# DEFORMATIONS IN ELASTIC DIELECTRICS

by

Kulbir Singh

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# APPROVAL

Name :                 Kulbir Singh

Degree :               Master of Science

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Examining Committee :

Chairman :    C. Villegas

M. Singh
Senior Supervisor

C. Y. Shen

G. Bojadziev

S. Trehan
External Examiner
Professor
Department of Mathematics
Simon Fraser University

Date Approved : December 6, 1988

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# ABSTRACT

The theories governing finite, small but finite, and infinitesimal deformations in elastic dielectrics are reviewed. The exact solutions for homogeneous, isotropic elastic dielectrics are listed. The completeness of the exact solutions when the dielectric is either compressible or incompressible is demonstrated for the finite deformations as well as for infinitesimal deformations. A set of new exact solutions in the small finite deformation theory, previously unknown, are obtained.

# DEDICATION

TO MY PARENTS

# ACKNOWLEDGMENT

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# CHAPTER 1

## 1.    INTRODUCTION

The elastic behaviour of dielectric materials in the presence of electrostatic forces has been a subject matter of study for the last three decades. In 1956, R.A.Toupin [1] formulated a unified theory governing deformations of elastic dielectrics. In this theory, the interaction between the mechanical forces and the electrostatic forces was postulated from the continuum view point for the first time. Toupin assumed the existence of a stored energy function and derived its form through the principle of virtual work. The constitutive equations obtained from such a form then stipulate local and Maxwell stresses as well as polarisation fields distributed continuously throughout the continuum. A few years later, Eringen [2] presented another formulation with almost the same basic features.

The interactions between electrostatic and mechanical forces have also been described by some other expressions based on different hypotheses. Unfortunately, none of these hypotheses answered some of the important questions with regard to the inter–relationships among the electric field, the dielectric displacement field and the mechanical stresses unambiguously. Keeping this in view, Singh and Pipkin [3] formulated a theory of elastic dielectrics based on a single stress hypothesis encompassing both the electrostatic and mechanical forces. We have reproduced that theory in Section 3 of this presentation.

In the finite theory of elastic dielectrics Singh and Pipkin [3] formulated what is known as a controllable state. The deformation and either the electric field or the dielectric displacement field are prescribed initially and it is then verified that such a state can be supported without any body force or distributed charge in every homogenous, isotropic,

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elastic dielectric. The knowledge of the stored energy function as to how it depends on the six invariants of the strain and the electrical field is not required in a controllable state. Such states can therefore be employed experimentally to determine the form of the stored energy function for various dielectric solids. Having thus known the stored energy function, other boundary value problems which are not controllable can then be solved in finite elastic dielectrics.

There is a moderately large class of controllable states in incompressible elastic dielectrics. These are listed in Section 3. Singh and Pipkin [3] proved the completeness of the set of these states. However, when the dielectric is compressible, Singh [4] demonstrated that only homogeneous deformations accompanied with homogenous electrostatic forces can constitute a controllable state.

The controllable states are usually difficult to experiment with because the most general form of the stored energy function is quite cumbersome. Certain polynomial approximations of the stored energy function have been considered by Toupin [1] and Eringen [2], where the polarisation was taken as the independent variable. Singh [5] formulated the various polynomial approximations of the stored energy by assuming the total electric field as the independent variable. We have presented this theory of small but finite deformations in Section 4. When the electrical effects are not present it is shown that the approximate forms of the stored energy function reduce to the well known strain energy forms of Mooney – Rivlin materials [6] and Neo – Hookean solids [7] in finite elasticity.

Considerable analytical simplification can be gained by using suitable approximations to the stored energy function. The first order finite approximation, which reduces to Mooney – Rivlin form [6] in the absence of electrical effects, can be applied to solve problems in which the principal stretches are small and the electrical field strength sufficiently weak. It is to be expected that with approximate forms of the stored energy

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function, the set of controllable states would be considerably larger as compared to when the stored energy function is purely arbitrary. Singh and Trehan [8] proved that when the dielectric is compresible, the only controllable state within the first approximation is still the combination of the homogeneous deformation with the uniform electrostatic field. It then simply follows that only homogenous controllable states are possible for all higher order approximations of the stored energy function. However, when the dielectric is incompressible, there are controllable states which are controllable in approximate theories but not controllable in the general theory.

In this presentation we have attempted to find the controllable states within the formulation of the first order and the second order approximations for incompressible, homogeneous, isotropic, elastic dielectrics. These states, which can not be solved on the basis of a completely arbitrary stored energy function, are presented in Sections 7 and 8. These sections form the original work done for this thesis.

# CHAPTER 2

## 2.    CONTINUUM ELECTROSTATICS

We consider a deformable elastic dielectric continuum that occupies volume V and bounded by surface S.  If we apply mechanical forces and an applied electric field, the body will be deformed and polarised.  The deformation of the body is quasi-static which means that the deformation is so slow that at any instant of time the external forces are in equilibrium with the mechanical and electrical forces inside the dielectric.  We take into account only the electro -mechanical effects and exclude all gravitational and inertial body forces.  It is also to be noted that the assumptions and the equations of continuum electrostatics and mechanics that we write in this section are not dependent on the nature and composition of the material of the dielectric and the media surrounding it.

## 2.1    ELECTROSTATICS

We shall assume the existence of an electric field $\underline{E}$, with the dimensions of force per unit charge, in the continuum as well as in the medium surrounding it.  The electric field being conservative, we have

$$\oint_C \underline{E}.\underline{dx} = 0 \quad , \tag{2.1}$$

where C is any arbitrary closed curve, and $\underline{dx}$ is the vector arc element along C.

Corresponding to the electric field, we shall assume there exists a dielectric displacement field or flux, with the dimensions of charge per unit area, such that

$$\oiint_S \underline{D}.\underline{n} \, dS = Q \quad , \tag{2.2}$$

where S is an arbitrary closed surface, $\underline{n}$ the unit outward normal to S, and Q the total charge enclosed by S.

We consider an arbitrary volume V enclosed by a surface S. We shall assume that the total effect of mechanical and electrical forces due to deformation of the continuum occupying the volume V is statically equivalent to a resultant force $F_i$ and a resultant moment $G_i$ about the origin, both of which can be expressed in terms of a stress distribution $t_i$ in the forms

$$F_i = \oiint_S t_i \, dS \quad , \qquad G_i = \oiint_S e_{ijk} x_j t_k \, dS \quad , \tag{2.3}$$

where $x_i$ are Cartesian coordinates. We have excluded the gravitational and inertial body forces. Also, excluded are surface couples and body couples.

From (2.1), two results follow. Firstly, there exists an electrostatic potential $\Phi$ such that

$$\underline{E} = - \underline{\nabla} \, \Phi \tag{2.4}$$

and secondly, the tangential component of the field $\underline{E}$ is continuous across S :

$$e_{ijk} (E_j^+ - E_j^-) \, n_k = 0 \quad , \tag{2.5}$$

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where $E_j^+$ is evaluated on the side of outward normal $n_i$ and $E_j^-$ being evaluated on the other side of the surface S.

From (2.2), we obtain

$$\underline{\nabla} \cdot \underline{D} = q \quad , \tag{2.6}$$

where q is the charge density per unit volume.

$$(D_i^+ - D_i^-) \, n_i = w \quad , \tag{2.7}$$

where w denotes the charge per unit area of S.

In the formulation of theory and problems in this presentation, we shall set $q = 0$ inside the dielectric and $w = 0$ on the dielectric surface.

## 2.2    EQUILIBRIUM EQUATIONS

Equations of static equilibrium of an arbitrary region with volume V enclosed by a surface S demands

$$\oiint_S t_i \, dS + \oiiint_V \rho \, f_i \, dV = 0 \quad , \tag{2.8}$$

$$\oint_S e_{ijk} x_j t_k \, dS + \oiiint_V \rho \, e_{ijk} x_j f_k \, dV = 0 \quad , \qquad (2.9)$$

where $\rho \, f_i$ represents the body force, other than electrical effects, per unit volume. We have neglected the surface or body couples.

In particular, if V is chosen as an elementary tetrahedron whose three surfaces are parallel to the coordinate axes, then equations (2.8) and (2.9) yield

$$t_i \, (\underline{x} , n) = \sigma_{ji} \, (\underline{x}) \, n_j \quad , \qquad (2.10)$$

where $\sigma_{ji}$ denotes the stress matrix. With the use of the Divergence Theorem, and equation (2.10), equations (2.8) and (2.9) give

$$\sigma_{ij} = \sigma_{ji} \qquad (2.11)$$

and

$$\sigma_{ij,j} + \rho \, f_i = 0 \qquad . \qquad (2.12)$$

It also follows from (2.8) and (2.9) that if a mechanical force $T_i$ per unit area is applied to the surface of the dielectric

$$T_i = (\sigma_{ij}^- - \sigma_{ij}^+) \, n_j \quad , \qquad (2.13)$$

where $\sigma_{ij}^+$ denotes the stress outside the surface S and $\sigma_{ij}^-$ denotes the stress inside the surface S of the dielectric. In the absence of electrical effects, at the surface of the body, one takes $\sigma_{ij}^+$ to be zero. In the present theory, $\sigma_{ij}^+$ shall be the Maxwell stress present everywhere in the medium surrounding the dielectric.

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## 2.3    CONSTITUTIVE EQUATIONS

The flux $D_i$ is directly proportional to the electric field $E_i$ in free space surrounding the dielectric :

$$D_i = \varepsilon\, E_i \quad , \qquad\qquad (2.14)$$

where $\varepsilon$ is the physical constant for the free space.

Maxwell stress $M_{ij}$ is the stress $\sigma_{ij}^{+}$ in the free space given by the expression

$$M_{ij} = \varepsilon\, E_i\, E_j - \frac{\varepsilon}{2} E_k\, E_k\, \delta_{ij} \; . \qquad\qquad (2.15)$$

These two constitutive equations describe the charge free surroundings around the dielectric bodies.

Let us consider a generic particle situated at $X_A$ in the undeformed state $(A = 1, 2, 3)$. Its position in the deformed state will be given by the relation

$$x_i = x_i\,(X_A), \qquad (i = 1, 2, 3) \qquad\qquad (2.16)$$

where $x_i$ and $X_A$ refer to a fixed Cartesian System.

The deformation gradients

$$x_{i,A} = \frac{\partial x_i}{\partial X_A}$$

give the measure of deformation the particle has been subjected to. We will assume that $D_i$ and $\sigma_{ij}$ at a point are functions of $E_i$ and $x_{i,A}$ at that point. That is,

$$D_i = f_i (x_{p,A}, E_q) \ , \quad \sigma_{ij} = \phi_{ij} (x_{p,A}, E_q) \ . \tag{2.17}$$

These relations show that corresponding to one value of deformations gradients and the electric field we will have a unique state of flux and stress.

Now the principle of material indifference requires that when the deformed body is subjected to a rigid rotation along with the field $E_i$ into a new orientation with respect to the coordinate frame $x$, then the force system which includes stress and flux should also undergo the same rotation so as to remain fixed with respect to the body. This restriction demands that the equations (2.17) should be of the form

$$D_i = x_{i,A} F_A (x_{p,P} \ x_{p,Q}, \ x_{p,P} E_p) \ ,$$
$$\sigma_{ij} = x_{i,A} x_{j,B} F_{AB} (x_{p,P} \ x_{p,Q}, \ x_{p,P} E_p) \ . \tag{2.18}$$

For an isotropic medium which is in its undeformed, field free state, equation (2.18) becomes

$$D_i = (A_0 \sigma_{ij} + A_1 g_{ij} + A_2 g_{ik} g_{kj}) E_j \tag{2.19}$$

and

$$\sigma_{ij} = \phi_0 \delta_{ij} + S_{ij} \ , \tag{2.20}$$

where

$$S_{ij} = \phi_1 g_{ij} + \phi_2 g_{ik} g_{kj} + E_i (\phi_3 \delta_{jk} + \phi_4 g_{jk} + \phi_5 g_{jn} g_{nk}) E_k +$$
$$+ E_j (\phi_3 \delta_{jk} + \phi_4 g_{ik} + \phi_5 g_{in} g_{nk}) E_k \tag{2.21}$$

and $g_{ij}$ is the finger strain tensor

$$g_{ij} = x_{i,B} \, x_{j,B} \quad . \tag{2.22}$$

Here, the functions A and $\phi$ are dependent on the following six invariants

$$I_1 = g_{ij} \quad , \quad I_2 = g_{ij} \, g_{ij} \quad , \quad I_3 = E_i \, E_i \quad ,$$

$$I_4 = E_i \, g_{ij} \, E_j \quad , \quad I_5 = E_i \, g_{ij} \, g_{jk} \, E_k \quad , \tag{2.23}$$

$$I_6 = \det g_{ij} \quad . \tag{2.24}$$

For incompressible dielectrics, the invariant $I_6$ is unity for all deformations. In that case, A and $\phi$ are functions of five invariants given by (2.23) and pressure p will be generated as a reaction to the condition of no volume change. Hence, for incompressible materials we will have to write the following equation in place of (2.20)

$$\sigma_{ij} = - p \, \delta_{ij} + S_{ij} \quad , \tag{2.25}$$

where p is arbitrary and $S_{ij}$ is given by (2.21)

There are certain problems which are easier to solve with $D_i$ as the independent variable rather than $E_i$. Interchanging the roles of $E_i$ with $D_i$ in equations (2.17), equations (2.19), (2.20) and (2.21) take the form

$$E_i = (\Omega_0 \, \delta_{ij} + \Omega_1 \, g_{ij} + \Omega_2 \, g_{ik} \, g_{kj}) \, D_j \quad , \tag{2.26}$$

$$\sigma_{ij} = - p \, \delta_{ij} + S_{ij} \quad , \tag{2.27}$$

where

$$S_{ij} = \psi_1 \, g_{ij} + \psi_2 \, g_{ik} \, g_{kj} + D_i \, (\psi_3 \, \delta_{jk} + \psi_4 \, g_{jk} + \psi_5 \, g_{jn} \, g_{nk}) \, D_k +$$

$$+ D_j \, (\psi_3 \, \delta_{ik} + \psi_4 \, g_{ik} + \psi_5 \, g_{in} \, g_{nk}) \, D_k \quad . \qquad (2.28)$$

For incompressible materials, functions $\Omega$ and $\psi$ will be dependent on the invariants

$$J_1 = g_{ii} \quad , \qquad J_2 = g_{ij} \, g_{ij} \quad , \qquad J_3 = D_i \, D_i \quad ,$$

$$J_4 = D_i \, g_{ij} \, D_j \quad , \qquad J_5 = D_i \, g_{ij} \, g_{jk} \, D_k \quad . \qquad (2.29)$$

# CHAPTER 3

3.    THEORY OF FINITE EXACT DEFORMATIONS IN ELASTIC DIELECTRICS

For solving some of the problems of finite deformations in elastic dielectrics, we make use of the inverse method. The deformation and the electrostatic forces are prescribed at the outset and then it is proved that this combination of deformation and the electrostatic forces will result in a controllable state of the material. A controllable state is that state which can be supported without mechanical body force or distributed charge in every homogeneous, isotropic, incompressible, elastic dielectric. Deformations which produce controllable states are called exact deformations.

The solutions of the problems of exact deformations are useful in the experimental determination of the strain – energy function as mentioned earlier in the introduction section. Singh [4] proved that when the dielectric is compressible, any homogenous deformation combined with a uniform electric field or a uniform flux shall form a controllable state and that a homogenous controllable state is the only one possible. However, for incompressible dielectrics, Singh and Pipkin [3] found a large number of controllable states, besides a homogenous state. The authors also proved in [3] that such a set of controllable states forms a complete set. We shall now reproduce all the controllable states for an incompressible, homogeneous, isotropic dielectric.

3.1    BASIC EQUATIONS OF A CONTROLLABLE STATE WITH ELECTRIC
       FIELD AS THE INDEPENDENT VARIABLE

The deformation mapping is:

$$x_i = x_i (x_A) \tag{3.1}$$

The electric field is conservative,

$$E_{i,j} = E_{j,i} \qquad \text{everywhere.} \tag{3.2}$$

The flux is solenoidal

$$D_{i,i} = 0 \qquad \text{everywhere.} \tag{3.3}$$

In the absence of a mechanical body force, equilibrium condition yields

$$p_{,i} = S_{ij,j} \qquad \text{inside the dielectric,} \tag{3.4}$$

where $S_{ij}$ is given by (2.21) and (2.28) depending upon whether the electric field $\underline{E}$ or the dielectric displacement field $\underline{D}$ is considered as the independent variable.

In the medium surrounding the dielectric,

$$\sigma_{ij} = M_{ij} = \varepsilon\, E_i\, E_j - \frac{\varepsilon}{2} E_k\, E_k\, \delta_{ij} \quad , \tag{3.5}$$

where it is clear that equilibrium equation $\sigma_{ij,j} = 0$ is identically satisfied by $M_{ij}$.

At the charge free surface of the dielectric, the boundary conditions are

$$e_{ijk} (E_j^{(0)} - E_j)\, n_k = 0 \quad , \tag{3.6}$$

$$(D_i^{(0)} - D_i)\, n_i = 0 \quad , \tag{3.7}$$

$$T_i = (\sigma_{ij}^{\,0} - \sigma_{ij})\, n_j \quad , \tag{3.8}$$

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where $E_i^{(0)}$, $D_i^{(0)}$ and $\sigma_{ij}^{(0)}$ are evaluated in the medium surrounding the dielectric and $E_i$, $D_i$, $\sigma_{ij}$ are evaluated inside the dielectric.

## 3.2   HOMOGENOUS DEFORMATION IN A UNIFORM FIELD

Consider the deformation of an infinite slab, bounded by the surfaces $X_3 = \pm\, h$ in the undeformed state. Let $\lambda_1, \lambda_2, \lambda_3$ be the extention ratios in the coordinate directions $x_1$, $x_2$, $x_3$, respectively and $k_1$, $k_2$ be the shear amounts in the $x_1$ and $x_2$ directions. Let us also assume that the particle is at point $X_a$ in the undeformed state and then moves to the point $x_i$. The deformation mapping will be

$$x_1 = \lambda_1 X_1 + k_1 \lambda_3 X_3 \ , \qquad x_2 = \lambda_2 X_2 + k_2 \lambda_3 X_3 \ , \qquad x_3 = \lambda_3 X_3 \ ,$$

where for incompressibility $\lambda_1 \lambda_2 \lambda_3 = 1$.

## 3.3   EXPANSION AND EVERSION OF A SPHERICAL SHELL IN A RADIAL FIELD

The deformation mapping considered here is

$$r(R) = \pm (R^3 - R_a^3 \pm r_a^3)^{1/3} \ , \qquad \theta = \pm\, \Theta \ , \qquad \phi = \Phi \ , \qquad (3.9)$$

where a material particle initially at $(R, \Theta, \Phi)$ in spherical polar coordinates has moved to $(r, \theta, \phi)$ after the deformation. Here, $R_a$ and $r_a$ are the internal radii of the shell in the initial and the deformed states, respectively. Expansion or contraction of the shell is indicated by the positive sign, whereas the negative sign in (3.9) indicates eversion of the shell. The physical components of the strain $\underset{\sim}{g}$ are

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$$g_{rr} = \left(\frac{dr}{dR}\right)^2 \quad, \qquad g_{\theta\theta} = g_{\phi\phi} = \left(\frac{r}{R}\right)^2 \quad, \qquad g_{r\theta} = g_{\theta\phi} = g_{\phi r} = 0 \quad .(3.10)$$

This deformation which is volume preserving, can be combined with the prescribed dielectric displacement field

$$D_r = \frac{Q}{4\pi r^2} \quad, \qquad D_\theta = D_\phi = 0 \qquad\qquad (3.11)$$

and it shall form a controllable state. The field equations (3.2) to (3.5) as well as the boundary conditions (3.6) to (3.8) are all satisfied.

## 3.4 CYLINDRICALLY SYMMETRIC DEFORMATIONS OF A TUBE IN A RADIAL FIELD OF FLUX

We consider the deformation of a cylindrical tube described by the mapping

$$r = (AR^2 + B)^{1/2} \quad, \quad \theta = C\Theta + DZ \quad, \qquad z = E\Theta + FZ \quad, \qquad\qquad (3.12)$$

where the particle initially at $(R,\Theta,Z)$ in cylindrical coordinates moves to the position $(r,\theta,z)$. The incompressibility condition requires $A\,(CF - DE) = 1$. The physical components of the strain tensor $\underset{\sim}{g}$ are

$$g_{rr} = \left(\frac{AR}{r}\right)^2 \quad, \qquad g_{\theta\theta} = \left(\frac{Cr}{R}\right)^2 + (Dr)^2 \quad,$$

$$g_{zz} = \left(\frac{E}{R}\right)^2 + F^2 \quad, \quad g_{r\theta} = g_{rz} = 0 \quad, \qquad g_{\theta z} = \frac{CFR}{R^2} + FDR \quad . \qquad\qquad (3.13)$$

This deformation can be combined with the prescribed dielectric displacement field

$$D_r = \frac{Q}{2\pi r} \quad , \qquad D_\theta = D_z = 0 \qquad\qquad (3.14)$$

to form a controllable state. All the field equations and the boundary conditions governing a controllable state, as outlined in the beginning of Section 3, are properly met.

## 3.5   CYLINDRICALLY SYMMETRIC DEFORMATIONS OF A TUBE SECTOR IN A HELICAL ELECTRIC FIELD

The deformation (3.12) can also be considered for controllability when it is combined with an initially prescribed helical electric field

$$E_r = 0 \quad , \qquad E_\theta = \frac{H}{r} \quad , \qquad E_z = \text{constant} \quad .$$

The field equations as well as the boundary conditions governing a controllable state are all satisfied.

## 3.6   DEFORMATIONS OF A CUBOID IN A UNIFORM FIELD OF FLUX

Consider the deformations of a tube wall section, initially bounded by the surfaces $R = R_a$ , $R = R_b$, $\theta = \pm \Theta_0$ and $z = \pm Z_0$ in cylindrical polar coordinates. For this deformation the particle initially at the point $(R,\Theta,Z)$ moves to the position $(x,y,z)$ in a cartesian system according to the following relations:

$$x = AR^2 \quad , \qquad y = B\Theta \quad , \qquad z = \frac{Z}{2AB} + C\Theta \quad . \qquad\qquad (3.15)$$

The components of the finger strain $\underset{\sim}{g}$ are

$$g_{xx} = 4A^2R^2 \ , \qquad g_{xy} = g_{xz} = 0 \ , \qquad g_{yy} = (\tfrac{B}{R})^2 \ ,$$
$$g_{yz} = \frac{BC}{R^2} \ , \qquad g_{zz} = (\tfrac{C}{R})^2 + (\tfrac{1}{2AB})^2 \ .$$

This mapping is volume preserving. With this deformation, we shall prescribe a uniform dielectric displacement field in the x – direction given by

$$D_x = \text{constant} \ , \qquad\qquad D_y = D_z = 0$$

both inside as well as outside the dielectric. It is shown in [3] that such a state is controllable.

## 3.7 DEFORMATIONS OF A CUBOID IN A ELECTRIC FIELD

The deformations given by (3.15) when combined with the prescribed uniform electric field

$$E_x = 0 \ , \qquad\qquad E_y = \text{constant} \ , \qquad\qquad E_z = \text{constant}$$

also forms a controllable state [3].

## 3.8 FLEXURAL DEFORMATIONS OF A BLOCK IN A RADIAL FIELD OF FLUX

For this class of deformations, the particle occupies the final position $(r,\theta,z)$ in cylindrical polar coordinates from its original position $(X,Y,Z)$ in a cartesian system according to the mapping

$$r = AX^{1/2} \quad , \quad \theta = BY \quad , \quad z = \frac{2Z}{A^2B} + CY \quad . \tag{3.16}$$

We assume that the block is infinitely long in the $Z$ – direction and is bounded by two plane surfaces $X$ = constant and two plane surfaces $Y$ = constant before the deformation. Two plane surfaces $X$ = constant become the internal and external cylindrical boundaries of the tube whereas the planes $Y = \pm\frac{\pi}{B}$ are mapped into $\theta = \pm\pi$ as a result of the deformation.

The physical components of strain $\underset{\sim}{g}$ in the cylindrical system $(r,\theta,z)$ are found to be

$$g_{rr} = \frac{A^4}{4r^2} , \qquad g_{r\theta} = g_{rz} = 0 ,$$

$$g_{\theta\theta} = B^2r^2 , \qquad g_{\theta z} = BCr , \qquad g_{zz} = C^2 + \left(\frac{2}{A^2B}\right)^2 \quad .$$

It can be verified that the deformation (3.16) is volume preserving. If the dielectric displacement field

$$D_r = \frac{Q}{2\pi r} , \qquad D_\theta = D_z = 0 \tag{3.17}$$

is prescribed, then such a field combined with the deformation (3.16) generates a controllable state.

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## 3.9   FLEXURAL DEFORMATIONS OF A BLOCK IN A HELICAL ELECTRIC FIELD

If with the deformation (3.16), an electric field of the form

$$E_r = 0 , \qquad E_\theta = \frac{H}{r} , \qquad E_z = \text{constant} \tag{3.18}$$

is prescribed, then such a state of deformation and the field also forms a controllable state.

## 3.10   AZIMUTHAL SHEAR OF A CUBOID IN A UNIFORM AXIAL FIELD

In this deformation, the particle originally at the position $(R,\Theta,Z)$ takes up the position $(r,\theta,z)$ such that

$$r = AR , \qquad \theta = B\log R + C\Theta , \qquad z = \frac{Z}{A^2 C} . \tag{3.19}$$

The components of strain $\underset{\sim}{g}$ are

$$g_{rr} = A^2 , \qquad g_{r\theta} = A^2 B , \qquad g_{rz} = 0 ,$$
$$g_{\theta\theta} = A^2 (B^2 + C^2) , \quad g_{\theta z} = 0 , \qquad g_{zz} = \frac{1}{A^4 C^2} . \tag{3.20}$$

The incompressibility condition is identically satisfied.

With the strains (3.20), we prescribe the electric field

$$E_r = E_\theta = 0 , \qquad E_z = \text{constant} . \tag{3.21}$$

The deformation (3.19) and the electric field (3.21) yield a controllable state.

# CHAPTER 4

## 4. SMALL FINITE DEFORMATION THEORIES

The early development in the area of small finite deformations in finite elasticity theory was initiated by Murnaghan and Rivlin. Singh [5] formulated a similar theory of small finite deformations for elastic dielectrics. He assumed the stored energy to be a function of deformation gradients and the applied electric field. Basic equations which relate stress, strain, electric field and dielectric displacement field were obtained. Special polynomial forms of the stored energy function were derived. Two other researchers, Toupin [1] and Eringen [2], have also worked on elastic dielectric problems by considering certain polynomial approximations of the strain energy function. Their treatment of the problems is different from that of Singh's.

## 4.1 CONSTITUTIVE EQUATIONS

Once again, the constitutive equations applicable to the charge free medium surrounding the dielectric are the following:

$$D_i = \varepsilon E_i \quad , \tag{4.1}$$

$$M_{ij} = \sigma_{ij}^{+} = \varepsilon \left( E_i E_i - \frac{1}{2} E_k E_k \delta_{ij} \right) \quad . \tag{4.2}$$

Here, $\varepsilon$ is the physical constant for free space and $\sigma_{ij}^{+}$ is the Maxwell's stress tensor. It is obvious that $M_{ij}$ identically satisfies the equilibrium equations without body forces.

As we are only concerned with the homogenous and perfectly elastic dielectric bodies, we shall assume there exists a stored energy function W in the dielectric. This stored energy function is assumed to depend on the electric field $E_i$ and the deformation gradients

$$W = W (x_{i,a}, E_k) \qquad (4.3)$$

Following the procedure outlined in [5], we obtain the following constitutive equations

$$\sigma_{ij} = \rho \frac{\partial W}{\partial \left(\frac{\partial x_i}{\partial X_k}\right)} \frac{\partial x_j}{\partial X_k} + \rho \frac{\partial W}{\partial E_i} E_j \qquad (4.4)$$

and

$$D_i = \rho \frac{\partial W}{\partial E_i} \qquad (4.5)$$

where $\rho$ is the mass density measured in the deformed configuration.

The principle of material indifference places some restriction on the form of W. According to this principle if a dielectric body experiences a rigid rotation together with the field $E_i$, then the force system will undergo the same rotation. This restriction requires stored energy to be of the form

$$W = W \left(\frac{\partial x_k}{\partial X_p} \frac{\partial x_k}{\partial X_q}, \frac{\partial x_i}{\partial X_j} E_i\right) \quad . \qquad (4.6)$$

If we take a dielectric which is isotropic in its undeformed, field free state, then the stored energy function W has to be a function of the six scalar invariants

$$W = W (I_1, I_2, I_3, I_4, I_5, I_6, ) \quad , \qquad (4.7)$$

where

$$I_1 = g_{ii} \quad , \qquad I_2 = \frac{1}{2}(g_{ii} g_{jj} - g_{ij} g_{ij}) \quad , \qquad I_3 = \det g_{ij} \quad ,$$

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$$I_4 = E_i \, E_i \quad , \quad I_5 = E_i \, g_{ij} \, E_j \quad , \qquad\qquad I_6 = E_i \, g_{ij} \, g_{jk} \, E_k \quad . \quad (4.8)$$

Substitution of W given by (4.7) into the constitutive equations (4.4) and (4.5) and the use of Cayley –Hamilton theorem reduces the constitutive equations for an isotropic, homogenous, elastic dielectric to the form

$$\begin{aligned}
\sigma_{ij} = \frac{2\rho_0}{\sqrt{I_3}} \Big[ & \left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2}\right) g_{ij} - \frac{\partial W}{\partial I_2} g_{ij}{}^2 + I_3 \frac{\partial W}{\partial I_3} g_{ij} \\
& + \frac{\partial W}{\partial I_4} E_i \, E_j + \frac{\partial W}{\partial I_5} \left(g_{ik} \, E_k \, E_j + g_{jk} \, E_k \, E_i\right) \\
& + \frac{\partial W}{\partial I_6} \left(g_{ik}{}^2 \, E_k \, E_j + g_{jk}{}^2 \, E_k \, E_i + g_{ik} \, g_{jp} \, E_k \, E_p\right) \Big]
\end{aligned} \qquad (4.9)$$

and

$$D_i = \frac{2\rho_0}{\sqrt{I_3}} \left(\frac{\partial W}{\partial I_4} \delta_{ij} + \frac{\partial W}{\partial I_5} g_{ij} + \frac{\partial W}{\partial I_6} g_{ij}{}^2\right) E_j \quad , \qquad (4.10)$$

where

$$g_{ij}{}^2 = g_{ik} \, g_{kj} \qquad \text{and } \rho = \frac{\rho_0}{\sqrt{I_3}} \quad .$$

Here, $\rho_0$ is the mass density of the undeformed dielectric.

## 4.1    APPROXIMATE THEORIES

Assuming that the stored energy function W $(I_1, I_2, I_3, I_4, I_5, I_6)$ can be expressed as a polynomial in the invariants $I_k$, we may write

$$W = \sum_{\alpha\beta\gamma\delta\lambda\mu} A_{\alpha\beta\gamma\delta\lambda\mu} \, (I_1 - 3)^\alpha (I_2 - 3)^\beta (I_3 - 1)^\gamma I_4{}^\delta \, I_5{}^\lambda \, I_6{}^\mu \quad . \qquad (4.11)$$

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Here, $A_{\alpha\beta\gamma\delta\lambda\mu}$ represent material constants.

Different approximate forms of W are obtained depending on the number of terms we retain in (4.11), furnishing us with various approximate deformation theories for the homogeneous, isotropic, elastic dielectric.

Let $e_1$, $e_2$, $e_3$, denote the principal extentions at a point P of the dielectric body and $E_1$, $E_2$, $E_3$ the components of the electric field referred to the principal directions of strain at P. Then the invariants (4.8) can be written as

$$I_1 = (1+e_1)^2 + (1+e_2)^2 + (1+e_3)^2 \ ,$$

$$I_2 = (1+e_1)^2(1+e_2)^2 + (1+e_2)^2(1+e_3)^2 + (1+e_3)^2(1+e_1)^2 \ ,$$

$$I_3 = (1+e_1)^2(1+e_2)^2(1+e_3)^2 \ ,$$

$$I_4 = E_1^2 + E_2^2 + E_3^2 \ ,$$

$$I_5 = (1+e_1)^2 E_1^2 + (1+e_2)^2 E_2^2 + (1+e_3)^2 E_3^2 \ ,$$

$$I_6 = (1+e_1)^4 E_1^2 + (1+e_2)^4 E_1^2 + (1+e_3)^4 E_3^2 \ . \tag{4.12}$$

The series (4.11) can be written in a better form by defining a new set of invariants $J_k$

$$J_1 = I_1 - 3 \ ,$$

$$J_2' = (I_2 - 3) - 2\,(I_1 - 3) \ ,$$

$$J_3 = (I_3 - 1) - (I_2 - 3) + (I_1 - 3) \ ,$$

$$J_4 = I_4 \ ,$$

$$J_5 = I_5 - I_4 \ ,$$

$$J_6 = I_6 - 2\,(I_5 - I_4) - I_4 \ . \tag{4.13}$$

Like invariants $I_k$, the invariants $J_k$ also form a complete set for an isotropic dielectric. In terms of the new invariants $J_k$,

$$W = \sum_{\alpha\beta\gamma\delta\lambda\mu} B_{\alpha\beta\gamma\delta\lambda\mu} \, J_1{}^{\alpha} \, J_2{}^{\beta} \, J_3{}^{\gamma} \, J_4{}^{\delta} \, J_5{}^{\lambda} \, J_6{}^{\mu} \quad , \tag{4.14}$$

where $B_{\alpha\beta\gamma\delta\lambda\mu}$ represent material constants.

From (4.12) and (4.13), we observe that the invariants $J_k$ have the property

$$J_1 = 0 \; (e_i) \quad ,$$
$$J_2 = 0 \; (e_i{}^2) \quad ,$$
$$J_3 = 0 \; (e_i{}^3) \quad ,$$
$$J_4 = 0 \; (E_i{}^2) \quad ,$$
$$J_5 = 0 \; (e_k E_i{}^2) \; ,$$
$$J_6 = 0 \; (e_k{}^2 \, E_i{}^2) \quad . \tag{4.15}$$

The polynomial (4.14) can give us different approximate forms of W depending on the order of the principal extentions and powers of the electric field we choose to retain in the expression. These approximate forms of W shall be satisfying the principle of material indifference and thus give a complete theory. Such forms will be invariant under all rigid rotations of the dielectric and the electric field. They can be compared to the approximate theories of finite elasticity like Mooney – Rivlin materials and the Neo – Hookean materials.

## 4.2    FIRST APPROXIMATION

For small principle extentions and weak electric fields, the first approximation is defined by retaining in W only terms involving principle extentions $e_i$ up to second powers, terms involving the electric field to second powers in components $E_i$ and product terms of the

type $e_i E_k^2$. With this definition (4.14) and (4.15) give the following form of the energy function :

$$W = a_0 + a_1 J_1 + a_2 J_2 + a_3 J_1^2 + a_4 J_4 + a_5 J_5 + a_6 J_1 J_4 \quad , \quad (4.16)$$

where $a_0, a_1, \ldots, a_6$ are material constants.

Since we want the stored energy W as well as stresses to vanish when in the field free undeformed state, we shall take $a_0 = a_1 = 0$.

Hence (4.16) reduces to

$$W = a_2 J_2 + a_3 J_1^2 + a_4 J_4 + a_5 J_5 + a_6 J_1 J_4 \quad . \quad\quad\quad (4.17)$$

In the absence of an electric field, equation (4.17) reduces to Mooney – Rivlin form in finite elasticity

$$W = a_2 J_2 + a_3 J_1^2 \quad .$$

If we further neglect terms higher than second in the displacement gradients $\dfrac{\partial u_i}{\partial x_j}$, the field components $E_k$ and also product terms of order higher than $E_k^2 \dfrac{\partial u_i}{\partial X_j}$, then (4.17) takes the form

$$W = 2 a_2 (e_{ii} e_{jj} - e_{ij} e_{ij}) + 2 a_3 e_{ii} e_{jj} + a_4 E_i E_i + 2 a_5 e_{ij} E_i E_j + 2 a_6 e_{ii} E_j E_j \quad , \quad (4.18)$$

where $e_{ij}$ is the infinitesimal strain tensor

$$e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) \quad .$$

The W given by (4.18) is the stored energy function of the classical coupled theory of electrostriction.

It will not be out of place to mention here that the classical theory of electrostriction is not a complete theory because unlike W in (4.17), the W given by (4.18) does not allow arbitrary rigid rotations of the dielectric together with the electric field.

Substituting (4.17) in (4.9) and (4.10), we obtain the constitutive equations of the first order finite deformation theory of isotropic, homogenous, elastic dielectrics :

$$\sigma_{ij} = \frac{2\rho_0}{\sqrt{I_3}} \left[ \left\{ a_2 + \left( a_2 + 2a_3 \right) J_1 + a_6 J_4 \right\} g_{ij} - a_2 \, g_{ij}^{\,2} + \left( a_4 - a_5 \right) E_i \, E_j \right.$$
$$\left. + \left\{ g_{ik} \, E_k \, E_j + g_{jk} \, E_k \, E_i \right\} \right]$$

and

$$D_i = \frac{2\rho_0}{\sqrt{I_3}} \left[ \left( a_4 - a_5 \right) \delta_{ij} + a_6 J_1 \, \delta_{ij} + a_5 \, g_{ij} \right] E_j$$

## 4.3    SECOND APPROXIMATION

Second approximation to W is defined by retaining terms in series (4.14) of power less than or equal to three in the principal extensions and electric, field components and product terms of order lower than or equal to $e_k^{\,2} \, E_i^{\,2}$. The corresponding expression for W is

$$W = a_2 J_2 + a_3 J_1^{\,2} + a_4 J_4 + a_5 J_5 + a_6 J_1 J_4 + a_7 J_1^{\,3} + a_8 J_1 J_2 + a_9$$
$$J_3 + a_{10} J_2 J_{10} + a_{11} J_1^{\,2} J_4 + a_{12} J_6 + a_{13} J_1 J_5 \quad . \tag{4.19}$$

As before, the a,'s are material constants.

It is to be noted that in the absence of an electric field, equation (4.19) will yield the Murnagham stored energy form for finite elasticity theory.


## 4.4    INCOMPRESSIBLE DIELECTRIC


So far our analysis is based on the fact that the elastic dielectrics are homogeneous and isotropic. But if, in addition, the dielectric is also incompressible, then the invariant $I_3 = 1$ and the stored energy W is a function only of the five invariants :

$$W = W \; (I_1, I_2, I_4, I_5, I_6)$$

The condition of incompressibility $I_3 = 1$ in terms of principal extensions $e_i$ is

$$(1 + e_i) \; (1 + e_2) \; (1 + e_3) = 1 \quad . \tag{4.20}$$

Introducing a new scalar invariant $J_2'$

$$J_2' = J_2 - J_1 \quad , \tag{4.21}$$

the polynomial expansion (4.14) for W can be written as

$$W = \sum_{\alpha\beta\gamma\delta\lambda} B_{\alpha\beta\gamma\delta\lambda} \; J_1{}^\alpha \; J_2'{}^\beta \; J_4{}^\gamma \; J_5{}^\delta \; J_6{}^\lambda \quad . \tag{4.22}$$

Hence, for first approximation of small finite deformations, W of (4.17) is replaced by

$$W = b_1 J_1 + b_2 J_4 + b_3 J_5 \; . \tag{4.23}$$

Here, the b's are physical constants of the material.

For the second approximation the form of W, instead of (4.19), is

$$W = b_1 J_1 + b_2 J_4 + b_3 J_5 + b_4 J_2{}' + b_5 J_1 J_4 + b_6 J_6 \; . \tag{4.24}$$

In dealing with conservative systems, condition of incompressibility produces a pressure p as a reaction. With the form of W as in (4.23), the constitutive equations (4.9) and (4.10) for the first approximation for an incompressible, homogeneous, isotropic, elastic dielectric are :

$$\sigma_{ij} = -p\, \delta_{ij} + C_1\, g_{ij} + C_2\, E_i\, E_j + C_3\, (g_{ik}\, E_k E_i + g_{jk}\, E_k\, E_i), \tag{4.25}$$

$$D_i = C_2\, E_i + C_3\, g_{ij}\, E_j \; . \tag{4.26}$$

Here, p is an arbitrary pressure and the constants C's are the material constants.

# CHAPTER 5

5.    DEFORMATIONS NON–CONTROLLABLE IN FINITE THEORY BUT
      CONTROLLABLE IN APPROXIMATE THEORIES

The constitutive equations (4.9) and (4.10) with the most general form of strain energy are much more restrictive than the ones when the approximate forms of the strain energy function are used. The controllable states with the most general form of W for compressible dielectric are only homogeneous states [4]. When the dielectric is incompressible, the complete set of controllable states are listed in Section 3. It is quite obvious that such states shall be automatically controllable when approximate forms of the stored energy function are employed. However, there could be states which are controllable in the approximate theories but not controllable in the general theory. Singh [5] found one such deformation – simultaneous extension and shear of a cylindrical annulus for an incompressible, homogeneous, isotropic, elastic dielectric which is not controllable with the general form of W but is controllable when the first approximation for W (4.17) is used. In this presentation, our aim is to find more controllable states in this category.

# CHAPTER 6

6. COMPRESSIBLE DIELECTRICS

Singh and Trehan [8] proved that when the dielectric is compressible, only homogeneous states are controllable for all possible polynomial approximations of the stored energy function.

# CHAPTER 7

## 7.    INCOMPRESSIBLE DIELECTRICS

We shall employ here the first approximation for W given by (4.17) leading to the corresponding constitutive relations (4.25) and (4.26).  The field equations (3.2), (3.3) and (3.4), as well as the boundary conditions (3.6), (3.7) and (3.8) remain the same.

## 7.1    SIMULTANEOUS EXTENSION AND TORSION OF A CYLINDER

We consider a tube of homogeneous, incompressible, elastic dielectric with internal radius $R_a$ and external radius $R_b$.  The particle originally located at the point $(R,\Theta,Z)$ in cylindrical polar coordinates moves to the position $(r,\theta,z)$ according to the relation :

$$r = \sqrt{A}R \ , \quad \theta = \Theta + DZ \ , \quad z = FZ \ , \tag{7.1}$$

where A, D, F are constants.

Incompressibility condition requires $A\,F = 1$.

The mapping (7.1) is a special case of the mapping (3.12).  However, we aim to combine this deformation with the electric field different from the one given by (3.15).

We assume that a radial electric field is imposed by placing the tube between the plates of a coaxial cylindrical condenser.  The form of this electric field inside the dielectric is

$$E_r = \frac{K}{r} \ , \qquad E_\theta = 0 \ , \qquad E_z = 0 \ . \tag{7.2}$$

The physical components of strain for (7.1) are

$$g_{rr} = A \quad , \qquad g_{\theta\theta} = \left(\frac{1}{R^2} + D^2\right) r^2 \quad , \qquad g_{zz} = F^2 \quad ,$$

$$g_{r\theta} = g_{rz} = 0 \quad , \qquad g_{\theta z} = FDR \quad . \tag{7.3}$$

From (4.26), we get

$$D_r = C_2 E_r + A\, C_3 E_r \;=\; C_2 \frac{K}{r} + AC_3 \frac{K}{r} \;=\; \frac{K}{r}(C_2 + AC_3) \quad ,$$

$$D_\theta = 0 \quad , \qquad\qquad D_z = 0 \quad .$$

The boundary condition (3.7) demands

$$(\underline{D}^0 - \underline{D}).n = 0$$

or

$$(D_r{}^0 - D_r) = 0 \quad .$$

Hence,

$$D_r{}^0 = \frac{K}{r}(C_2 + AC_3) \quad , \tag{7.4}$$

and the continuity condition $(\underline{E}^0 - \underline{E}).t = 0$ at the boundary of the dielectric demands that

$$E_\theta = E_\theta{}^0 = 0$$

and

$$E_z = E_z{}^0 = 0 \quad .$$

For the surrounding medium outside the dielectric,

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$$D_i{}^0 = \varepsilon \, E_i{}^0 \quad ,$$

which means

$$D_\theta{}^0 = \varepsilon \, E_\theta{}^0 = 0 \quad ,$$

$$D_z{}^0 = \varepsilon \, E_z{}^0 = 0 \quad .$$

From (7.4),

$$E_r{}^0 = \frac{K}{\varepsilon r}(C_2 + AC_3) \quad .$$

We can summarise the various components of the electric field and the electric flux as

$$E_r = \frac{K}{r} \quad , \qquad\qquad E_\theta = 0 \quad , \qquad\qquad E_z = 0 \quad ,$$

$$E_r{}^0 = \frac{L}{r} \quad , \qquad\qquad E_\theta{}^0 = 0 \quad , \qquad\qquad E_z{}^0 = 0 \quad , \qquad (7.5)$$

where

$$L = \frac{K}{\varepsilon}(C_2 + AC_3) \quad .$$

$$D_r = \frac{K}{r}(C_2 + AC_3) \quad , \qquad D_\theta = 0 \quad , \qquad D_z = 0 \quad ,$$

$$D_r{}^0 = \frac{K}{r}(C_2 + AC_3) \quad , \qquad D_\theta{}^0 = 0 \quad , \qquad D_z{}^0 = 0 \qquad (7.6)$$

It can be easily verified that the electric flux given by (7.6) is solenoidal and the electric field given by (7.5) is conservative.

Using (7.2) and (7.3) in (4.25), we get

$$\sigma_{rr} = -p + C_1 A + (C_2 + 2C_3 A)\frac{K^2}{r^2} \quad ,$$

$$\sigma_{\theta\theta} = -p + C_1 r^2\left(\frac{1}{R^2} + D^2\right) \quad ,$$

$$\sigma_{zz} = -p + C_1 F^2 \quad ,$$

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$$\sigma_{r\theta} = 0 \quad , \qquad \sigma_{rz} = 0 \quad , \qquad \sigma_{\theta z} = FDC_1 r \quad . \qquad (7.7)$$

Equilibrium equations without body forces in cylindrical coordinate system are given by

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \quad ,$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} = 0 \quad ,$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} = 0 \quad . \qquad (7.8)$$

Use of (7.7) in the above equations will satisfy the last two equations (7.8). The first equation will yield the pressure p :

$$p\,(r) = C_1 D^2 \frac{r^2}{2} + (C_2 + 2AC_3)\frac{K^2}{2r^2} + Y_1 \quad , \qquad (7.9)$$

where $Y_1$ is a constant of integration.

In the medium surrounding the dielectric we can find the Maxwell stresses from the equation (2.15)

$$\sigma_{rr}{}^0 = -\sigma_{\theta\theta}{}^0 = -\sigma_{zz}{}^0 = \frac{K^2}{2\varepsilon r^2}(C^2 + AC_3)^2 \quad ,$$

$$\sigma_{r\theta}{}^0 = \sigma_{\theta z}{}^0 = \sigma_{rz}{}^0 = 0 \quad . \qquad (7.10)$$

Therefore, in order to support the deformation (7.1) in the presence of field given by (7.5), we will have to apply the tractions to the curved surfaces $r = \sqrt{A}\, R_a$ and $r = \sqrt{A}\, R_b$ according to the formula :

$$t_i = (\sigma_{ij} - \sigma_{ij}{}^0)\, n_j$$

Using (7.7), (7.9) and (7.10) in the above relation, we obtain

$$T_r\,(r = \sqrt{A}R_a) = \frac{1}{2}C_1 D^2 A R_a{}^2 + (C_2 + 2AC_3)\frac{K^2}{2AR_a{}^2}$$
$$- \frac{K^2}{2\varepsilon AR_a{}^2}(C_2 + AC_3)^2 + Y_2 \quad , \tag{7.11}$$

$$T_\theta\,(r = \sqrt{A}R_a) = 0 \quad ,$$

$$T_z\,(r = \sqrt{A}R_a) = 0 \quad . \tag{7.12}$$

By setting the radial component $T_r$ in (7.11) of the applied traction to zero, constant $Y_2$ can be evaluated. Similar expressions like (7.11) and (7.12) can be obtained for surface tractions at the curved surface $r = \sqrt{A}\, R_b$

## 7.2 CYLINDRICALLY SYMMETRICAL DEFORMATIONS OF A TUBE IN A UNIFORM ELECTRIC FIELD ALONG THE AXIS

Consider an exactly similar tube as in section 7.1. Let this tube obey the following deformation relations

$$r = (AR^2 + B)^{1/2} \quad , \quad \theta = C\Theta \quad , \quad z = FZ \quad , \tag{7.13}$$

where the constants A, B, ..., F satisfy the incompressibility condition $ACF = 1$. Components of strain for (7.13) are :

$$g_{rr} = \left(\frac{AR}{r}\right)^2 \quad , \qquad g_{\theta\theta} = \left(\frac{Cr}{R}\right)^2 \quad , \qquad g_{zz} = F^2 \quad ,$$

$$g_{r\theta} = g_{rz} = g_{\theta z} = 0 \quad . \tag{7.14}$$

Let us superimpose an electric field

$$E_r = 0 \quad , \qquad\qquad E_\theta = 0 \quad , \qquad\qquad E_z = K \tag{7.15}$$

on the deformation (7.13) inside the dielectric.

The dielectric displacement field corresponding to the above mentioned prescribed electric field from (4.26) is

$$D_r = 0 \quad , \qquad\qquad D_\theta = 0 \quad , \qquad\qquad D_z = C_2 K + C_3 F^2 K \quad .$$

From the boundary condition

$$(\underline{D}^0 - \underline{D}).n = 0 \quad , \tag{7.16}$$

we get

$$D_r{}^0 = D_r = 0 \quad .$$

The boundary condition

$$(\underline{E}^0 - \underline{E}).t = 0 \tag{7.17}$$

will be fulfilled if

$$E_\theta{}^0 = E_\theta = 0 \quad , \qquad\qquad E_z{}^0 = E_z = K \quad .$$

Using the relation

$$D_i = \varepsilon\, E_i$$

for the medium surrounding the dielectric, we get,

$$D_\theta{}^0 = 0 \quad , \qquad D_z{}^0 = \varepsilon\, E_z{}^0 = \varepsilon\, K \quad .$$

Summarising, the fields are

$$E_r = 0 \quad , \qquad E_\theta = 0 \quad , \qquad E_z = K \quad ,$$
$$E_r{}^0 = 0 \quad , \qquad E_\theta{}^0 = 0 \quad , \qquad E_z{}^0 = K \quad , \qquad (7.18)$$

and

$$D_r = 0 \quad , \qquad D_\theta = 0 \quad , \qquad D_z = C_2 K + C_3 F^2 K \quad ,$$
$$D_r{}^0 = 0 \quad , \qquad D_\theta{}^0 = 0 \quad , \qquad D_z{}^0 = \varepsilon K \quad . \qquad (7.19)$$

The fields given by (7.18) and (7.19) satisfy $\nabla \times E = 0$ and $\nabla . D = 0$, respectively. As usual, we use (4.25) in combination with (7.18) and (7.19) for the calculation of stresses. We obtain

$$\sigma_{rr} = -\,p + C_1 \left(\frac{AR}{r}\right)^2 \quad ,$$

$$\sigma_{\theta\theta} = -\,p + C_1 \left(\frac{Cr}{R}\right)^2 \quad ,$$

$$\sigma_{zz} = -\,p + C_1 F^2 + C_2 K^2 + 2C_3 F^2 K^2 \quad ,$$

$$\sigma_{r\theta} = 0 \quad , \qquad \sigma_{rz} = 0 \quad , \qquad \sigma_{\theta z} = 0 \quad . \qquad (7.20)$$

Equilibrium equations without body forces in cylindrical coordinate system are :

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \quad , \quad \text{(a)}$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} = 0 \quad , \quad \text{(b)} \qquad \text{(7.21)}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} = 0 \quad . \quad \text{(c)}$$

The stresses (7.20) satisfy the last two equations. Substituting, stress components from (7.20) into the first equilibrium equation, we obtain the pressure p as

$$p\,(r) = C_1 \left( \frac{A^2 R^2}{2r^2} - \frac{C^2 r^2}{2R^2} \right) + Y_3 \quad ,$$

where $Y_3$ is a constant of integration.

From equation (2.15), the Maxwell stress components are

$$-\sigma_{rr}{}^0 = -\sigma_{\theta\theta}{}^0 = \sigma_{zz}{}^0 = \frac{\varepsilon K^2}{2} \quad ,$$

$$\sigma_{r\theta}{}^0 = \sigma_{\theta z}{}^0 = \sigma_{rz}{}^0 = 0 \quad .$$

The deformation (7.13) can be supported by the surface tractions at the curved surfaces $r = (AR_a{}^2 + B)^{1/2}$ and $r = (AR_b{}^2 + B)^{1/2}$. These surface tractions can be evaluated from the relation :

$$T_i = (\sigma_{ij} - \sigma_{ij}{}^0)\, n_j \qquad \qquad \text{(7.22)}$$

$$T_r \left[ r = (AR_a{}^2 + B)^{1/2} \right] = C_1 \left[ \frac{A^2 R^2}{2(AR_a{}^2 + B)} - \frac{C^2(AR_a{}^2 + B)}{2R^2} \right] + \frac{\varepsilon K^2}{2} + Y_4 \quad ,$$

$$T_\theta = 0 \quad , \qquad\qquad T_z = 0 \quad .$$

Constant $Y_4$ can be found by setting $T_r$ equal to zero. Similar expressions for surface tractions can be obtained at the surface $r = (AR_b{}^2 + B)^{1/2}$

## 7.3 CYLINDRICALLY SYMMETRICAL DEFORMATIONS OF A TUBE IN A UNIFORM AXIAL ELECTRIC FLUX

Let us now superimpose a uniform axial dielectric displacement field

$$D_r = 0 \quad , \qquad D_\theta = 0 \quad , \qquad D_z = L \qquad (7.23)$$

inside the dielectric on the deformation (7.13) of section (7.2)

From the relation

$$E_i = P_2\, D_i + P_3\, g_{ij}\, D_j \quad ,$$

we get

$$E_r = 0 \quad , \qquad E_\theta = 0 \quad , \qquad E_z = P_2 L + P_3 F^2 L \quad . \qquad (7.24)$$

Equation (7.14) gives us

$$D_r{}^0 = 0 \qquad (7.25)$$

and (7.15) gives us

$$E_\theta{}^0 = 0 \quad , \qquad\qquad E_z{}^0 = P_2 L + P_3 F^2 L \quad , \qquad (7.26)$$

From (7.24),

$$E_r{}^0 = 0 \quad . \tag{7.27}$$

From (7.25),

$$D_\theta{}^0 = 0 \quad , \qquad\qquad D_z{}^0 = \varepsilon \, (P_2 L + P_3 F^2 L) \quad . \tag{7.28}$$

In other words,

$$
\begin{aligned}
D_r &= 0 \quad , & D_\theta &= 0 \quad , & D_z &= L \quad , \\
D_r{}^0 &= 0 \quad , & D_\theta{}^0 &= 0 \quad , & D_z{}^0 &= \varepsilon \, (P_2 L + P_3 F^2 L) \quad , \tag{7.29} \\
E_r &= 0 \quad , & E_\theta &= 0 \quad , & E_z &= P_2 L + P_3 F^2 L \quad , \\
E_r{}^0 &= 0 \quad , & E_\theta{}^0 &= 0 \quad , & E_z{}^0 &= P_2 L + P_3 F^2 L \quad . \tag{7.30}
\end{aligned}
$$

It can be verified that the fields (7.29) and (7.30) satisfy $\nabla . D = 0$ and $\nabla \times E = 0$,

respectively.

For the calculation of stress components we make use of the relation :

$$\sigma_{ij} = - p \, \delta_{ij} + P_1 g_{ij} + P_2 \, D_i \, D_j + P_3 \, (g_{ik} \, D_k \, D_j + g_{jk} \, D_k \, D_i) \tag{7.31}$$

Using (7.14) and (7.29) in (7.31) we get :

$$
\begin{aligned}
\sigma_{rr} &= - p + P_1 \left(\frac{AR}{r}\right)^2 \quad , \\
\sigma_{\theta\theta} &= - p + P_1 \left(\frac{Cr}{R}\right)^2 \quad , \\
\sigma_{zz} &= - p + P_1 F^2 + P_2 L^2 + 2P_3 F^2 L^2 \quad , \\
\sigma_{r\theta} &= \sigma_{rz} = \sigma_{\theta z} = 0 \quad . \tag{7.32}
\end{aligned}
$$

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It is found that the stresses (7.32) satisfy the equilibrium equations and the first of them furnishes the pressure p :

$$p\,(r) = \frac{P_1}{2}\left(\frac{AR}{r}\right) - \frac{P_1}{2}\left(\frac{Cr}{R}\right)$$

The relation for finding Maxwell stress components is

$$\sigma_{ij} = \varepsilon\, D_i\, D_j - \frac{\varepsilon}{2}\, D_k\, D_k\, \delta_{ij}\quad.$$

This will yield

$$-\sigma_{rr}^{\,0} = -\sigma_{\theta\theta}^{\,0} = \sigma_{zz}^{\,0} = \frac{\varepsilon}{2}\, L^2\quad,$$

$$\sigma_{r\theta}^{\,0} = \sigma_{\theta z}^{\,0} = \sigma_{rz}^{\,0} = 0\quad.$$

From (7.22) we get the desired surface tractions

$$T_r\left[r = (AR_a^2 + B)^{1/2}\right] = \frac{P_1}{2}\left[\frac{A^2 R^2}{AR_a^2 + B} - \frac{C(R_a^2 + B)}{R^2}\right] + \frac{\varepsilon}{2}\, L^2 + Y_5$$

,

$$T_\theta = 0\quad, \qquad\qquad T_z = 0\quad.$$

Constant $Y_5$ can be found by setting $T_r$ equal to zero. Same way surface tractions can be evaluated on the surface $r = (AR_b^2 + B)^{1/2}$

## 7.4    DEFORMATIONS OF A CUBOID IN A UNIFORM ELECTRIC FIELD

Let us consider a tube wall section bounded by the surfaces $R = R_a$, $R = R_b$, $\Theta = \pm\,\Theta_0$ and $Z = \pm\,Z_0$ in cylindrical polar coordinates. The particle originally at the point $(R,\,\Theta,\,Z)$

moves to the location (x, y, z) in a cartesian system as follows :

$$x = AR^2 \ , \qquad y = B\Theta \ , \qquad z = \frac{z}{2AB} + C\Theta \ . \qquad (7.33)$$

The strain components corresponding to this deformation are found to be

$$g_{xx} = 4A^2R^2 \ , \qquad g_{xy} = g_{xz} = 0 \ , \qquad g_{yy} = \left(\frac{B}{R}\right)^2 \ ,$$
$$g_{yz} = \frac{BC}{R^2} \ , \qquad g_{zz} = \left(\frac{C}{R}\right)^2 + \left(\frac{1}{2AB}\right)^2 \ . \qquad (7.34)$$

Let the prescribed electric field inside the dielectric be

$$E_x = H \ , \qquad E_y = 0 \ , \qquad E_z = 0 \ . \qquad (7.35)$$

From equation (4.26), the components of dielectric displacement field are

$$D_x = (C_2 + 4C_3A^2R^2) \ H \ , \qquad D_y = 0 \ , \qquad D_z = C_3 \frac{BC}{R^2} H \ . \qquad (7.36)$$

It is further assumed that the x − axis is perpendicular to the top and bottom faces of the cube. Then the boundary conditions

$$(\underline{D}^0 - \underline{D}) \ .n = 0 \qquad (7.37)$$

and

$$(\underline{E}^0 - \underline{E}) \ .t = 0 \qquad (7.38)$$

furnish

$$D_x{}^0 = (C_2 + 4C_3A^2R^2) \ H \ , \qquad (7.39)$$
$$E_x{}^0 = \frac{1}{\varepsilon} \ (C_2 + 4 \ C_3A^2R^2) \ H \qquad (7.40)$$

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and

$$E_y{}^0 = 0 \quad , \qquad E_z{}^0 = 0 \quad . \tag{7.41}$$

Equatioin (7.40) gives,

$$D_y{}^0 = 0 \quad , \qquad D_z{}^0 = 0 \quad . \tag{7.42}$$

Rewriting equations (7.35), (7.36), (7.39), (7.40), (7.41), and (7.42)

$$
\begin{aligned}
E_x &= H \quad , & E_y &= 0 \quad , & E_z &= 0 \quad , \\
E_x{}^0 &= \frac{H}{\varepsilon}\,(C_2 + 4C_3 A^2 R^2) \quad , & E_y{}^0 &= 0 \quad , & E_z{}^0 &= 0 \quad , \\
D_x &= (C_2 + 4C_3 A^2 R^2)\,H \quad , & D_y &= 0 \quad , & D_z &= C_3\frac{BC}{R^2}\,H \quad , \\
D_x{}^0 &= (C_2 + 4C_3 A^2 R^2)\,H \quad , & D_y{}^0 &= 0 \quad , & D_z{}^0 &= 0 \quad .
\end{aligned}
\tag{7.43}
$$
$$\tag{7.44}$$

The fields (7.43) and (7.44) satisfy the field equations $\nabla \times E = 0$ and $\nabla.D = 0$,

respectively.

Using equations (4.25), (7.34) and (7.35) for the calculation of components of stress, we

get

$$
\begin{aligned}
\sigma_{xx} &= -p + 4C_1 A^2 R^2 + C_2 H^2 + 8C_3\,(A^2 R^2 H^2) \quad , \\
\sigma_{yy} &= -p + C_1 \left(\frac{B}{R}\right)^2 \quad , \\
\sigma_{zz} &= -p + C_1 \left[\left(\frac{C}{R}\right)^2 + \left(\frac{1}{2AB}\right)^2\right] \quad , \\
\sigma_{xy} &= 0 \quad , \quad \sigma_{xz} = 0 \quad , \quad \sigma_{yz} = C_1 \frac{BC}{R^2} \quad .
\end{aligned}
\tag{7.45}
$$

Equilibrium equations without body forces in cartesian coordinates are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \quad , \qquad\qquad (a)$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0 \quad , \qquad\qquad (b)$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \quad . \qquad\qquad (c) \qquad (7.46)$$

We note that stress components (7.45) are only functions of $x$. Using equation (7.45) in (7.46) we find that (b) and (c) are identically satisfied. Condition (a) gives pressure $p$ which is assumed to be a function of variable $x$ :

$$p(x) = A(C_1 + 8C_3H^2)x + Y_6 \quad ,$$

where $Y_6$ is a constant of integration.

Using equation (2.15) we obtain the following components of Maxwell's stress

$$\sigma_{xx}{}^0 = -\sigma_{yy}{}^0 = -\sigma_{zz}{}^0 = \frac{\varepsilon}{2}H^2 \quad ,$$

$$\sigma_{xy}{}^0 = \sigma_{xz}{}^0 = \sigma_{yz}{}^0 = 0 \quad .$$

Equation (7.22) will give the surface tractions on faces $x_a = A R_a{}^2$ and $x_b = A R_b{}^2$ :

$$T_x = A^2(C_1 + 8C_3H^2)R_a{}^2 - \frac{\varepsilon}{2}H^2 + Y_7 \quad ,$$

$$T_y = 0 \quad , \qquad\qquad T_z = 0 \quad ,$$

where $Y_7$ is a constant of integration and can be found by equating $T_x$ equal to zero. Similarly we can find the surface tractions on the other face $x_b = A R_b{}^2$

## 7.5    DEFORMATIONS OF A CUBOID IN A UNIFORM FIELD OF FLUX :

We now consider the deformation (7.33) and superimpose on it the dielectric displacement field

$$D_x = 0 \quad , \qquad D_y = M \quad , \qquad D_z = 0 \qquad\qquad (7.47)$$

inside the dielectric.

The electric field components are calculated from

$$E_i = P_2 \, D_i + P_3 \, g_{ij} \, D_j \quad .$$

They are

$$E_x = 0 \quad , \qquad E_y = P_2 \, M + P_3 \left(\tfrac{B}{R}\right)^2 M \quad , \qquad E_z = P_3 \frac{BC}{R^2} M \quad . \qquad (7.48)$$

It is assumed that the x-axis is perpendicular to the top and bottom faces of the cube. Using equations (7.37) and (7.38), we get

$$D_x{}^0 = 0$$

and therefore

$$E_x{}^0 = 0 \quad , \qquad\qquad\qquad\qquad (7.49)$$

$$E_y{}^0 = M \left[P_2 + P_3 \left(\tfrac{B}{R}\right)^2\right] \quad , \qquad\qquad E_z{}^0 = P_3 M \frac{BC}{R^2} \quad .$$

Also, then

$$D_y{}^0 = \varepsilon \, M \left[P_2 + P_3 \left(\tfrac{B}{R}\right)^2\right]$$

and

$$D_z{}^0 = \varepsilon \, P_3 M \frac{BC}{R^2} \quad . \qquad\qquad\qquad (7.50)$$

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In summary, the electrostatic fields are

$$D_x = 0 \quad , \qquad D_y = M \quad , \qquad\qquad D_z = 0 \quad ,$$

$$D_x{}^0 = 0 \quad , \qquad D_y{}^0 = \varepsilon\, M \left[ P_2 + P_3 \left(\tfrac{B}{R}\right)^2 \right], \quad D_z{}^0 = \varepsilon\, P_3 M \,\frac{BC}{R^2} \qquad\qquad (7.51)$$

$$E_x = 0 \quad , \qquad E_y = M \left[ P_2 + P_3 \left(\tfrac{B}{R}\right)^2 \right], \qquad E_z = P_3 M \,\frac{BC}{R^2} \quad ,$$

$$E_x{}^0 = 0 \quad , \qquad E_y{}^0 = M \left[ P_2 + P_3 \left(\tfrac{B}{R}\right)^2 \right], \qquad E_z{}^0 = P_3 M \,\frac{BC}{R^2} \quad . \qquad\qquad (7.52)$$

The fields given by (7.51) and (7.52) satisfy the field equations $\nabla . D = 0$ and $\nabla \times E = 0$, respectively.

Using equations (7.31), (7.34) and (7.47), the stresses are given by

$$\sigma_{xx} = -p + 4P_1 A^2 R^2 \quad ,$$

$$\sigma_{yy} = -p + P_1 \left(\tfrac{B}{R}\right)^2 + P_2 M^2 + 2P_3 \left(\tfrac{B}{R}\right)^2 M^2 \quad ,$$

$$\sigma_{zz} = -p + P_1 \left[ \left(\tfrac{C}{R}\right)^2 + \left(\tfrac{1}{2AB}\right)^2 \right] \quad ,$$

$$\sigma_{xy} = 0 \quad , \qquad \sigma_{xz} = 0 \quad , \qquad \sigma_{yz} = (P_1 + P_3 M^2)\frac{BC}{R^2} \quad . \qquad (7.53)$$

Stress components (7.53) are functions of $x$ only. Using equation (7.53) in (7.46), the last two equilibrium equations are identically satisfied. Equation (7.46) (a) gives pressure p which we can assume to be a function of variable $x$ only :

$$p\,(x) = 4P_1 A x + Y_8 \quad ,$$

where $Y_8$ is a constant of integration.

With (2.15), Maxwell stress components are

$$-\sigma_{xx}{}^0 = \sigma_{yy}{}^0 = -\sigma_{zz}{}^0 = \frac{\varepsilon}{2} M^2 \quad ,$$

$$\sigma_{xy}{}^0 = \sigma_{xz}{}^0 = \sigma_{yz}{}^0 = 0 \quad .$$

Equation (7.22) gives the surface tractions on faces $x_a = A\, R_a{}^2$, $x_b = A\, R_b{}^2$

$$T_x = 4P_1 A^2 R_a{}^2 - \frac{\varepsilon}{2} M^2 + Y_9 \quad ,$$

$$T_y = 0 \quad , \qquad T_z = 0 \quad .$$

$Y_9$ is a constant of integration and can be found by setting $T_x$ equal to zero.

Surface tractions on the other face $x_b = A\, R_b{}^2$ can be found in a similar way.

## 7.6  FLEXURAL DEFORMATIONS OF A BLOCK IN A RADIAL ELECTRIC FIELD

In this deformation, the particle moves from its original position (X, Y, Z) in cartesian

system to (r, θ, z) in cylindrical polar coordinates as follows :

$$r = AX \quad , \qquad \theta = BY \quad , \qquad z = \frac{2Z}{A^2 B} + CY \quad . \tag{7.54}$$

It is assumed that before the deformation, the block is infinitely long in the z – direction

and is bounded by two plane surfaces $X^2$ = constant and two plane surfaces $Y^2$ = constant.

Two plane surfaces $X^2$ = constant become the internal and external cylindrical boundaries

of the tube and the planes $Y = \pm \frac{\pi}{B}$ become $\theta = \pm \pi$ as a result of deformation.

The strain components corresponding to the mapping (7.54) are

$$g_{rr} = A^2 \quad , \qquad g_{\theta\theta} = r^2 B^2 + C^2 \quad , \qquad g_{zz} = \left(\frac{2}{A^2 B}\right)^2 \quad ,$$

$$g_{r\theta} = 0 \quad , \qquad g_{rz} = 0 \quad , \qquad\qquad g_{\theta z} = \frac{2C}{A^2 B} \quad . \qquad (7.55)$$

Let us superimpose a radial field of the form

$$E_r = \frac{K}{r} \quad , \qquad\qquad E_\theta = 0 \quad , \qquad\qquad E_z = 0 \quad . \qquad (7.56)$$

From equation (4.25), the components of dielectric displacement field are

$$D_r = \frac{K}{r}(C_2 + A^2 C_3) \quad , \qquad D_\theta = 0 \quad , \qquad D_z = 0 \quad . \qquad (7.57)$$

From equation (7.16),

$$D_r^{\ 0} = \frac{K}{r}(C_2 + A^2 C_3) \quad . \qquad\qquad\qquad (7.58)$$

Therefore,

$$E_r^{\ 0} = \frac{K}{\varepsilon r}(C_2 + A^2 C_3) \quad . \qquad\qquad\qquad (7.59)(a)$$

From equation (7.17),

$$E_\theta^{\ 0} = 0 \quad , \qquad\qquad E_z^{\ 0} = 0 \quad . \qquad\qquad (7.59)(b)$$

Therefore,

$$D_\theta^{\ 0} = 0 \quad , \qquad\qquad D_z^{\ 0} = 0 \quad . \qquad\qquad (7.59)(c)$$

Rewriting equations (7.56), (7.57), (7.58) and (7.59)

$$E_r = \frac{K}{r} \quad , \qquad\qquad\qquad E_\theta = 0 \quad , \qquad E_z = 0 \quad ,$$

$$E_r^0 = \frac{K}{\varepsilon r}(C_2 + A^2 C_3) \quad , \qquad E_\theta^0 = 0 \quad , \qquad E_z^0 = 0 \quad , \qquad (7.60)$$

$$D_r = \frac{K}{r}(C_2 + A^2 C_3) \quad , \qquad D_\theta = 0 \quad , \qquad D_z = 0 \quad ,$$

$$D_r^0 = \frac{K}{r}(C_2 + A^2 C_3) \quad , \qquad D_\theta^0 = 0 \quad , \qquad D_z^0 = 0 \quad . \qquad (7.61)$$

The electric field found above is conservative and the electric flux is solenoidal.

Using equations (4.25), (7.55) and (7.56), the stress components are

$$\sigma_{rr} = -p + C_1 A_2 + C_2 \frac{K^2}{r^2} + 2C_3 \frac{K^2}{r^2} \quad ,$$

$$\sigma_{\theta\theta} = -p + C_1 (r^2 B^2 + C_2) \quad ,$$

$$\sigma_{zz} = -p + C_1 \left(\frac{2}{A^2 B}\right)^2 \quad ,$$

$$\sigma_{r\theta} = 0 \quad , \qquad \sigma_{rz} = 0 \quad , \qquad \sigma_{\theta z} = 0 \quad . \qquad (7.62)$$

If we substutute (7.62) in (7.21) we will find that (b) and (c) are satisfied, where as (a) will give us the pressure p :

$$p\,(r) = C_1 A^2 \ln r + \frac{K^2}{r^2}\left(\frac{1}{2} C_2 + C_3\right) + Y_{10} \quad ,$$

where $Y_{10}$ is a constant of integration.

Using (4.2) and (7.60) (b), Maxwell stress components are:

$$\sigma_{rr}^0 = -\sigma_{\theta\theta}^0 = -\sigma_{zz}^0 = (C_2 + A^2 C_3) \frac{K^2}{2\varepsilon r^2} \quad ,$$

$$\sigma_{r\theta}^0 = \sigma_{\theta z}^0 = \sigma_{rz}^0 = 0 \quad .$$

The surface tractions required on the surface $r = A\,X_a$ to support this state are found to be

$$T_r\,(r = A\,X_a) = C_1 A^2 \ln\,(A\,X_a) + \frac{K^2}{A^2 X_a^{\,2}}\,(\tfrac{1}{2}C_2 + C_3)$$

$$- (C_2 + A^2 C_3)\,\frac{K^2}{2\varepsilon A^2 X_a^{\,2}} \quad,$$

$$T_\theta\,(r = A\,X_a) = 0 \quad, \qquad\qquad T_z\,(r = A\,X_a) = 0 \quad.$$

Similar expressions for surface traction can be found for the surface $(r = A\,X_b)$

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# CHAPTER 8

## 8. SECOND APPROXIMATION OF SMALL FINITE THEORY OF HOMOGENEOUS, ISOTROPIC, ELASTIC DIELECTRIC

The field equations are, as before,

$$E_{i,j} - E_{j,i} = 0 \quad,$$

$$D_{i,i} = 0 \quad,$$

$$\sigma_{ij,j} = 0$$

and the boundary conditions on the surface of the dielectric are

$$(\underline{D}^0 - \underline{D}).\underline{n} = 0 \quad,$$

$$(\underline{E}^0 - \underline{E}).\underline{t} = 0 \quad.$$

The surface tractions on the boundary of the dielectric can be determined by the relation

$$T_i = (\sigma_{ij} - M_{ij})\, n_j$$

## 8.1 CONSTITUTIVE EQUATIONS

For second approximation, the stored energy function W has the form

$$W = b_1 J_1 + b_2 J_4 + b_3 J_5 + b_4 J_2{}' + b_5 J_1 J_4 + b_6 J_6 \quad.$$

In terms of the invariants $I_1$, $I_2$, $I_3$, we use equations (4.13) and (4.21) to get

$$W = b_1 (I_1 - 3) + b_2 I_4 + b_3 (I_5 - I_4) + b_4 \left[ (I_2 - 3) - 2 (I_1 - 3) - (I_1 - 3) \right]$$
$$+ b_5 (I_1 - 3) I_4 + b_6 \left[ I_6 - 2 (I_5 - I_4) - I_4 \right] \quad . \qquad (8.1)$$

Simplifying equation (8.1) and rearranging terms,

$$W = b_5 I_1 I_4 - 3 b_4 I_1 + b_1 I_1 + b_4 I_2 + b_2 I_4 - b_3 I_4 - 3 b_5 I_4 + b_6 I_4 + b_3 I_5$$
$$- 2 b_6 I_5 + b_6 I_6 + 6 b_4 - 3 b_1$$
$$= (b_1 + b_5 I_4 - 3 b_4) I_1 + b_4 I_2 + (b_2 - b_3 - 3 b_5 + b_6) I_4 + (b_3 - 2 b_6) I_5$$
$$+ b_6 I_6 + 6 b_4 - 3 b_1 \quad .$$

This means,

$$\frac{\partial W}{\partial I_1} = b_1 - 3 b_4 + b_5 I_4 \quad ,$$
$$\frac{\partial W}{\partial I_2} = b_4 \quad , \qquad \frac{\partial W}{\partial I_3} = 0 \quad ,$$
$$\frac{\partial W}{\partial I_4} = b_5 I_1 + b_2 - b_3 - 3 b_5 + b_6 \quad ,$$
$$\frac{\partial W}{\partial I_5} = b_3 - 2 b_6 \quad ,$$
$$\frac{\partial W}{\partial I_6} = b_6 \quad . \qquad (8.2)$$

The stress strain relations given earlier, are given by

$$\sigma_{ij} = \frac{2 \rho_0}{\sqrt{I_3}} \left[ \left( \frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2} \right) g_{ij} - \frac{\partial W}{\partial I_2} g_{ij}^2 + I_3 \frac{\partial W}{\partial I_3} \delta_{ij} \right.$$

$$+ \frac{\partial W}{\partial I_4} E_i E_j + \frac{\partial W}{\partial I_5} \left( g_{ik} E_k E_j + g_{jk} E_k E_i \right)$$

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$$+ \frac{\partial W}{\partial I_6} \left( g_{ik}{}^2 E_k E_j + g_{jk}{}^2 E_k E_i + g_{ik} g_{jp} E_k E_p \right) \Big] \quad . \qquad (4.9)$$

Also,

$$I_1 = g_{ii} \quad , \qquad\qquad I_2 = \frac{1}{2} \left( g_{ii} g_{jj} - g_{ij} g_{ij} \right) \quad ,$$

$$I_3 = \det g_{ij} = 1 \quad , \qquad I_4 = E_i E_i \quad ,$$

$$I_5 = E_i g_{ij} E_j \quad , \qquad I_6 = E_i g_{ij} g_{jk} E_k \quad . \qquad\qquad (4.8)$$

Substituting from (8.2) and (4.8) into (4.9) we obtain

$$\sigma_{ij} = 2\rho_0 \Big[ \left( b_1 - 3\, b_4 + b_5 E_i E_i + b_4 g_{ii} \right) g_{ij} - b_4 g_{ik} g_{kj}$$

$$+ \left( b_5 g_{ii} + b_2 - b_3 - 3\, b_5 + b_6 \right) E_i E_j + (b_3 - 2\, b_6) \left( g_{ik} E_k E_j + g_{jk} E_k E_i \right)$$

$$+ b_6 \left( g_{ik}{}^2 E_j E_k + g_{jk}{}^2 E_i E_k \right) + b_6 g_{ik} g_{jl} E_k E_l \Big] \quad .$$

If we set

$$2\rho_0 \left( b_1 - 3\, b_4 \right) = C_1,$$

$$2\rho_0 b_5 = C_2,$$

$$2\rho_0 b_4 = C_3,$$

$$2\rho_0 \left( b_2 - b_3 - 3\, b_5 + b_6 \right) = C_4,$$

$$2\rho_0 \left( b_3 - 2\, b_6 \right) = C_5,$$

$$2\rho_0 b_6 = C_6 \quad ,$$

we obtain,

$$\sigma_{ij} = -\, p\, \delta_{ij} + C_1 g_{ij} + C_2 \left( g_{ij} E_k E_k + g_{kk} E_i E_j \right) + C_3 \left( g_{kk} g_{ij} - g_{ik} g_{kj} \right) + C_4 E_i E_j$$

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$$+C_5 \left(g_{ik} \, E_j \, E_k + g_{jk} \, E_k \, E_i\right) + C_6 \left(g_{ik}^{\ 2} \, E_j \, E_k + g_{jk}^{\ 2} \, E_k \, E_i + g_{ik} \, g_{jl} \, E_k \, E_l\right) \quad . \quad (8.3)$$

Here, p represents an arbitrary pressure and C's are the material constants.

The dielectric displacement field $D_i$ from (4.10) is

$$D_i = \frac{2\rho_0}{\sqrt{I_3}} \left(\frac{\partial W}{\partial I_4} \, \delta_{ij} + \frac{\partial W}{\partial I_5} \, g_{ij} + \frac{\partial W}{\partial I_6} \, g_{ij}^{\ 2}\right) E_j$$

Using equations (8.2) and (4.8)

$$D_i = 2\rho_0 \Big[ \left(b_5 \, g_{ii} + b_2 - b_3 - 3 \, b_5 + b_6\right) \delta_{ij} + \left(b_3 - 2 \, b_6\right) g_{ij}$$
$$+ \, b_6 \, g_{ij}^{\ 2} \Big] E_j$$

$$= 2\rho_0 \, b_5 \, g_{kk} \, \delta_{ij} \, E_j + 2 \, \rho_0 \left(b_2 - b_3 - 3 \, b_5 + b_6\right) \delta_{ij} \, E_j$$
$$+ 2\rho_0 \left(b_3 - 2 \, b_6\right) g_{ij} \, E_j + 2\rho_0 \, b_6 \, g_{ij}^{\ 2} \, E_j,$$

or

$$D_i = C_2 \, g_{kk} \, E_i + C_4 \, E_i + C_5 \, g_{ij} \, E_j + C_6 \, g_{ij}^{\ 2} \, E_j \quad . \quad (8.4)$$

Outside the dielectric, however,

$$\sigma_{ij} = M_{ij} = \varepsilon \, E_i \, E_j - \frac{\varepsilon}{2} \, E_k \, E_k \, \delta_{ij} \qquad (2.15)$$

and

$$D_i = \varepsilon \, E_i \qquad (2.14)$$

## 8.2 CYLINDRICALLY SYMMETRICAL DEFORMATIONS OF A TUBE IN A UNIFORM ELECTRIC FIELD ALONG THE AXIS

We shall consider the deformation given by the mapping

$$r = (AR^2 + B)^{1/2} \quad , \quad \theta = C\Theta \quad , \quad z = FZ \qquad (8.5)$$

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in cylindrical polar coordinates.

Components of the finger strain tensor $\underset{\sim}{g}$ are

$$g_{rr} = \left(\frac{AR}{r}\right)^2 \quad , \qquad g_{\theta\theta} = \left(\frac{Cr}{R}\right)^2 \quad , \qquad g_{zz} = F^2 \quad ,$$

$$g_{r\theta} = g_{rz} = g_{\theta z} = 0 \quad . \tag{8.6}$$

We combine with the deformation (8.6) the electric field

$$E_r = 0 \quad , \qquad E_\theta = 0 \quad , \qquad E_z = K \quad . \tag{8.7}$$

The dielectric displacement field corresponding to the above prescribed electric field can be obtained from the relation (8.4)

$$D_r = 0 \quad , \qquad D_\theta = 0 \quad , \qquad D_z = K \, (C_2 F^2 + C_4 + C_5 F^2 + C_6 F^4) $$

For satisfying the boundary condition

$$(\underline{D}^0 - \underline{D}).n = 0 \quad , \tag{8.8}$$

we need

$$D_r{}^0 = D_r = 0.$$

The second boundary condition

$$(\underline{E}^0 - \underline{E}).t = 0 \tag{8.9}$$

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will be fulfilled if we choose

$$E_\theta{}^0 = E_\theta = 0 \quad \text{and} \quad E_z{}^0 = E_z = K$$

From the relation

$$D_i = \varepsilon\, E_i$$

we get,

$$D_\theta{}^0 = 0 \quad , \quad D_z{}^0 = \varepsilon\, E_z{}^0 = \varepsilon\, K \quad \text{and} \quad E_r{}^0 = 0$$

In summary,

$$E_r = 0 \quad , \qquad E_\theta = 0 \quad , \qquad E_z = K \quad ,$$
$$E_r{}^0 = 0 \quad , \qquad E_\theta{}^0 = 0 \quad , \qquad E_z{}^0 = K \quad , \qquad\qquad (8.10)$$
$$D_r = 0 \quad , \qquad D_\theta = 0 \quad , \qquad D_z = K\,(C_2 F^2 + C_4 + C_5 F^2 + C_6 F^4)$$
$$D_r{}^0 = 0 \quad , \qquad D_\theta{}^0 = 0 \quad , \qquad D_z{}^0 = \varepsilon\, K \quad . \qquad\qquad (8.11)$$

The fields given by (8.10) and (8.11) satisfy the field equations $\nabla \times E = 0$ and $\nabla.D = 0$,

respectively.

The components of stress from (8.3) are :

$$\sigma_{rr} = -\,p + C_1 \left(\frac{AR}{r}\right)^2 \quad ,$$

$$\sigma_{\theta\theta} = -\,p + C_1 \left(\frac{Cr}{R}\right)^2 \quad ,$$

$$\sigma_{zz} = -\,p + C_1 F^2 + C_4 K^2 + K_2 F^2 \left(2C_2 + 2C_5 + 3C_6 F^2\right) \quad ,$$

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$$\sigma_{r\theta} = 0 \quad , \qquad\qquad\qquad \sigma_{rz} = 0 \quad ,$$

$$\sigma_{\theta z} = K^2 F^2 \left( C_5 + C_6 F^2 \right) \quad . \tag{8.12}$$

The strsses (8.12) satisfy the last two equilibrium equations (7.21) (b) and (7.21) (c). Substituting (8.12) into the first equilibrium equation, we obtain the pressure p as

$$p = -\frac{C_1}{r} (AR)^2 - \frac{1}{2} \left( \frac{AR}{r} \right)^2 - \frac{1}{2} \left( \frac{Cr}{R} \right)^2 + Y_{11}$$

Here, $Y_{11}$ is a constant of integration

Using equation (2.15), the Maxwell stress components are

$$-\sigma_{rr}{}^0 = -\sigma_{\theta\theta}{}^0 = \sigma_{zz}{}^0 = \frac{\varepsilon K^2}{2} \quad ,$$

$$\sigma_{r\theta}{}^0 = \sigma_{\theta z}{}^0 = \sigma_{rz}{}^0 = 0 \quad .$$

The deformation (8.5) can be supportaed by the surface tractions at the curved surfaces
$r = (A \, R_a{}^2 + B)^{1/2}$ and $r = (A \, R_b{}^2 + B)^{1/2}$. These surface tractions can be evaluated from the relation (7.22)

$$T_r \left[ r = (A \, R_a{}^2 + B)^{1/2} \right] = -\frac{C_1 \, (A \, R)^2}{(A \, R_a{}^2 + B)^{1/2}} - \frac{1}{2} \left( \frac{AR}{r} \right)^2 - \frac{1}{2} \left( \frac{Cr}{R} \right)^2$$

$$+ \frac{\varepsilon K^2}{2} + Y_{12} \quad ,$$

$$T_\theta = 0 \quad , \qquad\qquad\qquad T_z = 0 \quad .$$

The constant $Y_{12}$ can be found by setting $T_r$ equal to zero. Similar expressions for surface tractions can be obtained at the surface $r = (A \, R_b{}^2 + B)^{1/2}$

## 8.3 CYLINDRICALLY SYMMETRICAL DEFORMATIONS OF A TUBE IN A UNIFORM DIELECTRIC DISPLACEMENT FIELD

Here, we combine with the deformation (8.6), the dielectric displacement field

$$D_r = 0 \quad , \qquad D_\theta = 0 \quad , \qquad D_z = K \quad . \tag{8.13}$$

The electric field corresponding to (8.13) dielectric displacement field can be obtained from the relation

$$E_i = P_2 \, g_{ii} \, D_i + P_4 \, D_i + P_5 \, g_{ij} \, D_j + P6 \, g_{ij}{}^2 \, E_j \quad ,$$

which is,

$$E_r = 0 \quad , \qquad E_\theta = 0 \quad , \qquad E_z = K \, (P_2 F^2 + P_4 + P_5 F^2 + P_6 F^4)$$

Equation (8.9) will be satisfied if

$$E_\theta{}^0 = E_\theta = 0 \quad , \qquad E_z{}^0 = E_z = K \, (P_2 F^2 + P_4 + P_5 F^2 + P_6 F^4) \quad .$$

Equation (8.8) will be satisfied if

$$D_r{}^0 = D_r = 0 \quad .$$

From the relation

$$D_i = \varepsilon \, E_i \quad ,$$

we get,

$$D_\theta{}^0 = 0 \quad , \qquad\qquad D_z{}^0 = \varepsilon \, K \, (P_2 F^2 + P_4 + P_5 F^2 + P_6 F^4)$$

and

$$E_r{}^0 = 0$$

In summary,

$$E_r = 0 \quad , \qquad E_\theta = 0 \quad , \qquad E_z = K \, (P_2 F^2 + P_4 + P_5 F^2 + P_6 F^4) \quad ,$$

$$E_r{}^0 = 0 \quad , \qquad E_\theta{}^0 = 0 \quad , \qquad E_z{}^0 = K \, (P_2 F^2 + P_4 + P_5 F^2 + P_6 F^4) \quad , \qquad (8.14)$$

$$D_r = 0 \quad , \qquad D_\theta = 0 \quad , \qquad D_z = K \quad ,$$

$$D_r{}^0 = 0 \quad , \qquad D_\theta{}^0 = 0 \quad , \qquad D_z{}^0 = \varepsilon K \, (P_2 F^2 + P_4 + P_5 F^2 + P_6 F^4) \qquad (8.15)$$

The fields given by (8.14) and (8.15) satisfy the field equations $\nabla \times E = 0$ and $\nabla . D = 0$, respectively.

The components of stress can be obtained from the equation

$$\sigma_{ij} = - p \, \delta_{ij} + P_1 \, g_{ij} + P_2 \left( g_{ij} \, D_i \, D_i + g_{ii} \, D_i \, D_j \right) + P_3 \left( g_{ii} \, g_{ij} - g_{ik} \, g_{kj} \right) + P_4 \, D_i \, D_j$$

$$+ P_5 \left( g_{ik} \, D_j \, D_k + g_{jk} \, D_k \, D_i \right) + P_6 \left( g_{ik}{}^2 \, D_i \, D_k + g_{ik}{}^2 \, D_j \, D_k + g_{ik} \, g_{jl} \, D_k \, D_l \right) \quad .$$

Here, p and the constants P's have the usual meaning.

$$\sigma_{rr} = - p + P_1 \left( \frac{AR}{r} \right)^2 \quad ,$$

$$\sigma_{\theta\theta} = - p + P1 \left( \frac{Cr}{R} \right)^2 \quad ,$$

$$\sigma_{zz} = - p + P_1 F^2 + P_4 K^2 + K^2 F^2 \left( 2P_2 + 2P_5 + 3P_6 F^2 \right) \quad ,$$

$$\sigma_{r\theta} = 0 \quad , \qquad \sigma_{rz} = 0 \quad , \qquad \sigma_{\theta z} = K^2 F^2 \left( P_5 + P_6 F^2 \right) \quad . \ (8.16)$$

The stresses (8.16) satisfy the last two equilibrium equations (7.21) (b) and (7.21) (c). The first equilibrium equation yield the pressure p as :

$$p = -\frac{P_1}{r} (AR)^2 - \frac{1}{2} \left(\frac{AR}{r}\right)^2 - \frac{1}{2} \left(\frac{Cr}{R}\right)^2 + Y_{13} \quad .$$

Here, $Y_{13}$ is a constant of integration.

Using equation (2.15), the Maxwell stress components are

$$-\sigma_{rr}{}^0 = -\sigma_{\theta\theta}{}^0 = \sigma_{zz}{}^0 = \frac{\varepsilon K^2}{2} \quad ,$$

$$\sigma_{r\theta}{}^0 = \sigma_{\theta z}{}^0 = \sigma_{rz}{}^0 = 0 \quad .$$

The deformation (8.5) can be supported by the surface tractions at the curved surfaces $r = (AR_a{}^2 + B)^{1/2}$ and $r = (AR_b{}^2 + B)^{1/2}$. These surface tractions can be evaluated from the relation (7.22)

$$T_r \left[ r = (A R_a{}^2 + B)^{1/2} \right] = -\frac{P_1 (AR)^2}{(AR_a{}^2 + B)^{1/2}} - \frac{1}{2} \left(\frac{AR}{r}\right)^2 - \frac{1}{2} \left(\frac{Cr}{R}\right)^2$$
$$+ \frac{\varepsilon K^2}{2} + Y_{14} \quad ,$$

$$T_\theta = 0 \quad , \qquad\qquad\qquad T_z = 0 \quad .$$

The constant $Y_{14}$ can be found by setting $T_r$ equal to zero. Similar expressions for surface tractions can be obtained at the surface $r = (A R_b{}^2 + B)^{1/2}$

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