PHILOSOPHICAL HYPERGRAPHICS:

SOME APPLICATIONS TO PHILOSOPHY OF THE THEORY OF HYPERGRAPHS

by

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Abstract

This thesis demonstrates the applicability of hypergraphs to philosophical problems. I employ and enrich the theory of transverse hypergraphs, the colouring theory of hypergraphs, and the novel *harmonic* theory of hypergraphs. I also demonstrate that the relationship between the latter two theories is one of logical duality.

Because this thesis consists of a number of distinct articles, each representing a thematically diverse application of hypergraph theory to philosophy, it is difficult to speak of a unifying thread, except insofar as I may explain the general modus operandi in highly schematic terms.

To that end, common to all of the articles is the exemplification of the following: A problem is given whereby there is a collection of objects and a question has arisen as to whether these objects stand in a particular relationship to one another. I use hypergraphs to represent the objects. A key feature of the objects is then modelled using either chromatic or harmonic number, or the notion of a transverse hypergraph. Lastly, properties of, or relations between chromatic number, harmonic number, or the notion of a transverse hypergraph, are shown to entail a solution to the problem.

The main results in this thesis are summarized as follows: (1) It is possible to design a non-statistical polling technique which forms the basis of a representative political system. (2) The conditions under which a malfunction of a technical system is identical with its diagnosis can be characterized using equivalent maximality and minimality conditions on harmonic and chromatic number. (3) An axiomatization exists of extent of Wittgenstein's notion of family resemblance. (4) Taxonomic properties of identity can be discerned by exploring the mathematical relationship between diachronicity and synchronicity. (5) A new axiomatization of a class of weakly aggregative modal logics can be found by dualizing chromatic number, and exploiting harmonic number. (6) Completeness for classes of weakly aggregative and non-normal modal logics can be simplified by dualizing neighborhood semantics. (7) There is a relevant inference relation which is dual to the paraconsistent n-forcing relation, and which can be represented as a restriction of the classical $\vdash$.

Keywords: harmonic theory of hypergraphs; representativeness; modal logic; diagnosis; family resemblance; systematics

Subject Terms: Hypergraphs; Duality(Logic); Modality(Logic); Relevance logic; Conceptual structures information theory; Knowledge representation information theory

For Ray and my family.

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Table of Contents

Approval ii

Abstract iii

Dedication v

Acknowledgements vi

Table of Contents vii

List of Figures x

Introduction 1

0.1 Preamble . . . 1

0.1.1 Topic . . . 1

0.1.2 Rationale . . . 2

0.2 Hypergraphs . . . 3

0.2.1 Terminology . . . 3

0.2.2 q-Colourability . . . 6

0.2.3 q-Harmonicity . . . 7

0.2.4 Logical Duality . . . 8

0.3 Contents . . . 10

0.3.1 Democratic Harmonics . . . 10

0.3.2 Self-dual Malfunctions . . . 11

0.3.3 An Axiomatization of Family Resemblance . . . 11

0.3.4 On the Duality of Synchrony and Diachrony: A Dynamic Theory of Identity 12

0.3.5 Harmonics for Hypergraphs . . . 13

0.3.6 A Dualization of Neighborhood Structures . . . 14

0.3.7 On Imploding: the Logic of (In)Vacuity . . . 15

Bibliography . . . 16

1 Democratic Harmonics **19**

1.1 Introduction . . . 20

1.2 k-Harmonic Hypergraphs . . . 23

1.3 A Democratic Polling Scheme 25
1.3.1 Vertex Criticality 28
1.3.2 Saturation 29
Bibliography 30

2 Self-dual Malfunctions **31**
2.1 Introduction 31
2.1.1 Applications 33
2.2 Conflict Sets and Diagnoses 34
2.3 Hypergraphs and Transverse Hypergraphs 36
2.3.1 Logical Duality 36
2.4 Self-duality 39
2.4.1 Harmonic Maximality 40
2.4.2 Chromatic Minimality 41
2.4.3 Chromatic Maximality 42
2.5 Conclusion 43
Bibliography 44

3 An Axiomatization of Family Resemblance **46**
3.1 The Basic Idea 46
3.2 Its Realization 49
3.3 Composite Families 54
3.4 Taxonomic Hierarchies 57
3.5 Resemblance Revisited 61
Bibliography 62

4 On the Duality of Synchrony and Diachrony: A Dynamic Theory of Identity **63**
4.1 Introduction 64
4.2 Dual views 66
4.2.1 A Formal Model 66
4.2.2 (Weak) Duality 68
4.2.3 An Intermediate Duality 71
4.2.4 Duality and Potential 72
4.3 Harmonic number 73
4.3.1 A Measure of Similarity 73
4.3.2 Chromatic Number 75
4.4 Conclusion 77
Bibliography 78

5 Harmonics for Hypergraphs **80**
5.1 Introduction 81
5.2 Hypergraphs and Harmonics 82
5.2.1 An Application to Hypergraph Theory 83
5.2.2 Logical Duality 86
5.2.3 Functional Completeness 87

5.3 Completeness . . . 88
Bibliography . . . 90

6 A Dualization of Neighbourhood Structures **92**
6.1 Introduction . . . 92
6.2 Hyperframes . . . 93
6.3 Normal Hyperframes . . . 96
6.3.1 $(n+1)$-ary Relational Frames and n-Bounded Hyperframes . . . 97
6.4 Weakly Aggregative Modal Logic . . . 99
Bibliography . . . 101

7 On Imploding: the Logic of (In)Vacuity **102**
7.1 Introduction . . . 103
7.2 q-Folding . . . 106
7.2.1 Compactness . . . 109
7.3 Harmonic Number . . . 111
7.4 The Rules . . . 114
Bibliography . . . 117

List of Figures

0.2.1 A representation of a graph. 5

0.2.2 A representation of a hypergraph. 6

0.2.3 Another representation of a hypergraph 6

0.2.4 A combination of visual styles. 7

1.1.1 A transversal s for the set of demes 20

3.1.1 The family of games. 47

3.3.1 A composite 55

3.3.2 An m-colouring of $\mathcal{F} = \{e_1, e_2, e_3, e_4\}$. 56

3.4.1 The family of domestic animals. 57

3.4.2 A tree of families. 58

4.2.1 The instance graph for individual $\mathcal{I}$. 67

4.2.2 The property graph for individual $\mathcal{I}$. 68

4.2.3 Weak duality 70

4.3.1 An m-colouring of $\mathsf{H} = \{e_1, e_2, e_3, e_4\}$. 76

5.1.1 The truth condition for $\Box$ on $(n + 1)$-ary relational frames. 81

6.2.1 The truth condition for $\Box$ on hyperframes. 94

6.3.1 The truth condition for $\Box$ on $(n + 1)$-ary relational frames. 98

7.2.1 The truth condition for $\Box$ on $(q + 1)$-ary relational frames. 106

7.2.2 The truth condition for $\Diamond$ on $(q + 1)$-ary relational frames. 108

Introduction

0.1 Preamble

0.1.1 Topic

This monograph consists of a selection of articles which demonstrate the utility of applying the mathematical theory of hypergraphs to various philosophical problems. The part of this theory with which we will be mainly concerned, the *harmonic theory of hypergraphs*, is unique to this thesis, and evolved from a joint study of well developed theories in graph theory, in particular, the colouring theory of graphs and the theory of transverse graphs.

Why hypergraphs? Successes in this particular intersection of mathematics with philosophy have for the most part been restricted to mathematical logic. They include (a) the development of a rule of inference which, in contrast to classical logic, enables one to draw non-arbitrary conclusions from an inconsistent collection of premises [20][7], (b) the development of a class of weak modal logics, and a non-standard semantical framework for modal logic [18][19][6][5] (cf. also [8]), (c) a proof that the $\mathbf{K_n}$ modal logics are complete with respect to the class of all $(n+1)$-ary relational frames [1][23], and (d) a simplification of the completeness proof for the $\mathbf{K_n}$ logics [14].

Here, in addition to considering issues which are mainly, *prima facie*, and *ut nunc*, of logical interest, I widen the purview of philosophical hypergraphics, including within its domain questions and topics of a more central philosophical character. Below, I briefly, and as non-technically as possible, sketch the content of the articles included in this thesis. To that end, it will be useful to introduce some important theorems, as well as some terminology which is common to the articles, notwithstanding minor notational deviations which occur among them.

0.1.2 Rationale

One could question the rationale for this thesis on the basis that, properly construed, mathematics has little or no role in non-logical philosophy—that mathematical philosophy is mathematics if anything, and not philosophy at all. But this raises the thorny issues of what the purpose of philosophy is, or should be, and of what philosophy is, in the first place. What Einstein and Infeld say of physics is also a plausible view of what the aim of philosophy should be. They write that an important part of the purpose of physics is to "raise new questions, new possibilities, to regard old problems from a new angle"(p. 92)[4].

Constricting the angle from which a problem is viewed has the effect of polarizing the light that one sheds on it. Any answer to the problem formulated from this perspective will lie in the same dimly lit corner within which the problem is first formulated. Einstein and Infeld note, "[t]he formulation of a problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skill."(p. 92)[4] But the point is that broadening the angle from which a problem is viewed has the advantage of dispersing a greater spectrum of light; it therefore has the potential to generate greater intellectual illumination.

This is a basic issue of good intellectual, and therefore, ideally, academic, hygiene. It is not to the benefit of philosophy, or any academic discipline, to restrict the angles from which we view problems to acute, rather than oblique ones. Orthogonal approaches should be valued too. By broadening the arcs which subtend various philosophical problems, this thesis demonstrates that philosophy has much to gain by including the theory of hypergraphs and its methods within its framework.

Indeed, many of the questions which were asked by the philosophers of antiquity now have solutions in mathematics and physics. As Russell notes:

> Zeno was concerned, as a matter of fact, with three problems, each presented by motion, but each more abstract than motion, *and capable of a purely arithmetical treatment.* These are the problems of the infinitesimal, the infinite, and continuity. *To state clearly the difficulties involved, was to accomplish perhaps the hardest part of the philosopher's task.* This was done by Zeno. From him to our own day, the finest intellects of each generation in turn [tried to solve] the problems, but achieved, broadly speaking, nothing. In our own time, however, three men—Weierstrass, Dedekind, and Cantor—have

> not merely advanced the three problems, but have completely solved them. The solutions, for those acquainted with mathematics, are so clear as to leave no longer the slightest doubt or difficulty....Of the three problems, that of the infinitesimal was solved by Weierstrass; the solution of the other two was begun by Dedekind, and definitely accomplished by Cantor.(p. 64)[17][emphasis mine]

From the fact that a problem is first formulated in philosophy, narrowly conceived, it doesn't follow that that is where we should remain in order to find its solution, or even merely a possible solution—one of what are perhaps many. Inasmuch as logic is a branch of philosophy, philosophical problems often have multiple solutions. This is because, for example, a theorem can often be proved in a variety of ways. If distinct proofs are distinct solutions then clearly there are philosophical problems with multiple solutions. To take an example from physics, Kaku observes that:

> ...Einstein's equations gave new insights into such ancient questions [as], is there an end to the universe? If the universe ends with a wall, then can we ask the question, what lies beyond the wall?...[O]ne might state that the universe is infinite in three dimensions. There is no brick wall in space that represents the end of the universe; a rocket sent into space will never collide with some cosmic wall. However, there is the possibility that the universe might be finite in four dimensions. (If it were a four dimensional ball, or hypersphere, you might conceivably travel completely around the universe and come back to where you started. In this universe, the farthest object you can see with a telescope is the back of your head.)(pp.137-8)[11]

Here again it can be seen that the answers to questions that are philosophical in origin, regarding the finiteness of the universe in this case, can be profitably sought by employing tools that are common to other domains of inquiry, for example, the tools of physics (and mathematics, particularly cosmic topology). The main purpose of this monograph is to show that the theory of hypergraphs is another such tool.

0.2 Hypergraphs

0.2.1 Terminology

Consider the kinds of graphs that children learn to read in secondary school, for example, two-dimensional space graphs, which consist of two axes, a vertical y-axis, representing distance, and an intersecting horizontal x-axis, also representing distance, with points plotted in various places between the two axes. To each point γ there corresponds a pair of numbers (u, v) which tells us that if we were to draw a straight line through position u on the x-axis, perpendicular to the x-axis, and a straight line through position v on the y-axis, perpendicular to the y-axis, then the two lines would intersect exactly at point γ. Considered in this way, a graph is just a collection of pairs (u, v) whose elements are ordered. But now what happens if we add a z-axis, as is required, for example, in representations of three-dimensional space? Then our graph can be represented as a collection of triples. In fact for any finite number n ($n \geq 2$) we can say that a graph can be represented as a collection of n-tuples[1] whose elements are ordered. For convenience, let us refer to the collections of which a graph consists as its *edges*. Then, generalizing this tendency towards higher dimensionality, rather than requiring that all edges of a graph have the same number of elements, that is, the same *width*, we can allow the width of edges to vary. We can also choose to disregard the order of the elements of the edges, and we can allow that the edges are infinitely wide, and that there are infinitely many of them. In this way we obtain a kind of 'hyper-dimensional' graph, or a *hypergraph*, which reduces, in the case where all of its edges are pairs, to the notion of a graph which is common in the mathematical theory called *graph theory*. Thus, a *hypergraph* is a collection of collections of numbers or variables, or more simply still, a collection of collections. For the most part we will be considering finite collections of finite collections, but we allow for the existence of *infinite hypergraphs*, that is, hypergrahs which are either infinitely long, or which contain at least one edge that is infinitely wide. A *finite hypergraph* is one that is composed of a finite collection of finite collections.

One drawback of the notion of a hypergraph is that we lose the comparatively easy visual representation of the graphs of graph theory. But there are other representations. In the case where all of the edges of a hypergraph are pairs, as occurs for instance with $\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,1\}\}$, we may draw Figure 0.2.1. The dots represent what are called the *vertices* of the hypergraph—these are just the elements of the edges. A line between two vertices represents that there is an edge which is composed of them. This is a standard visual representation of a graph in graph theory. Since the order of the vertices appearing in an edge is irrelevant to such graphs,[2] their positions with respect to one another may vary. That is, there are an indefinitely large number of distinct visual representations, following the conventions noted here, of any particular hypergraph which

[1] A 2-tuple is a pair, a 3-tuple is a triple, a 4-tuple is a quadruple, etc.

[2] Graphtheorists distinguish between *directed graphs* and graphs: a graph is essentially a collection of pairs; a directed graph is a collection of pairs in which the order of the elements matters.

happens to consist exclusively of pairs.

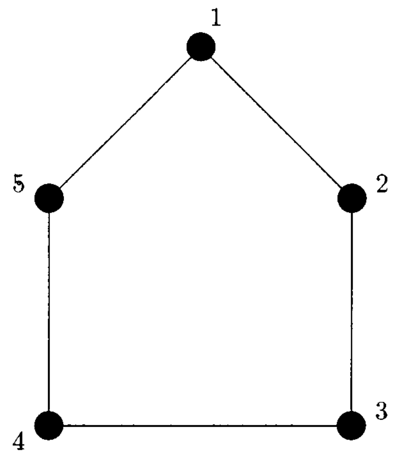

Figure 0.2.1: A representation of a graph.

When we move to hypergraphs that are not graphs (of graph theory), visual representations are more difficult. Let H be the hypergraph $\{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$. Then because at least one edge of H is a triple, we can no longer use the convenient representation common in graph theory without generating an ambiguity. To see this, consider Figure 0.2.2. Here we indicate that an edge comprises some collection of vertices by drawing a membrane around exactly those vertices. Notice that if we replace the membrane enclosing 2, 3 and 4 with lines connecting these vertices, our representation would fail to distinguish between the single edge $\{2,3,4\}$ and a collection of smaller edges consisting of pairs. A different kind of visual representation, of the same hypergraph, is depicted in Figure 0.2.3. For the most part this is our preferred representation, although we occasionally use a combination of the two styles. Here, a commonality among edges can be represented by a repetition of the common vertex, where vertices are now represented by their numeric labels. One advantage of the former style of representation, depicted in Figure 0.2.2, is that intersections among edges are more easily visualized. It is for this reason that we use a combination of the two methods, depending on which aspect(s) of a hypergraph we intend to make most salient in an illustration. For example, in Figure 0.2.4, the edges of the hypergraph H are not necessarily disjoint; the illustration is intended to depict the fact that the collection C has at least one vertex in common with every element of H.

Now that we have the notion of a hypergraph in hand, we can study its properties. One of the properties which is of fundamental importance to my research has to do with what are called the *colourability properties* of a hypergraph. It will be convenient in what follows to abbreviate

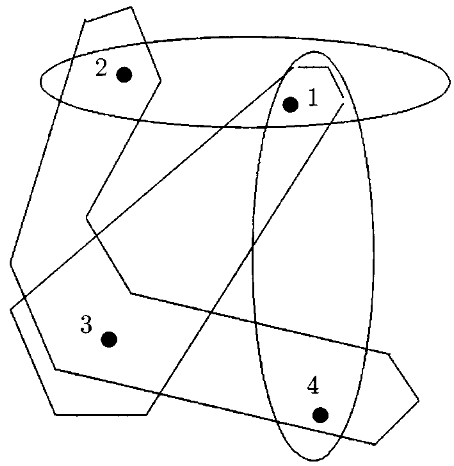

Figure 0.2.2: A representation of a hypergraph.

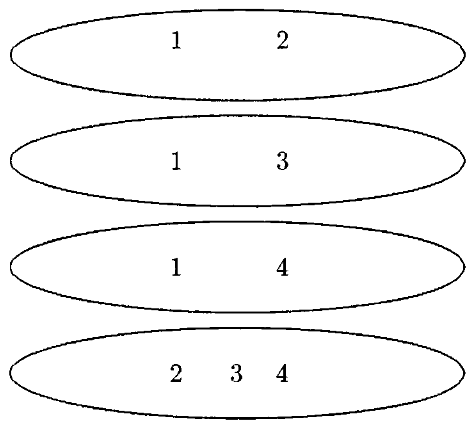

Figure 0.2.3: Another representation of a hypergraph

'hypergraph' to 'graph': there is no danger of ambiguity in doing so since every graph (of graph theory) is a hypergraph, and if it is specifically a graph whose members are all pairs which is intended by 'graph', then I add the appendix '(of graph theory)' to indicate this.

0.2.2 q-Colourability

Where q is some positive integer, a *q-colouring* of a graph is an assignment of each vertex to exactly one of q colours. Given a q-colouring of a graph H, we can ascertain things about its edges in relation to the q-colouring. For instance, of a given edge we can say whether all of its vertices are assigned to the same colour. If a graph H is such that there is a q-colouring of it where no edge has all of its vertices assigned to the same colour—that is to say, where no edge is *monochrome*—then

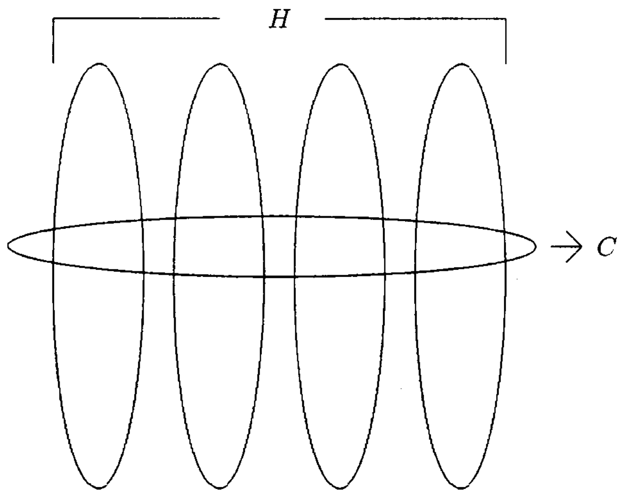

Figure 0.2.4: A combination of visual styles.

we say that H is *q-colourable*; otherwise we say that H is *q-uncolourable.* The *chromatic number* of a graph H, denoted '$\chi(H)$', is the smallest integer q such that H is q-colourable; if there is no number q such that H is q-colourable then we stipulate that the chromatic number of H is ∞—an arbitrarily high value.

For example, let $H = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,1\}\}$. Then $\chi(H) = 3$. This is because, since H has an odd number of vertices which are arranged in a cycle, H is not 2-colourable. But it is easy to see that by the addition of a third colour we can obtain a colouring which leaves no edge monochrome. For instance, assign vertices 1 and 3 to yellow, 2 and 4 to red, and 5 to green.

Alternatively, let H consist of the single 'unit' edge $\{1\}$. Then $\chi(H) = \infty$ since whenever the single vertex of H, 1, is assigned to a colour, some edge of H is monochrome.

0.2.3 q-Harmonicity

In addition to speaking of the colourability properties of a graph, we can talk about its *intersectival* properties; we refer to these as its *harmonic* properties. The harmonic properties of graphs constitute a hitherto largely unstudied class of properties of graphs in contrast to the well-entrenched theory of chromatics. A collection of edges of a graph is said to have a *non-empty intersection* when there is some vertex which appears in every edge in the collection; otherwise we say that the intersection of the collection is empty. We say that for a positive integer q, a graph is *q-wise* *intersecting* if and only if every collection of q edges from the graph has a non-empty intersection. The *harmonic number* of a graph H, indicated by '$\eta(H)$', is the smallest positive integer q such that H is not q-wise intersecting; if H happens to be q-wise intersecting for every positive integer q, then we say that $\eta(H)$ is ∞—again, an arbitrarily high value. If $\eta(H) > q$ then we say that H is *q-harmonic*.

For example, let $H = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$. Then $\eta(H) = 4$ because the intersection of H is empty while every triple of edges of H has a non-empty intersection. To understand why every triple of edges has a non-empty intersection, notice that should a triple of edges have an empty intersection, then the collection of vertices omitted by the edges of the triple would be identical to the collection consisting of the totality of the graph's vertices; but this is impossible because the graph itself consists of the collection of all triples of vertices, and thus a triple of edges can omit at most three of the four vertices of the graph.

To take another example, let $H = \{\{1,2,3\},\{1,2,4\},\{1,3,4\}\}$. Then $\eta(H) = \infty$. This is because the intersection of H is non-empty, and thus there is no smallest integer q for which there is an q-tuple of edges of H having an empty intersection.

0.2.4 Logical Duality

One of the fundamental contributions of this monograph to the theory of hypergraphs is the observation that harmonic number and chromatic number are *logically dual*—logically dual in the sense in which the *and*, or *conjunction*, represented by $\wedge$, of logic is dual to the *or*, or *disjunction*, represented by $\vee$, of classical, propositional, logic. To define the meaning of "duality" precisely, we need to interpret $\wedge$ and $\vee$. Where the letters p and q represent arbitrary sentences, and 'T' represents 'true' and 'F' represents 'false', recall that the truth table for $\vee$ is:

p	$\vee$	q
T	T	T
T	T	F
F	T	T
F	F	F

And the truth table for $\wedge$ is:

p	$\wedge$	q
T	T	T
T	F	F
F	F	T
F	F	F

Given this truth functional interpretation of $\vee$ and $\wedge$, they can be said to be logically dual in the following sense: by interchanging all occurrences of T's with F's and conversely, in either table, we obtain exactly the other, modulo the order of the rows.

It is possible to extend the above truth tables so that we can interpret disjunctions of n-terms ($n \geq 2$) and also n-termed conjunctions. We say that a disjunction is true just in case at least one of its disjuncts is; a conjunction is true just in case each of its conjuncts is. As a consequence of this convention, we can treat graphs as if they were sentences. For, let the vertices of a graph H be themselves sentences. Then we can conjoin the elements of the edges of H to form some number of conjunctions, which we may then disjoin. Call the result of such an operation performed on H, "$\vee\!\!\wedge H$". The dual operation thus yields $\wedge\!\!\vee H$, and $\vee\!\!\wedge H$ and $\wedge\!\!\vee H$ are accordingly dual.

Now the *transverse graph* of a graph figures crucially in our description of the duality of chromatic and harmonic number. To define this we need the notion of a *cover* for a graph. A *cover* for a graph is a collection of its vertices that has a non-empty intersection with every edge of the graph. A *minimal cover* for a graph is a cover for a graph that is not a proper superset of any of the graph's covers. The *transverse (hyper)graph* for a graph H, TH, is the collection of all minimal covers for H. As $\vee\!\!\wedge H$ and $\wedge\!\!\vee H$ are dual, derivatively we can understand H and TH as duals. This is because $\vee\!\!\wedge H$ is equivalent to, that is, is true (false) under exactly the same circumstances as, $\wedge\!\!\vee TH$, and similarly for $\wedge\!\!\vee H$ and $\vee\!\!\wedge TH$. To see why this is so, consider the circumstances under which $\vee\!\!\wedge H$ is true: $\vee\!\!\wedge H$ is true exactly when at least one of the edges of H has all of its elements true. But every edge of H has a non-empty intersection with every edge of TH (this follows straightaway from the definition of a transverse graph as a collection of covers). Therefore making all of the elements of some edge of H true makes at least one element of every edge of TH true, thereby making $\wedge\!\!\vee TH$ true as well. Now consider how to make $\wedge\!\!\vee TH$ true—this can be accomplished only by making at least one element of every edge of TH true, that is, by making all of the elements of some cover, call it c, for TH true. But c contains some edge of H, for suppose not: then some minimal cover for H has an empty intersection with c, which is impossible, since c is a cover for TH. Similar reasoning demonstrates the equivalence of $\wedge\!\!\vee H$ and $\vee\!\!\wedge TH$. But consequently, given that $\vee\!\!\wedge H$ and $\wedge\!\!\vee H$ are dual, $\wedge\!\!\vee TH$ is dual to $\wedge\!\!\vee H$, and $\vee\!\!\wedge TH$ is dual to $\vee\!\!\wedge H$. It is in this sense that we say that H and TH are dual. Therefore, we may conclude that chromatic number is dual to harmonic number because we have the following:

Theorem 0.2.1. *For any graph H, $\chi(H) = q$ if and only if $\eta(TH) = q$.*[3]

Theorem 0.2.1 is important for two reasons: first, it demonstrates a relationship between the novel harmonic theory of graphs and the well-established theory of chromatics for graphs; second, it is integral to many of the results proved later on in the monograph.

0.3 Contents

0.3.1 Democratic Harmonics

The first selection, *Democratic Harmonics*, consists of an employment of k-harmonic graphs in the construction of a dynamic system of electoral representativeness. Jennings and Schotch have shown the utility of k-uncolourable graphs when it comes to formalizing non-arbitrary inference from inconsistent premise sets [20]. Here I show how this application can be extended to the dual political context in which elected officials form policies which represent incohesive interest groups. By exploiting k-harmonic graphs, a non-statistical model for a polling technique is designed whose aim is to preserve electoral representativeness through deliberative legislative processes. In this respect the technique is dynamic, in addition to marking a departure from conventional, statistically-based models of representation. In political science, the formalisms for modelling representativeness tend to be drawn primarily from the theories of games, social choice and multi-dimensional geometries, with a focus on such issues as the equilibria of competing strategies. In contrast, this article uses a theory of graphs to model potential and desirable transfers of information among groups of elected officials.

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[3]Since Theorem 0.2.1 is proved in several locations in the remainder of this monograph, for simplicity's sake we omit its proof here.

0.3.2 Self-dual Malfunctions

In *Self-dual Malfunctions*, I consider the question whether there is any *problem*, or *malfunction* in a system, which is identical to the *solution* for this problem. A word here is in order about what is meant by "problem" and by "solution". Diagnostic problems require a differentiation between normally and abnormally behaving system components. Reiter's system of diagnosis from first principles [15] is an approach to diagnostics in which a *problem* can be said to be, roughly speaking, the collection of all minimal lists of system components the members of which it is inconsistent to suppose are functioning normally, in the presence of an observation of the system and a description of the system's normal behaviour. A *solution* consists of the collection of all minimal lists of system components which it is consistent to suppose the members of which are faulty while the other components are not. In this article, characterizing the class of malfunctions for which a problem is identical to its solution is shown to be reducible to characterizing the conditions under which, for an arbitrary graph H whose vertices are sentences,

$$\bigvee_{i=1}^{|H|} \bigwedge_{j=1}^{|e_i|} j \dashv\vDash \bigwedge_{i=1}^{|H|} \bigvee_{j=1}^{|e_i|} j.$$

Alternatively, this class of malfunctions can be characterized in terms of a maximality condition of harmonic number for graphs, in addition to an original maximality condition on chromatic number, as well as an equivalent minimality condition. In this way, characterizing the relationship between problems and their solutions contributes to a general theory of logical, and in particular, self, duality, as well as demonstrating how the harmonic theory of graphs can be applied to such a theory.

0.3.3 An Axiomatization of Family Resemblance

In *An Axiomatization of Family Resemblance*, inspired by Rosch and Mervis' seminal investigation into Wittgenstein's notion of *family resemblance* [16], Jennings and Nicholson collaboratively pursue the question whether *extent* of family resemblance can be axiomatized. Jennings further suggests that by invoking concepts from the harmonic theory of graphs it is possible to give a measure of the closeness of *family resemblance*, and to make precise the idea of a composite likeness. Nicholson's results in this paper confirm these suggestions, and include a completeness proof for extent of family resemblance which exploits the harmonic theory of graphs.

Family resemblance was introduced by Wittgenstein in [24] as part of an account of what constitutes possession of a general term, and what is required for its correct use. The cognitive significance of this account is documented by Rosch and Mervis in [16]. For Wittgenstein, as for Rosch and Mervis, what constitutes possession of a general term is not that one can identify some feature which is shared by all and only instances of the term; in place of such essentialist doctrines they argue that instances of a term are united by way of a network of similarities or intersections—this network is called the *family resemblance* of a concept. To take the example that Wittgenstein does, consider the general term "games": there is no single feature which is common to all games, unless it is a feature which is shared by some things which are not games. Therefore, it is not because we understand that there is something essentially game-like that enables us to use the term "game" correctly; instead, what accounts for something being a game is that it participates in the network of similarities possessed by all games—that is, it shares a family resemblance with other games.

Using a generalization of the harmonic number of a graph to model the family resemblance of a concept, Nicholson shows that for any positive integer m, for any general term possessing any level of family resemblance strictly greater than m, there is a taxonomical representation of the term whereby each subordinate taxon has family resemblance strictly greater than m. This is significant from a philosophical point of view because it shows that there is a taxonomic representation of any general term which is complete with respect to the notion of family resemblance envisaged by Wittgenstein.

0.3.4 On the Duality of Synchrony and Diachrony: A Dynamic Theory of Identity

In *On the Duality of Synchrony and Diachrony: A Dynamic Theory of Identity*, I use graphs to define a theory of identity which explains the relationship between diachronic and synchronic perspectives of an individual. Specifically, drawing on the duality of a graph and its transverse graph, a notion of *weak duality* emerges; I show that it is in this way that diachronicity and synchronicity are related.

This bifurcated representation of identity is useful for several reasons: First, it suggests how we can apply a weakened version of Leibniz's *Principle of the Indiscernibility of Identicals* (namely, that if x and y are identical then x has a property δ if and only if y has δ). On the wider notion, that an entity at one time differs in one property from an entity at a second time does not entail that the two entities are not identical. Second, by invoking the theory of harmonics for graphs, I obtain well-defined measures of inter- and intra-personal resemblance. Third, the theory of identity can be applied to issues of gender identity in relation to personal identity. By invoking the chromatics of graphs, I show that given a certain degree of richness in our taxonomy of the kinds of properties an individual can possess, if the collection of an individual's behaviors in his or her social settings is sufficiently cohesive, and there is an appropriate relation of duality between her or his synchronic and diachronic representations, then it is possible that some social role in which she or he engages will witness him or her behaving in both of two differently gendered ways—at the same time, no less.

0.3.5 Harmonics for Hypergraphs

In *Harmonics for Hypergraphs*, Jennings, Nicholson, and Sarenac take up a strictly logical aspect of philosophical hypergraphics, and show that there is a truth function which, modulo some simple propositional transformations, is capable of generating all of the graphs of harmonic number strictly greater than n. In particular, the truth function is called 'n over $n+1$', and is denoted by '$\frac{n}{n+1}$'. Where $[n+1]$ denotes a set $\{1, 2, ..., n+1\}$ of $n+1$ formulae ($n \geq 1$), $\frac{n}{n+1}(1, 2, ..., n+1)$ is the sentence $\bigwedge\bigvee\binom{[n+1]}{n}$, where for any integer $i \geq 1$, $\binom{[n+1]}{i}$ denotes the set of all i-tuple subsets of $[n+1]$. Thus for example, $\frac{3}{4}(1, 2, 3, 4) = (1 \vee 2 \vee 3) \wedge (1 \vee 2 \vee 4) \wedge (1 \vee 3 \vee 4) \wedge (2 \vee 3 \vee 4)$.

As an application of this result we prove the following: that for each $n \geq 1$, every sentence which is valid with respect to the class of all $(n+1)$-ary relational frames is provable as a theorem in the modal system $\mathbf{K^n}$, whose axioms and rules are:

$$[RN] : \vdash \alpha \quad \Rightarrow \; \vdash \Box\alpha \tag{0.3.1}$$

$$[RM] : \vdash \alpha \rightarrow \beta \quad \Rightarrow \; \vdash \Box\alpha \rightarrow \Box\beta \tag{0.3.2}$$

$$[RPL] : \vdash_{PL} \alpha \quad \Rightarrow \; \vdash \alpha \tag{0.3.3}$$

$$[MP] : \vdash \alpha \text{ and } \vdash \alpha \rightarrow \beta \quad \Rightarrow \; \vdash \beta \tag{0.3.4}$$

$$[US] : \vdash \alpha \text{ and } \beta \text{ is a substitution instance of } \alpha \quad \Rightarrow \; \vdash \beta \tag{0.3.5}$$

$$[K^n] : \vdash \Box p_1 \wedge ... \wedge \Box p_{n+1} \rightarrow \Box\frac{n}{n+1}(p_1, ..., p_{n+1}) \tag{0.3.6}$$

$(n+1)$-ary relational semantics are a generalization of the usual binary relational semantics for modal logic and represent the diagonalization restriction of the much more general algebraic work of Jónnson and Tarski in [9] and [10]. An $(n+1)$-ary relational frame is a pair (U, R) where U is a non-empty set and $R \subseteq U^{n+1}$ is an $(n+1)$-ary relational defined on U. The truth condition for $\Box$ on models defined on these structures, devised by Schotch and Jennings [19][18][6], is given:

$$\models^{\mathfrak{M}}_{x} \Box\alpha \Leftrightarrow . \forall y_1, ..., y_n, Rxy_1...y_n \Rightarrow \exists i \in [n] : \models^{\mathfrak{M}}_{y_i} \alpha.$$

In this paper, Jennings is responsible for framing the logical completeness problem as a functional completeness problem of n-harmonics, and Nicholson and Sarenac collaborated on a solution to the latter. Nicholson formulated both problems as part of a general, introductory, study of the theory of n-harmonics with applications to the theory of graphs more generally: by couching a portion of Berge's work on tranverse graphs [2] in the language of harmonicity, and introducing a maximality condition on harmonicity, Nicholson shows how to characterize the class of graphs which are identical to their transverse graphs—or which are, in other words, self-dual—using the language of harmonic number.

0.3.6 A Dualization of Neighborhood Structures

In *A Dualization of Neighborhood Structures* I simplify a result which is analogous to the completeness result of *Harmonics for Hypergraphs* by detouring around the functional completeness result pertaining to harmonic number which is therein exploited. I do this by using transverse graphs to alter the collection of structures of which the logical result is proved. Essentially this is done by augmenting the definition of a neighborhood, minimal, or Scott-Montague [21][13], model (found in [22] under the name 'neighborhood model', and in [3], under 'minimal model'; cf. also [12]) to incorporate the notion of a transverse graph in the truth condition assigned to the modal operator $\Box$. In particular, the condition is:

$$\models^{\mathfrak{M}}_{x} \Box\beta \text{ iff } \|\beta\|^{\mathfrak{M}} \text{ is a transversal for } \mathcal{H}(x),$$

where $\|\beta\|^{\mathfrak{M}}$ is the set of points in the model $\mathfrak{M}$ at which β is true, and $\mathcal{H}(x)$ is a graph assigned to x. A central virtue of this approach is that I obtain a simplified completeness proof—simplified, because it avoids a proof of chromatic compactness—for the K_n modal logics, whose axioms and rules are:

$$[RN] : \vdash \alpha \;\Rightarrow\; \vdash \Box\alpha \tag{0.3.7}$$

$$[RM] : \vdash \alpha \rightarrow \beta \;\Rightarrow\; \vdash \Box\alpha \rightarrow \Box\beta \tag{0.3.8}$$

$$[RPL] : \vdash_{PL} \alpha \;\Rightarrow\; \vdash \alpha \tag{0.3.9}$$

$$[MP] : \vdash \alpha \text{ and } \vdash \alpha \rightarrow \beta \;\Rightarrow\; \vdash \beta \tag{0.3.10}$$

$$[US] : \vdash \alpha \text{ and } \beta \text{ is a substitution instance of } \alpha \;\Rightarrow\; \vdash \beta \tag{0.3.11}$$

$$[K_n] : \vdash \Box p_1 \wedge ... \wedge \Box p_{n+1} \rightarrow \Box\frac{2}{n+1}(p_1, ..., p_{n+1}) \tag{0.3.12}$$

where $\frac{2}{n+1}(p_1, ..., p_{n+1})$ is the sentence $\bigvee_{1 \leq i < j \leq n+1} p_i \wedge p_j$. An additional virtue of this approach is that it enables a determination result for a denumerable sequence of non-normal logical systems. This is significant from a philosophical perspective for two reasons: First, non-normal logics have deontic motivations inasmuch as they do not impose infinite sets of obligations. A logic is normal when $\Box\alpha$ is a theorem whenever α is a theorem. Thus, if $\Box$ represents 'it is obligatory that', then in any deontic logic with an infinite number of theorems there is an infinite number of obligations.[4] Second, if we read $\Box$ as a necessity operator, then the existence of determined non-normal modal logics marks a conceptual divergence between logical validity and its classical, Aristotelian account. According to the classical account, an argument is valid when it is necessary that if the premises are true then the conclusion is true. But there are non-normal logics in which there are logically valid conditionals whose $\Box$ formulae are not theorems. This raises the philosophical question of how theoremhood in such systems should be understood, or alternatively, the question of what the $\Box$ operator represents in such systems.

0.3.7 On Imploding: the Logic of (In)Vacuity

In the final selection, *On Imploding: the Logic of (In)Vacuity*, I explore an inference relation definable using the harmonic number of graphs. In classical logic, if q is a tautology, then q may be inferred from any and every sentence p. Roughly, this is because if q must be true, then it is

[4]It is important to note, however, that the absence of normality does not guarantee the absence of an infinite number of obligations. Although the absence of normality is necessary, it is not sufficient for this end. What we really need is the rule:

$$\vdash \alpha \;\Rightarrow\; \vdash \neg\Box\alpha. \tag{0.3.13}$$

Jennings has suggested that what we really want is a variety of connexivist implication, which is a restriction of classical logic to contingencies.

impossible for p to be true while q is false, and therefore inferring q from p will always preserve truth—that is, necessarily, our conclusion q will be true if our premise p is. Nevertheless, some critics of classical logic maintain, it is not always correct to infer q from p in such circumstances. Take, for example, the case where q is irrelevant to p; contrary to the dictates of classical logic, according to these critics we ought to think that there is something wrong with the following kind of argument:

1. $E = mc^2$.

2. Therefore, either a square is a rectangle or it is not.

But consequently, we have to adopt a perspective somewhat distinct from, or at least narrower than, that of classical logic. This article presents one such approach. It begins with the idea common to Preservationist Logic(s) that there are other properties besides truth that we might want permissible inference to preserve. One of these properties is the absence of informational vacuity. Notice that since it is impossible to falsify the conclusion expressed by (2), (2) says very little, if anything, about the world. If we require that a conclusion we draw conveys at least as much information as our premise, then the inference from (1) to (2) is outlawed as fallacious. The question is then, "How can we construct a system of inference general enough to satisfy this constraint for any argument whatsoever?" In this article, by exploiting the theory of harmonics, and restricting ourselves to arguments with singleton premise sets, we present one solution. We also prove that the collection of inferences which are correct in relation to this new schema comprise a sub-collection of the inferences which are correct in classical logic. That is to say, the logic we introduce can be seen as a restriction of classical inference to a variety of relevant inference.

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Chapter 1

Democratic Harmonics

D. Nicholson[1]

Abstract

This article implements the duals of k-uncolourable hypergraphs in the construction of a dynamic system of electoral representativeness. Jennings and Schotch have shown the utility of k-uncolourable hypergraphs when it comes to formalizing non-arbitrary inference from inconsistent premise sets [6]. Here it is shown how this application can be extended to the dual political context in which elected officials form policies which represent incohesive interest groups. By exploiting k-harmonic graphs, a non-statistical model for a democratic polling technique is designed whose aim is to preserve electoral representativeness through deliberative legislative processes. In this respect the technique is dynamic, in addition to marking a departure from conventional, statistically-based models of representation. In political science, the formalisms for modelling representativeness tend to be drawn primarily from theories of games, social choice and multi-dimensional geometries, with a focus on such issues as the equilibria of competing strategies. In contrast, this article uses a theory of hypergraphs to model potential and desirable transfers of information among groups of elected officials.

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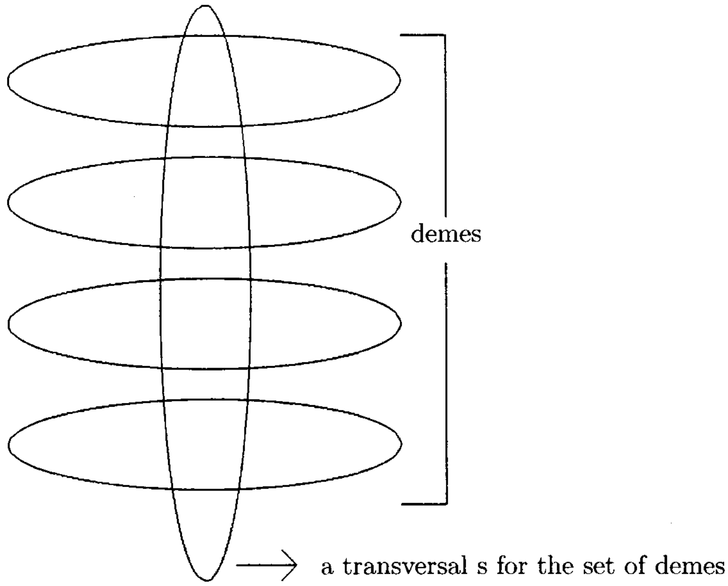

Figure 1.1.1: A transversal s for the set of demes

1.1 Introduction

It is trite political theory that the integrity of a democracy can be safeguarded only insofar as the representativeness of its governing bodies can be ensured. In general, the less representative a legislature is, the more opportunity there exists for private interest groups to affect public policy for the purpose of furthering their own aims. Arguably, the democratic changes in Athens of 462 B.C. were designed to stop this kind of dilution of the sovereignty of the popular assembly, the *ekklesia*, by requiring the random selection of a 500 membered body, the *boulē*, whose main function was to oversee what matters were to appear for debate in the public forum [2]. The idea was that each of the local *demes*, the constituent villages of Attica, was to supply a number of *bouleutai* to the *boulē* which was considered proportionate to the size of its population. Thus, as the *boulē* began to be characterized by a preponderance of wealthy and familially-related members, we find historical evidence that ways had been discovered to circumvent the randomness of the selection procedure. Nevertheless, in theory we can say that the *boulē* represented the Athenian electorate, since it consisted of a randomly selected transversal for the hypergraph H of its demes. (See Figure 1.1.1.)

Definition 1.1.1. A *hypergraph* H is any finite non-empty set of finite non-empty sets e, where $\bigcup_{i=1}^{|\mathsf{H}|} e_i \in \mathsf{H}$, abbreviated '$\cup\mathsf{H}$', is the set of *vertices* of H, and the elements of H are said to be its *edges.*

Definition 1.1.2. Let H be a hypergraph, and let s be a set. Then s is a *transversal* for H iff $\forall e \in \mathsf{H}, s \cap e \neq \emptyset$.

In contrast to the ancient Athenian system, modern democracies *elect* what amounts to the transversal of constituencies of which their governing body mainly consists. But, unless wealth selects uniquely for competence, it is arguable that one consequence of this contrast is that modern democracies tend to favour less representative governments, in that wealthier citizens are more likely to popularize their views among an electorate, and are therefore more likely to gain access to governing positions. Mind you, it's probably false that lack of membership in a group entails an inability to represent that group's interests, but in any case, the extent to which a legislature does not possess an efficient medium for the dissemination of incohesive electoral preferences *does* seem to suggest a kind of measure for the representativeness of its policies. One could also include such factors as a penchant for sophistry, bickering, and miscommunication on the part of officials among those which tend to dilute representativeness. In effect, what these remarks suggest is that mere "rep-by-pop"[2] is an insufficient basis for the construction of a democratic legislative process—that there is a kind of *dynamic* representativeness, which really ought to be preserved through a legislature's deliberations.

To illustrate, we suggest a "top-down" revision of standard democratic legislative structures, as opposed to a "bottom-up" analysis which focuses on the way that representatives are selected. To that end, a *level of divisiveness* will be introduced which can be used to measure the relative difference between the number of mutually incohesive political perspectives among an electorate with respect to some issue, and the number of possible outcomes for that issue which are being debated in a legislative house. The point is that if the size of the former variable is significantly larger than that of the latter, then any policy formed as a result of such debate is to that extent potentially lacking in responsiveness to the public will. And, as a policy tends to lack in responsiveness to the public will, so too does its representativeness of the public will tend to diminish. Indeed, as when for example a single party dominates the legislature, not only is doubt thereby cast upon the representativeness of its decisions, but the putative democratic virtue of public political debate is rendered moot—especially within a system in which party members cannot with impunity fail to "toe the party line". Unless the voices of dissent within an electorate are heard and echoed in the legislature, there is no difference, for practical purposes, between a legislature composed of a

[2] Or, "representation by population".

transversal of constituencies, and a legislature selected to favour the members of the parties whose platforms favour the status quo. In this way, the structure of a legislative system—specifically, those aspects of its structure which pertain to the dissemination of views about possible outcomes for issues being debated—may negatively affect its representativeness. Therefore, a democratic scheme may be judged better or worse accordingly as it serves or does not serve to diminish such effects.

Suggestively, there appears to be a dual structural similarity between the political phenomenon under discussion and the preservationist treatment of classical inference from inconsistent premise sets: Preservationist logicians have shown that by measuring what is called the *level of coherence* of a set Σ of sentences, it is possible to restrict inferences from Σ to those sentences which, when added to Σ, do not necessarily trivialize the resulting set, and in fact preserve the level of coherence of Σ [6]. In particular, the inference relation employed is called "n-forcing". Similarly, one could argue, so too ought we to try to minimize the difference between the number of competing political perspectives in an electorate, and the number of vessels provided in the legislature for the representation of these perspectives, by preserving the number of competing political perspectives in the form of the number of outcomes for an issue that are being debated. For although arbitrary inference from an inconsistent database is classically permissible, it doesn't follow that any such inference ought to be made.[3] Similarly, that a policy is decided upon by a democratically *formed* body does not entail that the policy itself is representative, in the sense that it was formed as a result of political debate in which the various, potentially dissenting, faces of the public will have had voice.

The preservationist strategy exploits the notion of what in the logic literature [6][4] has been called a *k-trace*—an object which for our purposes we can understand simply as a *k-uncolourable hypergraph.*

Definition 1.1.3. Let Σ be a set, and let k be a positive integer. Then the *set of k-partitions* of Σ, $\Pi_k(\Sigma)$, is the set $\{\{c_1, ..., c_k\} \mid \bigcup_{i=1}^{k} c_i = \Sigma \text{ and } c_h \cap c_i = \emptyset \ (1 \le h < i \le k)\}$. The *chromatic number* of a hypergraph H, $\chi(\mathsf{H})$, is then defined:

$$\chi(\mathsf{H}) := \begin{cases} min\ k \in \mathbb{Z}^+ : \exists \pi \in \Pi_k(\cup \mathsf{H}) : \forall e \in \mathsf{H}, c \in \pi, e \nsubseteq c & \text{if this limit exists;} \\ \infty & \text{otherwise.} \end{cases} \quad (1.1.1)$$

If a hypergraph H is such that $\chi(\mathsf{H}) > k$ for some $k \ge 1$, then we say that H is k-uncolourable; H is k-colourable, else.

[3] Notwithstanding that some, or even all, inferences from an inconsistent database might be permissible.

Definition 1.1.4. Let Σ be a set of sentences. Then the *coherence level* of Σ, $l(\Sigma)$, is defined:

$$l(\Sigma) := \begin{cases} min\ k \in \mathbb{Z}^+ : \exists \pi \in \Pi_k(\Sigma) : \forall c \in \pi, c \nvdash \bot & \text{if this limit exists;} \\ \infty & \text{otherwise.} \end{cases} \quad (1.1.2)$$

Definition 1.1.5. Let Σ be a set of sentences, let α be a sentence, and let $n \in \mathbb{Z}^+$. Then Σ *n-forces* $\alpha, \Sigma \,[\!\vdash_n \alpha$, iff $\exists \mathsf{H} \subseteq \wp(\Sigma) :\ \chi(\mathsf{H}) > n$, and $\forall e \in \mathsf{H},\ e \vdash \alpha$.

It is easy to see that the closure under n-forcing of a set Σ, where $l(\Sigma) = n$, has a coherence level no greater than n; the teleology of n-forcing is to minimize the difference between the coherence level of the input and that of the closure of the input under level-forcing. But a structure that is dual to this scheme seems to be exactly what we should want in a democratic political system: conceptualizing the input to a democratic process as an electorate, and the output as the members of a legislative body who *effectively* represent the interests of their constituents, we should want the divisiveness of the output to be at least as great as that of the input. Whence the polling technique presented in this article: it exploits a class of objects called *k-harmonic hypergraphs*, which are dual to k-uncolourable hypergraphs. This approach represents a departure from the kinds of formal strategies conventionally used in political science; there, the formalisms tend to be drawn primarily from theories of games, social choice and multi-dimensional geometries, with a focus on such issues as the equilibria of competing strategies. In contrast, this article uses a theory of hypergraphs to model potential and desirable transfers of information among groups of elected officials.

1.2 k-Harmonic Hypergraphs

We establish the lemmas required to provide a logically dual characterization of k-uncolourability for hypergraphs.

Definition 1.2.1. Let H be a hypergraph. Then the *transverse hypergraph* $Tr\mathsf{H}$ for H is the set of all minimal transversals for H.

Definition 1.2.2. Let H be a hypergraph whose vertices are sentences. Let $\bigvee\!\bigwedge \mathsf{H}$ be the result of first conjoining the vertices in elements of H, and then disjoining the resultant conjunctions. Then $\bigvee\!\bigwedge \mathsf{H}$ is the *formulation* of H, and the corresponding sentence $\bigwedge\!\bigvee \mathsf{H}$ (interchanging all occurrences of $\vee$ and $\wedge$ in the preceding sentence) is the *dual formulation* of H.

Theorem 1.2.1. $\forall H, TrTrH \subseteq H$ [5].

Proof. Let $e \in TrTr\mathsf{H}$. Suppose that $e \notin \mathsf{H}$. Since each edge f of H is a transversal for $Tr\mathsf{H}$, it follows that $\forall f \in \mathsf{H}, e \not\supseteq f$. But therefore, $\exists g \in Tr\mathsf{H}$ such that $g \cap e = \emptyset$, which is impossible since $e \in TrTr\mathsf{H}$. □

Definition 1.2.3. Let H be a hypergraph. Then H is *simple* if $\forall e, f \in \mathsf{H}, e \not\supset f$.

Theorem 1.2.2. *$\forall H, H$ is simple only if $H = TrTrH$* [1].

Proof. By Theorem 1.2.1 it follows that $TrTr\mathsf{H} \subseteq \mathsf{H}$. To show that $\mathsf{H} \subseteq TrTr\mathsf{H}$ if H is simple, assume that H is simple, and let $e \in \mathsf{H}$. Then e is a transversal for $Tr\mathsf{H}$. If $e \notin TrTr\mathsf{H}$ then $\exists f \subset e$ such that $f \in TrTr\mathsf{H}$. But $TrTr\mathsf{H} \subseteq \mathsf{H}$ (Theorem 1.2.1) and thus H is not simple, contrary to assumption. Whence $\mathsf{H} \subseteq TrTr\mathsf{H}$. □

Corollary 1.2.3. *$\forall H, \models \bigwedge\!\!\bigvee H \leftrightarrow \bigvee\!\!\bigwedge TrH$, and $\models \bigwedge\!\!\bigvee TrH \leftrightarrow \bigvee\!\!\bigwedge H$.*

Proof. The result can be proved using Theorem 1.2.1. □

Corollary 1.2.3 illustrates a sense in which for any hypergraph H, H and $Tr\mathsf{H}$ are logically dual to one another: since $\bigwedge\!\!\bigvee \mathsf{H}$ and $\bigvee\!\!\bigwedge \mathsf{H}$ are dual (in the sense in which $\vee$ and $\wedge$ are logical duals) Corollary 1.2.3 entitles us to assert that $\bigwedge\!\!\bigvee \mathsf{H}$ and $\bigwedge\!\!\bigvee Tr\mathsf{H}$ are dual, as well as $\bigvee\!\!\bigwedge \mathsf{H}$ and $\bigvee\!\!\bigwedge Tr\mathsf{H}$. Consequently, using the notion of a transverse hypergraph, we may devise a notion which is dual to k-uncolourability. We introduce this dualized notion as an independent class of objects, and then establish duality with the class of k-uncolourable hypergraphs.

Definition 1.2.4. Let H be a hypergraph, and let $k \in \mathbb{Z}^+$. By "$\binom{\mathsf{H}}{k}$" we denote the set of all k-tuple subsets of H, and we say that H is *k-wise intersecting* if $\forall B \in \binom{\mathsf{H}}{k}, \cap B \neq \emptyset$. The *harmonic number* of a hypergraph H, $\eta(\mathsf{H})$, is then defined:

$$\eta(\mathsf{H}) := \begin{cases} min\ k \in \mathbb{Z}^+ : \mathsf{H} \text{ is not } k\text{-wise intersecting} & \text{if this limit exists;} \\ \infty & \text{otherwise.} \end{cases} \tag{1.2.1}$$

If a hypergraph H is such that $\eta(\mathsf{H}) > k$ for some $k \geq 1$, then we say that H is k-harmonic; H is not k-harmonic, else.

The following theorem illustrates the sense in which k-harmonic hypergraphs are dual to k-uncolourable hypergraphs:

Theorem 1.2.4. *$\forall H, \eta(H) > k \Leftrightarrow \chi(TrH) > k$.*

Proof.

[$\Rightarrow$] Suppose that $\chi(Tr\mathsf{H}) = j \leq k$. Then $\exists \pi \in \Pi_j(\cup\mathsf{H})$ such that $\forall e \in Tr\mathsf{H}, \forall c \in \pi, e \not\subseteq c$. Let $B = \{\cup\mathsf{H} - c \mid c \in \pi\}$. Then $\forall b \in B, b$ is a transversal for $Tr\mathsf{H}$. But $\cap B = \emptyset$. Therefore for some $h \leq j$, there is an h-tuple of edges of $TrTr\mathsf{H}$ with an empty intersection. That is, $\eta(TrTr\mathsf{H}) \leq h \leq j \leq k$. But $TrTr\mathsf{H} \subseteq \mathsf{H}$ (Theorem 1.2.1). Whence $\eta(\mathsf{H}) \leq k$. Therefore $\eta(\mathsf{H}) > k$ only if $\chi(Tr\mathsf{H}) > k$.

[$\Leftarrow$] Suppose that $\eta(\mathsf{H}) = j \leq k$. Then there is a j-tuple of edges of H, $\{e_1, ..., e_j\}$ such that $\cap\{e_1, ..., e_j\} = \emptyset$. Let $\delta = \{\cup\mathsf{H} - e_i \mid 1 \leq i \leq j\}$. Then $\forall d \in \delta, \forall e \in Tr\mathsf{H}$, $e \not\subseteq d$ because $\forall e \in \mathsf{H}, e$ is a transversal for $Tr\mathsf{H}$. Therefore from δ one may construct a partition $\pi \in \Pi_j(\cup\mathsf{H})$ such that $\forall e \in Tr\mathsf{H}, \forall c \in \pi, e \not\subseteq c$. Therefore $\chi(Tr\mathsf{H}) \leq j \leq k$. Therefore $\chi(Tr\mathsf{H}) > k$ only if $\eta(\mathsf{H}) > k$.

□

1.3 A Democratic Polling Scheme

Assume that the nation's electorate is divided into constituencies in some conventional way. Assume further that each constituency elects to a legislative body (call it Parliament) some number of representatives proportionate to its population. While Parliament is thus composed of a cross-section of eligible voters, the question still remains as to whether or not the decisions made by Parliament adequately represent the public will. In cases where Parliament is dominated by a single party, the answer to this question is often argued to be "no". Consequently, it is sometimes argued, when a single party does hold a majority of the seats in the legislative assembly, political debate is moot.

Intuitively, one way around this alleged difficulty is to add an extra layer of structural complexity to the decision-making which occurs in Parliament, by reiterating the "transversal–formation" process by which elected officials essentially gain entry to the house in the first place. In this way, if it is a cross-section of Parliament which ends in shaping the legislative process, the predominance of the occurrence of the members of a single party in Parliament may not entail its domination of policy formation. But to design such an elaboration of the usual democratic scheme, we have to decide at least two things:

- where, structurally speaking, the re-iteration of the transversal–formation is to occur, and
- what is the hypergraph of which a transversal will be formed?

One possible solution is as follows:

1. Define the *level of divisiveness* of Parliament (P) with respect to an issue I, $d_I(P)$, as the least $k \in \mathbb{Z}^+$ such that $\exists \pi \in \Pi_k(\Sigma)$ such that $\forall c \in \pi, \forall x, y \in c$, x and y favour the same possible outcome with respect to I. At least initially, the level of divisiveness of the electorate should be smaller than or equal to the level of divisiveness of Parliament.

2. Form a *Parliamentary k-harmonic hypergraph* H, where $\cup\mathsf{H} = P$, and where k is set to $d_I(P)$. This is a decomposition of Parliament into $|\mathsf{H}|$ *Parliamentary committees*, every k-tuple collection of which having at least one member of Parliament in common.

3. Stipulate that debate over the issue I is to occur within the Parliamentary committees, after which each member of Parliament will cast a single vote. This stage is called the *initial vote*. Note that where $d_I(\cup\mathsf{H}) = j$ for this stage, the initial vote induces a j-partition π of $\cup\mathsf{H}$ such that $\forall c \in \pi, \forall x, y \in c, x$ and y agree about I.

4. Form $Tr\mathsf{H}$, and if $e \in Tr\mathsf{H}$ is a subset of some cell of π, count each of the initial votes of the elements of e together as a single vote. This stage is called the *final vote*. In fact, these are the only votes that directly affect the final decision about the issue under consideration. We thereby stipulate that all and only members' initial votes will be counted each time their vote is identical with the initial vote of every other member of an element of $Tr\mathsf{H}$ in which they happen to mutually appear.

Using the theories of k-harmonic hypergraphs and k-uncolourable hypergraphs, we now describe how and why it is that this particular scheme achieves the democratic virtues adduced above. To this end, the first question to consider is perhaps the most basic: "Why k-harmonic hypergraphs?"

In general, it is because a k-harmonic hypergraph is k-wise-intersecting. Consequently, if $k > 1$, then $\forall e \in \mathsf{H}$, e is a transversal for H. I.e., H is *self-representative.* So by decomposing Parliament into a k-harmonic hypergraph where $k = d_I(\cup\mathsf{H}) > 1$, we decompose a large representative body into a set of smaller ones which, because they are smaller, provide more fertile ground for effective debate and communication without preventing any single member of Parliament from communicating with any other.[4] Moreover, since it follows that $Tr\mathsf{H}$ is k-uncolourable if H is k-harmonic (Theorem 1.2.4), we have that $\exists e \in Tr\mathsf{H}$ such that $\forall x, y \in e$, x and y agree about I. One advantage of the scheme is therefore that some final vote will be counted.

[4] At least indirectly.

This is, however, peripheral to the main advantage of a polling scheme designed in this way, namely its structural enforcement of the democratic virtue of inter-interest group debate and consensus. For on this scheme the voting power of individual members of Parliament is filtered to the elements of a set $Tr\mathsf{H}$ of representative samples from Parliament, any one of which may contain members of competing parties. In this way, that a single party controls a majority of Parliamentary seats, does not entail that it has the ability to dominate the legislative process for the purposes of its own agenda. Indeed, because the structures which have legislative power may contain members of competing parties, it is in the interest of members of competing parties who appear in the same parliamentary committees to arrive at consensus; else they risk losing their legislative power altogether. To illustrate this point, we exploit the *restriction* of a hypergraph.

Definition 1.3.1. Let H be a hypergraph; let $S \subseteq \cup\mathsf{H}$. Then the *restriction* of H to S, $\mathsf{H}[S]$, is the hypergraph whose edge set is $\{e \in \mathsf{H} \mid e \subseteq S\}$.

Theorem 1.3.1. *$\forall H, H$ is k-uncolourable only if $\forall e \in TrH$, $H[e]$ is $(k-1)$-uncolourable.*

Proof. Let $e \in Tr\mathsf{H}$. If $H[e]$ is $(k-1)$-colourable then $\exists \pi \in \Pi_{k-1}(e)$ such that $\forall f \in \mathsf{H}[e]$, $\forall c \in \pi, f \nsubseteq c$. But $\forall f \in \mathsf{H}, f \nsubseteq \cup\mathsf{H} - e$. Therefore, $\exists \pi'(= \pi \cup \{\cup\mathsf{H} - e\}) \in \Pi_k(\cup\mathsf{H})$ such that $\forall e \in \mathsf{H}, \forall c \in \pi', e \nsubseteq c$. That is, $\chi(\mathsf{H}) \leq k$. □

Thus, if H is a parliamentary k-harmonic hypergraph, then we have that $\forall e \in \mathsf{H}$, if $d_I(e) \leq k-1$, then the members of some subset e' of e such that $e' \in Tr\mathsf{H}$ will mutually agree on some possible outcome for I, and consequently, some subset of e will have its final vote counted. But more than that, it would seem that as $d_I(e)$ *decreases*, the more likely it is that the number of subsets e' of e which have their final votes counted *increases*. To understand this, consider that if $d_I(e) = 1$, then $\forall e' \in (Tr\mathsf{H})[e]$, the members of e' have their final votes counted at least once. Or, if $d_I(e) = n$, and it turns out that $\chi((Tr\mathsf{H})[e]) \leq n$, then it is possible that no subsets e' of e with $e' \in Tr\mathsf{H}$ will be such that their members will have their final votes counted. Thus, by exploiting various properties of k-harmonic hypergraphs, it seems possible to design a legislative system imbued with a structural incentive for its constituent committees to reduce their respective levels of divisiveness. *Vertex-critical* k-harmonic hypergraphs therefore seem to be particularly helpful in this context, given that we have: if H is a vertex-critical k-harmonic hypergraph ($k > 1$), then $\forall e \in \mathsf{H}$, $\chi((Tr\mathsf{H})[e]) = k$.

1.3.1 Vertex Criticality

Definition 1.3.2. A hypergraph H is a *vertex-critical k-harmonic hypergraph* if H is a simple k-harmonic hypergraph, and $\forall x \in \cup \mathsf{H}, \exists j \leq k, \exists B \in \binom{\mathsf{H}}{j} : \cap B = \{x\}$.

Definition 1.3.3. A hypergraph H is *k-vertex critical* if $\chi(\mathsf{H}) > k$, and $\forall S \subset \cup \mathsf{H}, S \neq \emptyset \Rightarrow \chi(\mathsf{H}[S]) \leq k$.

Theorem 1.3.2. *A hypergraph H is a vertex-critical k-harmonic hypergraph* $\Leftrightarrow$ *TrH is k-vertex critical.*

Proof.

[$\Rightarrow$] Assume that H is a vertex-critical k-harmonic hypergraph. Then by Theorem 1.2.4, $\chi(Tr\mathsf{H}) > k$. We want to show that $\forall S_{\neq\emptyset} \subset \cup \mathsf{H}$, $\chi((Tr\mathsf{H})[S]) \leq k$. Let $S_{\neq\emptyset} \subset \cup \mathsf{H}$ be arbitrary. Let $x \in S$. Since H is a vertex critical k-harmonic hypergraph, $\exists j \leq k$, $\exists B \in \binom{\mathsf{H}}{j}$ such that $\cap B = \{x\}$. But $\forall e \in Tr\mathsf{H}, \forall b \in B, e \not\subseteq \cup \mathsf{H} - b$. Therefore $\chi((Tr\mathsf{H})[\cup \mathsf{H} - \{x\}]) \leq j \leq k$, and so by the downward monotonicity of colourability, $\chi((Tr\mathsf{H})[S]) \leq k$.

[$\Leftarrow$] Assume that $Tr\mathsf{H}$ is k-vertex critical. Then $\eta(\mathsf{H}) > k$ by Theorem 1.2.4. We want to show that $\forall x \in \cup \mathsf{H}$, $\exists j \leq k$, $\exists B \in \binom{\mathsf{H}}{j}$ such that $\cap B = x$. But since $Tr\mathsf{H}$ is k-vertex critical, we know that $\forall x \in \cup \mathsf{H}$, $\exists \pi \in \Pi_k(\cup \mathsf{H} - \{x\})$ such that $\forall e \in Tr\mathsf{H}, \forall c \in \pi, e \not\subseteq c$. But then $\cap\{\cup \mathsf{H} - c \mid c \in \pi\}(= C) = \{x\}$ and $\forall d \in C, \exists e \in \mathsf{H}$ such that $e \subseteq d$. Further, $\cap\{e \in \mathsf{H} \mid \exists d \in C$ such that $e \subseteq d\}(= D) = \{x\}$; otherwise, since $\forall f \in Tr\mathsf{H}, \forall g \in \{\cup \mathsf{H} - e \mid e \in D\}, f \not\subseteq g$, it follows that $\chi(Tr\mathsf{H}) \leq k$, contrary to our initial assumption. Therefore $\cap D = \{x\}$.

□

Theorem 1.3.3. *If H is simple, k-vertex critical and* $|H| \geq 2$, *then* $\forall e \in TrH, \chi(H[e]) = k$.

Proof. Assume that H is k-vertex critical. Then $\chi(\mathsf{H}) > k$. But therefore, $\forall e \in Tr\mathsf{H}, \chi(\mathsf{H}[e]) > k-1$ (Theorem 1.3.1). Now suppose that $\chi(\mathsf{H}[e]) > k$. Since H is simple and $|\mathsf{H}| \geq 2$, $\cup \mathsf{H} - e \neq \emptyset$. Therefore H is not k-vertex critical if $\chi(\mathsf{H}[e]) > k$, because $\chi(H[e]) > k$. Whence $\chi(\mathsf{H}[e]) \leq k$ and thus $\chi(\mathsf{H}[e]) = k$. □

Theorem 1.3.4. *If H is a vertex-critical k-harmonic hypergraph and* $k > 1$ *then* $\forall e \in H$, $\chi((TrH)[e]) = k$.

Proof. Assume that H is a vertex-critical k harmonic hypergraph where $k > 1$. Then $Tr\mathsf{H}$ is k-vertex critical (Theorem 1.3.2). Moreover, $|Tr\mathsf{H}| \geq 2$; for otherwise, $\mathsf{H} = \{\{x_1\}, ..., \{x_m\}\}$ for some $m \geq 1$, in which case $k = 1$. But then $\forall e \in TrTr\mathsf{H}, \chi((Tr\mathsf{H})[e]) = k$ (Theorem 1.3.3). But $\mathsf{H} = TrTr\mathsf{H}$ since H is simple (Theorem 1.2.2). Whence $\forall e \in \mathsf{H}, \chi((Tr\mathsf{H})[e]) = k$. □

1.3.2 Saturation

A *saturated* k-harmonic hypergraph is one that is in a certain sense *maximal* with respect to harmonic number. Therefore, in addition to vertex-critical hypergraphs, saturated k-harmonic hypergraphs, because they are maximally self-representative, also seem well-suited to the application we are considering, relative to harmonic number strictly greater than k. Moreover, on the practical side of things, it is a theorem that $\forall k > 1$, $\forall j > k$, there is a saturated k-harmonic hypergraph with j vertices. Because of this theorem, and particularly the algorithmic structure of its proof, a "random–hypergraph–generator" which decomposes Parliament into a saturated harmonic hypergraph of the appropriate level and size would not be too inefficient to implement. We don't go into details of the computational complexity of such a procedure, but simply point out that once an isomorphism class of any particular hypergraph is known, it remains only to enumerate randomly the $|\cup\mathsf{H}|$ members of Parliament in order to decompose the legislature appropriately [3]. We close with the formalism necessary to establish these latest results.

Definition 1.3.4. A hypergraph H *extends* a hypergraph H', $H \rhd H'$, if $\mathsf{H} \supset \mathsf{H}'$ and $\forall e \in \mathsf{H} - \mathsf{H}', \forall e' \in \mathsf{H}', e \not\supseteq e'$.

Definition 1.3.5. A hypergraph H is a *saturated k-harmonic hypergraph* if H is a simple k-harmonic hypergraph, and $\forall \mathsf{H}' \rhd \mathsf{H}$, if H' is simple then H' is not k-harmonic.

Theorem 1.3.5. *$\forall k > 1$, $\forall j > k$, $\exists H$ such that H is a saturated k-harmonic hypergraph with j vertices.*

Proof. We begin by defining a procedure which given any saturated k-harmonic hypergraph G having j vertices and at least two edges, generates a saturated k-harmonic hypergraph H with $j+1$ vertices ($1 < k < j$): Let $e \in Tr\mathsf{G}$ be arbitrary. Let x be a vertex not appearing in $\cup\mathsf{G}$. Let $f = \cup\mathsf{G} - e$, and let G' be $\mathsf{G} \cup \{f \cup \{x\}\}$. Further if $g \in \mathsf{G}'$ and $\exists B \in \binom{\mathsf{G}'}{k-1}$ such that $g \in B$ and $\cap B = e$, then replace $g \in \mathsf{G}'$ with $g \cup \{x\}$ to obtain the hypergraph G''. Then G'' is simple and k-harmonic: That G'' is simple follows from the fact that a new vertex x was added to f, and if x is added to any other edge $g \in \mathsf{G}$ then $e \subseteq g$; thus if $f \cup \{x\} \subseteq g \cup \{x\}$ then $g = \cup\mathsf{G}$, contrary to the simplicity of G.

That G'' is k-harmonic can be seen in the following way: Since $\eta(\mathsf{G}) > k$, to prove that $\eta(\mathsf{G}'') > k$ it suffices to show that $\forall B \in \binom{\mathsf{G}''-\{f\cup\{x\}\}}{k-1}$, $\cap B \cap (f \cup \{x\}) \neq \emptyset$; so, suppose not. Then $\cap B \subseteq e$. But, in that case $\cap B = e$. For suppose that $\cap B \subset e$. Let $\cap B = b$. Then $\forall g \in Tr\mathsf{G}, \forall d \in \{b\} \cup \{\cup\mathsf{G} - b' \mid b' \in B\}$, $g \not\subseteq d$. That is, $Tr\mathsf{G}$ is k-colourable, which contradicts our supposition that G is k-harmonic, given Theorem 1.2.4. So $\cap B = e$. But if $\cap B = e$, then by construction, x is added to every element of B. That is, $x \in \cap B$, which is absurd since $x \notin e$. Whence $\eta(\mathsf{G}'') > k$.

Now to obtain a k-saturated graph H from G'', extend G'' using elements of $\wp(\cup\mathsf{G}'')$ in such a way so as to preserve simplicity and k-harmonicity. We can say that if H is the largest simple hypergraph such that H extends G'', and $\eta(\mathsf{H}) > k$, then H is a saturated k-harmonic hypergraph having $j+1$ vertices. Whence, given a saturated k-harmonic hypergraph with j vertices, and having at least 2 edges, we may construct one with $j+1$ vertices. But as $\forall k > 1$, there are saturated k-harmonic hypergraphs with $k+1$ vertices, namely, the set of all k-tuple subsets of a $(k+1)$-membered set, our theorem follows straightaway.

□

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Chapter 2

Self-dual Malfunctions

D. Nicholson

Abstract

In this article, we use a maximality condition on the harmonic number of a hypergraph to derive an original maximality condition on chromatic number. We also show the relevance of these conditions to Reiter's theory of diagnosis from first principles [12], by using them, and a related minimality condition on chromatic number, to characterize the class of self-dual system malfunctions. Characterizing the class of self-dual malfunctions reduces to characterizing the conditions under which, for an arbitrary finite hypergraph $H = \{e_1, e_2, ..., e_i, ..., e_m\}$ whose vertices are sentences,

$$\bigvee_{i=1}^{|m|} \bigwedge_{j=1}^{|e_i|} \alpha_j \;\dashv\vDash\; \bigwedge_{i=1}^{|m|} \bigvee_{j=1}^{|e_i|} \alpha_j.$$

This research illustrates the applicability of the theory of harmonics for hypergraphs to diagnostic theory, as well as to a general theory of logical duality.

2.1 Introduction

Diagnostic problems require a differentiation between normally and abnormally behaving system components. But given a malfunction in a system, two kinds of diagnostic questions can be asked. First,

1. What went wrong? That is, which component(s) is (are) misbehaving?

And second,

2. Why is it broken? Or, which components are responsible for the problem?

Pretheoretically, one might assume that the answers to these questions coincide. But this assumption results in an expectation which can be responsible for mistaken or missed diagnoses. Such is the case, for example, when the signs of a disease are confused with the disease itself. Given an observation of psychopathology, for instance, one could cite the elements of the nervous system responsible for regulating brain chemicals in response to 1. But to assume that this answer is adequate with respect to 2 risks missing a diagnosis of an underlying anatomical condition, such as a tumour, which is causing the pathology. Reiter's model of diagnosis in a technical system, a form of *diagnosis from first principles* [12], is an environment in which the nature of this fallacy is made plain. Using this model, an appropriate answer to 1 is the set of all minimal lists of system components the members of which it is inconsistent to suppose are functioning normally, in the presence of an observation of the system and a description of the system's normal behaviour. An adequate answer to 2 consists of the set of all minimal lists of system components which it is consistent to suppose the members of which are faulty while also supposing that the other components are not. In this article, characterizing the class of system malfunctions for which the answers to 1 and 2 are identical is shown to be reducible to characterizing the conditions under which, for an arbitrary finite hypergraph $H = \{e_1, e_2, ..., e_i, ..., e_m\}$ whose vertices are sentences,

$$\bigvee_{i=1}^{m} \bigwedge_{j=1}^{|e_i|} \alpha_j \dashv\vDash \bigwedge_{i=1}^{m} \bigvee_{j=1}^{|e_i|} \alpha_j .$$

Consequently, because of the duality of $\wedge$ and $\vee$, characterizing the relationship between answers to 1 and 2 contributes to a general theory of logical duality. Alternatively, the class of malfunctions for which the answers to 1 and 2 are identical can be characterized in terms of a dual formulation of a maximality measure of chromatic number for hypergraphs. This fact demonstrates the applicability of the harmonic theory of hypergraphs both to diagnosis from first principles as well as to a general theory of logical duality.

2.1.1 Applications

In this article we focus on the system of diagnostics known as 'diagnosis from first principles'. As Reiter notes, there are two divergent approaches in the literature. He writes:

> In the first approach, often referred to as diagnosis from first principles, one begins with a description of some system—a physical device or real-world setting of interest, say—together with an observation of the system's behaviour. If this observation conflicts with the way the system is meant to behave, one is confronted with a diagnostic problem, namely, to determine those system components which, when assumed to be functioning abnormally, will explain the discrepancy between the observed and correct system behaviour. For solving this diagnostic problem from first principles, the only available information is the system description, i.e. its design or structure, together with the observation(s) of the system behaviour. In particular, no heuristic information about system failures is available, for example, of the kind "When the system exhibits such and such aberrant behaviour, then in 90% of these cases, such and such components have failed." Notable examples of approaches to diagnostic reasoning from first principles are [4][5][6][7][10][11].
>
> Under the second approach to diagnostic reasoning, which might be described as the experiential approach, heuristic information plays a dominant role. The corresponding diagnostic reasoning systems attempt to codify the rules of thumb, statistical intuitions, and past experience of human diagnosticians considered experts in some particular task domain. The structure or design of the corresponding real world system being diagnosed is only weakly represented, if at all. Successful diagnoses stem from the codified experience of the human expert being modeled, rather than from what is often referred to as "deep" knowledge of the system being diagnosed. A notable example of such an approach to diagnosis from experience is the MYCIN system [2]. [12]

Here we take the first approach, and, using Reiter's theory in particular, we tacitly focus on technical systems representable in a language which satisfies the following conditions:

1. every sentence has exactly one of two values upon interpretation, and
2. the set of constants of the language includes $\wedge$, $\vee$, and $\neg$, which are given their usual Boolean interpretations.

Accordingly, this theory has the potential to accommodate a wide range of diagnostic problems. Reiter notes that

> time varying digital hardware have[sic] natural representations in a temporal logic [9] and this might form the basis for a diagnostic reasoning system for these devices. Similarly, time varying physiological properties are central to certain kinds of medical diagnosis tasks [13]. Database logic has been proposed for representing many forms of databases [8] so that violation of database integrity constraints might profitably be viewed as a diagnostic reasoning problem with database logic providing the system description language [12].

2.2 Conflict Sets and Diagnoses

As defined in [12], a *technical system* S is a pair $(SD,\ COMP = \{c_1, ..., c_n\})$, where SD is a set of sentences describing a system, and $COMP$ is a finite set of *components.* For any system S, SD describes how the system normally behaves. Introducing the set OBS, a finite set of sentences representing an observation of S, and given a unary predicate 'AB', interpreted "is behaving abnormally", a determination that S is malfunctioning, given OBS, is a determination that

$$OBS \cup SD \cup \{\neg AB(c_1), ..., \neg AB(c_n)\} \vdash \bot.$$

This statement is a representation of the fact that the system is faulty. Typically, however, when confronted with a malfunction, concern is focused on (minimal) sets of components which are behaving abnormally. Whence a *conflict set*[1] for (S, OBS) is a set $e \subseteq COMP$ such that:

$$OBS \cup SD \cup \{\neg AB(c) \mid c \in e\} \vdash \bot,$$

and a *minimal conflict set* is a conflict set e such that $\forall f \subset e$,

$$OBS \cup SD \cup \{\neg AB(c) \mid c \in f\} \not\vdash \bot.$$

But from merely having identified a least set of such "problematic" components in a system that is behaving badly, where a problem is represented as a logical inconsistency, it does not follow

[1] The notion of a conflict set is due to de Kleer [5].

that one has identified a least set e of components for which the assumption that *each member of e is behaving abnormally, and every component in $COMP - e$ is behaving normally*, resolves the problem.

To illustrate, let S be a system with components c_1, c_2, c_3, c_4, and let OBS be an observation of a malfunction, which is consistent with a consistent system description SD. Suppose further that $\{c_1, c_2\}$ and $\{c_3, c_4\}$ are the minimal conflict sets for the malfunction. Then we have:

$$OBS \cup SD \cup \{\neg AB(c_1), \neg AB(c_2)\} \vdash \bot \tag{2.2.1}$$

$$OBS \cup SD \cup \{\neg AB(c_3), \neg AB(c_4)\} \vdash \bot \tag{2.2.2}$$

But there are four sets of components such that the assumption that each member is behaving abnormally, while every other component is not, is consistent, namely, $\{c_1, c_4\}, \{c_1, c_3\}, \{c_2, c_3\}$, and $\{c_2, c_4\}$. In other words, and taking the set $\{c_1, c_4\}$ as an example, we have:

$$OBS \cup SD \cup \{AB(c_1), AB(c_4)\} \cup \{\neg AB(c_2), \neg AB(c_3)\} \not\vdash \bot \tag{2.2.3}$$

and

$$OBS \cup SD \cup \{AB(c_1)\} \cup \{\neg AB(c_2), \neg AB(c_3), \neg AB(c_4)\} \vdash \bot \tag{2.2.4}$$

$$OBS \cup SD \cup \{AB(c_4)\} \cup \{\neg AB(c_1), \neg AB(c_2), \neg AB(c_3)\} \vdash \bot \tag{2.2.5}$$

$$OBS \cup SD \cup \{\neg AB(c_1), \neg AB(c_2), \neg AB(c_3), \neg AB(c_4)\} \vdash \bot. \tag{2.2.6}$$

Thus the notion of a (minimal) conflict set, by itself, is insufficient for diagnosing a malfunction, where, intuitively speaking, a *diagnosis* is a conjecture that certain of the components are faulty, and the others are not [12]. Specifically, a *diagnosis* for (S, OBS) is a minimal set $e \subseteq COMP$ such that:

$$OBS \cup SD \cup \{AB(c) \mid c \in e\} \cup \{\neg AB(c) \mid c \in COMP - e\} \not\vdash \bot.$$

Given a malfunction in a system, and the set of all minimal conflict sets for it, the paramount diagnostic question is therefore: How does one generate the set of all diagnoses? As shown in [12], this problem is reducible to that of generating the transverse hypergraph TH for an arbitrary hypergraph H.

2.3 Hypergraphs and Transverse Hypergraphs

If $V_{\neq\emptyset}$ is a set and $E = \{e_1, e_2, ..., e_i, ...\} \subseteq 2^V$ then the pair (V, E) is a *hypergraph* H with V the *vertex* set of H and E the set of *edges* of H. Since for most purposes V can be taken to be $\bigcup_{i=1} e_i$, abbreviated '$\cup H$', H can be identified with E, and we can refer to the edges of H by speaking of its elements. If $\forall e, f \in H, e \not\supset f$, then H is a *simple* hypergraph. If H is a hypergraph and S is a set then S is a *transversal* for H if S has a non-empty intersection with every edge of H. A *minimal transversal* for H is a transversal for H which is not a proper superset of any transversal. The *transverse hypergraph* of H, TH, is the set of all minimal transversals for H.

Proposition 2.3.1. *A set e is a diagnosis for (S, OBS) iff $COMP - e$ is a maximal subset of $COMP$ which is not a superset of any conflict set.*

Proof. The result is proved in [12]. □

Theorem 2.3.2. *If H is the set of all minimal conflict sets for (S, OBS), then TH is the set of all diagnoses* [Adapted from [12]].

Proof. Assume that H is the set of all minimal conflict sets for (S, OBS). Then the set of relative complements of TH-edges with respect to $COMP$ is the set of all maximal subsets of $COMP$ which are not supersets of any conflict sets. Therefore, from Proposition 2.3.1 it follows that TH is the set of all diagnoses for (S, OBS). □

Theorem 2.3.2 exemplifies the relevance of the theory of transverse hypergraphs to diagnosis from first principles. In what follows, this connexion is further exploited in the formulation of independent and dual characterizations of the class of system malfunctions for which the set of all minimal conflict sets is identical to the set of diagnoses.

2.3.1 Logical Duality

The transverse hypergraph of a hypergraph is its dual. To demonstrate this we formulate hypergraphs as sentences. We assume in what follows that all hypergraphs are *finite*, that is, they have finite vertex sets.

If the vertices of a hypergraph H are sentences, then the $\vee$*-formulation* of H, $F^{\vee}(H)$, is defined:

$$F^{\vee}(H) := \bigvee_{i=1}^{|H|} \bigwedge_{j=1}^{|e_i|} \alpha_j \in e_i$$

and the $\wedge$-*formulation* of H, $F^{\wedge}(H)$, is defined

$$F^{\wedge}(H) := \bigwedge_{i=1}^{|H|} \bigvee_{j=1}^{|e_i|} \alpha_j \in e_i.$$

Lemma 2.3.3. *$\forall H, e \in H, \exists f \in TTH$ such that $e \supseteq f$, $\forall H, TTH \subseteq H$, and $\forall H$, if H is simple then $H = TTH$.*

Proof. Let H be a hypergraph. Since every edge of H is a transversal for TH we have $\forall e \in H, \exists f \in TTH$ such that $e \supseteq f$.

Suppose that $e \in TTH$, but that $e \notin H$. Since every edge of H is a transversal for TH, $\forall f \in H, e \not\supseteq f$. Therefore every element of H has an element that is not a member of e. So $\exists f \in TH$ such that $f \cap e = \emptyset$, which is impossible because $e \in TTH$. Whence $\forall H, TTH \subseteq H$.

Assume now that H is simple, and let $e \in H$. We know that e is a superset of some minimal transversal f for TH. Suppose that $e \supset f$. Because H is simple, and because $TTH \subseteq H$, $\forall g \in H, f \not\supseteq g$. So every element of H possesses an element that is not in f. Therefore $\exists g \in TH$ such that $g \cap f = \emptyset$, which is impossible because $f \in TTH$. Whence $e = f$, and $H \subseteq TTH$ if H is simple. Consequently, $H = TTH$ if H is simple.

□

Theorem 2.3.4. *$\forall H, F^{\vee}(H) \dashv\models F^{\wedge}(TH)$ and $F^{\wedge}(H) \dashv\models F^{\vee}(TH)$.*

Proof. Theorem 2.3.4 is easily proved using Lemma 2.3.3. □

Now $F^{\vee}(H)$ is dual to $F^{\wedge}(H)$. The notion of duality which is meant here is not just the familiar one from Church [3] whereby one obtains what he calls the *principal dual* of a sentence by replacing each occurrence of a connective with its dual, where the *dual of a connective* may be obtained by interchanging all occurrences of 1's and 0's in its truth table. Thus, $\vee$ and $\wedge$ are dual, and therefore $F^{\vee}(H)$ and $F^{\wedge}(H)$ are principal duals. By 'duality' we mean to extend the notion of duality in Church, which is specific to connectives, to entire formulae, so that one formula, α, is *dual* to another, β, iff α is equivalent to the truth function obtained by interchanging all occurrences of 1's and 0's in the truth table for β. Whence we have the following principle of duality:

Proposition 2.3.5. *If a sentence α is dual to a sentence β, and α is truth functionally equivalent to a sentence δ, and β is truth functionally equivalent to a sentence γ, then δ is dual to γ.*

Using Theorem 2.3.4 and Proposition 2.3.5 we have:

Theorem 2.3.6. *$\forall H, F^{\vee}(H)$ is dual to $F^{\vee}(TH)$ and $F^{\wedge}(H)$ is dual to $F^{\wedge}(TH)$.*

Theorem 2.3.6 illustrates the sense in which H and TH are dual, for any hypergraph H. In this sense of duality, the *chromatic number* of a hypergraph can be said to be dually expressed as its *harmonic number.*

If n is a positive integer, let '$[n]$' abbreviate '$\{1, 2, ..., n\}$'. Then if H is any hypergraph, a function $f : \cup H \rightarrow [n]$ is an *n-colouring* of H $(n \geq 1)$ if $\forall e \in H, \exists x, y \in e : f(x) \neq f(y)$. That is, a function $f : \cup H \rightarrow [n]$ is an n-colouring of H if it assigns a colour to every vertex in such a way that no edge is monochrome.

The *chromatic number* of H, $\chi(H)$, is defined:

$$\chi(H) := \begin{cases} \mathit{min}\ n \in \mathbb{Z}^+ : \text{ there is an } n\text{-colouring of } H & \text{if this limit exists;} \\ \infty & \text{otherwise.} \end{cases}$$

If $\chi(H) \leq n$ then H is *n-colourable*; else H is *n-uncolourable.* An *n-chromatic* hypergraph is one whose chromatic number is n.

For any hypergraph H, the *harmonic number* of H, $\eta(H)$, is defined:

$$\eta(H) := \begin{cases} \mathit{min}\ n \in \mathbb{Z}^+ : \exists G \in \binom{H}{n} : \cap G = \emptyset & \text{if this limit exists;} \\ \infty & \text{otherwise,} \end{cases} \tag{2.3.1}$$

where $\binom{H}{n}$ is the set of all n-tuple subsets of H, and for any hypergraph G, '$\cap G$' abbreviates '$\bigcap_{i=1} g_i \in G$'. If $\eta(H) = n$ then H is *n-harmonic.*

Theorem 2.3.7. *$\forall H, n \geq 1, \chi(H) = n \Leftrightarrow \eta(TH) = n$.*

Proof. It is sufficient to show that $\forall H, m \geq 0, \chi(H) > m$ iff $\eta(TH) > m$. To that end:

[$\Rightarrow$] Assume that $\eta(TH) \leq m$. Then $\exists l \leq m, \exists A \in \binom{TH}{l}$ such that $\cap A = \emptyset$. Let $B = \{\cup H - a \mid a \in A\}$. Then B induces an l-colouring of H. Therefore $\chi(H) \leq m$.

[$\Leftarrow$] Assume that $\chi(H) \leq m$. Then $\exists l \leq m$ such that there is a function $f : \cup H \rightarrow [l]$ such that $\forall e \in H, \forall i \in [l], e \not\subseteq \{x \in \cup H \mid f(x) = i\}$. Let $A = \{\{x \in \cup H \mid f(x) = i\} \mid i \in [l]\}$. Let $B = \{\cup H - a \mid a \in A\}$. Then $\forall b \in B, b$ is a transversal for H. Further, $\cap B = \emptyset$. Therefore $\exists k \leq l, \exists C \in \binom{TH}{k}$ such that $\forall c \in C, \exists b \in B, c \subseteq b$ and $\cap C = \emptyset$. Whence $\eta(TH) \leq m$.

□

2.4 Self-duality

For a simple hypergraph H, the identity of H and TH, or, the *self-duality* of H, can be expressed as the truth functional equivalence of the dual sentences $F^{\vee}(H)$ and $F^{\wedge}(H)$. That is,

Theorem 2.4.1. *$\forall H$, if H is simple then $H = TH \Leftrightarrow F^{\vee}(H) \ \dashv\models\ F^{\wedge}(H)$.*

Proof. Assume that H is simple.

[$\Rightarrow$] Suppose that $H = TH$. Suppose that $F^{\vee}(H)$ is true on some valuation ν. Then $F^{\wedge}(TH)$ is true on ν (Theorem 2.3.4), in which case, since $H = TH$, $F^{\wedge}(H)$ is true on ν.

Suppose now that $F^{\wedge}(H)$ is true on ν. Then $F^{\wedge}(TH)$ is true on ν. But then $F^{\vee}(H)$ is true on ν as well (Theorem 2.3.4).

[$\Leftarrow$] Assume that $F^{\vee}(H)$ is truth functionally equivalent to $F^{\wedge}(H)$. Let $e \in H$. Suppose that every element of e is true on some valuation ν. Then $F^{\wedge}(H)$ is true on ν, so e is a transversal for H. Therefore $\exists f \in TH$ such that $e \supseteq f$. Now let $e \in TH$, and and suppose that every element of e is true on some valuation ν. Then $F^{\vee}(TH)$ is true on ν. So $F^{\wedge}(H)$ is true on ν (Theorem 2.3.4), and thus $F^{\vee}(H)$ is true on ν, by assumption. Whence $\exists f \in H$ such that $e \supseteq f$. So every edge of H contains an edge of TH, and every edge of TH contains an edge of H. Therefore since both H and TH are simple, $H = TH$. □

Because of Theorems 2.4.1 and 2.3.2, we may call a system malfunction for which the set of all minimal conflict sets is identical to the set of all diagnoses a *self-dual malfunction*. Because of Theorem 2.4.1, by characterizing the class of self-dual malfunctions in terms of chromatic and harmonic properties we demonstrate an application of the theories of harmonics and chromatics to a general theory of logical duality.

In [1], Berge shows that if H is a simple hypergraph, then $H = TH$ iff H is pairwise intersecting and 2-uncolourable, i.e., iff $\chi(H) > 2$ and $\eta(H) > 2$. More specifically we can show:

Theorem 2.4.2. *$\forall H$, if H is simple then $H = TH \Leftrightarrow$ either $H = \{\{x\}\}$ or $\chi(H) = \eta(H) = 3$.*

To prove this theorem we require the notion of a *partition* of a set: Let n be a positive integer, and let S be a set. Then the set of *n-partitions of S*, $\Pi_n(S)$ is defined:

$$\Pi_n(S) := \{\pi = \{c_1, ..., c_n\} \mid \forall i, j\ (1 \leq i < j \leq n), c_i \cap c_j = \emptyset \text{ and } \bigcup_{i=1}^{n} c_i = S\}$$

Proof. Let H be simple.

[$\Rightarrow$] Assume that $H = TH$, and that $\forall x \in \cup H, H \neq \{\{x\}\}$. Then $\eta(H) > 2$ because every edge of H is a transversal for H. Also, $\chi(H) > 2$ else there is a 2-partition π of H such that $\forall c \in \pi, e \in H, e \not\subseteq c$, in which case each $c \in \pi$ is a transversal for H, in which case $\exists A \in \binom{H}{2}$ such that $\cap A = \emptyset$, contrary to $\eta(H) > 2$. Now since $H = TH$, $\forall e \in H, x \in e, \exists A \in \binom{H}{2} : e \in A$ and $\cap A = \{x\}$. Moreover, since $\forall x \in \cup H, H \neq \{\{x\}\}, |H| \geq 2$ because $|TH| > 2$. Let $e \in H$ be arbitrary, and let $x \in e$. Then $\exists f \in H : x \notin f$. Otherwise $\{x\} \in TH$, contrary to the simplicity of H. $\therefore \eta(H) = 3$. But then $\chi(TH) = 3$ (Theorem 2.3.7, Lemma 2.3.3), in which case $\chi(H) = 3$.

[$\Leftarrow$] Assume that $\chi(H) = \eta(H) = 3$. Let $e \in H$. Since $\eta(H) > 2$, e is a transversal for H, so $\exists f \subseteq e$ such that $f \in TH$. Suppose that $f \subset e$. Since $\chi(H) > 2$, $\eta(TH) > 2$ (Theorem 2.3.7). Therefore f is a transversal for TH. So $\exists g \subseteq f$ such that $g \in TTH$. But $TTH \subseteq H$ (Lemma 2.3.3). Therefore $g \in H$, contrary to the simplicity of H. Whence $e = f$ and $H \subseteq TH$.

Now let $e \in TH$. Since $\eta(TH) > 2$ (Theorem 2.3.7), e is a transversal for TH. So exists $f \subseteq e$ such that $f \in H$, since $TTH \subseteq H$ (Lemma 2.3.3). Suppose that $f \subset e$. Then $\exists g \in TTH$ such that $g \cap f = \emptyset$. But $H = TTH$ (Lemma 2.3.3). Whence $\eta(H) \leq 2$, contrary to the assumption that $\eta(H) = 3$. Therefore $e = f$ and $TH \subseteq H$. Consequently, $H = TH$.

Lastly, suppose that $H = \{\{x\}\}$. Then trivially $H = TH$. Therefore $H = TH$. ☐

In fact, the class of hypergraphs identical to their transverse hypergraphs can be characterized independently of chromatic properties by exploiting a maximality condition imposed on harmonic number. This class can also be characterized chromatically, independently of harmonic properties, by dualizing the harmonic maximality condition.

2.4.1 Harmonic Maximality

If H is n-harmonic then H is *maximally n-harmonic* if for any set S such that $\forall e \in H, S \not\supseteq e$, it follows that $\eta(H \cup \{S\}) < n$.

Theorem 2.4.3. *Let H be simple. Then $H = TH \Leftrightarrow$ either $H = \{\{x\}\}$, for some x, or H is maximally 3-harmonic.*

Proof. Assume that H is simple.

[⇒] Assume that $H = TH$ and that $H \neq \{\{x\}\}$, for any $x \in \cup H$. Let S be such that $\forall e \in H, S \not\supseteq e$: Then $\exists f \in TH(= H) : f \cap S = \emptyset$. $\therefore \eta(H \cup \{S\}) \leq 2$. But from Theorem 2.4.2 we have $\eta(H) = 3$. Whence H is maximally 3-harmonic.

[⇐] Assume that H is maximally 3-harmonic. Let e be in H. Then since $\eta(H) > 2, \exists f \subseteq e$ such that $f \in TH$. Suppose $f \subset e$. Then $\eta(H \cup \{f\}) = \eta(H)$, which is absurd by the maximality of H with respect to η. Whence $H \subseteq TH$.

Now let $e \in TH$. Then by the maximality of H, $\exists f \in H$ such that $e \supseteq f$. If $e \supset f$ then, since $\eta(H) > 2$, e is not a minimal transversal for H. $\therefore TH \subseteq H$ and $H = TH$.

If $H = \{\{x\}\}$ then trivially $H = TH$. Thus we may conclude that $H = TH$.

□

2.4.2 Chromatic Minimality

Self-dual malfunctions can be characterized independently of harmonic properties by dualizing harmonic maximality, by means of which we obtain a minimality condition on chromatic number.

If S is a set and n is a positive integer, the set of *n-decompositions of* S, $\Delta_n(S)$ is:

$$\Delta_n(S) := \{\delta = \{d_1, ..., d_n\} \mid \bigcup_{i=1}^{n} d_i = S\}.$$

If H and G are hypergraphs then G *defeats* H if $\forall e \in H, g \in G, e \not\subseteq g$. Thus, $\forall n \geq 1, H, \chi(H) > n$ iff $\forall \delta \in \Delta_n(\cup H)$, δ does not defeat H. If H is n-chromatic then H is *minimally n-chromatic* if $\forall e \in H, \exists \delta \in \Delta_{n-2}(\cup H - e)$ such that δ defeats H. In other words, a hypergraph H is minimally n-chromatic if $\chi(H) = n$ and $\forall e \in H$, deleting the vertices of e from H results in a hypergraph G with $\chi(G) = n - 2$. Lovász is reputed to have conjectured that if a graph (of graph theory)[2] G is minimally n-chromatic then its edge set comprises the set of all pairs from an n-membered set [14] (p. 191).

Theorem 2.4.4. *$\forall H, n \geq 1$, H is minimally n-chromatic iff TH is maximally n-harmonic.*

Proof.
[⇒] Assume that H is minimally n-chromatic, for some $n \geq 1$. We have TH is n-harmonic using Theorem 2.3.7. Now let S be a set such that $\forall e \in TH, S \not\supseteq e$. Then $\exists f \in TTH$ such that $S \cap f = \emptyset$.

[2] A graph (of graph theory) is a hypergraph whose edges are pairs.

But $TTH \subseteq H$ (Lemma 2.3.3). So $f \in H$. Since H is minimally n-chromatic, $\exists\delta \in \Delta_{n-2}(\cup H - f)$ such that δ defeats H. Let $A = \{\cup H - d \mid d \in \delta\}$. Then where $m \leq n-2, \exists B \in \binom{TH}{m} : \forall a \in A, \exists b \in B : a \supseteq b$. But $S \cap \cap B = \emptyset$. Therefore $\exists C \in \binom{TH \cup \{S\}}{l}$, where $l \leq n-1$, such that $\cap C = \emptyset$. Therefore $\eta(TH \cup \{S\}) < n$, whence TH is maximally n-harmonic.

[$\Leftarrow$] Assume that TH is maximally n-harmonic. We have H is n-chromatic using Theorem 2.3.7. Let $e \in H$. Since TH is maximally n-harmonic, $\eta(TH \cup \{(\cup H) - e\}) < n$. Therefore where $m \leq n-2, \exists A \in \binom{TH}{m} : \cap A \cap ((\cup H) - e) = \emptyset$. Let $B = \{(\cup H) - a \mid a \in A\}$. Then $B \in \Delta_m((\cup H) - e)$ and B defeats H. Therefore $\exists\delta \in \Delta_{n-2}((\cup H) - e) : \delta$ defeats H. Therefore H is minimally n-chromatic. □

Theorem 2.4.5. *$\forall H$, if H is simple then $H = TH \Leftrightarrow$ either $H = \{\{x\}\}$ for some x, or H is minimally 3-chromatic.*

Proof. Let H be simple.

[$\Rightarrow$] Assume that $H = TH$. Then either $\exists x : H = \{\{x\}\}$, or H is maximally 3-harmonic (Theorem 2.4.3). If H is maximally 3-harmonic then TH is maximally 3-harmonic, whence H is minimally 3-chromatic (Theorem 2.4.4).

[$\Leftarrow$] Assume that either $H = \{\{x\}\}$, for some x, or H is minimally 3-chromatic. If H is minimally 3-chromatic then TH is maximally 3-harmonic (Theorem 2.4.4). Therefore $TH = TTH = H$ (Theorem 2.4.3 and Lemma 2.3.3). □

2.4.3 Chromatic Maximality

Although chromatic minimality appears to be a minimality condition, for simple hypergraphs it is equivalent to a maximality condition. Chromatic maximality yields another characterization of the class of self-dual malfunctions.

If G and H are hypergraphs then G *subsumes* H, $G \sqsupseteq H$, if $\forall e \in G, \exists f \in H : e \supseteq f$; G *properly subsumes* H, $G \sqsupset H$, if $G \sqsupseteq H$, and $G \neq H$. If H is a hypergraph and n is a positive integer, then H is *maximally n-chromatic* if H is n-chromatic and $\forall G \sqsupset H$, if G is simple then $\chi(G) < n$.

Theorem 2.4.6. *$\forall H$, if H is simple then H is minimally n-chromatic $\Leftrightarrow$ H is maximally n-chromatic.*

Proof. Let H be simple.

[$\Rightarrow$] Assume that H is minimally n-chromatic. Let G be any simple hypergraph such that $G \sqsupset H$. Now if $G \sqsupset H$ then either $G \subset H$ or $\exists g \in G, h \in H : g \supset h$. Suppose that $G \subset H$. Let $e \in H - G$. Since H is minimally n-chromatic, $\exists \delta \in \Delta_{n-2}(\cup H - e) : \delta$ defeats H. Therefore $\delta \cup \{e\} \in \Delta_{n-1}(\cup H)$ and $\delta \cup \{e\}$ defeats G, in which case $\chi(G) \leq n-1$. So suppose that $\exists g \in G, h \in H : g \supset h$. Since H is minimally n-chromatic, $\exists \delta \in \Delta_{n-2}(\cup H - h) : \delta$ defeats H. Therefore $\delta \cup \{h\} \in \Delta_{n-1}(\cup H)$ and $\delta \cup \{h\}$ defeats G. Where $\delta \cup \{h\} = \{h, d_1, d_2, ..., d_{n-2}\}$, let δ' be the result of adding any elements of $\cup G - \cup H$ to d_{n-2}. Then δ' defeats G, in which case $\chi(G) \leq n-1$. Therefore H is maximally n-chromatic.

[$\Leftarrow$] Assume that H is maximally n-chromatic. Suppose that H is not minimally n-chromatic. Then $\exists e \in H$ such that $\forall \delta \in \Delta_{n-2}(\cup H - e)$, δ does not defeat H. Notice that $\exists \delta \in \Delta_{n-1}(\cup H) : \delta$ defeats $H - \{e\}$; else since $H - \{e\}$ is simple and $H - \{e\} \sqsupset H$, H is not maximally n-chromatic. Let $A = \{d \in \delta \mid \delta \in \Delta_{n-1}(\cup H), \delta$ defeats $H - \{e\}$ and $e \subsetneq d\}$. Let $G = H - \{e\} \cup TTA$. Then $\chi(G) \geq n$, G is simple, and $G \sqsupset H$, in which case H is not maximally n-chromatic, contrary to assumption. Therefore H is minimally n-chromatic. □

Theorem 2.4.7. *$\forall H$, if H is simple then $H = TH \Leftrightarrow$ either $H = \{\{x\}\}$ for some x, or H is maximally 3-chromatic.*

Proof. Assume that H is simple.

[$\Rightarrow$] Assume that $H = TH$. Then using Theorem 2.4.5, $H = \{\{x\}\}$ for some x or H is minimally 3-chromatic. Therefore either $H = \{\{x\}\}$ or H is maximally 3-harmonic, given Theorem 2.4.6.

[$\Leftarrow$] Assume that either $H = \{\{x\}\}$, for some x, or H is maximally 3-chromatic. If H is maximally 3-chromatic then H is minimally 3-chromatic (Theorem 2.4.6). But if H is minimally 3-chromatic or $H = \{\{x\}\}$, then $H = TH$ (Theorem 2.4.5). Thus $H = TH$.

□

2.5 Conclusion

In this article we have shown a relationship between Reiter's theory of diagnosis from first principles and the harmonic theory of hypergraphs. In particular, we have shown that a malfunction for a system (S, OBS) is self-dual iff the set H of all of its minimal conflict sets is either identical to the unit set of a unit set, or is maximally 3-harmonic. Furthermore, we have shown that the harmonic maximality of a transverse hypergraph TH is dual to both the chromatic minimality and maximality of the hypergraph H. Thus, self-duality can also be characterized in terms of a minimality condition on chromatic number, as well as a maximality condition. Future, hypergraphic, research in this area is suggested by the following questions:

1. Is Lovász' conjecture, as reported in [14] (p. 191), true? That is, if a graph G is minimally n-chromatic then is it the case that G is the complete graph on n vertices? Dually, this question amounts to:
2. If a hypergraph H is maximally n-harmonic, then is H the set of all $(n-1)$-tuple subsets of an n-membered set?
3. Is it the case that $\forall n \geq 1, \forall p \geq n$, there is a maximally n-harmonic (minimally (maximally) n-chromatic) hypergraph having p vertices?
4. How many maximally n-harmonic (minimally (maximally) n-chromatic) hypergraphs are there having a vertex set of size p $(p \geq n \geq 1)$?

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Chapter 3

An Axiomatization of Family Resemblance

D. Nicholson and R.E. Jennings[1]

Abstract

We invoke concepts from the theory of hypergraphs to give a measure of the closeness of family resemblance, and to make precise the idea of a composite likeness. It is shown that for any positive integer m, for any general term possessing any extent of family resemblance strictly greater than m, there is a taxonomical representation of the term whereby each subordinate taxon has an extent of family resemblance strictly greater than m.

3.1 The Basic Idea

The idea of family resemblance was introduced by Wittgenstein [3] as an ingredient of his account of what constitutes possession of a concept, and what is required for the application of a general term. The account is intended to be more satisfactory than corresponding accounts that rely upon the apprehension of essential properties:

[1]This work is to appear in the Journal of Applied Logic. Permission to include this article was granted by both R.E. Jennings and by the publisher of the journal.

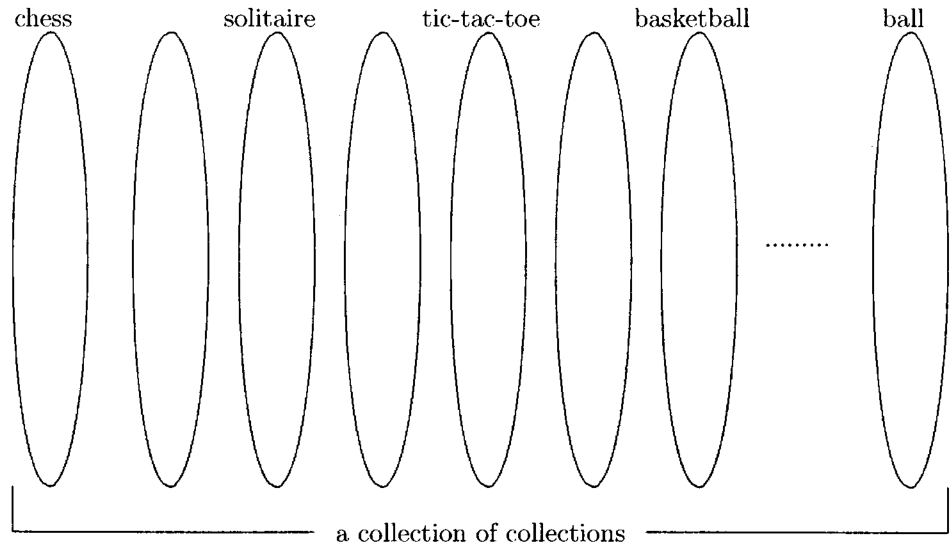

Figure 3.1.1: The family of games.

> 66. . . . Consider for example the proceedings we call games. . .if you look at them you will not see something that is common to *all*, but similarities, relationships, and a whole series of them at that. . . look for example at board games with their multifarious relationships. Now pass to card-games; here you find many correspondences to the first group, but many common features drop out, and others appear. When we pass next to ball games, much that is common is retained, but much is lost.–Are they all 'amusing'? Compare chess with noughts and crosses. Or is there always winning and losing, or competition between players? In ball games there is winning and losing, but when a child throws his ball at the wall and catches it again, this feature has disappeared. . .
>
> And the result of this examination is: we see a complicated network of similarities overlapping and criss-crossing: sometimes overall similarities, sometimes similarities of detail.
>
> 67. I can think of no better expression to characterize these similarities than "family resemblances"; for the various resemblances between members of a family: build, features, colour of eyes, gait, temperament, etc. etc. overlap and criss-cross in the same way.–And I say 'games' form a family. [3]

A *family*, as Wittgenstein envisaged the notion, can be represented as a collection of collections of properties or attributes, satisfying unspecified intersection requirements. (See Figure 3.1.1.) We may assume that the properties in question are pairwise-independent, that is, that for pairs of these properties, having the one does not entail having the other. Of course there may well be properties shared by all the members of a family, for biological families, the property of being a biological item or having a common ancestor would be one such, but this property, even if it is not entailed by other properties of the collection, does not seem to be part of Wittgenstein's conception. If Rosch and Mervis have got it right [2], the problem with such properties is that not that they are shared by every member of the family, but that they are shared with members of other distinct families to the same extent. And if Wittgenstein had in mind to explicate what he took to be a common notion, then the research reported in [2] would seem to bear him out. At any rate, there are good grounds for restricting our attention to families $\mathcal{F}$ having the property that all pairs of properties in $\cup\mathcal{F}$ are independent. Wittgenstein:

> But if someone wished to say: "There is something common to all these constructions–namely the disjunction of all their common properties"–I should reply: Now you are playing with words. One might as well say: "Something runs through the whole thread–namely the continuous overlapping of those fibres". [3], para. 67.

Again, one must not try to be more precise than we have been in the general account as to what these intersection requirements are, since the collection of families is itself a family. One family might be characterized by one intersection property, another by another. In what follows, therefore, we cannot claim to do justice to the vagueness of the general notion: in the nature of the case there could never be grounds for a claim that one had.

In [2], Rosch and Mervis confirm the hypothesis that "members of categories which are considered most prototypical are those with most attributes in common with other members of the category and least attributes in common with other categories" [2]. Accordingly, [2] represents the first empirical documentation of the existence in natural language categories of such general structural relationships as Wittgenstein posits. They write:

> [W]e viewed natural semantic categories as networks of overlapping attributes; the basic hypothesis was that members of a category come to be viewed as prototypical of the category as a whole in proportion to the *extent* to which they bear a family resemblance to (have attributes which overlap those of) other members of the category. [2][emphasis ours]

This notion of the *extent*, or *level* as we sometimes say, of family resemblance is what this paper is about. Wittgenstein's account suggests, and Rosch and Mervis's confirms, that there is a logic of categories and general terms which resists conventional essentialist representation. This raises the question of whether there is an adequate non-essentialist formal representation. In this article, we propose a model of category structure that is intended to approximate what Wittgenstein, and Rosch and Mervis, have in mind; it is such that any concept, possessing any extent of family resemblance above a certain degree can be represented as a taxon, in a hierarchy of concepts, subordinating only taxa which also possess an extent of family resemblance above this degree. To show this we introduce a derivational system consisting of a base set of properties together with a collection of rules for generating taxa. Along the way we show how to define a measure of closeness of family resemblance, and we illustrate the relationship between family resemblance and the mathematical theory of hypergraphs by making precise the notion of a composite likeness.

3.2 Its Realization

A collection of objects, X forms a family, $\mathcal{F}$, in virtue of some set, $\mathcal{P}$, of properties, such that $\forall x \in X, \exists \mathcal{P}' \subseteq \mathcal{P} : \forall \varphi \in \mathcal{P}', \varphi x$. But typically, the application of the term *family* requires that $\mathcal{P}$ be sufficiently small in relation to X, that q-tuples of objects ($0 < q \leq |X|$) of X share properties from $\mathcal{P}$. Hence the informal notion of *family resemblance*, the physically apparent *harmoniousness* of families, drawn, as it were, from a restricted palette of features. Informally, on this account, a family is represented as a collection of collections of properties that, to some extent, overlap. The harmoniousness of a family lies in the character of this overlap.

Definition 3.2.1. Let $\mathcal{P}$ be a set of properties. Then a set $\mathcal{F}$ is a *family* on $\mathcal{P}$ if $\mathcal{F}_{\neq\emptyset} \subseteq 2^{\mathcal{P}}$ and $\emptyset \notin \mathcal{F}$ (See Figure 3.1.1).[2]

A word is in order about the consistency of this set-theoretic conception of a family with Wittgenstein's view of the indeterminacy of concepts. There are at least two aspects of this indeterminacy; one is that some concepts are "unbounded"; a second, related, issue involves cases which are not clearly instances of a general term [3]. There is a difference between a collection with indeterminately many members, and a collection in which membership is indeterminate. A set, however, is typically construed as being a collection for which both membership and cardinality are fixed.

[2]Reference to $\mathcal{P}$ is omitted when context is sufficiently disambiguating.

Part of this apparent incongruity can be resolved by allowing the set $\mathcal{P}$ of properties to be indefinitely large. In this way the size of a family may be indefinite, and the issue of the boundaries of a concept needn't for practical purposes arise. As for the second kind of indeterminacy, apparently pertaining to vagueness, the model we present is intended as a discrete approximation of a potentially continuous phenomena, as, for example, a binomial distribution can be used to approximate a normal distribution, and therefore the model shouldn't be expected to replicate exactly the natural continuity of general terms.

Now as 'family resemblance' refers to a pattern of intersections among the members of a family, there are two dimensions along which the general notion may be analyzed, and which our account must make salient if it is to be adequate with the respect to the notion envisaged by Wittgenstein. In addition to the frequency with which an overlap of attributes between items occurs, one can speak of the *thickness* of the overlap, or the number of elements of which the overlap is comprised. This latter quality can be expressed as a generalization of the former, which we measure using the *harmonic number* of a family.

Definition 3.2.2. If S is a set and q is a positive integer, we write '$\binom{S}{q}$' for the set of all q-tuple subsets of S. If $\mathcal{F}$ is a family then the *harmonic number* of $\mathcal{F}$, $\eta(\mathcal{F})$, is defined:

$$\eta(\mathcal{F}) := \begin{cases} min\ n \in \mathbb{Z}^+ : \exists \mathcal{G} \in \binom{\mathcal{F}}{n} : \cap \mathcal{G} = \emptyset & \text{if this limit exists;} \\ \infty & \text{otherwise.} \end{cases} \tag{3.2.1}$$

If $\eta(\mathcal{F}) > n$ then we say that $\mathcal{F}$ is *n-harmonic.*

For example, let

$$\mathcal{F} = \{\{1,2\},\{1,3\},\{2,3\}\}$$

Then $\eta(\mathcal{F}) = 3$. This is because every pair of elements of $\mathcal{F}$ has a non-empty intersection while there is a triple of edges, namely, $\mathcal{F}$ itself, whose intersection is empty. Another family whose harmonic number is 3 is the following:

$$\{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$$

Note that if a finite family $\mathcal{F}$ is such that $\cap\mathcal{F} = \emptyset$, then for some positive integer m where $2 \leq m \leq |\mathcal{F}|$, $\eta(\mathcal{F}) = m$, and if $\cap F \neq \emptyset$ then $\eta(\mathcal{F}) = \infty$.

Definition 3.2.3. A family $\mathcal{F}$ is *trivial* if $\cap\mathcal{F} \neq \emptyset$.

For example, $\{\{1\}\}$ is trivial, as is the family $\{e \subseteq \mathbb{Z}^+ \mid 8 \in e\}$.

Proposition 3.2.1. $\forall\mathcal{F}, [\mathcal{F}$ *is not trivial* $\Leftrightarrow \forall x \in \cup\mathcal{F}, \exists e \in \mathcal{F} : e \subseteq \cup\mathcal{F} - \{x\}]$.

Generalizing n-harmonics we have:

Definition 3.2.4. If $\mathcal{F}$ is a family and n is a positive integer, the *n(-harmonic) saturation number* of $\mathcal{F}$, $\sigma_n(\mathcal{F})$, is defined:

$$\sigma_n(\mathcal{F}) := min\ m \geq 1 : \exists k \in [n], \exists\mathcal{G} \in \binom{\mathcal{F}}{k} : |\cap\mathcal{G}| < m$$

where for any positive integer n, '$[n]$' abbreviates '$\{1, 2, ..., n\}$'. If $\sigma_n(\mathcal{F}) > m$ then $\mathcal{F}$ is *m n-(harmonically) saturated.*

The idea behind the n-saturation number of a family $\mathcal{F}$ is this: Informally, let the *thickness* (*thinness*) of a k-tuple be the size of its intersection. Then the n-saturation number of $\mathcal{F}$ is 1 larger than the thickness of the thinnest k-tuple of $\mathcal{F}$ for all $k \in [n]$. It can be seen to follow from this informal reading of '$\sigma_n(\mathcal{F})$' that if $\sigma_n(\mathcal{F}) > m \geq 1$ then for each $k \in [n]$ $(n \geq 1)$, every k-tuple subset of $\mathcal{F}$ is at least m thick. The converse is also true:

Proposition 3.2.2. $\forall\mathcal{F}, n \geq 1, m \geq 1,\ [\sigma_n(\mathcal{F}) > m \Leftrightarrow \forall k \in [n], \forall\mathcal{G} \in \binom{\mathcal{F}}{k}, |\cap\mathcal{G}| \geq m]$.

For example, consider the following families:

$$\mathcal{F}_1 = \{\{1,2\},\{1,3\},\{2,3\}\}$$
$$\mathcal{F}_2 = \{\{1,2,4\},\{2,3,5\},\{3,4,1\},\{4,5,2\},\{5,1,3\}\}$$
$$\mathcal{F}_3 = \{\{1,2,3\}\}$$
$$\mathcal{F}_4 = \{\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{2,3,4,5\}\}$$

If $n = 1$ then $\sigma_n(\mathcal{F}_1) = 3, \sigma_n(\mathcal{F}_2) = 4 = \sigma_n(\mathcal{F}_3)$, and $\sigma_n(\mathcal{F}_4) = 5$. In addition we have $\sigma_2(\mathcal{F}_1) = 2, \sigma_2(\mathcal{F}_2) = 2, \sigma_2(\mathcal{F}_3) = 4, \sigma_2(\mathcal{F}_4) = 4, \sigma_3(\mathcal{F}_2) = 1, \sigma_3(\mathcal{F}_4) = 3, \sigma_4(\mathcal{F}_1) = 1 = \sigma_5(\mathcal{F}_4)$, and $\sigma_4(\mathcal{F}_4) = 2$.

Immediate from the definitions of harmonic number and harmonic saturation, the next proposition illustrates the sense in which m n-saturation is a generalization of n-harmonicity:

Proposition 3.2.3. $\forall\mathcal{F}, n \geq 1,\ [\eta(\mathcal{F}) > n \Leftrightarrow \sigma_n(\mathcal{F}) > 1]$.

The following theorem asserts that if $\mathcal{F}$ is a non-trivial n-harmonic family, $\mathcal{F}$ is m n-saturated only if, for all $k \in [n]$, the thickness of k-tuples of $\mathcal{F}$ increases as k decreases.

Theorem 3.2.4. $\forall \mathcal{F}, n \geq 1, m \geq 1$, *[if $\mathcal{F}$ is not trivial then* $[\sigma_n(\mathcal{F}) > m \Rightarrow \forall i, \forall \mathcal{G} \in \binom{\mathcal{F}}{n-i}, |\cap \mathcal{G}| \geq i+1 \ (0 \leq i \leq n-1)]]$.

Proof. Assume that $\mathcal{F}$ is not trivial and that $\sigma_n(\mathcal{F}) > m \geq 1$. Let i be arbitrary $(0 \leq i \leq n-1)$. Suppose that $\mathcal{G} \in \binom{\mathcal{F}}{n-i}$ is such that $|\cap \mathcal{G}| < i+1$. Let $\cap \mathcal{G} = \{x_1, x_2, ..., x_h\} (h \leq i)$. Let $\mathcal{J} = \{\cup \mathcal{F} - \{x_m\} \mid m \in [h]\}$. From Proposition 3.2.1, since $\mathcal{F}$ is not trivial, $\forall e \in \mathcal{J}, \exists f \in \mathcal{F} : f \subseteq e$. But $\cap(\mathcal{G} \cup \mathcal{J}) = \emptyset$. $\therefore \exists g \in [(n-i)+h] \cap [n], \exists \mathcal{H} \in \binom{\mathcal{F}}{g}$ such that $\mathcal{G} \subseteq \mathcal{H}$ and $\cap \mathcal{H} = \emptyset$. $\therefore \eta(F) \leq n$, which is absurd, given Proposition 3.2.3.

□

The notions of harmonic number and harmonic saturation represent two dimensions in terms of which extent of family resemblance can be analysed. One, corresponding to harmonic number, refers to the frequency with which attributes are shared among the members of a family; the second, corresponding to harmonic saturation, pertains to the extent to which attributes are shared, relative to a given frequency. As a result of its dyadic character, there are some families which are apparently not comparable with respect to family resemblance. Consider the case, for example, where a family $\mathcal{F}_1$ has a lower harmonic number than a family $\mathcal{F}_2$ while for some $k \geq 1$, every k-tuple subset of $\mathcal{F}_1$ is thicker than all, or even most, k-tuple subsets of $\mathcal{F}_2$. For example, let

$$\begin{aligned}\mathcal{F}_1 =& \{\{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3\}, \{d_1, d_2, d_3, a_1, a_2, a_3, b_1, b_2, b_3\},\\ & \{d_1, d_2, d_3, a_1, a_2, a_3, c_1, c_2, c_3\}, \{d_1, d_2, d_3, b_1, b_2, b_3, c_1, c_2, c_3\}\}\\ \mathcal{F}_2 =& \{\{1,2,3,4\}, \{1,2,3,5\}, \{1,3,4,5\}, \{2,3,4,5\}, \{1,2,4,5\}\}.\end{aligned}$$

Then $\eta(\mathcal{F}_1) = 4$ and $\sigma_3(\mathcal{F}_1) > 3$, and $\eta(\mathcal{F}_2) = 5$, while $\sigma_3(\mathcal{F}_2) = 3$. Because of the inherent vagueness of 'family resemblance' it would seem that prima facie we have no grounds for saying of either family that it possesses a greater (lesser) degree of family resemblance than the other, nor that their respective levels of family resemblance are equal. Such considerations suggest that if we are to measure family resemblance, then our gauge should be relativized to a given frequency. Accordingly we propose:

Definition 3.2.5. For a family $\mathcal{F}$ and positive integer n, the *n-resemblance* of $\mathcal{F}$ is $\sigma_n(\mathcal{F})$.

Proposition 3.2.5. $\forall \mathcal{F}, n \geq 1, 1 \leq \sigma_n(\mathcal{F}) \leq |\cup \mathcal{F}| + 1$. *If $\mathcal{F}$ is not trivial then* $\sigma_n(\mathcal{F}) \leq |\cup \mathcal{F}|$.

Proof. Suppose that $\sigma_n(\mathcal{F}) > |\cup\mathcal{F}| + 1$. Then $\forall k \in [n]$, every k-tuple of $\mathcal{F}$ is at least $|\cup\mathcal{F}| + 1$ thick, which is absurd. Now suppose that $\mathcal{F}$ is not trivial, and let $\sigma_n(\mathcal{F}) > |\cup\mathcal{F}|$. Then $\forall k \in [n]$, every k-tuple subset of $\mathcal{F}$ is at least $|\cup\mathcal{F}|$ thick. But this can be so only if $|\mathcal{F}| = 1$, in which case $\mathcal{F}$ is trivial, contrary to supposition. □

Using harmonic saturation we can define a relation of *closeness* of family resemblance:

Definition 3.2.6. A family $\mathcal{F}$ *more closely n-resembles* $\mathcal{G}_1$ *than* $\mathcal{G}_2$ if $\sigma_n(\mathcal{F} \cup \mathcal{G}_1) > \sigma_n(\mathcal{F} \cup \mathcal{G}_2)$.

Thus, for example, where

$$\mathcal{F} = \{\{5,3,4\},\{6,3,4\}\}$$
$$\mathcal{G}_1 = \{\{6,5,1,2,3\},\{6,5,1,2,4\},\{1,3,4\},\{2,3,4\}\}$$
$$\mathcal{G}_2 = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$$

$\mathcal{F}$ more closely 3-resembles $\mathcal{G}_1$ than $\mathcal{G}_2$ because $\sigma_3(\mathcal{F} \cup \mathcal{G}_1) = 2$ and $\sigma_3(\mathcal{F} \cup \mathcal{G}_2) = 1$. Also, $\mathcal{F}$ more closely 2-resembles $\mathcal{G}_1$ than $\mathcal{G}_2$ since $\sigma_2(\mathcal{F} \cup \mathcal{G}_1) = 3$ and $\sigma_2(\mathcal{F} \cup \mathcal{G}_2) = 2$. However, $\mathcal{F}$ does not more closely 1-resemble $\mathcal{G}_1$ than $\mathcal{G}_2$ since $\sigma_1(\mathcal{F} \cup \mathcal{G}_1) = \sigma_1(\mathcal{F} \cup \mathcal{G}_2) = 4$.

Alternatively, we can define a measure for the similarity of families with respect to family resemblance:

Definition 3.2.7. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be families. Then $\mathcal{F}_1$ *d-n-resembles* $\mathcal{F}_2 \Leftrightarrow \sigma_n(\mathcal{F}_1 \cup \mathcal{F}_2) \geq \sigma_n(\mathcal{F}_1) - d$ $(0 \leq d \leq \sigma_n(\mathcal{F}_1))$.

For instance, let

$$\mathcal{F}_1 = \{\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{2,3,4,5\}\}$$
$$\mathcal{F}_2 = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}.$$

Then $\sigma_3(\mathcal{F}_1) = 3, \sigma_3(\mathcal{F}_2) = 2$, and $\sigma_3(\mathcal{F}_1 \cup \mathcal{F}_2) = 2$. Therefore the least value of d such that $\mathcal{F}_1$ d-3-resembles $\mathcal{F}_2$ is 1. In general, the higher the least value of d is, the greater the degree with which $\mathcal{F}_2$ attenuates the n-resemblance of $\mathcal{F}_1$ when the two families are juxtaposed. To take another example, let

$$\mathcal{F}_1 = \{\{1,2,5,6\},\{3,4,7,8\}\}$$
$$\mathcal{F}_2 = \{\{1,2,3,4\},\{2,3,4,5\}\}$$

Then 3 is the least value of d such that $\mathcal{F}_2$ d-2-resembles $\mathcal{F}_1$ because $\sigma_2(\mathcal{F}_1 \cup \mathcal{F}_2) = 1$ and $\sigma_2(\mathcal{F}_2) = 4$, while $\mathcal{F}_1$ 0-2-resembles $\mathcal{F}_2$ because $\sigma_2(\mathcal{F}_1) = 1$.

The preceding example shows that $d-n$-resemblance is not symmetrical. If $d-n$-resemblance were taken to model exactly our conversational understanding of 'resemblance' this result is surprising. But no mathematical theory of any interest is as particular as the instances that motivate it. There is no such theory, for even if no such cases ever arose in conversational uses this would demonstrate only that conversational features were particular instances of the general case. Much as the absence of the so-called paradoxical inferences from conversational uses of *if...then...* does not of itself demonstrate that the conditional cannot be modelled by the material $\supset$. In any case, non-symmetry does constrain conversational uses. It is an historical non-symmetry in the structure of the verb (*re-semble*) itself. In general, the observance of non-symmetry serves conversationally to mark the distinction between originals and their (sometimes later) simulacra. Compare

1. He looks like his father.

2. *His father looks like him.

1. He resembles Napoleon.

2. *Napoleon resembles him.

Proposition 3.2.6. *$\forall \mathcal{F}_1, \mathcal{F}_2, n \geq 1, \mathcal{F}_1$ 0-n-resembles $\mathcal{F}_2 \Leftrightarrow \sigma_n(\mathcal{F}_1 \cup \mathcal{F}_2) = \sigma_n(\mathcal{F}_1)$.*

3.3 Composite Families

In literature, the term *composite* is applied to fictional characters who comprise traits of numerous source figures. In our account of family resemblance we apply the term to what could be regarded as the formal counterpart of such a fictional character. The idea of a composite, mathematically realized, is the lynch pin connecting the study of families as conceived by Wittgenstein and empirically studied by Rosch and Mervis to the well-established mathematical theory of hypergraphs. In fact, save that for local purposes we understand $\mathcal{P}$ as a set of properties, the language of *families* in our account could yield its place to the language of *hypergraphs*, and the *harmonics of families* could be understood abstractly as defining hitherto unstudied properties of hypergraphs. In fact, the fundamental features that ground the idea of a family are dual to the characteristics that define one of the principal mathematical interests in hypergraphs: the properties relating to *chromaticity*.

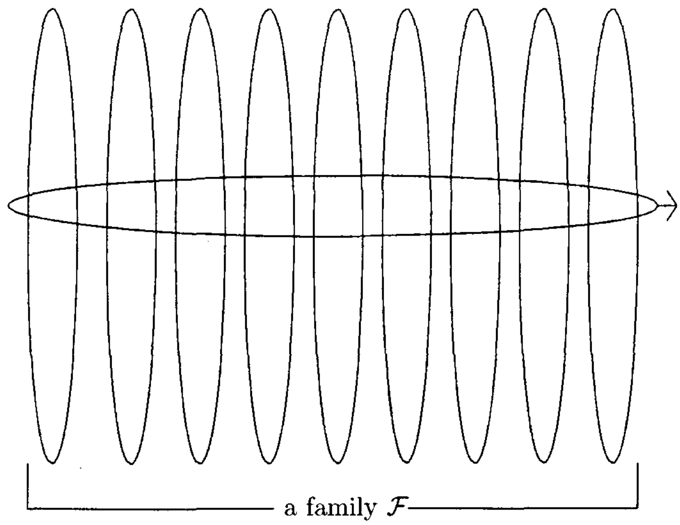

Figure 3.3.1: A composite

Definition 3.3.1. A set c is a *composite* of a family $\mathcal{F}$ iff $\forall e \in \mathcal{F}, c \cap e \neq \emptyset$ (see Figure 3.3.1); c is a *minimal composite* of $\mathcal{F}$ iff c is a composite of $\mathcal{F}$ and no proper subset of c is a composite of $\mathcal{F}$. The *composite family* $\mathcal{C}(\mathcal{F})$ of $\mathcal{F}$, which may be written '$\mathcal{CF}$', is the set of all minimal composites of $\mathcal{F}$.

Proposition 3.3.1. $\forall \mathcal{F}, [\mathcal{CC}(\mathcal{F}) \subseteq \mathcal{F}]$, *and* $[\forall e \in \mathcal{F}, \exists f \in \mathcal{CC}(\mathcal{F}) : e \supseteq f]$.

Definition 3.3.2. If $\mathcal{F}$ is a family and m is a positive integer, a function $f : \cup\mathcal{F} \to [m]$ is an *m-colouring* of $\mathcal{F}$ if $\forall e \in \mathcal{F}, k \in [m], e \not\subseteq \{x \in \cup\mathcal{F} \mid f(x) = k\}$ (see Figure 3.3.2). If there is an m-colouring of $\mathcal{F}$ then we say that $\mathcal{F}$ is *m-colourable*; $\mathcal{F}$ is *m-uncolourable*, else. The *chromatic number* of $\mathcal{F}$, $\chi(\mathcal{F})$, is defined:

$$\chi(\mathcal{F}) := \begin{cases} min\ m \in \mathbb{Z}^+ : \mathcal{F} \text{ is } m\text{-colourable} & \text{if this limit exists;} \\ \infty & \text{otherwise.} \end{cases} \tag{3.3.1}$$

The chromatic number of a family can also be defined in terms of the set of decompositions of its union.

Definition 3.3.3. Let S be a set and m a positive integer. The *set of m-decompositions* of S, $\Delta_m(S)$, is defined:

$$\Delta_m(S) := \{\delta = \{d_1, ..., d_m\} \mid \bigcup_{i=1}^{m} d_i = S\}$$

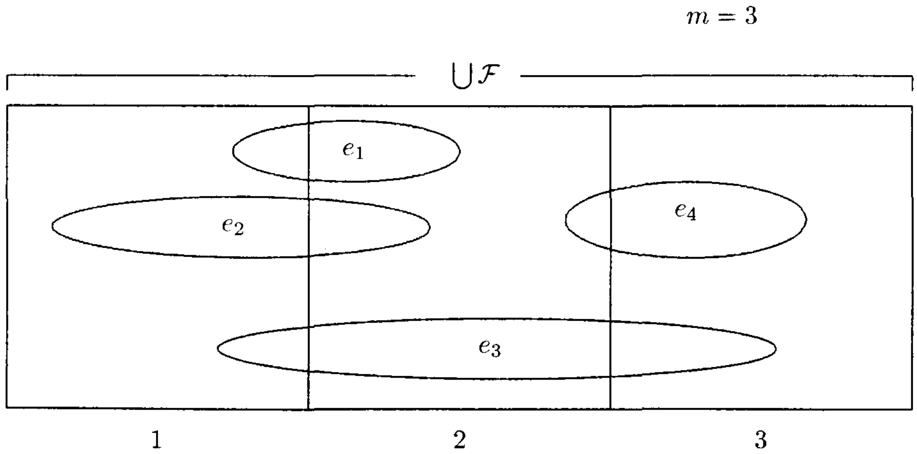

Figure 3.3.2: An m-colouring of $\mathcal{F} = \{e_1, e_2, e_3, e_4\}$.

Proposition 3.3.2. $\forall \mathcal{F}, \forall m \geq 1, [\chi(\mathcal{F}) > m \Leftrightarrow \forall \delta \in \Delta_m(\cup\mathcal{F}), \exists d \in \delta, e \in \mathcal{F} : e \subseteq d]$.

Definition 3.3.4. Let $\mathcal{F}$ be a family and S a subset of $\cup\mathcal{F}$. The *restriction of* $\mathcal{F}$ *to* S, $\mathcal{F}[S]$, is the family defined:

$$\mathcal{F}[S] := \{e \in \mathcal{F} \mid e \subseteq S\}$$

Theorem 3.3.3. $\forall \mathcal{F}, n \geq 1, m \geq 1, [\sigma_n(\mathcal{F}) > m \Leftrightarrow. \forall S, |S| < m \Rightarrow \chi((\mathcal{C}(\mathcal{F})[\cup\mathcal{F} - S])) > n]$.

Proof. Let $\mathcal{F}$ be an arbitrary family, and let m and n be arbitrary positive integers.

[$\Rightarrow$] Let $S \subseteq \cup\mathcal{F}$ be such that $|S| < m$ and $\mathcal{C}(\mathcal{F})[\cup\mathcal{F} - S]$ is n-colourable. Then $\exists \delta \in \Delta_n(\cup\mathcal{F} - S)$ such that $\forall e \in \mathcal{C}(\mathcal{F}), \forall d \in \delta, e \nsubseteq d$ (Proposition 3.3.2). Let $\mathcal{G} = \{\cup\mathcal{F} - d \mid d \in \delta\}$. Then $\forall g \in \mathcal{G}, g$ is a composite of $\mathcal{C}(\mathcal{F})$, and $\cap\mathcal{G} = S$. Therefore $\exists k \in [n], \exists \mathcal{J} \in \binom{\mathcal{CC}(\mathcal{F})}{k} : \cap\mathcal{J} \subseteq S$. But $\mathcal{CC}(\mathcal{F}) \subseteq \mathcal{F}$ (Proposition 3.3.1). Therefore $\sigma_n(\mathcal{F}) \leq m$.

[$\Leftarrow$] Let $\sigma_n(\mathcal{F}) = k \leq m$. Then $\exists j \in [n], \exists \mathcal{G} \in \binom{\mathcal{F}}{j} : |\cap\mathcal{G}| < k \leq m$. Let $\delta = \{\cup\mathcal{F} - g \mid g \in \mathcal{G}\}$. Then $\delta \in \Delta_j(\cup\mathcal{F} - \cap\mathcal{G})$. But $\forall e \in \mathcal{C}(\mathcal{F})[\cup\mathcal{F} - \cap\mathcal{G}], \forall d \in \delta, e \nsubseteq d$ (Proposition 3.3.1). Therefore $\mathcal{C}(\mathcal{F})[\cup\mathcal{F} - \cap\mathcal{G}]$ is n-colourable (Proposition 3.3.2). But $|\cap\mathcal{G}| < m$. Therefore $\exists S \subseteq \cup\mathcal{F} : |S| < m$ and $\chi(\mathcal{C}(\mathcal{F})[\cup\mathcal{F} - S]) \leq n$. ☐

For the case $m = 1$, what Theorem 3.3.3 amounts to in the presence of Proposition 3.2.3 is the assertion that n-harmonicity in a family is equivalent to the n-uncolourability of its composite family. If $m \geq 1$ then Theorem 3.3.3 asserts that the n-uncolourability of a composite family is preserved under the deletion of fewer than m elements from $\cup\mathcal{F}$ if $\mathcal{F}$ is m n-harmonically saturated.

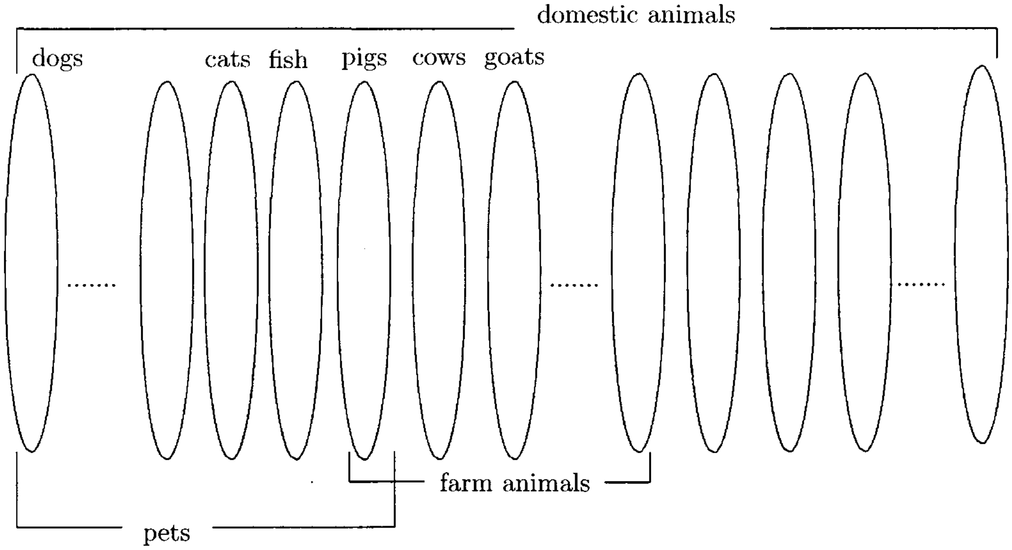

Figure 3.4.1: The family of domestic animals.

3.4 Taxonomic Hierarchies

Rosch and Mervis documented the cognitive significance of distinct families within more generic groupings of items [2]. As an example, the family of 'pets' lies within, or is subordinated by, the more general category of 'domestic animals', a category which also comprises families of otherwise subordinated non-human creatures. Similarly, the concepts 'dog', 'cat', 'rabbit' are members of the superordinating category 'pets'. (See Figure 3.4.1.)

For Rosch and Mervis, as for Wittgenstein, what accounts for the subordination of a concept within a more general category is not that there is some single criterion possessed by all and only members of the category, but rather that there is a network of shared attributes, or intersections. This network is the family resemblance of the category, whose extent we have represented using the concept of the n-resemblance of a family $\mathcal{F}$, which refers to $\sigma_n(\mathcal{F})$, the n-saturation number of $\mathcal{F}$.

Now granted that subordinate categories inherit the extent of family resemblance of superordinating ones, is it true that for every category, with any level of family resemblance, there is a taxonomic representation which preserves this fact? (See Figure 3.4.2. The arrows represent the superordination relation.) Below, we show one way that a subordination relation can be structured so that this is answered affirmatively.

Definition 3.4.1. A *taxon* is a taxonomic group of any rank, including all the subordinate groups; it is any group of organisms or populations considered to be sufficiently distinct from other such

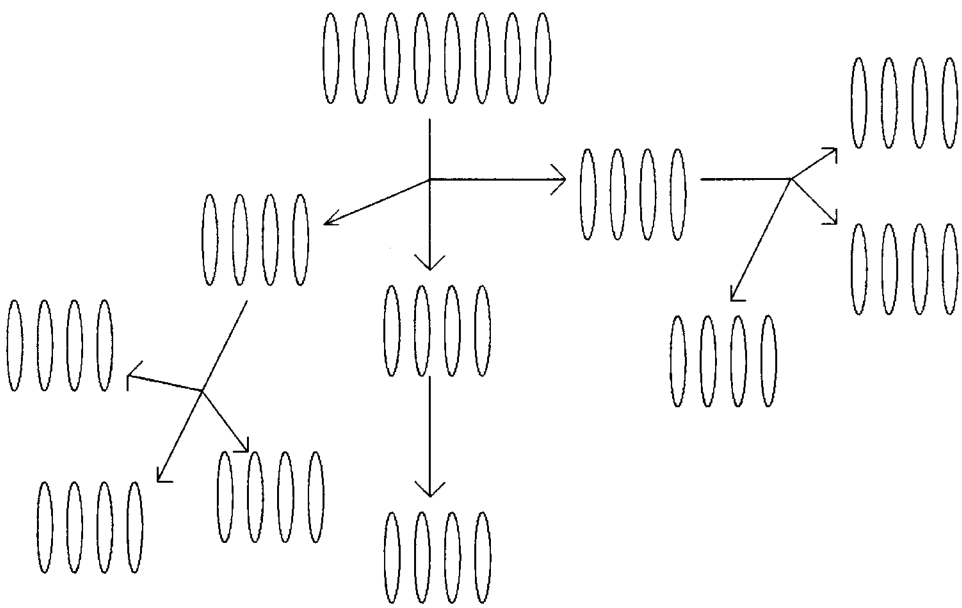

Figure 3.4.2: A tree of families.

groups to be treated as a separate unit. The (taxonomic) *rank* of a taxon is its position in a hierarchy of classification. [1]

To formalize what is meant by the *rank* of a taxon, we employ the notion of an *m-n-derivation.* Intuitively, this can be understood analogously to a proof in a logical system, substituting hypergraph and set-theoretic operations for rules of inference. Essentially we take iterations of these operations to structure the subordination relation among taxa. The system as a whole will be shown to be sound and complete with respect to n-resemblance strictly greater than $m \geq 1$.

Definition 3.4.2. Let $\mathcal{F}$ and $\mathcal{G}$ be families. If every element of $\mathcal{F}$ is a superset of an element of $\mathcal{G}$, then $\mathcal{F}$ *subsumes* $\mathcal{G}$, written '$\mathcal{F} \sqsupseteq \mathcal{G}$', or '$\mathcal{G} \sqsubseteq \mathcal{F}$'.

Proposition 3.4.1. $\forall m \geq 1, n \geq 1, \forall \mathcal{F}, \mathcal{G}$, *[if* $\sigma_n(\mathcal{F}) > m$ *and* $\mathcal{G} \sqsupseteq \mathcal{F}$ *then* $\sigma_n(\mathcal{G}) > m$*].*

Given our intention to devise a system which is sound with respect to n-resemblance, Proposition 3.4.1 licenses the following rule 'upward subsumption':

$$[\uparrow\sqsupseteq]: \text{ given } \mathcal{F}, \text{ if } \mathcal{G} \sqsupseteq \mathcal{F}, \text{ obtain } \mathcal{G} \tag{3.4.1}$$

Definition 3.4.3. Let $S = \mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_i, ..., \mathcal{F}_q$ be a sequence of q families $(1 < q)$, and let T be a set. Then if $n \geq 1$, T *n-covers* S if $\exists\{\mathcal{F}_1, ..., \mathcal{F}_n\} \in \binom{S}{n}$ such that $\forall i \in [n]$, $\exists e \in \mathcal{F}_i : T \supseteq$

e; T *minimally n-covers* S if T n-covers S, and no proper subset of T n-covers S. We write '$\frac{n}{q}(\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_q)$' for the set of all minimal n-covers of S.

Theorem 3.4.2. $\forall m \geq 1, n \geq 1$, *for any sequence* $\mathcal{G}_1, ..., \mathcal{G}_{n+1}$ *of families, if for each* $i \in [n+1]$, $\sigma_n(\mathcal{G}_i) > m$, *then* $\sigma_n(\frac{n}{n+1}(\mathcal{G}_1, ..., \mathcal{G}_{n+1})) > m$.

Proof. Let $\sigma_n(\frac{n}{n+1}(\mathcal{G}_1, ..., \mathcal{G}_{n+1})) = k \leq m$. Then $\exists j \in [n], \mathcal{F} \in \binom{\frac{n}{n+1}(\mathcal{G}_1, ..., \mathcal{G}_{n+1})}{j} : |\cap \mathcal{F}| < m$. By definition, each $e \in \mathcal{F}$ is a superset of an element from every member of some n-tuple subset of $\mathcal{G}_1, ..., \mathcal{G}_{n+1}$. By a pigeonhole argument, $\exists i \in [n+1]$ such that $\forall e \in \mathcal{F}$, $\exists f \in \mathcal{G}_i$: $e \supseteq f$. So if $\sigma_n(\mathcal{G}_i) > m$ then $|\cap \mathcal{F}| \geq m$: a contradiction. Whence $\exists i \in [n+1], \sigma_n(\mathcal{G}_i) \leq m$. □

Because $\frac{n}{n+1}$ preserves n-resemblance strictly greater than $m \geq 1$ (Theorem 3.4.2), in addition to rule $[\uparrow\sqsupseteq]$, we also therefore have 'n over $n+1$':

$$[\frac{n}{n+1}]: \text{ Given } \mathcal{G}_1, ..., \mathcal{G}_{n+1}, \text{ obtain } \frac{n}{n+1}(\mathcal{G}_1, ..., \mathcal{G}_{n+1}) \ (n \geq 1). \quad (3.4.2)$$

Our final rule is intended to license type-raising for m-tuples from a base set $\mathcal{P}$ of properties:

$$[m]: \text{ From } \{x_1, x_2, ..., x_m\} \subseteq \mathcal{P} \text{ obtain } \{\{x_1, x_2, ..., x_m\}\} \ (m \geq 1). \quad (3.4.3)$$

Definition 3.4.4. If $n \geq 1$ and $m \geq 1$, an *m-n-derivation* of a family $\mathcal{F}$ from a set $\mathcal{P}$ is a finite sequence of families on $\mathcal{P}$, ending with $\mathcal{F}$, where each family is obtained either from preceding ones by an application of $[\frac{n}{n+1}]$ or $[\uparrow\sqsupseteq]$, or from $\mathcal{P}$ by an application of $[m]$.

Theorem 3.4.3. $\forall \mathcal{F}, n \geq 1, m \geq 1$, *there is an m-n-derivation of* $\mathcal{F} \Rightarrow \sigma_n(\mathcal{F}) > m$.

Proof. It is sufficient to prove that $[\frac{n}{n+1}]$ and $[\uparrow\sqsupseteq]$ preserve n-resemblance strictly greater than m, and that $\forall x_1, x_2, ..., x_m \in \mathcal{P}, \sigma_n(\{\{x_1, x_2, ..., x_m\}\}) > m$. We have already shown the former (Proposition 3.4.1 and Theorem 3.4.2); the latter follows from the fact that if $\{x_1, ..., x_m\} \subseteq \mathcal{P}$, then $\sigma_n(\{\{x_1, ..., x_m\}\}) = m + 1$. □

Theorem 3.4.4. $\forall \mathcal{F}, n \geq 1, m \geq 1, \sigma_n(\mathcal{F}) > m \Rightarrow$ *there is an m-n-derivation of* $\mathcal{F}$.

Proof. Let $m \geq 1$ and $n \geq 1$ be arbitrary. Let $\mathcal{F}$ be an arbitrary family on a set $\mathcal{P}$ of properties such that $\sigma_n(\mathcal{F}) > m$. We induce on $|\mathcal{F}|$. For the basis, let $|\mathcal{F}| \leq n$. Then $|\cap \mathcal{F}| \geq m$. Let $\{x_1, ..., x_m\} \subseteq \cap\mathcal{F}$. Then $\mathcal{F} \sqsupseteq \{\{x_1, ..., x_m\}\}$. Therefore there is an m-n-derivation of $\mathcal{F}$ from $\mathcal{P}$ using $[m]$ and an application of $[\uparrow\sqsupseteq]$.

Now let $|\mathcal{F}| \geq n+1$. Where $\{e_1, ..., e_{n+1}\} \subseteq \mathcal{F}$, define:

$$\mathcal{F}_i := \mathcal{F} - \{e_i\} \qquad (i \in [n+1])$$

Then $\forall i \in [n+1], \sigma_n(\mathcal{F}_i) > m$, by the downward monotonicity of n-resemblance strictly greater than m. The hypothesis of induction therefore allows us to assert that for each $i \in [n+1]$, there is an m-n-derivation of $\mathcal{F}_i$ from $\mathcal{P}$. But $\forall e \in \mathcal{F}$, e is an n-cover for $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_{n+1}$. Therefore $\mathcal{F} \sqsupseteq \frac{n}{n+1}(\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_{n+1})$. Whence, there is an m-n-derivation of $\mathcal{F}$ from $\mathcal{P}$, namely, the sequence consisting of the m-n-derivation of $\mathcal{F}_1$, followed by the m-n-derivation of $\mathcal{F}_2$, ..., followed by the m-n-derivation of $\mathcal{F}_{n+1}$, followed by an application of $\frac{n}{n+1}$, and terminated by an application of $[\uparrow\sqsupseteq]$. □

With the notion of an m-n-derivation in hand, we may now speak of the *rank* of a taxon.

Definition 3.4.5. The *rank* of a family $\mathcal{F}$, relative to a given m-n-derivation $\mathfrak{D}$ of a family $\mathcal{G}$, $\rho_{\mathfrak{D}}(\mathcal{F})$, is the position of $\mathcal{F}$ in $\mathfrak{D}$.

We also introduce the notion of a *proper* (m-n-)derivation to distinguish between useful and irrelevant applications of rules. We shall not have occasion here to provide a comprehensive analysis of relevance with respect to derivations, and rely on what is, we hope, a shared prima facie intuition with the reader.

Definition 3.4.6. An m-n-derivation $\mathfrak{D} = (\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_i, ..., \mathcal{G}_q)$ of a family $\mathcal{G}_q$ $(q \geq 1)$ is *proper* if $\forall i \in [q], \mathfrak{D} - (\mathcal{G}_i)$ is not an m-n-derivation of $\mathcal{G}_q$.

Evidentally, Theorem 3.4.4 may be restated in terms of proper derivations. Definition 3.4.6 enables us to formalize a concept of taxonomical subordination.

Definition 3.4.7. A family $\mathcal{F}$ *m-n-subordinates* a family $\mathcal{G}$ iff there is a proper m-n-derivation of $\mathcal{F}$ in which $\mathcal{G}$ appears.

Theorem 3.4.5. $\forall \mathcal{F}, m \geq 1, n \geq 1, \sigma_n(\mathcal{F}) > m \Rightarrow \exists q \geq 1$: *there is a representation of $\mathcal{F}$ as a taxon of rank q where $\mathcal{F}$ m-n-subordinates only taxa of n-resemblance strictly greater than m.*

Proof. Theorem 3.4.5 is immediate from the fact that $\forall \mathcal{F}$, if $\sigma_n(\mathcal{F}) > m$ then there is a proper m-n-derivation $\mathfrak{D}$ of $\mathcal{F}$ such that $\forall \mathcal{G}$, if $\mathcal{G}$ is in $\mathfrak{D}$ then $\sigma_n(\mathcal{G}) > m$ (Theorems 3.4.3 and 3.4.4). □

3.5 Resemblance Revisited

The notion of resemblance inherent in Definition 3.2.6 suggests an ordering of families which is distinct from the subordination ordering described in Section 3.4. Let the notation:

$$x <_y z$$

denote that z more closely resembles y than x does, where x,y and z are now families. Then we can ask such questions as: Is there a property ϕ such that $\forall y$, $<_y$ has ϕ? Properties like transitivity, irreflexity, antisymmetry, connectivity, etc., seem like natural candidates for interrogation in this context.

It also seems natural to ask whether, for a given application, the notion of resemblance captured by Definition 3.2.6 is adequate. This suggests that we ask questions about different relations of resemblance. In this paper we have explicitly considered only two: 'resembles more closely than', and '*d*-*n*-resembles'. Implicitly, however, our paper suggests a broad array of measures of resemblance. We can say, e.g., that two families resemble one another to the extent to which they possess the same proof complexity (as this is defined is Section 3.4). Alternatively, harmonic number can be used as a measure of resemblance; we can say that $\mathcal{F}_1$ resembles $\mathcal{F}_2$ iff $\eta(\mathcal{F}_1) = \eta(\mathcal{F}_2)$. In this way we can talk about the equivalence class of families that resemble a given family $\mathcal{F}$, where we define $[\mathcal{F}]_\eta := \{\mathcal{G} \mid \eta(\mathcal{F}) = \eta(\mathcal{G})\}$. A similar point can be made about subsumption, defined in Definition 3.4.2. Let $S(\mathcal{G})$ be the set $\{\mathcal{F} \mid \mathcal{F} \sqsubseteq \mathcal{G}\}$, and say that $\mathcal{G} < \mathcal{F}$ iff $S(\mathcal{G}) \subset S(\mathcal{F})$. Then if $\mathcal{G} < \mathcal{F}$ it follows that $\mathcal{G} \sqsubseteq \mathcal{F}$.

In the general case, although it is difficult to say of what exactly it is that resemblance between families, for a given context, consists, at the very least we can say that it is some kind of overlapping, or sharing, of properties, or attributes. Supposing that we could define this quality of 'overlapping', and measure it, we could then compare it vis-à-vis pairs of families. In turn this would enable us to define a relativized ternary relation, so to make sense of the notion of 'more closely resembles than'.

For example, where x,y, and z are families, we could say whether or not $overlap(y, x) < overlap(y, z)$, and we could define:

$$x <_y z \text{ iff } overlap(y, x) < overlap(y, z).$$

The same questions would then arise regarding the ordering this ternary relation imposes on the collection of families, and the properties of this ordering.

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Chapter 4

On the Duality of Synchrony and Diachrony: A Dynamic Theory of Identity

D. Nicholson[1]

Abstract

In this paper we provide an analysis of the relationship between diachrony and synchrony, which entails a dynamic extensional mereological theory of identity. By 'extensional mereological' we mean that it is assumed that an individual, at any given time, is the sum of the properties and relations which he or she instantiates; by 'dynamic' we mean that change is incorporated as a fundamental part of the theory of identity which is proposed. In what follows, the theory of hypergraphs, particularly the theory of transverse hypergraphs, is used to prove that the relationship between diachrony and synchrony is one of 'weak duality'. What is meant by 'duality' here (strong duality), stems from the duality between the logical operators 'or' and 'and' of propositional logic. Furthermore, by invoking a quasi-Leibnizian identity principle, viz., that no distinct properties of the same kind are possessed by an individual at any specific time, an intermediate duality between

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[1]This work is to appear in the Proceedings for the 2007 International Conference on Artificial Intelligence and Pattern Recognition. Permission to include the article was granted by the publisher.

diachronic and synchronic perspectives is represented. It is also proved that there is a subclass of individuals whose diachronic and synchronic perspectives are strongly dual. Then, by exploiting the harmonic theory of hypergraphs, it is shown how to devise well-defined measures of inter- and intra-personal resemblance. Lastly, using the colouring theory of hypergraphs, we consider an application to gender identity. It is proved that given a certain richness in our taxonomy of properties, if the collection of an individual's behaviours in his or her respective social settings is sufficiently cohesive, and there is an appropriate relation of duality between his or her synchronic and diachronic representations, then it is possible that some social role in which he or she engages will witness him or her behaving in both of two differently gendered ways–at the same time, no less.

4.1 Introduction

Accounting for change in a single individual over time is a common obstacle for identity theorists. To what extent must something change in order to become something else? To what extent must someone change in order to become someone else? The questions are related but not identical. Prima facie, if Leibniz's *Principle of the Indiscernibility of Identicals*[2] is right, then if I'm the same person I was six years ago, I should have all and only the properties I did then. But, as many would claim, the fact that I do not does not entail that I am a different person. For example, although I have a few more grey hairs on my head, I'm still the person I was in 2001, modulo other more or less trivial changes. Apparently, the crux of what appears to be a conflict between Leibniz's principle—as applied to the (dis)continuity of personhood through time—and our intuitions about personal identity, has to do with which properties of a person we are to take as being constitutive of personal identity; for clearly, not all properties which we apply to persons are strictly relevant to their identities as persons. There are some properties which are, seemingly, more (or less) relevant to the preservation of personhood than others.

Each of us is familiar enough with the cognitively devastating effects of Alzheimer's Disease to know that we possess attributes whose loss renders questionable whether we continue to imbue the corporeal vessel with which we began this life.[3] Or, to take a less extreme example, consider the issue of gender. If an individual, say a male, changes his legally assigned gender, without making other changes, we probably wouldn't hesitate to consider him the same person as he was prior to

[2]The principle is: If x and y are identical entities, then x has a property δ if and only if y has δ.

[3]No dualism about mental states is intended by this language. The theory to follow is neutral about the nature of mental states.

the change. But what if he adopts a so-called feminine lifestyle, and modifies his body by taking estrogen supplements and having genital surgery, etc. Some people might argue that he has become a different person. Certainly prior to 1930 in Canada it would seem natural that many people would think that gender is partially constitutive of personal identity, because an individual undergoing such changes would effectively be losing his personhood by doing so, from a legal perspective.[4]

What we seem to have then, is that there are some properties which are clearly relevant to the preservation of personal identity through time, there are some which are clearly not relevant, and there are some which are clearly relevant in some contexts, but not necessarily all. This suggests a relativized continuum of relevance—that whether or not some property is relevant to the conservation of personal identity can in some, if not all, contexts be a matter of degree. Considerations such as these suggest what appears to be the more basic question:

To what extent must a collection of properties be preserved in order for an entity to preserve its identity through time?

For presumably, e.g., merely changing one's legal status vis-à-vis gender is not sufficient for transforming one's personal identity into another. But what if the legal change is accompanied by dramatic biological changes, including any cognitive alterations which may accompany exposure to unusually high doses[5] of (biologically opposite) sex hormones?

To take another example, some argue that an unmedicated schizophrenic who is experiencing psychosis can become another person. That is to say, the individual demonstrates such dramatic changes to his or her personality that we are no longer willing to identify him or her with his or her prior (or healthy) self. But how ill must he or she be in order for this transformation to take place? As with many illnesses, the onset of psychosis, its signs and symptoms, can be gradual.

In what follows we employ hypergraphic tools in the construction of a dynamic theory of identity, that is, one which incorporates change as a fundamental part of its structure, by simultaneously incorporating diachronic and synchronic perspectives on an individual. The theory is not restricted

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[4]In Canada, women were not legally recognized as 'persons' until 1930 [2].

[5]'Unusual' relative to those not undergoing sex changes, or taking hormones for (other?) medical reasons such as prostate or breast cancer, for instance. (Whether trans-genderism or transsexualism is a medical issue is controversial among those undergoing sex changes, notwithstanding that gender dysphoria, or gender identity disorder, is classified as a psychiatric condition by the Diagnostic and Statistical Manual of Mental Disorders:

> "Gender Identity Disorder" (GID) is a diagnostic category in the Diagnostic and Statistical Manual of Mental Disorders (DSM), published by the American Psychiatric Association. The DSM is regarded as the medical and social definition of mental disorder throughout North America and strongly influences the[sic] The International Statistical Classification of Diseases and Related Health Problems published by the World Health Organization. GID currently includes a broad array of gender variant adults and children who may or may not be transsexual and may or may not be distressed or impaired. [1])

to personal identity, but it can be instructive to keep this in mind as a special subcase. One virtue of applying hypergraphs in this way is that we obtain a mathematical measure of the degree to which an entity resembles other entities, including itself at an earlier time. This is useful, moreover, not only because it suggests a solution to the dilemma posed by Leibniz's Principle, but also because it lays theoretical groundwork for a concept of identity with potential applications in the social sciences. To illustrate this we consider the question of gender and its relationship to personal identity.

4.2 Dual views

Whereas diachronicity refers to change over time, a synchronic image is a representation at a single time. Thus, a snapshot is roughly synchronic, and a video recording which lasts for some non-empty non-singleton interval is diachronic. The first issue we consider can be framed by the following question: "What is the relationship between synchronic and diachronic views of the same individual?" The theory of hypergraphs is used to explain this relationship.

4.2.1 A Formal Model

A *hypergraph* H, is a non-empty family of non-empty sets $\{e_1, e_2, ..., e_i, ...\}$. Each set e_i is called an *edge* of H, and the collection of elements of edges of H is called the set of *vertices* of H. Hypergraphs can be infinite, or finite; a *finite* hypergraph has only finitely many vertices; an *infinite* hypergraph has either infinitely many edges, or an edge which is infinitely long. It will be convenient to abbreviate hypergraph to *graph.* We'll begin with a formal definition of a synchronic perspective of an individual $\mathcal{I}$. For any given moment in time, or *instant*, as we'll call it, s, we call the *property function*, $\mathcal{P}_{\mathcal{I}}(s)$ the set of properties of $\mathcal{I}$, at instant s, and we let γ be a set of instants. Then the *instance graph* for $\mathcal{I}$, relative to γ, viz., '$\mathsf{E}_\gamma(\mathcal{I})$', is just $\mathcal{P}_{\mathcal{I}}[\gamma]$, that is, it is the graph:

$$\mathsf{E}_\gamma(\mathcal{I}) := \{\mathcal{P}_{\mathcal{I}}(s) \mid s \in \gamma\}. \tag{4.2.1}$$

Where context allows we drop subscripts and reference to a particular individual $\mathcal{I}$. An instance graph E then, can be construed as a set of synchronic images, the edges of which may be ordered in accordance with the parameter γ, where γ determines how many, and which, such images are to be given. (See Figure 4.2.1.) For convenience we sometimes refer to the edges of E by 'instants',

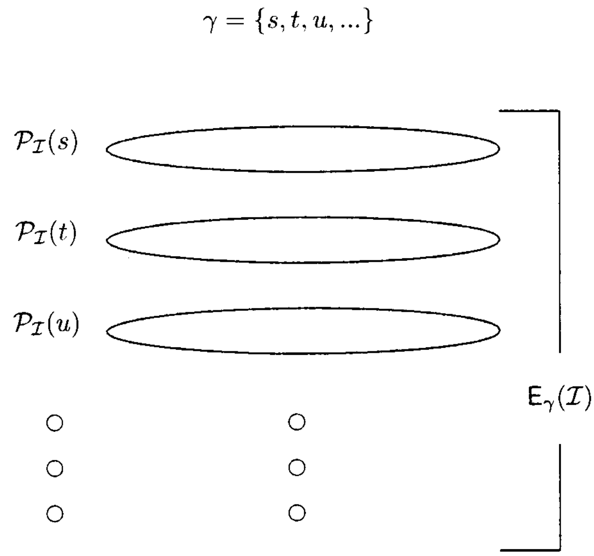

Figure 4.2.1: The instance graph for individual $\mathcal{I}$.

where confusion with the elements of γ is unlikely to be significant. We also stipulate that for each pair of instants s and t in γ, there is a function $\mu : \mathcal{P}(s) \rightarrow \mathcal{P}(t)$ such that $\mu(\alpha) = \beta$ only if α and β are properties of the same *kind*. Nothing more is meant by this, however, than is required to ensure that, in effect, each edge of E displays a complete picture of the individual, relative to the selected parameters. In other words, we stipulate that each kind of property which is available for representation is in fact represented in every instant of E. For example, if $\mathcal{P}(s)$ includes the individual's gender, then so would $\mathcal{P}(t)$—the property kind in this case being *gender*. Two properties can be thought of as being of the same kind, if they have been grouped together as such under some selected rubric.[6]

We now show how to relate a diachronic picture of an individual $\mathcal{I}$ to the instance graph E for $\mathcal{I}$.

Let κ be the set $\{\pi_1, \pi_2, ..., \pi_t, ...\}$ of distinct kinds of properties instantiated in the edges of instance graph E, and define a function σ from κ to the power set of vertices of E, where for each t $(1 \leq t)$:

$$\sigma(\pi_t) := \{x \mid x \text{ is of kind } \pi_t\}. \tag{4.2.2}$$

[6]When clean snow melts, it becomes colourless. This might be thought to constitute a counterexample to the proposed schema, but not if we allow for a kind of 'negative' property, which indicates the absence of any instance of a given property kind. We could, for example, call it 'non'.

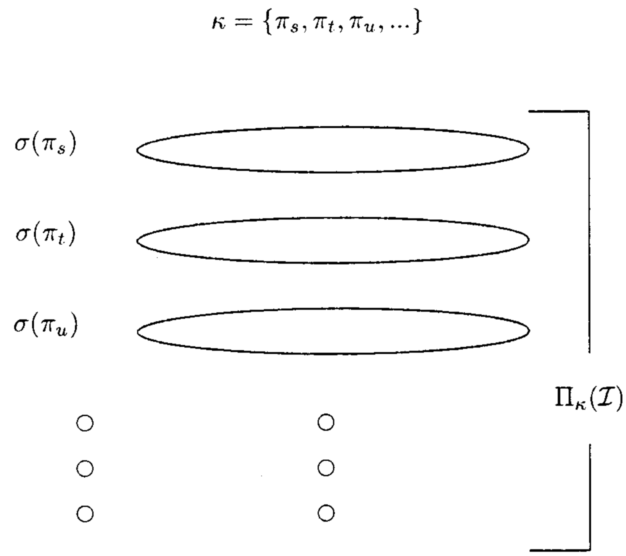

Figure 4.2.2: The property graph for individual $\mathcal{I}$.

In effect σ sorts the properties instantiated in instants of E into distinct kinds which are named in κ. And the *property graph* for individual $\mathcal{I}$ relative to κ is the graph:

$$\Pi_\kappa(\mathcal{I}) := \{\sigma(\pi_t) \mid \pi_t \in \kappa\}. \tag{4.2.3}$$

(See Figure 4.2.2.) Again, we drop subscripts and reference to $\mathcal{I}$ when context allows, and speak simply of the property graph Π. A property graph $\Pi = \{k_1, k_2, ...\}$ then, can be understood as a set of diachronic images of an individual, relative to single kinds of properties—diachronic in the sense that the elements of an edge k of Π can be ordered using a temporal ordering of the set γ, thereby providing an image of the change a property kind π_t undergoes across the instants in E, ordered similarly.

4.2.2 (Weak) Duality

The relationship between an instance graph E and its corresponding property graph Π, can be explained mathematically by utilizing the notion of the *dual* of a graph: Where H is any graph, a *cover* for H is a set e with a non-empty intersection with every edge of H; a *minimal cover* e for H is a cover for H such that $\forall f \subset e$, f is not a cover for H. The *dual* of H, H^d, is the set of all minimal covers for H. If e is a cover for a graph H then we also say '*e covers* H'.

Proposition 4.2.1. $\forall H, H^{dd} \subseteq H$. [4, 3]

Proof. Let $e \in \mathsf{H}^{dd}$. Suppose that $e \notin \mathsf{H}$. Note that $\forall f \in \mathsf{H}, f$ is a cover for H^d. Therefore, $\forall f \in \mathsf{H}, e \not\supseteq f$, else e is not a minimal cover for H^d. But then $\exists g \in \mathsf{H}^d$ such that $g \cap e = \emptyset$, which is impossible if $e \in \mathsf{H}^{dd}$. Whence $e \in \mathsf{H}$. □

For any H, H and H^d are said to be dual because of a correspondence between H and H^d, respectively, and pairs of logically dual formulae of propositional logic—logically dual in the sense in which the $\vee$ and $\wedge$ truth functions of propositional logic are dual to one another, viz., that from the graph of one, one may obtain the other's graph, modulo the order of the rows, by interchanging all 1's and 0's. In the same way, letting the vertices of a graph H be sentences, we may say that

$$\bigvee_{p=1}^{|\mathsf{H}|} \bigwedge_{q=1}^{|e_p|} x_q \in e_p \in \mathsf{H} \text{ is dual to } \bigwedge_{p=1}^{|\mathsf{H}|} \bigvee_{q=1}^{|e_p|} x_q \in e_p \in \mathsf{H}. \tag{4.2.4}$$

But since for any graph G, $\mathsf{G}^{dd} \subseteq \mathsf{G}$ (Proposition 4.2.1), it follows that

$$\bigvee_{p=1}^{|\mathsf{H}|} \bigwedge_{q=1}^{|e_p|} x_q \in e_p \in \mathsf{H} \;\;\dashv\models \bigwedge_{r=1}^{|\mathsf{H}^d|} \bigvee_{s=1}^{|e_r|} x_s \in e_r \in \mathsf{H}^d, \text{ and} \tag{4.2.5}$$

$$\bigvee_{r=1}^{|\mathsf{H}^d|} \bigwedge_{s=1}^{|e_r|} x_s \in e_r \in \mathsf{H}^d \;\;\dashv\models \bigwedge_{p=1}^{|\mathsf{H}|} \bigvee_{q=1}^{|e_p|} x_q \in e_p \in \mathsf{H}. \tag{4.2.6}$$

Therefore

$$\bigvee_{p=1}^{|\mathsf{H}|} \bigwedge_{q=1}^{|e_p|} x_q \in e_p \in \mathsf{H} \text{ is dual to } \bigvee_{r=1}^{|\mathsf{H}^d|} \bigwedge_{s=1}^{|e_r|} x_s \in e_r \in \mathsf{H}^d. \tag{4.2.7}$$

And similarly,

$$\bigwedge_{p=1}^{|\mathsf{H}|} \bigvee_{q=1}^{|e_p|} x_q \in e_p \in \mathsf{H} \text{ is dual to } \bigwedge_{r=1}^{|\mathsf{H}^d|} \bigvee_{s=1}^{|e_r|} x_s \in e_r \in \mathsf{H}^d. \tag{4.2.8}$$

Whence we say that H and H^d are dual in this strong sense of logical duality.

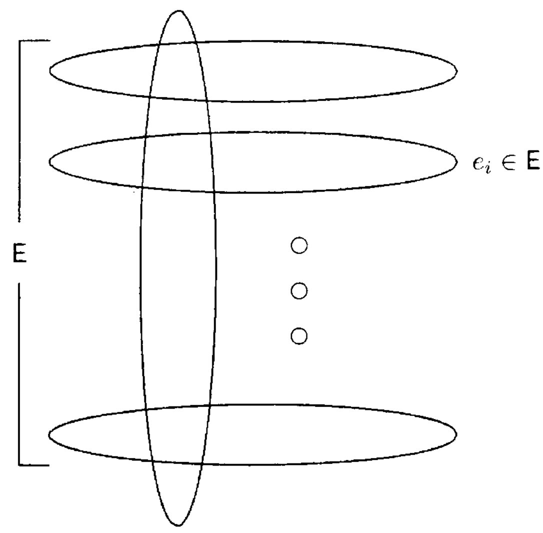

Figure 4.2.3: Weak duality

But by thus replacing truth-functional equivalence with entailment in Conditions 4.2.5 and 4.2.6, we obtain a sense of *weak duality* for graphs. We also thereby obtain a description of the logical relationship between an instance graph E and its corresponding property graph Π. Graph-theoretically weak duality between hypergraphs amounts to the condition that *every edge of either is a cover for the other*; in other words, if H and G are graphs then H and G are *weakly dual* if and only if:

$$\forall e \in \mathsf{H}, \exists f \in \mathsf{G}^d : e \supseteq f \text{ and } \forall f \in \mathsf{G}, \exists e \in \mathsf{H}^d : f \supseteq e. \tag{4.2.9}$$

(See Figure 4.2.3.) If H and G are graphs then H is *dual* to G if and only if $\mathsf{H} = \mathsf{G}^d$.

Proposition 4.2.2. *$\forall H, G$, if H is dual to G then H and G are weakly dual.*

Proof. Assume that H is dual to G—i.e., assume that $\mathsf{H} = \mathsf{G}^d$. Let $e \in \mathsf{H}$. Then e covers G. Now let $f \in \mathsf{G}$. Then $\forall g \in \mathsf{G}^d$, $g \cap f \neq \emptyset$. I.e., f covers G^d, in which case f covers H, by our initial assumption. □

Note that just because a graph H is dual to a graph G, it doesn't follow that G is dual to H. For example, let $\mathsf{H} = \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$, and let $\mathsf{G} = \{\{1,2\},\{3,4\},\{1,2,3,4\}\}$. Then $\mathsf{H} = \mathsf{G}^d$, but $\mathsf{G} \neq \mathsf{H}^d$. Symmetry of duality does obtain, however, if we restrict our attention to simple graphs. A graph H is *simple* if no two distinct edges e, f, belonging to H are such that

$e \subseteq f$.

Proposition 4.2.3. *$\forall H, G$, if H and G are simple, then $H = G^d \Leftrightarrow G = H^d$.*

Proof. Suppose that H and G are simple.

[⇒] Assume that $\mathsf{H} = \mathsf{G}^d$. Let $e \in \mathsf{G}$. Then e covers G^d, in which case e covers H. Suppose that $\exists f$ such that $f \subset e$ and $f \in \mathsf{H}^d$. Then $f \in \mathsf{G}^{dd}$, by substitution into our assumption. So $f \in \mathsf{G}$ (Proposition 4.2.1), which contradicts our supposition that H is simple.

[⇐] This direction follows by symmetry.

□

Theorem 4.2.4. *$\forall H, G$, H and G are weakly dual if and only if:*

$$\bigvee_{p=1}^{|H|} \bigwedge_{q=1}^{|e_p|} x_q \in e_p \in H \models \bigwedge_{r=1}^{|G|} \bigvee_{s=1}^{|e_r|} x_s \in e_r \in G, \text{ and} \tag{4.2.10}$$

$$\bigvee_{r=1}^{|G|} \bigwedge_{s=1}^{|e_r|} x_s \in e_r \in G \models \bigwedge_{p=1}^{|H|} \bigvee_{q=1}^{|e_p|} x_q \in e_p \in H. \tag{4.2.11}$$

Proof. The result is easily demonstrated using propositional logic. □

Theorem 4.2.5. *If E is an instance graph and Π the corresponding property graph then E and Π are weakly dual.*

Proof. The result follows from our requirement that every instant include at least one property of every kind. □

4.2.3 An Intermediate Duality

Now, by invoking a quasi-Leibnizian identity principle, a different relation between E and Π can be specified vis-à-vis duality. For example, if E satisfies the condition that *at any instant if properties α and β are of the same kind then they are identical*, then E and Π can be shown to satisfy:

$$\mathsf{E} \subseteq \Pi^d, \text{ and} \tag{4.2.12}$$

$$\Pi \subseteq \mathsf{E}^d. \tag{4.2.13}$$

That is, E and Π satisfy a condition which is intermediate between weak duality and duality simpliciter. The condition is quasi-Leibnizian only because it is similar to a generalization of a version of Leibniz's principle of the Identity of Indiscernibles[7] stated for properties and property kinds (and relative to instants), as opposed to entities and properties.

Theorem 4.2.6. *If* E *is an instance graph and* Π *the corresponding property graph then: if for every edge* $e \in \mathsf{E}$*, if* $x \in e, y \in e$*, and* $x \neq y$*, then* x *and* y *are of different kinds, then,* $\mathsf{E} \subseteq \Pi^d$*, and* $\Pi \subseteq \mathsf{E}^d$*.*

Proof. Assume that for every edge $e \in \mathsf{E}$, if $x \in e, y \in e$, and $x \neq y$, then x and y are of different kinds. Let $e \in \mathsf{E}$. Then e is a cover for Π by Theorem 4.2.5. Suppose that e is not minimal; let $f \subset e$ be such that $f \in \Pi^d$. Then by a pigeonhole argument, $\exists x, y \in e, x \neq y$, and x and y are properties of the same kind, contrary to assumption. $\therefore e \in \Pi^d$, and $\mathsf{E} \subseteq \Pi^d$. By parallel reasoning it follows that $\Pi \subseteq \mathsf{E}^d$.

□

4.2.4 Duality and Potential

Although Conditions 4.2.12 and 4.2.13 do not clearly entail that Π is dual to E or conversely, we can characterize a subclass of graphs satisfying these conditions, which are dual. Intuitively, the notion of duality for instance graphs and their corresponding property graphs suggests an idea of *maximum actualized potential* on the part of an individual. For if Π is dual to E, that is, if $\Pi = \mathsf{E}^d$, then given that $\forall \mathsf{H}, \mathsf{H}^{dd} \subseteq \mathsf{H}$ (Proposition 4.2.1), it follows that every minimal selection of properties of distinct kinds is an instant of E and thus also a synchronic image of the individual $\mathcal{I}$.

And similarly, if E is not dual to Π, i.e., $E \neq \Pi^d$, but Conditions 4.2.12 and 4.2.13 are satisfied, then there is a minimal cover for Π which is not an instant of E—in other words, there is a way of instantiating properties of $\mathcal{I}$ which $\mathcal{I}$ has not realized; in this way, the individual retains some growth potential, be it for better or worse. We can make this notion of an individual's potential precise by defining a growth parameter ξ: Let E be an instance graph, and let E *maximized* be the graph:

$$\mathsf{E}^m := \Pi^d. \tag{4.2.14}$$

Then the *growth potential* of E is:

$$\xi(\mathsf{E}) := |\mathsf{E}^m| - |\mathsf{E}|. \tag{4.2.15}$$

[7]The principle is: If entities x and y are such that for any property δ, x has δ iff y has δ, then x and y are identical.

Theorem 4.2.7. *For any instance graph* E *with its corresponding property graph* Π*, if* $E \subseteq \Pi^d$ *then* E *is dual to* Π *if and only if* $\xi(\mathsf{E}) = 0$.

Proof. Assume that $\mathsf{E} \subseteq \Pi^d$.

[$\Rightarrow$] Suppose that $\mathsf{E} = \Pi^d$. Then $\mathsf{E} = \mathsf{E}^m$, in which case $|\mathsf{E}^m| - |\mathsf{E}| = \xi(\mathsf{E}) = 0$.

[$\Leftarrow$] Let $\xi(\mathsf{E}) = 0$. Then $|\mathsf{E}^m| - |\mathsf{E}| = 0$. But by assumption $\mathsf{E} \subseteq \Pi^d$. So $\mathsf{E} \subseteq \mathsf{E}^m$, and therefore $\mathsf{E} = \mathsf{E}^m = \Pi^d$. □

Theorem 4.2.8. *For any instance graph* E *with its corresponding property graph* Π*, if* E *and* Π *are simple then* $\Pi = E^d \Rightarrow \xi(E) = 0$.

Proof. Assume that E and Π are simple and that $\mathsf{E} \subseteq \Pi^d$. Assume further that $\Pi = E^d$. If E is simple then it can be shown that $\mathsf{E} = \mathsf{E}^{dd}$ using Proposition 4.2.1. But in that case $|\Pi^d| = |\mathsf{E}^{dd}| = |\mathsf{E}|$, in which case $|\Pi^d| - |\mathsf{E}| = 0$—that is, $\xi(\mathsf{E}) = 0$. □

Theorem 4.2.9. *For any instance graph* E *with its corresponding property graph* Π*, if* $E \subseteq \Pi^d$ *and* E *and* Π *are simple then* $\xi(E) = 0 \Rightarrow \Pi = E^d$.

Proof. Assume the antecedents and that $\xi(\mathsf{E}) = 0$. Then $|\Pi^d| - |E| = 0$. But $\mathsf{E} \subseteq \Pi^d$, whence $E = \Pi^d$. But E and Π are simple, by assumption. Therefore $\Pi = E^d$ (Proposition 4.2.3). □

Corollary 4.2.10. *For any instance graph* E *with its corresponding property graph* Π*, if* E *and* Π *are simple and* $E \subseteq \Pi^d$ *then* $\Pi = E^d \Leftrightarrow \xi(E) = 0$.

4.3 Harmonic number

4.3.1 A Measure of Similarity

The notion of the *harmonic number* of a hypergraph can be employed to define a measure of intra- and interpersonal resemblance. If H is a graph, we say that H is *m-wise intersecting*, for $m \geq 1$, if every m edges of H share a common element. The *harmonic number* of H, $\eta(\mathsf{H})$, is the smallest integer n such that H is not n-wise intersecting. For example, if $\mathsf{H} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$, then $\eta(\mathsf{H}) = 3$. This is because every pair of edges of H share a common element, and there is also a triple of H-edges with an empty intersection, for instance, $\{\{1,3\},\{1,4\},\{2,3,4\}\}$. In the case where H is a graph such that $\cap\mathsf{H} \neq \emptyset$, there is no finite number m such that H is not m-wise intersecting. In such cases we use '∞' to refer to a number which is arbitrarily large and set

$\eta(\mathsf{H}) := \infty$. Since a graph is a collection of non-empty sets, the minimum harmonic number for any graph is 2.

The harmonic number of a graph H is an expression of the internal cohesion, or cohesiveness, of H, and, prima facie it seems reasonable to suppose that for many applications, individuals would tend to have instance graphs exhibiting a large degree of this cohesiveness; to suppose otherwise generates a view of identity whereby an individual changes radically from one instant to the next, altering most or all of its properties. A similar prima facie case can be made for the union of instance graphs drawn from various subclasses of individuals; it would seem that in many cases, e.g., family relations, the union would exhibit harmonic number above some roughly predictable degree.

To make these ideas more precise we can use harmonic number to define a relation of *closeness of resemblance*, and of *d-resemblance*, or 'resemblance of degree *d*': Given instance graphs $\mathsf{E}_1, \mathsf{E}_2$, and $\mathsf{E}_3, \mathsf{E}_2$ *more closely resembles* E_1 than E_3 does if:

$$\eta(\mathsf{E}_1 \cup \mathsf{E}_2) > \eta(\mathsf{E}_1 \cup \mathsf{E}_3). \tag{4.3.1}$$

And, where $d \geq 1$,

$$\mathsf{E}_1 \ d\text{-resembles } \mathsf{E}_2 \text{ iff } \eta(\mathsf{E}_1 \cup \mathsf{E}_2) > \eta(\mathsf{E}_1) - d. \tag{4.3.2}$$

Intuitively, if E_1 d-resembles E_2 then d can be thought of as a measurement of how much of the harmonic number of E_1 is attenuated or lost when E_1 is united with E_2, the smallest value of d satisfying the statement 'E_1 d-resembles E_2' giving the most accurate reading in this regard. In order to deal with graphs with arbitrarily large harmonic number, we allow that d can refer to the arbitrarily high value '∞'. Thus, for example, letting $\mathsf{E}_1 = \{\{1,2\}\}$, and $\mathsf{E}_2 = \{\{3,4\}\}$, we have $\eta(\mathsf{E}_1 \cup \mathsf{E}_2) = 2 > (\eta(\mathsf{E}_1) = \infty) - \infty$, in which case E_1 is said to ∞-resemble E_2.

We now state some elementary propositions regarding d-resemblance.

Proposition 4.3.1. *An instance graph E_1* 1-resembles *instance graph E_2 if and only if $\eta(E_1 \cup E_2) = \eta(E_1)$.*

Proposition 4.3.2. *For instance graphs E_1 and E_2, if $d \geq \eta(E_1) - 1$ then E_1 d-resembles E_2.*

Proposition 4.3.3. *For any pair of graphs E_1 and E_2, there is some value d such that E_1 d-resembles E_2.*

Proposition 4.3.4. *If an instance graph E_1 d-resembles instance graph E_2 then E_1 $(d+1)$-resembles*

E_2.

To find an application for d-resemblance we have only to look as far as the question of which this article is the progeny. One may object to using Leibniz's Principle of the Indiscernibility of Identicals in discussions of the preservation of identity through time, on the grounds that a change in an entity's properties does not imply that it is discontinuous with itself. The principle in question can be formulated thus:

$$\forall x, y[x = y \Rightarrow \forall\delta(\delta x \Leftrightarrow \delta y)] \tag{4.3.3}$$

Using the theory of hypergraphs this statement can be reformulated to make allowances for change through time; in particular, we can say that:

$$\forall x, y[x = y \Rightarrow \mathsf{E}(x) \ d\text{-resembles } \mathsf{E}(y)] \tag{4.3.4}$$

where the value for d is chosen in accordance with standards appropriate to some particular application.

4.3.2 Chromatic Number

Returning to the issue of the quasi-Leibnizian condition, akin to the Identity of Indiscernibles, that *no two properties of the same kind appear in the same instant*,[8] we now consider its relationship to harmonic number. In fact we will show that for any instance graph E, in the event that $\eta(\mathsf{E}) > 2$, this condition cannot be satisfied if the property kinds corresponding to E each have at least two members, and if Π is the dual of E. To prove this we exploit a notion logically dual to harmonic number, viz., that of *chromatic number.*

If H is a graph and $m > 1$, then H is *m-colourable* if there is a partition of $\cup\mathsf{H}$ into m pairwise disjoint mutually exhaustive sets, none of which is a superset of any edge of H. Such a partition is called an m-colouring of H. (See Figure 4.3.1.) The *chromatic number* of H, $\chi(\mathsf{H})$, is the least integer n for which H is n-colourable. In the event that H is not n-colourable for any finite n, as, for example, occurs with $\mathsf{H} = \{\{1\}\}$, then we set $\chi(\mathsf{H}) := \infty$.

Lemma 4.3.5. $\forall H, \eta(H) > n \Leftrightarrow \chi(H^d) > n$. [4]

[8] See section 4.2.3.

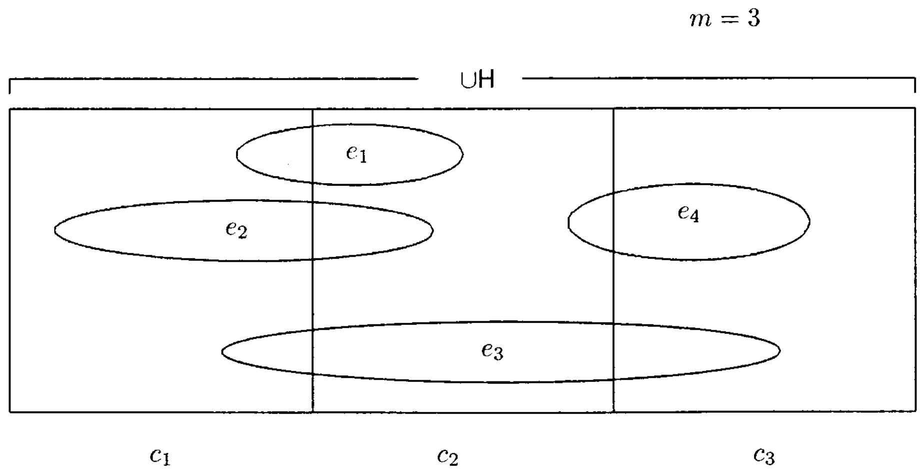

Figure 4.3.1: An m-colouring of $\mathsf{H} = \{e_1, e_2, e_3, e_4\}$.

Proof.

[$\Rightarrow$] Assume that $\chi(H^d) \leq n$. Then there is an n-colouring $\{e_1, ..., e_n\}$ of H^d. Let $\{f_1, ..., f_n\} = \{\cup H - e_i \mid 1 \leq i \leq n\}$. Then $\forall i, f_i$ covers H^d, and $\cap\{f_1, ..., f_n\} = \emptyset$ $(1 \leq i \leq n)$. Therefore $\exists j \leq n$ such that there is a j-tuple of edges of H^{dd} with an empty intersection. But $H^{dd} \subseteq H$ (Proposition 4.2.1). Whence $\eta(H) \leq j \leq n$.

[$\Leftarrow$] Suppose that $\eta(H) \leq n$. Then for some $j \leq n$, there is a j-tuple $\{e_1, ..., e_j\}$ of edges of H with an empty intersection. Let $\{f_1, ..., f_j\} = \{\cup H - e_i \mid 1 \leq i \leq j\}$. Then $\cup\{f_1, ..., f_j\} = \cup H$, else $\cap\{e_1, ..., e_j\} \neq \emptyset$, and $\forall g \in H^d, \forall i, g \not\subseteq f_i$ $(1 \leq i \leq j)$, else for some i, e_i does not cover H^d $(1 \leq i \leq j)$. Therefore, using $\{f_1, ..., f_j\}$ we may construct a j-colouring of H^d by deleting repetitions of vertices occurring in the elements of $\{f_1, ..., f_j\}$. That is, $\chi(H^d) \leq j \leq n$. □

For any graph H and set s, the *subgraph of* H *induced by* s is the graph:

$$\mathsf{H}[s] := \{e \in \mathsf{H} \mid e \subseteq s\}. \tag{4.3.5}$$

Lemma 4.3.6. $\forall H, n \geq 2, \chi(H) > n \Rightarrow \forall e \in H^d, \chi(H[e]) > n - 1.$

Proof. Assume that $n \geq 2$ and that $\chi(\mathsf{H}) > n$. Let $e \in \mathsf{H}^d$. Suppose that $\mathsf{H}[e]$ is $(n-1)$-colourable. Then there is an $(n-1)$-partition of e into pairwise disjoint mutually exhaustive sets $\{f_1, ..., f_{n-1}\}$ such that $\forall i, \forall g \in \mathsf{H}[e], g \not\subseteq f_i$ $(1 \leq i \leq n-1)$. But then $\{f_1, ..., f_{n-1}, \cup\mathsf{H} - e\}$ is an n-partition of $\cup\mathsf{H}$ into pairwise disjoint mutually exhaustive sets such that $\forall g \in \mathsf{H}$, g is not a subset of any element of the partition. Therefore H is n-colourable, contrary to assumption. □

Theorem 4.3.7. *If $\eta(E) > 2$, $\forall f \in \Pi, |f| \geq 2$, and Π is the dual of E—i.e., $\Pi = E^d$, then $\exists e \in E, \exists d \in \Pi$ such that $|e \cap d| \geq 2$.*

Proof. Assume that $\eta(\mathsf{E}) > 2$, $\forall f \in \Pi, |f| \geq 2$, and Π is the dual of E. Since $\eta(\mathsf{E}) > 2$, we have $\chi(\mathsf{E}^d) > 2$. So $\forall g \in \mathsf{E}^{dd}, \chi(\mathsf{E}^d[g]) > 1$ (Lemma 4.3.6). But $\mathsf{E}^{dd} \subseteq \mathsf{E}$ (Proposition 4.2.1). Therefore $\exists e \in \mathsf{E}, \exists d \in \mathsf{E}^d$ such that $e \supseteq d$. But by assumption, $\mathsf{E}^d = \Pi$, and $\forall f \in \Pi, |f| \geq 2$. Whence $\exists e \in \mathsf{E}, \exists d \in \Pi$ such that $|e \cap d| \geq 2$. □

4.4 Conclusion

To conclude, let us take gender as an example of a property which is relevant to the preservation or integrity of personal, or self-identity. Let us say that what is meant by 'the instantiation (or preservation) of an exclusive gender' is the instantiation of some one kind of gender (i.e., to the exclusion of other genders) at a fixed time. (Many people probably think that they satisfy this constraint for all fixed times in their lives.) For the sake of argument, let us allow that an individual can change genders over time, but let us also exclude the possibility that he or she possesses no gender. One could then argue as follows:

> The preservation of an exclusive gender is necessary for the preservation of self-identity because humans—in order for their sense of identity to develop normally—are necessarily social beings, self-identity is essentially an expression of social dimensionality—of adopting different roles in different social settings—and at the intersection of any non-empty collection of sets of behaviours for different social contexts we find highly specific, and differing, gender-appropriate mores.

The point behind this passage is not so much the argument as its conclusion, and the language in which the inference is couched. The language suggests a hypergraphic representation to which the formalism of harmonics can be applied. Indeed, as an individual's behaviours in a social circumstance can be encoded as a collection of properties of the individual, we can also apply the formalism of instance graphs.

The question before us then, is *whether exclusivity of gendered behaviour of a particular kind, at a fixed time, is a requirement for an associated instance graph to be representative of a single individual.*

In our consideration of the chromatics of hypergraphs we have seen that if $\eta(\mathsf{E}) > 2$, every property kind has at least two elements, and the property graph Π is the dual of E, then some instant of E will include two distinct properties of the same kind (Theorem 4.3.7). We may therefore conclude that given a certain richness in our classification schema, if the collection of an individual's behaviours in his or her respective social settings is sufficiently cohesive, and there is an appropriate relation of duality between her or his synchronic and diachronic representations, then it is possible that some social role in which she or he engages will witness him or her behaving in both of two differently gendered ways—at the same time, no less. But this is a striking result, for it lays well-defined theoretical groundwork for theorizing about gender in terms which move beyond those enmeshed in standard bivalent thinking about sex, whereby a person is either male, or female, not both, and not neither. The existence of hermaphroditic, or intersexed, individuals alone should suffice for biological evidence against this pervasive view.[9] The present result shows that we needn't pigeonhole people into exactly one of the two categories 'male', 'female' – that adopting a broader, well-defined theory of gender is feasible.

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[9] Consider, for example, the existence of people with so-called 'mosaic genetics': some cells have XX chromosomes, whereas others have XY [5].

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Chapter 5

Harmonics for Hypergraphs

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Abstract

A truth-functional formulation, $\frac{n}{n+1}$, of the set $\binom{[n+1]}{n}$ $(n \in Z^+)$ of all n-tuple subsets of an $n+1$ membered set is shown to be functionally complete with respect to the formulation of the elements of a new class of structures, the class of all $(n+1)$-harmonic hypergraphs. As a corollary we prove that the system of logic defined by:

$\vdash \alpha \;\Rightarrow \vdash \Box\alpha$

$\vdash \alpha \to \beta \;\Rightarrow \vdash \Box\alpha \to \Box\beta$

$\vdash_{PL} \alpha \;\Rightarrow \vdash \alpha$

$\vdash \alpha \to \beta$ and $\vdash \alpha \;\Rightarrow \vdash \beta$

$\vdash \alpha$ and β is a substitution instance of $\alpha \;\Rightarrow \vdash \beta$

$\vdash \Box\alpha_1 \wedge \Box\alpha_2 \wedge ... \wedge \Box\alpha_{n+1} \to \Box\frac{n}{n+1}(\alpha_1, \alpha_2, ..., \alpha_{n+1})$

where $\frac{n}{n+1}(\alpha_1, \alpha_2, ..., \alpha_{n+1})$ is $\bigwedge_{i=1}^{n+1} \bigvee_{f=1}^{n} \alpha_f \in e_i, e_i \in \binom{[\alpha_i]}{n}$, is complete with respect to the modal logics of $(n+1)$-ary relational frames. In addition we consider some hypergraph theoretic applications of harmonic number, particularly in the domain of transversal hypergraphs.

[1]Permission to include this article was granted by both R.E. Jennings and D. Sarenac.

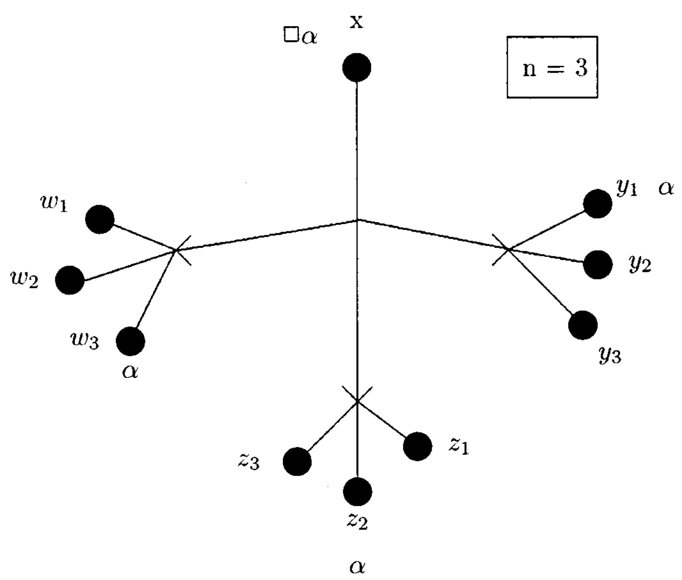

Figure 5.1.1: The truth condition for □ on $(n+1)$-ary relational frames.

5.1 Introduction

In this article we introduce a new class of structures, the class of $(n+1)$-harmonic hypergraphs, and demonstrate a connection between this and a class of non-standard modal logics, in particular, the logics of $(n+1)$-ary relational frames. Although the conception is implicit in Jonnson and Tarski [9, 10], recent interest in $(n+1)$-ary relational frames and their modal logics stems from Schotch and Jennings [6] [7] [12] [13]. The fundamentals of $(n+1)$-ary relational semantics are a generalization of those of Chellas [3]. If $\mathcal{U} \neq \emptyset$ and $\mathcal{R} \subseteq \mathcal{U}^{n+1}$, then $\mathfrak{F} = \langle \mathcal{U}, \mathcal{R} \rangle$ is an $(n+1)$-ary relational frame. $\mathfrak{M} = \langle \mathfrak{F}, \mathcal{V} \rangle$ is an $(n+1)$-ary model on $\mathfrak{F}$ if $\mathcal{V}$ maps the set of atoms to $\wp(\mathcal{U})$. $\mathcal{V}$ is extended to $[[\circ]]^{\mathfrak{M}}$ in the usual way for Boolean connectives and to the modal connective □ by

$$[[\Box\alpha]]^{\mathfrak{M}} = \{x \in \mathcal{U} \mid \forall \langle y_1, y_2, ..., y_n \rangle \in \mathcal{U}^n, \mathcal{R}xy_1y_2...y_n \Rightarrow \exists j \in [n] : \mathfrak{M} \models_{y_j} \alpha\},$$

where $[n]$ for any positive integer n abbreviates the set $\{1, 2, ..., n\}$. (See Figure 5.1.1.)

The introduction of $(n+1)$-ary frame theory introduced numerous new questions for correspondence theory, since formulae such as

$$[K] : \Box\alpha_1 \wedge \Box\alpha_2 \rightarrow \Box(\alpha_1 \wedge \alpha_2)$$

which were trivially valid in the binary idiom have no such status in the more generalized setting, and consistency formulae such as

$$[D] : \Box\alpha \rightarrow \Diamond\alpha \text{ and}$$

$$[G] : \Diamond\Box\alpha \rightarrow \Box\Diamond\alpha$$

that correspond to elementary classes of binary frames are not first order definable in the generalized idiom [5] [8]. But the first reliable general completeness proof for the class of $(n+1)$-ary relational logics appeared in Apostoli and Brown [1]. A simplified proof is given in [11] (cf. [14] for an algebraic treatment). It is now known that, as supposed all along, the class of $(n+1)$-ary relational frames is completely axiomatized by the modal system K_n:

$$[RN] : \ \vdash \alpha \ \Rightarrow \vdash \Box\alpha$$

$$[RM] : \ \vdash \alpha \rightarrow \beta \ \Rightarrow \vdash \Box\alpha \rightarrow \Box\beta$$

$$[RPL] : \ \vdash_{PL} \alpha \ \Rightarrow \vdash \alpha$$

$$[US] : \ \vdash \alpha \text{ and } \beta \text{ is a substitution instance of } \alpha \Rightarrow \vdash \beta$$

$$[MP] : \ \vdash \alpha \text{ and } \vdash \alpha \rightarrow \beta \Rightarrow \vdash \beta$$

$$[K_n] : \ \vdash \Box\alpha_1 \wedge \Box\alpha_2 \wedge ... \wedge \Box\alpha_{n+1} \rightarrow \Box\frac{2}{n+1}(\alpha_1, \alpha_2, ..., \alpha_{n+1})$$

where $\frac{2}{n+1}(\alpha_1, \alpha_2, ..., \alpha_{n+1})$ is $\bigvee_{i=1} \bigwedge_{f=1}^{2} \alpha_f \in e_i$ for $e_i \in \binom{[\alpha_i]}{2}$ (for any set S and positive integer n, $\binom{S}{n}$ denotes the set of all n-tuple subsets of S). Here, we replace $[K_n]$ with

$$[K^n] : \ \vdash \Box\alpha_1 \wedge \Box\alpha_2 \wedge ... \wedge \Box\alpha_{n+1} \rightarrow \Box\frac{n}{n+1}(\alpha_1, \alpha_2, ..., \alpha_{n+1})$$

where $\frac{n}{n+1}(\alpha_1, \alpha_2, ..., \alpha_{n+1})$ is $\bigwedge_{i=1}^{n+1} \bigvee_{f=1}^{n} \alpha_f \in e_i, e_i \in \binom{[\alpha_i]}{n}$, and prove completeness as a corollary of the functional completeness of $\frac{n+1}{n}$ with respect to the logical formulations of all $(n+1)$-harmonic hypergraphs.

5.2 Hypergraphs and Harmonics

If $V_{\neq\emptyset}$ is a set and $E = \{e_1, e_2, ..., e_i, ...\} \subseteq 2^V$ then the pair (V, E) is a *hypergraph* H with V the *vertex* set of H and E the set of *edges* of H. Since for most purposes V can be taken to be $\bigcup_{i=1} e_i$, abbreviated '$\cup H$', H can be identified with E, and we can refer to the edges of H by speaking of its elements. A *finite hypergraph* is one whose vertex set is finite. Thus, if H is a finite hypergraph then $|H|$ is finite and its edges are finitely long. A *simple hypergraph* H is one satisfying the condition that $\forall e, f \in H$, $e \not\supset f$. For any hypergraph H, the *harmonic number* of H, $\eta(H)$, is defined:

$$\eta(H) := \begin{cases} min\ n \in \mathbb{Z}^+ : \exists G \in \binom{H}{n} : \cap G = \emptyset & \text{if this limit exists;} \\ \infty & \text{otherwise,} \end{cases} \tag{5.2.1}$$

where if $G = \{e_1, ..., e_i, ...\}$ is a hypergraph, then '$\cap G$' abbreviates '$\bigcap_{i=1} e_i$'. We say that H is $(n+1)$-*harmonic* if $\eta(H) = n + 1$.

5.2.1 An Application to Hypergraph Theory

A hypergraph theoretic motivation for studying η can be found in the work of Berge on transverse hypergraphs [2]. If H is hypergraph and S is a set then S is a *transversal* for H if $\forall e \in H, S \cap e \neq \emptyset$; S is a *minimal transversal* for H if S is a transversal for H and no proper subset of S is a transversal for H. The *transverse hypergraph* of H, TH, is the set of all minimal transversals of H.

Theorem 5.2.1. *$\forall H, TTH \subseteq H$.*

Proof. Let H be a hypergraph, and let $e \in TTH$. Since every edge f of H is a transversal for TH, $\forall f \in H, \exists g \in TTH$ such that $f \supseteq g$. So suppose now that $e \notin H$. Then every edge f of H has an element that is not in e. Therefore, $\exists g \in TH$ such that $g \cap e = \emptyset$, which is impossible since $e \in TTH$. Whence $e \in H$. □

Theorem 5.2.2. *$\forall H$, if H is simple then $H = TTH$.*

Proof. Let H be a simple hypergraph, and let $e \in H$. Now e is a transversal for TH, so $\exists f \in TTH$ such that $e \supseteq f$. Suppose that $e \supset f$. Notice that since H is simple, $f \notin H$. Also, because H is simple, we have $\forall g \in H, f \not\supseteq g$. Therefore every element of H contains an element that is not in f—that is, $\exists h \in TH$ such that $h \cap f = \emptyset$, which is impossible since $f \in TTH$. Therefore $e = f$, and $H \subseteq TTH$. From the preceding theorem we have the converse, namely that $TTH \subseteq H$, in which case $H = TTH$, as desired. □

Berge shows that if H is a simple hypergraph then $H = TH$ iff H is pairwise intersecting and 2-uncolourable [2]. In the language of harmonics this amounts to the claim that for a simple hypergraph H, $H = TH$ iff $\eta(H) > 2$ and H is 2-uncolourable.

If H is any hypergraph, and n is a positive integer, then a function $f : \cup H \rightarrow [n]$ is an *n-colouring* of H $(n \geq 1)$ if $\forall e \in H, \exists x, y \in e : f(x) \neq f(y)$. The *chromatic number* of H, $\chi(H)$, is defined:

$$\chi(H) := \begin{cases} min\ n \in \mathbb{Z}^+ : \text{ there is an } n\text{-colouring of } H & \text{if this limit exists;} \\ \infty & \text{otherwise.} \end{cases}$$

If $\chi(H) \leq n$ then H is *n-colourable*; else H is *n-uncolourable.* An *n-chromatic* hypergraph is one whose chromatic number is n.

Thus, Berge has shown that for any simple hypergraph H, $H = TH$ iff $\chi(H) > 2$ and $\eta(H) > 2$ [2]. More specifically we can show that $\forall H$, if H is simple then $H = TH \Leftrightarrow$ either $H = \{\{x\}\}$ or $\chi(H) = \eta(H) = 3$. To prove this theorem we require the notion of a *partition* of a set, and a lemma. To those ends, let n be a positive integer, and let S be a set. Then the set of *n-partitions of* S, $\Pi_n(S)$ is defined:

$$\Pi_n(S) := \{\pi = \{c_1, ..., c_n\} \mid \forall i, j \ (1 \leq i < j \leq n), c_i \cap c_j = \emptyset \text{ and } \bigcup_{i=1}^{n} c_i = S\}$$

Lemma 5.2.3. $\forall H, n \geq 0, \eta(H) = n + 1 \Leftrightarrow \chi(TH) = n + 1.$

Proof. It is sufficient to show that $\forall H, m \geq 0, \eta(H) > m$ iff $\chi(TH) > m$. To that end:

[$\Rightarrow$] Assume that $\chi(TH) \leq m$. Then there is an m-colouring of TH, so there is an m-partition π of $\cup H$ such that $\forall e \in H, \forall c \in \pi, e \not\subseteq c$. Let $A = \{\{(\cup H) - c\} \mid c \in \pi\}$. Then $\forall a \in A, \exists e \in TTH$ such that $e \subseteq a$. But $TTH \subseteq H$ (Theorem 5.2.1). Therefore, $\forall a \in A, \exists e \in H$ such that $e \subseteq a$. Also, $\cap A = \emptyset$. Therefore $\exists l \in [m], \exists B \in \binom{H}{l}$ such that $\cap B = \emptyset$. That is, $\eta(H) \leq l \leq m$.

[$\Leftarrow$] Assume that $\eta(H) \leq m$. Then for some $l \in [m]$, $\exists A \in \binom{H}{l}$ such that $\cap A = \emptyset$. Let $B = \{\{(\cup H) - a\} \mid a \in A\}$. Then $\forall e \in TH, \forall b \in B, e \not\subseteq b$. But B induces an l-colouring of TH. Whence $\chi(TH) \leq l \leq m$.

□

Theorem 5.2.4. $\forall H$, *if* H *is simple then* $H = TH \Leftrightarrow$ *either* $H = \{\{x\}\}$ *or* $\chi(H) = \eta(H) = 3$.

Proof. Let H be simple.

[$\Rightarrow$] Assume that $H = TH$, and that $\forall x \in \cup H, H \neq \{\{x\}\}$. Then $\eta(H) > 2$ because every edge of H is a transversal for H. Also, $\chi(H) > 2$ else there is a 2-partition π of H such that

$\forall c \in \pi, e \in H, e \not\subseteq c$, in which case each $c \in \pi$ is a transversal for H, in which case $\exists A \in \binom{H}{2}$ such that $\cap A = \emptyset$, contrary to $\eta(H) > 2$. Now since $H = TH$, $\forall e \in H, x \in e, \exists A \in \binom{H}{2} : e \in A$ and $\cap A = \{x\}$. Moreover, since $\forall x \in \cup H, H \neq \{\{x\}\}, |H| \geq 2$ because $|TH| > 2$. Let $e \in H$ be arbitrary, and let $x \in e$. Then $\exists f \in H : x \notin f$. Otherwise $\{x\} \in TH$, contrary to the simplicity of H. $\therefore \eta(H) = 3$. But then $\chi(TH) = 3$, using Lemma 5.2.3, in which case $\chi(H) = 3$.

[$\Leftarrow$] Assume that $\chi(H) = \eta(H) = 3$. Let $e \in H$. Since $\eta(H) > 2$, e is a transversal for H, so $\exists f \subseteq e$ such that $f \in TH$. Suppose that $f \subset e$. Since $\chi(H) > 2$, $\eta(TH) > 2$ (Theorem 5.2.2 and Lemma 5.2.3). $\therefore f$ is a transversal for TH. So $\exists g \subseteq f$ such that $g \in TTH$. But $TTH \subseteq H$ (Theorem 5.2.1). $\therefore g \in H$, contrary to the simplicity of H. Whence $e = f$ and $H \subseteq TH$.

Now let $e \in TH$. Since $\eta(TH) > 2$ (Theorem 5.2.2 and Lemma 5.2.3), e is a transversal for TH. So exists $f \subseteq e$ such that $f \in H$, since $TTH \subseteq H$ (Theorem 5.2.1). Suppose that $f \subset e$. Then $\exists g \in TTH$ such that $g \cap f = \emptyset$. But $H = TTH$ (Theorem 5.2.2). Whence $\eta(H) \leq 2$, contrary to the assumption that $\eta(H) = 3$. Therefore $e = f$ and $TH \subseteq H$.

Lastly, suppose that $H = \{\{x\}\}$. Then $H = TH$, trivially. Therefore $H = TH$. □

In fact, the class of hypergraphs identical to their transverse hypergraphs can be characterized independently of chromatic properties by exploiting a maximality condition imposed on η. If H is n-harmonic then H is *maximally n-harmonic* if for any set S such that $\forall e \in H, S \not\supseteq e$, it follows that $\eta(H \cup \{S\}) < n$.

Theorem 5.2.5. *Let H be simple. Then $H = TH \Leftrightarrow$ either $H = \{\{x\}\}$, for some x, or H is maximally 3-harmonic.*

Proof. Assume that H is simple.

[$\Rightarrow$] Assume that $H = TH$ and that $H \neq \{\{x\}\}$, for any $x \in \cup H$. Let S be such that $\forall e \in H, S \not\supseteq e$. Then $\exists f \in TH(= H) : f \cap S = \emptyset$. $\therefore \eta(H \cup \{S\}) \leq 2$. But from Theorem 5.2.4 we have $\eta(H) = 3$. Whence H is maximally 3-harmonic.

[$\Leftarrow$] Assume that H is maximally 3-harmonic. Let e be in H. Then since $\eta(H) > 2, \exists f \subseteq e$ such that $f \in TH$. Suppose $f \subset e$. Then contrary to hypothesis, H is not maximal with respect to η, since $\eta(H \cup \{f\}) = \eta(H)$. Whence $H \subseteq TH$.

Let $e \in TH$. Then by the maximality of H, $\exists f \in H$ such that $e \supseteq f$. If $e \supset f$ then, since $\eta(H) > 2$, e is not a minimal transversal for H. $\therefore TH \subseteq H$ and $H = TH$. □

In addition to indicating a hypergraph theoretic application of η, Theorem 5.2.5 has the virtue of yielding a logical one as well. This is because it characterizes the conditions under which the logical formulation of a simple hypergraph is self-dual.

5.2.2 Logical Duality

If H is a finite hypergraph then H can be formulated as a truth function in any propositional language whose connectives include $\vee$ and $\wedge$, where these are given standard, Boolean, interpretations. This is because the vertices of H can be assigned propositional variables: Let $VAR = \{p_1, p_2, ..., p_i, ...\}$ be a denumerable set of propositional variables. Then where H is a hypergraph, let $g : \cup H \rightarrow VAR$ be any function such that $\forall i \in \cup H, g(i) = p_i$.

Now let $H = \{e_1, ..., e_m\}$ be a finite hypergraph. Then the *formulation* of H, $F(H)$ is defined:

$$F(H) := \bigwedge_{i=1}^{m} \bigvee_{h=1}^{|e_i|} p_h. \tag{5.2.2}$$

The *dual formulation* of H, $F^d(H)$ is defined:

$$F^d(H) := \bigvee_{i=1}^{m} \bigwedge_{h=1}^{|e_i|} p_h. \tag{5.2.3}$$

The dual formulation of H has been so named with Church's notion of the *principle dual* of a formula in mind [4]. The principle dual of a formula is obtained by replacing each occurrence of a connective with an occurrence of the dual of the connective. The dual of a connective is obtained by interchanging all 1's and 0's in its truth definition. Thus, for any finite hypergraph H, $F(H)$ and $F^d(H)$ are principal duals. By 'duality' however, we mean a broader notion than that in Church; here, a formula α is *dual* to a formula β iff α is equivalent to the result of interchanging all occurrences of 1's and 0's in the truth table for β. Whence we have the following principle of duality:

Proposition 5.2.6. *If a sentence α is dual to a sentence β, and α is truth functionally equivalent to a sentence δ, and β is truth functionally equivalent to a sentence γ, then δ is dual to γ.*

Proposition 5.2.6 enjoined with the following theorem suggests that for any finite hypergraph H, there is a sense in which we can think of H and its transverse hypergraph TH as being dual. Since we are now dealing with truth functional formulations of hypergraphs, in this section we restrict our attention to finite hypergraphs.

Theorem 5.2.7. $\forall H, F(H) \;\; \dashv\models \;\; F^d(TH)$ & $F^d(H) \;\; \dashv\models \;\; F(TH)$.

Proof. Theorem 5.2.7 is easily proved using Theorem 5.2.1. □

Corollary 5.2.8. *$\forall H$, $F^d(TH)$ is dual to $F^d(H)$, and $F(H)$ is dual to $F(TH)$.*

Corollary 5.2.8 illustrates the sense in which H and TH, for arbitrary H, are dual. In the same vein, chromatic number is dual to harmonic number. For Theorem 5.2.2 and Lemma 5.2.3 entail:

1. an $(n+1)$-chromatic simple hypergraph is the transverse hypergraph of an $(n+1)$-harmonic hypergraph, and
2. an $(n+1)$-harmonic simple hypergraph is the transverse hypergraph of an $(n+1)$-chromatic hypergraph.

This suggests that our interest in hypergraphs runs dual to one of the principal mathematical interests in hypergraphs, namely, that pertaining to chromatic number.

With an eye towards proving completeness for the modal system K^n with respect to the class of $(n+1)$-ary relational frames, we now show that the truth function $\frac{n}{n+1}$ is complete with respect to the formulations of all $(n+1)$-harmonic hypergraphs.

5.2.3 Functional Completeness

For a set $\{\alpha_1, \alpha_2, ..., \alpha_{n+1}\}$ of formulae, and a positive integer n, the truth function $\frac{n}{n+1}$ is defined:

$$\frac{n}{n+1}(\alpha_1, \alpha_2, ..., \alpha_{n+1}) \;:=\; F(\binom{[\alpha_i]}{n})$$

We intend to prove that if Σ is an arbitrary non-empty set of formulae closed under $\frac{n}{n+1}$ and propositional implication, then Σ contains the formulation of every finite hypergraph H such that $\eta(H) > n$ and $\cup H \subseteq \Sigma$. As a corollary it follows that implication enjoined with $\frac{n}{n+1}$ is complete with respect to the formulation of transverse hypergraphs of finite n-uncolourable hypergraphs—a fact which is to play a prominent role in the completeness proof for K^n.

We induce on $|H|$. For the basis, let $|H|$ be 1. Then where $H = \{e = \{p_1, p_2, ..., p_k\}\}, F(H) \doteq p_1 \vee p_2 \vee ... \vee p_k$; thus $F(H) \in \Sigma$ since $e \subseteq \Sigma$ (using $\vdash p \rightarrow p \vee q$).

Now let $|H| = t \geq 1$. Suppose that $t < n + 1$. Then $\cap H \neq \emptyset$, else $\eta(H) \leq n$. Let p_i be in $\cap H$. Then $\vdash p_i \rightarrow F(H)$, whence $F(H) \in \Sigma$.

Assume $t \geq n + 1$. Where $\{e_1, e_2, ..., e_{n+1}\}$ is a subset of H, define:

$$H_i := H - \{e_i\} \quad (1 \leq i \leq n + 1)$$

Then $\eta(H_i) > n$ for each i $(1 \leq i \leq n + 1)$, in which case $F(H_i) \in \Sigma$. Now for any pair i, j $(1 \leq i \neq j \leq n + 1), H_i \cup H_j = H$; therefore $F(H_i) \wedge F(H_j) \;\dashv\models\; F(H)$, and therefore $\vdash F^d(\binom{[F(H_i)]}{2}) \rightarrow F(H)$. But $\vdash \frac{n}{n+1}(F(H_1), F(H_2), ..., F(H_{n+1})) \rightarrow F^d(\binom{[F(H_i)]}{2})$. Whence $\vdash \frac{n}{n+1}(F(H_1), F(H_2), ..., F(H_{n+1})) \rightarrow F(H)$, in which case $F(H) \in \Sigma$, and we have:

Theorem 5.2.9. *If a non-empty set Σ of formulae is closed under implication and $\frac{n}{n+1}$, then $\forall H$, if $\cup H$ is a finite subset of Σ and $\eta(H) > n$, then $F(H) \in \Sigma$.*

5.3 Completeness

Theorem 5.2.9 is pivotal in the following proof that the system K^n:

$$[RN] : \vdash \alpha \Rightarrow \vdash \Box\alpha \tag{5.3.1}$$

$$[RM] : \vdash \alpha \rightarrow \beta \Rightarrow \vdash \Box\alpha \rightarrow \Box\beta \tag{5.3.2}$$

$$[RPL] : \vdash_{PL} \alpha \Rightarrow \vdash \alpha \tag{5.3.3}$$

$$[US] : \vdash \alpha \text{ and } \beta \text{ is a substitution instance of } \alpha \Rightarrow \vdash \beta \tag{5.3.4}$$

$$[MP] : \vdash \alpha \text{ and } \vdash \alpha \rightarrow \beta \Rightarrow \vdash \beta \tag{5.3.5}$$

$$[K^n] : \vdash \Box\alpha_1 \wedge \Box\alpha_2 \wedge ... \wedge \Box\alpha_{n+1} \rightarrow \Box\frac{n}{n+1}(\alpha_1, \alpha_2, ..., \alpha_{n+1}) \tag{5.3.6}$$

completely axiomatizes the logics of $(n+1)$-ary relational frames. We begin by defining a canonical model $\mathfrak{M} = \langle \mathfrak{F}, \mathcal{V} \rangle$, where $\mathfrak{F} = \langle \mathcal{U}, \mathcal{R} \rangle$ is the canonical frame defined as follows:

$$\mathcal{U} \text{ is the set of all maximal } K^n\text{-consistent sets, and} \tag{5.3.7}$$

$$\mathcal{R} \text{ is a subset of } \mathcal{U}^{n+1}, \text{ where } \forall x \in \mathcal{U}, \forall \langle y_1, ..., y_n \rangle \in \mathcal{U}^n, \tag{5.3.8}$$

$$\mathcal{R}xy_1...y_n \text{ iff } \{y_1, ..., y_{n+1}\} \text{ induces an } n\text{-decomposition of } \{\alpha \mid \mathfrak{M} \models_x \Box\alpha\}. \tag{5.3.9}$$

Let '$\Box(x)$' denote the set $\{\alpha \mid \mathfrak{M} \models_x \Box\alpha\}$. Our aim is to prove the fundamental theorem:

$$\forall x \in \mathcal{U}, \alpha, \ \mathfrak{M} \models_x \alpha \Leftrightarrow \alpha \in x. \tag{5.3.10}$$

The proof is by an induction on the complexity of α, and we omit all but the hardest case, viz., to show that $\Box\beta \notin x \Rightarrow \mathfrak{M} \not\models_x \Box\beta$.

Assume that $\Box\beta \notin x$. By the definition of $\mathcal{R}$, it is sufficient to prove

$$\exists \pi \in \Pi_n(\Box(x)) : \forall c \in \pi, c \cup \{\neg\beta\} \nvdash_{K^n} \bot. \tag{5.3.11}$$

For in that case, there is an x-related n-tuple of points $y_1, y_2, ..., y_n$, none of which is in $[[\beta]]^{\mathfrak{M}}$, and thus $\mathfrak{M} \not\models_x \Box\beta$, as desired. Moreover, to prove 5.3.11, it is sufficient to prove:

$$\forall \Sigma \subset \Box(x), \text{ if } \Sigma \text{ is finite, then } \exists \pi \in \Pi_n(\Sigma) : \forall c \in \pi, c \cup \{\neg\beta\} \nvdash_{K^n} \bot. \tag{5.3.12}$$

Otherwise, if 5.3.12 is true, but 5.3.11 is false, then the hypergraph H whose edges consist of all least finite $\Box(x)$-subsets e such that $e \cup \{\neg\beta\} \vdash_{K^n} \bot$ is n-uncolourable. Chromatic number, however, is compact for hypergraphs whose edges are finitely long. That is, $\forall H$, if every edge of H is finitely long, then $\chi(H) \leq n$ iff $\forall G \subseteq H$, if G is finite then $\chi(G) \leq n$.[2] Therefore, there is a finite hypergraph $G \subseteq H$ such that $\chi(G) > n$. But then $\cup G$ is a finite subset of $\Box(x)$ for which there is no n-partition π such that $\forall c \in \pi, c \cup \{\neg\beta\} \nvdash_{K^n} \bot$, which is absurd by hypothesis. Whence we aim to show that 5.3.12 is true.

[2]Chromatic compactness is provable using the compactness of propositional logic.

Suppose that 5.3.12 is false—suppose that there is no n-partition π of Σ such that $\forall c \in \pi, c \cup \{\neg\beta\} \nvdash_{K^n} \bot$. Then by drawing from some cell in each n-partition of Σ, we can form a simple hypergraph H where $\forall e \in H, e$ is a minimal set such that $e \vdash_{K^n} \beta$. Since H is simple, by Theorem 5.2.2, $H = TTH$. Accordingly, because $\chi(H) > n$, from Theorem 5.2.3 it follows that $\eta(TH) > n$. Whence, since $\cup H \subset \Box(x)_{\neq\emptyset}$, and $\Box(x)$ is closed under implication, by the functional completeness of $\frac{n}{n+1}$ we have $F(TH) \in \Box(x)$ (Theorem 5.2.9). But $\vdash F(TH) \to \beta$ (Theorem 5.2.7). So by the presence of $[RM]$, $\Box\beta \in x$, which is absurd. QED

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Chapter 6

A Dualization of Neighbourhood Structures

D. Nicholson

Abstract

A dualization of neighborhood semantics is introduced for the purpose of obtaining a simplified proof that weakly aggregative modal logic is complete with respect to $(n+1)$-ary relational frames. This new class of quasi-semantic structures exploits the theory of transverse hypergraphs. Unlike the other proofs in the literature, the one included here does not cite chromatic, or colouring, compactness. Along the way we prove completeness for a denumerable class of non-normal modal logics, which have deontic, as well as philosophical logical, motivations.

6.1 Introduction

The search for a completeness proof for Jennings and Schotch's weakly aggregative modal logic lasted for nearly twenty years (cf. [4][6][3][7]) before its goal was attained in 1995 by Apostoli and Brown in [1], and also independently, algebraically, by Urquhart in [9]. Apostoli and Brown's proof was subsequently simplified, in 2000, by Nicholson, Jennings and Sarenac in [5]. But both proofs exploit the compactness of colouring for hypergraphs whose edges are finitely long. *Chromatic* *compactness*, also called *colouring compactness*, is the claim that $\forall H$, if H is a hypergraph then H is k-colourable ($k \geq 1$) iff every finite $G \subseteq H$ is k-colourable.[1] A *hypergraph* is a family of sets, called *edges*. A hypergraph H is *k-colourable* iff there is a partition of its union, called its *vertex set*, into k pairwise disjoint, mutually exhaustive sets, or cells, such that no edge of H is a subset of any cell. The simplifying thrust of Nicholson *et al.* in [5] raises the question whether there is an even simpler completeness proof, one which avoids citing chromatic compactness. In this paper an affirmative answer to this question is demonstrated, by invoking the theory of what are dubbed *hyperframes*. The theory of hyperframes implements the theory of transverse hypergraphs and consists essentially of a dualization of the neighborhood semantics for modal logic explored by Segerberg in [8], and referred to as 'minimal models' by Chellas in [2].

6.2 Hyperframes

A *hyperframe* $\mathfrak{F}$ is pair $(\mathcal{U}, \mathcal{H})$ where $\mathcal{U}$ is a non-empty set (the *universe* of the frame) and $\mathcal{H}$ is a *hypergraph function* from $\mathcal{U}$ to $\wp\wp(\mathcal{U})$. Accordingly, for each $x \in \mathcal{U}, \mathcal{H}(x)$ is a *hypergraph* on $\mathcal{U}$: a family of subsets of $\mathcal{U}$, where the subsets are called the *edges* of the hypergraph, and the elements of the edges are called the *vertices* of the hypergraph. For each x in $\mathcal{U}$, $\mathcal{H}(x)$ is called the *hypergraph on x* (relative to $\mathfrak{F}$). A hyperframe is thus, in essence, a neighborhood frame, as the latter is defined in [8], for example. But a model on a hyperframe is distinct from a model on a neighborhood frame when it comes to interpreting $\Box$.

If $\mathfrak{F} = (\mathcal{U}, \mathcal{H})$ is a hyperframe and $\mathcal{V} : Nat \rightarrow \wp(\mathcal{U})$ is a valuation function, then $(\mathcal{U}, \mathcal{H}, \mathcal{V})$ is a *(hyper)model* $\mathfrak{M}$ on $\mathfrak{F}$. Truth at a point x in a model $\mathfrak{M} = (\mathcal{U}, \mathcal{H}, \mathcal{V})$, with respect to the language of a standard propositional logic, is defined in the standard way for Boolean connectives. To interpret the unary necessity operator $\Box$, we use the notion of the *transversal* of a hypergraph.

Definition 6.2.1. If $\mathcal{H}$ is a hypergraph on a non-empty set $\mathcal{U}$, and S is a subset of $\mathcal{U}$, S is a *transversal* for $\mathcal{H}$ iff $\forall \mathcal{E} \in \mathcal{H}, S \cap \mathcal{E} \neq \emptyset$.

It follows from this definition that $\emptyset \in \mathcal{H}$ iff $\mathcal{H}$ has no transversals, and $\mathcal{H} = \emptyset$ iff, vacuously, every subset of $\mathcal{U}$ is a transversal for $\mathcal{H}$.

[1] Colouring compactness is provable from the compactness of propositional logic.

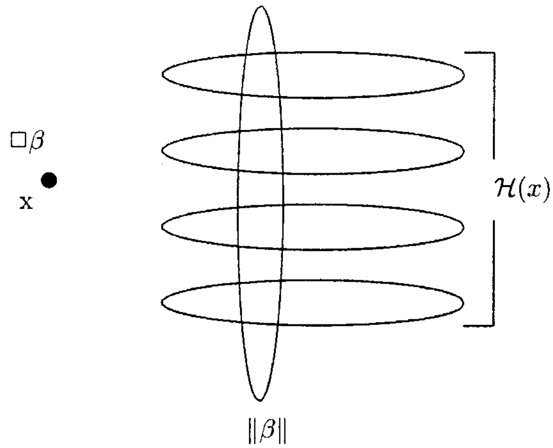

Figure 6.2.1: The truth condition for □ on hyperframes.

The truth condition for □ is:

$$\models^{\mathfrak{M}}_{x} \Box\beta \text{ iff } \|\beta\|^{\mathfrak{M}} \text{ is a transversal for } \mathcal{H}(x).$$

(See Figure 6.2.1.) Introducing $\Diamond\beta$ as an abbreviation for $\neg\Box\neg\beta$, we therefore have:

$$\models^{\mathfrak{M}}_{x} \Diamond\beta \text{ iff } \exists\mathcal{E} \in \mathcal{H}(x) : \|\beta\|^{\mathfrak{M}} \supseteq \mathcal{E}$$

If $\mathfrak{M} = (\mathcal{U}, \mathcal{H}, \mathcal{V})$ is a model and $\forall x \in \mathcal{U}, \models^{\mathfrak{M}}_{x} \alpha$, then α is *valid on* $\mathfrak{M}$, written '$\models^{\mathfrak{M}} \alpha$'; if for every valuation function $\mathcal{V}$, α is valid on $(\mathcal{U}, \mathcal{H}, \mathcal{V})$ then α is *valid on* $\mathfrak{F} = (\mathcal{U}, \mathcal{H})$, indicated by '$\mathfrak{F} \models \alpha$'. If α is valid on every member of a class $\mathfrak{C}$ of frames, then α is *valid with respect to* $\mathfrak{C}$, '$\mathfrak{C} \models \alpha$'.

The logic determined by the class $\mathfrak{C}$ of all hyperframes is axiomatized by $\mathbf{N_q}$,[2] the system defined by:

$$[RR]: \vdash \alpha \to \beta \Rightarrow \vdash \Box\alpha \to \Box\beta \tag{6.2.1}$$

$$[PL]: \vdash_{PL} \alpha \Rightarrow \vdash \alpha \tag{6.2.2}$$

$$[MP]: \vdash \alpha \to \beta \;\&\; \vdash \alpha \Rightarrow \vdash \beta \tag{6.2.3}$$

$$[US]: \vdash \alpha \;\&\; \beta \text{ is a substitution instance of } \alpha \Rightarrow \vdash \beta \tag{6.2.4}$$

[2]The name of this system has been drawn from [3].

It is easy to check that $\mathbf{N_q}$ is sound with respect to $\mathfrak{C}$; to prove completeness we use a Henkin construction which capitalizes on the theory of transverse hypergraphs.

Definition 6.2.2. Let $\mathcal{H}$ be a hypergraph on a set $\mathcal{U}$. A set $\mathcal{E} \subseteq \mathcal{U}$ is a *minimal transversal* for $\mathcal{H}$ if $\mathcal{E}$ is a transversal for $\mathcal{H}$ and $\forall \mathcal{E}' \subset \mathcal{E}, \mathcal{E}'$ is not a transversal for $\mathcal{H}$. The *transverse hypergraph* for $\mathcal{H}$, $T(\mathcal{H})$, or just $T\mathcal{H}$ for convenience, is the set of all minimal transversals for $\mathcal{H}$.

Proposition 6.2.1. *For any hypergraph $\mathcal{H}$ on a set $\mathcal{U}$, $\mathcal{H} = \emptyset$ iff $T\mathcal{H} = \{\emptyset\}$, and $\emptyset \in \mathcal{H}$ iff $T\mathcal{H} = \emptyset$.*

Proof. [$\Rightarrow$] Assume that $\mathcal{H} = \emptyset$. Then every subset of $\mathcal{U}$ is a transversal for $\mathcal{H}$; but then if $\mathcal{E} \neq \emptyset$, $\exists \mathcal{E}' \subset \mathcal{E}$ such that $\mathcal{E}'$ is a transversal for $\mathcal{H}$. So every minimal transversal for $\mathcal{H}$ is empty. And, $\emptyset \in T\mathcal{H}$ because $\emptyset$ has no proper subsets. Thus $T\mathcal{H} = \{\emptyset\}$. [$\Leftarrow$] Assume now that $T\mathcal{H} = \{\emptyset\}$. Then $\forall \mathcal{E} \in \mathcal{H}, \emptyset \cap \mathcal{E} \neq \emptyset$. Therefore $\mathcal{H} = \emptyset$.

[$\Rightarrow$] Suppose that $\emptyset \in \mathcal{H}$. Then $\forall \mathcal{E} \in T\mathcal{H}$, $\mathcal{E} \cap \emptyset \neq \emptyset$. Whence $T\mathcal{H} = \emptyset$. [$\Leftarrow$] Lastly, suppose that $T\mathcal{H} = \emptyset$. If $\mathcal{H} = \emptyset$ then by the above reasoning $\emptyset \in T\mathcal{H}$. So $\mathcal{H} \neq \emptyset$. If, then, $\emptyset \notin \mathcal{H}$, $\exists \mathcal{E}_{\neq \emptyset}$ such that $\mathcal{E} \in T\mathcal{H}$, which is absurd. Therefore $\emptyset \in \mathcal{H}$. □

Definition 6.2.3. A hypergraph $\mathcal{H}$ is *simple* if $\forall \mathcal{E}, \mathcal{E}' \in \mathcal{H}$, $\mathcal{E} \not\subset \mathcal{E}'$.

Proposition 6.2.2. *For any hypergraph $\mathcal{H}$ on a set $\mathcal{U}$, $TT\mathcal{H} \subseteq \mathcal{H}$. If $\mathcal{H}$ is simple then $\mathcal{H} = TT\mathcal{H}$.*

Proof. Let $\mathcal{H}$ be a hypergraph on $\mathcal{U}$, and let $\mathcal{E} \in TT\mathcal{H}$. Suppose that $\mathcal{E} \notin \mathcal{H}$. Note that $\forall \mathcal{E}' \in \mathcal{H}, \mathcal{E}'$ is a transversal for $T\mathcal{H}$. Therefore $\forall \mathcal{E}' \in \mathcal{H}, \exists x \in \mathcal{E}'$ such that $x \notin \mathcal{E}$, in which case $\exists \mathcal{E}' \in T\mathcal{H}$ such that $\mathcal{E} \cap \mathcal{E}' = \emptyset$, contrary to the assumption that $\mathcal{E} \in TT\mathcal{H}$. Therefore $TT\mathcal{H} \subseteq \mathcal{H}$.

Now let $\mathcal{E} \in \mathcal{H}$, and assume that $\mathcal{H}$ is simple. Since $\mathcal{E}$ is a transversal for $T\mathcal{H}$, $\exists \mathcal{E}' \subseteq \mathcal{E}$ such that $\mathcal{E}' \in TT\mathcal{H}$. But the above reasoning shows that $TT\mathcal{H} \subseteq \mathcal{H}$. Therefore, since $\mathcal{H}$ is simple, $\mathcal{E}' = \mathcal{E}$, i.e., $\mathcal{E} \in TT\mathcal{H}$, whence $\mathcal{H} \subseteq TT\mathcal{H}$. □

If $\mathbf{L}$ is a modal logic and α is a sentence, the *proof set* for α in $\mathbf{L}$, $|\alpha|_{\mathbf{L}}$, is the set of all maximal $\mathbf{L}$-consistent sets of which α is a member. The *canonical frame* for $\mathbf{L}$ is the structure $\mathfrak{F}_{\mathbf{L}} = (\mathcal{U}_{\mathbf{L}}, \mathcal{H}_{\mathbf{L}})$ where $\mathcal{U}_{\mathbf{L}}$ is the class of all maximal $\mathbf{L}$-consistent sets of formulae, and $\forall x \in \mathcal{U}_{\mathbf{L}}$,

$$\mathcal{H}_{\mathbf{L}}(x) = T(\{|\gamma|_{\mathbf{L}} : \Box\gamma \in x\}).$$

The *canonical model* for $\mathbf{L}$, $\mathfrak{M}_{\mathbf{L}}$, is the triple $(\mathcal{U}_{\mathbf{L}}, \mathcal{H}_{\mathbf{L}}, \mathcal{V}_{\mathbf{L}})$ where $\mathcal{V}_{\mathbf{L}}$ is defined:

$$\forall n \in Nat, x \in \mathcal{V}_{\mathbf{L}}(n) \text{ iff } p_n \in x.$$

Theorem 6.2.3. *Let* $\mathbf{L}$ *be any logic closed under* $[RR], [PL], [US]$ *and* $[MP]$. *Then*

$$\forall \alpha, x \in \mathcal{U}_{\mathbf{L}}, \|\alpha\|^{\mathfrak{M}_{\mathbf{L}}} = |\alpha|_{\mathbf{L}}.$$

Proof. We show that $\forall x \in \mathcal{U}_{\mathbf{L}}, \alpha$ is true at x iff $\alpha \in x$. The proof is by induction on the complexity of α. We omit all but the case for $\alpha = \Box\beta$.

Suppose that $x \in \|\Box\beta\|$. Then $\|\beta\|$ is a transversal for $\mathcal{H}_{\mathbf{L}}(x)$, in which case, by the hypothesis of induction, so is $|\beta|$. Therefore $\exists \mathcal{E} \subseteq |\beta|$ such that $\mathcal{E} \in T(\mathcal{H}_{\mathbf{L}}(x))$. Whence $\mathcal{E} \in \{|\gamma| : \Box\gamma \in x\}$ (Proposition 6.2.2). So let $\mathcal{E} = |\gamma|$. Then $|\gamma| \subseteq |\beta|$ and $\Box\gamma \in x$. But then $\vdash \gamma \rightarrow \beta$, and thus $\vdash \Box\gamma \rightarrow \Box\beta$ (by $[RR]$). Therefore $\Box\beta \in x$.

Suppose now that $\Box\beta \in x$. Then $\forall \mathcal{E} \in \mathcal{H}_{\mathbf{L}}(x), \mathcal{E} \cap |\beta| \neq \emptyset$. I.e., $|\beta|$ is a transversal for $\mathcal{H}_{\mathbf{L}}(x)$. By the induction hypothesis, $|\beta| = \|\beta\|$. Whence $x \in \|\Box\beta\|$. □

Corollary 6.2.4. *The system* $\mathbf{N_q}$ *is determined by the class of all hyperframes.*

6.3 Normal Hyperframes

The logic $\mathbf{N_q}$ is not normal because there is at least one theorem of $\mathbf{N_q}$ whose $\Box$ formula is not a theorem. E.g., $\vdash_{\mathbf{N_q}} \top$ while $\nvdash_{\mathbf{N_q}} \Box\top$. Since there is a hypermodel $\mathfrak{M}$ containing a point x such that $\emptyset \in \mathcal{H}(x)$, it follows that $\mathfrak{C} \not\models \Box\top$, and hence $\nvdash_{\mathbf{N_q}} \Box\top$ (Corollary 6.2.4). By similar, dual, reasoning there is a model $\mathfrak{M}$ containing a point x such that $\forall\alpha, \models^{\mathfrak{M}}_{x} \Diamond\alpha$, and thus $\models^{\mathfrak{M}}_{x} \Diamond\bot$. Hyperframes therefore provide an opportunity for the systematic investigation of non-normal logics, that is, logics that are not closed under the rule:

$$[RN]: \vdash \alpha \Rightarrow \vdash \Box\alpha \tag{6.3.1}$$

Definition 6.3.1. If $\mathfrak{F} = (\mathcal{U}, \mathcal{H})$ is a hyperframe then $\mathfrak{F}$ is *normal* if $\forall x \in \mathcal{U}, \emptyset \notin \mathcal{H}(x)$. A model is normal if it is based on a normal hyperframe.

This is significant from a philosophical perspective for two reasons: First, non-normal logics have deontic motivations insofar as we would like to not have an infinite number of obligations. A logic is normal when $\Box\alpha$ is a theorem whenever α is a theorem. Thus, if $\Box$ represents 'it is obligatory that', then in any deontic logic with an infinite number of theorems there is an infinite number of obligations.[3] Second, if we read $\Box$ as a necessity operator, then the existence of determined non-normal modal logics marks a conceptual divergence between logical validity and its classical, Aristotelian account. According to the classical account, an argument is valid when it is necessary that if the premises are true then the conclusion is true. But there are non-normal logics in which there are logically valid conditionals whose $\Box$ formulae are not theorems. This raises the philosophical question of how theoremhood in such systems should be understood, or alternatively, the question of what the $\Box$ operator represents.

Theorem 6.3.1. *Let* $\mathbf{L}$ *be a logic which is closed under* $[RR], [RN], [US], [PL]$*, and* $[MP]$*. Then the canonical hyperframe for* $\mathbf{L}$ *is normal.*

Proof. Let $x \in \mathcal{U}_{\mathbf{L}}$, and suppose that $\emptyset \in \mathcal{H}_{\mathbf{L}}(x)$. Then by Proposition 6.2.1, $T\mathcal{H}_{\mathbf{L}}(x) = \emptyset$. But $\mathcal{H}_{\mathbf{L}}(x) = T(\{|\gamma| : \Box\gamma \in x\})$; therefore $T\mathcal{H}_{\mathbf{L}}(x) = \{|\gamma| : \Box\gamma \in x\} = \emptyset$ (Proposition 6.2.2). I.e., $\forall\gamma, \Box\gamma \notin x$. But $\vdash \top$, and so by $[RN]$, $\vdash \Box\top$, in which case $\Box\top \in x$, an absurdity. $\Box$

6.3.1 $(n+1)$-ary Relational Frames and n-Bounded Hyperframes

An $(n+1)$*-ary relational frame* $(n \geq 1)$ is a pair $(\mathcal{U}, \mathcal{R})$ where $\mathcal{U}$ is a non-empty set and $\mathcal{R} \subseteq \mathcal{U}^{n+1}$. If $\mathcal{V}$ is a function from Nat to $\wp(\mathcal{U})$ then the triple $(\mathcal{U}, \mathcal{R}, \mathcal{V})$ is an $(n+1)$*-ary relational model* based on the frame $(\mathcal{U}, \mathcal{R})$. Truth and validity relative to $(n+1)$-ary relational frames and models are as defined for hyperframes and models, with the exception that truth at a point x for $\Box$ formulae is defined:

$$\models^{\mathfrak{M}}_{x} \Box\alpha \text{ iff } \forall\langle y_1, ..., y_n\rangle \in \mathcal{R}(x), \exists i \in [n] : \models^{\mathfrak{M}}_{y_i} \alpha,$$

where for any positive integer n, $[n]$ denotes $\{1, 2, ..., n\}$, and if $\mathcal{R} \subseteq \mathcal{U}^{n+1}$, then $\mathcal{R}(x) = \{\langle y_1, ..., y_n\rangle : \langle x, y_1, ..., y_n\rangle \in \mathcal{R}\}$. (See Figure 6.3.1.)

Definition 6.3.2. A hyperframe $\mathfrak{F} = (\mathcal{U}, \mathcal{H})$ is *n-bounded*, for $n \geq 1$, if $\forall x \in \mathcal{U}, \forall\mathcal{E} \in \mathcal{H}(x), |\mathcal{E}| \leq n$. A model is n-bounded if it is based on an n-bounded hyperframe.

[3] It is important to note, however, that the absence of normality does not guarantee the absence of an infinite number of obligations. Although the absence of normality is necessary, it is not sufficient for this end. What we really need is the rule:

$$\vdash \alpha \;\Rightarrow\; \vdash \neg\Box\alpha. \quad (6.3.2)$$

Jennings has suggested that what we really want is a variety of connexivist implication, which is a restriction of classical logic to contingencies.

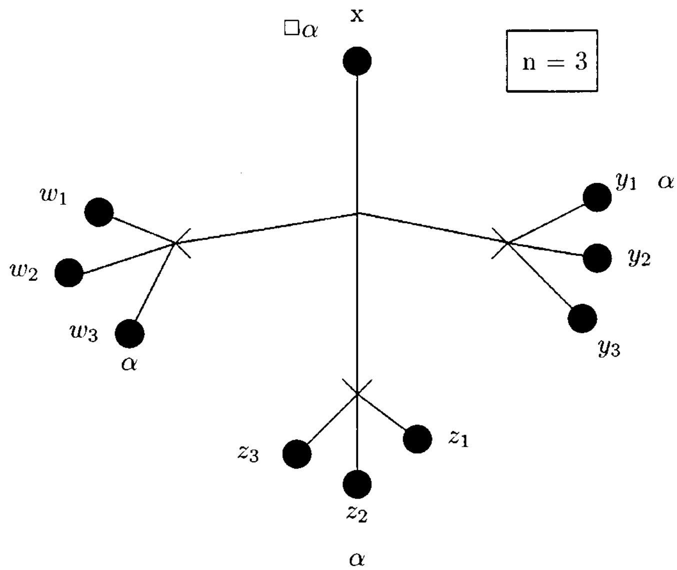

Figure 6.3.1: The truth condition for □ on $(n+1)$-ary relational frames.

Definition 6.3.3. Let $\mathfrak{F} = (\mathcal{U}, \mathcal{H})$ be an n-bounded normal hyperframe, and let the relation $\mathcal{R} \subseteq \mathcal{U}^{n+1}$ be defined pointwise:

$$\mathcal{R}(x) := \{\langle y_1, ..., y_m, \underbrace{y_i, ..., y_i}_{n-m \ times} \rangle :$$

$$\{y_1, ..., y_m\} \in \mathcal{H}(x) \ \& \ i \in [m]\}$$

Then the *$(n+1)$-ary relational transformation of* $\mathfrak{F}$ is the $(n+1)$-ary relational frame $\mathfrak{F}^* = (\mathcal{U}, \mathcal{R})$.

It is easy to see that for any model $\mathfrak{M}$ on an n-bounded normal hyperframe there is an equivalent $(n+1)$-ary relational model. That is:

Theorem 6.3.2. *For every normal n-bounded hyperframe $\mathfrak{F} = (\mathcal{U}, \mathcal{H})$, every model $\mathfrak{M}^* = (\mathcal{U}, \mathcal{R}, \mathcal{V})$ on the $(n+1)$-ary relational transformation $\mathfrak{F}^* = (\mathcal{U}, \mathcal{R})$ of $\mathfrak{F}$ is pointwise equivalent to the hypermodel $\mathfrak{M} = (\mathcal{U}, \mathcal{H}, \mathcal{V})$, that is,*

$$\forall x \in \mathcal{U}, \forall \alpha, \ \models^{\mathfrak{M}}_{x} \alpha \Leftrightarrow \models^{\mathfrak{M}^*}_{x} \alpha$$

Proof. The (omitted) proof is by induction on the complexity of α. □

6.4 Weakly Aggregative Modal Logic

A consequence of the theory of n-bounded normal hyperframes is a simple proof of the completeness of the system $\mathbf{K_n}$ with respect to the class of all $(n+1)$-ary relational frames. The system $\mathbf{K_n}$ $(n \geq 1)$ is defined:

$$[RR] : \vdash \alpha \rightarrow \beta \Rightarrow \vdash \Box\alpha \rightarrow \Box\beta \tag{6.4.1}$$

$$[RN] : \vdash \alpha \Rightarrow \vdash \Box\alpha \tag{6.4.2}$$

$$[PL] : \vdash_{PL} \alpha \Rightarrow \vdash \alpha \tag{6.4.3}$$

$$[MP] : \vdash \alpha \rightarrow \beta \;\&\; \vdash \alpha \Rightarrow \vdash \beta \tag{6.4.4}$$

$$[US] : \vdash \alpha \;\&\; \beta \text{ is a substitution instance of } \alpha \Rightarrow \vdash \beta \tag{6.4.5}$$

$$[K_n] : \vdash \Box\alpha_1 \wedge \ldots \wedge \Box\alpha_{n+1} \rightarrow \Box \bigvee_{1 \leq i < j \leq n+1} \alpha_i \wedge \alpha_j \tag{6.4.6}$$

$\mathbf{K_1}$ is just the Kripke system $\mathbf{K}$. For each $n > 1$, $\mathbf{K_n}$ is *weakly aggregative* because it replaces the strong aggregation principle $[K](= [K_1])$ with the weaker $[K_n]$.

It would appear that the completeness proof herein is a simplification of the other proofs in the literature, found in [1] and [5], as both of these rely heavily on the colouring theory of hypergraphs, reference to which is omitted in the present proof. In particular, the previous proofs exploit colouring compactness, the claim that if H is a hypergraph each of whose edges is finitely long, then H is k-colourable iff every finite subgraph of H is k-colourable. In contrast, the crucial lemma used here, in addition to Theorem 6.3.1, is that the canonical hyperframe for any logic that includes $[K_n]$ $(n \geq 1)$ is n-bounded.

For convenience, we introduce the convention that if $t_1, t_2, \ldots, t_{n+1}$ are sets $(n \geq 1)$, then $\frac{2}{n+1}(t_i)_{i \in [n+1]}$ denotes $\bigcup_{1 \leq i < j \leq n+1} t_i \cap t_j$, and if $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ are sentences then $\frac{2}{n+1}(\alpha_i)_{i \in [n+1]}$ represents $\bigvee_{1 \leq i < j \leq n+1} \alpha_i \wedge \alpha_j$.

Lemma 6.4.1. *$\forall n \geq 1$, if $\mathcal{H}$ is a hypergraph such that $\forall \mathcal{E} \in \mathcal{H}, |\mathcal{E}| \leq n$ then whenever $t_1, t_2, \ldots, t_{n+1}$ are transversals for $\mathcal{H}$, so is $\frac{2}{n+1}(t_i)_{i \in [n+1]}$.*

Proof. Suppose that $\forall \mathcal{E} \in \mathcal{H}$, $|\mathcal{E}| \leq n$ and that $\forall i \in [n+1], t_i$ is a transversal for $\mathcal{H}$. Suppose further that $\frac{2}{n+1}(t_i)_{i \in [n+1]} \cap \mathcal{E} = \emptyset$, for some $\mathcal{E} \in \mathcal{H}$. By a pigeonhole argument, $\exists i, j \in [n+1] (i \neq j)$ such that $t_i \cap t_j \cap \mathcal{E} \neq \emptyset$, which is absurd since $t_i \cap t_j \subseteq \frac{2}{n+1}(t_i)_{i \in [n+1]}$. □

Lemma 6.4.2. *Let $\mathcal{H}$ be a simple hypergraph such that $\exists n \geq 1, \exists \mathcal{E} \in \mathcal{H}, |\mathcal{E}| > n$. Then there are $n+1$ transversals for $\mathcal{H}$, $t_1, t_2, ..., t_{n+1}$, such that $\frac{2}{n+1}(t_i)_{i\in[n+1]}$ is not a transversal for $\mathcal{H}$.*

Proof. Let $\mathcal{H}$ be a simple hypergraph and let $n \geq 1$ be an arbitrary integer such that for some $\mathcal{E} \in \mathcal{H}, |\mathcal{E}| > n$. Suppose that $\mathcal{E} = \{x_1, x_2, ..., x_i, ..., x_{n+1}, ...\}$. Then it is possible to construct $n+1$ transversals for $\mathcal{H}$, $t_1, t_2, ..., t_{n+1}$ such that $\forall i \in [n+1]$, $t_i \cap \mathcal{E} = \{x_i\}$; otherwise $\exists \mathcal{E}' \in \mathcal{H}$ such that $\mathcal{E}' \subset \mathcal{E}$, contrary to the assumption that $\mathcal{H}$ is simple. But then $\frac{2}{n+1}(t_i)_{i\in[n+1]} \cap \mathcal{E} = \emptyset$.

□

Theorem 6.4.3. $\forall n \geq 1$, *if* $\mathbf{L}$ *is a modal logic which is closed under [RR], [PL], [US], and modus ponens, then if $[K_n] \in \mathbf{L}$, it follows that the canonical frame for* $\mathbf{L}$ *is n-bounded.*

Proof. Assume that $\mathbf{L}$ is a modal logic which satisfies the antecedent conditions, including that $[K_n] \in \mathbf{L}$ for some $n \geq 1$. We show that $\forall x \in \mathcal{U}_\mathbf{L}$, $\forall \mathcal{E} \in \mathcal{H}_\mathbf{L}(x)$, $|\mathcal{E}| \leq n$. Suppose not. Let $x \in \mathcal{U}_\mathbf{L}$ be such that $\mathcal{E} \in \mathcal{H}_\mathbf{L}(x)$ and $|\mathcal{E}| > n$. From Lemma 6.4.2 it follows that there are $n+1$ transversals for $\mathcal{H}_\mathbf{L}(x)$, $t_1, t_2, ..., t_{n+1}$, such that $\frac{2}{n+1}(t_i)_{i\in[n+1]}$ is not a transversal for $\mathcal{H}_\mathbf{L}(x)$. But since t_i is a transversal for $\mathcal{H}_\mathbf{L}(x)$ $(i \in [n+1])$, $\exists \mathcal{E}_i \subseteq t_i$ such that $\mathcal{E}_i \in T(\mathcal{H}_\mathbf{L}(x))$, i.e., from Proposition 6.2.2, $\mathcal{E}_i \in \{|\gamma| : \Box\gamma \in x\}$. So let $\mathcal{E}_i = |\gamma_i|$ $(i \in [n+1])$, where $\Box\gamma_i \in x$. Since $[K_n] \in \mathbf{L}$, and $\mathbf{L}$ is closed under uniform substitution, $\Box\gamma_1 \wedge ... \wedge \Box\gamma_{n+1} \rightarrow \Box\frac{2}{n+1}(\gamma_i)_{i\in[n+1]} \in x$. Therefore $\Box\frac{2}{n+1}(\gamma_i)_{i\in[n+1]} \in x$, and thus $\models^{\mathfrak{M}_L}_x \Box\frac{2}{n+1}(\gamma_i)_{i\in[n+1]}$ (Theorem 6.2.3), in which case $\|\frac{2}{n+1}(\gamma_i)_{i\in[n+1]}\|^{\mathfrak{M}_\mathbf{L}}$ is a transversal for $\mathcal{H}_\mathbf{L}(x)$. But:

$$\begin{aligned}
\|\frac{2}{n+1}(\gamma_i)_{i\in[n+1]}\|^{\mathfrak{M}_\mathbf{L}} &= \frac{2}{n+1}(\|\gamma_i\|^{\mathfrak{M}_\mathbf{L}})_{i\in[n+1]} \\
&= \frac{2}{n+1}(|\gamma_i|)_{i\in[n+1]} \\
&= \frac{2}{n+1}(\mathcal{E}_i)_{i\in[n+1]}
\end{aligned}$$

And $\frac{2}{n+1}(\mathcal{E}_i)_{i\in[n+1]} \subseteq \frac{2}{n+1}(t_i)_{i\in[n+1]}$. Therefore $\frac{2}{n+1}(\mathcal{E}_i)_{i\in[n+1]}$ is not a transversal for $\mathcal{H}_\mathbf{L}(x)$, which is absurd. □

Since $\forall n \geq 1$, $\mathbf{K_n}$ is sound with respect to the class of all normal n-bounded hyperframes (see Lemma 6.4.1), given Theorems 6.3.1 and 6.4.3 we have:

Corollary 6.4.4. $\forall n \geq 1$, *the modal system* $\mathbf{K_n}$ *is determined by the class of all normal n-bounded hyperframes.*

In closing we have:

Theorem 6.4.5. *$\forall n \geq 1$, the system $\mathbf{K_n}$ is complete with respect to the class of all $(n+1)$-ary relational frames.*

Proof. Assume that α is valid with respect to the class of all $(n+1)$-ary relational frames. Then α is valid on the $(n+1)$-ary relational transformation of the canonical hyperframe $\mathfrak{F}_{\mathbf{K_n}}$ for $\mathbf{K_n}$ (Theorems 6.3.1, 6.4.3). Therefore, by Theorem 6.3.2, α is valid on the canonical model $\mathfrak{M}_{\mathbf{K_n}}$ for $\mathbf{K_n}$, whence $\vdash_{\mathbf{K_n}} \alpha$. □

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Chapter 7

On Imploding: the Logic of (In)Vacuity

D. Nicholson

Abstract

Jennings and Schotch introduced the K_q modal systems ($q \geq 1$) axiomatizable by a weakening of Scott's Rule for K. Semantically, Scott's rule in effect asserts that the set of necessities at a point is closed under classical provability. The weakened rule asserts that the set of necessities at a point is closed under a paraconsistent inference relation that Jennings and Schotch called *q-forcing*. In this paper a corresponding question is raised about the closure conditions on the set of possibilities at a point. This leads to a relevant inference relation, *q-folding*, the dual of q-forcing, which can be defined in terms of the *harmonicity* of families of sets of data. This inference relation is relevant because it preserves informational content of a set, and thereby blocks the irrelevant inference to arbitrary sets whose members cannot all be false at the same time. It is proved that q-folding represents a restriction of the classical provability relation.

7.1 Introduction

It is well-known that Jennings and Schotch's paper 'On Detonating' represents an important contribution to the literature of paraconsistent (and relevance) logic [7]. In it, they describe a non-dialetheic approach to paraconsistent logic that has been called 'weakly aggregative' because *ex falso quodlibet* is blocked as a result of a weakening of $\wedge$-Introduction. This weakening enables a distinction to be made not only between such sets as

$$\{P \wedge \neg P\} \text{ and} \tag{7.1.1}$$

$$\{P, \neg P\} \tag{7.1.2}$$

(only the former is explosive in Jennings and Schotch's system), but also between such sets as

$$\{P, P \rightarrow \neg Q, Q\} \text{ and} \tag{7.1.3}$$

$$\{P \wedge Q, \neg P \wedge Q, \neg Q\}. \tag{7.1.4}$$

One difference between the latter two sets with respect to consistency is that it is easier to partition the first of the two into consistent subsets. One needs to divide the second set into at least three parts to achieve the consistency of each part, whereas a two-partition suffices for the first set. What this means from the point of view of classical inference is that it is easier to reason classically but non-trivially from the first set, modulo an inference relation that pays attention to partitions into classical cells. This idea is formalized in [7] in terms of a measure of the level of the consistency of set. But while the dual issue pertaining to *verum ex quodlibet* is raised by Schotch and Jennings, its details are omitted. The purpose of this paper is to provide those missing details. The general question at issue can be put thus:

> Given a set Σ of data and a formulae α, is it the case that α proves Σ?

The question is abductive in nature, and the answer illustrates that the need to mitigate the explosiveness of sets of formulae is not a uniquely paraconsistent imperative. There is a more general issue of relevance at stake. Besides inconsistency, there is an alternative vehicle for the trivialization of inferential closure, namely, informational vacuity. For if $\emptyset \vdash \Sigma$ and Σ is closed under the rule $[\alpha \vdash \Sigma \Rightarrow \alpha \in \Sigma]$ then Σ is the universal set of formulae. That is to say, classical logic draws no principled distinctions among sentences with respect to a set proved by the empty set, a *vacuous set*, for short. This is a straightforward generalization to a multiple conclusion setting of the principle of implosion for classical logic, *verum ex quodlibet*, that anything implies a tautology.

But there is something to be said for the idea that $\{P \vee \neg P\}$ is more vacuous, is less inferentially useful, contains less information than say, the set $\{P \rightarrow Q, Q \rightarrow R\}$, even though the empty set entails each. In this paper, an inference relation called *q-folding* is developed which preserves distinctions between sets with respect to their informational content, by preserving what is called the *vacuity level* φ of a conclusion set. Intuitively, this can be understood as the level of the absence of informational content of a set. For any set Σ, if $\top$ is in Σ then $\varphi(\Sigma)$ is arbitrarily high. In general, $\varphi(\Sigma)$ is a function of the number of parts into which Σ must be divided so that no part is entailed by the empty set.

If Δ is any set and p is a positive integer, the set of *p-partitions* of Δ is defined:

$$\Pi_p(\Delta) := \{\{c_1, ..., c_p\} \mid \bigcup_{m \in [p]} c_m = \Delta \ \ \& \ c_m \cap c_n = \emptyset \ (1 \leq m \neq n \leq p)\},$$

where if p is any integer the notation '$[p]$' abbreviates '$\{1, 2, ..., p\}$'. Moreover, a set Σ of formulae is *falsifiable* iff $\emptyset \nvdash \Sigma$ iff there is a valuation on which every element of Σ is false.[1] If Σ is a set of formulae, the *vacuity level* of Σ is defined:

$$\varphi(\Sigma) := \begin{cases} min\ q \geq 1 : \exists \pi \in \Pi_q(\Sigma) : \forall c \in \pi, c \text{ is falsifiable} & \text{if this limit exists;} \\ \infty & \text{otherwise.} \end{cases} \tag{7.1.5}$$

Thus for example, $\varphi(\{\top\}) = \infty$, and $\varphi(\{P, \neg P\}) = 2$. Similarly,

$$\varphi(\{P \rightarrow Q, Q \rightarrow R\}) = 2,$$

and

$$\varphi(\{P, \neg P \vee Q, P \rightarrow \neg Q\}) = 3.$$

Modelling the relevant inference relation 'q-folding' on a closure condition for the set of formulae true a point x in a $(q+1)$-ary model for modal logic ($q \geq 1$), it is shown that the class of pairs $\langle \alpha, \Sigma \rangle$ such that α q-folds Σ is finitely axiomatizable. In this respect the article follows suit with [7], where a corresponding result for the dual, paraconsistent, inference relation 'q-forcing', is proved.

[1] No loss of generality results in what follows from alternating between semantic and syntactic representations of falsifiability.

However, q-folding is not relevant simpliciter. For one thing, any inconsistent formula α q-folds any set Δ whatsoever, for every $q \geq 1$. But relevant and paraconsistent systems alike notoriously want for some quality or other: a principle which leads to triviality is excluded, but at the cost either of including another principle with equal, or at least other, burdensome consequences, or of excluding some feature which someone considers essential to any system worthy of the name 'logic'.

A representative sample of such evaluations can be found in [4], where Priest and Routley describe several approaches to paraconsistency and relevance together with alleged inadequacies. For example, the 'connectional' approach to relevance embodied in Parry systems, in which antecedents and consequents of correct inference satisfy a variable sharing requirement, suffers, according to Priest and Routley, from its retention of Disjunctive Syllogism [4] (see also [5]). da Costa's 'positive plus' approach to paraconsistency rejects the negative explosion paradox $\alpha \rightarrow (\neg\alpha \rightarrow \beta)$, but in retaining $\alpha \rightarrow (\beta \rightarrow \alpha)$, sacrifices parts of negation theory, and consequently does not possess a 'true' negation [4]. A similar remonstrance is made against the weakly-adjunctive approach to paraconsistency, a representative of which is described below in Schotch and Jennings' system: *ex falso quodlibet* is blocked only because so-called 'true' conjunction is done away with [4].

But these kinds of evaluations, specifically, ones which seek to undermine the motivation for research based on some particular approach to paraconsistency or relevance, seem ill-conceived. Just because there is not a perfect fit between a formal system and an application does not mean that other applications will not be forthcoming. As Lu puts it, "[i]t is not always possible for us to foresee how a formal theory will find applications" [2]; deciding that an approach is not worth pursuing because it falls short with respect to one (intended) application is accordingly premature. The history of paraconsistency and relevance is arguably young enough that formal research in its domain can be justifiably undertaken with the motivation of being primary research.

In any case, the point in devising the putatively relevant system which is presented below was not to show that there is a 'truly' relevant logic which is the dual of a paraconsistent logic, but rather that there is a paraconsistent logic whose dual has some interesting properties of relevance. To date, no one has undertaken a systematic analysis of the dual of n-forcing. Let the present article be counted as a first.

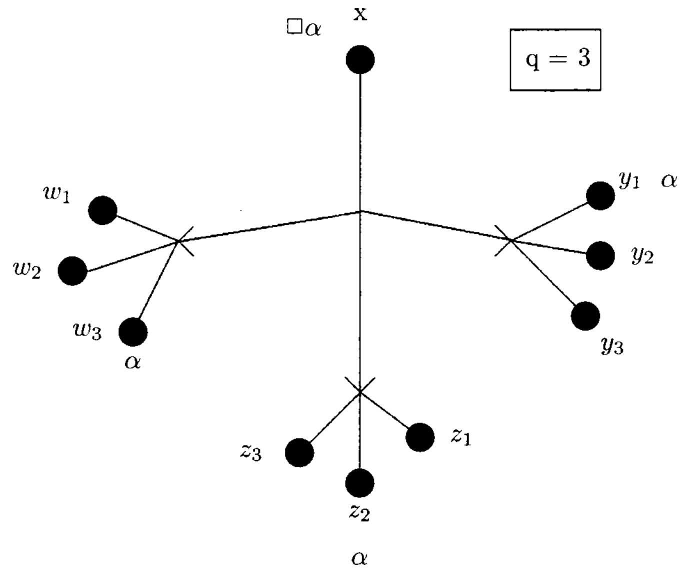

Figure 7.2.1: The truth condition for □ on $(q+1)$-ary relational frames.

7.2 q-Folding

In [6] it is shown how to derive the model theory for the K_q ($q \geq 1$) modal logics from the paraconsistent inference relation 'q-forcing', from which the inference relation is again recoverable. For any positive integer q, the K_q *modal logic* is the logic determined by the class of $(q+1)$-ary relational frames [1][3]. A $(q+1)$*-ary relational frame* for modal logic is a pair $\langle U, R\rangle$ where U is a non-empty set of points, and $R \subseteq U^{q+1}$ is a $(q+1)$-ary relation on U. To obtain a model on a frame, a valuation function V from formulae to subsets of U is defined, standard for Boolean connectives, and for □ given as:

$$x \in V(\Box\alpha) \Leftrightarrow . \ \forall y_1, ..., y_q, Rxy_1, ..., y_q, \Rightarrow \exists i \in [q] \text{ such that } y_i \in V(\alpha). \qquad (7.2.1)$$

(See Figure 7.2.1.) If Σ is a set of formulae and α is a formula then Σ *q-forces* α, $\Sigma[\vdash_q \alpha$, if $\forall\pi \in \Pi_q(\Sigma), \exists c \in \pi$ such that $c \vdash \alpha$. Thus, if $q = 1$ then $[\vdash_q$ is just $\vdash$. If $q > 1$ then $[\vdash_q$ is paraconsistent because it distinguishes between an inconsistent set and a set that contains an inconsistent formula, by preserving what is called the *coherence level* of a set Σ, $\kappa(\Sigma)$. If Σ is a set of formulae, the *coherence level* of Σ is defined:

$$\kappa(\Sigma) := \begin{cases} min\ q \geq 1 : \exists \pi \in \Pi_q(\Sigma) : \forall c \in \pi, c \not\vdash \emptyset & \text{if this limit exists;} \\ \infty & \text{otherwise.} \end{cases} \tag{7.2.2}$$

That is, as $\varphi(\Sigma)$ is the size of the least partition of Σ into falsifiable sets, dually, $\kappa(\Sigma)$ is the size of the least partition of Σ into classically consistent sets. Thus $\kappa(\{P \wedge \neg P\}) = \infty$ and $\kappa(\{P, \neg P\}) = 2$. Note that while $\forall \alpha, q \geq 1, P \wedge \neg P \ [\vdash_q \alpha$, it is false that $\forall \alpha, \{P, \neg P\} \ [\vdash_2 \alpha$. In general, $\forall \Sigma, \alpha$, if $\kappa(\Sigma) = q \geq 1$ then $\Sigma \ [\vdash_q \alpha$ only if $\kappa(\Sigma, \alpha) \leq q$.[2] Whence the paraconsistency of q-forcing.

Now if $\mathfrak{M}$ is a $(q+1)$-ary model, and x is a point in $\mathfrak{M}$, let $X^{\mathfrak{M}}$ be the set $\{\alpha \mid \alpha \text{ is true at } x\}$. Further, if $*$ is a unary connective and Σ is a finite set of formulae, let $*[\Sigma]$ be the set $\{*\phi \mid \phi \in \Sigma\}$. If $*$ is an associative binary connective and $\Sigma = \{\phi_1, \phi_2, ..., \phi_m\}$ is finite then $*[\Sigma] := \phi_1 * \phi_2 * ... * \phi_m$. On a standard binary relational model $\mathfrak{M}$ for kripkean modal logic, which on the K_q scheme is a $(q+1)$-ary model for $q = 1$, if $x \in \mathfrak{M}$ then $X^{\mathfrak{M}}$ is closed under the following rule:

$$\frac{\wedge[\Box[\Sigma]] \in X^{\mathfrak{M}} \ \&\ \Sigma \vdash \alpha}{\Box\alpha \in X^{\mathfrak{M}}} \tag{7.2.3}$$

Dually, where '$\Diamond\alpha$' abbreviates '$\neg\Box\neg\alpha$', the set $X^{\mathfrak{M}}$ is also closed under:

$$\frac{\Diamond\alpha \in X^{\mathfrak{M}} \ \&\ \alpha \vdash \Sigma}{\vee[\Diamond[\Sigma]] \in X^{\mathfrak{M}}} \tag{7.2.4}$$

As discussed in [1], replacing the classical $\vdash$ in Condition 7.2.3 with the paraconsistent q-forcing relation, $[\vdash_q$, yields closure for $X^{\mathfrak{M}}$ in the logic K_q $(q \geq 1)$. That is, for all $q \geq 1$, for any point x in a $(q+1)$-ary model $\mathfrak{M}$, $X^{\mathfrak{M}}$ is closed under:

$$\frac{\wedge[\Box[\Sigma]] \in X^{\mathfrak{M}} \ \&\ \Sigma[\vdash_q \ \alpha}{\Box\alpha \in X^{\mathfrak{M}}} \tag{7.2.5}$$

Here it is shown that the inference relation which can replace the classical $\vdash$ in Condition 7.2.4 to yield closure for $X^{\mathfrak{M}}$ in the K_q logics is both relevant, because of its relationship with the level of vacuity of a set, as well as finitely axiomatizable.

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[2]For any set S and item x, '$S \cup \{x\}$' is abbreviated 'S, x'.

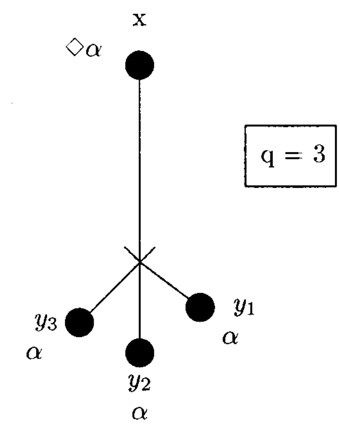

Figure 7.2.2: The truth condition for $\Diamond$ on $(q+1)$-ary relational frames.

From the truth condition for $\Box$ on $(q+1)$-ary models, the truth condition for $\Diamond$ is:

$$x \in V(\Diamond\alpha) \Leftrightarrow \exists y_1, ..., y_q, Rxy_1, ..., y_q \text{ and } \forall m \in [q], y_m \in V(\alpha). \tag{7.2.6}$$

(See Figure 7.2.2.) As a result, $X^{\mathfrak{M}}$, for an arbitrary point x in a $(q+1)$-ary model, is not necessarily closed under Condition 7.2.4 if $q \geq 2$. But by dualizing the definition of $[\vdash_q$, it is possible to formulate an appropriate substitute: $\forall \alpha, \Sigma, q \geq 1$, α *q-folds* Σ, $\alpha \Vdash\!\vdash_q \Sigma$, if $\forall \pi \in \Pi_q(\Sigma), \exists c \in \pi$ such that $\alpha \vdash c$. Then for all $q \geq 1$, for any point x in any $(q+1)$-ary model, $X^{\mathfrak{M}}$ is closed under:

$$\frac{\Diamond\alpha \in X^{\mathfrak{M}} \;\&\; \alpha \Vdash\!\vdash_q \Sigma}{\vee[\Diamond[\Sigma]] \in X^{\mathfrak{M}}} \tag{7.2.7}$$

Proof. Let $\Diamond\alpha$ be in $X^{\mathfrak{M}}$. Then $\exists y_1, ..., y_q$ such that $Rxy_1...y_q$ and $\forall j \in [q], \alpha$ is true at y_j. Now suppose that $\forall \phi \in \Sigma, \Diamond\phi \notin X^{\mathfrak{M}}$. Then $\forall \phi \in \Sigma, \exists j \in [q]$ such that ϕ is false at y_j. Let π be the set $\{c_j \subseteq \Sigma \mid \phi \in c_j \text{ iff } \phi \text{ is false at } y_j\}$. Then $\forall c \in \pi, \alpha \nvdash c$. Therefore $\alpha \not\Vdash\!\vdash_q \Sigma$. So if $\alpha \Vdash\!\vdash_q \Sigma$ then $\exists \phi \in \Sigma$ such that $\Diamond\phi \in X^{\mathfrak{M}}$, in which case $\vee[\Diamond[\Sigma]] \in X^{\mathfrak{M}}$. $\square$

The connection between q-folding and level of vacuity of a set can be made clear by generalizing the dual correspondent to the classical notion of the satisfiability of a set Σ, i.e., the ability to simultaneously falsify each member of Σ. If $q \geq 1$, then Σ is *q-falsifiable* iff $\exists \pi \in \Pi_q(\Sigma) : \forall c \in \pi, c$

is falsifiable iff $\varphi(\Sigma) \leq q$. The idea is that q-falsifiability is to q-folding as falsifiability is to the classical $\vdash$. If Σ is a set of formulae and α is a formula then let $\neg\alpha \vee [\Sigma]$ be the set $\{\neg\alpha \vee \beta \mid \beta \in \Sigma\}$. Then:

Theorem 7.2.1. *$\forall\alpha, \Sigma, q \geq 1, \alpha \Vdash_q \Sigma$ iff $\neg\alpha \vee [\Sigma]$ is not q-falsifiable.*

Proof.

[$\Rightarrow$] Assume that $\neg\alpha \vee [\Sigma]$ is q-falsifiable ($q \geq 1$). Then there is a partition $\pi \in \Pi_q(\neg\alpha \vee [\Sigma])$ such that $\forall c \in \pi, c$ is falsifiable. Therefore for each $c \in \pi$ there is a valuation which assigns α to true and every element of Σ disjoined with $\neg\alpha$ in c to false. Whence there is a partition $\pi' \in \Pi_q(\Sigma)$ such that $\forall c \in \pi', \alpha \not\vdash c$.

[$\Leftarrow$] If $\alpha \not\Vdash_q \Sigma$ then there is a partition $\pi \in \Pi_q(\Sigma)$ such that $\forall c \in \pi$, there is a valuation which assigns α to true and every element of c to false. By disjoining $\neg\alpha$ with each element of c for each $c \in \pi$ while preserving distinctions between the c's, we obtain a q-partition of $\neg\alpha \vee [\Sigma]$, each of whose elements is falsifiable. □

Furthermore, q-folding preserves the vacuity of a set. That is:

Theorem 7.2.2. *$\forall\Sigma, \alpha, q \geq 1$, if $\varphi(\Sigma) = q$ and $\alpha \Vdash_q \Sigma$ then $\varphi(\Sigma, \alpha) \leq q$.*

Proof. Let $\varphi(\Sigma) = q$ and suppose that $\alpha \Vdash_q \Sigma$. Then $\forall\pi \in \Pi_q(\Sigma), \exists c \in \pi$ such that $\alpha \vdash c$. $\therefore \exists\pi \in \Pi_q(\Sigma) : \forall c \in \pi, \emptyset \not\vdash c$ and $\exists c \in \pi$ such that $\alpha \vdash c$. $\therefore \emptyset \not\vdash c \cup \{\alpha\}$. Whence $\varphi(\Sigma, \alpha) \leq q$. □

Consequently, q-folding is relevant in the sense that only formulae which preserve level of vacuity q-fold a vacuous set. E.g., notwithstanding the classical, truth functional basis of the relation, the irrelevant inference from an arbitrary formula α to the set $\{P, \neg P\}$ is blocked. In this way, the attenuation of informational content of a set may be kept in check while still permitting classical inference.

7.2.1 Compactness

The authors of [1] exploit the connection between coherence level and the colouring theory of hypergraphs to establish the compactness of κ and $[\vdash_q$. A set A is a *hypergraph* if A is a non-empty family of non-empty sets, a_m, called *edges*. If $\bigcup_{m\in|A|} a_m$ is finite then A is a *finite* hypergraph. If A and B are hypergraphs and $A \subseteq B$ then A is a *subgraph* of B; A is a *finite subgraph* of B,

$A \subseteq_{fin} B$,[3] if $A \subseteq B$ and A is finite. If Σ is a set then $H(\Sigma)$ is the set $\{A \mid \Sigma \supseteq \bigcup_{m \in |A|} a_m\}$ of *hypergraphs on* Σ. A *q-colouring* ($q \geq 1$) of a hypergraph A on Σ is a function $f : \Sigma \to [q]$ such that $\forall a \in A, \exists x, y \in a$ such that $f(x) \neq f(y)$. The *chromatic number* of a hypergraph A, $\chi(A)$, is defined:

$$\chi(A) := \begin{cases} min\ q \geq 1 : \text{there is a } q\text{-colouring of } A & \text{if this limit exists;} \\ \infty & \text{otherwise.} \end{cases} \tag{7.2.8}$$

Thus, if $\chi(A) > q$ for any hypergraph A, then $\forall \pi \in \Pi_q(\bigcup_{m \in |A|} a_m)$, $\exists a \in A, c \in \pi : a \subseteq c$. If $\chi(A) > q$ then we say that A is *q-uncolourable*; otherwise A is *q-colourable*.

Now as φ is dual to κ, we may use a strategy similar to that adopted in [1] to demonstrate that q-folding and q-falsifiability are compact. In each case the proof relies upon the compactness of q-colourability for hypergraphs with exclusively finite edges:

Theorem 7.2.3. *$\forall A, q \geq 1$, if each edge of A is finite then $\chi(A) > q$ iff $\exists B \subseteq_{fin} A$ such that $\chi(B) > q$.*

Proof. This can be established using the compactness of propositional logic. □

Theorem 7.2.4. *$\forall \Sigma, q \geq 1, \Sigma$ is q-falsifiable iff every finite subset of Σ is q-falsifiable.*

Proof. The direction from left to right is trivial. For the converse, following [1], we define G_Σ as the canonical hypergraph on $\Sigma_{\neq \emptyset}$ whose edges comprise the set $\{e \subseteq_{fin} \Sigma \mid \emptyset \vdash e\}$. Then $\varphi(\Sigma) \leq q$ iff $\chi(G_\Sigma) \leq q$. So assume that $\varphi(\Sigma) > q$. Then $\chi(G_\Sigma) > q$. Whence $\exists B \subseteq_{fin} G_\Sigma$ such that $\chi(B) > q$. Let Γ be $\bigcup_{m \in |B|} b_m$. Then $\Gamma \subseteq_{fin} \Sigma$ and $\varphi(\Gamma) > q$. □

Theorem 7.2.5. *$\forall \alpha, \Sigma, q \geq 1, \alpha \Vdash_q \Sigma$ iff $\exists \Gamma \subseteq_{fin} \Sigma$ such that $\alpha \Vdash_q \Gamma$.*

Proof. The direction from right to left is trivial given the right invariance of $\vdash$ under supersets. For the converse, assume that $\alpha \Vdash_q \Sigma$. Then $\forall \pi \in \Pi_q(\Sigma), \exists c \in \pi$ such that $\alpha \vdash c$. So $\forall \pi \in \Pi_q(\Sigma), \exists c \in \pi, d \subseteq_{fin} c$ such that $\alpha \vdash d$. Let A be the set $\{d \subseteq_{fin} c \mid c \in \pi \in \Pi_q(\Sigma)\ \&\ \alpha \vdash d\}$. Then $\chi(A) > q$ and by colouring compactness (Theorem 7.2.3), $\exists B \subseteq_{fin} A$ such that $\chi(B) > q$ where $\forall b \in B, \alpha \vdash b$. Let Γ be the set $\bigcup_{m \in |B|} b_m$. Then $\forall \pi \in \Pi_q(\Gamma), \exists c \in \pi$ such that $\alpha \vdash c$. Whence $\alpha \Vdash_q \Gamma$, where Γ is finite. □

[3] The notation '$\subseteq_{fin}$' is also used to indicate that one set is a finite subset of another set.

The proof that q-folding is compact (Theorem 7.2.5) illustrates that q-folding may be given the following finitary definition:

$$\alpha \Vdash_q \Sigma \Leftrightarrow \exists A \in H(\Sigma) \text{ such that } A \text{ is finite, } \chi(A) > q, \text{ and } \forall a \in A, \alpha \vdash a. \tag{7.2.9}$$

For this reason we may assume in what follows, without loss of generality, that hypergraphs are finite unless otherwise specified. This finitary definition of q-folding is useful moreover because using the notion of a *tranverse hypergraph* it permits a dual definition of q-folding in terms of the *harmonic number* of a hypergraph. We can then exploit this dual definition in a finite axiomatization of a restriction of $\vdash$ which represents q-folding.

7.3 Harmonic Number

If A is a hypergraph, an *intersector* for A is a set b such that $\forall a \in A, b \cap a \neq \emptyset$. A *minimal intersector* for A is an intersector which is not a proper superset of any intersector. The *transverse hypergraph* of A, $T(A)$ is the set of all minimal intersectors for A. As theorems we have $\forall A, T(T(A)) \subseteq A$ [3], and for any hypergraph A, $\forall a \in A, \exists b \in T(T(A))$ such that $b \subseteq a$. Whence $\forall A, \chi(A) > q$ iff $\chi(T(T(A)) > q$. There is a straightforward sense in which a hypergraph A and its tranverse $T(A)$ are logical duals. This is illustrated by formulating hypergraphs as sentences. Let A be a hypergraph with $\bigcup_{i \in |A|} a_i$ a collection of sentences. Define the $\vee$-*formulation* of A to be:

$$F^{\vee}(A) := \bigvee_{m \in |A|} \bigwedge_{j \in |a_m|} \alpha_j, \tag{7.3.1}$$

and the $\wedge$-*formulation* of A:

$$F^{\wedge}(A) := \bigwedge_{m \in |A|} \bigvee_{j \in |a_m|} \alpha_j. \tag{7.3.2}$$

Then:

Theorem 7.3.1. *$\forall A, F^{\vee}(A) \;=\!\mid\models\; F^{\wedge}(T(A))$, and $F^{\wedge}(A) \;=\!\mid\models\; F^{\vee}(T(A))$* [3].

Consequently, because $F^{\vee}(A)$ and $F^{\wedge}(A)$ are logically dual, so too are $F^{\vee}(A)$ and $F^{\vee}(T(A))$, as well as $F^\wedge(A)$ and $F^\wedge(T(A))$. It is in this sense that A and $T(A)$ are logically dual.

Because of Theorem 7.3.1 and Condition 7.2.9, q-folding may be defined in terms of the $\vee$-formulation of the transverse hypergraph of a q-uncolourable hypergraph A. This suggests the following question: *Is there a characterization of q-folding in terms of the $\vee$-formulation of a hypergraph which precludes reference to the chromatic number of a transverse hypergraph?* The answer is 'yes' because there is a dual characterization of chromatic number in terms of what is called the *harmonic number* of a hypergraph. If A is a hypergraph the *harmonic number* of A, $\eta(A)$ is defined:

$$\eta(A) := \begin{cases} min\ q \geq 1 : \exists B \in \binom{A}{q} : \bigcap_{m \in [q]} b_m \in B = \emptyset & \text{if this limit exists;} \\ \infty & \text{otherwise,} \end{cases} \tag{7.3.3}$$

where for any set A and positive integer q, $\binom{A}{q}$ denotes the set of all q-membered subsets $\{b_1, b_2, ..., b_q\}$ of A. If $\eta(A) > q$ then we say that A is *q-harmonic.* From the logical duality of hypergraphs and their transverse hypergraphs it follows that harmonic number and chromatic number are logically dual given the following theorem:

Theorem 7.3.2. $\forall A, q \geq 1, \eta(A) > q \Leftrightarrow \chi(T(A)) > q$ [3].

As a result we have:

Theorem 7.3.3. $\forall \Sigma, \alpha, q \geq 1$, *the following conditions are equivalent:*

1. $\alpha \Vdash_q \Sigma$,
2. $\exists A \in H(\Sigma), \chi(A) > q$ *and* $\forall a \in A, \alpha \vdash a$,
3. $\exists A \in H(\Sigma), \chi(A) > q$, *and* $\alpha \vdash F^\wedge(A)$, *and*
4. $\exists A \in H(\Sigma), \eta(A) > q$ *and* $\alpha \vdash F^\vee(A)$.

Proof.

[(1) $\Rightarrow$ (2)] Assume that $\forall \pi \in \Pi_q(\Sigma), \exists c \in \pi : \alpha \vdash c$. Let $A = \{c \in \pi \mid \pi \in \Pi_q(\Sigma)\ \&\ \alpha \vdash c\}$. Then $\chi(A) > q$ and $\forall a \in A, \alpha \vdash a$.

[(2) $\Rightarrow$ (3)] Let $A \in H(\Sigma)$ be such that $\chi(A) > q$ and $\forall a \in A, \alpha \vdash a$. Let $B = \{b \subseteq_{fin} a \mid a \in A\ \&\ \alpha \vdash b\}$. Then by colouring compactness (Theorem 7.2.3), $\exists C \subseteq_{fin} B$ such that $\chi(C) > q$ and $\alpha \vdash F^\wedge(C)$.

[(3) $\Rightarrow$ (4)]. Let $A \in H(\Sigma)$ be such that $\chi(A) > q$ and $\alpha \vdash F^{\wedge}(A)$. From Theorem 7.3.2 it follows that $\eta(T(A)) > q$, and from Theorem 7.3.1 it follows that $\alpha \vdash F^{\vee}(T(A))$.

[(4) $\Rightarrow$ (1)] Assume that $A \in H(\Sigma), \eta(A) > q$ and $\alpha \vdash F^{\vee}(A)$. Then $\alpha \vdash F^{\wedge}(T(A))$(Theorem 7.3.1), so $\forall b \in T(A), \alpha \vdash b$. But $\chi(T(A)) > q$ (Theorem 7.3.2). So $\forall \pi \in \Pi_q(\Sigma), \exists c \in \pi, b \in T(A)$ such that $b \subseteq c$, and therefore $\alpha \vdash c$.

□

It follows from Theorem 7.3.3 that $\vee$-formulating q-harmonic hypergraphs on a set Σ preserves $\varphi(\Sigma)$:

Theorem 7.3.4. *$\forall \Sigma, q \geq 1,\ \varphi(\Sigma) \leq q, B \in H(\Sigma)$ and $\eta(B) > q \Rightarrow \varphi(\Sigma, F^{\vee}(B)) \leq q$.*

Proof. Let $\varphi(\Sigma) = r \leq q$ and $\eta(B) > q$. Then $\eta(B) > r$. But $F^{\vee}(B) \vdash F^{\vee}(B)$. Therefore, we have $\varphi(\Sigma, F^{\vee}(B)) \leq r \leq q$, from Theorems 7.2.2 and 7.3.3. □

Theorem 7.3.4 plays an important role in finitely axiomatizing $\Vdash_q$. Its significance is this: although the closure of a set Σ under $\vee$-Introduction does not preserve level of vacuity (recall that $\varphi(\{p \vee \neg p\}) = \infty$), closing Σ under $\vee$-formulations of q-harmonic hypergraphs does. But for any set Σ, if $q \geq 1$ then $\alpha \Vdash_q \Sigma$ iff there is a q-harmonic hypergraph B on Σ such that $\alpha \vdash F^{\vee}(B)$ (Theorem 7.3.3). To represent $\Vdash_q$ as a restriction of classical implication, it is therefore sufficient to finitely axiomatize the $\vee$-formulations of q-harmonic hypergraphs without using $\vee$-Introduction. As a replacement for $\vee$-Introduction, the truth function $F^{\vee}\binom{\{\alpha_i\}}{q}$ is adopted for a $q+1$-membered set $\{\alpha_1, ..., \alpha_{q+1}\}$ of sentences,[4] which taken together with singular implications excluding $\vee$-Introduction, is functionally complete with respect to the $\vee$-formulation of all q-harmonic hypergraphs. Some intermediate results are required prior to a demonstration of this claim.

Let $A = \{A_1, A_2, ..., A_r\}$ be a set of r hypergraphs ($r \geq 1$), and let S be a set. Then S *n-covers* A ($r \geq n \geq 1$) if $\exists \{A_1, ..., A_n\} \subseteq A$ such that $\forall i$ ($1 \leq i \leq n$), $\exists a \in A_i : S \supseteq a$. In words: the set S n-covers A iff S contains an edge from each of n elements of A; S *minimally n-covers A* if S n-covers A, and no proper subset of S n-covers A. We write '$\frac{n}{r}(A_1, A_2, ..., A_r)$' for the set of minimal n-covers of A when $|A| = r$.

[4] If $\{x_1, ..., x_r\}$ is any r-membered set ($r \geq 1$), the notation '$\{x_i\}$' ($i \in [r]$) is sometimes used as an abbreviation for '$\{x_1, ..., x_r\}$', where context makes clear that '$\{x_i\}$' does not indicate a singleton set.

Theorem 7.3.5. *Let $A_1, ..., A_{q+1}$ be hypergraphs $(q \geq 1)$. Then:*

$$F^{\vee}\binom{\{F^{\vee}(A_i)\}}{q} \;\dashv\models\; F^{\vee}\frac{q}{q+1}(A_1, ..., A_{q+1}).$$

Theorem 7.3.6. *If $B_1, ..., B_{q+1}$ are hypergraphs and for each i $(1 \leq i \leq q+1)$, $\eta(B_i) > q$, then $\eta(\frac{q}{q+1}(B_1, ..., B_{q+1})) > q$.*

Proof. Let $A \in \binom{\frac{q}{q+1}(B_1, ..., B_{q+1})}{q}$. By a pigeonhole argument, there is some hypergraph B_i $(1 \leq i \leq q+1)$ such that $\forall a \in A, \exists b \in B_i$ such that $b \subseteq a$. But $\eta(B_i) > q$. $\therefore \bigcap_{j \in [q]}\{b_j \in B_i \mid \exists a \in A : b_j \subseteq a\} \neq \emptyset$. $\therefore \bigcap_i^q a_i \in A \neq \emptyset$. Whence $\eta(\frac{q}{q+1}(B_1, ..., B_{q+1})) > q$. □

7.4 The Rules

The inference relation q-folding is finitely axiomatizable. To prove this, a restriction of the classical $\vdash$, '$\Vdash_q$' (to be read 'q-proves'), is introduced, and the representation theorem for q-folding is proved, namely, $\forall \alpha, \Sigma, q \geq 1$, $\alpha \mathrel{|\!|\!\vdash_q} \Sigma$ iff $\alpha \Vdash_q \Sigma$.

The following structural rules are adopted:

[Pres $\vdash$]	$\alpha \vdash \beta \Rightarrow \alpha \Vdash_q \beta$
[Ref]	$\alpha \in \Sigma \Rightarrow \alpha \Vdash_q \Sigma$
[Mon(i)]	$\dfrac{\alpha \Vdash_q \Sigma \quad \Delta \supseteq \Sigma \quad \varphi(\Delta) \leq q}{\alpha \Vdash_q \Delta}$
[Mon(ii)]	$\dfrac{\alpha \Vdash_q \Sigma \quad \beta \vdash \alpha}{\beta \Vdash_q \Sigma}$
[Tran]	$\dfrac{\alpha \Vdash_q \Sigma, \beta \quad \beta \Vdash_q \Sigma}{\alpha \Vdash_q \Sigma}$

The following are the rules of inference:

$[\Vdash_q \wedge]$ $$\frac{\beta \Vdash_q \Sigma, \alpha_1,\ \beta \Vdash_q \Sigma, \alpha_2, \ldots, \beta \Vdash_q \Sigma, \alpha_{q+1}}{\beta \Vdash_q \Sigma, F^{\wedge}\binom{\{\alpha_i\}}{q}}$$

$[\wedge \Vdash_q]$ $$\frac{\alpha \Vdash_q \Sigma}{\alpha \wedge \beta \Vdash_q \Sigma}$$

$[\Vdash_q \vee]$ $$\frac{\beta \Vdash_q \Sigma, \alpha}{\beta \Vdash_q \Sigma, \alpha \vee \delta}$$

$[\vee \Vdash_q]$ $$\frac{\alpha_1 \Vdash_q \Sigma,\ \alpha_2 \Vdash_q \Sigma, \ldots, \alpha_{q+1} \Vdash_q \Sigma}{F^{\vee}\binom{\{\alpha_i\}}{q} \Vdash_q \Sigma}$$

It follows as a consequence of the preservational profile of $\Vdash_q$ that $\Vdash_q$ is, to some extant, relevant. Unlike in classical logic, where a vacuous set Σ is proved by any sentence, if $\varphi(\Sigma) > 1$ then whether or not $\beta \Vdash_q \Sigma$ depends on two factors: (a) whether or not $\beta \vdash \Sigma$, and (b) whether or not $\varphi(\Sigma, \beta) \leq q$ if $\varphi(\Sigma) \leq q$. To prove this, it suffices to show that $\Vdash_q$ is φ-sound—that for all $q \geq 1$, if $\alpha \Vdash_q \Sigma$ then $\varphi(\Sigma, \alpha) \leq \varphi(\Sigma) \leq q$. And since q-folding is φ-sound (Theorem 7.2.2), for this it is enough to show that for all $q \geq 1$, $\alpha \Vdash_q \Sigma$ only if $\alpha \Vvdash_q \Sigma$.

Theorem 7.4.1. $\forall \Sigma, \alpha, q \geq 1, \alpha \Vdash_q \Sigma \Rightarrow \alpha \Vvdash_q \Sigma$.

Proof. We begin with the structural rules. Note that q is intended to float at the level of Σ. I.e., if $\alpha \Vdash_q \Sigma$ then $\varphi(\Sigma) = q$.

[Pres$\vdash$]. If $\alpha \vdash \beta$ then $\alpha \vdash F^{\vee}\{\{\beta\}\}$. But $\eta(\{\{\beta\}\} > q$ for all $q \geq 1$. Whence $\alpha \Vvdash_q \beta$.

[Ref]. $\forall \alpha, \alpha \vdash F^{\vee}\{\{\alpha\}\}$. But $\{\{\alpha\}\} \in H(\Sigma)$ if $\alpha \in \Sigma$. $\therefore \alpha \Vvdash_q \Sigma$ if $\alpha \in \Sigma$.

[Mon(i)]. Assume that $\alpha \Vvdash_q \Sigma$, that $\Delta \supseteq \Sigma$, and that $\varphi(\Delta) \leq q$. Then $\alpha \Vvdash_q \Delta$ because $H(\Delta) \supseteq H(\Sigma)$.

[Mon(ii)]. If $\beta \vdash \alpha$ and $\alpha \vdash F^{\vee}(A)$ where $A \in H(\Sigma)$ and $\eta(A) > q$, then $\beta \vdash F^{\vee}(A)$ where $A \in H(\Sigma)$ and $\eta(A) > q$. $\therefore \beta \Vvdash_q \Sigma$.

[Tran]. Assume that $\alpha \Vvdash_q \Sigma, \beta$ and that $\beta \Vvdash_q \Sigma$. Then from Theorem 7.3.3 it follows that $\exists A \in H(\Sigma, \beta)$ where $\chi(A) > q$, and such that $\alpha \vdash F^{\wedge}(A)$, and $\exists B \in H(\Sigma)$ where $\chi(B) > q$, and where $\beta \vdash F^{\wedge}(B)$. Using A and B we construct a hypergraph $D \in H(\Sigma)$ where $\chi(D) > q$ and

$\alpha \vdash F^{\wedge}(D)$: For each edge $a \in A$ such that $\beta \in a$, replace a with $(a - \{\beta\}) \cup b$, for each $b \in B$. Then:

1. $\alpha \vdash F^{\wedge}(D)$. Suppose not. Let α be true on a valuation on which $F^{\wedge}(D)$ is false. Then all elements of some $d \in D$ are false. $\therefore d \notin A$. So for some $a \in A, b \in B, d = (a - \{\beta\}) \cup b$. $\therefore \beta$ is true, else $\alpha \nvdash F^{\wedge}(A)$. $\therefore F^{\wedge}(B)$ is true, which is impossible if all elements of $b \in B$ are false.

2. $\chi(D) > q$. Suppose not. Let $\pi \in \Pi_q(\Sigma)$ be such that $\forall d \in D, c \in \pi, d \nsubseteq c$. But for some $c \in \pi, b \in B, b \subseteq c$, and adding β to c produces a q-partition of Σ, β. So $\exists a \in A$ such that $a \subseteq c \cup \{\beta\}$, where $\beta \in a$. $\therefore (a - \{\beta\}) \cup b \subseteq c$, which is absurd.

And now for the rules of inference:

$[\Vdash_q \wedge]$. For each i $(1 \leq i \leq q+1)$, let $B_i \in H(\Sigma, \alpha_i)$ be such that $\eta(B_i) > q$ and $\beta \vdash F^{\vee}(B_i)$. Let B_i' be the result of replacing each occurrence of α_i in B_i with $F^{\wedge}(\binom{\{\alpha_i\}}{q})$. Let δ be $F^{\vee}(\binom{\{F^{\vee}(B_i')\}}{q})$. Then $\delta \dashv\models F^{\vee}\frac{q}{q+1}(B_1', ..., B_{q+1}')$ (from Theorem 7.3.5). So given Theorem 7.3.6, it suffices to show that $\beta \models \delta$. Suppose not. Let β be true on a valuation and δ false. If δ is false then for some pair i, j $(1 \leq i < j \leq q+1), F^{\vee}(B_i')$ and $F^{\vee}(B_j')$ are false. So an intersector for each has all of its elements false. But β is true and $\beta \vdash F^{\vee}(B_i)$ and $\beta \vdash F^{\vee}(B_j)$. So $F^{\wedge}(\binom{\{\alpha_i\}}{q})$ is false. So for some q-tuple $\alpha_1, ..., \alpha_q$, each element is false. $\therefore$ either $F^{\vee}(B_i)$ or $F^{\vee}(B_j)$ is false, which is absurd. $\therefore \beta \models \delta$.

$[\wedge \Vdash_q]$. Assume that $\alpha \Vvdash_q \Sigma$. Then $\exists A \in H(\Sigma)$ such that $\chi(A) > q$ and $\forall a \in A$, $\alpha \vdash a$. Whence by the left downward monotonicity of $\vdash$ for single formulae, $\alpha \wedge \beta \vdash a, \forall a \in A$.

$[\Vdash_q \vee]$. Assume that $\exists A \in H(\Sigma, \alpha)$ where $\chi(A) > q$ and $\forall a \in A, \beta \vdash a$. Let B be the result of replacing each occurrence of α in A with $\alpha \vee \delta$. Then $\chi(B) > q$ and $\forall b \in B, \beta \vdash b$, by the right upward monotonicity of $\vdash$ for single formulae on the left.

$[\vee \Vdash_q]$. Assume that for each i $(1 \leq i \leq q+1), \alpha_i \vdash F^{\vee}(B_i)$ where $\eta(B_i) > q$ and $B_i \in H(\Sigma)$. Let δ be $F^{\vee}(\binom{\{F^{\vee}(B_i)\}}{q})$. Then $\delta \dashv\models F^{\vee}\frac{q}{q+1}(B_1, ..., B_{q+1})(= \gamma)$ (given Theorem 7.3.5). But $F^{\vee}(\binom{\{\alpha_i\}}{q}) \models \delta$, whence $F^{\vee}(\binom{\{\alpha_i\}}{q}) \models \gamma$.

□

Theorem 7.4.1 amounts to a demonstration not only that $\Vdash_q$ preserves vacuity level, but also that $\Vdash_q$ is sound with respect to $\Vvdash_q$. Taken together with the following corresponding completeness result we have the representation theorem for q-folding.

Theorem 7.4.2. *$\forall\Sigma, \alpha, q \geq 1$, if $\alpha \Vdash\!\vdash_q \Sigma$ then $\alpha \Vdash_q \Sigma$.*

Proof. Assume that $\alpha \Vdash\!\vdash_q \Sigma$ where $\varphi(\Sigma) \leq q$. Then $\exists B \in H(\Sigma)$ such that $\eta(B) > q$ and $\alpha \vdash F^{\vee}(B)$. We induce on $|B|$ to show that $\forall B$, if $\eta(B) > q$ and $B \in H(\Sigma)$ then $F^{\vee}(B) \Vdash_q \Sigma$.

For the basis, $|B| = 1$. Then $B = \{\{1, 2, ..., k\}\}$ $(k \geq 1)$ and $F^{\vee}(B)$ is $1 \wedge 2 \wedge ... \wedge k$ with $i \in \Sigma$ $(1 \leq i \leq k)$. $\therefore F^{\vee}(B) \vdash i$. $\therefore F^{\vee}(B) \Vdash_q i$ (using [Pres $\vdash$]). $\therefore F^{\vee}(B) \Vdash_q \Sigma$ (from [Mon(i)]).

For the inductive step, assume $|B| = r > 1$. Suppose that $r \leq q$. Then $\bigcap_{m \in |B|} b_m \neq \emptyset$. Let $i \in \bigcap_{m \in |B|} b_m$. Then $F^{\vee}(B) \vdash i$. $\therefore F^{\vee}(B) \Vdash_q i$ (from [Pres $\vdash$]). So using [Mon(i)], $F^{\vee}(B) \Vdash_q \Sigma$.

Now let $|B| = r \geq q + 1$. Let $\{b_1, ..., b_{q+1}\} \subseteq B$ and define $B_i := B - \{b_i\}$ $(1 \leq i \leq q + 1)$. Then by the hypothesis of induction, $F^{\vee}(B_i) \Vdash_q \Sigma$. So from [$\vee \Vdash_q$] we have $F^{\vee}(\binom{F^{\vee}(B_i)\}}{q}) \Vdash_q \Sigma$. But $F^{\vee}(\binom{F^{\vee}(B_i)\}}{q}) \dashv\models F^{\vee}(B)$. $\therefore F^{\vee}(B) \Vdash_q F^{\vee}(\binom{F^{\vee}(B_i)\}}{q})$ (from [Pres $\vdash$]). So using [Mon(i)] and Theorems 7.2.2 and 7.4.1, $F^{\vee}(B) \Vdash_q F^{\vee}(\binom{F^{\vee}(B_i)\}}{q}), \Sigma$. $\therefore F^{\vee}(B) \Vdash_q \Sigma$ (using [Tran]).

This completes the induction, whence we have $F^{\vee}(B) \Vdash_q \Sigma$. But now since $\alpha \vdash F^{\vee}(B), \alpha \Vdash_q F^{\vee}(B)$ (using [Pres $\vdash$]). So $\alpha \Vdash_q F^{\vee}(B), \Sigma$ (using [Mon(i)]). So using [Tran], $\alpha \Vdash_q \Sigma$ because $F^{\vee}(B) \Vdash_q \Sigma$. Therefore $\alpha \Vdash\!\vdash_q \Sigma$ only if $\alpha \Vdash_q \Sigma$ if $\varphi(\Sigma) \leq q$. □

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