

IRREDUCIBLE CHARACTERS OF $GL_2(\mathbb{Z}/p^2\mathbb{Z})$

by

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Irreducible Characters of $GL_2(\mathbb{Z}/p^2\mathbb{Z})$

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Abstract

Representation theory studies the structure of a finite group G by looking at the set of homomorphisms ρ from G to the group of automorphisms of a complex vector space V . The character χ of a representation (ρ, V) of G is the complex valued function $\chi(g) = \text{tr}(\rho(g))$ for $g \in G$. In general, it is less complicated to work with the characters of a group G than with the representations themselves. Fortunately, a representation is uniquely determined by its character.

This thesis focusses on characters of the group $G = GL_2(\mathbb{Z}/p^2\mathbb{Z})$, the general linear group of 2×2 invertible matrices over the local ring $R = \mathbb{Z}/p^2\mathbb{Z}$. In particular, we study $GL_2(\mathbb{Z}/p^2\mathbb{Z})$ directly without resorting to $SL_2(\mathbb{Z}/p^2\mathbb{Z})$, the subgroup of G of elements with determinant 1. Let R^\times denote the group of units of R , and let μ and ν be irreducible characters of R^\times . We construct the character $\text{Ind}_B^G \omega_{\mu\nu}$ of G , where B is the Borel (or upper triangular) subgroup in G and $\omega_{\mu\nu} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \mu(a)\nu(d)$. In this thesis, we determine the decomposition of $\text{Ind}_B^G \omega_{\mu\nu}$, for all pairs of characters $\{\mu, \nu\}$ of R^\times , into a direct sum of irreducible characters. Since all representations of a finite group G are composed as the direct sum of irreducible representations, this information can be used to find further characters of G .

To my mother

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Chapter 1

Introduction

Representation theory studies the structure of a finite group G by looking at the set of homomorphisms ρ from G to the group of automorphisms of a complex vector space V ; each of these homomorphisms is called a *representation*. The underlying objects of interest in representation theory are *irreducible representations*; all representations can be formed as the direct sum of irreducible representations.

Character theory simplifies the study of representations; the *character* χ of a representation (ρ, V) of G is the complex valued function $\chi(g) = \text{tr}(\rho(g))$, the trace of $\rho(g)$ for $g \in G$. In general, it is less complicated to work with the characters of a group G than with the representations themselves. Fortunately, a representation is fully determined by its character.

One focus of character theory is in determining the complete set of irreducible representations of groups of invertible $n \times n$ matrices, and in particular 2×2 matrices. The three primary groups of interest are: $GL_n(R)$, the group of invertible $n \times n$ matrices over the ring R ; $SL_n(R)$, the subgroup of matrices with determinant 1; and $LF_n(R) = SL_n(R)/\pm I$, with I the $n \times n$ identity matrix. These groups are called the *general linear group*, the *special linear group*, and the *linear fractional group*, respectively.

The literature on representations of these groups spans the entire 20th century. In 1896, Frobenius determined the irreducible characters of $LF_2(\mathbb{Z}/p\mathbb{Z})$ for a prime p [3], while in 1907, Schur determined those of $SL_2(\mathbb{Z}/p\mathbb{Z})$ [16]. Around 1933, both Praetorius [14] and Rohrbach [15] independently determined the characters of $SL_2(\mathbb{Z}/p^2\mathbb{Z})$. In 1955, Green

determined the characters of $GL_n(\mathbb{F}_q)$ with \mathbb{F}_q the finite field with q elements¹ [5] and in 1966 and 1967 Tanaka gives a classification of representations of $SL_2(\mathbb{Z}/p^m\mathbb{Z})$ [22, 23]. Hecke [7, 6], Kloosterman [8] and McQuillan [12], all performed work related to the representations of the groups $LF_2(\mathbb{Z}/n\mathbb{Z})$ while both Stienberg [21] and Silberger [18, 19, 20] determined and classified representations of finite general linear groups, projective, and special linear groups. In 1972, Kutzko determined the irreducible characters for the groups $LF_2(\mathbb{Z}/p^m\mathbb{Z})$ and $SL_2(\mathbb{Z}/p^m\mathbb{Z})$, where p is a prime greater than 3 [9, 10]. For n not divisible by 2 or 3, Kutzko's results can be extended to $SL_2(\mathbb{Z}/n\mathbb{Z})$, since $SL_2(\mathbb{Z}/n\mathbb{Z})$ is equal to the direct product of $SL_2(\mathbb{Z}/p_i^{m_i}\mathbb{Z})$ where $n = \prod p_i^{m_i}$ [9].

In 1977, Nobs classified the irreducible representations of $GL_2(\mathbb{Z}_p)$, with \mathbb{Z}_p the ring of integers in the field of p -adic rationals. Further, Nobs determined all the irreducible representations of $GL_2(\mathbb{Z}_p)$ and more specifically of $GL_2(\mathbb{Z}/p^n\mathbb{Z})$; though he does not present the character values of these representations the dimensions and numbers of each type of irreducible representation are included [13].

This thesis focusses on characters of the group $G = GL_2(\mathbb{Z}/p^2\mathbb{Z})$; in particular, we study $GL_2(\mathbb{Z}/p^2\mathbb{Z})$ directly without resorting to $SL_2(\mathbb{Z}/p^2\mathbb{Z})$. Let R be the local ring $\mathbb{Z}/p^2\mathbb{Z}$, with maximal ideal $\mathfrak{m} = p\mathbb{Z}/p^2\mathbb{Z}$, and denote the group of units of R by R^\times . Let B be the Borel, or upper-triangular, subgroup of G . Then for μ and ν irreducible characters of R^\times , a character $\omega_{\mu\nu}$ is defined on B such that $\omega_{\mu\nu} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \mu(a)\nu(d)$. The character $\text{Ind}_B^G \omega_{\mu\nu}$ of G is produced by inducing $\omega_{\mu\nu}$ from B to G .

In this thesis, we determine the decomposition of $\text{Ind}_B^G \omega_{\mu\nu}$ into a direct sum of irreducible characters. Chapter 2 presents an overview of representation and character theory, including the definitions and theorems used throughout. Chapter 3 presents the conjugacy classes of G and determines the size of each class; this chapter also presents the irreducible characters of $GL_2(\mathbb{Z}/p\mathbb{Z})$ that will be used to define characters of $GL_2(\mathbb{Z}/p^2\mathbb{Z})$. Chapter 4 determines the decomposition of $\text{Ind}_B^G \omega_{\mu\nu}$ into a direct sum of irreducible characters; this decomposition divides into four cases that depend on μ and ν , so the irreducible components of $\text{Ind}_B^G \omega_{\mu\nu}$ are determined separately for each case.

Supplementary calculations used throughout Chapter 4 are included in Appendices A and B; the values of $\text{Ind}_B^G \omega_{\mu\nu}$ on G are determined in Appendix A, while all inner products are calculated in Appendix B.

¹A construction of the characters of $GL_2(\mathbb{F}_p)$ can be found in [4, 11].

Chapter 2

Overview of Representation Theory

This chapter presents an overview of the basic definitions and results of representation theory that are used throughout the remainder of this thesis. Also included are some examples to help illuminate this discussion.

The following properties and definitions have been taken from Fulton and Harris' *Representation Theory* [4] and Serre's *Linear Representations of Finite Groups* [17].

2.1 Representation Theory

Throughout this text we assume that a group G is finite and will denote the order of G by $|G|$.

Definition 2.1 ([4], p.3) A **representation** of a finite group G on a finite-dimensional complex vector space V is a homomorphism $\rho : G \rightarrow GL(V)$ of G to the group of automorphisms of V .

A representation is determined by the pair (ρ, V) , but it is understood that G acts on V via the map ρ . Thus, in the following chapter, when one of the pair (ρ, V) is understood, we will refer to the other as the representation.

Example 2.2 ([17], p.4) A representation of degree 1 of a group G is a homomorphism $\rho : G \rightarrow \mathbb{C}^\times$, where \mathbb{C}^\times denotes the multiplicative group of nonzero complex numbers. Since each element of G has finite order, the values $\rho(s)$ of ρ are roots of unity; in particular, we have $|\rho(s)| = 1$. If we take $\rho(s) = 1, \forall s \in G$, we obtain a representation of G which is called the **trivial** (or **unit**) **representation**.

Definition 2.3 ([4], p.4) A subspace W of V is **invariant** under ρ if for each $w \in W$ and for all $g \in G$, $\rho(g) \cdot w \in W$. A **subrepresentation** of a representation V is a vector subspace W of V which is invariant under the action of G .

Definition 2.4 ([4], p.4) A representation V is **irreducible** if there is no proper nonzero invariant subspace W of V .

Theorem 2.5 ([17], p.5,7) Let V , W and Y be representations of G . Then both the direct sum $V \oplus W$ and the tensor product $V \otimes W$ define representations of G . Further,

$$V \otimes (W \oplus Y) = (V \otimes W) \oplus (V \otimes Y).$$

Theorem 2.6 ([4], p.6) If W is an invariant subspace of a representation V of a finite group G , then there is a complimentary invariant subspace W' of V such that $V = W \oplus W'$.

Using induction on the degree of the representation V , this theorem yields the following crucial result.

Corollary 2.7 ([17], p.7) Any representation is a direct sum of irreducible representations.

The next Lemma describes the degree to which this decomposition is unique.

Theorem 2.8 (Schur's Lemma) ([4], p.7) If V and W are irreducible representations of G , and $\varphi : V \rightarrow W$ is a G -module homomorphism, then:

1. either φ is an isomorphism, or $\varphi = 0$;
2. if $V = W$, then $\varphi = \lambda \cdot I$ for some $\lambda \in \mathbb{C}$, where I is the identity.

Hence, for any representation V of a finite group G , there is a decomposition

$$V = a_1 V_1 \oplus \dots \oplus a_k V_k$$

where the V_i are distinct irreducible representations and the a_i are the number of times they appear as invariant subspaces in V . The decomposition of V into a direct sum of the k factors is unique, as are the V_i and their multiplicities a_i .

Theorem 2.9 ([17], p.18) If $\{W_i : i = 1, \dots, h\}$ is the set of all distinct irreducible representations of a group G , with n_i their degrees, then $\sum_{i=1}^h n_i^2 = |G|$.

Theorem 2.10 ([17], p.19) *The number of distinct irreducible representations of a group G is equal to the number of conjugacy classes of G .*

Corollary 2.11 ([17], p.25) *G is abelian if and only if all of the irreducible representations of G have degree 1.*

Proof: Let G be an abelian group. Since each element of an abelian group is its own conjugacy class, by Theorem 2.10 there are $|G|$ distinct irreducible representations W_i of G . Let n_i be the degrees of the W_i 's. Then Theorem 2.9 tells us that $\sum_{i=1}^{|G|} n_i^2 = |G|$; however, as each $n_i \geq 1$, this implies that all the $n_i = 1$. The converse argument is similar. \square

Recalling from Example 2.2, we see that for an abelian group G each irreducible representation ρ of G is a homomorphism from G to the $|G|^{\text{th}}$ complex roots of unity.

Theorem 2.12 *For any subgroup H of an abelian group G ,*

$$\sum_{g \in H} \rho(g) = \begin{cases} |H|, & \rho \text{ is trivial on all of } H; \\ 0, & \text{otherwise.} \end{cases}$$

Proof: If ρ is trivial on H , then the result is obvious. Assume ρ is non-trivial on H . Then $\text{im}(H)$, the image of H under ρ , is an abelian group with order $s = |H|/|\ker(\rho)| > 1$. Each element of $\text{im}(H)$ is a root of the polynomial $x^s - 1$, and by noting the degree, these are all the roots of $x^s - 1$. By expansion, or by using the binomial theorem we see that the x^{s-1} coefficient is $-\sum_{\alpha \in \text{im}(H)} \alpha = 0$. Thus, the sum $\sum_{h \in H} \rho(h) = |\ker(\rho)| \sum_{\alpha \in \text{im}(H)} \alpha = 0$ as required. \square

2.2 Character Theory

The study of representation theory is greatly simplified by the use of characters. Character theory uses the fact that the map $\rho(g)$ on V is determined by its eigenvalues.

Definition 2.13 ([17], p.10) *If V is a representation of G , its **character** χ_V is the complex-valued function on the group G defined by the trace of $\rho(g)$ on V , denoted by*

$$\chi_V(g) = \text{tr}(\rho(g)).$$

Notice that for the character χ_ρ of a representation ρ of degree 1, $\chi_\rho(g) = \rho(g) \forall g \in G$, thus Example 2.2 and Theorem 2.12 can be rewritten replacing ρ with χ_ρ .

Theorem 2.14 ([17], p.16) *Two representations with the same character are isomorphic.*

Theorem 2.15 ([17], p.10) *If χ_V is the character of a representation (ρ, V) of degree n , then:*

1. $\chi_V(1) = n$;
2. $\chi_V(t^{-1}st) = \chi_V(s)$, $s, t \in G$;
3. $\chi_V(s^{-1}) = \chi_V(s)^*$, $s \in G$,

where c^* denotes the complex conjugate of c . A \mathbb{C} valued function on G displaying the second property is called a **class function**.

Theorem 2.16 ([17], p.11) *Let V and W be representations of G . Then,*

$$\chi_{V \oplus W} = \chi_V + \chi_W \text{ and } \chi_{V \otimes W} = \chi_V \cdot \chi_W.$$

Since each representation V of G is composed of irreducibles, a *character table* of G can be constructed; this table lists irreducible characters of G and their values on each conjugacy class of G .

An inner product is defined on the set of all class functions on G by

$$\langle \chi, \varphi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \varphi(g)^*. \quad (2.1)$$

In particular, if χ and φ are characters of G ,

$$\langle \chi, \varphi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \varphi(g^{-1}). \quad (2.2)$$

Often, we will be taking the inner product of a character with itself, which becomes

$$\langle \chi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2 \quad (2.3)$$

([17], p.14)

Remark 2.17 ([17], p.15) *The characters of the irreducible representations of a group G form an orthonormal basis (with respect to the inner product (2.1)) for the complex vector space of all class functions on G . Any class function on G that is a positive integer linear combination of irreducible characters is itself a character of a representation of G .*

Theorem 2.18 ([4], p.17) *A representation V is irreducible if and only if $(\chi_V, \chi_V) = 1$. In fact, if V decomposes into irreducibles as*

$$V = a_1 V_1 \oplus \cdots \oplus a_k V_k,$$

then $(\chi_V, \chi_V) = \sum a_i^2$ and $a_i = (\chi_V, \chi_{V_i})$.

Definition 2.19 ([4], p.33) *Let H be a subgroup of G , and consider the set of left cosets of H in G , G/H . The representation ρ of G in V is **induced** by the representation ρ of H in W if*

$$V = \bigoplus_{\sigma \in G/H} \sigma \cdot W.$$

We denote this as $V = \text{Ind}_H^G W$.

Theorem 2.20 ([17], p.30) *Let W be a linear representation of H , and let $x \simeq_G g$ mean that x is conjugate to g via some $s \in G$. There exists a linear representation V of G which is induced by W and is unique up to isomorphism. Further, if \mathcal{R} is a set of representatives for G/H and $C_G(g)$ is the centralizer of g in G , then the character of V can be calculated using the equivalent formulae:*

$$\text{Ind}_H^G \chi(g) = \sum_{\substack{r \in \mathcal{R} \\ r^{-1}gr \in H}} \chi_H(r^{-1}gr) \quad (2.4)$$

$$= \frac{1}{|H|} \sum_{\substack{s \in G \\ s^{-1}gs \in H}} \chi_H(s^{-1}gs) \quad (2.5)$$

$$= \frac{1}{|H|} |C_G(g)| \sum_{\substack{h \in H \\ h \simeq_G g}} \chi_H(h). \quad (2.6)$$

Chapter 3

Background on $GL_2(\mathbb{Z}/p^2\mathbb{Z})$

Throughout the remaining chapters, let G and \bar{G} denote the general linear groups $GL_2(\mathbb{Z}/p^2\mathbb{Z})$ and $GL_2(\mathbb{Z}/p\mathbb{Z})$ respectively, for p an odd prime. Let R be the local ring $\mathbb{Z}/p^2\mathbb{Z}$ with maximal ideal $\mathfrak{m} = p\mathbb{Z}/p^2\mathbb{Z}$ and group of units R^\times , and let \bar{R} be the field $\mathbb{Z}/p\mathbb{Z}$ with group of units \bar{R}^\times . Conjugation of an element s by an element t will mean $t^{-1}st$.

Before we determine characters of G , it will be useful to first discuss the conjugacy class structure of G . In Section 3.1 we determine the conjugacy classes of G , as well as the size of each class and the number of classes of each type. This information will be used throughout Chapter 4, where we determine the characters $\text{Ind}_B^G \omega_{\mu\nu}$ of G and the irreducible characters involved in the decomposition of $\text{Ind}_B^G \omega_{\mu\nu}$.

There is a homomorphism $\pi : G \rightarrow \bar{G}$ such that each entry of a matrix $g \in G$ is reduced mod \mathfrak{m} . For this reason, a representation or character of \bar{G} becomes a representation or character of G by first allowing the map π to act on the elements of G . Section 3.2 describes the characters of \bar{G} that will be used to determine characters of G in Chapter 4.

3.1 Conjugacy Classes of $GL_2(\mathbb{Z}/p^2\mathbb{Z})$

Consider the map

$$\begin{array}{ccc} \pi : G & \twoheadrightarrow & \bar{G} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}, \end{array}$$

where for $\alpha \in R$, $\bar{\alpha}$ denotes reduction mod \mathfrak{m} . As the homomorphism π from G to \bar{G} is surjective, the first isomorphism theorem tells us that $\bar{G} \simeq G/\ker(\pi)$ and hence the order of G is $|G| = |\bar{G}||\ker(\pi)|$. It is clear that the kernel in G of the map π is

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : a, d = 1 \pmod{\mathfrak{m}}, b, c = 0 \pmod{\mathfrak{m}} \right\}.$$

This subgroup has order p^4 ; we will see in Section 3.2 that $|\bar{G}| = p(p-1)^2(p+1)$, and therefore we have $|G| = p^5(p-1)^2(p+1)$.

Theorem 3.1 [1] *Suppose that ϵ is a fixed non-square in R^\times , the group of units of R . Then G partitions into conjugacy classes with representatives as listed below. Further, each representative belongs to a distinct class of G , and these are all the conjugacy classes of G . (Note that if H is a quotient of R^\times then by $\beta \in H$ we mean that the β are chosen from a complete set of inequivalent representatives in R^\times of H .)*

$$\text{Type I} \quad I_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}_{\alpha \in R^\times}$$

$$\text{Type B} \quad B_\alpha = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}_{\alpha \in R^\times}$$

$$\text{Type T} \quad T_{\alpha, \delta} = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}_{\alpha, \delta \in R^\times, \alpha \not\equiv \delta \pmod{\mathfrak{m}}}$$

$$\text{Type T'} \quad T'_{\alpha, \beta} = \begin{pmatrix} \alpha & \epsilon\beta \\ \beta & \alpha \end{pmatrix}_{\alpha \in R, \beta \in R^\times / \{\pm 1\}}$$

$$\text{Type RT} \quad RT_{\alpha, \beta} = \begin{pmatrix} \alpha & p\beta \\ \beta & \alpha \end{pmatrix}_{\alpha \in R^\times, \beta \in (R/\mathfrak{m})^\times / \{\pm 1\}}$$

$$\text{Type RT'} \quad RT'_{\alpha, \beta} = \begin{pmatrix} \alpha & p\epsilon\beta \\ \beta & \alpha \end{pmatrix}_{\alpha \in R^\times, \beta \in (R/\mathfrak{m})^\times / \{\pm 1\}}$$

$$\text{Type } RI \quad RI_{\alpha,\beta} = \begin{pmatrix} \alpha & p\beta^2 \\ p & \alpha \end{pmatrix}_{\alpha \in R^\times, \beta \in (R/\mathfrak{m})^\times / \{\pm 1\}}$$

$$\text{Type } RB \quad RB_\alpha = \begin{pmatrix} \alpha & 0 \\ p & \alpha \end{pmatrix}_{\alpha \in R^\times}$$

$$\text{Type } RI' \quad RI'_{\alpha,\beta} = \begin{pmatrix} \alpha & p\epsilon\beta^2 \\ p & \alpha \end{pmatrix}_{\alpha \in R^\times, \beta \in (R/\mathfrak{m})^\times / \{\pm 1\}}$$

Proof:

For types I, B, T, T', RT, RT' and RB , two different representatives from the same type are not conjugate as they do not have both the same trace and the same determinant. Further, representatives from types T, T', RT and RT' have discriminants $(\alpha - \delta)^2, \epsilon\beta^2, p\beta^2$ and $p\epsilon\beta^2$ respectively, where the discriminant of the characteristic polynomial of an element in G is calculated as $\Delta(g) = \text{tr}(g)^2 - 4\det(g)$; thus, representatives from any two of these types cannot be conjugate. Matrices of type I, B, RI, RB and RI' all have discriminant 0, and hence are not conjugate to those of types T, T', RT and RT' . Also, representatives from the different types I, B, RI, RB and RI' have centralizers of different orders (see Table 3.1), and hence cannot be conjugate. It is left to prove that two representatives of the same type for types RI or RI' are not conjugate, and that this is in fact all the classes of G .

To show that the classes of type RI are distinct, it is enough to show that if there exists a $g \in G$ such that $RI_{a,b} \cdot g = g \cdot RI_{c,d}$ then $a = c$, and $b = \pm d \pmod{\mathfrak{m}}$. Suppose that representatives $RI_{a,b}$ and $RI_{c,d}$ are conjugate via some $g = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \in G$. Then

$$RI_{a,b} \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} w & x \\ y & z \end{pmatrix} RI_{c,d}.$$

Since two conjugate matrices have the same trace, a must equal c , and therefore we have

$$\begin{pmatrix} aw + pb^2y & ax + pb^2z \\ pw + ay & px + za \end{pmatrix} = \begin{pmatrix} aw + px & wpd^2 + ax \\ ay + zp & ypd^2 + za \end{pmatrix}.$$

This implies that $b^2y = d^2y \pmod{\mathfrak{m}}$ and that $b^2z = d^2z \pmod{\mathfrak{m}}$, but as at least one of $y, z \in R^\times$ for g to be in G , we have that $b^2 = d^2 \pmod{\mathfrak{m}} \Rightarrow b = \pm d \pmod{\mathfrak{m}}$ as required. The argument is the same to show that the classes of type RI' are distinct.

That these are all the classes of G will be verified by noting that

$$\sum_{C_r \in \mathcal{S}} |C_r| = |G|,$$

where \mathcal{S} is the collection of all the different conjugacy classes C_r of G . The size $|C_r|$ of each class C_r is listed in Table 3.1; they are found using the orbit stabilizer theorem and Theorem 3.2. \square

Theorem 3.2 [1] *The centralizer for each class representative is:*

$$\begin{aligned} C_G(I_\alpha) &= G \\ C_G(B_\alpha) &= \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a \in R^\times, b \in R \right\} \\ C_G(T_{\alpha,\delta}) &= \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in R^\times \right\} \\ C_G(T'_{\alpha,\beta}) &= \left\{ \begin{pmatrix} a & b\epsilon \\ b & a \end{pmatrix} : (a, b) \neq (0, 0) \pmod{\mathfrak{m}} \right\} \\ C_G(RT_{\alpha\beta}) &= \left\{ \begin{pmatrix} a & bp \\ b & a \end{pmatrix} : a \in R^\times, b \in R \right\} \\ C_G(RT'_{\alpha,\beta}) &= \left\{ \begin{pmatrix} a & bp\epsilon \\ b & a \end{pmatrix} : a \in R^\times, b \in R \right\} \\ C_G(RI_{\alpha,\beta}) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a = d \pmod{\mathfrak{m}}, b = c\beta^2 \pmod{\mathfrak{m}}, a^2 - c^2\beta^2 \neq 0 \pmod{\mathfrak{m}} \right\} \\ C_G(RB_\alpha) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in R^\times, c \in R, a = d \pmod{\mathfrak{m}}, b = 0 \pmod{\mathfrak{m}} \right\} \\ C_G(RI'_{\alpha,\beta}) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a = d \pmod{\mathfrak{m}}, b = c\epsilon\beta^2 \pmod{\mathfrak{m}}, (a, c) \neq (0, 0) \pmod{\mathfrak{m}} \right\} \end{aligned}$$

Proof:

This can be verified by a direct matrix calculation. \square

Remark 3.3 *The orbit stabilizer theorem tells us that if g_1 is conjugate to g_2 in G , then $|C_G(g_1)| = |C_G(g_2)|$. By calculating the orders of the centralizers of Theorem 3.2 the order*

of the centralizer $C_G(g)$ for each $g \in G$ is determined; note that the order of the centralizer in G of an element g depends only on the **type** of conjugacy class it belongs to.

Table 3.1 summarizes the conjugacy class information of G . Included in this Table are the number of conjugacy classes of each type, the orders of the centralizers of each class representative (and hence the orders of the centralizers for each $g \in G$), and the size of each class (using the orbit stabilizer theorem). This information will be repeatedly used to determine the characters of G and to calculate the inner product of two characters.

From Table 3.1 there are $p(p-1)(p^2+p+1)$ conjugacy classes of G ; thus, by Theorem 2.10 there are exactly $p(p-1)(p^2+p+1)$ distinct irreducible representations, and hence characters, of G .

Throughout the remainder of this thesis, I will use the notation \mathcal{S} to be the collection of all the conjugacy classes C_r of G partitioned by the nine different types \mathcal{T} of conjugacy class, and r to mean the representative of the class C_r as chosen in Theorem 3.1

With this notation and using that all characters are class functions (2.2) and (2.3),

$$\langle \chi, \varphi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \varphi(g^{-1}) \quad \text{and} \quad \langle \chi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2$$

become

$$\langle \chi, \varphi \rangle = \frac{1}{|G|} \sum_{T \subset \mathcal{S}} \sum_{C_r \in T} |C_r| \chi(r) \varphi(r^{-1}) \quad (3.1)$$

and

$$\langle \chi, \chi \rangle = \frac{1}{|G|} \sum_{T \subset \mathcal{S}} \sum_{C_r \in T} |C_r| |\chi(r)|^2. \quad (3.2)$$

3.2 Characters of $GL_2(\mathbb{Z}/p\mathbb{Z})$

Recall that the reduction map $\pi : G \rightarrow \bar{G}$ is a surjective group homomorphism. A representation or character of \bar{G} can be extended to a representation or character of G by first allowing π to act on the elements of G . Such a character is called a *lift* of χ on \bar{G} to G .

Theorem 3.4 *Let $\rho_{\bar{G}}$ be a representation of \bar{G} and let ρ_G be the representation $\rho_{\bar{G}} \circ \pi$ on G . If $\rho_{\bar{G}}$ is an irreducible representation of \bar{G} then ρ_G is an irreducible representation of G .*

Proof: We will prove the equivalent statement that if ρ_G is a reducible representation of G , then $\rho_{\bar{G}}$ is a reducible representation of \bar{G} . Let V be the vector space associated with ρ_G , and suppose that ρ_G is reducible. Then V can be decomposed as $W_1 \oplus W_2$, for W_1 and W_2 non-trivial proper invariant subspaces of V . Thus for all $g \in G$ and for all $w_1 \in W_1$, $\rho_G(g)w_1 \in W_1$, or equivalently $\rho_{\bar{G}}(\pi(g))w_1 \in W_1$. However, π is surjective from G to \bar{G} , thus for all $\bar{g} \in \bar{G}$ and for all $w_1 \in W_1$, $\rho_{\bar{G}}(\bar{g})w_1 \in W_1$. Hence, W_1 is a non-trivial proper invariant subspace of V with respect to $\rho_{\bar{G}}$ and necessarily $\rho_{\bar{G}}$ is reducible. \square

The conjugacy classes and the irreducible characters of \bar{G} are calculated in [4, 11]. Let \bar{S} be the set of all conjugacy classes of \bar{G} partitioned into the four different types of classes \bar{T} . Let $\bar{C}_{\bar{r}}$ be a conjugacy class of \bar{G} with representative \bar{r} . Table 3-2 displays how \bar{G} is partitioned into conjugacy classes; from this table, the order of \bar{G} is counted to be $p(p-1)^2(p+1)$.

There are four types of irreducible characters of \bar{G} ; we will only be discussing the first three of these types. Let \bar{B} be the Borel subgroup of \bar{G} , given by

$$\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in (\mathbb{Z}/p\mathbb{Z})^\times, b \in (\mathbb{Z}/p\mathbb{Z}) \right\},$$

and let $\bar{\mu}$ and $\bar{\nu}$ be one-dimensional irreducible characters of \bar{R}^\times . The character $\psi_{\bar{\mu}}$ of \bar{G} is defined as the map

$$\bar{\mu} \circ \det : \bar{G} \rightarrow \mathbb{C}^\times,$$

and $\omega_{\bar{\mu}\bar{\nu}}$ is the character on \bar{B} such that

$$\omega_{\bar{\mu}\bar{\nu}} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \bar{\mu}(a)\bar{\nu}(d).$$

The values of the first three types of characters of \bar{G} are as given in Table 3-3, and can be found in [4, 11].

As previously mentioned, a character χ of \bar{G} becomes a character $\chi^G = \chi \circ \pi$ of G ; for this reason, Table 3-4 lists which class $\bar{C}_{\bar{r}}$ of \bar{G} $\pi(r)$ belongs to for each class representative r of G . For all classes except those of type RT and RT' , the element $\pi(r)$ in \bar{G} is already one of the class representatives of \bar{G} , as listed in Table 3-2; therefore, it is clear to which conjugacy class of \bar{G} they belong. As for representatives of type RI and RI' ,

$$\pi(RI_{\alpha,\beta}) = \pi(RI'_{\alpha,\beta}) = \begin{pmatrix} \bar{\alpha} & 0 \\ \bar{\beta} & \bar{\alpha} \end{pmatrix},$$

where, for $x \in R$, \bar{x} is the reduction of $x \bmod \mathfrak{m}$. Conjugating this by $\begin{pmatrix} 0 & \bar{\beta}^{-1} \\ 1 & 0 \end{pmatrix} \in \bar{G}$, we get the class representative $b_{\bar{\alpha}}$ of \bar{G} ; therefore, both $\pi(RI_{\alpha,\beta})$ and $\pi(RI'_{\alpha,\beta})$ belong to the class of $b_{\bar{\alpha}}$ of \bar{G} .

r	Number of $C_r \in \mathcal{T}$	$ C_r $	$ C_G(g) $
$I_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}_{\alpha \in R^\times}$	$p(p-1)$	1	$p^5(p-1)^2(p+1)$
$B_\alpha = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}_{\alpha \in R^\times}$	$p(p-1)$	$p^2(p-1)(p+1)$	$p^3(p-1)$
$T_{\alpha,\delta} = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}_{\substack{\{\alpha,\delta\} \subseteq R^\times \\ \alpha \not\equiv \delta \pmod{\mathfrak{m}}}}$	$\frac{p^2(p-1)(p-2)}{2}$	$p^3(p+1)$	$p^2(p-1)^2$
$T'_{\alpha,\beta} = \begin{pmatrix} \alpha & \epsilon\beta \\ \beta & \alpha \end{pmatrix}_{\substack{\beta \in R^\times / \{\pm 1\} \\ \alpha \in R}}$	$\frac{1}{2}p^3(p-1)$	$p^3(p-1)$	$p^2(p-1)(p+1)$
$RT_{\alpha,\beta} = \begin{pmatrix} \alpha & p\beta \\ \beta & \alpha \end{pmatrix}_{\substack{\beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \\ \alpha \in R^\times}}$	$\frac{1}{2}p(p-1)^2$	$p^2(p-1)(p+1)$	$p^3(p-1)$
$RT'_{\alpha,\beta} = \begin{pmatrix} \alpha & p\epsilon\beta \\ \beta & \alpha \end{pmatrix}_{\substack{\beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \\ \alpha \in R^\times}}$	$\frac{1}{2}p(p-1)^2$	$p^2(p-1)(p+1)$	$p^3(p-1)$
$RI_{\alpha,\beta} = \begin{pmatrix} \alpha & p\beta^2 \\ p & \alpha \end{pmatrix}_{\substack{\beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \\ \alpha \in R^\times}}$	$\frac{1}{2}p(p-1)^2$	$p(p+1)$	$p^4(p-1)^2$
$RB_\alpha = \begin{pmatrix} \alpha & 0 \\ p & \alpha \end{pmatrix}_{\alpha \in R^\times}$	$p(p-1)$	$(p-1)(p+1)$	$p^5(p-1)$
$RI'_{\alpha,\beta} = \begin{pmatrix} \alpha & p\epsilon\beta^2 \\ p & \alpha \end{pmatrix}_{\substack{\beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \\ \alpha \in R^\times}}$	$\frac{1}{2}p(p-1)^2$	$p(p-1)$	$p^4(p-1)(p+1)$

Table 3-1: Conjugacy information of G

\bar{r}	$ \bar{C}_r $	Number of $\bar{C}_r \in \bar{T}$
$a_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}_{x \in \bar{R}^\times}$	1	$p - 1$
$b_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}_{x \in \bar{R}^\times}$	$p^2 - 1$	$p - 1$
$c_{x,y} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}_{\substack{x,y \in \bar{R}^\times \\ x \neq y}}$	$p^2 + p$	$\frac{(p-1)(p-2)}{2}$
$d_{x,y} = \begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix}_{\substack{x,y \in \bar{R} \\ y \neq 0}}$	$p^2 - p$	$\frac{p(p-1)}{2}$

Table 3-2: The conjugacy classes of $GL_2(\mathbb{Z}/p\mathbb{Z})$: Shown are example representatives for a conjugacy class of each of the four types, as well as the size of each class and the number of classes of each type.

\bar{r}	$\psi_{\bar{\mu}}(r)$	$\varpi_{\bar{\mu}}(r) = \left(\text{Ind}_{\bar{B}}^{\bar{G}} \psi_{\bar{\mu}} _{\bar{B}} - \psi_{\bar{\mu}} \right)(r)$	$\text{Ind}_{\bar{B}}^{\bar{G}} \omega_{\bar{\mu}\bar{\nu}}(r)$
a_x	$\mu(x^2)$	$p\mu(x^2)$	$(p+1)\mu(x)\nu(x)$
b_x	$\mu(x^2)$	0	$\mu(x)\nu(x)$
$c_{x,y}$	$\mu(xy)$	$\mu(xy)$	$\mu(x)\nu(y) + \mu(y)\nu(x)$
$d_{x,y}$	$\mu(x^2 - \epsilon y^2)$	$-\mu(x^2 - \epsilon y^2)$	0

Table 3-3: Characters of $GL_2(\mathbb{Z}/p\mathbb{Z})$. Shown are the values of three types of irreducible characters of $\bar{G} = GL_2(\mathbb{Z}/p\mathbb{Z})$.

r	$r : \pi(r) \in \tilde{C}_r$
I_α	$a_{\bar{\alpha}}$
B_α	$b_{\bar{\alpha}}$
$T_{\alpha,\delta}$	$c_{\bar{\alpha},\bar{\delta}}$
$T'_{\alpha,\beta}$	$d_{\bar{\alpha},\bar{\delta}}$
$RT_{\alpha,\beta}$	$b_{\bar{\alpha}}$
$RT'_{\alpha,\beta}$	$b_{\bar{\alpha}}$
$RI_{\alpha,\beta}$	$a_{\bar{\alpha}}$
RB_α	$a_{\bar{\alpha}}$
$RI'_{\alpha,\beta}$	$a_{\bar{\alpha}}$

Table 3-4: The class in \tilde{G} of $\pi(r)$ for each class representative r of G .

Chapter 4

Decomposing $\text{Ind}_B^G \omega_{\mu\nu}$ into irreducible characters

In this chapter we determine the decomposition of the character $\text{Ind}_B^G \omega_{\mu\nu}$ of $GL_2(\mathbb{Z}/p^2\mathbb{Z})$, into irreducible characters. Recall from Section 3.2 that $\omega_{\bar{\mu}\bar{\nu}} \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{d} \end{pmatrix} = \bar{\mu}(\bar{a})\bar{\nu}(\bar{d})$ for characters $\bar{\mu}$ and $\bar{\nu}$ of \bar{R}^\times ; similarly, we construct $\omega_{\mu\nu} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \mu(a)\nu(d)$ for characters μ and ν of R^\times .

4.1 Introduction

The decomposition of $\text{Ind}_B^G \omega_{\mu\nu}$ into irreducibles breaks down into four cases, depending on the character $\mu\nu'$ of R^\times defined by $\mu\nu'(\alpha) = \mu(\alpha)\nu(\alpha^{-1})$ for all $\alpha \in R^\times$. In Case 1, $\mu = \nu$, so $\mu\nu'$ is the trivial character on R^\times . In Case 2 and Case 3, the character $\mu\nu'$ is the trivial character on the kernel in R^\times of the reduction mod \mathfrak{m} map. In Case 4, $\mu\nu'$ is not trivial on this kernel.

We treat the decomposition of $\text{Ind}_B^G \omega_{\mu\nu}$ separately within each case. In Case 1, $\text{Ind}_B^G \omega_{\mu\nu}$ is the direct sum of three distinct irreducible characters ψ_μ , ϑ_μ , and $\tilde{\vartheta}_\mu$. In Case 2, $\text{Ind}_B^G \omega_{\mu\nu}$ is the direct sum of two irreducible characters $\text{Ind}_B^{\tilde{G}} \omega_{\bar{\mu}\bar{\nu}} \circ \pi$ and $\varphi_{\mu\nu}$, where $\text{Ind}_B^{\tilde{G}} \omega_{\bar{\mu}\bar{\nu}}$ is the irreducible character of $\tilde{G} = GL_2(\mathbb{Z}/p\mathbb{Z})$ and $\varphi_{\mu\nu}$ is the remaining character found by subtraction. In Case 3, $\mu = \sigma\nu$ for σ trivial on the aforementioned kernel, and thus $\text{Ind}_B^G \omega_{\mu\nu} = (\text{Ind}_B^{\tilde{G}} \omega_{\sigma 1} \circ \pi \otimes \psi_\nu) + (\varphi_{\sigma 1} \otimes \psi_\nu)$ with both characters irreducible. Finally, in

Case 4, $\text{Ind}_B^G \omega_{\mu\nu}$ is an irreducible character of G .

Recall from Theorem 2.16 that for characters χ_1 and χ_2 of G , a character χ of G defined by

$$\chi(g) = \chi_1(g) \cdot \chi_2(g) \text{ or } \chi(g) = \chi_1(g) + \chi_2(g)$$

is the character of the tensor product or direct sum, respectively, of the representations of χ_1 and χ_2 . Throughout this chapter, a character μ of R^\times is understood to mean an irreducible character of R^\times ; since R^\times is an abelian group, there are $|R^\times| = p(p-1)$ distinct one-dimensional irreducible characters μ of R^\times by Theorems 2.10 and 2.11.

Two characters χ and λ are *inverse characters* of a group H if $\chi \cdot \lambda = 1$, the trivial character of H . Note that the characters $\mu\nu'$ and $\mu'\nu$ of R^\times , defined respectively by $\mu\nu'(\alpha) = \mu(\alpha)\nu(\alpha^{-1})$ and $\mu'\nu(\alpha) = \mu(\alpha^{-1})\nu(\alpha)$ for $\alpha \in R^\times$, are inverse characters of R^\times . Therefore $\mu\nu'$ and $\mu'\nu$ are simultaneously trivial or non-trivial on any subgroup of R^\times .

Lemma 4.1 *Define*

$$U_1 = \{\alpha \in R^\times : \alpha = 1 \bmod \mathfrak{m}\},$$

the kernel in R^\times of $\phi : R^\times \rightarrow \bar{R}^\times$, the surjective reduction mod \mathfrak{m} map. If μ is a character of R^\times that is trivial on U_1 , then $\mu = \bar{\mu} \circ \phi$ for some character $\bar{\mu}$ of \bar{R}^\times .

Proof: Suppose that μ is trivial on all of U_1 . Then for $\alpha \in R^\times$, $\mu(\alpha)$ depends only on the value of $\alpha \bmod \mathfrak{m}$:

$$\begin{aligned} \mu(\alpha + p) &= \mu(1 + \alpha^{-1}p)\mu(\alpha) \\ &= \mu(\alpha). \end{aligned}$$

Therefore, μ is a well defined class homomorphism on $R^\times/U_1 \cong \bar{R}^\times$, and thus defines a character $\bar{\mu}$ of \bar{R}^\times . \square

There are $p-1$ characters of \bar{R}^\times and hence, by Lemma 4.1, there are $p-1$ characters of R^\times that are trivial on the subgroup U_1 , with the remaining $(p-1)^2$ characters non-trivial on U_1 . This information is used to count the number of each type of irreducible representation of G constructed in this chapter.

4.2 The Characters $\text{Ind}_B^G \omega_{\mu\nu}$ of $GL_2(\mathbb{Z}/p^2\mathbb{Z})$

Let $D = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} : \alpha, \delta \in R^\times \right\}$ be the diagonal subgroup of G , and notice that $D \cong R^\times \times R^\times$. Therefore, we can construct a character $\mu \times \nu$ of D , where μ and ν are characters of R^\times with

$$\mu \times \nu \left(\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \right) = \mu(\alpha)\nu(\delta).$$

Observe that $B/U \cong D$, where $B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G : a, d \in R^\times, b \in R \right\}$ is the Borel subgroup of G , and U is the normal subgroup of B given by

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in R \right\}.$$

Hence, there is a surjective homomorphism θ from B to D with kernel U , given by

$$\begin{aligned} \theta : B &\twoheadrightarrow D \\ \begin{pmatrix} \alpha & b \\ 0 & \delta \end{pmatrix} &\mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}. \end{aligned}$$

So $\mu \times \nu$ can be extended to a character on B , denoted $\omega_{\mu\nu}$, with

$$\omega_{\mu\nu} \left(\begin{pmatrix} \alpha & b \\ 0 & \delta \end{pmatrix} \right) = \mu(\alpha)\nu(\delta).$$

The values of the character $\text{Ind}_B^G \omega_{\mu\nu}$ on G are determined in Appendix A and are included in Table 4-1. Observe from Table 4-1 that

$$\text{Ind}_B^G \omega_{\mu\nu} \cong \text{Ind}_B^G \omega_{\nu\mu},$$

regardless of whether or not $\mu\nu'$ is the trivial character on the subgroup U_1 of R^\times . By comparing of the values of $\text{Ind}_B^G \omega_{\mu_1\nu_1}$ on class types I , T and RI with those of $\text{Ind}_B^G \omega_{\mu_2\nu_2}$, it can be shown that $\text{Ind}_B^G \omega_{\mu_1\nu_1} \cong \text{Ind}_B^G \omega_{\mu_2\nu_2}$ if and only if the pair $\{\mu_1, \nu_1\}$ is the pair $\{\mu_2, \nu_2\}$. Therefore each unique pair $\{\mu, \nu\}$ produces a distinct character $\text{Ind}_B^G \omega_{\mu\nu}$ of G .

r	$\text{Ind}_B^G \omega_{\mu\nu}(r)$ for $\mu\nu'$ trivial on U_1	$\text{Ind}_B^G \omega_{\mu\nu}(r)$ for $\mu\nu'$ non-trivial on U_1
I_α	$p(p+1)\mu(\alpha)\nu(\alpha)$	$p(p+1)\mu(\alpha)\nu(\alpha)$
B_α	$p\mu(\alpha)\nu(\alpha)$	0
$T_{\alpha,\delta}$	$\mu(\alpha)\nu(\delta) + \mu(\delta)\nu(\alpha)$	$\mu(\alpha)\nu(\delta) + \mu(\delta)\nu(\alpha)$
$T'_{\alpha,\beta}$	0	0
$RT_{\alpha,\beta}$	0	0
$RT'_{\alpha,\beta}$	0	0
$RI_{\alpha,\beta}$	$2p\mu(\alpha)\nu(\alpha)$	$p[\mu(\alpha + \beta p)\nu(\alpha - \beta p) + \mu(\alpha - \beta p)\nu(\alpha + \beta p)]$
RB_α	$p\mu(\alpha)\nu(\alpha)$	$p\mu(\alpha)\nu(\alpha)$
$RI'_{\alpha,\beta}$	0	0

Table 4-1: The values of $\text{Ind}_B^G \omega_{\mu\nu}$ when $\mu\nu'$ is trivial on U_1 and when $\mu\nu'$ is not trivial on U_1 : These values are taken from Table A-3.

The inner product of $\text{Ind}_B^G \omega_{\mu\nu}$ with itself is determined in Appendix B. We see in Table B-1 that when $\mu\nu'$ is trivial on U_1 this inner product divides into two cases: when $\mu\nu' = 1$, and hence $\mu = \nu$, the inner product $\langle \text{Ind}_B^G \omega_{\mu\nu}, \text{Ind}_B^G \omega_{\mu\nu} \rangle = 3$; when $\mu\nu' \neq 1$ and $\mu\nu'$ is trivial on U_1 , $\langle \text{Ind}_B^G \omega_{\mu\nu}, \text{Ind}_B^G \omega_{\mu\nu} \rangle = 2$.

4.2.1 Case 1: $\mu\nu'$ trivial on R^\times

When $\mu\nu'$ is trivial on R^\times , we have $\mu = \nu$. We rewrite the values of the character $\text{Ind}_B^G \omega_{\mu\nu}$ of G by replacing ν with μ in the $\mu\nu'$ trivial on U_1 column in Table 4-1; these character values are included in Table 4-2.

By Table B-1, we see that $\langle \text{Ind}_B^G \omega_{\mu\mu}, \text{Ind}_B^G \omega_{\mu\mu} \rangle = 3$, and hence the character $\text{Ind}_B^G \omega_{\mu\mu}$ is not irreducible. Using the notation of Theorem 2.18,

$$3 = \langle \text{Ind}_B^G \omega_{\mu\mu}, \text{Ind}_B^G \omega_{\mu\mu} \rangle = \sum a_i^2 \Rightarrow \begin{cases} a_i = 1, & i = 1, \dots, 3; \\ a_i = 0, & \text{otherwise.} \end{cases}$$

Therefore, $\text{Ind}_B^G \omega_{\mu\mu}$ is the direct sum of three distinct irreducibles.

The Irreducible Component ψ_μ of $\text{Ind}_B^G \omega_{\mu\mu}$

The map

$$\psi_\mu = \mu \circ \det : G \rightarrow \mathbb{C}^\times$$

defines a character of G , with values as listed in Table 4-2. We see that ψ_μ is one-dimensional, and hence is necessarily an irreducible character of G . Table B-2 shows that $\langle \text{Ind}_B^G \omega_{\mu\mu}, \psi_\mu \rangle = 1$; therefore, ψ_μ is an irreducible component of the direct sum that composes $\text{Ind}_B^G \omega_{\mu\mu}$.

The Irreducible Component $\tilde{\vartheta}_\mu$ of $\text{Ind}_B^G \omega_{\mu\mu}$

Consider the character $\tilde{\vartheta}_\mu = \varpi_1 \circ \pi \cdot \psi_\mu$ of G . Recall that $\varpi_1 = \text{Ind}_{\tilde{B}}^{\tilde{G}} \psi_1|_{\tilde{B}} - \psi_1$ is one of the irreducible characters of $\tilde{G} = GL_2(\mathbb{Z}/p\mathbb{Z})$, and that π sends G to \tilde{G} by reduction mod \mathfrak{m} . The values of this character are in Table 4-2 and can be verified by multiplying the values of $\psi_\mu(r)$ with the values of $\varpi_1(\pi(r))$ from Table 3-3. Table 3-4 shows the class of \tilde{G} in which $\pi(r)$ belongs.

Table B-3 shows that $\langle \tilde{\vartheta}_\mu, \tilde{\vartheta}_\mu \rangle = 1$, and thus $\tilde{\vartheta}_\mu$ is an irreducible character of G . Further, Table B-4 shows that $\langle \text{Ind}_B^G \omega_{\mu\mu}, \tilde{\vartheta}_\mu \rangle = 1$; therefore, $\tilde{\vartheta}_\mu$ is an irreducible character of the direct sum that composes $\text{Ind}_B^G \omega_{\mu\mu}$.

The Irreducible Component ϑ_μ of $\text{Ind}_B^G \omega_{\mu\mu}$

We know that for each character μ of R^\times , the character $\text{Ind}_B^G \omega_{\mu\mu}$ of G is the sum of three distinct irreducible characters. Therefore for each $g \in G$, $\text{Ind}_B^G \omega_{\mu\mu}(g) = \psi_\mu(g) + \tilde{\vartheta}_\mu(g) + \vartheta_\mu(g)$; therefore, the values for the character ϑ_μ of G are equal to the values $\text{Ind}_B^G \omega_{\mu\mu}(g) - \psi_\mu(g) - \tilde{\vartheta}_\mu(g)$ for each $g \in G$. The values of ϑ_μ are included in Table 4-2.

The three characters ψ_μ , $\tilde{\vartheta}_\mu$ and ϑ_μ are distinct and irreducible, and their sum gives $\text{Ind}_B^G \omega_{\mu\mu}$; thus we have found all the irreducible characters involved in the decomposition of $\text{Ind}_B^G \omega_{\mu\mu}$. For each of the $p(p-1)$ distinct $\mu \in R^\times$, the character $\text{Ind}_B^G \omega_{\mu\mu}$ is also distinct. In addition, the det map from B to R^\times is surjective and thus characters of the form ψ_μ and $\tilde{\vartheta}_\mu$ are distinct for each $\mu \in R^\times$. As ϑ_μ is determined by subtraction, ϑ_μ is also distinct for each $\mu \in R^\times$. Therefore, there are $p(p-1)$ distinct irreducible characters of each of the types ψ_μ , $\tilde{\vartheta}_\mu$ and ϑ_μ .

r	$\text{Ind}_B^G \omega_{\mu\mu}(r)$	$\psi_\mu(r)$	$\tilde{\vartheta}_\mu(r)$	$\vartheta_\mu(r)$
I_α	$p(p+1)\mu(\alpha^2)$	$\mu(\alpha^2)$	$p\mu(\alpha^2)$	$(p^2-1)\mu(\alpha^2)$
B_α	$p\mu(\alpha^2)$	$\mu(\alpha^2)$	0	$(p-1)\mu(\alpha^2)$
$T_{\alpha,\delta}$	$2\mu(\alpha\delta)$	$\mu(\alpha\delta)$	$\mu(\alpha\delta)$	0
$T'_{\alpha,\beta}$	0	$\mu(\alpha^2 - \epsilon\beta^2)$	$-\mu(\alpha^2 - \epsilon\beta^2)$	0
$RT_{\alpha,\beta}$	0	$\mu(\alpha^2 - p\beta^2)$	0	$-\mu(\alpha^2 - p\beta^2)$
$RT'_{\alpha,\beta}$	0	$\mu(\alpha^2 - p\epsilon\beta^2)$	0	$-\mu(\alpha^2 - p\epsilon\beta^2)$
$RI_{\alpha,\beta}$	$2p\mu(\alpha^2)$	$\mu(\alpha^2)$	$p\mu(\alpha^2)$	$(p-1)\mu(\alpha^2)$
RB_α	$p\mu(\alpha^2)$	$\mu(\alpha^2)$	$p\mu(\alpha^2)$	$-\mu(\alpha^2)$
$RI'_{\alpha,\beta}$	0	$\mu(\alpha^2)$	$p\mu(\alpha^2)$	$-(p+1)\mu(\alpha^2)$
Number	$p(p-1)$	$p(p-1)$	$p(p-1)$	$p(p-1)$

Table 4-2: The irreducible characters that comprise $\text{Ind}_B^G \omega_{\mu\mu}$

4.2.2 Case 2: $\mu\nu'$ trivial on U_1 , non-trivial on R^\times and both μ and ν trivial on U_1 .

Assume both μ and ν are trivial on U_1 and that $\mu\nu'$ is trivial on U_1 but is non-trivial on R^\times , then $\mu \neq \nu$. Consider the character $\text{Ind}_B^{\tilde{G}} \omega_{\bar{\mu}\bar{\nu}} \circ \pi$ of G , where $\text{Ind}_B^{\tilde{G}} \omega_{\bar{\mu}\bar{\nu}}$ is the irreducible character of $\tilde{G} = GL_2(\mathbb{Z}/p\mathbb{Z})$ from Table 3-3 with $\bar{\mu}$ and $\bar{\nu}$ as in Lemma 4.1. Table 4-3 shows the values of $\text{Ind}_B^{\tilde{G}} \omega_{\bar{\mu}\bar{\nu}} \circ \pi$ on G , and Theorem 3.4 tells us that $\text{Ind}_B^{\tilde{G}} \omega_{\bar{\mu}\bar{\nu}} \circ \pi$ is an irreducible character of G . Further, Table B-5 shows that $\langle \text{Ind}_B^G \omega_{\mu\nu}, \text{Ind}_B^{\tilde{G}} \omega_{\bar{\mu}\bar{\nu}} \circ \pi \rangle = 1$; therefore, $\text{Ind}_B^{\tilde{G}} \omega_{\bar{\mu}\bar{\nu}} \circ \pi$ is an irreducible character in the direct sum that composes $\text{Ind}_B^G \omega_{\mu\nu}$. As $\text{Ind}_B^G \omega_{\mu\nu} = (\text{Ind}_B^{\tilde{G}} \omega_{\bar{\mu}\bar{\nu}} \circ \pi) + \varphi_{\mu\nu}$ for a character $\varphi_{\mu\nu}$ of G , the values of $\varphi_{\mu\nu}$ can be found by subtracting $\text{Ind}_B^{\tilde{G}} \omega_{\bar{\mu}\bar{\nu}} \circ \pi(g)$ from $\text{Ind}_B^G \omega_{\mu\nu}(g)$ for each $g \in G$. Table 4-3 summarizes these values.

There are $\frac{1}{2}(p-1)(p-2)$ pairs $\{\mu, \nu\} \subset R^\times$ such that $\mu \neq \nu$ and both μ and ν are trivial on U_1 . Therefore, there are $\frac{1}{2}(p-1)(p-2)$ characters $\text{Ind}_B^G \omega_{\mu\nu}$ where both μ and ν are trivial on U_1 , and for each there are distinct irreducible characters $\text{Ind}_B^{\tilde{G}} \omega_{\bar{\mu}\bar{\nu}}$ and $\varphi_{\mu\nu}$.

r	$\text{Ind}_B^G \omega_{\mu\nu}(r)$	$\text{Ind}_B^{\tilde{G}} \omega_{\bar{\mu}\bar{\nu}} \circ \pi(r)$	$\varphi_{\mu\nu}(r)$
I_α	$p(p+1)\mu(\alpha)\nu(\alpha)$	$(p+1)\mu(\alpha)\nu(\alpha)$	$(p^2-1)\mu(\alpha)\nu(\alpha)$
B_α	$p\mu(\alpha)\nu(\alpha)$	$\mu(\alpha)\nu(\alpha)$	$(p-1)\mu(\alpha)\nu(\alpha)$
$T_{\alpha,\delta}$	$\mu(\alpha)\nu(\delta) + \mu(\delta)\nu(\alpha)$	$\mu(\alpha)\nu(\delta) + \mu(\delta)\nu(\alpha)$	0
$T'_{\alpha,\beta}$	0	0	0
$RT_{\alpha,\beta}$	0	$\mu(\alpha)\nu(\alpha)$	$-\mu(\alpha)\nu(\alpha)$
$RT'_{\alpha,\beta}$	0	$\mu(\alpha)\nu(\alpha)$	$-\mu(\alpha)\nu(\alpha)$
$RI_{\alpha,\beta}$	$2p\mu(\alpha)\nu(\alpha)$	$(p+1)\mu(\alpha)\nu(\alpha)$	$(p-1)\mu(\alpha)\nu(\alpha)$
RB_α	$p\mu(\alpha)\nu(\alpha)$	$(p+1)\mu(\alpha)\nu(\alpha)$	$-\mu(\alpha)\nu(\alpha)$
$RI'_{\alpha,\beta}$	0	$(p+1)\mu(\alpha)\nu(\alpha)$	$-(p+1)\mu(\alpha)\nu(\alpha)$
Number	$\frac{1}{2}(p-1)(p-2)$	$\frac{1}{2}(p-1)(p-2)$	$\frac{1}{2}(p-1)(p-2)$

Table 4-3: The irreducible characters that comprise $\text{Ind}_B^G \omega_{\mu\nu}$ when both μ and ν are trivial on U_1 .

4.2.3 Case 3: $\mu\nu'$ trivial on U_1 , non-trivial on R^\times and μ or ν nontrivial on U_1

Suppose that $\mu\nu'$ is trivial on U_1 but that either μ or ν is non-trivial on U_1 . This will occur when $\mu = \sigma\nu$ for some character σ of R^\times that is trivial on U_1 ; note that as $\mu\nu'$ is non-trivial on R^\times , $\mu \neq \nu$ and thus $\sigma \neq 1$. The values of $\text{Ind}_B^G \omega_{\mu\nu}$ in this case are included in Table 4-4; these values are found by replacing $\mu(x)$ with $\sigma(x)\nu(x)$ in column 2 of Table 4-1, and by recognizing that $\sigma(x)$ is affected only by the value of $x \pmod{\mathfrak{m}}$.

As in the previous case, $\langle \text{Ind}_B^G \omega_{\mu\nu}, \text{Ind}_B^G \omega_{\mu\nu} \rangle = 2$ as shown in Table B-1. Observe that in this case $\text{Ind}_B^G \omega_{\mu\nu} = \text{Ind}_B^G \omega_{\sigma 1} \cdot \psi_\nu$ where again $\mu = \sigma\nu$. However, since both σ and 1 are trivial on U_1 , we have $\text{Ind}_B^G \omega_{\sigma 1} = \text{Ind}_B^{\tilde{G}} \omega_{\sigma 1} \circ \pi + \varphi_{\sigma 1}$ by Case 2. Therefore, $\text{Ind}_B^G \omega_{\mu\nu} = (\text{Ind}_B^{\tilde{G}} \omega_{\sigma 1} \circ \pi \cdot \psi_\nu) + (\varphi_{\sigma 1} \cdot \psi_\nu)$ by distribution. As $\text{Ind}_B^{\tilde{G}} \omega_{\sigma 1} \circ \pi \cdot \psi_\nu$ and $\varphi_{\sigma 1} \cdot \psi_\nu$ are both characters of G , they are positive integer linear combinations of irreducible characters; since they comprise $\text{Ind}_B^G \omega_{\mu\nu}$ which is the direct sum of only two irreducible characters, they are necessarily irreducible.

There are $(p-1)^2$ choices for ν not trivial on U_1 and $(p-2)$ choices for $\sigma \neq 1$ trivial on U_1 . Hence there are $\frac{1}{2}(p-2)(p-1)^2$ characters $\text{Ind}_B^G \omega_{\mu\nu}$ in Case 3. In addition, there are $\frac{1}{2}(p-2)(p-1)^2$ characters of the form $\text{Ind}_B^{\tilde{G}} \omega_{\sigma 1} \circ \pi \cdot \psi_\nu$ and $\frac{1}{2}(p-2)(p-1)^2$ characters of

r	$\text{Ind}_B^G \omega_{\mu\nu}(r)$	$\text{Ind}_B^{\tilde{G}} \omega_{\bar{\sigma}1} \circ \pi \cdot \psi_\nu(r)$	$\varphi_{\sigma 1} \cdot \psi_\nu(r)$
I_α	$p(p+1)\sigma(\alpha)\nu(\alpha^2)$	$(p+1)\sigma(\alpha)\nu(\alpha^2)$	$(p^2-1)\sigma(\alpha)\nu(\alpha^2)$
B_α	$p\sigma(\alpha)\nu(\alpha^2)$	$\sigma(\alpha)\nu(\alpha^2)$	$(p-1)\sigma(\alpha)\nu(\alpha^2)$
$T_{\alpha,\delta}$	$(\sigma(\alpha) + \sigma(\delta))\nu(\alpha\delta)$	$(\sigma(\alpha) + \sigma(\delta))\nu(\alpha\delta)$	0
$T'_{\alpha,\beta}$	0	0	0
$RT_{\alpha,\beta}$	0	$\sigma(\alpha)\nu(\alpha^2 - p\beta^2)$	$-\sigma(\alpha)\nu(\alpha^2 - p\beta^2)$
$RT'_{\alpha,\beta}$	0	$\sigma(\alpha)\nu(\alpha^2 - p\epsilon\beta^2)$	$\sigma(\alpha)\nu(\alpha^2 - p\epsilon\beta^2)$
$RI_{\alpha,\beta}$	$2p\sigma(\alpha)\nu(\alpha^2)$	$(p+1)\sigma(\alpha)\nu(\alpha^2)$	$(p-1)\sigma(\alpha)\nu(\alpha^2)$
RB_α	$p\sigma(\alpha)\nu(\alpha^2)$	$(p+1)\sigma(\alpha)\nu(\alpha^2)$	$-\sigma(\alpha)\nu(\alpha^2)$
$RI'_{\alpha,\beta}$	0	$(p+1)\sigma(\alpha)\nu(\alpha^2)$	$-(p+1)\sigma(\alpha)\nu(\alpha^2)$
Number	$\frac{1}{2}(p-2)(p-1)^2$	$\frac{1}{2}(p-2)(p-1)^2$	$\frac{1}{2}(p-2)(p-1)^2$

Table 4-4: The irreducible characters that comprise $\text{Ind}_B^G \omega_{\mu\nu}$ when $\mu\nu'$ is trivial on U_1 , $\mu = \sigma\nu$ where $\sigma \neq 1$ is trivial on U_1 and ν is not trivial on U_1 .

the form $\varphi_{\sigma 1} \cdot \psi_\nu$.

4.2.4 Case 4: $\mu\nu'$ is not trivial on U_1

Table B-6 shows that $\langle \text{Ind}_B^G \omega_{\mu\nu}, \text{Ind}_B^G \omega_{\mu\nu} \rangle = 1$ when $\mu\nu'$ is not trivial on U_1 . Therefore, in this case $\text{Ind}_B^G \omega_{\mu\nu}$ is irreducible; the values of this character are in Table 4-1. The number of such irreducible characters is equal to one half the number of ordered pairs in $R^\times \times R^\times$ minus the number of pairs $\{\mu, \nu\}$ involved in Cases 1, 2 and 3, namely $\frac{1}{2} [p^2(p-1)^2 - p(p-1) - (p-1)(p-2) - (p-1)^2(p-2)] = \frac{1}{2}p(p-1)^3$.

4.3 Conclusion

The complete list of irreducible characters of this chapter, along with the number of distinct characters of each type, are listed in Tables 4-5 and 4-6. This accounts for $\frac{1}{2}p(p-1)(p^2+3)$ of the $p(p-1)(p^2+p+1)$ irreducible characters of G .

r	$\psi_\mu(r)$	$\tilde{\vartheta}_\mu(r)$	$\vartheta_\mu(r)$	$\text{Ind}_B^G \omega_{\mu\nu}$
I_α	$\mu(\alpha^2)$	$p\mu(\alpha^2)$	$(p^2 - 1)\mu(\alpha^2)$	$(p^2 + p)\mu(\alpha)\nu(\alpha)$
B_α	$\mu(\alpha^2)$	0	$(p - 1)\mu(\alpha^2)$	0
$T_{\alpha,\delta}$	$\mu(\alpha\delta)$	$\mu(\alpha\delta)$	0	$\mu(\alpha)\nu(\delta) + \mu(\delta)\nu(\alpha)$
$T'_{\alpha,\beta}$	$\mu(\alpha^2 - \epsilon\beta^2)$	$-\mu(\alpha^2 - \epsilon\beta^2)$	0	0
$RT_{\alpha,\beta}$	$\mu(\alpha^2 - p\beta^2)$	0	$-\mu(\alpha^2 - p\beta^2)$	0
$RT'_{\alpha,\beta}$	$\mu(\alpha^2 - p\epsilon\beta^2)$	0	$-\mu(\alpha^2 - p\epsilon\beta^2)$	0
$RI_{\alpha,\beta}$	$\mu(\alpha^2)$	$p\mu(\alpha^2)$	$(p - 1)\mu(\alpha^2)$	$p [\mu(\alpha + p\beta)\nu(\alpha - p\beta) + \mu(\alpha - p\beta)\nu(\alpha + p\beta)]$
RB_α	$\mu(\alpha^2)$	$p\mu(\alpha^2)$	$-\mu(\alpha^2)$	$p\mu(\alpha)\nu(\alpha)$
$RI'_{\alpha,\beta}$	$\mu(\alpha^2)$	$p\mu(\alpha^2)$	$-(p + 1)\mu(\alpha^2)$	0
Number	$p(p - 1)$	$p(p - 1)$	$p(p - 1)$	$\frac{1}{2}p(p - 1)^3$

Table 4-5: Irreducible characters of $GL_2(\mathbb{Z}/p^2\mathbb{Z})$: Columns 2, 3 and 4 are the irreducible characters from Case 1, while column 5 is from Case 4.

r	$\text{Ind}_B^G \omega_{\mu\nu} \circ \pi(r)$	$\varphi_{\mu\nu}(r)$	$\text{Ind}_B^G \omega_{\delta 1} \cdot \psi_\nu(r)$	$\varphi_{\sigma 1} \cdot \psi_\nu(r)$
I_α	$(p+1)\mu(\alpha)\nu(\alpha)$	$(p^2-1)\mu(\alpha)\nu(\alpha)$	$(p+1)\sigma(\alpha)\nu(\alpha^2)$	$(p^2-1)\sigma(\alpha)\nu(\alpha^2)$
B_α	$\mu(\alpha)\nu(\alpha)$	$(p-1)\mu(\alpha)\nu(\alpha)$	$\sigma(\alpha)\nu(\alpha^2)$	$(p-1)\sigma(\alpha)\nu(\alpha^2)$
$T_{\alpha,\delta}$	$\mu(\alpha)\nu(\delta) + \mu(\delta)\nu(\alpha)$	0	$(\sigma(\alpha) + \sigma(\delta))\nu(\alpha\delta)$	0
$T'_{\alpha,\beta}$	0	0	0	0
$RT_{\alpha,\beta}$	$\mu(\alpha)\nu(\alpha)$	$-\mu(\alpha)\nu(\alpha)$	$\sigma(\alpha)\nu(\alpha^2 - p\beta^2)$	$-\sigma(\alpha)\nu(\alpha^2 - p\beta^2)$
$RT'_{\alpha,\beta}$	$\mu(\alpha)\nu(\alpha)$	$-\mu(\alpha)\nu(\alpha)$	$\sigma(\alpha)\nu(\alpha^2 - p\epsilon\beta^2)$	$\sigma(\alpha)\nu(\alpha^2 - p\epsilon\beta^2)$
$RI_{\alpha,\beta}$	$(p+1)\mu(\alpha)\nu(\alpha)$	$(p-1)\mu(\alpha)\nu(\alpha)$	$(p+1)\sigma(\alpha)\nu(\alpha^2)$	$(p-1)\sigma(\alpha)\nu(\alpha^2)$
RB_α	$(p+1)\mu(\alpha)\nu(\alpha)$	$-\mu(\alpha)\nu(\alpha)$	$(p+1)\sigma(\alpha)\nu(\alpha^2)$	$-\sigma(\alpha)\nu(\alpha^2)$
$RI'_{\alpha,\beta}$	$-(p+1)\mu(\alpha)\nu(\alpha)$	$-(p+1)\mu(\alpha)\nu(\alpha)$	$(p+1)\sigma(\alpha)\nu(\alpha^2)$	$-(p+1)\sigma(\alpha)\nu(\alpha^2)$
Number	$\frac{1}{2}(p-1)(p-2)$	$\frac{1}{2}(p-1)(p-2)$	$\frac{1}{2}(p-2)(p-1)^2$	$\frac{1}{2}(p-2)(p-1)^2$

Table 4-6: Irreducible characters of $GL_2(\mathbb{Z}/p^2\mathbb{Z})$: Columns 2 and 3 contain the irreducible characters of Case 2, while columns 4 and 5 are the irreducible characters from case 3.

Chapter 5

Conclusion

In this thesis, we have determined the decomposition of the character $\text{Ind}_B^G \omega_{\mu\nu}$ into a direct sum of irreducible characters of $G = GL_2(\mathbb{Z}/p^2\mathbb{Z})$. This task is divided into four cases that depend on the pair $\{\mu, \nu\}$ of irreducible characters of the ring $R^\times = (\mathbb{Z}/p^2\mathbb{Z})^\times$.

The first partition into cases occurred when we determined the values of $\text{Ind}_B^G \omega_{\mu\nu}$ on G . We discovered that these values depend upon whether or not the character $\mu\nu'$ of R^\times defined as $\mu(\alpha)\nu(\alpha^{-1})$ for $\alpha \in R^\times$ is trivial on the subgroup U_1 of R^\times , where U_1 is the kernel in R^\times of the reduction map $\phi : G \rightarrow GL_2(\mathbb{Z}/p\mathbb{Z})$.

A further division into cases occurred while taking the inner product of $\text{Ind}_B^G \omega_{\mu\nu}$ with itself when $\mu\nu'$ is trivial on U_1 . When $\mu = \nu$, this inner product is 3, and hence $\text{Ind}_B^G \omega_{\mu\nu}$ is the direct sum of three distinct irreducibles (Case 1). When $\mu\nu'$ is trivial on U_1 but not on R^\times , this inner product is 2, and $\text{Ind}_B^G \omega_{\mu\nu}$ is the direct sum of two irreducibles (Cases 2 and 3). When $\mu\nu'$ is not trivial on U_1 , the character $\text{Ind}_B^G \omega_{\mu\nu}$ is irreducible (Case 4). In Case 2, both μ and ν are trivial on U_1 , and $\text{Ind}_B^G \omega_{\mu\nu}$ decomposes as a character of $GL_2(\mathbb{Z}/p\mathbb{Z})$ extended to G , with the remaining irreducible character found by subtraction. In Case 3, μ and ν are not trivial on U_1 , but $\mu = \sigma\nu$ for some character $\sigma \neq 1$ of R^\times trivial on U_1 . In Cases 1, 2 and 3, irreducible characters of $GL_2(\mathbb{Z}/p\mathbb{Z})$ are used to produce irreducible characters of G .

A natural next step is to calculate the remaining characters of G , as the irreducible characters of Chapter 4 account for only $\frac{1}{2}p(p-1)(p^2+3)$ of the $p(p-1)(p^2+p+1)$ irreducible characters of G . In [9], Kutzko determined the irreducible characters of $SL_2(\mathbb{Z}/p^n\mathbb{Z})$, a subgroup of $GL_2(\mathbb{Z}/p^n\mathbb{Z})$. The construction of these characters for $n = 2$ might be considered to find further characters of G . In addition, Kutzko's characters of $SL_2(\mathbb{Z}/p^2\mathbb{Z})$ can be

induced to form characters of G ; however, prior to performing these calculations, we do not know how many of these will produce characters distinct from those we have already found.

In addition, for $G_n = GL_2(\mathbb{Z}/p^n\mathbb{Z})$ and μ and ν characters of the ring $(\mathbb{Z}/p^n\mathbb{Z})^\times$, a similar approach might be used inductively on n to determine the decomposition of the characters $\text{Ind}_{B_n}^{G_n} \omega_{\mu\nu}$. If the necessary characters of G_{n-1} are known, then we can use the same approach of extending these to characters of G_n and identifying these as characters in the direct sum composing $\text{Ind}_{B_n}^{G_n} \omega_{\mu\nu}$. Subtracting these known characters from $\text{Ind}_{B_n}^{G_n} \omega_{\mu\nu}$ will produce characters of G_n with smaller inner product; hopefully, these characters will be easier to decompose.

The characters and representations of G can be applied to number theory. In [1], a relation between the induced representations of $GL_2(\mathbb{Z}/p^2\mathbb{Z})$ is defined, which implies a relation between the jacobians of some modular curves of level p^2 . A similar relationship is discussed in [2] for $GL_2(\mathbb{F}_p)$ where \mathbb{F}_p is the finite field with p elements.

Appendix A

Character Values of $\text{Ind}_B^G \omega_{\mu\nu}$

In this appendix we determine the values of the character $\text{Ind}_B^G \omega_{\mu\nu}$ on G . As before, G will denote the group $GL_2(\mathbb{Z}/p^2\mathbb{Z})$, and the notation of Appendix A will be that of Chapter 4.

A.1 Preliminaries

Recall from Section 4.2 the Borel subgroup B of G ; this group has order $|B| = p^4(p-1)^2$. To find the character values $\text{Ind}_B^G \omega_{\mu\nu}$ on G , it will help to locate the $p^4(p-1)^2$ elements of B in G with respect to their conjugacy class in G . Table A-1 lists the elements $b \in B$ conjugate to each class representative r , along with an element $g \in G$ such that $g^{-1}rg = b$. Each of these elements of B are distinct; from Table A-2, we can see that there are $|B|$ of these elements, and hence all of B is found.

r	$b \in B : b \simeq_G r$	$g \in G : b = g^{-1}rg$
I_α	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$	$\forall g \in G$
B_α	$\begin{pmatrix} \alpha + kp & \gamma \\ 0 & \alpha - kp \end{pmatrix}_{\substack{\gamma \in R^\times \\ k=0, \dots, p-1}}$	$\begin{pmatrix} 1 & 0 \\ kp & \gamma \end{pmatrix}$
$T_{\alpha, \delta}$	$\begin{pmatrix} \alpha & \gamma \\ 0 & \delta \end{pmatrix}_{\gamma \in R}$ $\begin{pmatrix} \delta & \gamma \\ 0 & \alpha \end{pmatrix}_{\gamma \in R}$	$\begin{pmatrix} 1 & \frac{\gamma}{\alpha - \delta} \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ 1 & \frac{\gamma}{\delta - \alpha} \end{pmatrix}$
$T'_{\alpha, \beta}$	\emptyset	none
$RT_{\alpha, \beta}$	\emptyset	none
$RT'_{\alpha, \beta}$	\emptyset	none
$RI_{\alpha, \beta}$	$\begin{pmatrix} \alpha \pm p\beta & kp \\ 0 & \alpha \mp p\beta \end{pmatrix}_{k=0, \dots, p-1}$	$\begin{pmatrix} \pm\beta & k - \beta \\ 1 & \pm 1 \end{pmatrix}_{k \not\equiv 2\beta \pmod{m}}$ $\begin{pmatrix} \pm\beta & 2\beta \\ 1 & 0 \end{pmatrix}_{k \equiv 2\beta \pmod{m}}$
RB_α	$\begin{pmatrix} \alpha & np \\ 0 & \alpha \end{pmatrix}_{n=1, \dots, p-1}$	$\begin{pmatrix} 0 & n \\ 1 & 1 \end{pmatrix}$
$RI'_{\alpha, \beta}$	\emptyset	none

Table A-1: Elements of B in G : Elements of B in each conjugacy class C_r of G are found by conjugating C_r 's class representative r by a given element $g \in G$. Observe that each such $b \in B$ is distinct. The number of $b \in B$ found is counted in Table A-2, which verifies that these are all the elements of B .

r	Number of $C_r \in \mathcal{T}$	$ \{b \in B b \simeq_G r\} $	Product
I_α	$p(p-1)$	1	$p(p-1)$
B_α	$p(p-1)$	$p^2(p-1)$	$p^3(p-1)2$
$T_{\alpha,\delta}$	$\frac{1}{2}p^2(p-1)(p-2)$	$2p^2$	$p^4(p-1)(p-2)$
$T'_{\alpha,\beta}$	$\frac{1}{2}p^3(p-1)$	0	0
$RT_{\alpha,\beta}$	$\frac{1}{2}p(p-1)^2$	0	0
$RT'_{\alpha\beta}$	$\frac{1}{2}p(p-1)^2$	0	0
$RI_{\alpha,\beta}$	$\frac{1}{2}p(p-1)^2$	$2p$	$p^2(p-1)^2$
RB_α	$p(p-1)$	$(p-1)$	$p(p-1)^2$
$RI'_{\alpha,\beta}$	$\frac{1}{2}p(p-1)^2$	0	0
Total			$p^4(p-1)^2 = B $

Table A-2: Distribution of $b \in B$ across the conjugacy classes of G : Recall that \mathcal{S} is the set of all conjugacy classes C_r of G , and that \mathcal{S} is partitioned by the nine different types \mathcal{T} of classes. Included are the number of conjugacy classes of each type along with the orders of the sets $\{b \in B : b \simeq_G r\}$ for each class C_r . The sum $\sum_{C_r \in \mathcal{S}} |\{b \in B : b \simeq_G r\}| = \sum_{T \subset \mathcal{S}} \sum_{C_r \in T} |\{b \in B : b \simeq_G r\}|$ gives the total number of $b \in B$ found in Table A-1; note that this number is equal to the order of B , proving that these are all the elements of B .

A.2 Values of $\text{Ind}_B^G \omega_{\mu\nu}$ on G

From (2.6), we have

$$\text{Ind}_B^G \omega_{\mu\nu}(u) = \frac{1}{|B|} |C_G(u)| \sum_{\substack{b \in B \\ b \simeq_G u}} \omega_{\mu\nu}(u);$$

we use this expression, together with Table A-1, to determine the values of $\text{Ind}_B^G \omega_{\mu\nu}$. The classes $T'_{\alpha,\beta}$, $RT_{\alpha,\beta}$, $RT'_{\alpha,\beta}$ and $RI'_{\alpha,\beta}$ do not intersect B and hence their values under $\text{Ind}_B^G \omega_{\mu\nu}$ are 0; we isolate the calculations for the remaining classes below.

$$\boxed{\text{Ind}_B^G \omega_{\mu\nu}(I_\alpha)}$$

$$\begin{aligned} &= \frac{1}{|B|} |C_G(I_\alpha)| \sum_{\substack{b \in B \\ b \simeq_G I_\alpha}} \omega_{\mu\nu}(b) \\ &= \frac{p^5(p+1)(p-1)^2}{p^4(p-1)^2} \mu(\alpha) \nu(\alpha) \\ &= p(p+1) \mu(\alpha) \nu(\alpha) \end{aligned}$$

$$\boxed{\text{Ind}_B^G \omega_{\mu\nu}(B_\alpha)}$$

$$\begin{aligned} &= \frac{1}{|B|} |C_G(B_\alpha)| \sum_{\substack{b \in B \\ b \simeq_G B_\alpha}} \omega_{\mu\nu}(b) \\ &= \frac{p^3(p-1)}{p^4(p-1)^2} \sum_{\gamma \in R^\times} \sum_{k=0}^{p-1} \mu(\alpha + kp) \nu(\alpha - kp) \\ &= \frac{p^3(p-1)}{p^4(p-1)^2} p(p-1) \sum_{k=0}^{p-1} \mu(\alpha + kp) \nu(\alpha - kp) \end{aligned}$$

Notice that $(\alpha - kp) = \alpha^2(\alpha + kp)^{-1}$, and thus

$$\nu(\alpha - kp) = \nu(\alpha^2) \nu((\alpha + kp)^{-1}).$$

Recall from Section 4.1 the character $\mu\nu'$ and the subgroup U_1 of R^\times . In addition, recall from Theorem 2.12 that the sum of a non-trivial character over an abelian group is 0. Using

this, we have

$$\begin{aligned}
 \text{Ind}_B^G \omega_{\mu\nu}(B_\alpha) &= \nu(\alpha^2) \sum_{k=0}^{p-1} \mu\nu'(\alpha + kp) \\
 &= \nu(\alpha^2) \mu\nu'(\alpha) \sum_{k=0}^{p-1} \mu\nu'(1 + \alpha^{-1}kp) \\
 &= \mu(\alpha) \nu(\alpha) \sum_{x \in U_1} \mu\nu'(x),
 \end{aligned}$$

which decomposes into two cases.

Case 1: if $\mu\nu'$ is trivial on U_1 then

$$\text{Ind}_B^G \omega_{\mu\nu}(B_\alpha) = p\mu(\alpha)\nu(\alpha).$$

Case 2: if $\mu\nu'$ is non-trivial on U_1 then

$$\text{Ind}_B^G \omega_{\mu\nu}(B_\alpha) = 0.$$

$$\boxed{\text{Ind}_B^G \omega_{\mu\nu}(T_{\alpha,\delta})}$$

$$\begin{aligned}
 &= \frac{1}{|B|} |C_G(T_{\alpha,\delta})| \sum_{\substack{b \in B \\ b \simeq_G T_{\alpha,\delta}}} \omega_{\mu\nu}(b) \\
 &= \frac{p^2(p-1)^2}{p^4(p-1)^2} \sum_{\gamma \in R} (\mu(\alpha)\nu(\delta) + \mu(\delta)\nu(\alpha)) \\
 &= \frac{p^2(p-1)^2}{p^4(p-1)^2} p^2 (\mu(\alpha)\nu(\delta) + \mu(\delta)\nu(\alpha)) \\
 &= \mu(\alpha)\nu(\delta) + \mu(\delta)\nu(\alpha).
 \end{aligned}$$

$$\boxed{\text{Ind}_B^G \omega_{\mu\nu}(RI_{\alpha,\beta})}$$

$$\begin{aligned}
 &= \frac{1}{|B|} |C_G(RI_{\alpha,\beta})| \sum_{\substack{b \in B \\ b \simeq_G RI_{\alpha,\beta}}} \omega_{\mu\nu}(b) \\
 &= \frac{p^4(p-1)^2}{p^4(p-1)^2} \sum_{k=0}^{p-1} (\mu(\alpha + p\beta)\nu(\alpha - p\beta) + \mu(\alpha - p\beta)\nu(\alpha + p\beta)) \\
 &= p(\mu(\alpha + p\beta)\nu(\alpha - p\beta) + \mu(\alpha - p\beta)\nu(\alpha + p\beta)).
 \end{aligned}$$

Again we have $(\alpha - kp) = \alpha^2(\alpha + kp)^{-1}$, so $\nu(\alpha - kp) = \nu(\alpha^2)\nu((\alpha + kp)^{-1})$ and $\mu(\alpha - kp) = \mu(\alpha^2)\mu((\alpha + kp)^{-1})$. Therefore,

$$\text{Ind}_B^G \omega_{\mu\nu}(RI_{\alpha,\beta}) = p(\mu\nu'(\alpha + p\beta)\nu(\alpha^2) + \mu'\nu(\alpha + p\beta)\mu(\alpha^2)).$$

Case 1: if $\mu\nu'$ and $\mu'\nu$ are trivial on U_1 then

$$\text{Ind}_B^G \omega_{\mu\nu}(RI_{\alpha,\beta}) = p(\mu\nu'(\alpha)\nu(\alpha^2) + \mu'\nu(\alpha)\mu(\alpha^2)) = 2p\mu(\alpha)\nu(\alpha).$$

Case 2: if $\mu\nu'$ is not trivial on U_1 then

$$\text{Ind}_B^G \omega_{\mu\nu}(RI_{\alpha,\beta}) = p(\mu\nu'(\alpha + p\beta)\nu(\alpha^2) + \mu'\nu(\alpha + p\beta)\mu(\alpha^2)).$$

$$\boxed{\text{Ind}_B^G \omega_{\mu\nu}(RB_\alpha)}$$

$$\begin{aligned} &= \frac{1}{|B|} |C_G(RB_\alpha)| \sum_{\substack{b \in B \\ b \simeq_G RB_\alpha}} \omega_{\mu\nu}(b) \\ &= \frac{p^5(p-1)}{p^4(p-1)^2} \sum_{n=1}^{p-1} \mu(\alpha)\nu(\alpha) \\ &= p \mu(\alpha)\nu(\alpha). \end{aligned}$$

The results of the above calculations are summarized in Table A-3.

r	$\text{Ind}_B^G \omega_{\mu\nu}(r)$ for $\mu\nu'$ trivial on U_1	$\text{Ind}_B^G \omega_{\mu\nu}(r)$ for $\mu\nu'$ non-trivial on U_1
I_α	$p(p+1)\mu(\alpha)\nu(\alpha)$	$p(p+1)\mu(\alpha)\nu(\alpha)$
B_α	$p\mu(\alpha)\nu(\alpha)$	0
$T_{\alpha,\delta}$	$\mu(\alpha)\nu(\delta) + \mu(\delta)\nu(\alpha)$	$\mu(\alpha)\nu(\delta) + \mu(\delta)\nu(\alpha)$
$T'_{\alpha,\beta}$	0	0
$RT_{\alpha,\beta}$	0	0
$RT'_{\alpha,\beta}$	0	0
$RI_{\alpha,\beta}$	$2p\mu(\alpha)\nu(\alpha)$	$p[\mu(\alpha + \beta p)\nu(\alpha - \beta p) + \mu(\alpha - \beta p)\nu(\alpha + \beta p)]$
RB_α	$p\mu(\alpha)\nu(\alpha)$	$p\mu(\alpha)\nu(\alpha)$
$RI'_{\alpha,\beta}$	0	0

Table A-3: Values of $\text{Ind}_B^G \omega_{\mu\nu}$ on G : Summarized in this table are the character values $\text{Ind}_B^G \omega_{\mu\nu}$ both when the character $\mu\nu'$ is trivial and non-trivial on U_1 .

Appendix B

Inner Products of Characters of G

In this appendix, the inner products discussed in Chapter 4 are calculated. Recall (3.1) and (3.2),

$$\langle \chi, \varphi \rangle = \frac{1}{|G|} \sum_{T \subset \mathcal{S}} \sum_{C_r \in T} |C_r| \chi(r) \varphi(r^{-1}),$$

and

$$\langle \chi, \chi \rangle = \frac{1}{|G|} \sum_{T \subset \mathcal{S}} \sum_{C_r \in T} |C_r| |\chi(r)|^2,$$

where χ and φ are characters of G , \mathcal{S} is the set of all conjugacy classes C_r of G , and r is the class representative of C_r . Recall that \mathcal{S} is partitioned by the nine types of conjugacy classes T of G . Recall also that for a character χ of G , $\chi(g)^* = \chi(g^{-1})$ for all $g \in G$, where c^* denotes the complex conjugate of $c \in \mathbb{C}$. The characters $\mu\nu'$ and $\mu'\nu$, and the subgroup U_1 of R^\times are as defined in Section 4.1.

Many of the right hand entries of the following tables are easily calculated and can be verified by the reader. The more involved sums are indicated by \star and by \blacklozenge and are calculated at the end of this appendix.

$C_r \in \mathcal{T}$	$ C_r $	$\text{Ind}_B^G \omega_{\mu\nu}(r)$	$\sum_{C_r \in \mathcal{T}} C_r \text{Ind}_B^G \omega_{\mu\nu}(r) ^2$ $\mu\nu'$ trivial on R^\times	$\sum_{C_r \in \mathcal{T}} C_r \text{Ind}_B^G \omega_{\mu\nu}(r) ^2$ $\mu\nu'$ trivial on U_1 , and $\mu \neq \nu$
$I_\alpha \quad \alpha \in R^\times$	1	$p(p+1)\mu(\alpha)\nu(\alpha)$	$p^3(p-1)(p+1)^2$	$p^3(p-1)(p+1)^2$
$B_\alpha \quad \alpha \in R^\times$	$p^2(p-1)(p+1)$	$p\mu(\alpha)\nu(\alpha)$	$p^5(p-1)^2(p+1)$	$p^5(p-1)^2(p+1)$
$T_{\alpha,\delta} \quad \begin{smallmatrix} \{\alpha,\delta\} \subseteq R^\times \\ \alpha \neq \delta \pmod{\mathfrak{m}} \end{smallmatrix}$	$p^3(p+1)$	$\mu(\alpha)\nu(\delta) + \mu(\delta)\nu(\alpha)$	$2p^5(p-1)(p+1)(p-2)^\star$	$p^5(p-1)(p+1)(p-3)^\star$
$T'_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R \\ \beta \in R^\times / \{\pm 1\} \end{smallmatrix}$	$p^3(p-1)$	0	0	0
$RT_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p^2(p-1)(p+1)$	0	0	0
$RT'_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p^2(p-1)(p+1)$	0	0	0
$RI_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p(p+1)$	$2p\mu(\alpha)\nu(\alpha)$	$2p^4(p-1)^2(p+1)$	$2p^4(p-1)^2(p+1)$
$RB_\alpha \quad \alpha \in R^\times$	$(p-1)(p+1)$	$p\mu(\alpha)\nu(\alpha)$	$p^3(p-1)^2(p+1)$	$p^3(p-1)^2(p+1)$
$RI'_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p(p-1)$	0	0	0
Total			$3p^5(p-1)^2(p+1)$	$2p^5(p-1)^2(p+1)$

Table B-1: $\langle \text{Ind}_B^G \omega_{\mu\nu}, \text{Ind}_B^G \omega_{\mu\nu} \rangle$, $\mu\nu'$ trivial on U_1

$C_\tau \in \mathcal{T}$	$ C_\tau $	$\text{Ind}_B^G \omega_{\mu\mu}(r)$	$\psi_\mu(r)$	$\sum_{C_\tau \in \mathcal{T}} C_\tau (\text{Ind}_B^G \omega_{\mu\mu}(r)) (\psi_\mu(r^{-1}))$
$I_\alpha \quad \alpha \in R^\times$	1	$p(p+1)\mu(\alpha^2)$	$\mu(\alpha^2)$	$p^3(p-1)(p+1)^2$
$B_\alpha \quad \alpha \in R^\times$	$p^2(p-1)(p+1)$	$p\mu(\alpha^2)$	$\mu(\alpha^2)$	$p^5(p-1)^2(p+1)$
$T_{\alpha,\delta} \quad \begin{smallmatrix} \{\alpha,\delta\} \subseteq R^\times \\ \alpha \neq \delta \pmod{\mathfrak{m}} \end{smallmatrix}$	$p^3(p+1)$	$2\mu(\alpha\delta)$	$\mu(\alpha\delta)$	$2p^5(p-1)(p+1)(p-2)$
$T'_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R \\ \beta \in R^\times / \{\pm 1\} \end{smallmatrix}$	$p^3(p-1)$	0	$\mu(\alpha^2 - \epsilon\beta^2)$	0
$RT_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p^2(p-1)(p+1)$	0	$\mu(\alpha^2 - p\beta^2)$	0
$RT'_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p^2(p-1)(p+1)$	0	$\mu(\alpha^2 - p\epsilon\beta^2)$	0
$RI_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p(p+1)$	$2p\mu(\alpha^2)$	$\mu(\alpha^2)$	$2p^4(p-1)^2(p+1)$
$RB_\alpha \quad \alpha \in R^\times$	$(p-1)(p+1)$	$p\mu(\alpha^2)$	$\mu(\alpha^2)$	$p^3(p-1)^2(p+1)$
$RI'_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p(p-1)$	0	$\mu(\alpha^2)$	0
Total				$p^5(p-1)^2(p+1)$

Table B-2: $\langle \text{Ind}_B^G \omega_{\mu\mu}, \psi_\mu \rangle$

$C_r \in \mathcal{T}$	$ C_r $	$\tilde{\vartheta}_\mu(r)$	$\sum_{C_r \in \mathcal{T}} C_r \tilde{\vartheta}_\mu(r) ^2$
$I_\alpha \quad \alpha \in R^\times$	1	$p \mu(\alpha^2)$	$p^3(p-1)$
$B_\alpha \quad \alpha \in R^\times$	$p^2(p-1)(p+1)$	0	0
$T_{\alpha,\delta} \quad \begin{smallmatrix} \{\alpha,\delta\} \subseteq R^\times \\ \alpha \neq \delta \pmod{\mathfrak{m}} \end{smallmatrix}$	$p^3(p+1)$	$\mu(\alpha\delta)$	$\frac{1}{2}p^5(p-1)(p+1)(p-2)$
$T'_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R \\ \beta \in R^\times / \{\pm 1\} \end{smallmatrix}$	$p^3(p-1)$	$-\mu(\alpha^2 - \epsilon\beta^2)$	$\frac{1}{2}p^6(p-1)^2$
$RT_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p^2(p-1)(p+1)$	0	0
$RT'_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p^2(p-1)(p+1)$	0	0
$RI_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p(p+1)$	$p \mu(\alpha^2)$	$\frac{1}{2}p^4(p-1)^2(p+1)$
$RB_\alpha \quad \alpha \in R^\times$	$(p-1)(p+1)$	$p \mu(\alpha^2)$	$p^3(p-1)^2(p+1)$
$RI'_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p(p-1)$	$p \mu(\alpha^2)$	$\frac{1}{2}p^4(p-1)^3$
Total			$p^5(p-1)^2(p+1)$

Table B-3: $\langle \tilde{\vartheta}_\mu, \tilde{\vartheta}_\mu \rangle$

$C_r \in \mathcal{T}$	$ C_r $	$\text{Ind}_B^G \omega_{\mu\mu}(r)$	$\tilde{\vartheta}_\mu(r)$	$\sum_{C_r \in \mathcal{T}} C_r (\text{Ind}_B^G \omega_{\mu\mu}(r)) (\tilde{\vartheta}_\mu((r)^{-1}))$
$I_\alpha \quad \alpha \in R^\times$	1	$p(p+1)\mu(\alpha^2)$	$p\mu(\alpha^2)$	$p^3(p-1)(p+1)$
$B_\alpha \quad \alpha \in R^\times$	$p^2(p-1)(p+1)$	$p\mu(\alpha^2)$	0	0
$T_{\alpha,\delta} \quad \begin{smallmatrix} \{\alpha,\delta\} \subseteq R^\times \\ \alpha \neq \delta \pmod{\mathfrak{m}} \end{smallmatrix}$	$p^3(p+1)$	$2\mu(\alpha\delta)$	$\mu(\alpha\delta)$	$p^5(p-1)(p+1)(p-2)$
$T'_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R \\ \beta \in R^\times / \{\pm 1\} \end{smallmatrix}$	$p^3(p-1)$	0	$-\mu(\alpha^2 - \epsilon\beta^2)$	0
$RT_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p^2(p-1)(p+1)$	0	0	0
$RT'_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p^2(p-1)(p+1)$	0	0	0
$RI_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p(p+1)$	$2p\mu(\alpha^2)$	$p\mu(\alpha^2)$	$p^4(p-1)^2(p+1)$
$RB_\alpha \quad \alpha \in R^\times$	$(p-1)(p+1)$	$p\mu(\alpha^2)$	$p\mu(\alpha^2)$	$p^3(p-1)^2(p+1)$
$RI'_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p(p-1)$	0	$p\mu(\alpha^2)$	0
Total				$p^5(p-1)^2(p+1)$

Table B-4: $\langle \text{Ind}_B^G \omega_{\mu\mu}, \tilde{\vartheta}_\mu \rangle$

$C_r \in \mathcal{T}$	$ C_r $	$\text{Ind}_B^G \omega_{\mu\nu}(\tau)$	$\text{Ind}_B^G \omega_{\bar{\mu}\bar{\nu}} \circ \pi(\tau)$	$\sum_{C_r \in \mathcal{T}} C_r (\chi_1(\tau)) (\chi_2(\tau^{-1}))$
$I_\alpha \quad \alpha \in R^\times$	1	$p(p+1)\mu(\alpha)\nu(\alpha)$	$(p+1)\mu(\alpha)\nu(\alpha)$	$p^2(p-1)(p+1)^2$
$B_\alpha \quad \alpha \in R^\times$	$p^2(p-1)(p+1)$	$p\mu(\alpha)\nu(\alpha)$	$\mu(\alpha)\nu(\alpha)$	$p^4(p-1)^2(p+1)$
$T_{\alpha,\delta} \quad \begin{smallmatrix} \{\alpha,\delta\} \subseteq R^\times \\ \alpha \neq \delta \pmod{\mathfrak{m}} \end{smallmatrix}$	$p^3(p+1)$	$\mu(\alpha)\nu(\delta) + \mu(\delta)\nu(\alpha)$	$\mu(\alpha)\nu(\delta) + \mu(\delta)\nu(\alpha)$	$p^5(p-1)(p+1)(p-3)^\star$
$T'_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R \\ \beta \in R^\times / \{\pm 1\} \end{smallmatrix}$	$p^3(p-1)$	0	0	0
$RT_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p^2(p-1)(p+1)$	0	$\mu(\alpha)\nu(\alpha)$	0
$RT'_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p^2(p-1)(p+1)$	0	$\mu(\alpha)\nu(\alpha)$	0
$RI_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p(p+1)$	$2p\mu(\alpha)\nu(\alpha)$	$(p+1)\mu(\alpha)\nu(\alpha)$	$p^3(p-1)^2(p+1)^2$
$RB_\alpha \quad \alpha \in R^\times$	$(p-1)(p+1)$	$p\mu(\alpha)\nu(\alpha)$	$(p+1)\mu(\alpha)\nu(\alpha)$	$p^2(p-1)^2(p+1)^2$
$RI'_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p(p-1)$	0	$(p+1)\mu(\alpha)\nu(\alpha)$	0
Total				$p^5(p-1)^2(p+1)$

Table B-5: $\langle \text{Ind}_B^G \omega_{\mu\nu}, \text{Ind}_B^G \omega_{\bar{\mu}\bar{\nu}} \circ \pi \rangle$, μ, ν trivial on U_1 . We use the shorthand $\chi_1 = \text{Ind}_B^G \omega_{\mu\nu}$ and $\chi_2 = \text{Ind}_B^G \omega_{\bar{\mu}\bar{\nu}} \circ \pi$.

$C_\tau \in \mathcal{T}$	$ C_\tau $	$\text{Ind}_B^G \omega_{\mu\nu}(r)$	$\sum_{C_\tau \in \mathcal{T}} C_\tau \text{Ind}_B^G \omega_{\mu\nu}(r) ^2$
$I_\alpha \quad \alpha \in R^\times$	1	$p(p+1)\mu(\alpha)\nu(\alpha)$	$p^3(p-1)(p+1)^2$
$B_\alpha \quad \alpha \in R^\times$	$p^2(p-1)(p+1)$	0	0
$T_{\alpha,\delta} \quad \begin{smallmatrix} \{\alpha,\delta\} \subseteq R^\times \\ \alpha \neq \delta \pmod{\mathfrak{m}} \end{smallmatrix}$	$p^3(p+1)$	$\mu(\alpha)\nu(\delta) + \mu(\delta)\nu(\alpha)$	$p^5(p-1)(p+1)(p-2)^\star$
$T'_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R \\ \beta \in R^\times / \{\pm 1\} \end{smallmatrix}$	$p^3(p-1)$	0	0
$RT_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p^2(p-1)(p+1)$	0	0
$RT'_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p^2(p-1)(p+1)$	0	0
$RI_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p(p+1)$	$p \mu(\alpha + p\beta)\nu(\alpha - p\beta) + p \mu(\alpha - p\beta)\nu(\alpha + p\beta)$	$p^4(p-1)(p+1)(p-2)$
$RB_\alpha \quad \alpha \in R^\times$	$(p-1)(p+1)$	$p\mu(\alpha)\nu(\alpha)$	$p^3(p-1)^2(p+1)$
$RI'_{\alpha,\beta} \quad \begin{smallmatrix} \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})^\times / \{\pm 1\} \end{smallmatrix}$	$p(p-1)$	0	0
Total			$p^5(p-1)^2(p+1)$

Table B-6: $\langle \text{Ind}_B^G \omega_{\mu\nu}, \text{Ind}_B^G \omega_{\mu\nu} \rangle$, $\mu\bar{\nu}$ not trivial on U_1

$$\star = \sum_{\substack{T_{\alpha,\delta} \\ \{\alpha,\delta\} \subseteq R^\times \\ \alpha \neq \delta \pmod{\mathfrak{m}}}} p^3(p+1) \left(\mu(\alpha)\nu(\delta) + \mu(\delta)\nu(\alpha) \right) \left(\mu(\alpha^{-1})\nu(\delta^{-1}) + \mu(\delta^{-1})\nu(\alpha^{-1}) \right)$$

Case 1: $\mu = \nu$. In this case \star becomes

$$\begin{aligned} & \sum_{\substack{T_{\alpha,\delta} \\ \{\alpha,\delta\} \subseteq R^\times \\ \alpha \neq \delta \pmod{\mathfrak{m}}}} p^3(p+1) \left(2\mu(\alpha\delta)2\mu(\alpha^{-1}\delta^{-1}) \right) \\ &= 4p^3(p+1) \frac{1}{2} p^2(p-1)(p-2) \\ &= 2p^5(p-1)(p+1)(p-2). \end{aligned}$$

Otherwise, if $\mu \neq \nu$ we see that \star is equal to

$$\begin{aligned} & p^3(p+1) \sum_{\substack{T_{\alpha,\delta} \\ \{\alpha,\delta\} \subseteq R^\times \\ \alpha \neq \delta \pmod{\mathfrak{m}}}} \left(2 + \mu(\alpha)\nu(\delta)\mu(\delta^{-1})\nu(\alpha^{-1}) + \mu(\delta)\nu(\alpha)\mu(\alpha^{-1})\nu(\delta^{-1}) \right) \\ &= p^3(p+1) \left[2\frac{1}{2}p^2(p-1)(p-2) + \sum_{\substack{T_{\alpha,\delta} \\ \{\alpha,\delta\} \subseteq R^\times \\ \alpha \neq \delta \pmod{\mathfrak{m}}}} \left(\mu(\alpha)\nu(\delta)\mu(\delta^{-1})\nu(\alpha^{-1}) + \mu(\delta)\nu(\alpha)\mu(\alpha^{-1})\nu(\delta^{-1}) \right) \right] \end{aligned}$$

Notice that the set $\{T_{\alpha,\delta} : \{\alpha,\delta\} \subset R^\times, \alpha \neq \delta \pmod{\mathfrak{m}}\}$ is contained in D , the diagonal subgroup of G with elements denoted by $D_{\alpha,\delta}$. The sum over the $T_{\alpha,\delta}$'s can be replaced with the sum

$$\frac{1}{2} \left[\sum_D \left(\mu\nu'(\alpha)\mu'\nu(\delta) + \mu\nu'(\delta)\mu'\nu(\alpha) \right) - \sum_{\substack{D_{\alpha,\delta} \\ \alpha,\delta \in R^\times \\ \alpha = \delta \pmod{\mathfrak{m}}}} \left(\mu\nu'(\alpha)\mu'\nu(\delta) + \mu\nu'(\delta)\mu'\nu(\alpha) \right) \right],$$

multiplying by $1/2$ since $T_{\alpha,\delta} \simeq_G T_{\delta,\alpha}$. Therefore, \star becomes

$$\begin{aligned} & p^3(p+1) \left[p^2(p-1)(p-2) + \frac{1}{2} \sum_D \left(\mu\nu'(\alpha)\mu'\nu(\delta) + \mu\nu'(\delta)\mu'\nu(\alpha) \right) \right. \\ & \quad \left. - \frac{1}{2} \sum_{\substack{D_{\alpha,\delta} \\ \alpha,\delta \in R^\times \\ \alpha = \delta \pmod{\mathfrak{m}}}} \left(\mu\nu'(\alpha)\mu'\nu(\delta) + \mu\nu'(\delta)\mu'\nu(\alpha) \right) \right]. \end{aligned}$$

From Theorem 2.12, recall that for any non-trivial character χ of an abelian group G , $\sum_{g \in G} \chi(g) = 0$. As $\mu \neq \nu$ the character $\mu\nu' \times \mu'\nu$ is not trivial on D , and thus \star becomes:

$$\begin{aligned}
& p^3(p+1) \left[p^2(p-1)(p-2) + 0 - \frac{1}{2} \sum_{\substack{D_{\alpha,\delta} \\ \alpha, \delta \in R^\times \\ \alpha = \delta \pmod{m}}} (\mu\nu'(\alpha)\mu'\nu(\delta) + \mu\nu'(\delta)\mu'\nu(\alpha)) \right] \\
&= p^3(p+1) \left[p^2(p-1)(p-2) - \frac{1}{2} \sum_{\alpha \in R^\times} \sum_{k=0}^{p-1} (\mu\nu'(\alpha)\mu'\nu(\alpha + kp) + \mu\nu'(\alpha + kp)\mu'\nu(\alpha)) \right] \\
&= p^3(p+1) \left[p^2(p-1)(p-2) - \frac{1}{2} \sum_{\alpha \in R^\times} \sum_{k=0}^{p-1} \mu\nu'(\alpha)\mu'\nu(\alpha) (\mu'\nu(1 + \alpha^{-1}kp) + \mu\nu'(1 + \alpha^{-1}kp)) \right] \\
&= p^3(p+1) \left[p^2(p-1)(p-2) - \frac{1}{2} \sum_{\alpha \in R^\times} \sum_{x \in U_1} (\mu'\nu(x) + \mu\nu'(x)) \right].
\end{aligned}$$

The sum \star breaks down into two more cases.

Case 2 When $\mu\nu'$ and $\mu'\nu$ are non-trivial characters of U_1 , $\sum_{x \in U_1} [\bar{\mu}\nu(x) + \mu\bar{\nu}(x)] = 0$ and thus \star is

$$\begin{aligned}
& p^3(p+1)[p^2(p-1)(p-2) - 0] \\
&= p^5(p-1)(p+1)(p-2).
\end{aligned}$$

Case 3 When $\bar{\mu}\nu$ and $\mu\bar{\nu}$ are trivial characters of U_1 , $\sum_{x \in U_1} [\bar{\mu}\nu(x) + \mu\bar{\nu}(x)] = 2|U_1| = 2p$ and thus \star is

$$\begin{aligned}
& p^3(p+1)[p^2(p-1)(p-2) - \frac{1}{2}p(p-1)2p] \\
&= p^5(p-1)(p+1)(p-3).
\end{aligned}$$

In \blacklozenge the characters $\mu\nu'$ and $\mu'\nu$ are non-trivial on U_1 .

$$\begin{aligned} \blacklozenge &= p(p+1) \sum_{\substack{RI_{\alpha,\beta} \\ \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})/\{\pm 1\}}} \left| p \left[\mu(\alpha + p\beta)\nu(\alpha - p\beta) + \mu(\alpha - p\beta)\nu(\alpha + p\beta) \right] \right|^2 \\ &= p^3(p+1) \sum_{\substack{RI_{\alpha,\beta} \\ \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})/\{\pm 1\}}} \left(2 + \mu\nu'(\alpha + p\beta)\mu'\nu(\alpha - p\beta) + \mu\nu'(\alpha - p\beta)\mu'\nu(\alpha + p\beta) \right) \end{aligned}$$

Recognizing that $(\alpha + p\beta)^{-1} = \frac{\alpha - p\beta}{\alpha^2} = \alpha^{-1} - p\beta\alpha^{-2}$ and, $(\alpha^{-1} - p\beta\alpha^{-2})(\alpha - p\beta) = 1 - 2p\beta\alpha^{-1}$, \blacklozenge becomes

$$\begin{aligned} &p^3(p+1) \left[2\frac{1}{2}p(p-1)^2 + \sum_{\substack{RI_{\alpha,\beta} \\ \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})/\{\pm 1\}}} \mu\nu'(\alpha + p\beta)\mu'\nu(\alpha + p\beta) (\mu\nu'(1 - 2p\beta\alpha^{-1}) + \mu'\nu(1 - 2p\beta\alpha^{-1})) \right] \\ &= p^3(p+1) \left[p(p-1)^2 + \sum_{\substack{RI_{\alpha,\beta} \\ \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})/\{\pm 1\}}} (\mu\nu'(1 - 2p\beta\alpha^{-1}) + \mu'\nu(1 - 2p\beta\alpha^{-1})) \right] \\ &= p^3(p+1) \left[p(p-1)^2 + \sum_{\substack{RI_{\alpha,\beta} \\ \alpha \in R^\times \\ \beta \in (R/\mathfrak{m})/\{\pm 1\}}} (\mu\nu'(1 - 2p\beta\alpha^{-1}) + \mu'\nu(1 - 2p\beta\alpha^{-1})) \right] \\ &= p^3(p+1) \left[p(p-1)^2 + \sum_{\beta \in (R/\mathfrak{m})^\times/\{\pm 1\}} \left[\sum_{\xi \in R} (\mu\nu'(1 - 2p\beta\xi) + \mu'\nu(1 - 2p\beta\xi)) \right. \right. \\ &\quad \left. \left. - \sum_{\substack{\xi \in R \\ \xi \equiv 0 \pmod{\mathfrak{m}}}} (\mu\nu'(1 - 2p\beta\xi) + \mu'\nu(1 - 2p\beta\xi)) \right] \right]. \end{aligned}$$

Notice that only the value of $\xi \pmod{\mathfrak{m}}$ is involved in the above sum; equivalently,

$$\xi = \bar{\xi} \pmod{\mathfrak{m}} \Rightarrow \mu\bar{\nu}(1 - 2p\beta\xi) + \bar{\mu}\nu(1 - 2p\beta\xi) = \mu\bar{\nu}(1 - 2p\beta\bar{\xi}) + \bar{\mu}\nu(1 - 2p\beta\bar{\xi}).$$

Hence, the sum over $\xi \in R$ of $\mu\bar{\nu}(1 - 2p\beta\xi) + \bar{\mu}\nu(1 - 2p\beta\xi)$ is $p \sum_{u \in U_1} (\mu\bar{\nu}(u) + \bar{\mu}\nu(u))$. Notice also, when $\xi \equiv 0 \pmod{\mathfrak{m}}$ that $1 - 2p\beta\xi = 1$. Recall that in this case, $\mu\nu'$ and $\mu'\nu$ are

characters of R^\times that are not trivial on U_1 , so the sum $\sum_{u \in U_1} (\mu \bar{\nu}(u) + \bar{\mu} \nu(u)) = 0$. Now \blacklozenge becomes:

$$\begin{aligned}
&= p^3(p+1) \left[p(p-1)^2 + \sum_{\beta \in (R/\mathfrak{m})^\times / \{\pm 1\}} \left[p \sum_{u \in U_1} (\mu \bar{\nu}(u) + \bar{\mu} \nu(u)) - \sum_{\substack{\xi \in R \\ \xi \equiv 0 \pmod{\mathfrak{m}}}} 2 \right] \right] \\
&= p^3(p+1) \left[p(p-1)^2 + \frac{1}{2}(p-1)(0-2p) \right] \\
&= p^4(p+1)(p-1)(p-2).
\end{aligned}$$

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