ideals in $\mathcal{O}(X)$ and a bijective correspondence between the irreducible subvarieties of X and the prime ideals of $\mathcal{O}(X)$. Thus we obtain information about X from the maximal and prime ideals of the coordinate ring. This correspondence can be extended to some classes of noncommutative algebras, where the role of affine varieties is played by some category and one obtains information by studying the set of all prime ideals of the algebra A, which we denote by Spec(A).

The correct noncommutative analogue of the set of maximal ideals is the set of (left) *primitive* ideals. An ideal P of a ring R is said to be primitive if it is the annihilator of a simple (left) R-module M. In Proposition 2.1.6 we will show that in a ring R every primitive ideal is prime. Then in Remark 2.1.9 we will show that in a commutative ring an ideal is primitive if and only if it is maximal. However, the original motivation for the study of primitive ideals comes from representation theory.

Let G be a real *Lie group* and let $\mathcal{L}$ be the complex *Lie algebra* of G. We will give a precise definition of a Lie algebra in Section 2.4, but for now we can think of a complex Lie algebra as a complex vector space $\mathcal{L}$ with a binary operation $[\cdot,\cdot] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ called a *Lie bracket* that is not necessarily associative. Finding the finite-dimensional representations of G is almost entirely equivalent to finding the finite-dimensional representations of $\mathcal{L}$. The problem of finding the representations of $\mathcal{L}$ can be transformed into a problem of associative algebra by passage to the *universal enveloping algebra* $\mathcal{U}(\mathcal{L})$ of $\mathcal{L}$ (cf. Dixmier [7]). Since it is often difficult to find all irreducible representations of an algebra, as an intermediate step, Dixmier proposed that one should determine the primitive ideals and then for each primitive ideal P, try to find the irreducible modules whose annihilator is P.

We will see that the definition of a primitive ideal can be cumbersome and it is often difficult to determine the *primitive spectrum*, the set of all primitive ideals, from the definition alone. Dixmier proposed two different approaches, one algebraic and one topological, to characterize the primitive ideals in universal enveloping algebras. Let k be a field and let A be a k-algebra. The algebraic approach involves determining which prime ideals of A are *rational*. Again, we will provide a more precise definition in Section 2.4, but for now we can interpret the definition of a rational prime as a prime ideal $P \in A$ such that the center of A/P is algebraic over the base field k. The topological approach involves determining which prime ideals are *locally closed* in Spec(A), that is, where Spec(A) is equipped with the *Zariski topology* and a prime P is said to be locally closed if it is the intersection of an open and closed set in Spec(A).

Dixmier [8] and Moeglin [26], in 1977 and 1980 respectively, used the notions of rational and locally closed to prove that in the universal enveloping algebra of a complex finite-dimensional Lie algebra the conditions for which P is primitive, rational and locally closed are equivalent. This equivalence is referred to as the *Dixmier-Moeglin equivalence*. In 1980, Irving and Small [17] extended this result to finite-dimensional Lie algebras over arbitrary fields of characteristic zero. These results have motivated others to try to find a Dixmier-Moeglin equivalence for other rings. In 2000, Goodearl and Letzter [12] showed that the Dixmier-Moeglin equivalence is satisfied in certain *quantized coordinate rings*. We will give an example of a quantized coordinate ring in Example 2.2.3. In 2006, Goodearl [11] showed that certain *Poisson algebras* satisfy the Dixmier-Moeglin equivalence as well.

Our work lies in the study of *skew Laurent polynomial rings*. Let A be a k-algebra and let σ be a k-algebra automorphism of A. Then we can form the skew Laurent polynomial ring $A[z, z^{-1}; \sigma]$ by defining addition in the usual way and defining multiplication by $za = \sigma(a)z$ for $a \in A$. We will study skew Laurent polynomial rings in greater detail in Section 2.2. The following result of Bell, Rogalski and Sierra [3] pertaining to skew Laurent polynomial rings and the Dixmier-Moeglin equivalence is stated in Theorem 2.4.26 as follows. Let k be an uncountable algebraically closed field of characteristic zero, let A be a finitely generated commutative k-algebra and let σ be an automorphism of A. If $\dim(A) \leq 2$ and GKdim$(A[z, z^{-1}; \sigma]) < \infty$ then $A[z, z^{-1}; \sigma]$ satisfies the Dixmier-Moeglin equivalence. We will study GK dimension, short for *Gelfand-Kirillov dimension* and denoted GKdim(A) for a ring A, in greater detail in Section 2.3, but for now we can interpret GK dimension as the noncommutative analogue of *Krull dimension*. We will see in Theorem 2.3.10 that if k is a field and A is a finitely generated commutative k-algebra then in fact, the Krull dimension and GK dimension of A are the same.

The following motivating result of Zhang relating GK dimension and skew Laurent polynomial rings is stated in Theorem 2.3.15 as follows. Let A be a commutative k-algebra such that the field of fractions of A is a finitely generated field extension of k, and let σ be a k-algebra automorphism of A with skew Laurent polynomial ring $A[z, z^{-1}; \sigma]$. Then GKdim$(A[z, z^{-1}; \sigma]) =$ GKdim$(A) + 1$ if and only if there is a finite-dimensional subspace W of A such that $\sigma(W) = W$ and W generates A as a k-algebra. We say $A[z, z^{-1}; \sigma]$ has *low growth* if GKdim$(A[z, z^{-1}; \sigma]) =$ GKdim$(A) + 1$.

Let $A = \mathbb{C}[x_1, \ldots, x_d]$ be the polynomial ring over $\mathbb{C}$ and let σ be a $\mathbb{C}$-algebra automorphism of A. We will give a structure theorem for all low growth skew Laurent polynomial rings according to the Jordan form of the matrix representation of σ in Theorem 3.1.8. We will use Theorem 3.1.8 to obtain a Dixmier-Moeglin equivalence for the prime ideal (0) for low growth skew Laurent polynomial rings in Chapter 4.

In Section 2.5 we show that the skew Laurent polynomial ring $\mathbb{C}[x, y][z, z^{-1}; \sigma]$ where σ is the *Hénon map* given by

$$\sigma(x) = y + 1 - ax^2 \quad \text{and} \quad \sigma(y) = bx \quad \text{for } a, b \in \mathbb{C}$$

is an example of a skew Laurent polynomial ring which does not satisfy the Dixmier-Moeglin equivalence if certain conditions are placed upon a and b.

In Chapter 5 we use Theorem 3.1.8 to determine the *transcendence degree* of the center of the *quotient division ring* of $A[z, z^{-1}; \sigma]$ over $\mathbb{C}$ where the quotient division ring is the noncommutative analogue of the field of fractions. Finally, we give a short list of conjectures for future work.

Chapter 2

Preliminaries

2.1 Ring theory

In this section we will cover the necessary ring theoretic background material which we will use for the remainder of the paper. Most of these definitions and results are standard and are included for completeness.

Definition 2.1.1. A ring R is called *simple* if the only two sided ideals are (0) and R.

Example 2.1.2. The ring $M_n(\mathbb{C})$ of $n \times n$ matrices with complex entries is simple. To show this we will follow the proof in Lam [21, Example 3.1]. First let R be a ring and let I be a two-sided ideal of $M_n(R)$. Then $I = M_n(J)$ with J a two-sided ideal of R. If J is an ideal of R then $M_n(J)$ is an ideal of $M_n(R)$. Now suppose I is any ideal in $M_n(R)$, and let S be the set of all the $(1,1)$-entries of matrices in I. It is easy to verify that S is an ideal in R, and we are done if we can show that $I = M_n(S)$. For any matrix $M = (m_{ij})$ we have the identity

$$E_{ij} M E_{kl} = m_{jk} E_{il}, \tag{2.1}$$

where the set $\{E_{ij}\}$ denotes the matrix units. Assume $M \in I$. If we let $i = l = 1$, (2.1) shows that $m_{jk} E_{11} \in I$, and so $m_{jk} \in S$ for all j, k. Hence $I \subseteq M_n(S)$. Conversely, take any $(a_{ij}) \in M_n(S)$. If we show that $a_{il} E_{il} \in I$ for all i, l, then $(a_{ij}) \in I$ from (2.1). Let $M = (m_{ij}) \in I$ such that $a_{il} = m_{11}$. Then, for $j = k = 1$, (2.1) gives

$$a_{il} E_{il} = m_{11} E_{il} = E_{i1} M E_{1l} \in I.$$

Thus $I = M_n(S)$ is a two-sided ideal of $M_n(R)$. Since $\mathbb{C}$ is a field, the only (two-sided) ideals of $\mathbb{C}$ are (0) and $\mathbb{C}$ and therefore the only two-sided ideals of $M_n(\mathbb{C})$ are (0) and $M_n(\mathbb{C})$. Hence $M_n(\mathbb{C})$ is simple.

Definition 2.1.3. A ring R is said to be a *left primitive* ring (or for short a *primitive* ring), if there exists a left maximal ideal $\mathcal{M}$ such that if $xR \subseteq \mathcal{M}$ then $x = 0$. Alternatively, we define a primitive ring to be a ring for which there exists a faithful simple left R-module. That is, there exists a left R-module M such that $RM \neq (0)$, the only submodules of M are (0) and M and for $r \in R$ if $rM = (0)$ then $r = 0$. An ideal I of R is a *left primitive* ideal (shortly, *primitive* ideal) of R if R/I is a primitive ring.

Definition 2.1.4. An ideal P in a ring R is said to be a *prime ideal* if $P \neq R$ and for ideals $I, J \subseteq R$, if

$$IJ \subseteq P \text{ then } I \subseteq P \text{ or } J \subseteq P.$$

A ring R is said to be a *prime ring* if (0) is a prime ideal.

Example 2.1.5. Let D be a division ring. Suppose D has a nonzero left ideal, $I \subseteq D$. Since $I \neq (0)$, there exists a nonzero element $a \in I$. Let 1 be the identity in D. Every nonzero element in D is left-invertible, so there exists $a^{-1} \in D$ such that $a^{-1} \cdot a = 1$. Thus D has only two ideals, D and (0), so D is simple.

Let $M = D$ be a left D-module. Since every nonzero element of D has a left inverse, the only submodules of M are (0) and M. For all $r \in D$ if $rM = (0)$ then $r = 0$ since D has no zero divisors. Thus D has a faithful simple module and is therefore a primitive ring.

Suppose that I, J are two ideals of D and that $IJ \subseteq (0)$. Since D has no zero divisors this implies $I = (0)$ or $J = (0)$ and hence $I \subseteq (0)$ or $J \subseteq (0)$. Thus (0) is a prime ideal and D is a prime ring. We generalize this example in the following proposition.

Proposition 2.1.6. *If R is a simple ring then it is primitive. If R is a primitive ring then it is prime.*

Proof. Suppose $R \neq (0)$ is a simple ring and R is not primitive. Let $\mathcal{M}$ be a maximal left ideal and let $M = R/\mathcal{M}$ be a left R-module. Since $\mathcal{M}$ is maximal, M is simple. Suppose $rM = (0)$ and $r \neq 0$. Then $rR \subseteq \mathcal{M}$ and hence $RrR \subseteq R\mathcal{M} = \mathcal{M}$. Thus RrR is a proper two-sided ideal and so $r = 0$, a contradiction.

Now suppose R is primitive but not prime. Let $I, J \neq (0)$ be two-sided ideals of R such that $IJ = (0)$. Since R is primitive it has a faithful simple R-module M. Thus $IM, JM \neq (0)$, so $IM = JM = M$. Hence $M = I(JM) = (0)M = (0)$, a contradiction. $\square$

Example 2.1.7. The ring $T = \text{End } V$ of all linear transformations of a countably infinite-dimensional vector space V over a division ring D is an example of a ring which is primitive and prime but not simple. Let I be the set of linear transformations $T_i \in T$ such that $\text{im}(T_i)$ is a finite-dimensional subspace of V. We have that $I \neq (0)$ and I is proper since V is infinite-dimensional. Let $T_1, T_2 \in I$ then $\text{im}(T_1 + T_2) \subseteq \text{im}(T_1) + \text{im}(T_2)$ which is finite-dimensional. Let $S \in T$ then $\text{im}(T_1 S) \subseteq \text{im}(T_1)$ and $\text{im}(ST_1) \subseteq S\text{im}(T_1)$, both of which are finite-dimensional. Hence I is a proper nonzero ideal of T.

T is primitive since for any nonzero vector $v \in V, Tv = V$ and thus V is a simple T-module. V is faithful because $0 \in T$ is the only linear transformation which sends all vectors of V to $0 \in V$.

Example 2.1.8. A commutative domain D which is not a field is an example of a ring which is prime but not primitive. Since D is a domain there are no zero divisors and (0) is a prime ideal of D. Hence D is a prime ring. Let a be a nonzero element of D such that a is not a unit. Then aD is a principal, nontrivial ideal of D, so D is not simple. Now suppose that D is primitive. Then D has a faithful simple module M. Thus $M = D/\mathcal{M}$ for some maximal two-sided ideal of D. If $\mathcal{M} \neq (0)$ then there exists a nonzero $x \in \mathcal{M}$. Then $xD \subseteq \mathcal{M}$ so $D/\mathcal{M}$ is not faithful, a contradiction.

Remark 2.1.9. Let R be a commutative ring, let I be an ideal of R and let $S = R/I$ be the quotient ring of I in R. Then I is a primitive ideal if and only if it is maximal. Suppose I is maximal then if $xS \subseteq I$ then $x = 0$ and thus I is primitive. Suppose I is a primitive ideal then S has a faithful simple module M. Thus $M = S/\mathcal{M}$ for some maximal ideal $\mathcal{M}$ of S. Since $\mathcal{M}M = 0$ it follows that $\mathcal{M} = 0$ and if (0) is a maximal ideal of S then S is a field and thus I is a maximal ideal of R.

Definition 2.1.10. A prime ring R is *special* if there exists a non-nilpotent element $r \in R$ such that for every nonzero ideal I of R, $r^n \in I$ for some $n \in \mathbb{N}$.

Example 2.1.11. We will show that $\mathbb{C}[[x]]$ is a special ring. If I is a nonzero proper ideal then we can pick a nonzero $a \in I$. Then $a = \sum_{i=m}^{\infty} a_i x^i$ with $a_m \neq 0$ for some $m > 0$. Let

$$b = \sum_{i=1}^{\infty} \frac{a_{m+i}}{a_m} x^{i-1}.$$

Then $a = a_m x^m (1 + bx)$. Hence

$$x^m = a a_m^{-1} \sum_{i=0}^{\infty} (-1)^i (bx)^i \in I.$$

So every nonzero ideal contains a power of x. Thus $\mathbb{C}[[x]]$ is a special ring.

Definition 2.1.12. Let k be a field and let A be a k-algebra. Let f be a noncommutative nonzero polynomial in $k\{x_1, x_2, \ldots, x_d\}$, the free algebra over k in the noncommutative variables $x_1, \ldots, x_d$ for some d. If $f(a_1, \ldots, a_d) = 0$ for all $a_1, \ldots, a_d \in A$ then A is said to be a *polynomial identity algebra* or *P.I. algebra*.

Example 2.1.13. The ring of 2×2 matrices $M_2(R)$ over the commutative ring R is a P.I. ring. Let $A, B \in M_2(R)$, then the trace of $AB - BA = 0$. By the Cayley-Hamilton Theorem every matrix satisfies its own characteristic polynomial. The characteristic polynomial of a 2×2 matrix A is $\lambda^2 - \operatorname{tr}(A)\lambda + \det(A)$. Hence $(AB - BA)^2 = -\det(AB - BA)I_2$ and thus $(AB - BA)^2$ is a scalar matrix and is therefore in the center of $M_2(R)$. Hence $(AB - BA)^2$ commutes with every matrix $C \in R$. Thus the relation

$$(AB - BA)^2 C - C(AB - BA)^2 = 0$$

holds for all $A, B, C \in M_2(R)$. Hence $M_2(R)$ is a P.I. ring.

Definition 2.1.14. Let R be a ring. An element $a \in R$ is said to be *normal* if $aR = Ra$.

Remark 2.1.15. In any ring R we have that 0 and any unit are normal elements as well as any central elements. We will give an example of a nonzero normal element which is not a unit or central in Chapter 3.

2.2 Skew polynomial rings

This section concerns skew polynomials over a commutative ring A, in an indeterminate z. If we let σ be an automorphism of A then we can construct the following ring $A[z;\sigma]$. An element in $A[z;\sigma]$ can be written as $\sum_i a_i z^i$ with $a_i \in A$ and it is understood that the summation runs over a finite sequence of nonnegative integers i. We define addition in the usual way

$$\left(\sum_i a_i z^i\right) + \left(\sum_i b_i z^i\right) = \sum_i (a_i + b_i) z^i,$$

and we define multiplication by

$$za = \sigma(a)z.$$

We summarize this in the following definition.

Definition 2.2.1. Let A be a ring and let σ be an automorphism of A. Suppose that
(a) R is a ring containing A as a subring.
(b) z is an element of R.
(c) R is a free left A-module with basis $\{1, z, z^2, \ldots\}$.
(d) $za = \sigma(a)z$ for all $a \in A$.
If $R = A[z;\sigma]$ satisfies (a)-(d) then R is said to be a *skew polynomial ring* over A.

If we invert z then we obtain the following ring which will be our main interest for the remainder of this paper.

Definition 2.2.2. Let A be a ring and let σ be an automorphism of A. Then we can form the *skew (twisted) Laurent polynomial ring* $A[z, z^{-1};\sigma]$. This is the ring of Laurent polynomials

$$\sum_{n=-m_1}^{m_2} a_n z^n \qquad a_n \in A, \quad m_1, m_2 \in \mathbb{N}$$

with multiplication given by $za = \sigma(a)z$ and $z^{-1}a = \sigma^{-1}(a)z^{-1}$ for all $a \in A$.

The remainder of this section will be devoted solely to the study of skew Laurent polynomial rings. Most of the definitions and results we present are standard, whenever this is not the case references are given.

Example 2.2.3. Consider the following example from Goodearl and Warfield [13, Example 1O]. Let k be a field and let k^* be the multiplicative group of nonzero elements in k. Let $q \in k^*$. Then the k-algebra $\mathcal{O}_q((k^*)^2)$ is the quantized coordinate ring of $(k^*)^2$ with generators defined to be x, x^{-1}, y, y^{-1} such that

$$xx^{-1} = x^{-1}x = yy^{-1} = y^{-1}y = 1, \quad xy = qyx.$$

We will show that $\mathcal{O}_q((k^*)^2) = k[y, y^{-1}][x, x^{-1}; \sigma]$ where our commutative ring $A = k[y, y^{-1}]$ is an ordinary Laurent polynomial ring and σ is an automorphism of A given by $\sigma(y) = qy$.

Since $\mathcal{O}_q((k^*)^2)$ is a k-algebra it is closed under addition and multiplication so every element in $\mathcal{O}_q((k^*)^2)$ is of the form $\sum_{i,j\in\mathbb{Z}} c_{ij}y^i x^j$ where $c_{ij} \in k$ and the powers of y appear before the powers of x by applying the identity $xy = qyx$. Thus elements of this form can also be written as $\sum_{i\in\mathbb{Z}} c_i x^i$ where c_i is an element in the Laurent polynomial ring $k[y, y^{-1}]$ and σ is the automorphism of A given by $\sigma(y) = qy$. Thus $\mathcal{O}_q((k^*)^2)$ is equal to the skew Laurent polynomial ring $k[y, y^{-1}][x, x^{-1}; \sigma]$.

Example 2.2.4. Let $A = \mathbb{C}[x]$ and let σ be the automorphism of A given by $\sigma(x) = x + 1$. Then the ring $A[z, z^{-1}; \sigma]$ is a skew Laurent polynomial ring. We will return to this example later in this section.

Example 2.2.5. Let $A = \mathbb{C}[x, y]$ and let σ be the automorphism of A given by $\sigma(x) = y + 1 - ax^2$ and $\sigma(y) = bx$ for $a, b \in \mathbb{C}^*$. Then the ring $A[z, z^{-1}; \sigma]$ is a skew Laurent polynomial ring. The automorphism σ is called the Hénon map which we will study in greater detail in Section 2.5.

Definition 2.2.6. Let A be a ring and let σ be a ring automorphism of A. An ideal I of A is said to be σ-*stable* if $I = \sigma(I)$.

Proposition 2.2.7. *Let A be a commutative ring and let σ be a ring automorphism of A with infinite order. Then there is a bijection between the two-sided ideals of $A[z, z^{-1}; \sigma]$ and the σ-stable ideals of A given by*

$$I \subseteq A[z, z^{-1}; \sigma] \to I \cap A$$

and

$$I \subseteq A \to IA[z, z^{-1}; \sigma].$$

Proof. We have that the two-sided ideals A and (0) of $A[z, z^{-1}; \sigma]$ are σ-stable ideals of A. So, let I be a proper nonzero σ-stable ideal of A and let

$$J = \sum_{k=-\infty}^{\infty} I z^k.$$

Since $A[z, z^{-1}; \sigma]$ is closed under addition we need only consider an element $a \in A[z, z^{-1}; \sigma]$ of the form $a = a_i z^i$ for some $i \in \mathbb{Z}$. Then

$$aJ = a_i z^i \cdot \sum_{k=-\infty}^{\infty} I z^k = \sum_{k=-\infty}^{\infty} a_i z^i I z^k = \sum_{k=-\infty}^{\infty} a_i \sigma^i(I) z^{i+k} \subseteq \sum_{k=-\infty}^{\infty} I z^{i+k} = J.$$

$$Ja = \sum_{k=-\infty}^{\infty} I z^k \cdot a_i z^i = \sum_{k=-\infty}^{\infty} I \sigma^k(a_i) z^{k+i} \subseteq \sum_{k=-\infty}^{\infty} I z^{i+k} = J.$$

Thus J is a proper nonzero two-sided ideal of $A[z, z^{-1}; \sigma]$.

Conversely, suppose J is a proper nonzero two-sided ideal of $A[z, z^{-1}; \sigma]$. Pick $\sum_{i=0}^{d} a_i z^i \in J$ with $a_d \neq 0$ and d minimal. If $d = 0$ then we have $a_0 \neq 0 \in J$. Thus $J \cap A$ is a proper nonzero ideal of A. Since J is a two-sided ideal,

$$\sigma(J \cap A) = z^{-1}(J \cap A)z \subseteq J \cap A \quad \text{and} \quad J \cap A = z\sigma(J \cap A)z^{-1} \subseteq \sigma(J \cap A),$$

we have that $\sigma(J \cap A) = J \cap A$. Hence $J \cap A$ is a proper, nonzero σ-stable ideal of A.

Suppose $d > 0$ and that $a_0 + a_1 z + \cdots + a_d z^d \in J$. Then for all $x \in A$,

$$\begin{aligned} &x(a_0 + a_1 z + \cdots + a_d z^d) - (a_0 + a_1 z + \cdots + a_d z^d)\sigma^d(x) \\ &(xa_0 - a_0\sigma^d(x)) + (xa_1 - a_1\sigma^{d-1}(x))z + \cdots + (xa_d - a_d\sigma^0(x))z^d \in J. \end{aligned} \tag{2.2}$$

Since $\sigma^0(x) = x$, this has degree $d-1$. By the minimality of d, (2.2) must be equal to zero. Hence $xa_0 - a_0\sigma^d(x) = 0$ for all $x \in A$. We may assume that $a_0 \neq 0$, otherwise

$$(a_0 + a_1 z^1 + \cdots + a_d z^d)z^{-1} = a_1 + \cdots + a_{d-1}z^{d-1} \in J.$$

This contradicts the minimality of d. Thus $xa_0 = a_0\sigma^d(x)$ for all $x \in A$. Since A is a commutative domain and $a_0 \neq 0, x = \sigma^d(x)$ for all $x \in A$. This means that σ^d is the identity, a contradiction. $\square$

Proposition 2.2.8. *Let A be a commutative ring and let σ be a ring automorphism of A. Then the skew Laurent polynomial ring $A[z, z^{-1}; \sigma]$ is a Polynomial Identity ring if and only if σ has finite order.*

Proof. We refer the reader to Brown and Goodearl [5, Theorem I.14.1]. $\square$

Proposition 2.2.9. *If A is a commutative Noetherian ring and σ is a ring automorphism of A then an ideal I of A is σ-stable if and only if $\sigma(I) \subseteq I$.*

Proof. If I is σ-stable then $\sigma(I) = I$ implies $\sigma(I) \subseteq I$. If $\sigma(I) \subseteq I$ then we have that $I \subseteq \sigma^{-1}(I) \subseteq \sigma^{-2}(I) \subseteq \cdots$. Since A is a commutative Noetherian ring there exists a $m \in \mathbb{Z}$ such that $\sigma^{-m}(I) = \sigma^{-(m+1)}(I)$. This implies that $\sigma^{m+1}(\sigma^{-m}(I)) = \sigma^{m+1}(\sigma^{-(m+1)}(I))$. Hence $\sigma(I) = I$ and I is σ-stable. $\square$

While Proposition 2.2.9 might appear obvious, we have that the Noetherian hypothesis is necessary, which is demonstrated in the following example.

Example 2.2.10. Let $A = \mathbb{C}[x_n : n \in \mathbb{Z}]$ and let σ be an automorphism of A such that $\sigma(x_i) = x_{i+1}$. Then $I = (x_0, x_1, x_2, \ldots)$ has the property that $\sigma(I) \subseteq I$ but is not σ-stable since $x_0 \notin \sigma(I)$.

Definition 2.2.11. Let A be a ring and let σ be a ring automorphism of A. If (0) and A are the only σ-stable ideals then A is said to be a *σ-simple* ring.

Proposition 2.2.12. *Let A be a commutative domain and let σ be a ring automorphism of A. Then the skew Laurent polynomial ring $A[z, z^{-1}; \sigma]$ is simple if and only if A is σ-simple.*

Proof. Suppose $A[z, z^{-1}; \sigma]$ is simple and that A is not σ-simple. Then there is a proper nonzero σ-stable ideal contained in A. From Proposition 2.2.7 we have that this gives us a proper nonzero two-sided ideal of $A[z, z^{-1}; \sigma]$, a contradiction.

Conversely, suppose A is σ-simple and that $A[z, z^{-1}; \sigma]$ is not simple. Then there exists a proper nonzero two-sided ideal $J \subseteq A[z, z^{-1}; \sigma]$. Pick $\sum_{i=0}^{d} a_i z^i \in J$ with $a_d \neq 0$ and d minimal. If $d = 0$ then from Proposition 2.2.7 we obtain the proper nonzero σ-stable ideal $J \cap A \subseteq A$, a contradiction.

If $d > 0$ then from Proposition 2.2.7 we have that σ^d is the identity. Thus let a be a nonzero, non-unit in A and let $u = a \cdot \sigma(a) \cdot \sigma^2(a) \cdots \sigma^{d-1}(a)$. Then $\sigma(u) = u$ and $I = (u)$ is a proper nonzero σ-stable ideal, a contradiction. □

Example 2.2.13. Let $A = \mathbb{C}[x]$ and let σ be the $\mathbb{C}$-algebra automorphism of A given by $\sigma(x) = x + 1$. We will show that $A[z, z^{-1}; \sigma]$ is simple. By Proposition 2.2.12 it is enough to show that A is σ-simple. Assume A is not σ-simple. Then there exists a proper nonzero ideal $I \subsetneq A$. A is a PID so every ideal is of the form $(p(x))$ where $p(x) \in \mathbb{C}[x]$. Let $a(x)$ be a nonzero element in $(p(x)) \neq (0)$ with minimal degree d. If $d = 0$ then $a(x)$ is a unit in A and hence $(p(x))$ is not proper, a contradiction. So let $d > 0$ and let $a(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$. Since I is σ-stable,

$$\begin{aligned}\sigma(a(x)) - (a(x)) &= \big(a_d(x+1)^d + a_{d-1}(x+1)^{d-1} + \cdots + a_0\big) - (a_d x^d + \cdots + a_0) \\ &= a_d(d)x^{d-1} + \cdots + (a_d + a_{d-1} + \cdots + a_1) \in I.\end{aligned}$$

Since $a_d \neq 0$ we have that $\sigma(a(x)) - (a(x)) \neq 0$. Thus we have an element of smaller degree in I, a contradiction. Thus A is σ-simple.

Definition 2.2.14. Let A be a commutative ring and let σ be a ring automorphism of A. A is said to be a *σ-primitive ring* if there exists a maximal ideal of A that contains no nonzero σ-stable ideal.

Definition 2.2.15. Let A be a commutative ring and let σ be a ring automorphism of A. A σ-stable ideal $P \subseteq A$ is said to be *σ-prime* if for all the σ-stable ideals $I, J \in A$, if

$$IJ \subseteq P \text{ then } I \subseteq P \text{ or } J \subseteq P.$$

If (0) is a σ-prime ideal of A then A is a *σ-prime ring*.

Theorem 2.2.16. *Let A be a commutative ring and let σ be a ring automorphism of A. If A is a σ-simple ring then A is σ-primitive. If A is a σ-primitive ring then A is σ-prime.*

Proof. If A is σ-simple then the only σ-stable ideals of A are (0) and A. Hence the only σ-stable ideal contained in any maximal ideal is (0). Thus A is σ-primitive. Now assume A is σ-primitive but not σ-prime. Then there exist σ-stable ideals I and J such that $IJ \subseteq (0)$ but $I \not\subseteq (0)$ and $J \not\subseteq (0)$. Let $\mathcal{M}$ be a maximal ideal in A that contains no nonzero σ-stable ideals. Then we have that $IJ \subseteq (0) \subseteq \mathcal{M}$ with $I \not\subseteq \mathcal{M}$ and $J \not\subseteq \mathcal{M}$ since I and J are σ-stable ideals, but $\mathcal{M}$ is a prime ideal, a contradiction. □

Example 2.2.17. From Example 2.2.13 we know that the ring $A[z, z^{-1}; \sigma]$ with $A = \mathbb{C}[x]$ and $\sigma(x) = x+1$ is simple and that A is σ-simple. Hence A is also σ-primitive and σ-prime by Theorem 2.2.16.

Example 2.2.18. An example of a ring which is σ-prime but not prime is given in McConnell and Robson [25, Example 10.6.5].

Definition 2.2.19. Let A be a commutative ring and let σ be a ring automorphism of A. We say that A is *σ-special* if it is σ-prime and there exists a regular element $a \in A$ such that, for every nonzero σ-prime ideal I of A, there exists an n such that $a\sigma(a)\sigma^2(a)\cdots\sigma^n(a) \in I$.

The following are all results of Jordan [18, 19].

Proposition 2.2.20. *Let A be a commutative Noetherian domain and let σ be an automorphism of A. Then $A[z, z^{-1}; \sigma]$ is special if and only if A is σ-special.*

Proof. We refer the reader to Jordan [19, Lemma 2.7(i)]. □

Theorem 2.2.21. *Let A be a commutative Noetherian ring with a ring automorphism σ of infinite order then $A[z, z^{-1}; \sigma]$ is primitive if and only if A is σ-primitive or σ-special and σ has infinite order.*

Proof. We refer the reader to Jordan [19, Theorem 4.3]. □

Remark 2.2.22. In Chapter 4 and Chapter 5 we will prove the existence of skew Laurent polynomial rings which are primitive but not simple and from Proposition 2.2.12 and Theorem 2.2.21 this gives us the existence of skew Laurent polynomial rings which are σ-primitive but not σ-simple. Since we will be working with skew Laurent polynomial rings without zero divisors the results of Chapter 4 and Theorem 2.2.21 also give us the existence of skew Laurent polynomial rings that are σ-prime but not σ-primitive.

Proposition 2.2.23. *Let A be a commutative Noetherian domain which is affine over an uncountable field k and let σ be a k-automorphism of A. If A is σ-special then A is σ-primitive.*

Proof. We refer the reader to Jordan [19, Proposition 2.9]. □

Example 2.2.24. The ring of formal power series $A = \mathbb{C}[[x]]$ we considered in Example 2.1.11 is an example of a ring which is σ-prime and σ-special but not σ-primitive. We have that x is a non-nilpotent element such that for every nonzero ideal I of A, $x^n \in I$ for some $n \in \mathbb{N}$. Let σ be the $\mathbb{C}$-automorphism of A such that $\sigma(x) = 2x$. We have that A is σ-prime since A has no zero divisors. Since I is an ideal $2^{(n(n-1)/2)}x^n \in I$. We have that

$$2^{(n(n-1)/2)}x^n = x\sigma(x)\sigma^2(x)\cdots\sigma^{n-1}(x) \in I.$$

In particular, this holds for all σ-prime ideals so A is σ-special. Since every maximal ideal is σ-stable A is not σ-primitive.

Example 2.2.25. A ring which is σ-primitive but not σ-special is given in Jordan [18, Example 2].

2.3 Gelfand-Kirillov dimension

In this section we will provide the basic definitions and introductory results pertaining to Gelfand-Kirillov dimension with the goal of proving the final result, Theorem 2.3.15, which relates GK dimension and skew Laurent polynomial rings. This material predominantly comes from the book by Krause and Lenagan [20].

Definition 2.3.1. Let k be a field and let A be a finitely generated k-algebra. We say that a finite-dimensional subspace V of A is a *generating subspace* if $1_A \in V$ and every element of A is a k-linear combination of products of elements of V.

Suppose $1_A \in V$ and that V is spanned by $a_1 = 1, a_2, \ldots, a_m$. Let V^n denote the subspace spanned by all monomials in $a_1, \ldots, a_m$ of length n for $n \geq 1$. Then there is an ascending chain of subspaces

$$k = V^0 \subseteq V^1 \subseteq V^2 \subseteq \cdots \subseteq V^n \subseteq \cdots \subseteq \bigcup_{n=0}^{\infty} V^n = A$$

with $\dim_k(V^n) < \infty$, for all $n \in \mathbb{N}$. Thus if A is finite-dimensional then $A = V^n$ for some n.

Definition 2.3.2. Let ϕ be the set of all functions $f : \mathbb{N} \to \mathbb{R}^+$ which are eventually monotone increasing and positive valued. For $f, g \in \phi$ we have the relation $f \leq^* g$ if there exists an $m \in \mathbb{N}$ such that $f(n) \leq g(mn)$ for all but finitely many $n \in \mathbb{N}$. We have the equivalence relation $f \cong g$ if $f \leq^* g$ and $g \leq^* f$. For $f \in \phi$, the equivalence class $G(f) \in \phi/\cong$ is called the *growth* of f.

We define the function $d_V(n)$ to be the dimension of V^n over k, $\dim_k(V^n)$. However, $d_V(n)$ appears to depend on the choice of V. We will show in the following proposition that the choice of finite-dimensional generating subspaces does not affect the growth of $d_V(n)$.

Proposition 2.3.3. *Let k be a field and let A be a finitely generated k-algebra with finite-dimensional generating subspaces V and W. If $d_V(n)$ and $d_W(n)$ denote the dimensions of $\sum_{i=0}^n V^i$ and $\sum_{i=0}^n W^i$, respectively, then $G(d_V) = G(d_W)$.*

Proof. Since $A = \bigcup_{n=0}^\infty (V^0 + \cdots + V^n) = \bigcup_{n=0}^\infty (W^0 + \cdots + W^n)$, there exist positive integers s and t such that $W \subseteq \sum_{i=0}^s V^i$ and $V \subseteq \sum_{i=0}^t W^i$. Thus $d_W(n) \leq d_V(sn)$ and $d_V(n) \leq d_W(tn)$. Hence $d_V \cong d_W$. $\square$

This brings us to the definition of Gelfand-Kirillov dimension.

Definition 2.3.4. Let k be a field, let A be a finitely generated k-algebra and let V be a generating subspace of A. Then the *Gelfand-Kirillov dimension* of A is defined to be

$$\operatorname{GKdim}(A) = \limsup_{n\to\infty} \frac{\log \dim_k(V^n)}{\log n} = \limsup_{n\to\infty} \log_n(d_V(n)).$$

Next we will compute the GK dimension of several algebras. The first example has infinite GK dimension.

Example 2.3.5. Consider the free algebra $A = \mathbb{C}\{x, y\}$ on two generators. Then $V = \mathbb{C} \oplus \mathbb{C}x \oplus \mathbb{C}y$ is a generating subspace for A, and

$$d_V(n) = \dim_{\mathbb{C}}\left(\sum_{i=0}^n V^i\right) = 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1.$$

Thus

$$\operatorname{GKdim}(A) = \limsup_{n\to\infty} \frac{\log(2^n)}{\log n} = \infty.$$

Example 2.3.6. Consider the Weyl algebra $A = \mathbb{C}[x,y]/(xy - yx - 1)$ and let V be the generating subspace spanned by the images of $1, x$ and $y \in A$. The relation $xy = 1 + yx$ allows us to express any monomial of degree n in x and y as a linear combination of monomials of the form $x^i y^j$ with $i+j \leq n$. So a basis for V^n is given by $\{x^i y^j \mid i+j \leq n\}$. For each $1 \leq k \leq n$ there are $k+1$ different monomials of the form $x^i y^j$ such that $i+j = k$. Thus the number of monomials with $i + j \leq n$ is

$$\sum_{k=1}^{n}(k+1) = \frac{n(n-1)}{2} + n = \frac{n^2+n}{2}.$$

Thus

$$\text{GKdim}(A) = \limsup_{n\to\infty} \frac{\log((\frac{n}{2})(n+1))}{\log(n)} = \limsup_{n\to\infty} \frac{\log(n) - \log(2) + \log(n+1)}{\log(n)} = 2.$$

Next we will compute the GK dimension of a finitely generated commutative $\mathbb{C}$-algebra. However, first we will give the analogous definition of dimension for commutative rings which we define as the Krull dimension below.

Definition 2.3.7. The k-algebra A has *Krull dimension* m if there exists a chain of prime ideals $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_m$ of length m and there is no chain of greater length. If there exists a chain of prime ideals of A of arbitrary length then the Krull dimension is said to be infinite.

For prime ideals in general we have the following definition.

Definition 2.3.8. Let R be a ring and let P be a prime ideal. The supremum over all d of chains of prime ideals,

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_d = P,$$

is called the *height of* P.

The height of a prime ideal is not always finite. We demonstrate this in the following example.

Example 2.3.9. Consider the polynomial ring in infinitely many variables, $\mathbb{C}[x_1, x_2, x_3, \ldots]$ with the prime ideal $P = (x_1, x_2, x_3, \ldots)$. We have the following chain of ideals

$$(x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, x_2, \ldots, x_d) \subsetneq \cdots \subsetneq (x_1, x_2, x_3, \ldots) = P.$$

Hence P has infinite height. The ring $\mathbb{C}[x_1, x_2, x_3, \ldots]$ has infinite Krull dimension as well.

Theorem 2.3.10. *Let k be a field and let A be a commutative k-algebra. Then*
(a) $\mathrm{GKdim}(A)$ *is either infinite or a nonnegative integer.*
(b) If A is finitely generated, then $\mathrm{GKdim}(A)$ *is equal to the Krull dimension of A.*

Proof. We refer the reader to Krause and Lenagan [20, Theorem 4.5]. □

Example 2.3.11. Let $A = \mathbb{C}[x_1, \ldots, x_d]$. We know that

$$(0) \subsetneq (x_1) \subsetneq \cdots \subsetneq (x_1, \ldots, x_d),$$

is a chain of prime ideals of length d in A. Hence the Krull dimension of A is greater than or equal to d.

We have that $V = \mathbb{C} \oplus \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_d$ is a generating subspace of A. So a basis of V^n is given by all monomials $x_1^{i_1} \cdots x_d^{i_d}$ where $i_1 + \cdots + i_d \leq n$. The number of monomials $x_1^{i_1} \cdots x_d^{i_d}$ with $i_1 + \cdots + i_d \leq n$ is $\binom{n+d}{d}$ and

$$\binom{n+d}{d} \sim \frac{n^d}{d!} \quad \text{as } n \to \infty.$$

Thus

$$\mathrm{GKdim}(A) = \limsup_{n\to\infty} \frac{\log(\frac{n^d}{d!})}{\log n} = \limsup_{n\to\infty} \frac{d\log(n)}{\log(n)} - \frac{\log(d!)}{\log(n)} = d.$$

From Theorem 2.3.10 we have that the Krull dimension of A is d as well.

Finally, we provide the following characterization of possible values of the GK dimension of a k-algebra.

Theorem 2.3.12. *Let k be a field and let A be a finitely generated k-algebra. Then the possible values for the GK dimension of A are* $\{0\} \cup \{1\} \cup [2, \infty]$.

Proof. We have that GKdim(A) $= 0$ if and only if every finitely generated subalgebra of A is finite-dimensional (cf. Krause and Lenagan [20]). Krause and Lenagan [20, Proposition 1.4] gives us that GKdim(A) ≥ 1 for any algebra containing an infinite dimensional finitely generated subalgebra of A. A result of Bergman [4] which is also shown in Krause and Lenagan [20, Theorem 2.5] gives us that there is no algebra A with a GK dimension strictly between 1 and 2, Krause and Lenagan [20, Theorem 2.9] states that for every real number $r \geq 2$ there exists a two generator algebra $A = k\{x, y\}/I$ for some ideal I with

GKdim$(A) = r$. Finally, we showed that a finitely generated k-algebra can have infinite GK dimension in Example 2.3.5. $\square$

We now consider the GK dimension of skew Laurent polynomial rings.

Proposition 2.3.13. *Let k be a field, let A be a finitely generated k-algebra and let σ be a k-algebra automorphism of A. Then* $\mathrm{GKdim}(A[z, z^{-1}; \sigma]) \geq 1 + \mathrm{GKdim}(A)$.

Proof. Let V be a generating subspace of A. Then $W = V \oplus kz \oplus kz^{-1}$ is a generating subspace of $A[z, z^{-1}; \sigma]$. We have that $V^n \subseteq W^n$ and $kz^i \subseteq W^n$ for $1 \leq i \leq n$. Thus

$$V^n \oplus V^n z \oplus \cdots \oplus V^n z^n \subseteq W^{2n}.$$

Hence $\dim_k(W^{2n}) \geq \dim_k(V^n)(n+1)$. Thus we have that

$$\begin{aligned}
\mathrm{GKdim}(A[z, z^{-1}; \sigma]) &= \limsup_{n\to\infty} \frac{\log(\dim_k(W^n))}{\log(n)} \geq \limsup_{n\to\infty} \frac{\log(\dim_k(W^{2n}))}{\log(2n)} \\
&\geq \limsup_{n\to\infty} \frac{\log(\dim_k(V^n)) + \log(n+1)}{\log(n)} \\
&= \limsup_{n\to\infty} \left(\frac{\log(\dim_k(V^n))}{\log(n)} + 1 \right) = \mathrm{GKdim}(A) + 1. \quad \square
\end{aligned}$$

Now we will look at when we have a strict equality in Proposition 2.3.13.

Definition 2.3.14. Let A be a k-algebra over an algebraically closed field k and let σ be a k-algebra endomorphism of A. We say that σ is *locally algebraic* if every finite-dimensional k-vector subspace of A is contained in a σ-stable generating subspace of A.

Theorem 2.3.15. *Let A be a commutative k-algebra such that the field of fractions of A is a finitely generated field extension of k, and let σ be a k-algebra automorphism of A with skew Laurent polynomial ring $A[z, z^{-1}; \sigma]$. Then* $\mathrm{GKdim}(A[z, z^{-1}; \sigma]) = \mathrm{GKdim}(A) + 1$ *if and only if σ is locally algebraic.*

Proof. We refer the reader to Zhang [28, Theorem 1.1]. $\square$

Theorem 2.3.15 shows us that skew Laurent polynomials form an interesting subclass of rings of which to study the GK dimension. We give this subclass the following special name.

Definition 2.3.16. Let A be a commutative k-algebra and let σ be a k-algebra automorphism of A with skew Laurent polynomial ring $A[z, z^{-1}; \sigma]$. We say that $A[z, z^{-1}; \sigma]$ has *low growth* if $\mathrm{GKdim}(A[z, z^{-1}; \sigma]) = \mathrm{GKdim}(A) + 1$.

Next we will give an example of a skew Laurent polynomial ring which does not have low growth. However, first we will need the following result about the GK dimension of the group algebra $\mathbb{C}[G]$ of a nilpotent group G.

Theorem 2.3.17. (Bass-Guivarc'h) *Let G be a finitely generated nilpotent group with lower central series*

$$\{1\} = G_d \subseteq \cdots \subseteq G_2 \subseteq G_1 = G$$

such that the quotient group G_k/G_{k+1} is a finitely generated abelian group. Then

$$\mathrm{GKdim}(\mathbb{C}[G]) = \sum_{k \geq 1} k \,\mathrm{rank}(G_k/G_{k+1}),$$

where $\mathrm{rank}(G_k/G_{k+1})$ denotes the largest number of independent and torsion free elements of the abelian group.

Proof. We refer the reader to Bass [1] and Guivarc'h [14]. $\square$

Example 2.3.18. Let $A = \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ and let σ be an automorphism of A defined by $\sigma(x) = x$ and $\sigma(y) = xy$. Consider the skew Laurent polynomial ring $A[z, z^{-1}; \sigma]$ and let G be the group generated by x, y and z. Then $[x, y] = [x, z] = 1$ and $[y, z] = x^{-1}$, where $[g, h] = g^{-1}h^{-1}gh$.

If we let $u = z, v = y^{-1}$ and $w = x^{-1}$ then $G = \langle x, y, z\rangle = \langle u, v, w\rangle$ is the Heisenberg group with relations $[u, w] = [v, w] = 1$ and $[u, v] = w$. Then $G_1 = G$, $G_2 = [G, G] = \langle w\rangle$ and $G_3 = [G, G_2] = \{1\}$. Thus G is a nilpotent group with lower central series $\{1\} \subseteq \langle w\rangle \subseteq G$ with $G/\langle w\rangle \cong \mathbb{Z}^2$ by [9, Theorem 6.1.8] and $\langle w\rangle/\{1\} \cong \mathbb{Z}$. Then by the Bass-Guivarc'h formula in Theorem 2.3.17

$$\mathrm{GKdim}(G) = 1\,\mathrm{rank}(G/G_2) + 2\,\mathrm{rank}(G_2/G_3) = 1\,\mathrm{rank}(\mathbb{Z}^2) + 2\,\mathrm{rank}(\mathbb{Z}) = 4.$$

From the remark before 11.5 in Krause and Lenagan [20] we have that $\mathbb{C}[G] = A[z, z^{-1}; \sigma]$ and $\mathrm{GKdim}(\mathbb{C}[x^{\pm 1}, y^{\pm 1}]) = 2$. Thus $\mathrm{GKdim}(A) = 2$ and $\mathrm{GKdim}(A[z, z^{-1}; \sigma]) = \mathrm{GKdim}(A) + 2$.

2.4 The Dixmier-Moeglin equivalence

We begin this section by stating the celebrated result proved independently by J. Dixmier and C. Moeglin in [8] and [26] respectively.

Theorem 2.4.1. *Let $\mathcal{L}$ be a complex finite-dimensional Lie algebra and let $\mathcal{U}(\mathcal{L})$ be its enveloping algebra and let P be a prime ideal of $\mathcal{U}(\mathcal{L})$. Then the following are equivalent.*
1. P is rational.
2. P is locally closed in $\mathrm{Spec}(\mathcal{U}(\mathcal{L}))$.
3. P is primitive.

We will spend the remainder of this section providing the basic definitions and background necessary in order to understand the statement of the Dixmier-Moeglin equivalence for universal enveloping algebras and we will give examples of Dixmier-Moeglin equivalences for other rings. We will use this background in Chapter 4 to determine when we will obtain a Dixmier-Moeglin equivalence for low growth skew Laurent polynomial rings. We start with defining an *universal enveloping algebra*. To define this, we need to define a *Lie algebra*.

Definition 2.4.2. A *Lie algebra* $\mathcal{L}$ is a vector space over a field k with a multiplication which is usually termed a Lie bracket $[\cdot,\cdot]$ such that for $x, y, z \in \mathcal{L}$ and $c \in k$ we have

1. $[x, y] = -[y, x]$,
2. $[[x, y], z] + [[z, x], y] + [[y, z], x] = 0$,
3. $[x + cy, z] = [x, z] + c[y, z]$,
4. $[x, y + cz] = [x, y] + c[x, z]$.

Example 2.4.3. The algebra $\mathcal{L} = M_n(\mathbb{C})$ of $n \times n$ matrices over $\mathbb{C}$ is a Lie algebra with Lie bracket $[X, Y] = XY - YX$ for $X, Y \in M_n(\mathbb{C})$.

Note that a Lie algebra is not associative in general, meaning $[[x, y], z] \neq [x, [y, z]]$.

Definition 2.4.4. Let T be the *tensor algebra* of the vector space $\mathcal{L}$. Let

$$T = T^0 \oplus T^1 \oplus \cdots \oplus T^n \oplus \cdots ,$$

with $T^n = \mathcal{L} \otimes \mathcal{L} \otimes \cdots \otimes \mathcal{L}$ (n times) where the product in T is tensor multiplication. Let J be the two-sided ideal of T generated by the tensors

$$x \otimes y - y \otimes x - [x, y],$$

with $x, y \in \mathcal{L}$. The (unique) associative algebra T/J is the *universal enveloping algebra of* $\mathcal{L}$ which we denote by $\mathcal{U}(\mathcal{L})$.

We next define a *rational prime*. To define this we need to define a *quotient division ring*. In commutative ring theory, if A is a Noetherian domain one can invert the nonzero elements to form the field of fractions. We have the following analogous definition in the noncommutative case.

Definition 2.4.5. Let R b a ring. The classical *left quotient ring* of R is the left ring of fractions for R with respect to the set of all regular elements in R.

We are interested in the case where the quotient ring of R is also a division ring. We need the following definition of a *left Ore domain*.

Definition 2.4.6. Let R be a domain. If the nonzero elements of R form a *left Ore set*, that is, for each nonzero $x, y \in R$ there exists $r, s \in R$ such that $rx = sy \neq 0$. Then R is a *left Ore domain*.

Proposition 2.4.7. ***(Ore)*** *Let R be a ring. R has a classical left quotient ring which is a division ring if and only if R is a left Ore domain.*

Proof. We refer the reader to Goodearl and Warfield [13, Theorem 6.8]. □

Proposition 2.4.8. *Every left Noetherian domain is a left Ore domain.*

Proof. We refer the reader to Goodearl and Warfield [13, Corollary 6.7]. □

Proposition 2.4.7 and Proposition 2.4.8 show us that if A is a Noetherian domain then one can invert the regular elements of A to form a *quotient division ring*, which we denote $\mathrm{Fract}(A)$.

Definition 2.4.9. Let k be a field and let A be a Noetherian k-algebra. A prime ideal P of A is *rational* provided the center of $\mathrm{Fract}(A/P)$ is an algebraic extension of k.

Since we have already defined a primitive ideal in Section 2.1 it only remains to define a *locally closed* prime. To do this we will need the following topological background.

Definition 2.4.10. A subset L of a topological space X is said to be *locally closed* if there exists an open set U, containing L, such that L is closed in U. Equivalently, L is locally closed if and only if it is an intersection of an open set and a closed set in X.

Definition 2.4.11. The *prime spectrum* of a ring R, denoted $\mathrm{Spec}(R)$, is the set of all prime ideals of R.

The topology we will be using for the remainder of this paper is the following.

Definition 2.4.12. Let R be a ring. The *Zariski topology* on $\mathrm{Spec}(R)$ is constructed by taking the *Zariski-closed* sets to be

$$V(I) = \{P \in \ \mathrm{Spec}(R) \mid I \subseteq P\} \quad \text{for any ideal } I$$

and the *Zariski-open* sets to be

$$W(I) = \{P \in \ \mathrm{Spec}(R) \mid I \nsubseteq P\}.$$

Alternatively, when we are using the Zariski topology on Spec(R), we can interpret the definition of locally closed in terms of rings.

Definition 2.4.13. Let R be a ring. A prime ideal P of Spec(R) is *locally closed* in Spec(R) if P is a locally closed point of Spec(R), where R is equipped with the Zariski topology.

Next we will give equivalent conditions for a prime to be locally closed.

Lemma 2.4.14. *A prime ideal P in a ring R is locally closed in* Spec*(R) if and only if the intersection of all prime ideals properly containing P is an ideal properly containing P.*

Proof. We will follow the proof in Brown and Goodearl [5, Theorem II.7.7]. Let J be the intersection of all prime ideals properly containing P. Suppose $P \subsetneq J$, then $P \in V(P) \cap W(J)$ from Definition 2.4.11. Suppose there exists $Q \neq P \in V(P) \cap W(J)$. If $Q \in V(P)$ then $P \subsetneq Q$. Thus Q is an ideal that properly contains P and hence $J \subseteq Q$. This implies $Q \notin W(J)$. Hence $V(P) \cap W(J) = \{P\}$ and so $\{P\}$ is an intersection of an open set $W(J)$ and a closed set $V(P)$ in R. Thus P is locally closed.

Suppose P is locally closed. Then there exist ideals $I_1, I_2 \in R$ such that $V(I_1) \cap W(I_2) = \{P\}$. Then $P \in W(I_2)$ and $I_2 \subsetneq P$. Since $P \subseteq J$, $I_2 \subsetneq J$. Thus $P \subsetneq I_2 + P \subseteq J$. $\square$

We require the following definition to prove Lemma 2.4.16.

Definition 2.4.15. Let R be a ring and let P be a prime ideal of R. The ideal P is said to be *minimal* over an ideal I if there are no prime ideals strictly contained in P that contain I.

Lemma 2.4.16. (Noether) *Let R be a Noetherian ring. Then there are only finitely many minimal primes over a given ideal $I \subseteq R$.*

Proof. We follow the proof in Eisenbud [10, Exercise 1.2]. Suppose $I \subseteq R$ is an ideal such that there are infinitely many prime ideals containing I minimal with respect to inclusion. Since R is Noetherian, among the collection of all such I there is one that is maximal with respect to this property. We will denote this ideal J. The ideal J is not prime so there exist $f, g \notin J$ such that $fg \in J$. Let P be a prime minimal over J. Then either $f \in P$ or $g \in P$. So either P is minimal over (J, f) or (J, g). Thus either (J, f) or (J, g) is contained in infinitely many minimal primes, a contradiction. $\square$

Proposition 2.4.17. *Let k be an uncountable field and let A be a prime Noetherian, countably generated k-algebra with the descending chain condition on prime ideals. A has finitely many height one primes if and only if (0) is locally closed.*

Proof. Suppose A has finitely many height one primes, $\{P_1, \ldots, P_n\}$. Let $I = P_1 P_2 \cdots P_n$. If $I = (0)$ then $P_i = (0)$ for some i since (0) is a prime ideal, but then $P_i \subseteq (0)$ which contradicts the assumption that P_i is a height one prime. Thus $I \neq (0)$. Since we are assuming that A has the descending chain condition on prime ideals every nonzero prime ideal of A contains a height one prime. Then we have that $\{(0)\} = \mathrm{Spec}(A) \setminus V(I)$, which is an open set. $\mathrm{Spec}(A)$ is a closed set so $\{(0)\} = (\mathrm{Spec}(A) \setminus V(I)) \cap \mathrm{Spec}(A)$. Hence $\{(0)\}$ is an intersection of an open and closed set so (0) is locally closed.

Suppose (0) is locally closed in $\mathrm{Spec}(A)$. From Proposition 2.4.14 there exists an ideal $J \subseteq A$ such that $(0) \subsetneq J$ where J is the intersection of all height one prime ideals. So J is contained in all height one primes and by Lemma 2.4.16 there are only finitely many primes minimal over J. Thus there are only finitely many height one primes in A. □

Now we will give results relating rational, primitive and locally closed prime ideals. However, first we need to define the noncommutative *Nullstellensatz*, which requires the definition of a *Jacobson ring*.

Definition 2.4.18. A *Jacobson ring* is a ring R in which the intersection of all left maximal ideals of R/P is (0) for all prime ideals P in R.

In commutative algebra the Nullstellensatz over an algebraically closed field k can be stated as follows. Let I be an ideal over $A = k[x_1, \ldots, x_n]$, let $I(V(I))$ be the ideal of all polynomials in A which vanish on the affine variety of I and let $\mathrm{rad}(I)$ denote the *radical* of I. Then $I(V(I)) = \mathrm{rad}(I)$. In noncommutative algebra we have the following analogue of the commutative Nullstellensatz.

Definition 2.4.19. Let A be a Noetherian k-algebra. A satisfies the *Nullstellensatz over k* if A is a Jacobson ring and the endomorphism ring of every irreducible left A-module is algebraic over k.

The following theorem shows how the commutative Nullstellensatz can be used to characterize maximal ideals.

Theorem 2.4.20. *Let k be an algebraically closed field and let $A = k[x_1, \ldots, x_n]$. Then every maximal ideal of A is of the form $\mathcal{M}_p = (x_1 - a_1, \ldots, x_n - a_n)$ for some $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$. In particular the points of $\mathbb{A}^n$ are in one-to-one correspondence with the maximal ideals of A.*

Proof. Let $\mathcal{M}$ be a maximal ideal of A. Since every prime ideal is radical we have that $I(V(\mathcal{M})) = \mathcal{M}$ by the Nullstellensatz. If $p \in V(\mathcal{M})$ then $\mathcal{M} \subseteq \mathcal{M}_p$. Since $\mathcal{M}$ is assumed to be maximal $\mathcal{M} = \mathcal{M}_p$. The correspondence follows immediately. □

From Remark 2.1.9 we know that in a commutative ring primitive and maximal ideals are equivalent. So Theorem 2.4.20 also give us a characterization of the primitive ideals of polynomial rings over algebraically closed fields. However, in the noncommutative case we have the following implications.

Lemma 2.4.21. *Let A be a Noetherian k-algebra satisfying the Nullstellensatz over k. Then for all prime ideals of A, the following implications hold:*

$$P \textit{ is locally closed in } \operatorname{Spec}(A) \Rightarrow P \textit{ is primitive} \Rightarrow P \textit{ is rational.}$$

Proof. We will follow the proof in Brown and Goodearl [5, Lemma II.7.11]. Let P be a locally closed prime of A and let $\{P_i \mid i \in I\}$ be the set of all primitive ideals of A containing P. Since A satisfies the Nullstellensatz, A is a Jacobson ring and thus the intersection of maximal ideals of A/P is (0). Suppose $\cap_{i\in I}P_i = Q \supsetneq P$. Every maximal ideal in A/P is a primitive ideal containing P. Thus the intersection of all maximal ideals of A/P would not be zero. Hence $\cap_{i\in I}P_i = P$. By Lemma 2.4.14, the P_i can not all properly contain P otherwise $\cap_{i\in I}P_i \supsetneq P$. Thus some $P_i = P$, so P is primitive.

The proof of P is primitive implies P is rational is given in Brown and Goodearl [5, Lemma II.7.13]. □

Remark 2.4.22. An example of a ring where P is rational but not primitive is given in Irving [16] and an example of a ring where (0) is primitive but not locally closed is given in Lorenz [23].

The following Proposition is a useful way to determine whether a ring satisfies the Nullstellensatz.

Proposition 2.4.23. *If k is an uncountable field and R is a countably generated k-algebra then R satisfies the Nullstellensatz over k.*

Proof. This can be found in McConnell and Robson [25, Corollary 9.1.8]. □

Definition 2.4.24. Let k be a field and let A be a Noetherian k-algebra. If for all prime ideals of A the following three conditions: P is locally closed in $\operatorname{Spec}(A)$, P is primitive and P is rational are equivalent then A is said to satisfy the ***Dixmier-Moeglin equivalence***.

Example 2.4.25. From Brown and Goodearl [5, Corollary II.8.5] we have that the example we considered in Example 2.2.3, the quantized coordinate ring of $(k^*)^2$, $\mathcal{O}_q((k^*)^2)$ as well as other quantized coordinate rings satisfy the Dixmier-Moeglin equivalence. Goodearl and Letzter give other examples in [12].

We have shown in Example 2.2.3 that $\mathcal{O}_q((k^*)^2) = k[y^{\pm 1}][x, x^{-1}; \sigma]$ where σ is the automorphism of $k[y^{\pm 1}]$ given by $\sigma(y) = qy$ is a skew Laurent polynomial ring. We also have that this ring satisfies the Dixmier-Moeglin equivalence from the following theorem by Bell, Rogalski and Sierra [3].

Theorem 2.4.26. *Let k be an uncountable algebraically closed field of characteristic zero and let A be a finitely generated commutative k-algebra. Let σ be an automorphism of A. If $\dim(A) \leq 2$ and $\mathrm{GKdim}(A[z, z^{-1}; \sigma]) < \infty$ then $A[z, z^{-1}; \sigma]$ satisfies the Dixmier-Moeglin equivalence.*

Proof. We refer the reader to Bell, Rogalski and Sierra [3, Theorem 1.1]. □

In the following section we will give an example of a skew Laurent polynomial ring which does not satisfy the Dixmier-Moeglin equivalence and in Chapter 4 we will show that low growth skew Laurent polynomial rings satisfy the Dixmier-Moeglin equivalence for the prime ideal (0).

2.5 The Hénon map

In this section we will consider the Hénon map which is of interest to the study of dynamical systems. We will show that the Hénon map is an example of an automorphism of $\mathbb{C}^2$ which has a countably infinite set of periodic points and no σ^n-stable curves. We will show that the skew Laurent polynomial ring $\mathbb{C}[x, y][z, z^{-1}; \sigma]$ is an example of a ring which is primitive but (0) is not locally closed in $\mathrm{Spec}(\mathbb{C}[x, y][z, z^{-1}; \sigma])$ and hence the Dixmier-Equivalence is not satisfied.

Definition 2.5.1. The *Hénon map* is defined to be the map

$$\sigma : \mathbb{C}[x, y] \to \mathbb{C}[x, y]$$
$$\sigma(x) = y + 1 - ax^2 \quad \sigma(y) = bx,$$

in terms of polynomial rings or

$$\tau : \mathbb{C}^2 \to \mathbb{C}^2$$
$$\tau((x_0, y_0)) = (y_0 + 1 - ax_0^2, bx_0)$$

in terms of affine space where a, b are nonzero complex numbers.

Bedford and Smillie use the notion of *topological entropy* and *maximal entropy* for *polynomial diffeomorphisms* in studying the dynamics of the Hénon map. The background necessary to explain these results goes beyond the scope of this thesis but the interested reader can find them in Bedford and Smillie [2].

Proposition 2.5.2. *Let $\sigma : \mathbb{C}[x, y] \to \mathbb{C}[x, y]$ be the Hénon map defined above. Then σ is a $\mathbb{C}$-algebra automorphism.*

Proof. The Hénon map can be composed into the following three maps:

$$\begin{aligned}
\phi_1((x, y)) &= (x, 1 - ax^2 + y),\\
\phi_2((x, y)) &= (bx, y),\\
\phi_3((x, y)) &= (y, x).
\end{aligned}$$

such that $(\phi_3 \circ \phi_2 \circ \phi_1)(x, y) = (1 - ax^2 + y, bx) = \sigma$. The inverses of these maps are the following,

$$\begin{aligned}
\phi_1^{-1}((x, y)) &= (x, y - 1 + ax^2),\\
\phi_2^{-1}((x, y)) &= (\frac{1}{b}x, y),\\
\phi_3^{-1}((x, y)) &= (y, x).
\end{aligned}$$

where $\sigma^{-1} = (\phi_1^{-1} \circ \phi_2^{-1} \circ \phi_3^{-1})(x, y) = (\frac{1}{b}y, x - 1 + \frac{ay^2}{b^2})$. Since an inverse exists, the Hénon map is an automorphism. □

Theorem 2.5.3. *Let $a, b \in \mathbb{R}$ and let $\tau : \mathbb{C}^2 \to \mathbb{C}^2$ be defined as in Definition 2.5.1. If $a > \frac{(5+2\sqrt{5})(1+|b|)^2}{4}$ then the set $\{(x_0, y_0) \in \mathbb{C}^2 \mid \tau^n((x_0, y_0)) = (x_0, y_0)$ for some $n \in \mathbb{N}\}$ is countably infinite.*

Proof. Devaney and Nitecki show in [6] that if $a > \frac{(5+2\sqrt{5})(1+|b|)^2}{4}$ then τ will have maximal entropy and Bedford and Smillie show in [2, Theorem 1] that the set of fixed points of τ^n if τ has maximal entropy is exactly d^n elements where d is the algebraic degree of τ and $d \geq 2$. Since τ has algebraic degree 2, τ^n has exactly 2^n fixed points. Thus the set of periodic points is

$$\bigcup_{n=1}^{\infty} \{(x, y) \in \mathbb{C}^2 | \ \tau^n((x, y)) = (x, y) \text{ and for } 1 \leq j < n, \ \tau^j((x, y)) \neq (x, y)\},$$

which is a countable union of finite sets and hence countable. □

The following two standard background results, Theorem 2.5.4 and Theorem 2.5.5 will be necessary to prove the final result of this section.

Theorem 2.5.4. (Krull's height theorem) *Let A be a Noetherian ring and let $I = (a_1, \ldots, a_n)$ be a proper ideal generated by n elements of R. If P is a minimal prime ideal that is minimal over I then P has height at most n.*

Proof. We refer the reader to Matsumura [24, Theorem 13.5]. □

Theorem 2.5.5. *Let A be a Noetherian domain. A is a unique factorization domain if and only if every height one prime ideal is principal.*

Proof. We will follow the proof found in Matsumura [24, Theorem 20.1]. Suppose that A is a UFD and that P is a height one prime ideal. Let a be a nonzero element of P. Since A is a UFD we can express a as a product of prime elements $a = p_1 \cdots p_d$. Since P is a prime ideal at least one $p_i \in P$. If $p_i \in P$ then $(p_i) \subseteq P$. However, (p_i) is a nonzero prime ideal and P has height one. Thus $(p_i) = P$ and P is principal.

Conversely, suppose A is Noetherian with every height one prime ideal principal. Since A is Noetherian, every nonzero element $a \in A$ which is not a unit can be written as a product of finitely many irreducibles. Hence to prove A is a UFD it suffices to show an irreducible element a is a prime element. Let P be a minimal prime containing (a). Then by Krull's height theorem, the height of P is one. Thus we can write the ideal P as (b). Thus there exists a unit c in A such that $a = cb$. Since a is irreducible $(a) = (b) = P$, and thus a is a prime element. □

Before we can prove Lemma 2.5.8 we need the following result from Smith [27].

Theorem 2.5.6. *Let k be a field of characteristic zero and let $A = k[x, y]$. Let τ, π be automorphisms of $k[x, y]$ such that*

$$\tau(x) = \alpha x + \omega(y) \quad \text{and} \quad \tau(y) = \mu y + \nu$$

and

$$\pi(x) = y \quad \text{and} \quad \pi(y) = x,$$

for $\alpha, \mu, \nu \in k$ and $\omega(y) \in k[y]$ such that $\omega(y)$ has degree $d \geq 2$. Let $\sigma = \pi \circ \tau$. If $\sigma(f) = \lambda f$ for some $f \in k[x, y]$, $\lambda \in k$ then $f \in k$.

Proof. We refer the reader to (cf. Smith [27]). $\square$

Remark 2.5.7. If $k = \mathbb{C}, \alpha = 1, \omega(y) = 1 - ay^2, \mu = b$ and $\nu = 0$. Then

$$(\pi \circ \tau)(x) = \pi(x + 1 - ay^2) = y + 1 - ax^2$$

and

$$(\pi \circ \tau)(y) = \pi(by) = bx.$$

Then σ is the Hénon map.

Lemma 2.5.8. *Let $A = \mathbb{C}[x, y]$ and let $\sigma : A \to A$ be defined as in Definition 2.5.1. Then there are no height one σ^n-stable prime ideals in A.*

Proof. Let I be a nonzero, principal σ^n-stable ideal of A. Then $I = (f)$ with $f \in A$ such that f is not a unit. Thus $\sigma^n(f) = \lambda f$ for some $\lambda \in \mathbb{C}^*$. Let

$$g = f\sigma(f)\sigma^2(f) \cdots \sigma^{n-1}(f).$$

Since σ is an automorphism, units are mapped to units. Thus $\sigma(f)$ is not a unit since f is not a unit. Inductively, $\sigma^i(f)$ is not a unit for all $i \in \mathbb{N}$. A is a domain so a finite product of non-units is again a non-unit. Thus g is not a unit and g is nonzero.

We have

$$\begin{aligned}\sigma(g) &= \sigma(f\sigma(f) \cdots \sigma^{n-1}(f)) = \sigma(f)\sigma^2(f) \cdots \sigma^{n-1}(f)\sigma^n(f) \\ &= \lambda f\sigma(f) \cdots \sigma^{n-1}(f) = \lambda g.\end{aligned}$$

From Theorem 2.5.6 we have that $g \in \mathbb{C}^*$, a contradiction. A is a UFD so by Theorem 2.5.5 every height one prime ideal must be principal and the result follows. $\square$

Proposition 2.5.9. *Let $a, b \in \mathbb{R}$, let $A = \mathbb{C}[x, y]$ and let $\sigma : A \to A$ and $\tau : \mathbb{C}^2 \to \mathbb{C}^2$ be defined as in Definition 2.5.1. If $a > \frac{(5+2\sqrt{5})(1+|b|)^2}{4}$ then (0) is not locally closed in* $\mathrm{Spec}(A[z, z^{-1}; \sigma])$.

Proof. From Theorem 2.5.8 there are no σ^n-stable height one primes in A. Thus any σ^n-stable nonzero prime ideal must be height two and correspond to a σ^n-stable maximal ideal. From Theorem 2.5.3 there are a countably infinite number of τ^n-stable points $p \in \mathbb{C}^2$. This corresponds to a countably infinite number of σ^n-stable maximal ideals $\mathcal{M}_p$ of A.

For each σ^n-stable maximal ideal $\mathcal{M}_p$ we can form the σ-stable ideal I_n defined by $I_n = \cap_{j=0}^{n-1}\sigma^j(\mathcal{M}_p)$. Let f be a nonzero element of $\mathcal{M}_p$. Then $f\sigma(f)\cdots\sigma^{n-1}(f) \in I_n$ since $\sigma^j(f) \in \sigma^j(\mathcal{M}_p)$. Then $f\sigma(f)\cdots\sigma^{n-1}(f)$ is a nonzero element of I_n so $I_n \neq (0)$. Now suppose J_1, J_2 are σ-stable ideals of A such that $J_1J_2 \subseteq I_n \subseteq \mathcal{M}_p$. Since $\mathcal{M}_p$ is prime either $J_1 \subseteq \mathcal{M}_p$ or $J_2 \subseteq \mathcal{M}_p$. Without loss of generality assume $J_1 \subseteq \mathcal{M}_p$. Since J_1 is σ-stable $\sigma^j(J_1) = J_1$ and thus $J_1 \subseteq \cap_{j=0}^{n-1}\sigma^j(\mathcal{M}_p) = I_n$. Hence I_n is σ-prime so there exists a prime ideal $P \in A[z, z^{-1}; \sigma]$ such that $I_n = P \cap A$ by Proposition 2.2.7. By Theorem 2.5.3 for every $n \in \mathbb{N}$ there exists a σ^n-stable maximal ideal and thus a corresponding σ-prime ideal I_n. Hence there are a countably infinite number of distinct prime ideals in $A[z, z^{-1}; \sigma]$ corresponding to each σ-prime ideal I_n.

Let $P_n \in A[z, z^{-1}; \sigma]$ be a prime ideal such that $I_n = P_n \cap A$ and suppose that P_n is not minimal over (0). Then there exists a $J_n \in A$ such that $(0) \subsetneq J_n \subsetneq \cap_{j=0}^{n-1}\sigma^j(\mathcal{M}_p)$. So there must be a nonzero σ^n-stable prime ideal properly contained in $\mathcal{M}_p$. This ideal would have to be height one. By Proposition 2.5.8 there are no height one σ^n-stable prime ideals of A so P_n must be minimal over (0). So we have a countably infinite number of minimal primes over (0) and by Lemma 2.4.16 there can only be finitely many minimal primes over (0). By Proposition 2.4.17 (0) is locally closed in $\mathrm{Spec}(A[z, z^{-1}; \sigma])$ if and only if there are finitely many height one primes in $A[z, z^{-1}; \sigma]$, a contradiction. $\square$

Proposition 2.5.10. *Let $a, b \in \mathbb{R}$, let $A = \mathbb{C}[x, y]$ and let $\sigma : A \to A$ and $\tau : \mathbb{C}^2 \to \mathbb{C}^2$ be defined as in Definition 2.5.1. If $a > \frac{(5+2\sqrt{5})(1+|b|)^2}{4}$ then $A[z, z^{-1}; \sigma]$ is a primitive ring.*

Proof. From Proposition 2.5.3 we have countably many τ^n-stable points in $\mathbb{C}^2$. Thus there are uncountably many points which are not periodic. Let $p \in \mathbb{C}$ be a point which is not periodic. This point corresponds to a maximal ideal $\mathcal{M}_p \subseteq A$ such that $\sigma^j(\mathcal{M}_p) = \mathcal{M}_{\tau^j(p)}$. Suppose there exists a nonzero σ-stable ideal $I \subseteq \mathcal{M}_p$. Then for all $j \in \mathbb{Z}$ we have that $I = \sigma^j(I) \subseteq \sigma^j(\mathcal{M}_p) = \mathcal{M}_{\tau^j(p)}$. Hence $I \subseteq \bigcap_{j\in\mathbb{Z}} \mathcal{M}_{\tau^j(p)} = J$. There exists a surjective map $\phi : A/J \to \bigcap_{j=1}^{N} A/\mathcal{M}_{\tau^j(p)}$ and by the Chinese remainder theorem we have that

$$\bigcap_{j=1}^{N} A/\mathcal{M}_{\tau^j(p)} \cong A/\mathcal{M}_{\tau^1(p)} \oplus \cdots \oplus A/\mathcal{M}_{\tau^N(p)},$$

an N-dimensional vector space. Since this map is surjective for all $N \in \mathbb{N}$, A/J has infinite dimension over $\mathbb{C}$. Thus J has infinite codimension.

By Lemma 2.4.16 there can only be finitely many minimal primes over J and hence only finitely many minimal primes over I. We will denote them $P_1, \ldots, P_k$. If all P_i are height two primes then there would only be finitely many maximal ideals over J. We showed that J has infinite codimension, a contradiction. Hence at least one P_i is height one. Without loss of generality we can assume that P_1 has height one. We have that $I \subseteq P_i$ and since I is σ-stable we have that $\sigma(I) \subseteq P_i$. Thus σ must permute the prime ideals $P_1, \ldots, P_k$ and hence for some $1 \leq n \leq k$ we have that $\sigma^n(P_1) = P_1$. Then P_1 is a σ^n-stable height one prime. By Proposition 2.5.8 there are no σ^n-stable height one primes, a contradiction. □

From Proposition 2.5.9 and Proposition 2.5.10 we can conclude that the skew Laurent polynomial ring $A[z, z^{-1}; \sigma]$ does not satisfy the Dixmier-Moeglin equivalence.

Corollary 2.5.11. *Let $a, b \in \mathbb{R}$, let $A = \mathbb{C}[x, y]$ and let $\sigma : A \to A$ and $\tau : \mathbb{C}^2 \to \mathbb{C}^2$ be defined as in Definition 2.5.1. If $a > \frac{(5+2\sqrt{5})(1+|b|)^2}{4}$ then $\mathrm{GKdim}(A[z, z^{-1}; \sigma]) = \infty$.*

Proof. This follows directly from Proposition 2.5.10, Proposition 2.5.9 and Theorem 2.4.26. □

Chapter 3

Structure theory for low growth skew Laurent polynomial rings

3.1 Structure theory for low growth skew Laurent polynomial rings

In this section we will consider the skew (twisted) Laurent polynomial ring, $\mathbb{C}[x_1, \dots, x_d][z, z^{-1}; \sigma]$ where σ is a $\mathbb{C}$-algebra automorphism of $\mathbb{C}[x_1, \dots, x_d]$.

Let $A = \mathbb{C}[x_1, \dots, x_d]$ and let $\sigma : A \to A$ be a $\mathbb{C}$-algebra automorphism of A. We assume that the $(d+1)$-dimensional vector space $V = \mathbb{C} \oplus \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_d$ has the property that $\sigma(V) = V$. Then σ is a $\mathbb{C}$-linear automorphism of V such that for $1 \leq i, j \leq d$,

$$\sigma(x_i) = b_i + \sum_{j=1}^{d} c_{ij} x_j, \quad \text{and} \quad \sigma(1) = 1,$$

for some $c_{ij} \in \mathbb{C}$. Thus the matrix of σ relative to the basis $\{1, x_1, \dots, x_d\}$ is the $(d+1) \times (d+1)$ matrix

$$M_\sigma = \begin{pmatrix} 1 & b_1 & \dots & b_d \\ 0 & c_{11} & \dots & c_{d1} \\ 0 & c_{12} & \dots & c_{d2} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & c_{1d} & \dots & c_{dd} \end{pmatrix}.$$

The matrix M_σ is similar to a matrix in Jordan form. Let this matrix be denoted

$$J_\sigma = \begin{pmatrix} J_1(\lambda_1) & & & \\ & J_2(\lambda_2) & & \\ & & \ddots & \\ & & & J_k(\lambda_k) \end{pmatrix}.$$

where we let λ_i be the eigenvalue of the Jordan block $J_i(\lambda_i)$ with $\lambda_k = 1$ and we let the size of each Jordan block be m_i. So we have that $\sum_{i=1}^{k} m_i = d + 1$. Then $A[z, z^{-1}; \sigma]$ is a skew Laurent polynomial ring.

Let G denote the multiplicative group generated by $\lambda_1, \ldots, \lambda_k$ where each $\lambda_i \in \mathbb{C}^*$ with λ_i an eigenvalue of M_σ. Then G is a finitely generated abelian group and hence $G \cong \mathbb{Z}^r \oplus T$, with T a finite abelian group.

Definition 3.1.1. A subspace W of V is *σ-irreducible* if W cannot be decomposed as a direct sum $W_1 \oplus W_2$ of proper σ-stable subspaces of W.

Example 3.1.2. Suppose we have the skew Laurent polynomial ring $A = \mathbb{C}[x, y][z, z^{-1}; \sigma]$ where σ is a $\mathbb{C}$-algebra automorphism of A defined by $\sigma(x) = x + y$ and $\sigma(y) = y + 1$. Then the matrix of σ relative to the basis $\{1, x, y\}$ is the matrix in Jordan form

$$M_x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now consider the polynomial $f = \binom{y}{2} - x = \frac{1}{2}y^2 - \frac{1}{2}y - x \in \mathbb{C}[x, y]$.

$$\sigma(f) = \frac{1}{2}(y+1)^2 - \frac{1}{2}(y+1) - (x+y) = f.$$

Thus f is fixed by σ. We have that $f = \binom{y}{2} - x$ and $x = \binom{y}{2} - f$ so $\mathbb{C}[f, y] \subseteq \mathbb{C}[x, y]$ and $\mathbb{C}[x, y] \subseteq \mathbb{C}[f, y]$ so the two rings are equal.

Suppose that $W = \mathbb{C} \oplus \mathbb{C}f \oplus \mathbb{C}y$. Let $W_1 = \mathbb{C}f$ and $W_2 = \mathbb{C} \oplus \mathbb{C}y$. Then $\sigma(W_1) = W_1$ and $\sigma(W_2) = W_2$. Thus W_1 and W_2 are σ-stable subspaces of W so W is not σ-irreducible.

Our main goal is to prove a structure theorem for skew polynomial rings. This is Theorem 3.1.8, which appears at the end of this section. To prove Theorem 3.1.8 we will need the following results Lemma 3.1.3, Proposition 3.1.5, Lemma 3.1.6 and Theorem 3.1.7.

For Lemma 3.1.3 and Theorem 3.1.7 let $A = \mathbb{C}[y_1, \dots, y_d]$ be a finitely generated $\mathbb{C}$-algebra, let $V = \mathbb{C} \oplus \mathbb{C}y_1 \oplus \cdots \oplus \mathbb{C}y_d$ and let σ be a $\mathbb{C}$-algebra automorphism of A such that $\sigma(V) = V$ and the matrix of σ relative to the basis $\{1, y_1, \dots, y_d\}$ is similar to J_σ. Let the group of eigenvalues of J_σ be $G = \langle \lambda_1, \dots, \lambda_k \rangle$.

Let $e = y_1 \cdots y_d$ and suppose that $\sigma(y_i) = \lambda_i y_i$. We have that every element of $A[z, z^{-1}; \sigma]$ is of the form $\sum_{i=-m_1}^{m_2} a_i z^i$ for some $m_1, m_2 \in \mathbb{N}$ and $a_i \in A$. Since every λ_i is nonzero we have

$$e \sum_{i=-m_1}^{m_2} a_i z^i = \sum_{i=-m_1}^{m_2} a_i z^i \sigma^{-i}(e) = \sum_{i=-m_1}^{m_2} a_i (\lambda_1 \cdots \lambda_d)^{-i} z^i e \in A[z, z^{-1}; \sigma]e$$

and

$$\sum_{i=-m_1}^{m_2} a_i z^i e = \sum_{i=-m_1}^{m_2} a_i \sigma^i(e) z^i = e \sum_{i=-m_1}^{m_2} a_i (\lambda_1 \cdots \lambda_d)^i z^i \in eA[z, z^{-1}; \sigma].$$

Hence $eA[z, z^{-1}; \sigma] \subseteq A[z, z^{-1}; \sigma]e$ and $A[z, z^{-1}; \sigma]e \subseteq eA[z, z^{-1}; \sigma]$ so e is a normal element.

Now let $S = \{e^n | n \geq 0\}$ be the multiplicative set of nonnegative powers of e in $A[z, z^{-1}; \sigma]$. Since A is a commutative ring with 1 and e is not nilpotent, e becomes a unit in $S^{-1}A$. Then the ring $S^{-1}A$ is the localization of A at S and we define the ring

$$A_e[z, z^{-1}; \sigma] := S^{-1}A[z, z^{-1}; \sigma] = \mathbb{C}[y_1^{\pm 1}, \dots, y_d^{\pm 1}][z, z^{-1}; \sigma].$$

This gives us the inclusion map $A \hookrightarrow A_e$.

Lemma 3.1.3. *Let $A = \mathbb{C}[y_1, \dots, y_d]$ be a finitely generated $\mathbb{C}$-algebra and let σ be a $\mathbb{C}$-algebra automorphism of A such that $\sigma(V) = V$ and the matrix of σ relative to the basis $\{1, y_1, \dots, y_d\}$ is similar to J_σ with all Jordan blocks of size one. If G is torsion free then there exists a normal element $e \in A$ and we have that*

$$A_e[z, z^{-1}; \sigma] \cong \mathbb{C}[u_1^{\pm 1}, \dots, u_m^{\pm 1}][z, z^{-1}; \sigma][t_1^{\pm 1}, \dots, t_{d-m}^{\pm 1}],$$

such that $\sigma(u_i) = \mu_i u_i$ and $\mu_1, \dots, \mu_m$ generate a free abelian group of rank m.

Proof. Since every Jordan block of J_σ is of size one and $G = \langle \lambda_1, \dots, \lambda_k \rangle$ such that $\lambda_k = 1$ we must have that $k = d + 1$. Also we must have that V has a basis consisting of eigenvectors of J_σ. Let $y_1, \dots, y_d, y_{d+1} = 1$ be this basis. Then $\sigma(y_i) = \lambda_i y_i$ with $\lambda_i \in \mathbb{C}^*$.

We can identify $\mathbb{Z}^d$ with the group generated by $y_1^{\pm 1}, \ldots, y_d^{\pm 1}$ under multiplication and by assumption we have that $G = \langle \lambda_1, \ldots, \lambda_k \rangle$ is torsion free. Then we have the surjective map

$$\phi : \mathbb{Z}^d \to G \quad \text{given by} \quad \phi(y_i) = \lambda_i \quad \text{for } 1 \leq i \leq d.$$

Since G is free we have that the short exact sequence

$$0 \to \ker \phi \to \mathbb{Z}^d \to G \to 0$$

splits. Therefore there exists a section $s : G \to \mathbb{Z}^d$ such that $\phi \circ s$ is the identity on G and $\mathbb{Z}^d = \ker \phi \oplus s(G)$.

Let m be the rank of G and let $\mu_1, \ldots, \mu_m$ be a basis for G. We have that $\ker \phi \cong \mathbb{Z}^{d-m}$, so let $t_1, \ldots, t_{d-m}$ be a basis for $\ker \phi$. We have that each $t_i = y_1^{\alpha_{i,1}} \cdots y_d^{\alpha_{i,d}}$ for some $\alpha_{i,j} \in \mathbb{Z}$ with $i \leq d$ and

$$\sigma(t_i) = \sigma(y_1^{\alpha_{i,1}} \cdots y_d^{\alpha_{i,d}}) = \lambda_1^{\alpha_{i,1}} \cdots \lambda_d^{\alpha_{i,d}} y_1^{\alpha_{i,1}} \cdots y_d^{\alpha_{i,d}} = y_1^{\alpha_{i,1}} \cdots y_d^{\alpha_{i,d}} = t_i.$$

So each t_i is σ-fixed.

Since $\mathbb{Z}^d = \ker \phi \oplus s(G)$ we have that $t_i^{\pm 1}, \ldots, t_{d-m}^{\pm 1}, s(\mu_1)^{\pm 1}, \ldots, s(\mu_m)^{\pm 1}$ is a basis for $\langle y_1^{\pm 1}, \ldots, y_d^{\pm 1} \rangle$. If we let $u_i = s(\mu_i)$ then we have that

$$\begin{aligned} A_e[z, z^{-1}; \sigma] &= \mathbb{C}[y_1^{\pm 1}, \ldots, y_d^{\pm 1}][z, z^{-1}; \sigma] \\ &= \mathbb{C}[t_1^{\pm 1}, \ldots, t_{d-m}^{\pm 1}, u_1^{\pm 1}, \ldots, u_m^{\pm 1}][z, z^{-1}; \sigma] \\ &= \mathbb{C}[u_1^{\pm 1}, \ldots, u_m^{\pm 1}][z, z^{-1}; \sigma][t_1^{\pm 1}, \ldots, t_{d-m}^{\pm 1}]. \end{aligned}$$

such that $\sigma(u_i) = \mu_i u_i$ and $\mu_1, \ldots, \mu_m$ generate a free abelian group of rank m. $\square$

We demonstrate this case in the following example.

Example 3.1.4. Let $A = \mathbb{C}[y_1, y_2]$ and let σ be $\mathbb{C}$-algebra automorphism of A given by $\sigma(y_i) = \lambda_i y_i$ such that λ_i is not a root of unity and $\lambda_1^2 = \lambda_2$. Then the matrix of σ relative to the basis $\{1, y_1, y_2\}$ is the matrix in Jordan form

$$M_\lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $G = \langle \lambda_2 \rangle$ since $\lambda_2^2 = \lambda_1$. If we let $e = y_1 y_2$ then $A_e[z, z^{-1}; \sigma] = \mathbb{C}[y_1^{\pm 1}, y_2^{\pm 1}][z, z^{-1}; \sigma]$. Now consider the rational function $f := \frac{y_1^2}{y_2} \in A_e$. We have that

$$\sigma(f) = \sigma\left(\frac{y_1^2}{y_2}\right) = \frac{\lambda_1^2 y_1^2}{\lambda_2 y_2} = \frac{y_1^2}{y_2} = f.$$

Thus f is fixed by σ.

As above we can identify $\mathbb{Z}^2$ with the group generated by $y_1^{\pm 1}, y_2^{\pm 1}$ under multiplication. Then we have the surjective map

$$\phi : \mathbb{Z}^2 \to G \quad \text{given by} \quad \phi : y_i \mapsto \lambda_i.$$

We have the relation $(\lambda_1)^2(\lambda_2)^{-1} = 1$ so $\ker \phi$ is nontrivial. This relation corresponds to f so f is a basis for $\ker \phi$. Since G is free, the short exact sequence

$$0 \to \ker \phi \to \mathbb{Z}^2 \to G \to 0$$

splits and there exists a section $s : G \to \mathbb{Z}^2$ such that $s(\lambda_2) = y_2$ and $\mathbb{Z}^2 = \ker \phi \oplus s(G)$. Then $f^{\pm 1}, y_2^{\pm 1}$ is a basis for $\langle y_1^{\pm 1}, y_2^{\pm 1} \rangle$ and

$$A_e[z, z^{-1}; \sigma] = \mathbb{C}[f^{\pm 1}, y_2^{\pm 1}][z, z^{-1}; \sigma] = \mathbb{C}[y_2^{\pm 1}][z, z^{-1}; \sigma][f^{\pm 1}].$$

Let $A = \mathbb{C}[w_1, \ldots, w_{m-1}]$ be a finitely generated $\mathbb{C}$-algebra with $m > 2$ and let σ be a $\mathbb{C}$-algebra automorphism of A such that $\sigma(w_1) = w_1 + 1$ and $\sigma(w_i) = w_i + w_{i-1}$ for $2 \le i \le m - 1$. Then we can represent σ by the $m \times m$ matrix relative to the basis $\{1, w_1, \ldots, w_{m-1}\}$ as

$$M_1 = \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}.$$

Proposition 3.1.5. *Let $A = \mathbb{C}[w_1, \ldots, w_{m-1}]$ be a finitely generated $\mathbb{C}$-algebra with $m > 2$ and let σ be a $\mathbb{C}$-algebra automorphism of A such that the matrix of σ relative to the basis $\{1, w_1, \ldots, w_{m-1}\}$ is the $m \times m$ matrix M_1 with skew Laurent polynomial ring $A[z, z^{-1}; \sigma]$. Then for $2 \le k \le m - 1$ the polynomial*

$$p = \binom{w_1}{k} + \sum_{i=1}^{k-1} (-1)^i \frac{w_{i+1}}{k-1} \binom{w_1 - (i+1)}{k - (i+1)}$$

is σ-fixed.

Proof. For this proof we will make use of Pascal's rule, $\binom{n}{k}+\binom{n}{k+1}=\binom{n+1}{k+1}$.

$$\begin{aligned}
\sigma(p) &= \sigma\left(\binom{w_1}{k}+\sum_{i=1}^{k-1}(-1)^i\frac{w_{i+1}}{k-1}\binom{w_1-(i+1)}{k-(i+1)}\right)\\
&= \binom{w_1+1}{k}+\sum_{i=1}^{k-1}(-1)^i\frac{w_{i+1}+w_i}{k-1}\binom{w_1-i}{k-(i+1)}\\
&= \binom{w_1+1}{k}-\frac{w_1}{k-1}\binom{w_1-1}{k-2}-\frac{w_2}{k-1}\binom{w_1-1}{k-2}\\
&\quad+\sum_{i=2}^{k-1}(-1)^i\frac{w_{i+1}+w_i}{k-1}\binom{w_1-i}{k-(i+1)}\\
&= \binom{w_1}{k}-\frac{w_2}{k-1}\binom{w_1-1}{k-2}+\frac{w_2}{k-1}\binom{w_1-2}{k-3}\\
&\quad+\sum_{i=2}^{k-1}(-1)^i\frac{w_{i+1}}{k-1}\left(\binom{w_1-i}{k-(i+1)}-\binom{w_1-(i+1)}{k-(i+2)}\right)\\
&= \binom{w_1}{k}-\frac{w_2}{k-1}\binom{w_1-2}{k-2}+\sum_{i=2}^{k-1}(-1)^i\frac{w_{i+1}}{k-1}\binom{w_1-(i+1)}{k-(i+1)}\\
&= \binom{w_1}{k}+\sum_{i=1}^{k-1}(-1)^i\frac{w_{i+1}}{k-1}\binom{w_1-(i+1)}{k-(i+1)}=p. \quad \square
\end{aligned}$$

Proposition 3.1.5 gives us that every Jordan block of size $m\geq 2$ with eigenvalue 1 has $m-2$ σ-fixed elements. We will denote them as follows. For $1\leq k\leq m-2$ let

$$v_k := \binom{w_1}{k+1}+\sum_{i=1}^{k-1}(-1)^i\frac{w_{i+1}}{k}\binom{w_1-(i+1)}{k-1}+(-1)^k\frac{w_{k+1}}{k}. \tag{3.1}$$

Lemma 3.1.6. *If we define v_k as above then the rings $A=\mathbb{C}[w_1,\ldots,w_{m-1}]$ and $B=\mathbb{C}[w_1,v_1,\ldots,v_{m-2}]$ are equal.*

Proof. We will show that $A\subseteq B$ and $B\subseteq A$. Since $w_1\in A\cap B$ to prove this lemma it suffices to show that $v_k\in A$ for $1\leq k\leq m-2$ and $w_i\in B$ for $2\leq i\leq m-1$. From the equations in (3.1) for $1\leq k\leq m-2$ we have that v_k can be written as a polynomial in A. Thus $B\subseteq A$.

We will show by induction starting with $i = 2$ that each variable w_i can be expressed as some polynomial $f_i(w_1, v_1, \ldots, v_{i-1}) \in B$. Setting $i = 2$ in (3.1) gives $v_1 = \binom{w_1}{2} - w_2$, which we can rearrange to get $w_2 = \binom{w_1}{2} - v_1 = f_2$. Now assume that for all $i \leq s$ that w_i can be represented as an element in B which we will denote by f_i. From our equations in (3.1) we have,

$$v_s := \binom{w_1}{s+1} + \sum_{i=1}^{s-1} \frac{(-1)^i w_{i+1}}{s} \binom{w_1 - (i+1)}{s-1} + \frac{(-1)^s w_{s+1}}{s}, \tag{3.2}$$

and by our assumption, for $i \leq 2, w_i$ can be written as f_i. Thus (3.2) becomes

$$v_s = \binom{w_1}{s+1} + \sum_{i=1}^{s-1} \frac{(-1)^i f_{i+1}}{s} \binom{w_1 - (i+1)}{s-1} + \frac{(-1)^s w_{s+1}}{s},$$

which after rearranging gives

$$w_{s+1} = s(-1)^s \left(v_s - \binom{w_1}{s+1} - \sum_{i=1}^{s-1} \frac{(-1)^i f_{i+1}}{s} \binom{w_1 - (i+1)}{s-1} \right),$$

which is an element in B. By Proposition 3.1.5, $m - 2$ such v_k's exist. Hence for $2 \leq i \leq m-1$, w_i can be represented by a corresponding $f_i(w_1, v_1, \ldots, v_{i-1}) \in B$. Hence $A \subseteq B$. $\square$

Theorem 3.1.7. *Let $A = \mathbb{C}[x_1, \ldots, x_d]$ be a finitely generated $\mathbb{C}$-algebra and let σ be a $\mathbb{C}$-algebra automorphism of A such that $\sigma(V) = V$ and the matrix of σ relative to the basis $\{1, x_1, \ldots, x_d\}$ is similar to J_σ. If G is torsion free then there exists a normal element $e \in A$ such that*

$$A_e[z, z^{-1}; \sigma] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}][z, z^{-1}; \sigma][t_1^{\pm 1}, \ldots, t_p^{\pm 1}],$$

where $d = m + p$ and all the Jordan blocks of J_σ are of size one or

$$A_e[z, z^{-1}; \sigma] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}, x][z, z^{-1}; \sigma][t_1^{\pm 1}, \ldots, t_p^{\pm 1}],$$

where $d = m + p + 1$ and J_σ has at least one Jordan block of size two or more and $\sigma(y_i) = \lambda_i y_i, \sigma(x) = x + 1$ and $\lambda_1, \ldots, \lambda_m$ generate a free abelian group of rank m.

Proof. We have that $V = \mathbb{C} \oplus \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_d$ can also be expressed as $V = V_1 \oplus \cdots \oplus V_k$ where each V_i corresponds to $J_i(\lambda_i)$ and the size m_i of $J_i(\lambda_i)$ equals the dimension of V_i.

The case where all the Jordan blocks of J_σ are of size one is proved in Lemma 3.1.3 so now we will show the case where at least one Jordan block of J_σ is of size two or more. We will prove this case by induction on the number of Jordan blocks. First we will show that the theorem holds for one Jordan block $V_1 \subseteq V$ of size m. Then we will show that the theorem holds for $j+1$ Jordan blocks $V_1 \oplus \cdots \oplus V_{j+1} \subseteq V$ where $j+1 \leq k$ by first considering the case where $j+1 = k$ and then considering the case where $j+1 < k$.

Now we will show that the theorem holds for one Jordan block $V_1 \subseteq V$. Let V_1 have dimension m and let $\{y_0, \ldots, y_{m-1}\}$ be a basis for V_1 such that for some $\lambda \in \mathbb{C}^*$, $\sigma(y_0) = \lambda y_0$ and $\sigma(y_i) = \lambda y_i + y_{i-1}$ for $1 \leq i \leq m-1$. Thus the matrix of $\sigma_{|V_1}$ relative to the given basis is the $m \times m$ matrix in Jordan form

$$M_{\sigma_{|V_1}} = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}.$$

Then we have the $\mathbb{C}$-subalgebra

$$A_1 = \mathbb{C}[y_0, \ldots, y_{m-1}] \subseteq A.$$

We may assume that $\dim(V_1) \geq 2$ since the case where $\dim(V_1) = 1$ is proved in Lemma 3.1.3. We have that the normal element y_0 is an eigenvector of V_1 so we can let $e = y_0$ and form $A_{1_e}[z, z^{-1}; \sigma]$ by localizing A_1 at $S = \{e^n | n \geq 0\}$ to give

$$A_{1_e}[z, z^{-1}; \sigma] = \mathbb{C}[y_0^{\pm 1}, y_1, \ldots, y_{m-1}][z, z^{-1}; \sigma].$$

We can make the substitution $w_i = \frac{\lambda^i y_i}{y_0}$ for $1 \leq i \leq m-1$. Then

$$\sigma(w_1) = \sigma\left(\frac{\lambda y_1}{y_0}\right) = \frac{\lambda(\lambda y_1 + y_0)}{\lambda y_0} = \frac{\lambda y_1}{y_0} + 1 = w_1 + 1,$$

and for $2 \leq i \leq m-1$

$$\sigma(w_i) = \sigma\left(\frac{\lambda^i y_i}{y_0}\right) = \frac{\lambda^i(\lambda y_i + y_{i-1})}{\lambda y_0} = \frac{\lambda^i y_i}{y_0} + \frac{\lambda^{i-1} y_{i-1}}{y_0} = w_i + w_{i-1}.$$

Since the substitution above only involves inversion of y_0 we have that the rings $\mathbb{C}[y_0^{\pm 1}, y_1, \ldots, y_{m-1}]$ and $\mathbb{C}[y_0^{\pm 1}, w_1, \ldots, w_{m-1}]$ are equal. From Proposition 3.1.5 and Lemma 3.1.6 we have that

$$\begin{aligned} A_{1_e}[z, z^{-1}; \sigma] &= \mathbb{C}[y_0^{\pm 1}, w_1, \ldots, w_{m-1}][z, z^{-1}; \sigma] \\ &= \mathbb{C}[y_0^{\pm 1}, w_1, v_1, \ldots, v_{m-2}][z, z^{-1}; \sigma] \\ &= \mathbb{C}[y_0^{\pm 1}, w_1][z, z^{-1}; \sigma][v_1, \ldots, v_{m-2}]. \end{aligned}$$

Since $v_1, \ldots, v_{m-2}$ are σ-fixed and central elements are normal we can let $e' = y_0 v_1 \cdots v_{m-2}$ and form $A_{e'}[z, z^{-1}; \sigma]$ by localizing A_1 at $S = \{(e')^n | n \geq 0\}$ to give

$$A_{e'}[z, z^{-1}; \sigma] = \mathbb{C}[y_0^{\pm 1}, w_1][z, z^{-1}; \sigma][v_1^{\pm 1}, \ldots, v_{m-2}^{\pm 1}].$$

The result follows with $A_e := A_{e'}, x := w_1, t_i := v_i$ and $p := m - 2$.

Now assume that the theorem holds for $i \leq j$ Jordan blocks and let $W = V_1 \oplus \cdots \oplus V_j$ such that each V_i is σ-stable and σ-irreducible. Let $M_{\sigma_{|W}}$ be the matrix in Jordan form with j Jordan blocks by restricting σ to W and let $B \subseteq A$ be the $\mathbb{C}$-subalgebra generated by W. Then there exists an $e \in B$ that is normal such that we can form $B_e[z, z^{-1}; \sigma]$ by localizing B at $S = \{e^n | n \geq 0\}$ to give

$$B_e[z, z^{-1}; \sigma] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}][z, z^{-1}; \sigma][t_1^{\pm 1}, \ldots, t_q^{\pm 1}],$$

where $\dim(W) = m + q$ and all the Jordan blocks of $M_{\sigma_{|W}}$ are of size one or

$$B_e[z, z^{-1}; \sigma] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}, x][z, z^{-1}; \sigma][t_1^{\pm 1}, \ldots, t_q^{\pm 1}],$$

where $\dim(W) = m + q + 1$ and $M_{\sigma_{|W}}$ has at least one Jordan block of size two or more and $\sigma(y_i) = \lambda_i y_i$ and $\sigma(x) = x + 1$ such that $\lambda_1, \ldots, \lambda_m$ generate a free abelian group of rank m.

Now we will show that the theorem holds for $j + 1$ Jordan blocks. Suppose first that $j + 1 = k$ and $\dim(V_{j+1}) = n$. Since the k-th Jordan block of J_σ has an eigenvalue of one, $\{1, w_1, \ldots, w_{n-1}\}$ is a basis for V_{j+1} and the matrix of $\sigma_{|V_{j+1}}$ relative to the given basis is the $n \times n$ matrix M_1 with $\sigma(w_1) = w_1 + 1$ and $\sigma(w_i) = w_i + w_{i-1}$ for $2 \leq i \leq n - 1$.

If $n = 1$ then V_{j+1} is generated by $\{1\}$ so $A_e[z, z^{-1}; \sigma] = B_e[z, z^{-1}; \sigma]$. If $n \geq 2$ then by Proposition 3.1.5 and Lemma 3.1.6 we have that

$$\mathbb{C}[w_1, w_2^{\pm 1} \ldots, w_{n-1}^{\pm 1}] = \mathbb{C}[w_1, v_1^{\pm 1}, \ldots, v_{n-2}^{\pm 1}].$$

Since $v_1, \dots, v_{n-2}$ are σ-fixed and central elements are normal we can let $e' = ev_1 \cdots v_{m-2}$ and form $A_{e'}[z, z^{-1}; \sigma]$ by localizing A at $S = \{(e')^n | n \geq 0\}$. This gives us that

$$A_{e'}[z, z^{-1}; \sigma] \cong B_e[w_1][z, z^{-1}; \sigma][v_1^{\pm 1}, \dots, v_{n-2}^{\pm 1}].$$

If $M_{\sigma_{|W}}$ has all Jordan blocks of size one then

$$A_{e'}[z, z^{-1}; \sigma] \cong \mathbb{C}[y_1^{\pm 1}, \dots, y_m^{\pm 1}, w_1][z, z^{-1}; \sigma][v_1^{\pm 1}, \dots, v_{n-2}^{\pm 1}, t_1^{\pm 1}, \dots, t_q^{\pm 1}],$$

and the result follows with $A_e := A_{e'}$, $x := w_1$ and $p := q + n - 2$. If

$$B_e[z, z^{-1}; \sigma] \cong \mathbb{C}[y_1^{\pm 1}, \dots, y_m^{\pm 1}, x][z, z^{-1}; \sigma][t_1^{\pm 1}, \dots, t_q^{\pm 1}],$$

we have that $\sigma(x) = x + 1$ and $\sigma(w_1) = w_1 + 1$. Let $t_0 := x - w_1$ and let $e'' = e't_0$. Then we have that t_0 is σ-fixed and e'' is normal so we can form $A_{e''}[z, z^{-1}; \sigma]$ by localizing A at $S = \{(e'')^n | n \geq 0\}$. This gives us that

$$A_{e''}[z, z^{-1}; \sigma] \cong \mathbb{C}[y_1^{\pm 1}, \dots, y_m^{\pm 1}, x][z, z^{-1}; \sigma][t_0^{\pm 1}, v_1^{\pm 1}, \dots, v_{n-2}^{\pm 1}, t_1^{\pm 1}, \dots, t_q^{\pm 1}],$$

and the result follows with $A_e := A_{e''}$ and $p := q + n - 1$.

Now suppose $j + 1 < k$ and let $\{y_0, \dots, y_{n-1}\}$ be a basis for V_{j+1}. Then the matrix of $\sigma_{|V_{j+1}}$ relative to the given basis is the $n \times n$ matrix $M_{\sigma_{|V_{j+1}}}$ such that $\sigma(y_0) = \lambda_{j+1} y_0$ and $\sigma(y_i) = \lambda_{j+1} y_i + y_{i-1}$ for $1 \leq i \leq n - 1$.

Consider the group $H = \langle \lambda_1, \dots, \lambda_m, \lambda_{j+1} \rangle$. H is a subgroup of G and hence a free abelian group. Let $\{\mu_1, \dots, \mu_r\}$ be a basis for H where $r = m$ or $m + 1$. Let $A_{j+1} \subseteq A$ be the $\mathbb{C}$-subalgebra generated by $W \oplus V_{j+1}$.

As above we have that

$$\mathbb{C}[y_0^{\pm 1}, y_1, y_2^{\pm 1}, \dots, y_{n-1}^{\pm 1}] = \mathbb{C}[y_0^{\pm 1}, w_1, w_2^{\pm 1}, \dots, w_{n-1}^{\pm 1}] = \mathbb{C}[y_0^{\pm 1}, w_1, v_1^{\pm 1}, \dots, v_{n-2}^{\pm 1}].$$

We can let $e' = ey_0 v_1 \cdots v_{m-2}$ and since e is normal we can form $B_{e'} \subset A_e$ by localizing A_{j+1} at $S = \{(e')^n | n \geq 0\}$. It follows from Lemma 3.1.3, Proposition 3.1.5 and Lemma 3.1.6 that if $M_{\sigma_{|W}}$ has all Jordan blocks of size one then we have

$$\begin{aligned} B_{e'}[z, z^{-1}; \sigma] &\cong B_e[y_0^{\pm 1}, w_1, v_1^{\pm 1}, \dots, v_{n-2}^{\pm 1}][z, z^{-1}; \sigma] \\ &\cong \mathbb{C}[u_1^{\pm 1}, \dots, u_r^{\pm 1}, w_1][z, z^{-1}; \sigma][v_1^{\pm 1}, \dots, v_{n-2}^{\pm 1}, t_1^{\pm 1}, \dots, t_q^{\pm 1}], \end{aligned}$$

such that $\sigma(u_i) = \mu_i u_i$ and $\mu_1, \ldots, \mu_r$ generate a free abelian group of rank r, and the result follows.

If $M_{\sigma_{|W}}$ has at least one Jordan block of size two or more then we have

$$\begin{aligned} B_{e'}[z, z^{-1}; \sigma] &\cong B_e[y_0^{\pm 1}, w_1, v_1^{\pm 1}, \ldots, v_{n-2}^{\pm 1}][z, z^{-1}; \sigma] \\ &\cong \mathbb{C}[u_1^{\pm 1}, \ldots, u_r^{\pm 1}, x][z, z^{-1}; \sigma][v_1^{\pm 1}, \ldots, v_{n-2}^{\pm 1}, t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_q^{\pm 1}], \end{aligned}$$

such that $\sigma(u_i) = \mu_i u_i$ and $u_1, \ldots, u_r$ generate a free abelian group of rank r and t_0 is defined as in the $j + 1 = k$ case and the result follows. □

For Theorem 3.1.8 let $n \in \mathbb{N}$ and let the matrix in Jordan form similar to the matrix of σ^n relative to the basis $\{x_1, \ldots, x_d\}$ be denoted J_{σ^n}. Let $A[z^n, z^{-n}; \sigma^n]$ be a subring of $A[z, z^{-1}; \sigma]$ and let $G = \langle \lambda_1, \ldots, \lambda_k \rangle$ be as above.

Theorem 3.1.8. *Let $A = \mathbb{C}[x_1, \ldots, x_d]$ be a finitely generated $\mathbb{C}$-algebra and let σ be a $\mathbb{C}$-algebra automorphism of A. There exists an $n \in \mathbb{N}$ and a normal element $e \in A$ such that*

$$A_e[z^n, z^{-n}; \sigma^n] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}][z^n, z^{-n}; \sigma^n][t_1^{\pm 1}, \ldots, t_p^{\pm 1}],$$

where $d = m + p$ and all the Jordan blocks of J_{σ^n} are of size one or

$$A_e[z^n, z^{-n}; \sigma^n] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}, x][z^n, z^{-n}; \sigma^n][t_1^{\pm 1}, \ldots, t_p^{\pm 1}],$$

where $d = m + p + 1$ and J_{σ^n} has at least one Jordan block of size two or more and $\sigma^n(y_i) = \lambda_i^n y_i, \sigma^n(x) = x + 1$ and $\lambda_1^n, \ldots, \lambda_m^n$ generate a free abelian group of rank m.

Proof: Let σ be a $\mathbb{C}$-algebra automorphism of $A = \mathbb{C}[x_1, \ldots, x_d]$. Suppose the group of eigenvalues of M_σ generated by $\lambda_1, \ldots, \lambda_k$ is not torsion free. We have that $G \cong \mathbb{Z}^r \oplus T$ with

$$T = \oplus_{i=1}^s \mathbb{Z}/n_i\mathbb{Z} \quad \text{for some } n_i, s \in \mathbb{N}.$$

Let $n = \operatorname{lcm}(n_1, \ldots, n_s)$. Then $\langle \lambda_1^n, \ldots, \lambda_k^n \rangle$ is now a torsion free abelian group. If G is already a torsion free abelian group then $n = 1$. Since σ^n is a $\mathbb{C}$-algebra automorphism of A the subring $A[z^n, z^{-n}; \sigma^n] \subseteq A[z, z^{-1}; \sigma]$ satisfies the hypotheses of Theorem 3.1.7 so there is a normal element $e \in A$, such that A_e can be formed by inverting powers of e and we have that

$$\begin{array}{ccc} A[z, z^{-1}; \sigma] & & \\ \cup\!\!\mid & & \\ A[z^n, z^{-n}; \sigma^n] & \hookrightarrow & A_e[z^n, z^{-n}; \sigma^n], \end{array}$$

where

$$A_e[z^n, z^{-n}; \sigma^n] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}][z^n, z^{-n}; \sigma^n][t_1^{\pm 1}, \ldots, t_p^{\pm 1}],$$

if all the Jordan blocks of J_{σ^n} are of size one or

$$A_e[z^n, z^{-n}; \sigma^n] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}, x][z^n, z^{-n}; \sigma^n][t_1^{\pm 1}, \ldots, t_p^{\pm 1}],$$

if J_{σ^n} has at least one Jordan block of size two or more and $\sigma^n(y_i) = \lambda_i^n y_i, \sigma^n(x) = x + 1$ and $\lambda_1^n, \ldots, \lambda_m^n$ generate a free abelian group of rank m and the result follows. $\square$

Chapter 4

A Dixmier-Moeglin equivalence for skew Laurent polynomial rings

Our goal for this chapter is to use the structure theorems of low growth skew Laurent polynomial rings, Theorem 3.1.7 and Theorem 3.1.8 from the previous chapter, to show that low growth skew Laurent polynomial rings have a Dixmier-Moeglin equivalence.

4.1 Simplicity of certain skew Laurent polynomial rings

For this section we will let $A_1 = \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}]$ and let $A_2 = \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}, x]$ To prove the Dixmier-Moeglin equivalence we will first have to prove the following result about the simplicity of $A_1[z, z^{-1}; \sigma]$ and $A_2[z, z^{-1}; \sigma]$.

Proposition 4.1.1. *Let $\sigma : A_2 \to A_2$ be the $\mathbb{C}$-algebra automorphism given by $\sigma(x) = x+1$ and $\sigma(y_i) = \lambda_i y_i$ where $\lambda_1, \ldots, \lambda_m \in \mathbb{C}^*$ generate a free abelian group of rank m. Let $\widehat{\sigma}$ denote the restriction of σ to A_1. Then $A_1[z, z^{-1}; \widehat{\sigma}][t_1^{\pm 1}, \ldots, t_p^{\pm 1}]$ and $A_2[z, z^{-1}; \sigma][t_1^{\pm 1}, \ldots, t_p^{\pm 1}]$ are simple if and only if $p = 0$.*

Proof. We first handle the $A_2[z, z^{-1}; \sigma][t_1^{\pm 1}, \ldots, t_p^{\pm 1}]$ case. Assume $p = 0$ and that $A_2[z, z^{-1}; \sigma]$ is not simple. Then we have that A_2 is not σ-simple by Proposition 2.2.12. For the ring $\mathbb{C}[y_1, \ldots, y_m, x]$ let us fix a lexicographical monomial order $x > y_1 > \cdots > y_m$. We write $m <_{lex} m'$ if m is smaller than m' in the monomial order.

Let I be a proper nonzero σ-stable ideal of A_2. Then $I \cap \mathbb{C}[y_1, \ldots, y_m, x]$ is nonzero as well by clearing denominators. Let $a \in I \cap \mathbb{C}[y_1, \ldots, y_m, x] \neq (0)$ be the nonzero element with the smallest possible leading monomial in the monomial ordering. Let $y_1^{i_1} \cdots y_m^{i_m} x^t$ denote this leading monomial. Then

$$a = y_1^{i_1} \cdots y_m^{i_m} x^t + \sum c_{j_1,\ldots,j_m,s} \cdot y_1^{j_1} \cdots y_m^{j_m} x^s,$$

where the sum is over all monomials such that $y_1^{j_1} \cdots y_m^{j_m} x^s <_{lex} y_1^{i_1} \cdots y_m^{i_m} x^t$. We have that $\sigma(a) - \lambda_1^{i_1} \cdots \lambda_m^{i_m} a \in I$ and the coefficient of $y_1^{i_1} \cdots y_m^{i_m} x^t$ in $\sigma(a) - \lambda_1^{i_1} \cdots \lambda_m^{i_m} a$ is zero. If $\sigma(a) - \lambda_1^{i_1} \cdots \lambda_m^{i_m} a$ is nonzero then we have found an element in I with a smaller leading monomial in the monomial ordering, a contradiction.

If $t > 0$ we have that

$$\sigma(a) = \sum_{n=1}^{t} \binom{t}{n} \lambda_1^{i_1} \cdots \lambda_m^{i_m} y_1^{i_1} \cdots y_m^{i_m} x^t + \sum c_{j_1,\ldots,j_m,s} \lambda_1^{j_1} \cdots \lambda_m^{j_m} y_1^{j_1} \cdots y_m^{j_m} (x+1)^s.$$

If $c_{i_1,\ldots,i_m,n} = 0$ for $n < t$ then the monomial $y_1^{i_1} \cdots y_m^{i_m} x^n$ will have a nonzero coefficient in $\sigma(a) - \lambda_1^{i_1} \cdots \lambda_m^{i_m} a = 0$, a contradiction. If $c_{i_1,\ldots,i_m,n} \neq 0$ for some $n < t$ then let n' be the maximal n such that $c_{i_1,\ldots,i_m,n'}$ is nonzero. Then we have that the coefficient of $y_1^{i_1} \cdots y_m^{i_m} x^{n'}$ in $\sigma(a) - \lambda_1^{i_1} \cdots \lambda_m^{i_m} a$ is

$$\binom{t}{n'} \lambda_1^{i_1} \cdots \lambda_m^{i_m} + c_{i_1,\ldots,i_m,n'} \lambda_1^{i_1} \cdots \lambda_m^{i_m} - c_{i_1,\ldots,i_m,n'} \lambda_1^{i_1} \cdots \lambda_m^{i_m} \neq 0,$$

a contradiction.

If $t = 0$ we have that

$$\sigma(a) - \lambda_1^{i_1} \cdots \lambda_m^{i_m} a = \sum c_{j_1,\ldots,j_m,s} (\lambda_1^{j_1} \cdots \lambda_m^{j_m} - \lambda_1^{i_1} \cdots \lambda_m^{i_m}) y_1^{j_1} \cdots y_m^{j_m}.$$

If $\sigma(a) - \lambda_1^{i_1} \cdots \lambda_m^{i_m} a = 0$ then $\lambda_1^{j_1} \cdots \lambda_m^{j_m} = \lambda_1^{i_1} \cdots \lambda_m^{i_m}$ for all $y_1^{j_1} \cdots y_m^{j_m} <_{lex} y_1^{i_1} \cdots y_m^{i_m}$. Since $\lambda_1, \ldots, \lambda_m$ is a free abelian group of rank m it must be the case that $j_k = i_k$ for $1 \leq k \leq m$. Thus a has only one term, $y_1^{i_1} \cdots y_m^{i_m}$, which is a unit in A_1 and I is not a proper ideal, a contradiction.

Note that if $t = 0$ then $a \in I \cap \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}]$. Thus the argument above shows that if $p = 0$ then $A_1[z, z^{-1}; \widehat{\sigma}][t_1^{\pm 1}, \ldots, t_p^{\pm 1}]$ is simple.

Let $A_1[z, z^{-1}; \widehat{\sigma}][t_1^{\pm 1}, \ldots, t_p^{\pm 1}]$ and $A_2[z, z^{-1}; \sigma][t_1^{\pm 1}, \ldots, t_p^{\pm 1}]$ be simple and assume $p > 0$. Consider $t_p - a$ with $a \in \mathbb{C}^*$. Then $\sigma(t_p - a) = t_p - a$ and $\widehat{\sigma}(t_p - a) = t_p - a$

so $(t_p - a)$ is a proper nonzero σ-stable ideal of $A_2[t_1^{\pm 1}, \ldots, t_p^{\pm 1}]$ and a proper nonzero $\widehat{\sigma}$-stable ideal of $A_1[z, z^{-1}; \widehat{\sigma}][t_1^{\pm 1}, \ldots, t_p^{\pm 1}]$. Thus $A_1[t_1^{\pm 1}, \ldots, t_p^{\pm 1}]$ and $A_2[t_1^{\pm 1}, \ldots, t_p^{\pm 1}]$ are not $\widehat{\sigma}$-simple and σ-simple respectively and the result follows from Proposition 2.2.12. $\square$

4.2 A Dixmier-Moeglin equivalence for skew Laurent polynomial rings

For all of this section we will assume that $A = \mathbb{C}[x_1, \ldots, x_d]$ is a finitely generated $\mathbb{C}$-algebra and that σ is a $\mathbb{C}$-algebra automorphism of A. We will let J_{σ^n} be the $(d+1)\times(d+1)$ matrix in Jordan form that is similar to the matrix M_{σ^n} relative to the basis $\{1, x_1, \ldots, x_d\}$ such that the group of eigenvalues of J_{σ^n}, $\langle \lambda_1^n, \ldots, \lambda_k^n \rangle$, is torsion free. Then from Theorem 3.1.8 of the previous chapter we know there is an $n \in \mathbb{N}$ and a normal element $e \in A$ such that A_e can be formed by inverting powers of e and we have that

$$\begin{array}{ccc} A[z, z^{-1}; \sigma] & & \\ \cup\!| & & \\ A[z^n, z^{-n}; \sigma^n] & \hookrightarrow & A_e[z^n, z^{-n}; \sigma^n], \end{array}$$

where

$$A_e[z^n, z^{-n}; \sigma^n] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}][z^n, z^{-n}; \sigma^n][t_1^{\pm 1}, \ldots, t_p^{\pm 1}],$$

if all the Jordan blocks of J_{σ^n} are of size one or

$$A_e[z^n, z^{-n}; \sigma^n] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}, x][z^n, z^{-n}; \sigma^n][t_1^{\pm 1}, \ldots, t_p^{\pm 1}],$$

if J_{σ^n} has at least one Jordan block of size two or more and $\sigma^n(y_i) = \lambda_i^n y_i, \sigma^n(x) = x + 1$ and $\lambda_1^n, \ldots, \lambda_m^n$ generate a free abelian group of rank m.

In this section we will determine for what σ do we have that (0) is a primitive, locally closed and rational prime in each of $A[z, z^{-1}; \sigma], A[z^n, z^{-n}; \sigma^n]$ and $A_e[z^n, z^{-n}; \sigma^n]$. As before we let $A_1 = \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}]$ and let $A_2 = \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}, x]$.

Remark 4.2.1. We have that $\mathbb{C}$ is an uncountable field and $A[z, z^{-1}; \sigma], A[z^n, z^{-n}; \sigma^n]$ and $A_e[z^n, z^{-n}; \sigma^n]$ are all finitely generated Noetherian $\mathbb{C}$-algebras. Thus they all satisfy the Nullstellensatz by Proposition 2.4.23.

Theorem 4.2.2. *Let $A_e[z^n, z^{-n}; \sigma^n]$ be as above. Then the following are equivalent.*
1. $A_e[z^n, z^{-n}; \sigma^n] \cong A_1[z^n, z^{-n}; \sigma^n]$ or $A_e[z^n, z^{-n}; \sigma^n] \cong A_2[z^n, z^{-n}; \sigma^n]$.
2. $A_e[z^n, z^{-n}; \sigma^n]$ is simple.
3. $A_e[z^n, z^{-n}; \sigma^n]$ is primitive.
4. (0) is a rational prime.
5. (0) is locally closed in $\mathrm{Spec}(A_e[z^n, z^{-n}; \sigma^n])$.

Proof. Since $A_1[z^n, z^{-n}; \sigma^n] \subseteq A_1[z, z^{-1}; \sigma]$ and $A_2[z^n, z^{-n}; \sigma^n] \subseteq A_2[z, z^{-1}; \sigma]$. It follows from Proposition 4.1.1 that $(1) \Leftrightarrow (2)$.
$(2) \Rightarrow (3)$ This follows from 2.1.6.
$(3) \Rightarrow (4)$ If $A_e[z^n, z^{-n}; \sigma^n]$ is primitive then (0) is a primitive ideal and hence a rational prime by Lemma 2.4.21.
To prove $(4) \Rightarrow (1)$, we will prove the contrapositive. Assume $A_e[z^n, z^{-n}; \sigma^n] \ncong A_1[z^n, z^{-n}; \sigma^n]$ or $\ncong A_2[z^n, z^{-n}; \sigma^n]$. By Theorem 3.1.7 and Theorem 3.1.8 we have that

$$A_e[z^n, z^{-n}; \sigma^n] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}][z^n, z^{-n}; \sigma^n][t_1^{\pm 1}, \ldots, t_p^{\pm 1}],$$

if all the Jordan blocks of J_{σ^n} are of size one or

$$A_e[z^n, z^{-n}; \sigma^n] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}, x][z^n, z^{-n}; \sigma^n][t_1^{\pm 1}, \ldots, t_p^{\pm 1}],$$

if J_{σ^n} has at least one Jordan block of size two or more. This means that p must be nonzero and hence A_e has a σ^n-fixed element $t_p \in A_e$. Thus t_p is in the center of $\mathrm{Fract}(A_e[z^n, z^{-n}; \sigma^n]/(0))$ and is transcendental over $\mathbb{C}$. Hence (0) is not a rational prime.
$(5) \Rightarrow (3)$ This follows from Lemma 2.4.21.
$(2) \Rightarrow (5)$ If $A_e[z^n, z^{-n}; \sigma^n]$ is simple then the only prime ideal of $A_e[z^n, z^{-n}; \sigma^n]$ is (0). Hence $A_e[z^n, z^{-n}; \sigma^n]$ does not have any height one primes and thus (0) is locally closed in $\mathrm{Spec}(A_e[z^n, z^{-n}; \sigma^n])$ by Proposition 2.4.17. $\square$

We can now extend this equivalence to the case without the localization at e.

Theorem 4.2.3. *Let $A_e[z^n, z^{-n}; \sigma^n]$, J_{σ^n} and $A[z^n, z^{-n}; \sigma^n]$ be as above. Then we have that the following are equivalent.*
1. The eigenvalues of J_{σ^n} form a free abelian group of rank m and

$$A_e[z^n, z^{-n}; \sigma^n] \cong A_1[z^n, z^{-n}; \sigma^n],$$

if J_{σ^n} has all Jordan blocks of size one or

$$A_e[z^n, z^{-n}; \sigma^n] \cong A_2[z^n, z^{-n}; \sigma^n],$$

if J_{σ^n} has exactly one Jordan block of size two and the rest of size one.
2. $A[z^n, z^{-n}; \sigma^n]$ is primitive.
3. (0) is a rational prime in $A[z^n, z^{-n}; \sigma^n]$.
4. (0) is locally closed in $\operatorname{Spec}(A[z^n, z^{-n}; \sigma^n])$.

Proof. $(1) \Rightarrow (4)$ If (1) is true then by Theorem 4.2.2, $A_e[z^n, z^{-n}; \sigma^n]$ is simple. Suppose P is a proper nonzero prime ideal of $A[z^n, z^{-n}; \sigma^n]$. P must contain a unit in the ring $A_e[z^n, z^{-n}; \sigma^n]$ otherwise it would be a nonzero proper ideal of $A_e[z^n, z^{-n}; \sigma^n]$ which would contradict the simplicity of $A_e[z^n, z^{-n}; \sigma^n]$. Since e is a product of the eigenvectors $y_1, \ldots, y_k$ of J_{σ^n}, a unit in $A_e[z^n, z^{-n}; \sigma^n]$ which is not a unit in $A[z^n, z^{-n}; \sigma^n]$ is of the form $u = e^r = (y_1 \cdots y_k)^r$ for some $r \in \mathbb{N}$. Thus $u \in P$ for some $r \in \mathbb{N}$. Since P is a prime ideal and e is a normal element at least one of the y_i^r must be in P and hence $y_i \in P$ for some $1 \leq i \leq k$. Hence every prime ideal is contained in some (y_i), all of which are height one primes. Since there are only k of these prime ideals $A[z^n, z^{-n}; \sigma^n]$ must have only finitely many height one primes. Hence by Proposition 2.4.17, (0) is locally closed in $\operatorname{Spec}(A[z^n, z^{-n}; \sigma^n])$.
$(4) \Rightarrow (2) \Rightarrow (3)$ This follows from Lemma 2.4.21.
$(3) \Rightarrow (1)$ We will prove the contrapositive. Assume (1) is not true, by Theorem 3.1.7 and Theorem 3.1.8 we have that

$$A_e[z^n, z^{-n}; \sigma^n] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}][z^n, z^{-n}; \sigma^n][t_1^{\pm 1}, \ldots, t_p^{\pm 1}],$$

where all the Jordan blocks of J_{σ^n} are of size one or

$$A_e[z^n, z^{-n}; \sigma^n] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}, x][z^n, z^{-n}; \sigma^n][t_1^{\pm 1}, \ldots, t_p^{\pm 1}],$$

where J_{σ^n} has at least one Jordan block of size two or more. This means that p must be nonzero and hence A_e has a σ^n-fixed element $t_p \in A_e$. We also have that t_p is an element of $\operatorname{Fract}(A[z^n, z^{-n}; \sigma^n]/(0))$. Thus t_p is in the center of $\operatorname{Fract}(A[z^n, z^{-n}; \sigma^n]/(0))$ and is transcendental over $\mathbb{C}$. Hence (0) is not a rational prime of $A[z^n, z^{-n}; \sigma^n]$. $\square$

Note that in the last proof if $n = 1$ then we would have that the equivalence holds for $A[z, z^{-1}; \sigma]$, the desired result of this chapter. However this is not necessary. To show this we will need Theorem 4.2.4, a result of Letzter, and we will need to consider the three properties of the Dixmier-Moeglin equivalence for a ring R and prime ideal P which we will denote:
(A) P is left primitive.
(B) P is rational.
(C) P is locally closed in $\mathrm{Spec}(R)$.

Theorem 4.2.4. ***(Letzter)*** *Let k be a field and let R be a Noetherian k-algebra with finite GK-dimension. Let S be a finite free extension of R and let P be a prime ideal of R. Then we have the following two results:*
(1) If R has either of the properties $((A)$ implies $(B))$ or $((B)$ implies $(A))$, then S has the same property.
(2) R has the property $((A)$ implies $(C))$ if and only if S does.

Proof. (1) is proved in [22, Corollary 1.5] and (2) is proved in [22, Theorem 2.3] and [22, Theorem 2.4]. □

Theorem 4.2.5. *Let $A_e[z^n, z^{-n}; \sigma^n]$, $A[z^n, z^{-n}; \sigma^n]$, $A[z, z^{-1}; \sigma]$ and J_{σ^n} be as above. Then we have that the following are equivalent.*

1. The eigenvalues of J_{σ^n} form a free abelian group of rank m and

$$A_e[z^n, z^{-n}; \sigma^n] \cong A_1[z^n, z^{-n}; \sigma^n],$$

if J_{σ^n} has all Jordan blocks of size one or

$$A_e[z^n, z^{-n}; \sigma^n] \cong A_2[z^n, z^{-n}; \sigma^n],$$

if J_{σ^n} has exactly one Jordan block of size two and the rest of size one.
2. $A[z, z^{-1}; \sigma]$ is primitive.
3. (0) is a rational prime in $A[z, z^{-1}; \sigma]$.
4. (0) is locally closed in $\mathrm{Spec}(A[z, z^{-1}; \sigma])$.

Proof. We have that $A[z, z^{-1}; \sigma]$ is a finite free extension of $A[z^n, z^{-n}; \sigma^n]$. Since $A[z^n, z^{-n}; \sigma^n]$ has the property that if (0) is a primitive ideal then (0) is locally closed in $\mathrm{Spec}(A[z^n, z^{-n}; \sigma^n])$, from Theorem 4.2.4.2 this property also holds in $A[z, z^{-1}; \sigma]$. Since $A[z^n, z^{-n}; \sigma^n]$ has the property that if (0) is a rational ideal then (0) is a primitive ideal, from Theorem 4.2.4.1 this property also holds in $A[z, z^{-1}; \sigma]$. From Theorem 4.2.3 (0) is a rational ideal in $A[z^n, z^{-n}; \sigma^n]$ if and only if (1) is true. Since we have that $(4) \Rightarrow (2) \Rightarrow (3)$ by Lemma 2.4.21 as before, the result follows. $\square$

Chapter 5

Applications of the Dixmier-Moeglin result

5.1 Transcendence degree

In this section we will use the results of the previous chapter to determine for which A and for which σ the skew Laurent polynomial ring $A[z, z^{-1}; \sigma]$ is simple. We will also determine the transcendence degree of the center of the quotient division ring of $A[z, z^{-1}; \sigma]$.

Remark 5.1.1. *Let $n \in \mathbb{N}$. Since J_{σ^n} is similar to $(J_\sigma)^n$ the Jordan blocks of J_{σ^n} and J_σ have the same size. Therefore it is no loss of generality to replace σ by σ^n for our uses in this chapter in determining the size of the Jordan blocks.*

Theorem 5.1.2. *Let $d > 0$, let $A = \mathbb{C}[x_1, \ldots, x_d]$ be a finitely generated $\mathbb{C}$-algebra and let σ be a $\mathbb{C}$-algebra automorphism of A such that the matrix of σ relative to the basis $\{1, x_1, \ldots, x_d\}$ is similar to the matrix in Jordan form, J_σ. The skew Laurent polynomial ring $A[z, z^{-1}; \sigma]$ is simple if and only if $A = \mathbb{C}[x]$ and $J_\sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.*

Proof. We proved the reverse direction of this proof in Example 2.2.13. If $A[z, z^{-1}; \sigma]$ is not primitive then it can not be simple. From Theorem 4.2.5 we have that $A[z, z^{-1}; \sigma]$ is primitive if and only if there exists an $n \in \mathbb{N}$ such that $(J_\sigma)^n$ has at most one Jordan block of size two and the rest of size one and hence J_{σ^n} has at most one Jordan block of size two and the rest of size one. If J_{σ^n} has an eigenvalue other than one or an eigenvalue of one that does not correspond to the eigenvector 1 then there exists a $y \in A$ such that $\sigma(y) = \lambda y$. Then (y) is a σ-stable ideal, so $A[z, z^{-1}; \sigma]$ is not simple. Thus J_{σ^n} has only one Jordan block and it is of size two since $d > 0$. Thus $A = \mathbb{C}[x]$ and from Remark 5.1.1 it follows that $J_\sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. □

To determine the transcendence degree of the center of the quotient division ring of the ring $A[z, z^{-1}; \sigma]$ we will need the following definitions.

Definition 5.1.3. A subset S of a field ℓ is *algebraically independent* over a subfield k if the elements of S do not satisfy a non-trivial polynomial equation with coefficients in k.

Definition 5.1.4. The *transcendence degree* of a field extension ℓ/k is the largest cardinality of an algebraically independent subset of ℓ over k. We denote this as $\mathrm{tr.deg}_k(\ell)$.

Let $A = \mathbb{C}[x_1, \ldots, x_d]$ be a finitely generated $\mathbb{C}$-algebra and let σ be a $\mathbb{C}$-algebra automorphism of A such that the matrix of σ relative to the basis $\{1, x_1, \ldots, x_d\}$ is similar to the matrix in Jordan form, J_σ. Let each Jordan block of J_σ, $J_i(\lambda_i)$ have size m_i for $1 \leq i \leq k$ and let $\chi : J_i(\lambda_i) \to \mathbb{N}$ be such that $\chi(m_i) = 1$ if $m_i \geq 2$ and $\chi(m_i) = 0$ if $m_i = 1$. Then we have the following theorem.

Theorem 5.1.5. *Let $A = \mathbb{C}[x_1, \ldots, x_d]$ be a finitely generated $\mathbb{C}$-algebra and let σ be a $\mathbb{C}$-algebra automorphism of A such that the matrix of σ relative to the basis $\{1, x_1, \ldots, x_d\}$ is similar to the matrix in Jordan form, J_σ. Then*

$$\mathrm{tr.deg}_{\mathbb{C}}(Z(\mathrm{Fract}(A[z, z^{-1}; \sigma]))) = \sum_{i=1}^{k} (m_i - 1)\chi(m_i) + k - m - 1.$$

Proof. First we will show that

$$\mathrm{tr.deg}_{\mathbb{C}}(Z(\mathrm{Fract}(A[z, z^{-1}; \sigma]))) \geq \sum_{i=1}^{k} (m_i - 1)\chi(m_i) + k - m - 1.$$

From Theorem 3.1.8 we have an $n \in \mathbb{N}$ such that the subring $A[z^n, z^{-n}; \sigma^n] \subseteq A[z, z^{-1}; \sigma]$ has a normal element $e \in A$ such that A_e can be formed by inverting powers of e and we have that

$$\begin{array}{ccc} A[z, z^{-1}; \sigma] & & \\ \cup\!\shortmid & & \\ A[z^n, z^{-n}; \sigma^n] & \hookrightarrow & A_e[z^n, z^{-n}; \sigma^n], \end{array}$$

where

$$A_e[z^n, z^{-n}; \sigma^n] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}][z^n, z^{-n}; \sigma^n][t_1^{\pm 1}, \ldots, t_p^{\pm 1}],$$

if all the Jordan blocks of J_{σ^n} are of size one or

$$A_e[z^n, z^{-n}; \sigma^n] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}, x][z^n, z^{-n}; \sigma^n][t_1^{\pm 1}, \ldots, t_p^{\pm 1}],$$

if J_{σ^n} has at least one Jordan block of size two or more and $\sigma^n(y_i) = \lambda_i^n y_i, \sigma^n(x) = x + 1$ and $\lambda_1^n, \ldots, \lambda_m^n$ generate a free abelian group of rank m.

From Remark 5.1.1 it is no loss of generality to consider the Jordan blocks of J_{σ^n} instead of J_σ.

There are k eigenvalues of J_{σ^n} but the rank of the group $\langle \lambda_1^n, \ldots, \lambda_k^n \rangle = m$. Following the proof of Theorem 3.1.7 this gives us $k - m$ algebraically independent central elements in $\mathrm{Fract}(A[z, z^{-1}; \sigma])$ transcendental over $\mathbb{C}$. Thus

$$\mathrm{tr.deg}_{\mathbb{C}}(Z(\mathrm{Fract}(A[z, z^{-1}; \sigma]))) \geq k - m.$$

From Proposition 3.1.5 and Lemma 3.1.6 we have that for every Jordan block of size $m_i \geq 3$ we obtain $m_i - 2$ algebraically independent nontrivial central elements in $\mathrm{Fract}(A[z, z^{-1}; \sigma])$ transcendental over $\mathbb{C}$. Since there are k Jordan blocks in J_{σ^n} we have that

$$\mathrm{tr.deg}_{\mathbb{C}}(Z(\mathrm{Fract}(A[z, z^{-1}; \sigma]))) \geq \sum_{i=1}^{k} (m_i - 2)\chi(m_i) + k - m.$$

Following the proof of Theorem 3.1.7, if there were two Jordan blocks of size two or more then we obtained a central element in $\mathrm{Fract}(A[z, z^{-1}; \sigma])$ transcendental over $\mathbb{C}$ algebraically independent from any of the other nontrivial central elements. Thus

$$\begin{aligned} \mathrm{tr.deg}_{\mathbb{C}}(Z(\mathrm{Fract}(A[z, z^{-1}; \sigma]))) &\geq \sum_{i=1}^{k} (m_i - 2)\chi(m_i) + \sum_{i=1}^{k} \chi(m_i) - 1 + k - m \\ &= \sum_{i=1}^{k} (m_i - 1)\chi(m_i) + k - m - 1. \end{aligned}$$

From Theorem 3.1.7 and Theorem 3.1.8 we have that

$$\sum_{i=1}^{k}(m_i - 1)\chi(m_i) + k - m - 1 = p.$$

If

$$\operatorname{tr.deg}_{\mathbb{C}}(Z(\operatorname{Fract}(A[z, z^{-1}; \sigma]))) > p$$

where

$$A_e[z^n, z^{-n}; \sigma^n] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}][z^n, z^{-n}; \sigma^n][t_1^{\pm 1}, \ldots, t_p^{\pm 1}],$$

if all the Jordan blocks of J_{σ^n} are of size one or

$$A_e[z^n, z^{-n}; \sigma^n] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}, x][z^n, z^{-n}; \sigma^n][t_1^{\pm 1}, \ldots, t_p^{\pm 1}],$$

if J_{σ^n} has at least one Jordan block of size two or more then it must be the case that

$$\operatorname{tr.deg}_{\mathbb{C}}(Z(\operatorname{Fract}(A[z, z^{-1}; \sigma]))) > 0$$

where

$$A_e[z^n, z^{-n}; \sigma^n] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}][z^n, z^{-n}; \sigma^n],$$

if all the Jordan blocks of J_{σ^n} are of size one or

$$A_e[z^n, z^{-n}; \sigma^n] \cong \mathbb{C}[y_1^{\pm 1}, \ldots, y_m^{\pm 1}, x][z^n, z^{-n}; \sigma^n],$$

if J_{σ^n} has at least one Jordan block of size two or more. From Theorem 4.2.5 we have that in this case (0) is a rational prime of $A[z, z^{-1}; \sigma]$, but if

$$\operatorname{tr.deg}_{\mathbb{C}}(Z(\operatorname{Fract}(A[z, z^{-1}; \sigma]))) > 0$$

then this implies that $Z(\operatorname{Fract}(A[z, z^{-1}; \sigma]))$ has at least one nontrivial central element transcendental over $\mathbb{C}$, this contradicts the rationality of (0). □

5.2 Future directions

In this section we will provide a list of conjectures based on the results of Chapter 4 for future work in this area.

The full Dixmier-Moeglin equivalence result for universal enveloping algebras over $\mathbb{C}$ holds for any prime ideal P, but the result obtained in Theorem 4.2.5 only holds for the prime ideal (0). The proof of Theorem 4.2.5 should be adaptable to accommodate for all prime ideals of the finitely generated $\mathbb{C}$-algebra A. This gives the following conjecture.

Conjecture 5.2.1. *Let $A = \mathbb{C}[x_1, \dots, x_d]$ be a finitely generated $\mathbb{C}$-algebra and let σ be a $\mathbb{C}$-algebra automorphism of A such that σ restricts to a linear automorphism of the vector space $\mathbb{C} + \mathbb{C}x_1 + \cdots + \mathbb{C}x_d$. Then for all prime ideals P of the skew Laurent polynomial ring $A[z, z^{-1}; \sigma]$ the following are equivalent.*
1. P is a primitive ideal of $A[z, z^{-1}; \sigma]$.
3. P is a rational prime in $A[z, z^{-1}; \sigma]$.
4. P is locally closed in $\mathrm{Spec}(A[z, z^{-1}; \sigma])$.

Also, the original Dixmier-Moeglin equivalence, Theorem 2.4.1, was obtained for universal enveloping algebras over $\mathbb{C}$. In 1980 this result was extended by Irving and Small [17] to other fields. However, the proof of Theorem 4.2.5 uses Proposition 2.4.23 and Lemma 2.4.21 which require that the base field be uncountable. The theorem of Jordan canonical forms for a matrix M over a field k holds assuming all the eigenvalues of M are contained in k. This gives the following conjecture.

Conjecture 5.2.2. *Let k be an uncountable field and let $A = k[x_1, \dots, x_d]$ be a finitely generated k-algebra and let σ be a k-algebra automorphism of A such that σ restricts to a linear automorphism of the vector space $\mathbb{C} + \mathbb{C}x_1 + \cdots + \mathbb{C}x_d$ and all the eigenvalues of the matrix of σ relative to the basis $\{1, x_1, \dots, x_d\}$ are contained in an algebraic extension of k. Then for all prime ideals P of the skew Laurent polynomial ring $A[z, z^{-1}; \sigma]$ the following are equivalent.*
1. P is a primitive ideal of $A[z, z^{-1}; \sigma]$.
3. P is a rational prime in $A[z, z^{-1}; \sigma]$.
4. P is locally closed in $\mathrm{Spec}(A[z, z^{-1}; \sigma])$.

We referenced the result of Bell, Rogalski and Sierra in Section 2.4, Theorem 2.4.26. Let k be an uncountable algebraically closed field of characteristic zero, let A be a finitely generated commutative k-algebra and let σ be an automorphism of A. The theorem states that if $\dim(A) \leq 2$ and $\mathrm{GKdim}(A[z, z^{-1}; \sigma]) < \infty$ then $A[z, z^{-1}; \sigma]$ satisfies the Dixmier-Moeglin equivalence. Corollary 2.5.11 shows that the condition that the $\mathrm{GKdim}(A[z, z^{-1}; \sigma])$

be finite is necessary. Theorem 4.2.5 is a direct consequence of Theorem 2.4.26 if $\dim(A) \leq 2$, but we have shown that the Dixmier-Moeglin equivalence holds for the prime (0) for any $d < \infty$ and any $\mathbb{C}$-algebra automorphism σ such that the $(d+1)$-dimensional vector space $V = \mathbb{C} \oplus \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_d$ has the property that $\sigma(V) = V$. These results can be combined to give the following conjecture.

Conjecture 5.2.3. *Let k be an uncountable algebraically closed field of characteristic zero and let A be a finitely generated commutative k-algebra. Let σ be an automorphism of A. If $\dim(A) < \infty$ and $\mathrm{GKdim}(A[z, z^{-1}; \sigma]) < \infty$ then $A[z, z^{-1}; \sigma]$ satisfies the Dixmier-Moeglin equivalence.*

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