# COMPUTABLE ENUMERATION AND THE PROBLEM OF REPETITION

by

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# Abstract

The thesis is concerned with the question of characterizing those computably enumerable (c.e.) classes of computably enumerable sets which have a computable enumeration without repetition (an injective enumeration). This problem can be traced back to 1958, when Friedberg proved that the class of all computably enumerable sets can be injectively enumerated. We go beyond the scope of the literature by extending the study to the problem of characterizing the c.e. classes which are c.e. with a bounded number of repetitions and with finite repetitions.

An investigation of the question restricted to classes of cofinite sets leads to a satisfying answer in a special case but demonstrates the difficulties of the general problem.

The property of a class of being c.e. with at most finite repetitions is shown to behave more naturally than injective enumerability. We prove an extension theorem with a characterization as a corollary. We also show that the corresponding statement does not hold for the property of being c.e. with bounded repetitions.

By fixing an enumeration $\mathcal{C}$ of all computably enumerable classes it is possible to define index sets corresponding to properties of such classes. The index sets related to problems of enumerability with various constraints on the number of repetitions are shown to be complete at their natural level in the Arithmetical Hierarchy. For example

$$\{e : \mathcal{C}^{(e)} \text{ has a computable enumeration without repetition}\}$$

is $\Sigma_5$-complete, and

$$\{e : \mathcal{C}^{(e)} \text{ has a computable enumeration with finite repetitions}\}$$

is $\Sigma_6$-complete.

# Dedication

*For Tamara*

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# Contents

# Chapter 1

# Introduction

Consider a computer program that runs forever, possibly using an infinite amount of memory, and generating a sequence of infinitely many numbers. The set of numbers enumerated by the program is called *computably enumerable*. It is easy to alter the program so that the same numbers are enumerated without repetitions. The program is modified so that it maintains an internal list of the numbers generated so far. When a new number is generated it is written to output only if it has not already appeared.

Now consider a computer program that generates *infinitely many sequences of numbers* — the program generates infinitely many pairs of numbers and we understand that the first number of a pair is to be added to the sequence specified by the second number of the pair. We form a set from each of these sequences and say that the program enumerates the sequence of these sets. We form a class from all the sets and say that the program *computably enumerates* this class. The class is said to be *computably enumerable*. A basic result of recursion theory is that not all such classes are computably enumerable without repetitions.

The simplest example of a class that can be enumerated without repetitions is the class of all singletons $\{\{x\} : x \in \omega\}$. The simplest example of a computably enumerable class that cannot be enumerated without repetitions is

$$\{\{2x, 2x+1\} : x \in A\} \cup \{\{2x\}, \{2x+1\} : x \notin A\},$$

where $A$ is a computably enumerable set whose complement $\omega - A$ is not computably enumerable. This class is computably enumerable[1] because $A$ is. It is not computably enumerable without repetitions because otherwise $\omega - A$ would be computably enumerable.[2]

This thesis is concerned with one of the classical questions of recursion theory: which computably enumerable classes can be computably enumerated without repetitions?

## 1.1 Outline of the thesis

The known results regarding the subject of this thesis can be divided into two parts. One part consists of a variety of sufficient conditions for a class to be computably enumerable without repetitions. The second part consists of a collection of computably enumerable classes which are not computably enumerable without repetitions. After fixing the notation in the next section we will present both parts in Sections 1.3 and 1.4. The main difficulty of the characterization problem is the lack of suitable necessary conditions on classes with a computable enumeration without repetitions. We discuss this in Section 1.5. Section 1.6 deals with connections of injective enumerations to other areas of computability theory.

Let us define a class to be *$n$-computably enumerable*, *$\omega$-computably enumerable* if it has a computable enumeration in which each set occurs at most $n$ times, finitely many times, respectively. In Section 1.4 it will be seen that there are many cases with respect to the number of repetitions required in a computable enumeration of a computably enumerable class:

1. For every number $n > 1$ there are $(n+1)$-computably enumerable classes which are not $n$-computably enumerable.

2. There are classes which are not $n$-computably enumerable for any $n$ but are $\omega$-computably enumerable.

3. There are computably enumerable classes which are not $\omega$-computably enumerable – in every computable enumeration of such a class some set has to occur infinitely often.

[1] Let $V_{2x}$ be $\{2x\}$ or $\{2x, 2x+1\}$ according as $x \notin A$ or not and $V_{2x+1}$ be $\{2x+1\}$ or $\{2x+1, 2x\}$ according as $2x+1 \notin A$ or not. The class then is equal to $\{V_0, V_1, V_2, \dots\}$.

[2] Suppose there is a computable enumeration of this class with no repetitions. Then $x$ is in $\omega - A$ if and only if $2x$ and $2x+1$ are enumerated in different sets of this enumeration. Hence $\omega - A$ is computably enumerable, contradiction.

These results suggest that the property of a class to be computably enumerable without repetitions should be studied within the hierarchy $\mathcal{H}$. At the bottom of $\mathcal{H}$ are the 1-computably enumerable classes, followed by the 2-computably enumerable classes etc. and at the top are the classes which have computable enumerations with at most finite repetitions. For each of these properties it would be desirable to have a characterization. More specifically, we are interested in a characterization of the 1-computably enumerable classes which naturally generalizes across the hierarchy.

The results of this thesis fall into four groups. In Chapter 2 we roughly establish the form of the desired characterizations. In particular, we show that a first order characterization of the property "$n$-computably enumerable" in terms of computable predicates should be an existential property with four quantifier type alternations. It should be of the form $\exists\forall\exists\forall\exists \boldsymbol{R}$ where $\exists(\forall)$ means existential (universal) quantification, and $\boldsymbol{R}$ is a computable predicate. Then we show that a characterization of the property "$\omega$-computably enumerable" should be an existential property with five quantifier type alternations, i.e. of the form $\exists\forall\exists\forall\exists\forall \boldsymbol{R}$.

The characterization problem for classes of finite sets has found a solution due to Lachlan [15] (see Theorem 1.8). The restriction to classes of cofinite sets is studied in Chapter 3 as the next simplest case. We will construct three c.e. classes of cofinite sets with almost the same structure, two of which are 1-computably enumerable and the other not. This demonstrates the difficulties of the characterization problem. We also study a natural necessary property of 1-computably enumerable classes. It is sufficient for 1-computable enumerability in some cases, but fails in general. On the other hand, for classes with a designated bound on the co-cardinality we obtain a strong characterization.

The property of being $\omega$-computably enumerable turns out to behave more naturally than the properties "$n$-computably enumerable" with finite $n$. Chapter 4 presents an extension theorem for this property: the $\omega$-computable enumerability of a class $\mathcal{A}$ is shown to imply the $\omega$-computable enumerability for any c.e. class $\mathcal{C} \supseteq \mathcal{A}$, provided $\mathcal{C}$ and $\mathcal{A}$ stand in a certain relation to each other. As a corollary we obtain a characterization of the $\omega$-computably enumerable classes in terms of existence of a winning strategy for a certain game. We show that the natural analogue of this theorem fails when $n$ is finite.

## 1.2 Notation and definitions

We follow the standard notation of the literature on recursion theory, as can be found in [29, 22]. The set of natural numbers is denoted by $\omega$. Let $\varphi$ be a Gödel-numbering of the class of one place computable partial functions, and let $W_i = \text{Rng}(\varphi_i)$ be the $i$-th c.e. set. A class $\mathcal{C}$ of sets of natural numbers is called computably enumerable (c.e.), if its members are uniformly computably enumerable; this means there is a computable function $f$ such that $\{W_{f(i)} : i \in \omega\} = \mathcal{C}$. Equivalent definitions are:

1. There is a computable partial binary function $\alpha$ such that $\{\text{Rng}(\lambda s.\alpha(i,s)) : i \in \omega\} = \mathcal{C}$. Such a function $\alpha$ is then called an enumeration of $\mathcal{C}$ and $\alpha_i := \text{Rng}(\lambda n.\alpha(i,n))$, the $i$-th set of the enumeration.

2. There is an array $(\nu_s^{(i)})_{i,s\in\omega}$ of finite sets, such that the ternary predicate $x \in \nu_s^{(i)}$ is computable and $\{\bigcup_{s\in\omega} \nu_s^{(i)} : i \in \omega\} = \mathcal{C}$.

An enumeration is called *injective* if for different indices $i, j \in \omega$ the $i$-th and $j$-th set of the enumeration are different. Thus only infinite classes can have an injective enumeration.

A set $A$ is said to be repeated $x$ times by the enumeration $\alpha$ if the set $\{i : \alpha_i = A\}$ has cardinality $x$ ($x$ might be infinite). A computable enumeration $\alpha$ *witnesses that $\mathcal{A}$ is $n$-c.e. ($\omega$-c.e.)* if $\{i : \alpha_i = A\}$ has cardinality at most $n$ (is finite) for each $A \in \mathcal{A}$.

We use the following notations

| | |
|---|---|
| $\omega$ | the set of natural numbers |
| $P_n$ | the class of computable partial $n$-place functions |
| $R_n$ | the class of computable (total) $n$-place functions |
| $f(n)\downarrow, f(n)\uparrow$ | the partial function $f$ is defined at $n$, not defined at $n$, respectively |
| $\varphi$ | a Gödel-numbering of $P_1$, fixed throughout the thesis |
| $W_i$ | the $i$-th c.e. set $\text{Rng}(\varphi_i)$ |
| $\mathcal{W}$ | the class of all c.e. sets |
| $\Omega_i^{(e)}$ | the set $W_{\varphi_e(i)}$. If $\varphi_e(i)\uparrow$, then $\Omega_i^{(e)} = \emptyset$ |
| $\mathcal{C}^{(e)}$ | the class $\{\Omega_i^{(e)} : i \in \omega\}$ |

| | |
|---|---|
| $\langle\cdot,\dots,\cdot\rangle$ | fixed computable bijections between $\omega^n$ and $\omega$ |
| $x_{.n}$ | $x_n$, where $x = \langle x_1,\dots,x_n,\dots,x_m\rangle$ ($m$ being clear from the context). |
| $\langle n, A\rangle$ | the set $\{\langle n,a\rangle : a \in A\}, n \in \omega, A \subseteq \omega$ |
| $\langle n, m, A\rangle$ | the set $\{\langle n,m,a\rangle : a \in A\}, n, m \in \omega, A \subseteq \omega$ |
| $A_{.2}$ | the set $\{x \mid (\exists y)(\langle y,x\rangle \in A)\}$ |
| $A_{.3}$ | the set $\{x \mid (\exists y,z)(\langle y,z,x\rangle \in A)\}$ |
| $D$ | the canonical enumeration of the class of finite sets of numbers, such that $\{D_i : i \in \omega\} = \{A : A \subseteq_{\text{fin}} \omega\}$ |
| $\lvert A\rvert$ | the cardinality of $A$ |
| $2^A$ | the power set of $A$ |
| $\max(A)$ | the maximum of the set $A$. We let $\max(\emptyset) = -1$ |
| $A{\restriction}x$ | the set $A \cap \{0,\dots,x-1\}$ |
| $A_s$ | the set of numbers enumerated in $A$ after performing $s$ steps of its enumeration |
| $\widehat{A}$ | the smallest initial segment of $\omega$ containing $A$ |
| $A+n, n+A$ | the set $\{a+n : a \in A\}$ |
| $\Sigma^0_n$ | the class of sets $A$ of natural numbers for which there is a computable predicate $R$, such that $$A = \{x : (\exists a_1)(\forall a_2)\dots(Qa_n)(R(a_1,a_2,\dots,a_n,x))\},$$ where $Q$ denotes existential or universal quantification, according as $n$ is odd or even |
| $\Pi^0_n$ | the class of sets $A$ of natural numbers for which there is a computable predicate $R$, such that $$A = \{x : (\forall a_1)(\exists a_2)\dots(Qa_n)(R(a_1,a_2,\dots,a_n,x))\},$$ where $Q$ denotes existential or universal quantification, according as $n$ is even or odd |
| $\mathcal{P}_n$ | the set of classes $\mathcal{C} \subseteq \{X \subseteq \omega : \lvert\omega - X\rvert \le n\}$ |
| $\alpha \le_r \beta$ | $(\exists r \in R_1)(\forall i)(\alpha_i = \beta_{r(i)})$ |
| $L(\mathcal{C})$ | the upper semilattice $(\{[\alpha_i]_{\equiv_r} : \alpha \text{ enumerates } \mathcal{C}\}, \le_r)$. |

$(\Omega^{(e)})_{e\in\omega}$ is a computable enumeration of all computable enumerations of c.e. classes of c.e. subsets of $\omega$. $(\mathcal{C}^{(e)})_{e\in\omega}$ is a computable enumeration of all c.e. classes of c.e. subsets of $\omega$.

## 1.3 Sufficient conditions for injective enumerability

The study of the property of being injectively enumerable began in 1958 when Friedberg [2, Theorem 3] proved that $\mathcal{W}$, the class of all c.e. sets, can be computably enumerated without repetitions. His methods are fundamental for the proofs of many of the results of this section and also of this thesis. We give a short description of his approach so that the reader can understand the context of the sufficient conditions to follow.

Essentially, an injective enumeration $\psi$ of a given class $\mathcal{A}$ is constructed by copying the sets $\alpha_i$ of a given computable enumeration $\alpha$ of this class. In Friedberg's construction, $\alpha$ is chosen so that it enumerates $\mathcal{W}$. The enumeration $\psi$ has to satisfy:

1. $(\forall A \in \mathcal{A})(\exists i)(\psi_i = A)$ and
2. $(\forall i \neq j)(\psi_i \neq \psi_j)$.

The strategy to satisfy the first requirement is to assign a number $x$ to every number $i$. The number $x$ is called a *follower* of $i$. This is to designate the intention of enumerating the members of $\alpha_i$ in $\psi_x$. It is possible to monitor the enumeration of $\alpha_i$ and each time a number is enumerated in $\alpha_i$, it also is enumerated in $\psi_x$.

Of course, this action ignores the second requirement. If $\alpha_i = \alpha_{i'}$, then $i$ and $i'$ should not both have followers which last to the end of the construction. Therefore reassignment of certain followers during the course of the construction will certainly be necessary. The conflict between the two requirements is resolved by ensuring that only minimal $\alpha$-indices acquire permanent followers. The smallest index of a set in an enumeration is called a *minimal index*. A number $y$ is called a *permanent follower* of $i$, if $y$ is assigned to be the follower of $i$ at some stage of the construction, and does not change this status at later stages. The set of minimal indices of an enumeration is $\Sigma^0_2$ in general and therefore not accessible directly.

If $i$ is a minimal index, then there is a number $l$, such that $\alpha_i\restriction l \neq \alpha'_i\restriction l$ for all $i' < i$. Whenever a follower is assigned to a number $i$, the value of such a number $l$ is guessed in the variable $g(i)$.

At any stage $s$ it can occur that $\alpha_{i,s}\restriction g(i) = \alpha_{i',s}\restriction g(i)$ for some $i' < i$. In this case we release the present follower of $i$ and set up a new follower together with the higher guess $g(i) := g(i) + 1$. If $i$ is not a minimal index, every follower of $i$ is released, and so only minimal indices may receive permanent followers.

The problem is now reduced to determining a strategy for the released followers. We want to satisfy all requirements $P(i), Q(x)$ and $R(x)$:

$P(i)$. If $i$ is a minimal index, then $i$ has a permanent follower or $\alpha_i = \psi_x$ where $x$ is a released follower.

$Q(x)$. If $x$ is a released follower, then $\psi_x \in \mathcal{A}$.

$R(x)$. If $y < x$ then $\psi_y \neq \psi_x$.

The most important feature of Friedberg's method is to order these requirements linearly, for example:

$$P(0) < Q(0) < R(0) < P(1) < Q(1) < R(1) < P(2) < Q(2) < R(2) < \dots .$$

If in the course of the construction a conflict arises between two requirements then the lesser one in this order will prevail, and the other one will be injured. If we can arrange that such injuries occur only *finitely* often for each requirement, then the overall requirements of the construction will be satisfied. This approach has been called the *finite injury priority method* and is fundamental for all constructions in the area of this thesis. Moreover, it, together with sophisticated generalizations, is the central method of all of modern recursion theory.

In some cases, however, the requirements are not problematic. For example, if we start out with an enumeration for which the set of minimal indices is c.e., then the simple strategy of only assigning followers to the minimal indices is sufficient. For instance, the minimal indices of any computable enumeration of a class of graphs of computable functions form a computably enumerable set. Therefore all infinite c.e. classes of graphs of computable functions are injectively enumerable.

Friedberg's original construction has been refined and the reader can see in [22, p 230] how to treat the problem of released followers for the class of all c.e. sets.[3] In 1965, Pour-El and Putnam [27, Theorem 1] show that Friedberg's construction can be viewed as an embedding theorem: any

[3] Kummer has given the easiest proof of Friedberg's result, and we describe it on page 11.

infinite c.e. class has an extension by finite sets only, which is injectively enumerable. As the class of all c.e. sets is itself c.e. and already contains all finite sets, it is injectively enumerable. In particular they obtain the following theorem by making a slight change in Friedberg's construction.

**Theorem 1.1 (Pour-El, Putnam)** *Let $\mathcal{A}$ be an infinite c.e. class and $S$ an infinite c.e. set. Then there is an injectively enumerable class $\mathcal{B} \supseteq \mathcal{A}$ such that*

1. $X \in \mathcal{B} - \mathcal{A} \Rightarrow X$ *is finite and*

2. $X \in \mathcal{B} - \mathcal{A} \Rightarrow (\exists Y \in \mathcal{A})(X \subseteq Y \cup S)$.

A corollary of this theorem yields a sufficient condition for a class to have an injective enumeration:

**Corollary 1.2 (Pour-El, Putnam)** *Let $\mathcal{A}$ be an infinite c.e. class. If every finite subset of $\bigcup_{A \in \mathcal{A}} A$ is a member of $\mathcal{A}$, then $\mathcal{A}$ is injectively enumerable.*

*Proof.* Choose $S = \bigcup_{A \in \mathcal{A}} A$. □

This covers the c.e. classes containing all finite sets. For example, it follows that the class of all computable sets is injectively enumerable. Pour-El and Howard [26, Theorem 2] give the following structural criterion for injective enumerability in 1964. The condition asks for a computable partial function to provide information about the structure of a given class.

**Definition 1.3** [Pour-El, Howard] Let $\mathcal{A} \subseteq 2^\omega$ and denote the class of all finite subsets of members of $\mathcal{A}$ by $\mathcal{F}$. A function $h : \mathcal{F} \to \omega$ is called a *height function* for $\mathcal{A}$ if it satisfies the following three conditions:

1. $h$ is monotonic: $A \subseteq B \in \mathcal{F} \Rightarrow h(A) \leq h(B)$.

2. $h$ satisfies the ascending chain condition: for any sequence $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ of finite subsets of a member of $\mathcal{A}$ the sequence $h(A_0) \leq h(A_1) \leq h(A_2) \leq \cdots$ of associated heights is eventually constant.

3. For any $A \in \mathcal{F}$ there is $B \supset A$ in $\mathcal{F}$ such that $h(A) \neq h(B)$.

We identify a function $h : \mathcal{F} \to \omega$ with the partial function $h^* : \omega \to \omega$ defined by $h^*(i) := h(D_i)$.

**Theorem 1.4 (Pour-El, Howard)** *A c.e. class with a computable partial height function is injectively enumerable.*

From Theorem 1.4, Pour-El and Howard obtain the following sufficient conditions for injective enumerability [26, Corollary 1,3,4]:

**Corollary 1.5 (Pour-El, Howard)** *Let $\mathcal{A}$ be an infinite c.e. class.*

1. *If $\mathcal{A}$ consists only of finite sets and contains no member maximal with respect to inclusion, then $\mathcal{A}$ is injectively enumerable.*
2. *If $\mathcal{A}$ does not contain the set $\omega$, but every proper initial segment of $\omega$ has an extension in $\mathcal{A}$, then $\mathcal{A}$ is injectively enumerable.*
3. *If $\mathcal{A}$ is closed under finite unions but does not contain $\bigcup_{A\in\mathcal{A}} A$, then $\mathcal{A}$ is injectively enumerable.*

For 1, use the height function $h(i) := |D_i|$ and for 2 the height function

$$h(i) := 1 + \max\{x : \{0, \dots, x\} \subseteq D_i\},$$

where $\max(\emptyset) = 0$. The third part follows from the second by constructing a computable bijection between $\bigcup_{A\in\mathcal{A}} A$ and $\omega$. A more involved application of Theorem 1.4 can be found in Chapter 3, Theorem 3.7.

Theorem 1.4 was substantially strengthened by Lachlan in 1966. He defines [15, Definition 1] the property (E) for c.e. classes. Let $\mathcal{F}_i$ denote the class $\{W_j : j \in D_i\}$. A c.e. class $\mathcal{A}$ has the property (E) if there is a binary computable partial function $f \in P_2$, such that if $\mathcal{F}_i \subseteq \mathcal{A}$, then $f(i,j)$ is defined if and only if the class

$$(\mathcal{A} - \mathcal{F}_x) \cap \{X : X \supseteq D_j\}$$

is not empty, and then $f(i,j)$ is an $\Omega$-index of this class. In [15, Theorem 1, Lemma 5] and [16] Lachlan shows:

**Theorem 1.6 (Lachlan)** *An infinite c.e. class has the property (E) if and only if given a finite subclass $\mathcal{F} \subseteq \mathcal{A}$ the class $\mathcal{A} - \mathcal{F}$ is injectively enumerable uniformly in $\mathcal{F}$. Any c.e. class that has a computable partial height function satisfies (E).*

A criterion similar to those listed in Corollary 1.5 is given by Marchenkov in [20, Theorem 2].

**Theorem 1.7 (Marchenkov)** *Let $\mathcal{A}$ be a c.e. class. Suppose there is a c.e. subclass $\mathcal{B} \subseteq \mathcal{A}$ consisting of disjoint $\subseteq$-chains without maximal elements, such that every finite subset of a member of $\mathcal{A}$ has an extension in $\mathcal{B}$. Then $\mathcal{A}$ has an injective enumeration.*

All the conditions mentioned so far admit no obvious generalization to non-trivial sufficient conditions for $n$-computable enumerability or $\omega$-computable enumerability. The following two conditions do allow this, as they are both extension theorems of the form: if a class $\mathcal{A}$ has an injectively enumerable subclass $\mathcal{B}$ such that the pair $(\mathcal{A}, \mathcal{B})$ satisfies some property, then $\mathcal{A}$ is injectively enumerable.

The first one can be found in [15, Definition 2], and applies only to classes of finite sets. Lachlan defines the property (F) of c.e. classes. Informally[4], a c.e. class $\mathcal{C}$ has the property (F) if there is a winning strategy for Player II in the following game with up to $\omega$ rounds. At round $n$, player I specifies a finite set $F_n$. Then player II either specifies a c.e. set $V_n$ or does not specify any set. The game has $n$ rounds if and only if Player II specifies a set in rounds $n' < n$. Player II wins under the following two conditions.

1. If $V_n$ is specified at round $n$, then $F_n \subseteq V_n \in \mathcal{C}$ and $V_n \notin \{V_i : i < n\}$.

2. Player II specifies a set at round $n$ if and only if there exists $V$, $F_n \subseteq V \in \mathcal{C}$ such that $V \notin \{V_i : i < n\}$.

Otherwise Player I wins.

Condition (F) is necessary for injectively enumerable classes. From an injective enumeration $\alpha$ of $\mathcal{C}$ one can obtain a strategy as follows. Given finite sets $F_1, \dots, F_n$ determine the smallest $s$, if any, such that

$$\alpha_{0,s} \upharpoonright s, \dots, \alpha_{s,s} \upharpoonright s$$

contains $n$ extensions of $F_1, \dots, F_n$. At any stage of the search if an extension $\alpha_j$ for a set $F_i$ is found and $j$ is not reserved, reserve $j$ for $i$. If the search terminates, answer with the sequence of indices of $\alpha_{j_0}, \dots, \alpha_{j_n}$, where $j_i$ is the index reserved for $i$. In [15, Theorem 2] Lachlan shows:

[4] A formal definition is given on page 60.

**Theorem 1.8 (Lachlan)** *An infinite c.e. class of finite sets is injectively enumerable if and only if it satisfies (F).*

In Chapter 4 we show how this theorem can be seen as an extension theorem in the sense just described and generalize the property (F) and Theorem 1.8.

Another sufficient condition for injective enumerability in the form of an extension theorem is given by Kummer [7], [8, Extension Lemma] in 1989.

**Lemma 1.9 (Kummer)** *Let $\mathcal{A}$ be a c.e. class such that there are two disjoint c.e. subclasses $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ which cover $\mathcal{A}$ and satisfy*

1. *$\mathcal{A}_2$ is injectively enumerable and*
2. *every finite subset of a member of $\mathcal{A}_1$ has infinitely many extensions in $\mathcal{A}_2$.*

*Then $\mathcal{A}$ is injectively enumerable.*

The obvious generalization for $n$-computable enumerability also holds. The proof of this result is significantly simpler than the proofs of the other criteria. So it is remarkable that Kummer obtains Friedberg's theorem as an easy corollary [7, p 30], [8]. Choose

$$\mathcal{A}_1 := \{A : A \text{ is c.e. and has even or infinite cardinality}\}$$

and

$$\mathcal{A}_2 := \{A : A \text{ is finite with odd cardinality}\}.$$

$\mathcal{A}_1$ and $\mathcal{A}_2$ cover $\mathcal{W}$ and satisfy the other requirements of Lemma 1.9 to yield an injective enumeration of $\mathcal{W}$.

In 1965, Malt'sev and Wolf [19, 32] obtain the following criterion from a variation of Friedberg's construction.

**Theorem 1.10 (Malt'sev,Wolf)** *Let $\mathcal{A}$ be a c.e. class. If there is a canonically enumerable subclass $\mathcal{F}$ of finite sets such that*

1. *every member of $\mathcal{A}$ is the limit of an increasing sequence of sets from $\mathcal{F}$ and*

2. *every finite subset of a member of $\mathcal{A}$ has a proper extension in $\mathcal{F}$,*

*then $\mathcal{A}$ has an injective enumeration.*

Kummer [9, Theorem 6] contains the following generalization of this criterion:

**Theorem 1.11 (Kummer)** *If a c.e. class $\mathcal{A}$ has a canonically enumerable subclass $\mathcal{F}$ such that every finite subset of a member of $\mathcal{A} - \mathcal{F}$ has infinitely many extensions in $\mathcal{F}$, then $\mathcal{A}$ is injectively enumerable.*

Friedberg's original result can be obtained from each of these criteria. In most cases the application is clear. Only Theorems 1.4 and 1.6 cannot be applied to the class of all c.e. sets directly. A class containing the union of its members cannot have a height function and the classes $\mathcal{W} - \{W_i\}$ are not c.e. uniformly in $i$. However, $\mathcal{W} - \{\omega\}$ is easily seen to be c.e. and this class falls under Corollary 1.5.

## 1.4 Classes without injective enumerations

At the time of Friedberg's construction of an injective enumeration of the class of all c.e. sets, it was not clear whether all c.e. classes have injective enumerations or not — see the review [25] of Friedberg's paper [2]. The fact that there are c.e. classes without injective enumeration, whereas all infinite c.e. *sets* have computable enumerations without repetition, becomes plausible when one considers that equivalence of c.e. sets, i.e. the binary predicate $W_i = W_j$, is $\Pi^0_2$ and that equivalence of numbers, i.e. the predicate $i = j$, is clearly decidable.

The first examples of classes which have no injective enumeration can be found in [27, Theorem 2]. This paper also introduces the concept of $n$-computable enumerability for c.e. classes. Fix $n > 1$ and let $A$ be a c.e. set which is not computable. Define

$$\begin{aligned}\mathcal{A}_n := \{&\{ni\}, \{ni+1\}, \ldots, \{ni+(n-1)\} : i \notin A\} \cup \\ &\{\{ni, ni+1, \ldots, ni+(n-1)\} : i \in A\}.\end{aligned}$$

$\mathcal{A}_n$ is easily seen to be $n$-c.e. Given an $(n-1)$-computable enumeration of $\mathcal{A}_n$ one could compute $A$ — therefore $\mathcal{A}_n$ is not $(n-1)$-c.e. Another example is the class

$$\{\{x\} : x \notin A\} \cup \{\omega\}$$

which is computably enumerable, but not with at most finite repetitions [27, Theorem 4]. Suppose there is a computable enumeration $\gamma$ of this class in which the set $\omega$ appears finitely often. Then

$$\omega - A = \bigcup_{\gamma_i \neq \omega} \gamma_i$$

is computably enumerable, contradiction.

For the upper end of the hierarchy $\mathcal{H}$ we have the following three examples from [27]. By combining the classes $\mathcal{A}_n, n > 1$ we obtain a class of finite sets which is $\omega$-c.e. but not $n$-c.e. for any $n$ [27, Theorem 3]. Define

$$\mathcal{A} := \bigcup_{n>1} \{\langle n, A\rangle : A \in \mathcal{A}_n\}.$$

This class is c.e. with finite repetitions because the classes $\mathcal{A}_n$ are $n$-c.e. uniformly in $n$. It is not $n$-c.e. itself because at least one of its components is not.

Let $\mathcal{Q}$ be a $\Sigma^0_2$-predicate which is not $\Pi^0_2$. Then the class

$$\{\{x\} : \mathcal{Q}(x)\} \cup \{\{x, x+1\} : x \in \omega\}$$

is computably enumerable but not with at most finite repetitions [27, Theorem 6]. Any computable enumeration of this class must repeat at least one set infinitely often. Given any member, however, it is possible to computably enumerate the class such that this set is enumerated exactly once. This is because every member of the class is finite. For c.e. classes of infinite sets it is possible that any computable enumeration has to repeat every member infinitely often. In particular, Pour-El and Putnam show [27, Theorem 6a] that every element of the c.e. class

$$\{\{j : W_j = W_i \text{ is provable in Peano-Arithmetic}\} : i \in \omega\}^{5}$$

[5] We omit a proper formalization.

is repeated infinitely often in every computable enumeration of the class. In 1966, Young [33, Corollary 1] constructs a c.e. class which does not have any proper infinite c.e. subclass. See [1] for further elaboration.

The latest addition to this list is given by Kummer [8, Theorem 8]. He constructs a c.e. class $F$ of graphs of finite functions such that, for each $i$, $F$ is missing the graph of at most one finite function with $f(0) = i$, and such that $R_1 \cup F$ is computably enumerable but not injectively. Note that $F$ is injectively enumerable by Corollary 1.5 [10, p 69].

## 1.5 Necessary conditions for injective enumerability

The literature on the theory of c.e. classes does not contain any non-trivial necessary conditions for injective enumerability. In fact, there are not many *notions* defined for c.e. classes or computable enumerations of c.e. classes. However, there has been extensive study of the partial order $\leq_r$ defined on the collection of all computable enumerations of a c.e. class $\mathcal{C}$, see [17].

**Definition 1.12** Two enumerations $\alpha, \beta$ of a class $\mathcal{A}$ satisfy $\alpha \leq_r \beta$ if there is a computable function $r$ such that $\alpha_i = \beta_{r(i)}$ for all $i \in \omega$.

This relation is reflexive and transitive and so

$$\alpha \equiv_r \beta :\Leftrightarrow \alpha \leq_r \beta \text{ and } \beta \leq_r \alpha$$

is an equivalence relation, which is compatible with $\leq_r$. The resulting partial order is called $L(\mathcal{C})$. It forms an upper semilattice with supremum $[\alpha]_{\equiv_r} \oplus [\beta]_{\equiv_r}$ defined as $[\gamma]_{\equiv_r}$, where

$$\gamma_i := \begin{cases} \alpha_{i/2} & \text{if } i \text{ is even,} \\ \beta_{(i-1)/2} & \text{if } i \text{ is odd.} \end{cases}$$

Injective enumerations have minimal equivalence classes in this semilattice. Pour-El [24, Theorem 1] constructs two injective enumerations of $\mathcal{W}$ that are $\leq_r$-incomparable, thus showing that $\mathcal{L}(\mathcal{W})$ is not a lattice. Khutoretskii [13, Corollary 2] shows that there are infinitely many pairwise $\leq_r$-incomparable injective enumerations of $\mathcal{W}$, thus showing $\mathcal{L}(\mathcal{W})$ is infinite. An enumeration $\beta$ is called *positive* if the set $\{(i, j) : \beta_i = \beta_j\}$ is c.e. and *negative* if the set $\{(i, j) : \beta_i \neq \beta_j\}$ is c.e. An injective enumeration is both positive and negative, and positive enumerations are minimal in this partial order.

Khutoretskii [14, Corollary 1] gives a criterion for a c.e. class to have countably many pairwise non-equivalent minimal computable enumerations, each of which is not equivalent to a positive one. This criterion is satisfied by $\mathcal{W}$ [14, Corollary 2]. He also shows how to construct a positive enumeration of $\mathcal{W}$ [14, Example 2] such that no injective enumeration is below it with respect to $\leq_r$.

Every computable enumeration $\alpha$ of the class

$$\mathcal{A}_A := \{\{2x\}, \{2x+1\} : x \notin A\} \cup \{\{2x, 2x+1\} : x \in A\}$$

where $A$ is c.e., is positive[6] and therefore $\leq_r$-minimal. Because $\beta \leq_r \alpha$ for any computable enumerations[7] $\alpha, \beta$ of $\mathcal{A}_A$ all computable enumerations of $\mathcal{A}_A$ are also maximal — $|\mathcal{L}(\mathcal{A}_A)| = 1$. If $A$ is c.e. and not computable, $\mathcal{A}_A$ is not injectively enumerable. Thus the existence of a positive, minimal or maximal computable enumeration is not sufficient for injective enumerability. Indeed, the notions minimal, positive, and injective have little in common.

Any class with a negative computable enumeration is injectively enumerable, because the minimal indices of such an enumeration form a computably enumerable set (see page 7). The notion of negative enumeration has the following generalization:

**Definition 1.13** An enumeration $\beta$ is called *negative mod* $x$, where $x \in \omega \cup \{\omega\}$, if there is a c.e. set $A$ of pairs $(i, j)$ such that $\beta_i \neq \beta_j$ implies $(i, j) \in A$ and $|\{j \neq i : (i, j) \in A \text{ and } \beta_j = \beta_i\}| < x$ for every number $i$.

Observe that a class is $n$-computably enumerable if and only if it has a computable enumeration which is negative mod $n$, and similarly for computable enumerability with at most finite repetitions. This does give an equivalent definition of $n$-computable enumerability, but a very weak one.

Other simple necessary conditions for a c.e. class $\mathcal{A}$ to be injectively enumerable are

1. Any subclass $\mathcal{B}$, such that $\mathcal{A} - \mathcal{B}$ is finite, is c.e.

2. $\mathcal{A}$ has infinitely many c.e. subclasses.

[6] If $\alpha$ enumerates $\mathcal{A}_A$, $\alpha_i = \alpha_j$ if and only if $\alpha_i \cap \alpha_j \neq \emptyset$.
[7] Set $r(i) = ((\mu\langle n, s\rangle)(\alpha_{n,s} \cap \beta_{i,s} \neq \emptyset))_{.1}$.

3. $\mathcal{A}$ has an injectively enumerable subclass.

All of these conditions are satisfied, for example, by the c.e., non-injectively enumerable class $\mathcal{A}_A$, where $A$ is a c.e., non-computable set.

If we monitor an injective enumeration, we can generate a c.e. enumeration of the same class such that at any stage the present approximation to the enumeration is injective. We define

**Definition 1.14** A computable enumeration $\alpha$ is called *approximately injective* if for every $s > 0$, the finite sequence

$$\alpha_{0,s}, \alpha_{1,s}, \ldots, \alpha_{s,s}$$

has no repetition. (We allow numbers greater than $s$ to be enumerated at stage $s$)

Any injectively enumerable class is approximately injectively enumerable. But, of course any c.e. class of infinite sets is also approximately injectively enumerable, and so this notion does not yield a sufficient condition.

The failure of the necessary property (F) described in the previous section to be sufficient is discussed in Chapter 4. We investigate another necessary condition in Section 3.3. None of the sufficient conditions of the previous section is necessary, nor is their disjunction.

However, it is possible to "generate" all injective enumerations, and thus all injectively enumerable classes, in the following sense. From an injective enumeration $\alpha$ we can construct a computable function $f \in R_1$ such that

$$(\star) \qquad \alpha_{0,f(s)} \restriction f(s), \ldots, \alpha_{s,f(s)} \restriction f(s) \text{ has no repetition.}$$

For every $i$ there is a stage $s$ and a number $l$ such that $\alpha_{i,s} \restriction l = \alpha_i \restriction l \neq \alpha_{j,t} \restriction l$ for all $t \geq s$ and $j < i$. For every $s$, determine the smallest $t \geq s$ such that $\alpha_{0,t} \restriction t, \ldots, \alpha_{s,t} \restriction t$ contains no repetition. Set $f(s) = t$. On the other hand, given an enumeration $\gamma$ and a computable function $f$ we can construct an injective enumeration $\beta$ as follows. If at stage $s$, $(\star)$ is satisfied, let $\beta_{i,s} = \alpha_{i,s} \restriction s$ for all $i < s$. If at stage $s$, $(\star)$ is violated, enumerate in each $\beta_i$ only one more number, namely $i + s$, and stop the enumeration of $\beta_i$ for all $i \in \omega$.

Clearly, for any choice of computable enumerations $\alpha$ and computable functions $f \in R_1$, $\beta$ is an injective enumeration. For an injective enumeration $\alpha$ and an appropriate choice of $f$ we obtain that

$\alpha_i = \beta_i$ for all $i$. This gives an effective mapping from all pairs of computable enumerations of c.e. classes and computable partial functions into the class of computable enumerations of c.e. classes. Restricting the second argument to the computable functions the image is the class of all injective enumerations of c.e. classes.

## 1.6 Related research

The characterization problem has one restriction of special interest, namely to the classes of graphs of partial functions[8]. Friedberg's original construction can be easily changed to yield an injective enumeration of the class all computable partial functions [2, Corollary to Theorem 3]. As in the general case, non-trivial necessary conditions are not known. A relevant sufficient criterion is given by Kummer [9, Theorem 5].

**Theorem 1.15 (Kummer)** *Let $\mathcal{C} = \mathcal{S} \cup \mathcal{L}$ be a c.e. class of functions, such that $\mathcal{L} \subseteq R_1$ and $\mathcal{S} \subseteq P_1$. If $\mathcal{S}$ is c.e. and there is a canonically enumerable subclass $\mathcal{F} \subseteq \mathcal{S}$ such that every $f \in \mathcal{S}$ is the limit of an increasing sequence from $\mathcal{F}$, then $\mathcal{C}$ is injectively enumerable.*

At the end of Section 1.4 we already described Kummer's class $R_1 \cup F$ which does not have an injective enumeration. Owings extends Friedberg's result to metarecursion theory (see [28], also for definitions) in [23, Theorem 1,2]:

**Theorem 1.16 (Owings)** *There exists a meta-r.e. sequence $S(\alpha)$ $(\alpha < \omega_1)$ such that for every meta-r.e. set $W$ there is exactly one $\alpha$ for which $W = S(\alpha)$.*

**Theorem 1.17 (Owings)** *There is no meta-r.e. sequence $S(\alpha)$ $(\alpha < \omega_1)$ of $\Pi^1_1$ sets such that for each $\Pi^1_1$ set $A$ there is one and only one $\alpha$ for which $A = S(\alpha)$.*

In 1980, in [3, Corollary 1], Goncharov obtains:

**Theorem 1.18** *For every $k$ there is a c.e. class $\mathcal{A}$ such that $\mathcal{A}$ has* exactly *$k$ injective enumerations (up to $\equiv_r$).*

[8] For this section, we identify partial functions and their graphs.

He applies this to show that there are partial orderings and groups [4, 5], having exactly $k$ non-autoequivalent constructivizations. A *constructivization of a partial order* $(D; \leq_D)$ is a surjective mapping $\pi : \omega \to D$ such that the set $\{(n,m) : \pi(n) \leq_D \pi(m) : n, m \in \omega\}$ is computable. Two constructivizations $\pi, \pi'$ are *autoequivalent*, if there exist computable functions $f, g \in R_1$ and an automorphism $\psi : D \to D$ of $P$ such that $\pi = \psi \circ \pi' \circ g$ and $\pi \circ f = \psi \circ \pi'$. Similarly for groups.

Let us define two functions $f, g : \omega \to \omega$ to be *isomorphic*, if there is a bijection $\pi : \omega \to \omega$ such that $f = \pi \circ g \circ \pi^{-1}$. They are called *computably isomorphic*, if such a $\pi$ can be chosen computable. This defines a partition of the class of computable functions into equivalence classes of computably isomorphic functions. Using Theorem 1.18, Khusainov [6, Theorem 1] and Kummer [11, Theorem 5] show that for each $k$ there is a computable function having exactly $k$ computable isomorphism classes. More applications can be found in [11].

# Chapter 2

# Index set classifications

A set $A$ is said to be $\Sigma^0_n$-complete, if it is a $\Sigma^0_n$-set, and every $\Sigma^0_n$-set can be *reduced* to it. The latter is also expressed by saying that $A$ is $\Sigma^0_n$-*hard*. It means that for every $\Sigma^0_n$-set $B$ there is a computable function $r$ such that $x \in B$ if and only if $r(x) \in A$, for all $x \in \omega$. A basic fact of recursion theory is that a $\Sigma^0_n$-complete set cannot be in $\Pi^0_n$, $\Pi^0_m$ or $\Sigma^0_m$ for any $m < n$. By formulating the definitions of the properties '$n$-computable enumerability' and '$\omega$-computable enumerability' in the first order language of computable predicates it is easy to see that the index sets

$$\mathrm{CE}_n := \{e : \mathcal{C}^{(e)} \text{ is } n\text{-computably enumerable}\}$$

and

$$\mathrm{CE}_\omega := \{e : \mathcal{C}^{(e)} \text{ is } \omega\text{-computably enumerable}\}$$

are $\Sigma^0_5$ and $\Sigma^0_6$, respectively. The predicate $W_i = W_j$ is $\Pi^0_2$ in general, but also $\Sigma^0_2$, if $W_i, W_j$ are finite sets. Therefore the index set

$$\mathrm{CE}^{\mathrm{fin}}_n := \{e : \mathcal{C}^{(e)} \text{ is an } n\text{-computably enumerable class of finite sets}\}$$

is in $\Sigma^0_4$. In this chapter we will show that these index sets are complete at their respective levels. Therefore there can be no first order definition in terms of computable predicates of these properties with a simpler prefix than that of the natural definition. In fact, the reductions $r$ constructed for $S \in \Sigma^0_5$ (and $\Sigma^0_4$ for the classes $\mathrm{CE}^{\mathrm{fin}}_n$) satisfy for all $x$ the stronger statements

$x \in S \Rightarrow \mathcal{C}^{(r(x))}$ is injectively enumerable and

$x \notin S \Rightarrow \mathcal{C}^{(r(x))}$ is not $n$-c.e.

The properties $n$-c.e. and $\omega$-c.e. differ at a "cross over point". The properties of computable enumerations "having at most $n$ repetitions", "having at most finitely many repetitions" and "enumerating $\mathcal{C}$" ($\mathcal{C}$ is a fixed class[1]) are $\Sigma^0_3$, $\Pi^0_5$, $\Pi^0_4$, respectively. Thus the level of $\mathrm{CE}_n$, $n \in \omega$, in the Arithmetical Hierarchy is not determined by the complexity of expressing the boundedness of the repetitions, but by the complexity of expressing the property of enumerating a given class. For $\mathrm{CE}_\omega$ it is the other way round.

The classification of the index set $\mathrm{CE}_1$ was left open in [12], where it was shown that $\mathrm{CE}_1$ is $\Sigma^0_4$-hard. This article also states an example of a property whose natural definition does not agree with the completeness-level of its index set. Recall that $L(\mathcal{C})$ denotes the upper semilattice of the equivalence classes of computable enumerations of $\mathcal{C}$. From the natural definition

$$\{e : \mathcal{C}^{(e)} \subseteq R_1 \text{ and } L(\mathcal{C}^{(e)}) \text{ has a greatest element}\}$$

is a $\Sigma^0_5$-set. But by a result of Marchenkov [21, Theorem 3], $L(F)$ does not possess a greatest element for any class of computable functions $F$ with $L(F) > 1$. Therefore the index set is equal to

$$\{e : \mathcal{C}^{(e)} \subseteq R_1 \text{ and } |L(\mathcal{C}^{(e)})| = 1\}$$

which is $\Pi^0_4$-complete [12].

## 2.1 Overview of Chapter 2

In the next section we construct a class which is not $n$-c.e. The construction can be easily adapted to show that $\mathrm{CE}_n$ is $\Pi^0_4$-hard. This is done in Section 2.3. Based on this we show the $\Sigma^0_5$-hardness of $\mathrm{CE}_n$ in Section 2.4[2]. The proof of the $\Sigma^0_4$-completeness of the index sets $\mathrm{CE}_n^{\mathrm{fin}}$ turns out to be much simpler and is discussed in Section 2.5. The last section is devoted to the $\Sigma^0_6$-completeness of the index set $\mathrm{CE}_\omega$.

[1] Sets $\{e : \mathcal{C}^{(e)} = \mathcal{C}\}$ are not necessarily $\Pi^0_4$–complete – see [12].

[2] The construction for the case $n = 1$ can be found in [30].

## 2.2 A non-$n$-c.e. class

We define $[a, b] := \{0, 2, \ldots, 2a, 1, 3, \ldots, 2b+1\}$, $\mathrm{osup}(A) := \sup\{a : 2a+1 \in A\}$ and $\mathrm{esup}(A) := \sup\{a : 2a \in A\}$. Fix $i \in \omega$. Let

$$\mathcal{J}^{(i)} = \{\langle i, \omega\rangle\} \cup \{\langle i, [a, b]\rangle : a, b \in \omega\}.$$

Then we can effectively enumerate, uniformly in $i$, a class $\mathcal{A}^{(i)} \subseteq \mathcal{J}^{(i)}$ such that the conjunction of

$$\mathcal{A}^{(i)} = \mathcal{C}^{(i)} \cap \mathcal{J}^{(i)} \text{ and}$$

$$\mathcal{C}^{(i)} \cap \{A \subseteq \omega : (\exists z)(\langle i, z\rangle \in A)\} \subseteq \mathcal{J}^{(i)}$$

implies that $\Omega^{(i)}$ repeats some set more than $n$ times. From this condition it is clear that the c.e. union $\bigcup_{i\in\omega} \mathcal{A}^{(i)}$ is not $n$-c.e. The idea is to let $\mathcal{A}^{(i)}$ be either $\mathcal{J}^{(i)}$ or $\mathcal{J}^{(i)}$ with one finite set deleted. To obtain $\mathcal{A}^{(i)}$ we begin to enumerate the whole class $\mathcal{J}^{(i)}$. We also begin examining the sets $\Omega_j^{(i)}$, $j \in \omega$. If we find distinct $m_0, \ldots, m_n$ such that

$$\Omega_{m_0}^{(i)} \cap \langle i, \omega\rangle \neq \emptyset, \ldots, \Omega_{m_n}^{(i)} \cap \langle i, \omega\rangle \neq \emptyset$$

then we modify our enumeration of $\mathcal{A}^{(i)}$ in such a way that, if one of $\Omega_{m_j}^{(i)} \cap \langle i, \omega\rangle$ is finite, then one of the sets $\langle i, [a, b]\rangle$ is left out of $\mathcal{A}^{(i)}$. Of course, matters are arranged so that, if all sets $\Omega_{m_j}^{(i)}$, $0 \leq j \leq n$ are members of $\mathcal{J}^{(i)}$, then the set omitted is one of them.

Formally we define three partial computable functions $a, b, m \in P_1$ by the following construction, during which $m_0, \ldots, m_n$ may become defined.

*Step s:*

1. If some $m_j$, $0 \leq j \leq n$, is not defined, then do the following. Let $j$ be the least such that $m_j$ is undefined. Determine the smallest $x < s$, if any, such that $x \neq m_{j'}, j' < j$ and there is a number $\langle i, a\rangle \in \Omega_{x,s}^{(i)}$. Define $m_j := x$.

2. If $m_0, \ldots, m_n$ are all defined let $m = m(s) \in \{m_0, \ldots, m_n\}$ be the smallest such that $\mathrm{osup}((\Omega_{m,s}^{(i)})_{.2})$ is minimal. Define $b(s) := \mathrm{osup}((\Omega_{m,s}^{(i)})_{.2})$ and $a(s) := \mathrm{esup}((\Omega_{m,s}^{(i)})_{.2})$.

End of construction.

We verify the following lemmas. They will also be used in the following two sections.

**Lemma 2.1** *Suppose in the course of the construction $m_0, \dots, m_n$ all become defined and one of the sets $\Omega^{(i)}_{m_0}, \dots, \Omega^{(i)}_{m_n}$ is finite. Then the limits $\lim_s b(s) = b$ and $\lim_s a(s) = a$ exist and there is a set $A$ and an index $m$ such that esup$(A) = a$, osup$(A) = b$ and $\Omega^{(i)}_m = \langle i, A \rangle$.*

*Proof.* By hypothesis the set

$$O := \{o \in \omega : o = \text{osup}((\Omega^{(i)}_m)_{.2}) \text{ and } m \in \{m_0, \dots m_n\}\}$$

is not empty. Therefore there is a smallest $m \in \{m_0, \dots m_n\}$ such that $\text{osup}((\Omega^{(i)}_m)_{.2}) = \min(O)$. It follows that $\lim_s m(s) = m$, $\lim_s b(s) = \text{osup}((\Omega^{(i)}_m)_{.2})$ and $\lim_s a(s) = \text{esup}((\Omega^{(i)}_m)_{.2})$. For $A = (\Omega^{(i)}_m)_{.2}$ the conclusion is satisfied.□

**Lemma 2.2** *If the sequence $(b(s))_{s \in \omega}$ converges, then $\lim_s a(s) \in \omega \cup \{\infty\}$. If some $b(s) \downarrow$ then $\lim_s b(s) \in \omega \cup \{\infty\}$.*

*Proof.* If $(b(s))_{s \in \omega}$ converges then $\lim_s m(s) = m$ exists. It follows that $\lim_s a(s) = \text{esup}((\Omega^{(i)}_{m,s})_{.2})$. If $b(s) \downarrow$, then $m_1, \dots, m_n$ become defined, and $\lim_s b(s) = \min\{\text{osup}((\Omega^{(i)}_{m_x})_{.2}) : 0 \le x \le n\}$.

□

We define the class

$$\mathcal{A}^{(i)} := \{\langle i, \omega \rangle\} \cup \{A_{a,b,s_0} : a, b, s_0 \in \omega\},$$

where

$$A_{a,b,s_0} := \begin{cases} \langle i, [a, b] \rangle & \text{if for no } s > s_0 \text{ is } a(s) = a \text{ and } b(s) = b \\ \\ \langle i, \omega \rangle & \text{otherwise.} \end{cases}$$

By inspection, $\mathcal{A}^{(i)}$ is computably enumerable uniformly in $i$.

**Lemma 2.3** *If $\mathcal{A}^{(i)} = \mathcal{C}^{(i)} \cap \mathcal{J}^{(i)}$ and*

$$\mathcal{C}^{(i)} \cap \{A \subseteq \omega : (\exists z)(\langle i, z \rangle \in A)\} \subseteq \mathcal{J}^{(i)}$$

*then $\Omega^{(i)}$ repeats some set more than $n$ times.*

*Proof.* If the hypothesis holds, then eventually all $m_j, 0 \leq j \leq n$, are defined in the course of the construction of $a$ and $b$. Suppose every set is enumerated at most $n$ times by $\Omega^{(i)}$. Then one of the sets $\Omega^{(i)}_{m_0}, \ldots, \Omega^{(i)}_{m_n}$ is finite. By Lemma 2.1 $\lim_s a(s) = a$ and $\lim_s b(s) = b$ exist and $\langle i, [a, b]\rangle \in \mathcal{C}^{(i)}$. From the definition of $\mathcal{A}^{(i)}$, $\langle i, [a, b]\rangle \notin \mathcal{A}^{(i)}$. So $\mathcal{A}^{(i)} \neq \mathcal{C}^{(i)} \cap \mathcal{J}^{(i)}$, contradiction.□

Now we can define the c.e. class

$$\mathcal{A} := \bigcup_{i \in \omega} \mathcal{A}^{(i)}$$

and from the last lemma it follows that $\mathcal{A}$ is not $n$-c.e.

## 2.3 $\text{CE}_n$ is $\Pi^0_4$–hard

Let $[A, x]$ denote the union of all sets $[a, x]$ such that $a \in A$. The class $\mathcal{A}$ can be extended to

$$\mathcal{A} \cup \{\langle i, [\omega, b]\rangle | i, b \in \omega\}$$

which by Kummer's Lemma 1.9 has an injective enumeration. To recall the statement of this lemma: a class $\mathcal{B}$ has an injective enumeration, if there is a partition of $\mathcal{B}$ into c.e. classes $\mathcal{B}_1, \mathcal{B}_2$ such that $\mathcal{B}_1$ is injectively enumerable and every finite subset of a set in $\mathcal{B}_2$ has infinitely many extensions in $\mathcal{B}_1$. An index for an injective enumeration of $\mathcal{B}$ can be found uniformly in indices for an enumeration of $\mathcal{B}_2$ and an injective enumeration of $\mathcal{B}_1$.

Here the class $\{\langle i, [\omega, b]\rangle | i, b \in \omega\}$ is injectively enumerable (uniformly in $i$) and contains infinitely many extensions of every finite subset of a member of $\mathcal{A}$. Below we will demonstrate that the class

$$\mathcal{E}^{(i,x)} := \begin{cases} \mathcal{A}^{(i)} \cup \{\langle i, [\omega, b]\rangle | i, b \in \omega\} & \text{if } W_x \text{ is infinite,} \\ \\ \mathcal{A}^{(i)} & \text{if } W_x \text{ is finite.} \end{cases}$$

is computably enumerable uniformly in $i, x$. This allows us to show that $\text{CE}_n$ is $\Pi^0_4$-hard as follows. Let $S$ be a $\Pi^0_4$-set. We first find an adequate representation of $S$. We know that there is a $\Sigma^0_3$ predicate $Q$ such that $x \in S$ if and only if $(\forall i)(Q(i, x))$. Form the predicate $P(i, x) := (\forall i' \leq i)(Q(i', x))$. If $x \in S$ then $P(i, x)$ for all $i \in \omega$ and if $x \notin S$ then $(\exists i_0)(\forall i \geq i_0)(\neg P(i, x))$. By a standard fact of recursion theory (e.g. [29, p 61]) the bounded quantifier "$(\forall i' < i)$" may be ignored in counting quantifier complexity and so $P$ is a $\Sigma_3^0$ predicate like $Q$. Because the predicate "$W_x$ is infinite" is $\Pi_2^0$-complete (see e.g. [29, p 66]), there is a function $f \in R_3$ such that if $P(i,x)$ then $(\exists j)(W_{f(i,j,x)}$ is infinite) and if $\neg P(i,x)$ then $(\forall j)(W_{f(i,j,x)}$ is finite). Then $f$ satisfies

$$k \in S \Leftrightarrow (\forall i)(\exists j)(W_{f(i,j,k)} \text{ is infinite}), \text{ and}$$

$$(\forall j)(W_{f(i_0,j,k)} \text{ is finite}) \Rightarrow (\forall i \geq i_0)(\forall j)(W_{f(i,j,k)} \text{ is finite})$$

for all $i_0 \in \omega$ . Define $r \in R_1$ by the $S_n^m$-Theorem such that

$$\mathcal{C}^{(r(k))} = \bigcup_{i,j \in \omega} \mathcal{E}^{(i,f(i,j,k))}.$$

Suppose $k \in S$. Then for all $i$ there exists $j$ such that $W_{f(i,j,k)}$ is infinite, so

$$\mathcal{C}^{(r(k))} = \mathcal{A} \cup \{\langle i, [\omega, b]\rangle : i, b \in \omega\},$$

which has an injective enumeration. For $k \notin S$ we know that there is $i_0$ such that $W_{f(i,j,k)}$ is finite for all $i > i_0$ and $j \in \omega$ . If $\mathcal{C}^{(r(k))}$ is $n$-c.e. witnessed by $\Omega^{(i)}$, then without loss of generality $i > i_0$. But for $i > i_0$,

$$\mathcal{C}^{(r(k))} \cap \{A \subseteq \omega : (\exists z)(\langle i, z\rangle \in A)\} = \mathcal{A}^{(i)}$$

and this implies that $\Omega^{(i)}$ does not enumerate $\mathcal{C}^{(r(k))}$ with at most $n$ repetitions by Lemma 2.3.

**Lemma 2.4** *The class $\mathcal{E}^{(i,x)}$ as defined above is computably enumerable uniformly in $i, x$.*

*Proof.* For $A \subseteq \omega$ let $\widehat{A} := \{b : (\exists a)(a \geq b \text{ and } a \in A)\}$ be the smallest initial segment of $\omega$ containing $A$. Now define the following sets, using the functions $a, b$ as defined earlier:

$$B_{b,s_0} := \begin{cases} \langle i, [\widehat{W_x}, b]\rangle & \text{if } W_{x,s_0} \neq \emptyset \text{ and } b(s) \neq b \text{ for all } s > s_0, \\ \langle i, \omega\rangle & \text{otherwise.} \end{cases}$$

$$C_{b,s_0} := \begin{cases} \langle i, [\widehat{W}_x, b]\rangle & \text{if } |W_{x,s_0}| > a(s_0) \text{ and } a(s) = a(s_0) \text{ and} \\ & b(s) = b(s_0) \text{ for all } s \geq s_0, \\ \langle i, \omega \rangle & \text{otherwise.} \end{cases}$$

$$D_{b,s_0} := \begin{cases} \langle i, [\{n : (\exists t > s_0)(|W_{x,t}| > n \text{ and } a(t) > n)\}, b]\rangle & \text{if } b(s) = b(s_0) \text{ for all } s > \\ & s_0 \text{ and } a(s_0), |W_{x,s_0}| > 0, \\ \langle i, \omega \rangle & \text{otherwise.} \end{cases}$$

Let

$$\mathcal{F} = \{B_{b,s_0}, C_{b,s_0}, D_{b,s_0} : b, s_0 \in \omega\}.$$

It should be clear that $\mathcal{F}$ is computably enumerable uniformly in $i$ and $x$, is contained in

$$\mathcal{A}^{(i)} \cup \{\langle i, [\omega, b]\rangle : b \in \omega\}$$

and that if $W_x$ is finite, $\mathcal{F} \subseteq \mathcal{A}^{(i)}$. When $W_x$ is infinite, there are three cases. If the sequence $(b(s))_{s\in\omega}$ diverges then for every $b \in \omega$ there exists $s_0$ such that $b(s) \neq b$ for all $s > s_0$, so that $B_{b,s_0} = \langle i, [\omega, b]\rangle$. If $(b(s))_{s\in\omega}$ converges, by Lemma 2.2 either $(a(s))_{s\in\omega}$ converges and for every $b \in \omega$ one of the sets $C_{b,s_0}$ is equal to $\langle i, [\omega, b]\rangle$, or $\lim_s a(s)$ is infinite and for every $b \in \omega$ one of the sets $D_{b,s_0}$ is equal to $\langle i, [\omega, b]\rangle$. Now $\mathcal{E}^{(i,x)} = \mathcal{A}^{(i)} \cup \mathcal{F}$, and $\mathcal{F}$ is clearly computably enumerable uniformly in $i$ and $x$.□

## 2.4 $\mathrm{CE}_n$ is $\Sigma^0_5$-hard

The following constructions rely on the same approach as was presented above: we can arrange for the success of a diagonalization against a potential enumeration with at most $n$ repetitions to depend on the infiniteness of a given set. Let $S$ be a $\Sigma^0_5$-set. Then there is a $\Pi^0_4$ predicate $Q$ such that if $l \in S$ then $Q(i,l)$ for some $i \in \omega$ and if $l \notin S$ then $\neg Q(i,l)$ for all $i \in \omega$. Define the predicate $P(i,l) := (\exists i' \leq i)(Q(i',l))$. Using the same ideas as in the previous section, we can represent $S$ by a c.e. array of c.e. sets $V : \omega^4 \to 2^\omega$ such that for all $l \in \omega$

$l \in S$ if and only if $(\exists i)(\forall j)(\exists k)(V_{i,j,k,l}$ is infinite),

if $(\forall j)(\exists k)(V_{i_0,j,k,l}$ is infinite) then $(\forall i > i_0)(\forall j)(\exists k)(V_{i,j,k,l}$ is infinite), and

if $(\exists j_0)(\forall k)(V_{i,j_0,k,l}$ is finite), then $(\forall j > j_0)(\forall k)(V_{i,j,k,l}$ is finite).

Fix $i$ and $l$. Let

$$\mathcal{J}^{(i,j)} := \{\langle i, j, \omega\rangle\} \cup \{\langle i, j, [a, b]\rangle : a, b \in \omega\}.$$

The idea is to construct uniformly effectively from $i, l$ a c.e. class $\mathcal{C}^{(i,l)}$ such that

$\Omega^{(i)}$ is not an enumeration of $\mathcal{C}^{(i,l)}$ with at most $n$ repetitions

if and only if

$(\exists j_0)(\forall j \geq j_0)(\forall k)(V_{i,j,k,l}$ is finite).

For every $j \in \omega$ we enumerate a class $\mathcal{B}^{(i,j,l)}$ for which we try to arrange that if

- $\mathcal{C}^{(i)} \cap \{A \subseteq \omega : (\exists z)(\langle i, j, z\rangle \in A)\} \subseteq \mathcal{J}^{(i,j)}$ and
- $\mathcal{B}^{(i,j,l)} = \mathcal{C}^{(i)} \cap \mathcal{J}^{(i,j)}$

then $\Omega^{(i)}$ repeats one of its sets more than $n$ times .

This is the $j$-th attempt to diagonalize against $\Omega^{(i)}$. It will succeed if and only if there is a number $k \in \omega$ such that $V_{i,j,k,l}$ is infinite and no $k \in \omega$ such that $V_{i,j+1,k,l}$ is infinite. As in the previous section the whole class $\mathcal{J}^{(i,j)}$ is enumerated in $\mathcal{B}^{(i,j,l)}$, and a finite member of $\mathcal{J}^{(i,j)}$ may be deleted to effect the diagonalization — but only if there is an infinite set $V_{i,j,k,l}$. Also we adjoin the sets $\langle i, j, [\omega, b]\rangle$, $b \in \omega$ if there is an infinite set $V_{i,j+1,k,l}$, because such infinite sets allows us to enumerate $\mathcal{B}^{(i,j,l)}$ injectively uniformly in $i, j$ and $l$.

Of course, the infinite set $V_{i,j+1,k,l}$, while possibly frustrating the $j$-th attempt at diagonalizing against $\Omega^{(i)}$, enables the $(j+1)$-th attempt. The result is that the class $\mathcal{C}^{(i,l)} = \bigcup_{j\in\omega} \mathcal{B}^{(i,j,l)}$ is diagonalized against $\Omega^{(i)}$ as an enumeration with at most $n$ repetitions if there exists $j_0$ such that $V_{i,j_0+1,k,l}$ is finite for all $k \in \omega$. Note that $V_{i,j,k,l}$ is infinite for $j = 0$, so that there is at least one attempt for every $i \in \omega$.

Formally, given $i, j \in \omega$ and c.e. sequences of c.e. sets $X = (X_n)_{n\in\omega}$ and $Y = (Y_n)_{n\in\omega}$ we will construct c.e. classes $\mathcal{A}^{(i,j,X)}$ and $\mathcal{B}^{(i,j,X,Y)}$ satisfying the following:

i) $\mathcal{A}^{(i,j,X)} \subseteq \mathcal{J}^{(i,j)}$.

ii) If some $X_k$ is infinite, then either $\Omega^{(i)}$ repeats a set more than $n$ times or $\mathcal{A}^{(i,j,X)} \neq \mathcal{C}^{(i)} \cap \mathcal{J}^{(i,j)}$.

iii) $(\forall k)(X_k \text{ is finite}) \Rightarrow \mathcal{A}^{(i,j,X)} = \mathcal{J}^{(i,j)}$.

iv) $\mathcal{A}^{(i,j,X)}$ is injectively enumerable.

v) $(\forall k)(Y_k \text{ is finite}) \Rightarrow \mathcal{B}^{(i,j,X,Y)} = \mathcal{A}^{(i,j,X)}$.

vi) $(\exists k)(Y_k \text{ is infinite}) \Rightarrow \mathcal{B}^{(i,j,X,Y)} = \mathcal{A}^{(i,j,X)} \cup \{\langle i, j, [\omega, b]\rangle : b \in \omega\}$.

vii) An index of $\mathcal{A}^{(i,j,X)}$ can be computed effectively from $i, j$ and an index of $X$. An index of $\mathcal{B}^{(i,j,X,Y)}$ can be computed effectively from $i, j$ and indices of $X$ and $Y$.

Using these constructions, it can be shown that $\mathrm{CE}_n$ is $\Sigma^0_5$-hard as follows. Let $X(i,j,l) = (V_{i,j,k,l})_{k\in\omega}$ and $Y(i,j,l) = (V_{i,j+1,k,l})_{k\in\omega}$. We use $\mathcal{B}^{(i,j,l)}$ as an abbreviation for $\mathcal{B}^{(i,j,X(i,j,l),Y(i,j,l))}$ and $\mathcal{C}^{(i,l)}$ as an abbreviation for $\bigcup_{j\in\omega} \mathcal{B}^{(i,j,l)}$. Define $r$ as a computable function by the $S^m_n$-Theorem such that

$$\mathcal{C}^{(r(l))} := \bigcup_{i,j\in\omega} \mathcal{B}^{(i,j,l)}.$$

We will show that the function $r$ reduces $S$, the given $\Sigma^0_5$ set, to $\mathrm{CE}_n$.

Suppose $l \in S$. Then there is a number $i'$ such that

$$(\forall i \geq i')(\forall j)(\exists k)(V_{i,j,k,l} \text{ is infinite}).$$

Let $i_0$ denote the least such number $i'$. By vi) for all $i \geq i_0$ we have

$$\mathcal{C}^{(i,l)} = \bigcup_{j\in\omega} \mathcal{A}^{(i,j,X(i,j,l))} \cup \{\langle i, j, [\omega, b]\rangle : b \in \omega\}$$

which is injectively enumerable uniformly in $i$ by Kummer's Lemma 1.9. Consider a particular $i < i_0$. There is a number $j'$ such that $V_{i,j,k,l}$ is finite for all $j > j'$ and $k \in \omega$. Let $j_0(i)$ denote the least such number $j'$. By iii) and v) we see that for all $j > j_0(i)$, $\mathcal{B}^{(i,j)} = \mathcal{J}^{(i,j)}$, which is injectively enumerable uniformly in $i, j$. By iv) and v), $\mathcal{B}^{(i,j_0(i),l)}$ is injectively enumerable. Finally, from vi) and Lemma 1.9, $\mathcal{B}^{(i,j,l)}$ is injectively enumerable for each $j < j_0(i)$. Since, as $j$ varies, the classes $\mathcal{B}^{(i,j,l)}$ are pairwise disjoint, $\mathcal{C}^{(i,l)}$ is injectively enumerable. Since, as $i$ varies, the classes $\mathcal{C}^{(i,l)}$ are pairwise disjoint, $\mathcal{C}^{(r(l))}$ is injectively enumerable.

Suppose $l \notin S$. Then

$$(\exists j)(\forall k)(V_{i,j,k,l} \text{ is finite})$$

for all $i \in \omega$. Towards a contradiction suppose $\mathcal{C}^{(r(l))}$ is $n$-c.e., say by $\Omega^{(i_0)}$. From the choice of $V$, $V_{i_0,0,k,l} = \omega$ for all $k \in \omega$. Let $j_0$ be the greatest $j$ such that $V_{i_0,j_0,k,l}$ is infinite for some $k \in \omega$. From i), v) and vi)

$$\mathcal{C}^{(i_0)} \cap \{A \subseteq \omega : (\exists z)(\langle i_0, j_0, z\rangle \in A)\} = \mathcal{C}^{(r(l))} \cap \{A \subseteq \omega : (\exists z)(\langle i_0, j_0, z\rangle \in A)\} =$$
$$\mathcal{A}^{(i_0,j_0,Z)} \subseteq \mathcal{J}^{(i_0,j_0)}$$

where $Z$ denotes $(V_{i_0,j_0,k,l})_{k\in\omega}$. But from ii) we have $\mathcal{A}^{(i_0,j_0,Z)} \neq \mathcal{C}^{(i_0)} \cap \mathcal{J}^{(i_0,j_0)}$, contradiction.

**Lemma 2.5** *Given $i, j \in \omega$ and indices of c.e. sequences of c.e. sets $(X_n)_{n\in\omega}, (Y_n)_{n\in\omega}$, classes $\mathcal{A}^{(i,j,X)}$, $\mathcal{B}^{(i,j,X,Y)}$ can be constructed such that the properties i) ... vii) from above hold.*

*Proof.* First we construct functions $\overline{a}, \overline{b}, \overline{m} \in P_3$, similar to the functions $a, b, m$ from Section 2.2, which are designed to provide for the implicand of property ii). Then we define $a, b, m \in P_3$ and with these $\mathcal{A}^{(i,j,X)}$. The last part is the definition of $\mathcal{B}^{(i,j,X,Y)}$. We may assume without loss of generality that

- $(\forall k)(X_k, Y_k$ are initial segments of $\omega)$
- $(\forall k)(X_k$ infinite $\Rightarrow X_{k+1}$ infinite$)$
- $(\forall k)(Y_k$ infinite $\Rightarrow Y_{k+1}$ infinite$)$

The argument $k$ of $a(k,z,s), b(k,z,s)$ and $m(k,z,s)$ is used to guess which set $X_k$ is the first in the sequence $(X_k)_{k\in\omega}$ to be infinite and the argument $z$ is used to guess the cardinality of $X_{k-1}$, if such a set $X_k$ exists. We omit the arguments of $\overline{a}, \overline{b}, m_0, \ldots, m_n$ in the following construction:

*Step s*

1. If some $m_d$, $0 \leq d \leq n$ is not defined, then do the following. Let $d$ be the least such that $m_d$ is undefined. Determine the smallest $x < s$, if any, such that $x \neq m_{d'}, d' < d$ and there is a number $\langle i, j, 2y+1\rangle \in \Omega^{(i)}_{x,s}$ with $y > k + z$. Define $m_d := x$.

2. If all $m_0, \ldots, m_n$ are defined let $m \in \{m_0, \ldots, m_n\}$ be the smallest such that $\operatorname{osup}(\Omega^{(i)}_{m,s})_{.3}$ is minimal. Define $\bar{b} := \operatorname{osup}(\Omega^{(i)}_{m,s})_{.3}$ and $\bar{a} := \operatorname{esup}((\Omega^{(i)}_{m,s})_{.3})$.

End of construction.

Now the statements analogous to Lemmas 2.1, 2.2 also hold for $\lambda s.\bar{a}(k,z,s), \lambda s.\bar{b}(k,z,s)$ $(k, z$ fixed). We define $a$ by

$$a(k,z,s) := \begin{cases} \bar{a}(k,z,s) & \text{if } |X_{k-1,s}| = z \text{ and } s \in X_k, \\ \uparrow & \text{otherwise.} \end{cases}$$

and $b$ from $\bar{b}$ in the same way, where we assume $X_{-1} = \emptyset$. We have the equivalence

$$\left.\begin{array}{l} (k = \min\{k' : X_{k'} \text{ is infinite}\} \text{ and } z = |X_{k-1}|) \Leftrightarrow \\ (\text{for almost all } s \in \omega)(a(k,z,s) = \bar{a}(k,z,s) \text{ and } b(k,z,s) = \bar{b}(k,z,s)) \end{array}\right\} (\star)$$

From $a$ and $b$ we define the sets $A_{x,y,s_0}$ by

$$A_{x,y,s_0} := \begin{cases} \langle i,j,[x,y]\rangle & \text{if there are no } t > s_0, k, z \in \omega \text{ such that } a(k,z,t) = x \text{ and } \\ & b(k,z,t) = y, \\ \langle i,j,\omega\rangle & \text{otherwise,} \end{cases}$$

and the class

$$\mathcal{A}^{(i,j,X)} := \{\langle i,j,\omega\rangle\} \cup \{A_{x,y,s_0} : x, y, s_0 \in \omega\}.$$

To see that $\mathcal{A}^{(i,j,X)}$ is computably enumerable note that, whether $\langle i,j,[a,b]\rangle$ is in the class or not, depends only on the behaviour of $a(k,z,s)$ and $b(k,z,s)$ for $k + z < b$. By $(\star)$ and the statement corresponding to Lemma 2.2, it follows that $|\mathcal{J}^{(i,j)} - \mathcal{A}^{(i,j,X)}| \leq 1$.

Towards the definition of $\mathcal{B}^{(i,j,X,Y)}$, for all $i, j \in \omega$, from each c.e. sequence $X$ and c.e. set $Z$ we define

$$\mathcal{D}^{(i,j,X,Z)} := \begin{cases} \mathcal{A}^{(i,j,X)} \cup \{\langle i, j, [\omega, b]\rangle : b \in \omega\} & \text{if } Z \text{ is infinite,} \\ \mathcal{A}^{(i,j,X)} & \text{if } Z \text{ is finite.} \end{cases}$$

Then we set

$$\mathcal{B}^{(i,j,X,Y)} := \bigcup_{k \in \omega} \mathcal{D}^{(i,j,X,Y_k)}.$$

Using the same idea as in the proof of Lemma 2.4 we see that $\mathcal{D}^{(i,j,X,Z)}$ is c.e. uniformly in $i, j$, and indices of $X$ and $Z$. Hence $\mathcal{B}^{(i,j,X,Y)}$ is also c.e. uniformly.

Now property i) obviously holds. For property ii) note that by the equivalence ($\star$) the proof of Lemma 2.3 applies. Property iii) holds because if $X_k$ is finite for every $k \in \omega$, then $a(k, z, s)$ and $b(k, z, s)$ are undefined for all $k, z \in \omega$ and almost all $s \in \omega$, whence $\mathcal{A}^{(i,j,X)}$ is equal $\mathcal{J}^{(i,j)}$. Because of $|\mathcal{J}^{(i,j)} - \mathcal{A}^{(i,j,X)}| \leq 1$, iv) is satisfied. Properties v), vi) follow immediately from the definition of $\mathcal{B}^{(i,j,X,Y)}$. All constructions were carried out uniformly, so vii) is satisfied.□

## 2.5 The index set of $n$-c.e. classes of finite sets

Let $S$ be an arbitrary $\Sigma^0_4$-set. We show how to reduce $S$ to $\mathrm{CE}^{\mathrm{fin}}_n$. By routine manipulations similar to the ones described on page 23, we know that there is a computably enumerable array $(V_{i,j,k})_{i,j,k\in\omega}$ such that

$k \in S$ if and only if $(\exists i)(\forall j)(V_{i,j,k}$ is finite),

if $(\forall j)(V_{i_0,j,k}$ is finite) then $(\forall i > i_0)(\forall j)(V_{i_0,j,k}$ is finite), and

if $V_{i,j_0,k}$ is infinite, then $V_{i,j,k}$ is infinite for all $j > j_0$.

For all $i, j, z, s \in \omega$ define

$$g(i,j,z,s) = \begin{cases} \text{“yes”} & (\exists m_0, \ldots, m_n < s \text{ pairwise different}) \\ & (\langle i,j,z,0\rangle \in \Omega^{(i)}_{m_0,s}, \ldots, \langle i,j,z,n\rangle \in \Omega^{(i)}_{m_n,s}), \\ \text{“no”} & \text{otherwise.} \end{cases}$$

Define for all $x \leq n$ and all $i, j, k, z, t \in \omega$

$$A_{x,i,j,k,z,t} = \begin{cases} \{\langle i,j,z,0\rangle, \ldots, \langle i,j,z,n\rangle\} & \text{if there is } s > t \text{ such that } |V_{i,j,k,s}| > |V_{i,j,k,t}|, \\ & |V_{i,j-1,k,s}| = z \text{ and } g(i,j,z,s) = \text{“yes”}, \\ \\ \{\langle i,j,z,x\rangle\} & \text{otherwise,} \end{cases}$$

where $V_{i,-1,k}$ is understood to be the empty set for all $i, k \in \omega$.

Now the class

$$\mathcal{B}^{(k)} = \{A_{x,i,j,k,z,t} : x \leq n, i, j, z, t \in \omega\}$$

is computably enumerable uniformly in $k$ and contains only finite sets for every $k \in \omega$. By the $S_n^m$-Theorem there is a computable function $r$ such that $\mathcal{C}^{(r(k))} = \mathcal{B}^{(k)}$. We claim that, if $k \in S$ then $\mathcal{C}^{(r(k))}$ is injectively enumerable, and not $n$-c.e, otherwise.

Suppose that $k$ is not in $S$, and that $\Omega^{(i)}$ enumerates $\mathcal{B}^{(k)}$ with at most $n$ repetitions. Let $j$ be minimal such that $V_{i,j,k}$ is infinite and let $z = |V_{i,j-1,k}|$, which is finite. If $\{\langle i,j,z,0\rangle\}, \ldots, \{\langle i,j,z,n\rangle\}$ are all enumerated by $\Omega^{(i)}$, then for all $x \leq n$, $t \in \omega$ the set $A_{x,i,j,k,z,t}$ is equal to $\{\langle i,j,z,0\rangle, \ldots, \langle i,j,z,n\rangle\}$. If a set containing $\langle i,j,z,x\rangle$ is in $\mathcal{B}^{(k)}$ then it has to be equal to some set $A_{x,i,j,k,z,t}$. Therefore $\Omega^{(i)}$ is not an enumeration of $\mathcal{B}^{(k)}$. On the other hand, if one of the sets $\{\langle i,j,z,0\rangle\}, \ldots, \{\langle i,j,z,n\rangle\}$ is not enumerated by $\Omega^{(i)}$ then $g(i,j,z,s)$ is "no" for all $s$. Hence all of $\{\langle i,j,z,0\rangle\}, \ldots, \{\langle i,j,z,n\rangle\}$ belong to $\mathcal{B}^{(k)}$, contradiction. Therefore $\mathcal{B}^{(k)}$ is not $n$-c.e.

Suppose that $k$ is in $S$. We have to show that $\mathcal{B}^{(k)}$ is injectively enumerable. Let $i_0$ be maximal such that there is a number $j$ such that $V_{i,j,k}$ is infinite. For all $i > i_0$, all $j, z \in \omega$ there is a number $t$ such that $|V_{i,j,k,t}| = |V_{i,j,k}|$. Therefore the sets $\{\langle i,j,z,0\rangle\}, \ldots, \{\langle i,j,z,n\rangle\}$ are members of $\mathcal{B}^{(k)}$ and the class $\{\langle i,j,z,x\rangle : i > i_0, j, z \in \omega, x \leq n\}$ is injectively enumerable.

The class

$$\{\{\langle i,j,z,0\rangle, \ldots, \langle i,j,z,n\rangle\} : i > i_0, j, z \in \omega\} \cap \mathcal{B}^{(k)}$$

is computably enumerable and thus also injectively enumerable. Altogether,

$$\mathcal{B}_1^{(k)} := \{A_{x,i,j,k,z,t} : i > i_0, j, x, z, t \in \omega\}$$

is injectively enumerable.

For $i \leq i_0$ let $j_i$ denote the least $j$ such that $V_{i,j,k}$ is infinite. For all $i \leq i_0, j > j_i$ and $z \in \omega$ there is a stage $t$ such that $|V_{i,j-1,k,t}| > z$. So the sets $\{\langle i,j,z,0\rangle\}, \dots, \{\langle i,j,z,n\rangle\}$ are in $\mathcal{B}^{(k)}$ and we can injectively enumerate the class

$$\mathcal{B}_2^{(k)} = \{A_{x,i,j,k,z,t} : x \leq n, i \leq i_0, j > j_i, z \in \omega\}$$

by the same argument as was used for $\mathcal{B}_1^{(k)}$. Injectively enumerating

$$\mathcal{B}_3^{(k)} := \{A_{x,i,j,k,z,t} : x \leq n, i \leq i_0, j \leq j_i, z \in \omega\}$$

requires only finite information about the enumerations $\Omega^{(i)}, i \leq i_0$ and the sets $V_{i,j,k}, i \leq i_0, j \leq j_i$. Thus $\mathcal{B}^{(k)}$, being equal to the disjoint union $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$, is injectively enumerable.□

## 2.6 The index set $\mathbf{CE}_\omega$

Let $S$ be an arbitrary $\Sigma_6^0$-set. The following will show how $S$ reduces to $CE_\omega$. Let $\Gamma(i,j,k,l,x)$ abbreviate "$V_{i,j,k,l,x}$ is finite". One can show that we can choose an array $(V_{i,j,k,l,x})_{i,j,k,l,x\in\omega}$ such that the following five conditions are valid:

V1. $x \in S \Leftrightarrow (\exists i)(\forall j)(\exists k)(\forall l)(\Gamma(i,j,k,l,x))$.

V2. $[(\forall j)(\exists k)(\forall l)(\Gamma(i,j,k,l,x))$ and $i' > i] \Rightarrow (\forall j)(\exists k)(\forall l)(\Gamma(i',j,k,l,x))$.

V3. $(\exists j)(\forall k)(\exists l)(\neg\Gamma(i,j,k,l,x)) \Rightarrow (\exists! j)(\forall k)(\exists l)(\neg\Gamma(i,j,k,l,x))$.

V4. $(\forall l)(\Gamma(i,j,k,l,x)) \Rightarrow (\forall k' > k)(\forall l)(\Gamma(i,j,k',l,x))$.

V5. $(\exists l)(\neg\Gamma(i,j,k,l,x)) \Rightarrow (\exists! l)(\neg\Gamma(i,j,k,l,x))$.

To prove the theorem we will use:

**Lemma 2.6** *Given a computably enumerable array $(V_{k,l})_{k,l\in\omega}$ and an $\Omega$-index of a computable enumeration $\psi$ such that*

1. *if $(\forall l)(V_{k_0,l}$ is finite$)$ then $(\forall k > k_0)(\forall l)(V_{k,l}$ is finite$)$ and*

2. *if* $(\exists l)(V_{k,l}$ *is infinite) then there is only one such* $l$ *(for* $k$*),*

*we can uniformly construct an enumeration* $\gamma$ *of a class* $C$ *such that*

1. *if* $(\exists k)(\forall l)(V_{k,l}$ *is finite) then* $\gamma$ *repeats every set (except possibly the empty set) at most finitely often and*

2. *if* $(\forall k)(\exists l)(V_{k,l}$ *is infinite) then* $\psi$ *does not enumerate* $C$ *with at most finite repetitions, but there is an enumeration of* $C$ *with at most finite repetitions.*

the enumeration which is uniformly generated by the construction of the lemma applied to the array $(V_{i,j,k,l,x})_{k,l\in\omega}$ and the $\Omega-$index of the enumeration $\Theta^{(i,j)}$ defined by

$$\Theta_n^{(i,j)} := \{x : \langle i, j, x\rangle \in \Omega_n^{(i)}\}.$$

The latter can be obtained by the $S_n^m$-Theorem.

Let $\mathcal{C}^{(i,j,x)}$ denote the class enumerated by $\gamma^{(i,j,x)}$, omitting the empty set. Define the computable function $r$, again by using the $S_n^m$-Theorem, to satisfy

$$\mathcal{C}^{r(x)} = \bigcup_{i,j\in\omega} \langle i, j, \mathcal{C}^{(i,j,x)}\rangle.$$

In case $x \in S$ we see from the first conclusion of the lemma that for almost all pairs $(i, j)$ the members of the class $\mathcal{C}^{(i,j,x)}$ are enumerated with at most finite repetitions by $\gamma^{(i,j,x)}$, because from V2 and V3 there are only finitely many pairs $(i, j)$ such that

$$(\#) \qquad (\forall k)(\exists l)(V_{i,j,k,l,x} \text{ is infinite}).$$

For each pair satisfying $(\#)$ there exists an enumeration of $\mathcal{C}^{(i,j,x)}$ with finite repetitions from the second conclusion of the lemma. So the whole class $\mathcal{C}^{r(x)}$ has an enumeration with finite repetitions.

In the case that $x \notin S$ we know that

$$(\forall i)(\exists j)(\forall k)(\exists l)(V_{i,j,k,l,x} \text{ is infinite}).$$

Suppose $\Omega^{(i_0)}$ enumerates $\mathcal{C}^{r(x)}$ with finite repetitions. Then for the number $j_0$ such that

$$(\forall k)(\exists l)(V_{i_0,j_0,k,l,x} \text{ is infinite})$$

we derive a contradiction to the second conclusion of the lemma for the class $\mathcal{C}^{(i_0,j_0,x)}$.

*Proof of Lemma 2.6.* First, some remarks on the construction which follows. Broadly speaking there are two outcomes: either only finitely many actions are taken, or not. In the former case the construction is called *finite*, and in the latter *infinite*. It will be clear by inspection of step $s$ below that, when the construction is finite,

- all but a finite number of the $\gamma_i$'s are empty
- either one of the sets $\gamma_i$ is not enumerated by $\psi$, or almost all the sets $V_{k,l}$ are empty.

Hence the conclusions of the lemma are clear when the construction is finite.

For the rest we discuss what happens when the construction is infinite. For each pair $(k, l)$ with $V_{k,l} \neq \emptyset$ we will eventually appoint a *leader* $m$ and a *follower* $f$ for $(k, l)$. Each time a new leader is required we take the least even number $x$ such that $\gamma_x$ is still empty. This implies that no two pairs have the same leader. While $2i$ is not yet a leader, $\gamma_{2i}$ remains empty.

If step $s$ is not vacuous, then we enumerate $0, 1, \ldots, s-1$ into $\gamma_{2s+1}$. Otherwise, $\gamma_{2s+1} = \emptyset$.

Followers are chosen in such a way that no two pairs have the same follower. If $m$, $f$ are the respective leader, follower of some pair $(k, l)$, then $(m, f)$ is called a *proper pair*. The main property of followers is that, if $(m, f)$ is a proper pair, then at the end of the construction $\psi_f = \gamma_m$.

The construction ensures that all the numbers enumerated into members of $(\gamma_n)_{n\in\omega}$ by the end of step $s$ are less than $2^{s+1}$ and that at the end of step $s$, for each leader $m$, $\gamma_m$ has a member which is not in any other $\gamma_i$.

Without loss of generality we may assume that the effective simultaneous enumeration of the sets $V_{k,l}$, $k$, $l < \omega$, is such that, for each $s$, there is at most one pair $(k, l)$ such that $V_{k,l,s+1} \neq V_{k,l,s}$, and if there is such a pair then $k$, $l < s$. We now specify the construction.

*Step s.*

If there is no pair $(k, l)$ such that $V_{k,l,s+1} \neq V_{k,l,s}$, then pass immediately to the next step. Otherwise, proceed as follows with $k$, $l$ denoting the unique numbers such that $V_{k,l,s+1} \neq V_{k,l,s}$. We say that $(k, l)$ is *active* in step $s$.

A1. Does $(k, l)$ have a leader? If not, then let $x$ be the least even number such that $\gamma_x$ is still empty. Appoint $x$ the leader of $(k, l)$. In any case, let $m(k, l)$ denote the leader of $(k, l)$.

A2. For each leader $m$ enumerate the number $2^s + (m/2)$ in $\gamma_m$. (Note that $m/2 \leq s < 2^s$.) Look for the least $t > s$ such that $\psi_{f,t} = \gamma_m$ for each proper pair $(m, f)$ and $\psi_{y,t} = \gamma_{m(k,l)}$ for some $y < t$. The sets $\gamma_i$ remain fixed during this search. If $(k, l)$ does not yet have a follower, appoint the least such $y$ as the follower of $(k, l)$. Denote the follower of $(k, l)$ by $f(k, l)$.

A3. Enumerate $0, 1, \ldots, s - 1$ into $\gamma_{m(k,l)}$. Look for the least $u$ such that $\psi_{f(k,l),u} \supseteq \gamma_{m(k,l)}$. Again the sets $\gamma_i$ are held fixed.

A4. Enumerate $0, 1, \ldots, s - 1$ into $\gamma_{2s+1}$.

End of construction.

To see that the construction succeeds we establish four claims. Let $\gamma_{m,s}$ denote the finite set enumerated in $\gamma_m$ by the end of step $s$.

**Claim 1.** Let $(m, f)$ be a proper pair at the end of step $s$. Then $\gamma_{m,s} \subseteq \psi_f$.

*Proof.* By induction on $s$. If step $v$ is vacuous, then the conclusion for $s = v$ follows immediately by the induction hypothesis. If step $v$ is not vacuous, then $\gamma_{m,v} \subseteq \psi_f$ follows from the fact that $t$ is found in A2, and $u$ is found in A3, of step $v$. This is enough. ◫

**Claim 2.** If either $t$ is not found in A2 or $u$ is not found in A3 of step $s$, then $\psi$ does not enumerate $\mathcal{C}$.

*Proof.* Suppose $t$ is not found in A2. Consider a proper pair $(m, f)$. Let $x$ be a number which is in $\gamma_m$ at the end of step $s - 1$ but not in $\gamma_i$ for any $i \neq m$. Since $x < 2^s$, $x$ has the same property with respect to $\gamma_m$ after the actions taken in A2 of step $s$. Thus $\gamma_m$ is the only $\gamma_i$ with which $\psi_f$ can agree. If there is no $y$ such that $\psi_y \supseteq \gamma_{m(k,l)}$, then $\psi$ does not enumerate $\mathcal{C}$. So suppose $\psi_y \supseteq \gamma_{m(k,l)}$. Since $\gamma_{m(k,l)}$ is the only set $\gamma_i$ which contains $2^s + m(k, l)$, if $\psi_y \in \mathcal{C}$, then $\psi_y = \gamma_{m(k,l)}$. This is enough for A2, the argument for A3 is similar. ◫

Whenever the construction is finite, $\gamma_i = \emptyset$ for almost all $i$. From Claim 2 it is apparent that, if the construction is finite because the search in some instance of A2 or A3 is infinite, then $\mathcal{C}$ is not enumerated by $\psi$. The other way the construction can be finite is through every step being vacuous from some point on, in which case at most a finite number of the sets $V_{k,l}$ are non-empty. The conclusions of the lemma are immediate in both of these cases. So below we may assume that the construction is infinite. We see at once that for infinitely many $s$, $\gamma_{2s+1} = \{0, 1, \ldots, s-1\}$ by A4. Thus $C$ is injectively enumerable by Theorem 1.11.

**Claim 3.** Let $m_0$, $m_1$ be the leaders of distinct pairs $(k_0, l_0)$, $(k_1, l_1)$ such that $V_{k_0,l_0}$, $V_{k_1,l_1}$ are both finite. Then $\gamma_{m_0} \neq \gamma_{m_1}$ and at most one of $\gamma_{m_0}$, $\gamma_{m_1}$ is an initial segment of $\omega$.

*Proof.* As noted above, $m_0 \neq m_1$. Let $s$ be the greatest number such that either $V_{k_0,l_0,s+1} \neq V_{k_0,l_0,s}$ or $V_{k_1,l_1,s+1} \neq V_{k_1,l_1,s}$. Then $2^s + (m_0/2) \in \gamma_{m_0} - \gamma_{m_1}$, and $2^s + (m_1/2) \in \gamma_{m_1} - \gamma_{m_0}$.□

**Claim 4.** Let $m$ be the leader of $(k, l)$, $f$ the follower, and $V_{k,l}$ be infinite. Then $\psi_f = \gamma_m = \omega$.

*Proof.* From Claim 1, $\gamma_m \subseteq \psi_f$. For the opposite direction let $s > 0$ be any non-vacuous step at which $(m, f)$ is already a proper pair. Since the search in A2 of step $s$ is completed, $\psi_{f,s} \subseteq \psi_{f,t} \subseteq \gamma_m$. So $\psi_f \subseteq \gamma_m$ also. If $(k, l)$ is active at step $s$, then 0, 1, ... , $s - 1$ are enumerated into $\gamma_m$ in A3. This shows that $\gamma_m = \omega$. □

From Claim 3 we have the first conclusion of the lemma and from Claim 4 the second. This completes the proof.

# Chapter 3

# Injectively enumerable classes of cofinite sets

This chapter investigates the characterization problem for c.e. classes of cofinite sets. Section 3.1 deals with classes $\mathcal{C}$ such that $\mathcal{P}_n(\mathcal{C})$ holds for some $n \in \omega$, where $\mathcal{P}_n(\mathcal{C})$ means that $|\omega - A| \leq n$ for all $A \in \mathcal{C}$. Injective enumerability of classes satisfying $\mathcal{P}_1(\mathcal{C})$ will be characterized in terms of a notion of simplicity for subsets of $\omega$. More generally, it will be shown that, if $\mathcal{C}$ is an infinite c.e. class such that $\mathcal{P}_n(\mathcal{C})$, then $\mathcal{C}$ is injectively enumerable if and only if, for every $\mathcal{C}' \subseteq \mathcal{C}$, $|\mathcal{C} - \mathcal{C}'| \leq 2^{n-1}$ implies that $\mathcal{C}'$ is c.e. As this is a necessary condition for $k$-computable enumerability ($k \in \omega \cup \{\omega\}$), it follows from this that all notions $k$-c.e., $k \in (\omega - \{0\}) \cup \{\omega\}$, are equivalent for classes with $\mathcal{P}_n$.

In Section 3.2 three classes are constructed two of which are 1-c.e. and the other 2-c.e. but not 1-c.e., which have very similar structure. These three classes taken together illustrate very clearly the difficulty of finding a non-trivial necessary and sufficient condition for injective enumerability.

In Section 3.3 we find that neither of the injectively enumerable classes from the previous section satisfies any of the known sufficiency criteria. We look at a certain extractibility property which explains the injective enumerability of one of these classes and like classes. At the same time we show that this extractibility property does not yield a sufficient condition in general.

## 3.1 Classes with a bound on the co-cardinality

In this section we characterize injective enumerability for the c.e. classes such that $\mathcal{P}_n(\mathcal{C})$ holds for some bounding number $n$. First we look at classes $\mathcal{C}$ with $\mathcal{P}_1(\mathcal{C})$. Understanding injective enumerability for these classes is essential for the more general case.

### 3.1.1 Injective enumerability for classes with $\mathcal{P}_1$

Observe the following

**Proposition 3.1** *Let $\mathcal{C}$ be an infinite c.e. class such that $\mathcal{P}_1(\mathcal{C})$. Then*

1. *$\mathcal{C}$ is injectively enumerable if and only if $\mathcal{C} - \{\omega\}$ is c.e.*
2. *$\mathcal{C} - \{\omega\}$ is c.e. if and only if $\mathcal{C} - \{\omega\}$ has an infinite c.e. subclass.*

*Proof.* 1. If a c.e. class $\mathcal{C}$ satisfying $\mathcal{P}_1(\mathcal{C})$ is infinite then for every $x > 0$ the set $\{0, \dots, x\}$ has an extension different from $\omega$ in $\mathcal{C}$. By Part 2. of Corollary 1.5, page 9, computable enumerability of $\mathcal{C} - \{\omega\}$ implies injective enumerability of $\mathcal{C}$.

2. The 'only if' part is trivial. The 'if' part can be seen as follows. Let $\mathcal{S}$ be an infinite c.e. subclass of $\mathcal{C} - \{\omega\}$. From a pair $(e, i)$ we can effectively find an index of a computable enumeration of the class $\mathcal{S}_{e,i}$ defined by:

$$\mathcal{S}_{e,i} := \begin{cases} \mathcal{S} & \text{if } i \in W_e, \\ \\ \mathcal{S} \cup \{W_e\} & \text{otherwise.} \end{cases}$$

Now let $V$ be a c.e. set such that $\mathcal{C} = \{W_e : e \in V\}$. Then $\mathcal{C} - \{\omega\}$ is c.e. because it is equal to $\bigcup \{\mathcal{S}_{e,i} : e \in V, i \in \omega\}$. □

A class $\mathcal{C}$ with $\mathcal{P}_1(\mathcal{C})$ is determined by the set $\{x : \omega - \{x\} \in \mathcal{C}\}$ and whether $\omega$ is a member of $\mathcal{C}$. So the question of characterizing injective enumerability of such classes is a question about sets of natural numbers. And because of the observation above the question is:

For which sets of natural numbers $A$ such that the class $\{\omega - \{x\} : x \in A\} \cup \{\omega\}$ is computably enumerable[1], is the class $\mathcal{C}_A := \{\omega - \{x\} : x \in A\}$ also computably enumerable?

An answer can be given in terms of 1-*simple* sets:

**Definition 3.2** A set $A \subseteq \omega$ is called 1-*simple* if $A$ is coinfinite, c.e. and there is no disjoint weak array which covers $\omega - A$ each of whose members intersects $\omega - A$ in exactly one point.

**Theorem 3.3** *For an infinite set $A \subseteq \omega$ such that $\{\omega - \{x\} : x \in A\} \cup \{\omega\}$ is c.e. the class $\mathcal{C}_A := \{\omega - \{x\} : x \in A\}$ is c.e. if and only if there is an infinite subset of $A$ whose complement is c.e. but not* 1*-simple.*

*Proof.* To prove the 'if' part, suppose that $B \subseteq A$ is co-c.e., infinite and its complement is not 1-simple. Let $(F_i)_{i\in\omega}$ be a disjoint weak array witnessing that $\omega - B$ is not 1-simple. Then $\mathcal{B} := \{\omega - \{x\} : x \in B\}$ is enumerated by $(\gamma_n)_{n\in\omega}$ where $\gamma$ is defined by

$$\gamma_n := (\omega - B) \cup \bigcup_{i \neq n} F_i.$$

$\mathcal{B}$ is an infinite c.e. subclass of $\mathcal{C}_A$. By the second part of Proposition 3.1 it follows that $\mathcal{C}_A$ is c.e.

Let us turn to the 'only if' part. If $\mathcal{C}_A$ is c.e., then by Proposition 3.1 it is injectively enumerable, say by $\psi$. We want to show that there is an infinite co-c.e. subset of $A$ whose complement is not 1-simple. We carry out the following construction while simultaneously effectively enumerating the sets $\psi_i, i \in \omega$. Prior to stage $s$ we will have chosen distinct numbers $f_0, \dots, f_{s-1}$.

Stage $s$.

Part 1. Advance the enumeration of the sets $\psi_i$ $(i \in \omega)$ until the following is true: for each $i < s$, if $b_{i,s}$ denotes the least number not in $\psi_{f_i,s}$, then $b_{i,s} \in \psi_{f_j}$ for each $j < s, j \neq i$.

Part 2. Advance the enumeration of the sets $\psi_i$ $(i \in \omega)$ until $k \notin \{f_i : i < s\}$ is found such that $\{n : (\exists i < s)(n \leq b_{i,s})\} \subseteq \psi_k$. Set $f_s = k$.

End of construction.

[1]In fact, the sets $A$ for which $\{\omega - \{x\} : x \in A\} \cup \{\omega\}$ is c.e. are precisely the $\Sigma^0_2$-sets.

For each $i \in \omega$ let $b_i$ denote the unique number such that $\psi_{f_i} = \omega - \{b_i\}$. It is clear that for each $i$, $b_{i,s} = b_i$ for all sufficiently large $s$. Now we define $\overline{B} := \{b_i : i \in \omega\}$ and $F_i := \{b_{i,i+1}, b_{i,i+2}, \ldots\}$. Clearly $\overline{B} \subseteq A$ and it is easy to check that that the sets $F_i$ are pairwise disjoint and cover $\overline{B}$ and that $F_j \cap \overline{B} = \{b_j\}$. Further, $\omega - \overline{B}$ is c.e. because $n$ is in $\omega - \overline{B}$ if and only if there exists $s > n$ such that $n \notin \{b_{0,s}, \ldots, b_{s-1,s}\}$. This is enough.□

**Corollary 3.4** *Let $\mathcal{C}$ be an infinite c.e. class such that $\mathcal{P}_1(\mathcal{C})$. Then $\mathcal{C}$ is injectively enumerable if and only if there exists a c.e. coinfinite set $B \subseteq \omega$ which is not 1-simple such that $B \cup \{x : \omega - \{x\} \in \mathcal{C}\} = \omega$.*

*Proof.* Combine Proposition 3.1 and Theorem 3.3.□

**Theorem 3.5** *Let $H$ be a hyperhypersimple set. Then the c.e. class*

$$\mathcal{A}_{\overline{H}} := \{\omega - \{x\} : x \notin H\} \cup \{\omega\}$$

*is not injectively enumerable.*

*Proof.* Every coinfinite c.e. extension of $H$ is 1-simple.□

### 3.1.2 Injective enumerability for classes with $\mathcal{P}_n$

Note, that if $\mathcal{C}$ is a computably enumerable class with $\mathcal{P}_1(\mathcal{C})$ then $\mathcal{C} - \mathcal{D}$ is c.e. for every finite subclass $\mathcal{D}$ of $\mathcal{C}$ not containing $\omega$. If $\mathcal{C}$ is injectively enumerable or finite, this should be clear. Otherwise, $\mathcal{C}$ contains $\omega$ by Proposition 3.1. In this case, let $\alpha$ be a computable enumeration of $\mathcal{C}$. For $\mathcal{D} = \{\omega - \{d_0\}, \ldots, \omega - \{d_n\}\}$ we define

$$\beta_{\langle i,t\rangle} = \begin{cases} \alpha_i & \text{if } d_0, \ldots, d_n \in \alpha_{i,t}, \\ \\ \omega & \text{otherwise.} \end{cases}$$

Then $\beta$ is a computable enumeration of $\mathcal{C} - \mathcal{D}$. We use this observation to prove the following

**Theorem 3.6** *For every number $n > 0$ there is a c.e. class $\mathcal{C} \subseteq 2^\omega$ such that $\mathcal{P}_n(\mathcal{C})$ holds, $\mathcal{C}$ has no injective enumeration but every subclass $\mathcal{C}' \subseteq \mathcal{C}$ with $|\mathcal{C} - \mathcal{C}'| < 2^{(n-1)}$ is computably enumerable.*

*Proof.* Fix $n > 0$ and choose a hyperhypersimple set $H$. We use the class $\mathcal{A}_{\overline{H}}$ defined above in Theorem 3.5. Define

$$\mathcal{C} := \{n - 1 + A : A \in \mathcal{A}_{\overline{H}}\} \cup \{\omega - B : B \subseteq \{x \in \omega : x < n - 1\}\}.$$

Suppose $\alpha$ is an injective enumeration of $\mathcal{C}$. Then $\beta$ defined by

$$\beta_i := \{x : x + n - 1 \in \alpha_i\}$$

is a computable enumeration of $\mathcal{A}_{\overline{H}}$ in which the set $\omega$ occurs $2^{(n-1)}$ times. By the first part of Proposition 3.1, $\mathcal{A}_{\overline{H}}$ is injectively enumerable, contradiction.

For the second part of the statement, suppose $\mathcal{C}' \subseteq \mathcal{C}$ is such that $|\mathcal{C} - \mathcal{C}'| < 2^{(n-1)}$. Then there is a set $\omega - B \in \mathcal{C}'$ such that $B \subseteq \{x \in \omega : x < n - 1\}$. Let

$$S := \{x : (n - 1) + (\omega - \{x\}) \in \mathcal{C} - \mathcal{C}'\}.$$

Then $\gamma$ defined by

$$\gamma_x := \begin{cases} (n-1) + (\omega - \{x\}) & \text{if } x \notin H \cup S, \\ \\ \omega - B & \text{otherwise,} \end{cases}$$

is a computable enumeration which enumerates all sets from $\mathcal{C}'$, except possibly finitely many sets $\omega - B'$ such that $B' \subseteq \{x \in \omega : x < n - 1\}$. Hence $\mathcal{C}'$ is computably enumerable. □

Our next theorem characterizes the c.e. classes $\mathcal{C}$ with $\mathcal{P}_n(\mathcal{C})$ which are injectively enumerable. The general approach is to work with subclasses of $\mathcal{C}$ which admit a height-function. The bound hinted at in Theorem 3.6 turns out to be sharp:

**Theorem 3.7** *Let $\mathcal{C}$ be an infinite c.e. class such that we have $\mathcal{P}_n(\mathcal{C})$. Then $\mathcal{C}$ is injectively enumerable if and only if every subclass $\mathcal{C}' \subseteq \mathcal{C}$ such that $|\mathcal{C} - \mathcal{C}'| \leq 2^{(n-1)}$ is computably enumerable.*

*Proof.* Necessity is obvious. To prove sufficiency, suppose a class $\mathcal{C}$ is given, such that the hypothesis holds, i.e. $|\omega - A| \leq n$ for all $A \in \mathcal{C}$ and every $\mathcal{C}' \subseteq \mathcal{C}$ with $|\mathcal{C} - \mathcal{C}'| \leq 2^{n-1}$ is c.e. Define the relation $\sqsubseteq$ on finite sets $\sigma, \tau \subseteq \omega$ by:

$$\sigma \sqsubseteq \tau :\Leftrightarrow \sigma = \tau \restriction \max(\sigma) + 1.$$

Note that we let $\max(\emptyset) = -1$. Given a class $\mathcal{B} \subseteq 2^\omega$ we say that a finite set $\sigma \subseteq \omega$ is *$\mathcal{B}$-infinite-branching* if for every $a \in \omega$ there is a set $B \in \mathcal{B}$ such that $\sigma \sqsubseteq \omega - B$ and $\min(\omega - (B \cup \sigma)) > a$. A set is *minimal $\mathcal{B}$-infinite-branching* if it is $\mathcal{B}$-infinite-branching but contains no proper subset which is $\mathcal{B}$-infinite-branching.

**Lemma 3.8** *Let $\mathcal{B} \subseteq 2^\omega$ be a class such that $\mathcal{P}_m(\mathcal{B})$, where $m \geq 1$.*

1. *$\mathcal{B}$-infinite-branching sets have cardinality less than $m$.*
2. *There are only finitely many minimal $\mathcal{B}$-infinite-branching sets.*
3. *For almost every set in $\mathcal{B}$, its complement contains a $\mathcal{B}$-infinite-branching set.*

*Proof.* The first statement holds, because if $\sigma$ is $\mathcal{B}$-infinite-branching, then there exists $\tau$ such that $\sigma \sqsubseteq \tau$, $\omega - \tau \in \mathcal{B}$ and $\min(\tau - \sigma)$ is defined so that $|\sigma| < |\tau| \leq m$. We prove the second and third statement simultaneously by induction on $m$. The base case $m = 1$ is clear because either the empty set is the unique minimal $\mathcal{B}$-infinite-branching set or $\mathcal{B}$ is finite and there are no $\mathcal{B}$-infinite-branching sets. Suppose that the second and third statements hold for $m = k \geq 1$. Let $\mathcal{B} \subseteq 2^\omega$ satisfy $\mathcal{P}_{k+1}(\mathcal{B})$. Let

$$S := \{min(\omega - X) : X \in \mathcal{B} - \{\omega\}\}.$$

There are two cases. If $S$ is infinite, then the empty set is $\mathcal{B}$-infinite-branching and statements 2 and 3 clearly hold. If $S$ is finite, then the empty set is not $\mathcal{B}$-infinite-branching. For each $s \in S$, define

$$\mathcal{B}_s := \{X \cup \{s\} : X \in \mathcal{B} \text{ and } \min(\omega - X) = s\}.$$

Observe that $\sigma$ is $\mathcal{B}$-infinite-branching if and only if $\min(\sigma) = s$ and $\sigma - \{s\}$ is $\mathcal{B}_s$-infinite-branching for some $s \in S$. We have $\mathcal{P}_k(\mathcal{B}_s)$ for each $s \in S$. Applying the induction hypothesis, for each $s \in S$, there are at most a finite number of minimal $\mathcal{B}_s$-infinite-branching sets and all but a finite number of sets in $\mathcal{B}_s$ contain a minimal $\mathcal{B}_s$-infinite-branching set. Since $S$ is finite, statements 2 and 3 for $\mathcal{B}$ follow immediately. This completes the induction step and the proof of the lemma.□

The key to finding the injective enumeration of $\mathcal{C}$ is provided by:

**Lemma 3.9** *For any minimal $\mathcal{C}$-infinite-branching set $\sigma$ the class $\mathcal{C}_\sigma := \{A \in \mathcal{C} : \sigma \sqsubseteq \omega - A\}$ is injectively enumerable.*

*Proof.* It is enough to demonstrate that

$$\mathcal{C}'_\sigma := \mathcal{C}_\sigma - \{\omega - \sigma\}$$

is computably enumerable. This is because

$$h : \{D : D \text{ finite and } (\exists C \in \mathcal{C}'_\sigma)(D \subseteq C)\} \to \omega$$

defined by

$$h(D) := (\mu x)(x > \max(\sigma) \text{ and } x \notin D \text{ and } D \subseteq \omega - \sigma)$$

is a partial computable height-function for $\mathcal{C}'_\sigma$: because $\sigma$ is $\mathcal{C}$-infinite-branching, $h(D)$ is defined just if $D$ has an extension in $\mathcal{C}_\sigma$, and this also ensures that if $h(D)$ is defined, then there is $E$ such that $h(E)\downarrow, D \supset E$ and $h(D) \neq h(E)$. Monotonicity and satisfaction of the ascending chain condition for $h$ follow because $\omega - \sigma \notin \mathcal{C}'_\sigma$ and if $h(D) = x$ then $x = (\mu y)(y > \max(\sigma)$ and $y \notin D)$. By Theorem 1.4, $\mathcal{C}_\sigma$ is injectively enumerable.

We construct a computable enumeration of $\mathcal{C}'_\sigma$ as follows. Since $\mathcal{P}_n(\mathcal{C})$ holds and by the first part of Lemma 3.8 there are at most $2^{(n-1)}$ subsets of $\sigma$ and for

$$\mathcal{C}''_\sigma := \mathcal{C} - \{\omega - M : M \subseteq \sigma\}$$

we have $|\mathcal{C} - \mathcal{C}''_\sigma| \leq 2^{(n-1)}$. By the hypothesis of the theorem, $\mathcal{C}''_\sigma$ is computably enumerable. Let us look at sets $\sigma - \{n\}$ for members $n$ of $\sigma$. These sets are not infinite-branching, because $\sigma$ is minimal infinite-branching and so there must be a finite set $F > \max(\sigma)$ such that

$$(\forall A \in \mathcal{C}''_\sigma)[(\exists n)(\sigma - \{n\} \sqsubseteq \omega - A) \to (\exists x \in F)(x \notin A)].$$

With the help of such a set $F$ we can enumerate $\mathcal{C}'_\sigma$ as follows. Let $\gamma$ be an enumeration of $\mathcal{C}''_\sigma \cap \{A : \{0, \ldots, \max(\sigma)\} - \sigma \subseteq A\}$. Define

$$f_{s+1}(n) := \begin{cases} n & \text{if } \sigma \cap \gamma_{n,s} = \emptyset, \\ m & \text{if } s \text{ is least such that } \sigma \cap \gamma_{n,s} \neq \emptyset \text{ and where} \\ & \langle m, t\rangle \text{ is least such that } (F \cup \gamma_{n,s-1}) \subseteq \gamma_{m,t}, \\ f_s(n) & \text{otherwise.} \end{cases}$$

and

$$\beta_n := \bigcup_{s>0} \gamma_{f_s(n),s}.$$

Now $\beta$ enumerates $\mathcal{C}'_\sigma$. □

To complete the proof of the theorem, let us look at

$$m := \max\{x : (\exists\sigma)(\sigma \text{ is minimal } \mathcal{C}\text{-infinite-branching and } x \in \sigma)\}.$$

If $m = \max\{\emptyset\} = -1$ then $\emptyset$ is the only minimal $\mathcal{C}$-infinite-branching set, and Lemma 3.9 shows that $\mathcal{C}$ is injectively enumerable. So suppose $m \geq 0$. Below we will show that $\mathcal{C}^*$ is computably enumerable, where $\mathcal{C}^*$ denotes

$$\mathcal{C}^* := \{A \in \mathcal{C} : \omega - A \text{ has an element } > m \text{ and a } \mathcal{C}\text{-infinite-branching subset}\}.$$

As above, one verifies that the computable partial function $h$ defined by

$$h(D_i) := (\mu x > m)(x \notin D_i \text{ and } (\exists A \in \mathcal{C}^*)(D_i \subseteq A))$$

is a height-function for $\mathcal{C}^*$. It follows from Theorem 1.4 that $\mathcal{C}^*$ is injectively enumerable. By Lemma 3.8 the difference $\mathcal{C} - \mathcal{C}^*$ is finite. Hence the class $\mathcal{C}$ is injectively enumerable.

Why is $\mathcal{C}^*$ computably enumerable? For every minimal $\mathcal{C}$-infinite-branching set $\sigma$ Lemma 3.9 provides us with an enumeration of

$$\mathcal{C}^*_\sigma := \{A \in \mathcal{C} : \omega - A \text{ has an element } > m \text{ and } \sigma \sqsubseteq \omega - A\}.$$

Using these enumerations we can avoid enumerating sets whose complements have no $\mathcal{C}$-infinite-branching subset. For the purposes of the following construction we suppose given a list $L$ of the minimal $\mathcal{C}$-infinite-branching sets. Further, for each $\sigma$ in $L$ we suppose given an index of an injective enumeration $\theta^\sigma$ of $\mathcal{C}^*_\sigma$. Finally, given an enumeration $\psi$ of $\mathcal{C}$, we define:

$$B_{i,x} := \begin{cases} \psi_i & \text{if } (\forall s)(\{0,\dots,m\} - \psi_{i,s} \text{ has an infinite-branching subset and} \\ & x \notin \psi_{i,s}) \\ \theta^\sigma_m & \text{if } s \text{ is the least such that } \psi_{i,s} \text{ contains } x \text{ or meets every member} \\ & \text{of } L, \sigma \text{ is first in } L \text{ such that } \sigma \cap \psi_{i,s-1} \neq \emptyset \text{ and } \langle m,t\rangle \text{ is least} \\ & \text{such that } \theta^\sigma_{m,t} \supseteq \psi_{i,s-1}. \end{cases}$$

Then $\{B_{i,x} : i, x \in \omega, x > m\} = \mathcal{C}^*$ and $B_{i,x}$ is computably enumerable uniformly in $i$ and $x$. □

## 3.2 Three examples

Classes of cofinite sets not satisfying $\mathcal{P}_n$ allow the encoding of arbitrary c.e. *sequences* of classes of cofinite sets. We say that a sequence $(\mathcal{A}^{(i)})_{i\in\omega}$ of classes is computably enumerable, if there is a computable function $f$ such that $\mathcal{A}^{(i)} = \mathcal{C}^{(f(i))}$ for all $i \in \omega$. Let $D^{(i)} := \{0, 2, 4, \ldots, 2i, 2i+1\}$. We define a mapping

$$\xi \mapsto \mathcal{C}_\xi$$

from c.e. sequences of c.e. classes to c.e. classes as follows. For $\xi = (\mathcal{C}_i)_{i\in\omega}$ we set

$$\mathcal{C}_\xi := \{D^{(i)} \cup (X + (2i+3)) : i \in \omega \text{ and } X \in \mathcal{C}_i\}.$$

This mapping plays a key role below. Notice that $\mathcal{C}_\xi$ is c.e. because $\xi$ is. Moreover, $\mathcal{C}_\xi$ is $n$-c.e. if and only if the members of the sequence $\xi$ are $n$-c.e. uniformly in $i$. This is because for each set $Y$ in $\mathcal{C}_\xi$ one can compute the unique $i$ such that $Y = D^{(i)} \cup (X + (2i+3))$ for some member $X \in \mathcal{C}_i$.

The proof of Theorem 3.7 is not uniform. It requires information about the structure of the class and enumerations for certain subclasses. By Proposition 3.1 for infinite c.e. classes $\mathcal{C}$ of cofinite sets with $\mathcal{P}_1(\mathcal{C})$ it is sufficient for injective enumerability that an infinite subclass not containing $\{\omega\}$ be computably enumerable. The next lemma shows that such a class cannot be obtained uniformly. It is proved in Section 3.2.1.

**Lemma 3.10** *One can uniformly construct, given a c.e. set $V$, an injective enumeration $\beta$ of a class $\mathcal{B}_V$ such that $\mathcal{P}_1(\mathcal{B}_V)$, $\omega \in \mathcal{B}_V$, $|\{A : |\omega - A| = 1\} - \mathcal{B}_V| \leq 1$ and $V \notin \mathcal{B}_V - \{\omega\}$.*

Proposition 3.1 implies that it is sufficient for injective enumerability of c.e. classes $\mathcal{C}$ with $\mathcal{P}_1(\mathcal{C})$ to have an injectively enumerable subclass. But one cannot uniformly obtain an injective enumeration of a c.e. class $\mathcal{C}$ from an injective enumeration of a subclass. This is clear from the following lemma to be proved in Section 3.2.2.

**Lemma 3.11** *Uniformly in a computable enumeration $\psi$ we can injectively enumerate an infinite class $\mathcal{B}_\psi$ such that $\mathcal{P}_1(\mathcal{B}_\psi)$ and $\psi$ does not injectively enumerate $\mathcal{B}_\psi \cup \{\omega\}$.*

From Lemma 3.10 we obtain the uniformly injectively enumerable sequence $\rho := (\mathcal{B}_{W_i})_{i\in\omega}$. From Lemma 3.11 we obtain the two c.e. sequences $\sigma := (\mathcal{B}_{\Omega^{(i)}} \cup \{\omega\})_{i\in\omega}$ and $\tau := (\mathcal{B}_{\Omega^{(i)}})_{i\in\omega}$. Using the mapping $\xi \mapsto \mathcal{C}_\xi$ defined above we obtain classes $\mathcal{C}_\rho$, $\mathcal{C}_\sigma$ and $\mathcal{C}_\tau$. Clearly, $\mathcal{C}_\rho$, $\mathcal{C}_\sigma$ and $\mathcal{C}_\tau$ are all c.e. and $\mathcal{C}_\tau \subseteq \mathcal{C}_\sigma$. All of these classes have the same structure, every finite set that has an extension in one of them, having infinitely many extensions in all three. The difference $\mathcal{C}_\sigma - \mathcal{C}_\tau$ is included in the "trivial" class

$$\mathcal{C}_\omega := \{D^{(i)} \cup (\omega + (2i+3)) : i \in \omega\}.$$

Note that $\mathcal{C}_\omega \subseteq \mathcal{C}_\sigma$ because every member of the sequence $\sigma$ has $\omega$ as a member.

Because the members of the sequences $\tau$ and $\rho$ are uniformly injectively enumerable, $\mathcal{C}_\tau$ and $\mathcal{C}_\rho$ are injectively enumerable. But $\mathcal{C}_\sigma$ is not, because the members of $\sigma$ are not uniformly injectively enumerable. Suppose otherwise and let $f$ be a computable function such that $\Omega^{f(i)}$ injectively enumerates $\sigma_i$, the $i$-th member of $\sigma$. By the recursion theorem there is a number $i_0$ such that $\Omega^{(i_0)} = \Omega^{(f(i_0))}$. Then $\Omega^{(i_0)}$ injectively enumerates $\mathcal{B}_{\Omega^{(i_0)}} \cup \{\omega\}$, contradiction.

However $\mathcal{C}_\sigma$ is 2-c.e. This is because $\mathcal{C}_\omega$ is injectively enumerable, say by $\alpha$. Let $\beta$ injectively enumerate $\mathcal{C}_\tau$. Then $\gamma$ defined by

$$\gamma_n := \begin{cases} \alpha_{n/2} & \text{if } n \text{ is even,} \\ \beta_{(n-1)/2} & \text{otherwise} \end{cases}$$

only repeats sets in $\mathcal{C}_\omega$ and these at most twice.

### 3.2.1 Proof of Lemma 3.10

The idea of the proof is as follows. In stages 0, 1, 2, ... we will simultaneously enumerate the desired sets $\beta_i$. An important part in the construction is played by the injective finite function $g_s$ defined in stage $s$. The domain of $g_s$ is an initial segment of $\omega$, and

$$(\#) \qquad \operatorname{rng}(g_s) = \{\omega\} \cup (\operatorname{dom}(g_s) - \{n_i\}),$$

where we shall explain $n_i$ below. If $g_s(i) = j \in \omega$, our intention at the end of stage $s$ is to make $\beta_i = \omega - \{j\}$. If $g_s(i) = \omega$, our intention at the end of stage $s$ is to make $\beta_i = \omega$.

A crucial role is played by a sequence $(n_0, t_0), (n_1, t_1), \ldots$ defined as follows. Set $n_0 = t_0 = 0$. Suppose that $(n_i, t_i)$ has been defined. Let $s \geq t_i$ denote the least number if any such that

$$\min(\omega - V_s) < s \text{ and } n_i \in V_s \text{ and } |\operatorname{dom}(g_{t_i}) - V_s| \leq 1.$$

Set

$$n_{i+1} = \min(\omega - V_s),\ t_{i+1} = s + 1.$$

We call stages $t_0, t_1, \ldots$ the *critical stages*. In stage $t_i$ we guess that $V = \omega - \{n_i\}$, and this guess guides the way we define $g_s$ in every stage $s$, $t_i \leq s < t_{i+1}$. Thus in (#) above $i$ denotes the greatest number such that $t_i \leq s$.

There is another parameter which will be useful to us: for $i > 0$ such that $t_i \downarrow$, define $m_i = (g_{t_i-1})^{-1}(n_i)$. At stage $t_i$ we have to revise our target for $\beta_{m_i}$ because it was pointed at $\omega - \{n_i\}$ which is our new guess as to what $V$ is.

The details of the construction are as follows.

Stage 0. Set $g_0 = \{(0, \omega)\}$.

Stage $s + 1$. Let $i$ be the greatest number such that $t_i \leq s$. Let $l$ denote $|\text{dom}(g_s)|$.

Part I. There are two cases:

Case 1. $s + 1$ is not a critical stage. Define $g_{s+1} = g_s \cup \{(l, l)\}$.

Case 2. Otherwise. There are two subcases:

Subcase 2.1. $i = 0$ or $m_{i+1} \in \{0,\ m_i\}$. Let $g_{s+1}$ be obtained from $g_s$ by deleting the pairs

$$(m_{i+1}, n_{i+1}),\ ((g_s)^{-1}(\omega), \omega)$$

and adjoining the pairs

$$(m_{i+1}, l),\ \left((g_s)^{-1}(\omega), l+1\right),\ (l, n_s),\ (l+1, \omega).$$

Subcase 2.2. Otherwise. Let $g_{s+1}$ be obtained from $g_s$ by deleting the pairs

$$(0, g_s(0)),\ (m_{i+1}, n_{i+1}),\ ((g_s)^{-1}(\omega), \omega)$$

and adjoining the pairs

$$(m_{i+1}, l),\ \left((g_s)^{-1}(\omega), l+1\right),\ (0, l+2),\ (l, n_s),\ (l+1, \omega),\ (l+2, g_s(0)).$$

Part II. For each $j \in \text{dom}(g_{s+1})$ enumerate into $\beta_j$ every number $x < s$ such that $x \neq g_s(j)$.

End of construction.

For the verification we first establish some notation. Let $g$ denote $\lim_s g_s$. Note that $g(i) = \omega$ means that for all $x \in \omega$ there exists $y \in \omega$ such that $g_s(i) \downarrow$ and $g_s(i) > x$ for all $s > y$. Let $m$, $n$ denote $\lim_s m_s$, $\lim_s n_s$ respectively. For $i > 0$, let $u_i$ denote $t_i - 1$. Let $\beta_{j,s}$ denote the finite set of numbers which have been enumerated in $\beta_j$ by the end of stage $s$.

By induction on $s$, it is easy to check that $g_s$ is injective, $\mathrm{dom}(g_s)$ is an increasing initial segment of $\omega$ of size greater than $s$, and (#) holds. At the same time we see that for all $j \in \mathrm{dom}(g_s)$, $g_s(j) \notin \beta_{j,s}$.

To check the success of the construction we consider four cases.

**Case 1.** There exists $i$ such that $t_i \downarrow$ and $t_{i+1} \uparrow$.

For the discussion of this case let $i$ denote the unique such number. At every stage $s > t_i$, Case 1 holds in Part I. Since $t_{i+1} \uparrow$,

$$V = \omega \text{ or } |\omega - V| > 1 \text{ or } V = \omega - \{n_i\}.$$

Also, $g$ is a bijection from $\omega$ to $\{\omega\} \cup (\omega - \{n_i\})$. So $\beta$ injectively enumerates the class of all $X$ such that $|\omega - X| \leq 1$ and $X \neq \omega - \{n_i\}$.

For the rest we may assume that $t_i \downarrow$ for all $i$. Clearly, $n_i$ is increasing with $i$, and $V = \omega$.

Consider $x \in \mathrm{dom}(g_{u_i}) - \{0, m_{t_{i+1}}\}$. From stage $t_i$ and (#), $g_{t_i}(x) \in \mathrm{dom}(g_{t_i})$. Since $x \neq m_{t_{i+1}}$, $g_{t_i}(x) = g_{u_{i+1}}(x) \in V_{u_{i+1}}$. Hence $g_s(x) = g_{t_i}(x)$ for all $s \geq t_i$, and so

$$(*) \qquad x \in \mathrm{dom}(g_{u_i}) - \{0, m_{t_{i+1}}\} \Longrightarrow g(x) \downarrow = g_{t_i}(x) \neq \omega.$$

Consider $i > 0$ and $y < n_i$. Since $n_i < t_i$ and $y \in V_{u_i}$, $(g_s)^{-1}(y) = (g_{t_i})^{-1}(y)$ for all $s \geq t_i$. Hence

$$(**) \qquad (i > 0 \text{ and } y < n_i) \Longrightarrow y \in \mathrm{rng}(g).$$

**Case 2.** $m \downarrow$ and $0 < m < \omega$.

For all sufficiently large $i$, $m_i = m_{i-1} > 0$ and Subcase 2.1 occurs at stage $t_i$. Thus $g_s(0)$ changes only finitely often, $g(0) < \omega$, and $g(m) = \omega$. From (*) and (**), $g$ is a bijection from $\omega$ to $\omega \cup \{\omega\}$.

**Case 3.** $m \downarrow$ and $m = 0$.

For all sufficiently large $i$, $m_i = 0$ and Subcase 2.1 occurs at stage $t_i$. So $g(0) = \omega$. From (*), for each $x > 0$, $g(x) \downarrow$ and $g(x) < \omega$. So again $g$ is a bijection from $\omega$ to $\omega \cup \{\omega\}$.

**Case 4.** Otherwise.

For infinitely many $i$, Subcase 2.2 occurs at stage $t_i$. Thus $g(0) = \omega$. From (*), for all $x > 0$, $g(x) \downarrow$ and $g(x) < \omega$. Again, $g$ is a bijection from $\omega$ to $\omega \cup \{\omega\}$.

In each of the Cases 2, 3, 4, $V = \omega$ and $\beta$ injectively enumerates the class of all $X$ such that $|\omega - X| \leq 1$. This completes the proof.

### 3.2.2 Proof of Lemma 3.11

Given the computable enumeration $\psi$ we construct an injective enumeration $\beta$ of a class $\mathcal{B}$ for which the conclusion of the lemma holds. The class $\mathcal{B}$ will be equal to $\{A : |\omega - A| = 1\}$ or this class reduced by one member or this class extended by one element, namely $\omega$. For every $n \in \omega$ we rely on the $n$-strategy to ensure that

$$\{\psi_m : m > n\} \not\subseteq \mathcal{B} - \{\omega\}.$$

The strategy consists in choosing a number $p > n$ and leaving $\psi_p$ out of $\mathcal{B}$, if it is not $\omega$. If the strategy is successful, $\psi$ does not injectively enumerate $\mathcal{B} \cup \{\omega\}$. However, the $n$-strategy has an effect on $\mathcal{B}$ only if $n$ is minimal such that $\psi_n = \omega$.

Let $f : (\{-1\} \cup \omega) \times \omega \to \omega$ be a computable binary function such that

1. $f(x, s)$ is increasing in $x$ and non-decreasing in $s$,
2. $f(-1, s) = 0$,
3. $\lim_s f(x, s) = \infty$ if and only if $\psi_y = \omega$ for some $y \leq x$.

Using $f$ we construct a binary computable partial function $g$. The meaning of the function $g$ is that $\beta_m$ will be defined as $\omega - \{\lim_s g(m, s)\}$. The meaning of the function $f$ is that at stage $s + 1$ the $n$-strategy may assign $g(x, s + 1)$ a value different from $g(x, s)$ only if $f(n - 1, s) \leq x < f(n, s)$.

We say that the $n$-strategy *requires attention at stage* $s + 1$ *through* $j$ if there is a number $i$ such that each of the following holds.

C1. $f(n, s) < f(n, s + 1)$ and $(\forall x < n)[f(x, s) = f(x, s + 1)]$,

C2. $i = (\mu k > n)(\min(\omega - \psi_{k,s}) > \max\{g(x, s) : x < f(n - 1, s)\})$, C3. $f(n-1,s) \leq j < f(n,s)$ and $g(j,s) \downarrow = \min(\omega - \psi_{i,s})$,

C4. $\max\{g(x,s) : f(n-1,s) \leq x < g(j,s)\} \leq \min(\omega - (\psi_{i,s} \cup \{g(j,s)\}))$.

Note that from C2 there is only one possibility for $i$.

Construction of $g$.

Stage 0. Do nothing.

Stage $s+1$. There are two cases. Case 1. $s+1$ is odd.

Let $n$ be the least number such that the $n$-strategy requires attention. Denote $n$ by $n(s)$ if it exists. Let $i(s)$ denote the associated $i$ and $j(s)$ denote the least $j$ through which $n = n(s)$ requires attention. If $n(s) \downarrow$, set

$$g(j(s), s+1) = s+1 \text{ and } g(x, s+1) = g(j(s), s)$$

where $x$ is the least number such that $g(x,s) \uparrow$.

Case 2. $s+1$ is even.

Set $g(y, s+1) = \min(\omega - \mathrm{Rng}(\lambda x.g(x,s)))$ where $y$ is the least number such that $g(y,s) \uparrow$.

In either case, for all $k$, if $g(k,s) \downarrow$ and $g(k,s+1)$ has not been specified by the previous instructions, then let $g(k,s+1) = g(k,s)$.

End of construction.

Define

$$\beta_n := \{x : (\exists s > x)(g(n,s) \downarrow \neq x)\}$$

for all indices $n$ and let $\mathcal{B} = \{\beta_n : n \in \omega\}$. Then $\beta$ is defined uniformly in $\psi$ and we verify the following claims.

**Claim 1.** For $\mathcal{B}$ we have $\mathcal{P}_1(\mathcal{B})$.

*Proof.* If $g(n,s+1) \neq g(n,s) \downarrow$ then $g(n+1,s+1) = s+1$ so that $\beta_n$ is either $\omega - \{\lim_s g(n,s)\}$ or $\omega$ depending on whether the limit exists or not.□

**Claim 2.** Suppose $\psi$ enumerates $\omega$. Let $n_0$ be the least index such that $\psi_{n_0} = \omega$.

1. For every $m$ there is a stage $s$ such that if $g(m,t) \neq g(m,t-1)$ at a stage $t > s$ then $n_0$ receives attention at stage $t$.

2. There exists $i_0 > n_0$ such that for all sufficiently large $s$, $n(s) = n_0$ implies $i(s) = i_0$.

*Proof.* Let $f_0$ denote $\lim_s f(n_0 - 1, s)$. Fix $m$. By inspection of the construction, $g(x,s)\downarrow$, if $0 \leq x \leq [\frac{s}{2}] - 1$. Consider a stage $s_0$ such that

- $(\forall x < n_0)(\forall s > s_0)(f(x,s) = f(x,s_0))$,
- $g(x,s_0)\downarrow$ for all $x \leq \max(m, f_0)$, and
- $m < f(n_0, s_0)$.

By C1 no $n < n_0$ receives attention after stage $s_0$. Consider $s \geq s_0$ and $j$ such that $g(j,s+1) \neq g(j,s)\downarrow$. By inspection of the construction $s+1$ is odd, $n(s)\downarrow$, and $j = j(s)\downarrow$. By choice of $s_0$, $n_0 \leq n(s)$. Suppose $n(s) > n_0$. Then by C3 and the properties of $f$,

$$f(n_0, s_0) \leq f(n(s)-1, s) \leq j(s).$$

Since $m < f(n_0, s_0)$, $s = s_0$ witnesses the first part. A similar argument shows that for $m < f_0 = f(n_0 - 1, s_0)$, $g(m,s) = g(m,s_0)$ for all $s > s_0$. Thus, as $s$ increases through values greater or equal $s_0$ such that $n(s) = n_0$, by C2, $i(s)$ is non-increasing, since $\max\{g(x,s) : x < f(n_0 - 1, s)\}$ has a constant value. This is enough for the second part.□

**Claim 3.** If $\beta$ enumerates $\omega$ then so does $\psi$.

*Proof.* Suppose $\beta_j = \omega$. Note that $g(j,s)\downarrow$ for all $s \geq 2j+2$. By definition of $\beta_j$ there exist infinitely many $s$ such that $g(j,s+1) \neq g(j,s)$. By inspection of the construction the set

$$S = \{s : n(s)\downarrow, s \text{ even, and } j(s) = j\}$$

is infinite. From C1, C3 and the properties of $f$,

$$n(s) - 1 \leq f(n(s)-1, 0) \leq f(n(s)-1, s) \leq j$$

and

$$f(n(s), s) < f(n(s), s+1)$$

for all $s \in S$. So there exists $m \leq j+1$ such that $f(m,s) < f(m,s+1)$ for infinitely many $s$. Hence $\psi_m = \omega$ for some $m \leq j+1$.□

**Claim 4.** The enumeration $\beta$ is injective.

*Proof.* By inspection, $\lambda x.g(x,s)$ is injective on its domain. Thus it is sufficient to show that $\beta$ enumerates $\omega$ at most once. Fix $j$ such that $\beta_j = \omega$ and $k \neq j$. From the definition of $\beta_j$, $g(j,s+1) \neq g(j,s)$ for infinitely many $s$. Hence $\lim_s g(j,s) = \infty$. From Claim 3, $\psi$ enumerates $\omega$. Let $n_0$ denote the least $\psi$-index of $\omega$. From Claim 2 there exist $i_0$ and $s_0$ such that or all $s \geq s_0$, if either $g(j,s) \neq g(j,s+1)$ or $g(k,s) \neq g(k,s+1)$,then $n(s) = n_0$ and $i(s) = i_0$. Consider $s_1 \geq s_0$ such that $n(s_1) = n_0$ and $k < g(j,s_1) \neq g(j,s_1+1)$. From C4, $g(k,s_1) \in \psi_{i_0,s_1} \cup \{g(j,s_1)\}$. But $\lambda g(x,s_1)$ is injective, so $g(k,s_1) \in \psi_{i_0,s_1}$.

Consider $s > s_1$ and assume for induction that $g(k,s) = g(k,s_1)$. (Our discussion above shows that $g(k,s_1+1) = g(k,s_1)$.) If $g(k,s+1) \neq g(k,s)$, then $n(s) = n_0$, $j(s) = k$, and $g(k,s) \notin \psi_{i_0,s}$ by C3. This contradicts our finding above, since $g(k,s) = g(k,s_1)$ and $\psi_{i_0,s_1} \subseteq \psi_{i_0,s}$. So $g(k,s+1) = g(k,s)$ and the induction is complete. Clearly, since $\lim_s g(k,s)$ exists, $\beta_k \neq \omega$. □

**Claim 5.** $\psi$ does not injectively enumerate $\mathcal{B} \cup \{\omega\}$.

*Proof.* Let $g(x) := \lim_s g(x,s)$. As above let $n_0$ and $f_0$ denote the least $\psi$-index of $\omega$ and $\lim_s f(n_0 - 1, s)$ respectively. From C1, for all sufficiently large $s$ such that $n(s)\downarrow$, $n(s) \geq n_0$, and from C3, $j(s) \geq f_0$. Thus $g(m)\downarrow$ for all $m < f_0$. Let $i_0$ denote

$$(\mu k > n_0)(\min(\omega - \psi_k) > \max\{g(x) : x < f_0\}).$$

From C2, for all sufficiently large $s$, $n(s) \downarrow = n_0$ implies $i(s) = i_0$. Suppose that $\psi_{i_0} = \omega - \{p\}$.

Case 1. For infinitely many $s$, $n(s) \downarrow = n_0$. Then for all sufficiently large such $s$, $g(j(s),s) = p$ by C3, and $g(j(s),s+1) = s+1$ by the action in stage $s+1$. So there is no $x$ such that $g(x) = p$ — in other words, $\beta$ does not enumerate $\psi_{i_0}$.

Case 2. Otherwise. From Claim 2, $g(x)\downarrow$ for all $x$. For all sufficiently large $s$ such that C1 holds with $n = n_0$ we have: $f(n_0 - 1, s) = f_0$, $\min(\omega - \psi_{i_0,s}) = p$, $g(m,s) = g(m)$ for all $m < \max(f_0, p)$, and

$$\max\{g(x) : f_0 \leq x < p\} \leq \min(\psi_{i_0,s} \cup \{p\}).$$

So for all sufficiently large such $s$, there is no $m$ in the half open interval $[f_0, f(n,s))$ with $g(m,s) = p$. Otherwise, $n_0$ requires attention through some $j \leq m$ at stage $s$. We conclude as in Case 1 that $\beta$ does not enumerate $\psi_{i_0}$.

We conclude that either $\psi$ does not enumerate $\omega$, or $n_0 \downarrow$ and either $\psi_{i_0}$ does not have the form $\omega - \{p\}$ or $\beta$ does not enumerate $\psi_{i_0}$. Since $i_0 > n_0$, this is enough. □

**Claim 6.** If $\psi$ does not enumerate $\omega$, then $\mathcal{B} = \{A : |\omega - A| = 1\}$.

*Proof.* Suppose that $\psi$ does not enumerate $\omega$. From Claim 2, $g(x) \downarrow$ for all $x$. Fix $a \in \omega$. By inspection of the construction, for all sufficiently large $s$, $a \in \mathrm{Rng}(\lambda x.g(x,s))$ and

$$(\#) \qquad (\lambda x.g(x,s))^{-1}(a) \neq (\lambda x.g(x,s+1))^{-1}(a)$$

implies that $n(s) \downarrow$ and $g(j(s),s) = a$. Let $S$ denote the set of all $s$ such that $(\#)$ holds, $n(s) \downarrow$, and $g(j(s),s) = a$. From C2 and C3 for all $s \in S$,

$$\left.\begin{array}{l} a = g(j(s),s) = \min(\omega - \psi_{i(s),s}) > \\ \max\{g(x,s) : x < f(n(s)-1,s)\} \geq \\ f(n(s)-1) - 1 \geq n(s) - 2. \end{array}\right\} (\star)$$

But since $\psi$ does not enumerate $\omega$, $\lim_s f(n,s) \downarrow$ for all $n$. So for each $m$, $n(s) \downarrow = m$ for at most finitely many $s$ by C1. It follows that $S$ is finite which is enough. □

**Claim 7.** $|\{A : |\omega - A| = 1\} - \mathcal{B}| \leq 1$.

*Proof.* Let $n_0$ denote the least $\psi$-index of $\omega$. If $n_0 \uparrow$, we already have the desired conclusion from Claim 6. Fix $a \in \omega$. We pursue the same line of reasoning as in the proof of Claim 6 with the same notation. Suppose $S$ is infinite. By C1 $n_0 \leq n(s)$ for all sufficiently large $s \in S$. Since $\lim_s f(n_0,s) = \infty$ and $f(n_0,s) \leq f(n,s)$ for all $n \geq n_0$, it follows from $(\star)$ that $n(s) = n_0$ for all sufficiently large $s \in S$. Let $i_0$ be the value associated to $n_0$ by Claim 2. Again by $(\star)$, for all sufficiently large $s \in S$, $a = \min(\omega - \psi_{i_0,s})$. So $a = \min(\omega - \psi_{i_0})$. Since $i_0$ does not depend on $a$, this is enough. □

The lemma is proved.

## 3.3 Extracting injective enumerations from others

The existence of the classes $\mathcal{C}_\sigma$ and $\mathcal{C}_\rho$, constructed in Section 3.2 poses the question of how to separate classes of this form with and without an injective enumeration. Also the injective enumerability of $\mathcal{C}_\rho$ (and $\mathcal{C}_\tau$) cannot be explained from any of the known sufficient criteria for injective enumerability. The class $\mathcal{C}_\rho$ (and $\mathcal{C}_\tau$) does not satisfy (E), because the property of $\rho$ implies that

$$\mathcal{C}_\rho - \{D^{(i)} \cup (\omega + (2i+3)) : i \in \omega\}$$

is not computably enumerable. And no injectively enumerable subclass (which is not obtained from an injective enumeration of $\mathcal{C}_\rho$ (or $\mathcal{C}_\tau$)) can be found that would fit the other criteria mentioned in Section 1.3. This poses the question of finding a sufficient criterion for injective enumerability which covers these kinds of classes.

Both questions can be answered in terms of extracting injective enumerations from injective enumerations of larger and natural classes. Given any injective enumeration $\gamma$ and an injective computable function $g$, the enumeration $\gamma(g)$ described by

$$(\gamma(g))_n := \gamma_{g(n)}$$

is clearly injective. In this case we say that $\gamma(g)$ is *extracted* from $\gamma$ by $g$. Clearly, a class $\mathcal{C}$ of c.e. sets is injectively enumerable if and only if an enumeration of $\mathcal{C}$ can be extracted from an injective enumeration of an extension of $\mathcal{C}$. Let $\mathcal{C}$, $\mathcal{C}^*$ denote the class enumerated by $\gamma$,$\gamma(g)$ respectively. If $\mathcal{C}$ is a natural class, whose injective enumerability is obvious, we can explain the injective enumerability of $\mathcal{C}^*$ by the fact that $\mathcal{C}$ has many injective enumerations and from one of them, namely $\gamma$, we can extract an injective enumeration of $\mathcal{C}^*$, namely $\gamma(g)$, by $g$.

The class

$$\mathcal{S} := \{A - \{a\} : A \text{ is an initial segment of } \omega \text{ and } a \in A\} \cup \{\omega\}$$

is obviously injectively enumerable. It turns out that uniformly (modulo the information whether $\omega \in \mathcal{C}$) in an injective enumeration of a class $\mathcal{C}$ with $\mathcal{P}_1(\mathcal{C})$ we can construct an injective enumeration $\gamma$ of $\mathcal{S}$ and an injective computable function $g$ such that $g$ extracts an injective enumeration of $\mathcal{C}$ from $\gamma$:

**Theorem 3.12** *It is possible to construct, uniformly from an injective enumeration of a class $\mathcal{C}$ with $\mathcal{P}_1(\mathcal{C})$ and the information whether $\omega \in \mathcal{C}$, an injective enumeration $\gamma$ of $\mathcal{S}$ and an injective computable function $g$ which extracts an injective enumeration $\mathcal{C}$ from $\gamma$.*

*Proof.* Let $\psi$ be the given injective enumeration. If $\omega \in \mathcal{C}$ we copy $\psi$ into the even sets of $\gamma$. If $\omega \notin \mathcal{C}$ we set $\gamma_0 = \omega$ and copy $\psi$ in the positive even sets of $\gamma$. Let $g(n) := 2n$ for all $n$. The problem is to add the infinite sets of $(\mathcal{S} - \mathcal{C})$ in the sets $\gamma_{2n+1}$. For every $x$ we set up a strategy that enumerates $\omega - \{x\}$ in $\gamma$. The strategy consists in choosing an odd index $m$ and enumerating $\omega - \{x\} \restriction l$ for a suitable $l$ in $\gamma_m$. We keep copying $\omega - \{x\}$ into $\psi_m$, until a set $\psi_y$ appears to be $\omega - \{x\}$. Then we pause until $x$ is enumerated in $\psi_y$. When this happens, we return to copying $\omega - \{x\}$ into $\gamma_m$, until another set $\psi_{y'}$ appears to be $\omega - \{x\}$, and so on. It is possible to pursue all the $x$-strategies so that $\gamma$ is injective and enumerates $\omega - \{x\}$ if and only if $\psi$ does not. To avoid conflict between $x$-strategies one only needs to ensure that at every stage the finite sets enumerated by the strategies are different. The class $\mathcal{S}$ is rich enough to accomplish this. In the following construction we assume that for any $s \in \omega$, $|\{0, \ldots, s\} - \psi_{i,s}| \leq 1$ for all $i < s$.

Construction of $\gamma$.

Stage 0. Set $m_0 = 1$ and $\gamma_{2n} = \psi_n$ for all $n \in \omega$.

Stage s+1.

Part 1. For all $x$ such that $m_x$ is defined, in increasing order do the following.

If there is no $i < s$ such that $\psi_{i,s} = \{0, \ldots, s\} - \{x\}$ then enumerate $\{0, \ldots, l\} - \{x\}$ in $\gamma_{m_x}$ where $l$ is least such that $\{0, \ldots, l\} - \{x\} \neq \gamma_{i,s}$ for all $i \in \omega$.

Part 2. Set $m_{s+1}$ to be the least odd $n$ such that $\gamma_{n,s} = \emptyset$. For all pairs $(x, y)$ such that $\langle x, y \rangle < s$ and the set $\{0, \ldots, x\} - \{y\}$ is different from all sets $\gamma_{2k+1}$ enumerated so far, enumerate $\{0, \ldots, x\} - \{y\}$ in $\gamma_{2c+1}$ where $c$ is least such that $\gamma_{2c+1}$ is empty.

End of construction.

By the action taken in 1 we see that $\omega - \{x\} \notin \mathcal{C}$ if and only if $\gamma_{m_x} = \omega - \{x\}$. Part 2 ensures that all finite sets from $\mathcal{S}$ are enumerated as odd sets of $\gamma$. By the choice of $l$ in Part 1 and the pairs $(x, y)$ in Part 2, $\gamma$ is injective.□

This tells us which classes obtained from uniformly c.e. sequences $(\mathcal{B}_i)_{i\in\omega}$ such that $\mathcal{P}_1(\mathcal{B}_i)$ and $\omega \in \mathcal{B}_i$ for all $i \in \omega$, have an injective enumeration: they are those for which the classes $\mathcal{B}_i$ are uniformly extractable from injective enumerations of $\mathcal{S}$.

This approach does not yield an answer for the general question as is illustrated by:

**Theorem 3.13** *There is a class $\mathcal{C}$ with $\mathcal{P}_2(\mathcal{C})$, which is injectively enumerable but no computable enumeration of $\mathcal{C}$ is extractable from an injective enumeration of an extension $\mathcal{T}$ of $\mathcal{C}$ containing $\{A : |\omega - A| = 1\}$.*

We prove this by using the following lemma.

**Lemma 3.14** *Uniformly in $i, j, x \in \omega$ there is an injective enumeration of a class $\mathcal{C}_{i,j,x}$ with $\mathcal{P}_2(\mathcal{C}_{i,j,x})$ such that* $\min(\omega - C) = x$ *for all $C \in \mathcal{C}_{i,j,x}$ and if $\omega - \{x\} \in \mathcal{C}^{(j)}$ and $\Omega^{(j)}$ is injective then*

$$\omega - \{x\} \in \mathcal{C}_{i,j,x} \Leftrightarrow \omega - \{x\} \notin \{\Omega^{(j)}_{\varphi_i(n)} : n \in \omega\}.$$

From the properties of the classes $\mathcal{C}_{i,j,x}$ it follows that the class $\bigcup_{i,j\in\omega} \mathcal{C}_{i,j,\langle i,j\rangle}$ satisfies the requirements of the theorem.

*Proof of the Lemma.* We carry out the construction of the enumeration $\gamma$ below, which satisfies the statement of the lemma. For every number $n$ we set up a strategy which chooses a number $k$ and enumerates numbers in $\gamma_k$ in order to ensure

$$(\Omega^{(j)}_n = \omega - \{x\} \text{ and } n \notin \text{Rng}(\varphi_i)) \Leftrightarrow (\gamma_k = \omega - \{x\}).$$

Such a number $k$ is called an *attacker* for $n$. At every stage $s$, the set $\gamma_{k,s}$ is a finite set $\{0, \dots, z\} - \{x, u_k\}$ for a suitable $u_k$. If at stage $s$, $\Omega^{(j)}_n$ becomes more like $\omega - \{x\}$, then $u_k$ is increased. This process is modified in the following respect: if $m$ appears in $\text{Rng}(\varphi_{i,t})$ at stage $t$, then $u_k$ is not changed thereafter and $\gamma_k$ will be $\omega - \{x, u_k\}$.

Two further initiatives must be undertaken to avoid conflict between the strategies for $m$ and $n$, where $m < n$. Firstly, we must arrange that at any stage $u_k$ and $u_{k'}$ are different, where $k, k'$ are the attackers for $m, n$ respectively. Secondly, we must ensure that the $m$-strategy and $n$-strategy do not both enumerate $\omega - \{x\}$. This can be achieved by assigning higher priority to the $m$-strategy.

At any stage in which the set $\Omega_m^{(j)}$ gets closer to $\omega - \{x\}$, the $n$-strategy is reset and has to start over again, with a new attacker.

We say that a number $n$ *requires attention* at stage $s$ if

1. $n$ has an attacker and
2. $\max\{l : \{0, \dots, l\} - \{x\} \subseteq \Omega_{n,s+1}^{(j)}\} > \max\{l : \{0, \dots, l\} - \{x\} \subseteq \Omega_{n,s}^{(j)}\}$.

Construction of $\gamma$.

Stage 0. No $n$ has an attacker.

Stage $s + 1$.

Part 1. For all $n$ which require attention, in increasing order of $n$ do the following:

1. If $k$ is the attacker of $n$, and $k$ is in the state "attacking", then set $g(k, s + 1)$ to be the least number greater than $s + 1, x$ which is not in the range of $g$ so far. If in addition $k \in \text{Rng}(\varphi_{i,s})$ then let $k$ be in the state of "preventing" in all following stages.
2. If the attacker of $n$ is in the state of "preventing" then do the following for all $n' > n$ (in increasing order) that have an attacker. Let $k$ be least such that $g(k)$ is undefined so far. Release the present attacker and let $k$ be the new attacker. Set $g(k, s + 1)$ equal to the least number greater than $x$ which is not in the range of $g$ so far.

Part 2. Let $n > x$ be least such that $n$ is not in the range of $g$ so far. Let $y$ be least such that $g(y)$ is not defined so far. Define $g(y, s + 1) = n$.

Part 3. Let $k$ be the least such that $g(k)$ is undefined so far. Let $k$ be the attacker of $s$. Set $g(k, s + 1)$ equal to the least number greater than $x$ which is not in the range of $g$ so far. If $s \in \varphi_{i,t}$ then $s$ is in the state of "preventing", otherwise in the state of "attacking".

End of construction.

Define

$$\gamma_n = \{z \neq x : (\exists s > z)(g(n, s) \neq z)\}$$

and let $\mathcal{C}_{i,j,x} = \{\gamma_n : n \in \omega\}$.

**Verification.** By the assignments of values for $g(i, s)$ the enumeration $\gamma$ does not repeat any set $\omega - \{a, b\}$ for $a \neq b$, we have $\mathcal{P}_2(\mathcal{C}_{i,j,x})$ and $\min(\omega - C) = x$ for all $C \in \mathcal{C}$. If $\omega - \{x\} \notin \mathcal{C}^{(j)}$ then for every attacker $k$ the limit $\lim_s g(k, s)$ exists, and so $\gamma$ does not enumerate $\gamma - \{x\}$. If $\Omega_n^{(i)} = \omega - \{x\}$ and $n$ is the least such number, then there is a number $k$ such that $k$ is the attacker of $n$ at almost all stages. If $n \in \text{Rng}(\varphi_i)$ then $\omega - \{x\} \notin \mathcal{C}_{i,j,x}$ and otherwise it follows by the action taken in b) of Part 1. that only for $k$ is $\gamma_k = \omega - \{x\}$. □

# Chapter 4

# On Extension theorems

We define the following properties for computably enumerable classes.

**Definition 4.1** Let $n \in \omega \cup \{\omega\}$. A computably enumerable class $\mathcal{C}$ is said to have property ($G_n$) if it has an $n$-c.e. subclass which contains, for every finite set, the same number of extensions as $\mathcal{C}$.

The property ($G_n$) is clearly necessary for $n$-computable enumerability. In this chapter we first define properties ($F_n$), $n \in \omega \cup \{\omega\}$, as generalizations for the property (F), informally described on page 10. Lachlan's method for proving Theorem 1.8 is sufficient to prove that ($F_n$) and ($G_n$) imply $n$-computable enumerability for classes of finite sets. We show that ($G_n$) is equivalent to ($F_n$).

For arbitrary classes, ($G_1$) and ($F_1$) are not sufficient for injective enumerability: the non-injectively enumerable class $\mathcal{C}_\sigma$ constructed in Section 3.2 satisfies ($G_1$). This is witnessed by $\mathcal{C}_\tau$, for it contains infinitely many extensions for any finite set extended in $\mathcal{C}_\sigma$. Also, the non-injectively enumerable class $R_1 \cup F$ constructed by Kummer in [8], which was mentioned on page 14, satisfies ($G_1$) and ($F_1$) because, as mentioned, the class $F$ is injectively enumerable.

In Section 4.2 we show that ($G_\omega$) is sufficient for computable enumerability with finite repetitions. Therefore we obtain a characterization of computable enumerability along the lines of Theorem 1.8: a c.e. class is computably enumerable with at most finite repetitions if and only if it satisfies ($F_\omega$).

In Section 4.3 we construct an example to show that ($G_1$) is not sufficient for $n$-computable enumerability.

## 4.1 The properties ($F_n$)

We need the following preliminary definition:

**Definition 4.2** Let $\delta$ be an injective enumeration of ${}^{<\omega}\omega$ (the set of all finite sequences of natural numbers) such that the ternary predicate $x = \delta^{(i)}_n$ and the binary predicate $x = \mathrm{lh}(\delta^{(i)})$ are both computable. Here $\delta^{(i)}$ denotes the $i$-th member of $\delta$, $\mathrm{lh}(\delta^{(i)})$ the length, and $\delta^{(i)}_n$ the $n$-th member. For $\sigma \in {}^{<\omega}\omega$, and $j \in \omega$, $\sigma\hat{\ }j$ denotes the sequence obtained by concatenating $\sigma$ and $\langle j \rangle$.

**Definition 4.3** An infinite c.e. class $\mathcal{C}$ has the property ($F_n$), $n \in \omega$, if there is a partial computable function $\Phi$ such that

F1. if $\Phi(i)\downarrow$ then $\mathrm{lh}(\delta^{(\Phi(i))}) = \mathrm{lh}(\delta^{(i)})$,

F2. if $\delta^{(i)} \subseteq \delta^{(j)}$ and $\Phi(j)\downarrow$, then $\Phi(i)\downarrow$ and $\delta^{(\Phi(i))} \subseteq \delta^{(\Phi(j))}$,

F3. if $\Phi(i)\downarrow$ and $m < \mathrm{lh}(\delta^{(i)})$, then $D_{\delta^{(i)}_m} \subseteq W_{\delta^{(\Phi(i))}_m} \in \mathcal{C}$,

F4. if either $\Phi(i)\downarrow$ or $\delta^{(i)}$ is the empty sequence, and

$$(\star)\quad (\exists X \in \mathcal{C})[D_j \subseteq X \text{ and } (\forall x < \mathrm{lh}(\delta^{(i)}))(X \neq W_{\delta^{(\Phi(i))}_x})]$$

then $\Phi(k)\downarrow$, where $\delta^{(k)} = \delta^{(i)}\hat{\ }j$.

F5. if $\Phi(i)\downarrow$ and $m < \mathrm{lh}(\delta^{(i)})$, then $|\{x : x < m \text{ and } W_{\delta^{(\Phi(i))}_x} = W_{\delta^{(\Phi(i))}_m}\}| < n$,

**Definition 4.4** An infinite c.e. class $\mathcal{C}$ has the property ($F_\omega$), if there is a partial computable function $\Phi$ such that F1, F2, F3, and F4 hold, and if $f : \omega \to \omega$ is a function, such that $\delta^{(f(i))} \subset \delta^{(f(i+1))}$ and $\Phi \circ f$ is total, then $|\{m : (\exists i)(W_{\delta^{(\Phi(f(i)))}_m} = A)\}|$ is finite for all $A \in \mathcal{C}$.

($F_1$) agrees with Lachlan's condition (F). Lachlan [15, Lemma 6] shows that (F) is necessary for a class of c.e. sets to be c.e. without repetition. The proof also shows that ($F_n$) is necessary for $n$-computable enumerability ($n$ finite or infinite). Here we show

**Theorem 4.5** *The properties ($F_n$) and ($G_n$) are equivalent for $n \in \omega \cup \{\omega\}$, $n \neq 0$.*

*Proof of ($G_n$) $\Rightarrow$ ($F_n$).* Let $\mathcal{C}$ be a c.e. class and $\mathcal{D}$ be a c.e. subclass of $\mathcal{C}$ witnessing that $\mathcal{C}$ satisfies ($G_n$). Since ($F_n$) is necessary for $n$-computable enumerability, there exists $\Phi \in P_1$, which witnesses that $\mathcal{D}$ satisfies ($F_n$). We claim that $\Phi$ also witnesses that $\mathcal{C}$ satisfies ($F_n$). The only clause in Definition 4.3 (Definition 4.4) which needs checking, is F4. So suppose $\Phi(i)\downarrow$ or $\delta^{(i)}$ is the empty sequence, ($\star$) holds, and $\delta^{(k)} = \delta^{(i)\frown}j$. If ($\star$) has no witness in $\mathcal{D}$ then $D_j$ has more extensions in $\mathcal{C}$ than in $\mathcal{D}$, contradicting that ($G_n$) holds for $\mathcal{C}$ via $\mathcal{D}$. So ($\star$) has a witness in $\mathcal{D}$ and $\Phi(k)\downarrow$ as required.

*Proof of ($F_n$) $\Rightarrow$ ($G_n$).* Let $\Phi \in P_1$ witness that the c.e. class $\mathcal{C}$ satisfies ($F_n$). Abusing notation we treat $\Phi$ as a partial mapping from ${}^{<\omega}\omega$ into ${}^{<\omega}\omega$. Let $((a_s, b_s))_{s\in\omega}$ be a computable enumeration of $\omega \times \omega$ in which every pair occurs infinitely often. We effectively enumerate sequences $(\sigma_s)_{s\in\omega}$ and $(\tau_s)_{s\in\omega}$ with $\sigma_s, \tau_s \in {}^{<\omega}\omega$. For each $s$ we will have $\Phi(\sigma_s) = \tau_s$; $\Phi(\langle\rangle) = \langle\rangle$, by convention.

Step 0. Set $\sigma_0 = \tau_0 = \langle\rangle$.

Step $s+1$. There are two cases.

Case 1. $a_s \leq \mathrm{lh}(\sigma_s)$ and $\Phi_s((\sigma_s \restriction a_s)^\frown b_s)\downarrow$.

By the remark above and the properties of $\Phi$, $\Phi_s((\sigma_s \restriction a_s)^\frown b_s)$ has the form $(\tau_s \restriction a_s)^\frown i$, where $D_{b_s} \subseteq W_i \in \mathcal{C}$ and $W_i \notin \{W_{(\tau_s)_x} : x < a_s\}$. We speed up the enumeration of the sets $W_{(\tau_s)_x}$, $a_s \leq x < \mathrm{lh}(\tau_s)$, and the computation of $\Phi(\sigma_s{}^\frown b_s)$ until either

(a) $\Phi(\sigma_s{}^\frown b_s)\downarrow$, or

(b) we find $x$, $a_s \leq x < \mathrm{lh}(\tau_s)$, such that $D_{b_s} \subseteq W_{(\tau_s)_x}$.

Since $\Phi$ witnesses (F), either (a) or (b) eventually occurs. In case (a), set

$$\sigma_{s+1} := \sigma_s{}^\frown b_s, \tau_{s+1} := \Phi(\sigma_{s+1}).$$

In case (b) pass to Case 2.

Case 2. Otherwise. Set $\sigma_{s+1} := \sigma_s, \tau_{s+1} := \tau_s$.

End of construction.

Since $(\sigma_s)_{s\in\omega}$ is a $\subseteq$-chain, so is $(\tau_s)_{s\in\omega}$. Let $\tau$ denote $\bigcup_{s\in\omega} \tau_s$ and $\psi_i := W_{(\tau)_i}$. Then $\psi$ enumerates a class $\mathcal{B} \subseteq \mathcal{C}$ which repeats every set at most $n$-times if $n$ is finite, and finitely often if

$n = \omega$. Consider $a \in \omega$ and suppose that $b-1$ is greatest such that $\psi_{b-1}$ extends $D_a$, where we set $b = 0$ if $D_a$ has no extension in $\mathcal{B}$. Suppose that $\mathcal{C}$ has more extensions of $D_a$ than $\mathcal{B}$. Then for all $s$, $\Phi((\tau_s \upharpoonright b)\hat{\ } a) \downarrow$. For $s$ sufficiently large with $(a_s, b_s) = (a, b)$ this computation will be realized in stage $s+1$ and a new extension $\psi_{\mathrm{lh}(\tau_s)}$ of $D_a$ will be adjoined through Case 1(a), contradiction. Hence $\mathcal{B}$ and $\mathcal{C}$ have the same number of extensions of $D_a$ as required.□

## 4.2 An extension theorem

The strength of $(G_n)$ with respect to the number of repetitions required in a computable enumeration of a class satisfying it, is given by:

**Theorem 4.6** *If a c.e. class $\mathcal{C}$ satisfies $(G_\omega)$, witnessed by the class $\mathcal{B}$, then $\mathcal{C}$ is computably enumerable with finite repetitions. Moreover, there is a computable enumeration of $\mathcal{C}$ in which every member of $\mathcal{C} - \mathcal{B}$ appears once, and every member of $\mathcal{B}$ appears finitely often.*

*Proof.* The proof relies on the following lemma.

**Lemma 4.7** *Given a finite collection of c.e. sets $\mathcal{A} = \{\alpha_0, \ldots, \alpha_{n-1}\}$, a c.e. sequence of c.e. sets $(\beta_i)_{i\in\omega}$ and a c.e. set $A$ such that either*

1. *$A \in \mathcal{A} \cup \mathcal{B}$ or*
2. *for every finite $F \subseteq A$, $F \subseteq \beta_i$ for infinitely many $i$,*

*we can effectively enumerate an initial segment $I$ of $\omega$ and c.e. sets $\gamma_i$ ($i \in I$) such that*

- *L1. $\mathcal{C} \subseteq \{A\} \cup \mathcal{B}$,*
- *L2. if $A \notin \mathcal{A} \cup \mathcal{B}$, then $I$ is finite and there is a unique $i \in I$ such that $\gamma_i = A$,*
- *L3. if $A \in \mathcal{A}$, then $\mathcal{C} \subseteq \mathcal{B}$,*
- *L4. if every set occurs at most finitely often in the sequence $(\beta_i)_{i\in\omega}$, then the same is true of $(\gamma_i)_{i\in I}$.*

*Here $\mathcal{B}$ denotes $\{\beta_i : i \in \omega\}$ and $\mathcal{C}$ denotes $\{\gamma_i : i \in I\}$.*

Let $\alpha$ be a computable enumeration of a class $\mathcal{C}$ which satisfies ($\mathrm{G}_\omega$), and let $\beta$ be a computable enumeration which witnesses that $\mathcal{C}$ satisfies ($\mathrm{G}_\omega$). Let $\gamma^{(i)}$ be the enumeration constructed in the proof of the lemma, applied to the collection $\{\alpha_0, \ldots, \alpha_{i-1}, \beta_0, \ldots, \beta_{i-1}\}$, the sequence $\beta_i, \beta_{i+1}, \beta_{i+2}, \ldots$ and the c.e. set $\alpha_i$. Define the enumeration $\gamma'$ by

$$\gamma'_{\langle i,n \rangle} := \gamma^{(i)}_n$$

and let $\gamma$ be obtained from $\gamma'$ by deleting the empty set from $\gamma'$. By the statement of the lemma, $\gamma$ is a computable enumeration of $\mathcal{C} - \{\emptyset\}$ which repeats every set at most finitely often, and each set from $\mathcal{C} - \mathcal{B}$ at most once. The last step is to add $\{\emptyset\} \cap \mathcal{C}$ to $\gamma$, which clearly is uniformly possible.

*Proof of Lemma 4.7.* During the construction we will be simultaneously effectively enumerating the sets $\alpha_0, \ldots, \alpha_{n-1}$, $\beta_j$, $j \in \omega$ and $A$. At stage $s$ only numbers less than $s$ are enumerated. We will also be enumerating the desired initial segment $I$ in increasing order such that only numbers less than $s$ are enumerated at stage $s$.

At stage $s$ we define a mapping $p^s : I_s \to \omega$. We may also appoint followers for $A$ or dismiss them. If $p^s(i) = p^s(i')$ and $i \neq i'$, then $i$ and $i'$ are *companions* at the end of stage $s$. A number has at most one companion. Any follower in existence at the end of stage $s$ is in $I_s$. If $i$ is a follower and $i' \neq i$, then $i'$ is a follower if and only if $i$ and $i'$ are companions. But two numbers can be companions without being followers. We also use a variable called $l$. In the absence of action to the contrary values of $l$, $p^s$ and followers persist.

Stage 0. Set $l = 1$ and $I_0 = p^0 = \emptyset$. There is no follower.

Stage $s + 1$.

Case I. $s + 1$ is even.

Continue enumerating numbers less than $s + 1$ into the sets $\alpha_i$, $\beta_j$ and $A$, performing at least one step in the enumeration until one of the following occurs:

E1. $A \restriction l \in \{\alpha_0 \restriction l, \ldots, \alpha_{n-1} \restriction l\}$.

E2. There exists $j \notin \mathrm{Rng}(p^s)$ such that $A \subseteq \beta_j$.

E3. There exists $e \in I_s$ such that $\beta_{p^s(e)} = A$.

According to which of the three events is the first to occur take the following action:

A1. Dismiss any followers. Increase $l$ by one.

A2. Let $j$ be the first number found witnessing E2. Proceed according to the first of the following subcases which holds:

A2.1. There is no follower. Let $k$ denote $|I_s|$. Enumerate $k$ in $I$, appoint $k$ to be a follower, and set $p^{s+1}(k) = j$.

A2.2. There is a unique follower $i$. Set $p^{s+1}(i) = j$.

A2.3. There are followers $i, i'$ with $i' < i$. Dismiss $i'$ and set $p^{s+1}(i) = j$. $i$ becomes the unique follower.

A3. Let $e$ denote the first witness found for E3 with the proviso that, if $e$ has a companion $e'$, then $e' < e$. Proceed according to the first of the following cases which holds.

A3.1. There is no follower or $e$ is a follower. Do nothing.

A3.2. $e$ has a companion $e'$. Dismiss the existing followers, appoint $e$ and $e'$ as followers.

A3.3. Let $i$ denote the (larger) follower. If $i$ has a companion, dismiss it. Appoint $e$ to be a follower with $i$. Set $p^{s+1}(i) = p^s(e)$.

Case II. $s + 1$ is odd.

If $i$ is a follower at the end of stage $s$, enumerate $A_s$ in $\gamma_i$. If $p^s(i) = j$ and $i$ is not a follower, then enumerate $\beta_{j,s}$ in $\gamma_i$.

End of construction.

Note that by the second hypothesis of the lemma, one of the events E1, E2 and E3 must occur at every even stage. For $i \in I_s$, let $\gamma_{i,s}$ be the set of numbers enumerated in $\gamma_i$ by the end of stage $s$. We make the following two observations.

1. If $i$ is a follower at the end of an odd stage $s$, then $\gamma_{i,s} = A_s \subseteq \beta_{p^s(i),s}$.

2. If $p^s(i)$ is defined and $i$ is not a follower at the end of stage $s$, then $\gamma_{i,s} = \beta_{p^s(i),s}$.

This follows, because whenever $i$ is a follower at the end of stage $s$, $\beta_{p^s(i),s}$ contains $A_s$. Whenever $p^s(i)\downarrow$, $i$ is not a follower at the end of stage $s-1$, but is a follower at the end of stage $s$, then $A_s = \beta_{p^{s-1}(i),s}$.

**Claim 1.** For each $i \in I$ one of the following holds:

$P(i)$. eventually at the end of every stage $i$ is the unique follower or $i$ has a smaller companion.

$Q(i)$. eventually $i$ never has a smaller companion and is never the unique follower.

*Proof.* For induction suppose that the conclusion holds for all $i < n$. Suppose $n \in I_s$ and that each $i < n$ has settled down by the end of stage $s$. Suppose further that, at the end of stage $s$, $n$ is neither the unique follower nor has a smaller companion. Towards a contradiction suppose that stage $t = u + 1 > s$ is the least stage at which either $n$ is the unique follower or $n$ has a smaller companion. Now $n$ can become the unique follower at the end of stage $t$ only by A2.3 which implies that $n$ had a smaller companion at the end of stage $u$, contradiction. Hence $n$ acquires a smaller companion at stage $t$. This implies that A3.3 holds at stage $t$ with $n = e$ and $i < n$. But in this case at stage $t$, $i$ loses a smaller companion, which contradicts the choice of $s$. So there is in fact no stage $t > s$ at the end of which $n$ has either a unique follower or a smaller companion. □

**Claim 2.** Suppose that $i \in I_s$ is not the unique follower at the end of stage $s$ and that $i$ never has a smaller companion at any stage $t \geq s$. Then $p^t(i) = p^s(i)$ for all $t \geq s$.

*Proof.* By inspection of stage $s+1$, particularly A2.2, A2.3, and A3.3, if $p^{t+1}(i) \neq p^t(i)$, then either $i$ is the unique follower at the end of stage $t$ or $i$ has a smaller companion. The latter contradicts our hypothesis. But $i$ must have become the unique follower through an occurrence of A2.3 at a stage $u$, $s < u \leq t$. At the end of stage $u - 1$, $i$ had a smaller companion, contradiction. □

Let $p$ denote the partial function $\lim_{s\to\infty} p^s$. Claim 2 says that $Q := \{i \in I : Q(i)\} \subseteq dom(p)$. Since $p^s(i) = p^s(i')$ and $i \neq i'$ imply that $i$ and $i'$ are companions at the end of stage $s$, $p$ is injective on $Q$. Note that $\gamma_i = \beta_{p(i)}$ for all $i \in Q$.

Let $P$ denote $\{i \in I : P(i)\}$ and let $F$ denote the set of permanent followers. Note that $|F| \leq 2$, and $F = \emptyset$ if $A \in \mathcal{A}$ by the action taken for E1. Choose $c : (P - F) \to \omega$ such that for all $i \in P - F$, $c(i) < i$ and $c(i)$ is the companion of $i$ infinitely often. Clearly, $\mathrm{Rng}(c) \subseteq Q$ and therefore $\gamma_i = \gamma_{c(i)} = \beta_{p(c(i))}$ for $i \in P - F$.

**Claim 3.** Let $C \subseteq \omega$, $Z_1 = \{i \in dom(p) : \beta_{p(i)} = C\}$, and $Z_2 = \{i \in P - F : \beta_{p(c(i))} = C\}$. Then

$$\{i : \gamma_i = C\} = Z_1 \cup Z_2 \cup \begin{cases} F & \text{if } C = A, \\ \emptyset & \text{otherwise.} \end{cases}$$

*Proof.* "$\supseteq$" is clear, since $\gamma_i = \beta_{p(i)}$ if $i \in dom(p)$, $\gamma_i = \beta_{p(c(i))}$ if $i \in P - F$, and $\gamma_i = A$ if $i \in F$. We turn to the "$\subseteq$" part. Suppose $\gamma_i = C$. By Claim 1, $i$ is either in $Q$ or in $P$. For $i \in Q$, $i$ is in $dom(p)$ by Claim 2. If $i \in P$, either $i \in P - F$, and then $\gamma_i = \gamma_{c(i)} = \beta_{p(c(i))}$ or $i \in F$, so that $\gamma_i = A = C$. □

**Claim 4.**

1. If $i \neq j$ and $c(i) = c(j)$, then $\beta_{p(c(i))} = A$.
2. $|\{i \in Q : \beta_{p(i)} = A\}| \geq |\{i \in P - F : \beta_{p(c(i))} = A\}$

*Proof.* 1. Consider $i \in P - F$. Consider $s$ sufficiently large such that at the end of stage $s$, $c(i)$ is not the unique follower nor does it have a smaller companion, and $p^s(c(i)) = p(c(i))$. If $c(i)$ becomes the companion of $i$ at stage $s+1$, then A3.3 holds with $e = c(i)$ and $i$ the (larger) follower at the end of stage $s$. Hence $\beta_{p(c(i)),s} = A_s$. This is enough.

2. Let $X$ denote $\{i \in Q : \beta_{p(i)} = A\}$. If $X$ is infinite, there is nothing to prove. So suppose $X$ is finite. Let $Y$ be a finite subset of $c^{-1}(X)$. There exists $t_0$ such that at all stages greater than $t_0$ every number $y \leq \max(Y)$ has settled in the sense of Claim 1. After stage $t_0$ if $y \in Y$ has a smaller companion, then that companion is in $Q$. Further, there exists $t_1 > t_0$ such that for no $s \geq t_1$, $y \in Y$, and $k \in Q - X$, do we have $A_s = \beta_{p^s(k),s}$. After stage $t_1$, if $y \in Y$ acquires a smaller companion, that companion is in $X$. Thus there exists $t_2 > t_1$, such that for all $y \in Y$ any companion of $y$ at a stage greater or equal $t_2$ is in $X$. At the end of stage $t_2$ each $y \in Y$ is either the unique follower or has a smaller companion in $X$. So $|Y| \leq |X| + 1$. If $|Y| = |X| + 1$, then some $y \in Y$, say $y_0$, is the unique follower. At the first stage $u > t_2$ at which $y_0$ ceases to be the unique follower, $y_0$ must acquire a smaller companion. This is impossible because all members of $X$ are engaged by other members of $Y$. Hence $|Y| \leq |X|$ which completes the proof of the lemma. □

**Claim 5.** If $A \notin \mathcal{A} \cup \mathcal{B}$, then there is a permanent unique follower.

*Proof.* Clearly, $I$ is finite since all followers are dismissed at most finitely often. We first establish that at every sufficiently large stage there is a follower. Because $A \notin \mathcal{A}$, E1 occurs at only finitely many stages. Let $t_0$ be such that E1 does not hold at any stage $s+1 \geq t_0$. Suppose there is no follower at any stage greater than $t_0$. Then at every even stage $s+1 > t_0$ A3.1 is executed with $e$ not a follower and $p(i) = p^{t_0}(i)$ for all $i \in I$. Since $I$ is finite there exists $i \in I$ such that for infinitely many odd $s$, E3 holds at stage $s+1$ with $i = e$. Therefore $\beta_{p(i)} = A$, contradicting $A \notin \mathcal{B}$. So there is a stage $t' > t_0$ at the end of which there is a follower. Since E1 does not occur at any stage greater than $t'$, at every stage greater than $t'$ there is a follower.

Fix $t$ such that

- $I = I_t$,
- by the end of stage $t$ every $i \in I$ has settled down in the sense of Claim 1,
- for all $i \in Q$, $p^s(i) = p(i)$ for all $s \geq t$,
- for all $i \in Q$, $\beta_{p(i),s} = A_s$ never holds, and for $s \geq t$.
- there is a follower at every stage $s \geq t$.

Consider an even stage $s+1 > t$ at which E3 holds for $e$. Suppose $e$ has a companion $e' < e$. Then $Q(e')$, hence $p(e') = p^s(e') = p^s(e)$. So $A_s = \beta_{p^s(e),s} = \beta_{p(e'),s}$, contrary to the choice of $t$. Now suppose that $e$ has no companion and is not a follower. Since $Q(e)$, we have the same contradiction. Thus at any such stage $e$ is the unique follower and A3.1 is executed, i.e. there is no action.

Now consider an even stage $s+1 > t$ at which E2 holds. At the end of stage $s+1$ there is a unique follower. Further, if there is a unique follower $i$ at the end of stage $s$, then $i$ is the unique follower at the end of stage $s+1$.

From the observations above it is clear that there exists a unique follower at all stages greater than $t+2$. □

We are ready to demonstrate that the conclusion of the lemma is satisfied. L1 and L3 follow from Claim 3 and the fact that $F = \emptyset$ if $A \in \mathcal{A}$. L2 follows from Claim 5. Suppose the set $C$ is enumerated finitely often by $\beta$. We show that the sets $Z_1$ and $Z_2$ defined in Claim 3 are finite. (We already stated that $F$ is finite.) Recall that $p$ is injective on $Q$. For $i \in dom(p) - Q$ either $i \in F$ or

$i$ has the same smaller companion $j$ at almost all stages. A number can only have one companion at almost all stages. Altogether, $Z_1$ is finite.

If $C \neq A$, then by the first part of Claim 4, $c$ is injective on the set of $i \in P - F$ such that $\gamma_i = Z$. As $Q \subseteq \text{Rng}(c)$ and $p$ is injective on $Q$, $Z_2$ is finite. For $C = A$, the second part of Claim 4 shows that $Z_2$ is finite. Thus L4 is satisfied.□

The theorem is proved.□

**Corollary 4.8** *A c.e. class is computably enumerable with finite repetitions if and only if it satisfies* $(F_\omega)$.

## 4.3 No extension theorem for $n$-c.e.

Although the classes $R_1 \cup F$ and $\mathcal{C}_\tau$ described in Sections 1.4 and 3.2 do not have injective enumerations, they are 2-c.e. This is because they are *dense* in the following sense:

**Definition 4.9** A computably enumerable class $\mathcal{C}$ is *dense*, if any finite set that has one extension in $\mathcal{C}$ has infinitely many extensions in $\mathcal{C}$.

For a dense class $\mathcal{C}$ with $(G_1)$ set $\mathcal{A}_1 = \mathcal{C}$ and $\mathcal{A}_2 = \mathcal{B}$, where $\mathcal{B}$ witnesses that $\mathcal{C}$ has $(G_1)$. The method of proof of Martin Kummer's Lemma 1.9 yields an enumeration of $\mathcal{C}$ in which every member of $\mathcal{C} - \mathcal{B}$ occurs once, and every member of $\mathcal{B}$ occurs at most twice.[1] The paper [27] gives examples of c.e. classes of finite sets which are not $n$-c.e., but are $(n+1)$-c.e., and of c.e. classes of finite sets which are not $n$-c.e. for any number $n \in \omega$, but are computably enumerable with finite repetitions.

The following theorem strengthens these results, and shows that there is no extension theorem for the property of being $n$-c.e. corresponding to Theorem 4.6.

**Theorem 4.10** *There is a computably enumerable class $\mathcal{C}$ which has the property $(G_1)$ but is not $n$-c.e. for any number $n$.*

Note that such a class cannot be dense, has to have infinite members by Theorem 1.8 and has to be computably enumerable with finite repetitions by Theorem 4.6.

[1] This idea can be found in Kummer's dissertation [10, p 38].

*Proof.* First we define

**Definition 4.11** A c.e. class $\mathcal{C}$ has the property (G') if it has a subclass $\mathcal{B}$ which has an enumeration in which only the empty set appears more than once, and such that $\mathcal{B}$ has the same number of extensions of any finite set, as $\mathcal{C}$ does. A class is *k-c.e. (up to the empty set)* if it has an enumeration in which only the empty set is repeated more than $k$ times.

The proof of the theorem relies on Lemma 4.12:

**Lemma 4.12** *Given an $\Omega$-index $e$ of a computable enumeration $\psi$ and a number $n \geq 1$, it is possible to construct enumerations of classes $\mathcal{C}_{e,n}$ and $\mathcal{B}_{e,n}$ uniformly in $e, i$, such that $\psi$ does not witness that $\mathcal{C}_{e,n}$ is $n$-c.e. (up to the empty set)and $\mathcal{B}_{e,n}$ witnesses that $\mathcal{C}_{e,n}$ satisfies (G').*

The statement of the theorem follows like this. By the $S^m_n$-Theorem there is a computable binary function $r$ such that $\Omega^{r(i,n)}_m = \{z : \langle i, n, z\rangle \in \Omega^{(i)}_m\}$ for all $i, n, m \in \omega$.

Use the lemma to define

$$\mathcal{C} := \bigcup_{i,n\in\omega} \langle i, n, \mathcal{C}_{r(i,n),n}\rangle.$$

Clearly $\mathcal{C}$ satisfies the conclusion of the theorem.

We first describe how to prove the lemma for the case $n = 1$. Given $\psi = \Omega^{(i)}$, we construct an injective enumeration $\beta$ of a class $\mathcal{B}$ and an enumeration $\gamma$ of a class $\mathcal{C}$. We want to ensure that $\beta$ witnesses that $\mathcal{C}_{i,n} := \mathcal{B} \cup \mathcal{C}$ has the property (G'), and $\psi$ does not injectively enumerate $\mathcal{C}_{i,n}$.

The first step is to enumerate a number $a_0$ in the sets $\beta_0$ and $\gamma_0$. Then we wait until $a_0$ appears in a $\psi$-set (If such a set does not exist, $\psi$ is not an enumeration of $\mathcal{C} \cup \mathcal{B}$). The situation looks like this:

$$\beta_0 = \{a_0\} \qquad \gamma_0 = \{a_0\} \qquad \psi_{p_0} = \{a_0\}$$

The next step is to enumerate in $\beta_1$ a number $a_1 \neq a_0$. Again, we wait until $a_1$ appears in a $\psi$-set. The constellation now is:

$$\begin{array}{llllll} \beta_0 = \{a_0\} & \qquad \gamma_0 = \{a_0\} & \qquad \psi_{p_0} = \{a_0\} \\ \beta_1 = \{a_1\} & & \qquad \psi_{p_1} = \{a_1\} \end{array}$$

Now we extend the sets in $\beta$ and $\gamma$ as follows:

$$\begin{array}{lll} \beta_0 = \{a_0, b_0\} & \gamma_0 = \{a_0, a_1\} \\ \beta_1 = \{a_1, a_0\} & \end{array}$$

Here $b_0 \neq a_0, a_1$. We wait until $\psi_{p_1} = \{a_0, b_0\}$ and $\psi_{p_2} = \{a_1, a_0\}$. If this does not occur then $\psi$ either repeats $\beta_0$ or does not enumerate $\mathcal{C} \cup \mathcal{B}$. Otherwise the situation is:

$$\begin{array}{lll} \beta_0 = \{a_0, b_0\} & \gamma_0 = \{a_0, a_1\} & \psi_{p_0} = \{a_0, b_0\} \\ \beta_1 = \{a_1, a_0\} & & \psi_{p_1} = \{a_1, a_0\} \end{array}$$

We repeat this, and if $\psi$ enumerates what is enumerated by $\beta$ and $\gamma$ without duplication, the result after $m$ rounds is

$$\begin{array}{lll} \beta_0 = \{a_0, b_0\} & \gamma_0 = \{a_0, \dots, a_m\} & \psi_{p_0} = \{a_0, b_0\} \\ \beta_1 = \{a_1, a_0, b_1\} & & \psi_{p_1} = \{a_1, a_0, b_1\} \\ \beta_2 = \{a_2, a_1, a_0, b_2\} & & \psi_{p_2} = \{a_2, a_1, a_0, b_1\} \\ \vdots & & \vdots \\ \beta_{m-1} = \{a_{m-1}, \dots, a_0, b_{m-1}\} & & \psi_{p_{m-1}} = \{a_{m-1}, \dots, a_0, b_{m-1}\} \\ \beta_m = \{a_m, \dots, a_0, b_m\} & & \psi_{p_m} = \{a_m, \dots, a_0, b_m\} \\ \beta_{m+1} = \{a_{m+1}\} & & \psi_{p_{m+1}} = \{a_{m+1}\} \end{array}$$

Here all numbers $a_0, \dots a_{m+1}, b_0, \dots b_m$ are chosen to be pairwise different. If this is continued infinitely often, then $\gamma_0 = \{a_i : i \in \omega\}$ and this set is not enumerated by $\beta$ or $\psi$. Moreover every finite subset of $\gamma_0$ has an extension in $\beta$.

The strategy for arbitrary $n > 1$, relies on using the $(n-1)$-strategy, recursively. In order to give a formal description we define the following concepts. An *enumeration sequence* $\sigma$ is a sequence $(\sigma_i)_{i<L}$, where $L = L(\sigma) \in (\omega - \{0\}) \cup \{\omega\}$ and where each $\sigma_i$ is an *instruction* having one of the following forms:

| | |
|---|---|
| $\emptyset$ | "do nothing" |
| $(n, \beta, j)$ | "enumerate $n$ in $\beta_j$" |
| $(m, \gamma, k)$ | "enumerate $m$ in $\gamma_k$" |
| $\perp$ | "terminate" |

where $\sigma_i = \perp$ if and only if $L \in \omega$ and $i = L - 1$. We will be concerned only with enumeration sequences which are either finite or c.e. The enumerating sequence is called *terminating* if $L \in \omega$ and *non-terminating* otherwise. Let $\sigma, \tau$ be enumeration sequences; $\tau$ *extends* $\sigma$ if $\sigma$ terminates, $L(\sigma) < L(\tau)$, and $\tau_i = \sigma_i$ for all $i < L(\sigma) - 1$. With the enumerating sequence $\sigma$ we associate computable enumerations $\beta(\sigma), \gamma(\sigma)$ by

$$\beta(\sigma)_j = \{n : (\exists i < L(\sigma))(\sigma_i = (n, \beta, j))\}$$

$$\gamma(\sigma)_j = \{m : (\exists i < L(\sigma))(\sigma_i = (m, \gamma, k))\}.$$

By $\mathcal{B}(\sigma)$ we denote the class enumerated by $\beta(\sigma)$, and by $\mathcal{C}(\sigma)$ the union of the classes enumerated by $\beta(\sigma)$ and $\gamma(\sigma)$. We ignore occurrences of the empty set in $\beta(\sigma)$ and $\gamma(\sigma)$. There are a number of parameters associated with a terminating enumeration sequence $\sigma$. First, the length is denoted $L(\sigma)$. The *lower bound of* $\sigma$ denoted $\ell(\sigma)$ is the lesser of the least number occurring in any instruction of $\sigma$ ($\infty$ if every instruction is $\emptyset$ or $\perp$) and the least $i$ such that $\sigma_i \neq \emptyset$. The *upper bound* denoted $U(\sigma)$ is the greatest number occurring in any instruction of $\sigma$ (0 if every instruction is $\emptyset$ or $\perp$) and $L(\sigma)$. Furthermore,

$$b(\sigma) = (\mu x)(\forall y \geq x)(\beta(\sigma)_y = \emptyset),$$

$$c(\sigma) = (\mu x)(\forall y \geq x)(\gamma(\sigma)_y = \emptyset).$$

The enumeration sequence $\sigma$ is *good* if $\beta(\sigma)$ witnesses that $\mathcal{C}(\sigma)$ satisfies (G'). A terminating enumeration sequence $\sigma$ is *$n$-good with respect to $\psi$* if G0,...,G5 hold:

G0. $\sigma$ is good.

G1. $b(\sigma), c(\sigma) > 0$.

G2. $(\forall x < b(\sigma))(\beta(\sigma)_x \neq \emptyset \Rightarrow \beta(\sigma)_x - \bigcup_{y \neq x} \beta(\sigma)_y \neq \emptyset)$.

G3. there exists exactly one $i < b(\sigma)$ called the *critical index* for $\sigma$ such that

$$(\forall x < c(\sigma))(\gamma(\sigma)_x \neq \emptyset \Rightarrow \gamma(\sigma)_x = \beta(\sigma)_i).$$

$\beta(\sigma)_{i(\sigma)}$ is called the *critical set.*

G4. $\{\psi_{x,L(\sigma)} : (\exists y < b(\sigma))(\psi_{x,L(\sigma)} \cap \beta(\sigma)_y \neq \emptyset)\} = \{\beta(\sigma)_x : x < b(\sigma), \beta(\sigma)_x \neq \emptyset\}$.

G5. There are pairwise different $i_1, \dots, i_n$ such that $\psi_{i_x, L(\sigma)} = \beta(\sigma)_{i(\sigma)}$ for all $x = 1, \dots n$.

The critical set is denoted by $A(\sigma)$ and the critical index by $i(\sigma)$. Note that $A(\sigma) \neq \emptyset$.

Let $\Omega^{(e)} \restriction \mathcal{C}(\sigma)$ denote the enumeration $\psi$ defined by

$$\psi_i = \begin{cases} \Omega_i^{(e)} & \text{if } (\exists x \in \Omega_i^{(e)})(\exists X \in \mathcal{C}(\sigma))(X \cap \Omega_i^{(e)} \neq \emptyset), \\ \emptyset & \text{otherwise.} \end{cases}$$

**Lemma 4.13** *Given $e, N \in \omega$ and $n \geq 1$ we can effectively generate an enumeration sequence $\sigma$ such that $N < \ell(\sigma)$ and either*

*S1. $\sigma$ is terminating and $n$-good with respect to $\Omega^{(e)}$, or*

*S2. $\sigma$ is non-terminating, good, and $\mathcal{C}(\sigma)$ is not enumerated by $\Omega^{(e)} \restriction \mathcal{C}(\sigma)$ ignoring occurrences of $\emptyset$.*

**Lemma 4.14** *Given $e \in \omega$, $n \geq 1$ and a terminating enumeration sequence $\sigma$ which is 1-good with respect to $\Omega^{(e)}$, we can effectively generate an enumeration sequence $\tau$ extending $\sigma$ such that $\ell(\tau) = \ell(\sigma)$ and one of the following holds:*

*T1. $\tau$ is non-terminating, good, and $\mathcal{C}(\tau)$ is not enumerated by $\Omega^{(e)} \restriction \mathcal{C}(\tau)$ ignoring occurrences of $\emptyset$.*

*T2. $\tau$ is $(n+1)$-good with respect to $\Omega^{(e)}$.*

*T3. $\tau$ is 1-good with respect to $\Omega^{(e)}$, $i(\tau) \geq b(\sigma)$, $\beta(\tau)_x = \beta(\sigma)_x$ for all $x < b(\sigma)$ with $x \neq i(\sigma)$, $\beta(\tau)_{i(\sigma)} \neq A(\tau)$, $A(\sigma)$ is a proper subset of $A(\tau)$, and if $\Omega_{i,L(\sigma)}^{(e)} = A(\sigma)$ then $\Omega_{i,L(\tau)}^{(e)} \neq A(\tau)$.*

We prove Lemma 4.13 for $n$, assuming Lemma 4.14 holds for $n$, and prove Lemma 4.14 for $n+1$ assuming Lemma 4.13 holds for $n$; simultaneously we prove Lemma 4.13 for $n = 1$.

*Proof of Lemma 4.14 assuming Lemma 4.13 for $n$.* Given $e$ and $\sigma$, a terminating enumeration sequence which is 1-good with respect to $\Omega^{(e)}$, we construct an enumeration sequence $\tau$ as follows.

Step 1. Applying the construction of Lemma 4.13 find an enumeration sequence $\sigma'$ such that $U(\sigma) < \ell(\sigma')$ and $\sigma'$ satisfies the conclusion of Lemma 4.13 for $n$.

Step 2. Use $\sigma'$ to extend $\sigma$ to an enumeration sequence $\tau$ by letting

$$\tau_x = \begin{cases} \sigma_x & \text{if } x < L(\sigma) - 1, \\ \\ \sigma'_x & \text{if } L(\sigma) - 1 \leq x < L(\sigma'). \end{cases}$$

Step 3. Suppose $\sigma'$ is terminating. We adjoin instructions to $\tau$ to ensure that

- $\beta(\tau)_{i(\sigma)} = A(\sigma) \cup \{U(\sigma') + 1\}$,
- $\beta(\tau)_{i(\sigma')} = A(\sigma) \cup A(\sigma')$, and
- $(\gamma(\sigma_x) \neq \emptyset$ or $\gamma(\sigma')_x \neq \emptyset) \Rightarrow \gamma_x(\sigma_2) = A(\sigma) \cup A(\sigma')$.

Step 4. Extend $\tau$ by occurrences of $\emptyset$ until G4 holds and if that occurs, extend $\tau$ by $\perp$.

End of construction.

**Verification.** If for Step 2. $\sigma'$ is non-terminating, then $\tau$ is non-terminating and satisfies T1. Suppose Step 2. terminates. Note that for $j \neq i(\sigma), i(\sigma')$, if $\beta(\sigma)_j$ is non-empty, then $\beta(\tau)_j = \beta(\sigma)_j$, and if $\beta(\sigma')_j$ is non-empty, then $\beta(\tau)_j = \beta(\sigma')_j$. Also,

$$\gamma(\tau)_x \neq \emptyset \Rightarrow (\gamma(\sigma)_x \neq \emptyset \text{ or } \gamma(\sigma')_x \neq \emptyset).$$

If for the last step G4 never holds, $\tau$ is non-terminating and satisfies T1. So suppose $\tau$ is terminating. Because $\sigma'$ is $n$-good by S1, there are pairwise different numbers $i_1, \dots i_n$ such that $\Omega^{(e)}_{i_x, L(\sigma')} = A(\sigma')$ for $x = 1, \dots n$. By G4 and since $A(\tau)$ is the unique member of $\mathcal{C}(\tau)$ which includes $A(\sigma')$, $\Omega^{(e)}_{i_x, L(\tau)} = A(\tau)$ for $x = 1, \dots n$. Let $I = \{i : \Omega^{(e)}_{i, L(\sigma)} = A(\sigma)\}$, which is non-empty by assumption on $\sigma$, and none of $i_x \in I$ by assumption on $\sigma'$. Because G4 holds for $\tau$, $\Omega^{(e)}_{i, L(\tau)} \in \{A(\tau), A(\sigma) \cup \{U(\sigma') + 1\}\}$. Now there are two cases. Either there is an $i \in I$, such that $\Omega^{(e)}_i = A(\tau)$; then T2 holds. Or $\Omega^{(e)}_i = A(\sigma) \cup \{U(\sigma') + 1\}\}$ for all $i \in I$; then T3 holds. □

*Proof of Lemma 4.13 assuming Lemma 4.14 for* $n + 1$. Given $e$ and $N$ we construct $\sigma$ as follows. Set $\sigma_i = \emptyset$ for all $i \leq N + 1$ and $\sigma_{N+2} = (N + 1, \beta, N + 1)$, $\sigma_{N+3} = (N + 1, \gamma, N + 1)$. Extend $\sigma$ by occurrences of $\emptyset$ until G4 holds. If $n = 1$ extend $\sigma$ by $\perp$.

If $n > 1$, do the following:

P1. Let $\rho$ be the current sequence $\sigma$ extended by $\perp$.

P2. Set $i = L(\rho)$.

P3. Repeat, while $\tau_i \downarrow\neq \diamond$: set $\sigma_i = \tau_i$ and increase $i$ by one. Here, $\tau$ is obtained from Lemma 4.14 for $e$, $n+1$ and $\rho$.

P4. If $\tau$ is $(n+1)$-good with respect to $\Omega^{(e)}$, extend $\sigma$ by $\perp$. Otherwise continue at P1.

End of construction.

Now, if $\sigma$ is terminating and $n = 1$ holds, then S1 is satisfied. If $\sigma$ is terminating and $n > 1$, then S2 is satisfied, since the last $\tau$ obtained from Lemma 4.14, is $(n+1)$-good. Clearly, if $\sigma$ is non-terminating and $n = 1$, then $\sigma$ is good, and because G4 does not hold, S2 is satisfied. Suppose $\sigma$ is non-terminating and $n > 1$. There are two cases. Either one $\tau$ obtained from Lemma 4.14 satisfies T1 or all satisfy T3. In the first case it follows immediately that $\sigma$ satisfies S2. For the second case, let us denote by $\mathcal{T}$ the set of all $\tau$ obtained in P2. Then $\mathcal{T}$ is linearly ordered by "extends" and has a least element which we denote by $\tau_0$.

If we denote the smallest proper extension of $\tau_n$ in $\mathcal{T}$ by $\tau_{n+1}$, then $\tau_{n+1}$ is obtained by an application of Lemma 4.14 from $\tau_n'$. We have $\mathcal{T} = \{\tau_i : i \in \omega\}$. The sequence $\sigma$ is the union of all $\tau'$, such that $\tau'$ is the largest initial segment of some $\tau \in \mathcal{T}$ not containing $\perp$.

**Claim 1.** If $\gamma(\sigma)_i$ is not empty, then $\gamma(\sigma)_i = \bigcup_{\tau\in\mathcal{T}} A(\tau)$ and this set is infinite.

*Proof.* Let $\gamma(\sigma)_i, \gamma(\sigma)_j \neq \emptyset$. Then there is $n, m$ such that $\gamma(\tau_n)_i, \gamma(\tau_m)_j \neq \emptyset$. Without loss of generality, suppose $n > m$. Then $\gamma(\tau_k)_i = \gamma(\tau_k)_j = A(\tau_k)$ for all $k \geq n$, because $\tau_k$ satisfies G3. By T3, $A(\tau_n) \neq A(\tau_{n+1})$ and therefore $\bigcup_{\tau\in\mathcal{T}} A(\tau)$ is infinite.□

**Claim 2.** All $\beta(\sigma)_i$ are finite.

*Proof.* Let $\beta(\sigma)_i$ be non-empty. Then there is a smallest $n$ such that $\beta(\tau_n)_i$ is non-empty. If $\beta(\tau_n)_i \neq A(\beta_n)$ then, by T3, $\beta(\tau_k)_i = \beta(\tau_n)_i = \beta(\sigma)_i$ for all $k \geq n$. If $\beta(\tau_n)_i = A(\tau_n)$, then $\beta(\tau_{n+1})_i \neq A(\tau_{n+1})$, and again by T3, $\beta(\tau_{n+1})_i = \beta(\tau_k)_i = \beta(\sigma)$ for all $k \geq n+1$. As $\tau_n$ and $\tau_{n+1}$ are terminating, $\beta(\tau_n)_i, \beta(\tau_{n+1})_i$ are finite, and so $\beta(\sigma_n)_i$ is finite.□

**Claim 3.** $\beta(\sigma)$ is injective (up to the empty set).

*Proof.* Let $i \neq j$ such that $\beta(\sigma)_i, \beta(\sigma)_j$ are both non-empty. By the argument used for Claim 2 there is $n$ such that $\beta(\sigma)_i = \beta(\tau_n)_i$ and $\beta(\sigma)_j = \beta(\tau_n)_j$. Because $\tau_n$ satisfies G2, $\beta(\tau_n)_j \neq \beta(\tau_n)_i$. This is enough.□

**Claim 4.** Every finite subset of $\bigcup_{\tau \in \mathcal{T}} A(\tau)$ has infinitely many extensions in $\mathcal{B}(\sigma)$.

*Proof.* Let $F$ be a finite subset of $\bigcup_{\tau \in \mathcal{T}} A(\tau)$. Let $n$ be such that $F \subseteq A(\tau_n)$. By T3, $F \subseteq A(\tau_k)$ for all $k \geq n$. Let $k \geq n$. Because $\tau_k$ satisfies G3, $F \subseteq \beta(\tau_k)_{i(\tau_k)} \subseteq \beta(\sigma)_{i(\tau_k)}$. By T3, $i(\tau_{k+1}) \geq b(\tau_k) > \imath(\tau_k)$ and so $i(\tau_k) \neq i(\tau_{k'})$ for $k \neq k'$.□

**Claim 5.** $\bigcup_{\tau \in \mathcal{T}} A(\tau) \notin \mathcal{C}^{(e)}$, the class enumerated by $\Omega^{(e)}$.

*Proof.* Suppose $\Omega_i^{(e)} = \bigcup_{\tau \in \mathcal{T}} A(\tau)$. Let $s$ be least such that $\Omega_{i,s}^{(e)}$ is non-empty. By Claim 4 there is $x$ such that $\beta(\sigma_x) \supseteq \Omega_{i,s}^{(e)}$. By the argument used for Claim 2 there is $n$ such that $\beta(\sigma)_x = \beta(\tau_m)_x$ for all $m \geq n$. Let $m \geq n$ be such that $L(\tau_m) \geq s$. Then $\Omega_{i,L(\tau_m)}^{(e)} \cap \beta(\tau_m)_x \neq \emptyset$. By G4, $\Omega_{i,L(\tau_m)}^{(e)} = \beta(\tau_m)_j$ for some $j$.

Suppose $j \neq i(\tau_m)$. Then $\beta(\tau_m)_j = \beta(\tau_k)_j$ for all $k \geq m$. By G2 there is $b_0$ such that $b_0 \in (\beta(\tau_m)_j - \beta(\sigma)_{j'})$ for all $j' \neq j$. It follows that $b_0 \in \Omega_i^{(e)}$ and $b_0 \notin A(\tau_m) \subseteq \bigcup_{\tau \in \mathcal{T}} A(\tau)$, contradiction.

Suppose $j = i(\tau_m)$. By T3, $i$ is not an $\Omega_i^{(e)}$-index of $A(\tau_{m+1})$ with respect to $\tau_{m+1}$, contradiction.□ The lemma is proved.

To complete the proof of Lemma 4.12 and of the theorem, let $\beta = \beta(\sigma)$ and $\gamma = \gamma(\sigma)$ where $\sigma$ is obtained (by using the recursion theorem) from the constructions of Lemma 4.13 with $e$, $N = 0$ and $n$ and Lemma 4.14.□

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