A COMPUTATIONAL STUDY OF SOME CONTROL PROBLEMS IN ELECTRICALLY CONDUCTING FLOWS

by

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A Computational Study of Some Control Problems in Electrically Conducting Flows

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Abstract

This thesis deals with the computation of some control problems in an electrically conducting flow governed by incompressible Navier-Stokes equations and Maxwell's equations. We consider two different controls, namely electric current control and heat flux control, to study the optimal control techniques for their effectiveness in flow control problems. The methods and implementations can be extended to other controls such as velocity controls as well.

We use Lagrange multiplier techniques to derive the optimality systems. Finite element approximations are defined and computational methods are developed for both the steady and unsteady optimality systems. Computational experiments are conducted in some closed domains. Our computational experiments indicate that optimal control techniques seem to work very well for small or even for moderately high Reynolds' numbers.

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Contents

Chapter 1

Introduction

Theoretical and computational approaches in the control of fluid flows has recently received considerable attention both from the mathematical and engineering communities. In this thesis we consider some control problems in electrically conducting fluids. The type of electrically conducting fluid we consider is governed by the incompressible Navier-Stokes system coupled with Maxwell's equation. Proper formulation of control problems, development of robust algorithms to compute the solutions and the efficient implementation of these algorithms were the objectives of this study. We like to bring the readers attention to an experimental investigation of some control problems in electrically conducting flows given in [NB] and to a mathematical analysis of some relevent optimal control problems given in [HAM] and [HP].

First, we consider the problem of steering the velocity to a desired one in the flow domain by choosing an appropriate boundary current density. This control problem can be cast as a minimization problem with the cost function

$$\mathcal{J}_1(\mathbf{u}, g) = \frac{1}{2}\int_\Omega |\mathbf{u} - \mathbf{u}_d|^2 d\Omega + \frac{1}{2}\int_\Gamma |g|^2 d\Gamma,$$

where $\mathbf{u}_d$ is the desired velocity and the second term is essentially the cost of implementing the control, and subject to the constraint that the flow variables satisfy the governing equations, namely the incompressible Navier-Stokes equations and Maxwell equations. The uncontrolled flow, i.e., the flow with zero current density on the boundary, is depicted in Figs. 1.1 a) and b) and the desired flow we consider is given in Figs. 1.1 c) and d).

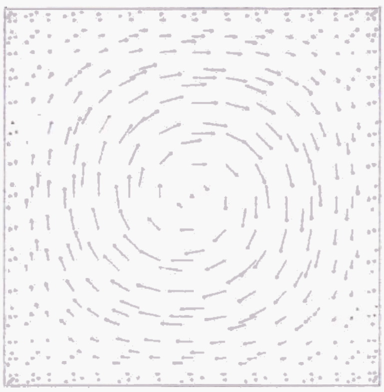

Figure 1.1 a) uncontrolled velocity field $\mathbf{u}_0$

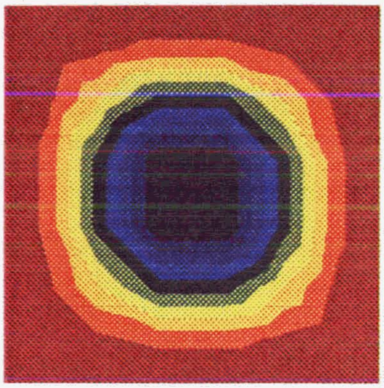

Figure 1.1 b) contours of $\mathbf{u}_0$

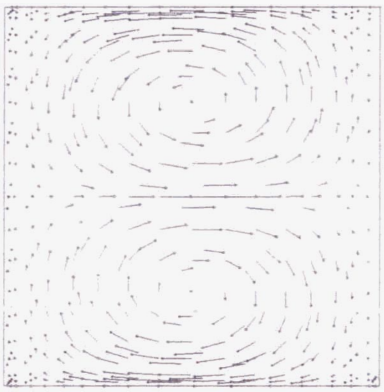

Figure 1.1 c) desired velocity field $\mathbf{u}_d$

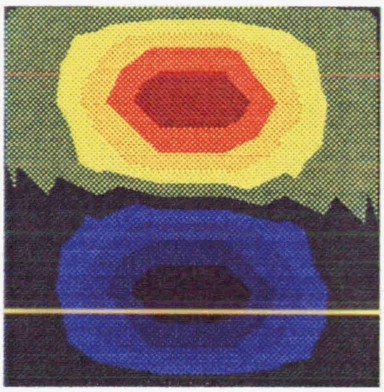

Figure 1.1 d) contours of $\mathbf{u}_d$

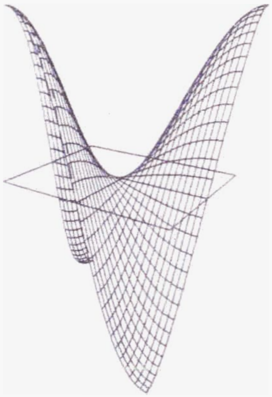

Figure 1.1 e) uncontrolled potential field ϕ_0

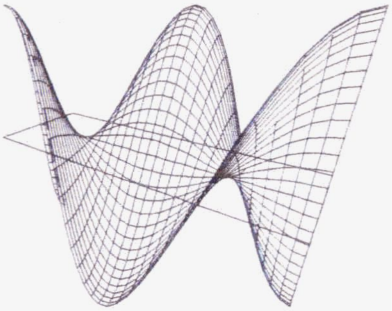

Figure 1.1 f) uncontrolled temperature field T_0

The second and third control problems are motivated essentially by the same objective, i.e. to obtain a desired potential distribution throughout the flow by choosing an appropriate boundary current density. In the second one we try to obtain a prescribed potential distribution and in the third one we want to obtain a uniform potential distribution in the flow. The appropriate cost functions then are

$$\mathcal{J}_2(\phi, g) = \frac{1}{2}\int_\Omega |\phi - \phi_d|^2 d\Omega + \frac{1}{2}\int_\Gamma |g|^2 d\Gamma$$

and

$$\mathcal{J}_3(\phi, g) = \frac{1}{2}\int_\Omega |\nabla\phi|^2 d\Omega + \frac{1}{2}\int_\Gamma |g|^2 d\Gamma.$$

An uncontrolled potential distribution is depicted in Fig.1.1 e).

The fourth and fifth control problems are similar to the last two except that we consider the temperature distribution instead of the potential distribution. The cost functions in this setting are

$$\mathcal{J}_4(T, g) = \frac{1}{2}\int_\Omega |T - T_d|^2 d\Omega + \frac{1}{2}\int_\Gamma |g|^2 d\Gamma$$

and

$$\mathcal{J}_5(T, g) = \frac{1}{2}\int_\Omega |\nabla T|^2 d\Omega + \frac{1}{2}\int_\Gamma |g|^2 d\Gamma.$$

An uncontrolled temperature distribution is given in Fig. 1.1 f).

We also consider these control problems for unsteady flows as well. The first, second and fourth problems involve matching the flow with a desired one at some fixed time levels. The third and fifth control problems involve maintaining a uniform distribution of the physical quantity considered at each time level.

In all the control problems we consider we write down the problems, after identifying the cost to be minimized, as minimization problems with equality constraints. Lagrange multiplier techniques are employed to convert the constrained minimization problems into unconstrained one. This leads us to optimality systems. To find the optimal controls, these optimality systems are solved. The solution procedure involves the following steps: finite dimensional discretization and solution of a finite dimensional nonlinear system. The problem is discretized using a mixed conforming Galerkin finite element method. The pressure and its adjoint counterpart are approximated by piecewise linear polynomials over triangles and all the other variables are approximated by piecewise quadratic polynomials over the same triangles. This particular selection of finite element spaces yields stable approximations to the pressure and its counterpart. Due to the nonlinearity in the state equations, we are lead to a nonlinear algebraic system to be solved for the unknown nodal values. Thus in the next step, we linearize this system using a Newton type linearization along with a simple linearization (simple linearization is necessary to find a good initial guess for Newton's method). Other linearization methods can also be used. Finally, a banded Gaussian elimination or an iterative method is employed to solve the linear systems of equations occurring in the solution process. We have also done some analysis for the numerical methods. In particular, convergence is shown and error estimates are derived.

In unsteady control problems, a solution of the optimality system involve a coupled forward-backward time evolution system. Due to the nonlinearity of the state equations, direct computation of the system can be costly. We therefore consider a numerical procedure to bypass this forward-backward time marching. We remark here that efficient methods to solve this forward-backward problem in the context of control of flows governed by the Navier-Stokes equations are currently underway.

Our computational procedure here requires time marching only forward in time. First we discretize the state equations using an implicit time discretization scheme, then in each time interval the state equation is steady and the Lagrange multiplier techniques are employed to derive the optimality system. We then solve the optimality system using the finite element procedure developed in the steady state setting.

This computational procedure for the unsteady flows requires both theoretical and numerical analysis since we do not know whether a solution obtained by this procedure is optimal nor do we know the cost considered decays monotonically in time. These interesting questions are currently being investigated. But our computational experiments conducted on the problems considered in this thesis and on the control of Navier-Stokes equations (not reported here, see [HR2]) indicate that the cost indeed decays.

Our computations give promising results for two dimensional domains at least in the moderate Reynolds number regime in both steady and unsteady flows. Higher dimensional problems, exterior problems and high Reynolds number flows are all identified as further research topics.

Finally, we give the plan of this thesis. The first two chapters deal with control problems in steady flows and the last two with control problems in unsteady flows.

In Chapter 1, we give a brief introduction of this thesis. Section 1.1 of this Chapter introduces the functional framework that will be used throughout this thesis.

In Chapter 2, we study steady control problems with electric current control. In Section 2.1 we give variational formulations of the constraints and derive the optimality system using the Lagrange multiplier technique. In Section 2.2, we define finite element approximations and derive error estimates. In Section 2.3, some solution methods for the discrete optimality systems are presented. Section 2.4 summarizes some mathematical results concerning the Optimality systems and finite element approximations. In Section 2.5, we present some computational results. Finally in Section 2.6, concluding remarks are given.

In Chapter 3, we again study some steady control problems but with heat flux controls. The plan of this Chapter is simply a copy of the previous one.

In Chapter 4, we study control problems in unsteady flows with electric current controls. Section 4.1 deals with the variational formulation of the constraint equations and the derivation of optimality systems. In Section 4.2 we report some computational results. Section 4.3 is the conclusion.

The plan of Chapter 5 is a copy of the previous Chapter with heat flux control. We conclude our Introduction Chapter with some notations. Throughout this thesis, $H^s(\mathcal{D})$, $s \in \mathbf{R}$, denotes the standard Sobolev space of order s with respect to the set $\mathcal{D}$, where $\mathcal{D}$ is either the flow domain Ω or its boundary Γ. Norms of functions belonging to $H^s(\Omega)$ and $H^s(\Gamma)$ are denoted by $\|\cdot\|_s$ and $\|\cdot\|_{s,\Gamma}$, respectively. Corresponding Sobolev spaces of vector valued functions will be denoted by $\mathbf{H}^s(\mathcal{D})$; e.g. $\mathbf{H}^1(\Omega) = [H^1(\Omega)]^d$. Norms for spaces of vector valued functions will be denoted by the same notation as that used for their scalar counterparts. For example, $\|\mathbf{v}\|^r_{L^r(\Omega)} = \sum_{j=1}^d \|v_j\|^r_{L^r(\Omega)}$ and where v_j, $j = 1, .., d$ denotes the components of $\mathbf{v}$. The following function spaces will be used frequently:

$$\mathbf{H}^1(\Omega) = \{v_j \in L^2(\Omega) : \frac{\partial v_j}{\partial x_k} \in L^2(\Omega) \quad \text{for } j,k = 1,\ldots,d\},$$

$$\mathbf{H}_0^1(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v} = 0 \quad \text{on} \quad \Gamma\},$$

$$L_0^2(\Omega) = \{p \in L^2(\Omega) : \int_\Omega p d\Omega = 0\},$$

and

$$\tilde{H}^1(\Omega) = H^1(\Omega) \cap L_0^2(\Omega).$$

The inner products in $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ are both denoted by $(\cdot,\cdot)$ and those in $L^2(\Gamma)$ and $\mathbf{L}^2(\Gamma)$ by $(\cdot,\cdot)_\Gamma$.

Chapter 2

Control of Steady Flows I: Electric Current Control

In this chapter we study some optimal control problems for a steady electrically conducting fluid. The control is the normal electric current on the flow boundary. In Section 2.1, we give the precise statement of the optimal control problems we consider. In Section 2.2, we derive the optimality system using the Lagrange multiplier technique. Some solution methods are presented in Section 2.3. Some mathematical results for the optimality system are given in Section 2.4. In Section 2.5, we present some computational results. Finally, in Section 2.6 we make some concluding remarks.

2.1 Statement of the Optimal Control Problems

We denote by Ω the flow region which is a bounded open container in $\mathbf{R}^2$ or $\mathbf{R}^3$ with a boundary Γ. The dimensionless equations governing the steady incompressible flow of an electrically conducting fluid in the presence of a magnetic field, when Maxwell's displacement currents are neglected, are given by the following; see [TC]:

$$\frac{1}{N}(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + (\mathbf{j}\times\mathbf{B}) + \frac{1}{M^2}\Delta\mathbf{u} + \mathbf{f} \quad \text{in } \Omega\,,$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\,,$$

$$\mathbf{j} = -\nabla\phi + (\mathbf{u} \times \mathbf{B}) \quad \text{in } \Omega,$$

$$\nabla \cdot \mathbf{j} = 0 \quad \text{in } \Omega,$$

$$\nabla \times \mathbf{B} = R_m \mathbf{j} \quad \text{in } \Omega,$$

and

$$\nabla \cdot \mathbf{B} = 0 \quad \text{in } \Omega$$

where $\mathbf{u}$ denotes the velocity field, p the pressure field, $\mathbf{j}$ the electric current density, $\mathbf{B}$ the magnetic field, and ϕ the electric potential. Also, N is the interaction number, M is the Hartmann number, and R_m is the magnetic Reynolds number.

We will deal with a special case in which the externally applied magnetic field is undisturbed by the flow; in particular, we assume that $\mathbf{B}$ is given. Such an assumption can be met in a variety of physical applications, e.g., in the modeling of electromagnetic pumps and the flow of liquid lithium for fusion reactor cooling blankets ([JW], [WH]).

Note that if the flow is two-dimensional, our convention in this thesis is that the applied magnetic field $\mathbf{B}$ is perpendicular to the flow plane, i.e., $\mathbf{B} = (0, 0, B(x, y))^T$, and that the cross product $\mathbf{u} \times \mathbf{B}$ is understood as $(u_1, u_2, 0)^T \times (0, 0, B(x, y))^T$.

Remark: The more general case in which B is unknown involves essentially a three dimensional system of equations governing the flow. Our computational methods considered in this thesis can be extended in a similar manner to the control problems in this three dimensional flow.□

In the above set of equations, we take divergence of the third equation and use the fourth equation to arrive at the following simplified system:

$$-\frac{1}{M^2}\Delta\mathbf{u} + \frac{1}{N}(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - (\mathbf{B} \times \nabla\phi) - (\mathbf{u} \times \mathbf{B}) \times \mathbf{B} = \mathbf{f} \quad \text{in } \Omega, \tag{2.1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \tag{2.2}$$

and

$$-\Delta\phi + \nabla \cdot (\mathbf{u} \times \mathbf{B}) = 0 \quad \text{in } \Omega.$$

We will replace the last equation with the following slightly more general one

$$-\Delta\phi + \nabla\cdot(\mathbf{u}\times\mathbf{B}) = k \quad \text{in } \Omega\,. \tag{2.3}$$

In (2.1)–(2.3), $\mathbf{B}$, $\mathbf{f}$ and k are given data.

The system (2.1)–(2.3) is supplemented with boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma \tag{2.4}$$

and

$$\frac{\partial\phi}{\partial n} = g \quad \text{on } \Gamma\,, \tag{2.5}$$

where g denotes the only control variable, namely, the normal electric current on Γ. Such a control can be effected by attaching electric sources with adjustable resistors to the electrode along the flow boundary. Although a normal electric current control is physically somewhat artificial (this could be achieved in practice only by insulating different small parts of an electrode from each other), it is mathematically more convenient than an electric potential control. The techniques to treat normal electric current controls are applicable to treat other types of controls and the solutions with a normal electric current control do indicate the behavior to expect in general. See [HM].

Our goal is to try to obtain a desired flow field by appropriately choosing the control – the normal electric current on Γ. Specifically we will investigate the following cases: matching a desired velocity field, matching a desired potential field, or minimizing the potential gradient. Mathematically, these tasks can be described, respectively, by the following optimal control setting: minimize the cost functional

$$\mathcal{K}(\mathbf{u},\phi,p,g) = \frac{1}{2\epsilon}\int_\Omega |\mathbf{u}-\mathbf{u}_d|^2 d\Omega + \frac{\delta}{2}\int_\Gamma |g|^2 d\Gamma\,, \tag{2.6}$$

or

$$\mathcal{M}(\mathbf{u},\phi,p,g) = \frac{1}{2\epsilon}\int_\Omega |\phi-\phi_d|^2 d\Omega + \frac{\delta}{2}\int_\Gamma |g|^2 d\Gamma\,, \tag{2.7}$$

or

$$\mathcal{N}(\mathbf{u},\phi,p,g) = \frac{1}{2\epsilon}\int_\Omega |\nabla\phi|^2 d\Omega + \frac{\delta}{2}\int_\Gamma |g|^2 d\Gamma\,, \tag{2.8}$$

subject to the constraints (2.1)–(2.5). Here $\epsilon > 0$ and $\delta > 0$ are positive parameters; $\mathbf{u}_d$ and ϕ_d are, respectively, the desired velocity field and potential field.

The minimization of the functional (2.6) or (2.7) or (2.8) subject to (2.1)–(2.5) is a special case of the following general optimal control setting:

$$\text{minimize the cost functional}$$
$$\mathcal{J}(\mathbf{u},\phi,p,g) = \mathcal{F}(\mathbf{u},\phi,p) + \frac{\delta}{2}\int_\Gamma |g|^2 d\Gamma \tag{2.9}$$
$$\text{subject to the constraints (2.1)–(2.5),}$$

where $\mathcal{F}(\mathbf{u},\phi,p)$ is a functional of $(\mathbf{u},\phi,p)$.

2.2 A Variational Formulation of the Constraints; An Optimality System of Equations

The variational formulation of the constraint equations is given as follows: seek $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, $p \in L_0^2(\Omega)$ and $\phi \in \widetilde{H}^1(\Omega)$ such that

$$\begin{aligned} \frac{1}{M^2}\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, d\Omega + \int_\Omega [\nabla\phi - (\mathbf{u}\times\mathbf{B})]\cdot[\nabla\psi - (\mathbf{v}\times\mathbf{B})]d\Omega + \frac{1}{N}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,d\Omega \\ - \int_\Omega p\nabla\cdot\mathbf{v}\,d\Omega = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\Omega + \int_\Omega k\,\psi\,d\Omega + \int_\Gamma g\,\psi\,d\Gamma \quad \forall(\mathbf{v},\psi)\in\mathbf{H}_0^1(\Omega)\times\widetilde{H}^1(\Omega) \end{aligned} \tag{2.10}$$

and

$$\int_\Omega q\nabla\cdot\mathbf{u}\,d\Omega = 0 \quad \forall q \in L_0^2(\Omega). \tag{2.11}$$

Or, equivalently, seek $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, $p \in L_0^2(\Omega)$ and $\phi \in \widetilde{H}^1(\Omega)$ such that

$$\begin{aligned} \frac{1}{M^2}\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\,d\Omega - \int_\Omega [\nabla\phi - (\mathbf{u}\times\mathbf{B})]\cdot(\mathbf{v}\times\mathbf{B})d\Omega + \frac{1}{N}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,d\Omega - \int_\Omega p\nabla\cdot\mathbf{v}\,d\Omega \\ = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\Omega \quad \forall\mathbf{v}\in\mathbf{H}_0^1(\Omega), \end{aligned} \tag{2.12}$$

$$\int_\Omega [\nabla\phi - (\mathbf{u}\times\mathbf{B})]\cdot(\nabla\psi)d\Omega = \int_\Omega k\,\psi\,d\Omega + \int_\Gamma g\,\psi\,d\Gamma \quad \forall\psi\in\widetilde{H}^1(\Omega) \tag{2.13}$$

and

$$\int_\Omega q\nabla\cdot\mathbf{u}\,d\Omega = 0 \quad \forall q \in L_0^2(\Omega). \tag{2.14}$$

Here the colon notation stands for the scalar product on $\mathbf{R}^{d\times d}$.

The precise mathematical statement of the optimal control problem (2.9) can now be given as follows:

$$\text{seek a } (\mathbf{u},p,\phi,g)\in \mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)\times \widetilde{H}^1(\Omega)\times L^2(\Gamma) \text{ such that the functional (2.9) is minimized subject to the constraints (2.1)-(2.5).} \tag{2.15}$$

We will turn the constrained optimization problem (2.15) into an unconstrained one by using Lagrange multiplier principles. (For mathematical theories of Lagrange multiplier principles, see, e.g., [VT].) We set $\mathcal{X} = \mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)\times \widetilde{H}^1(\Omega)\times L^2(\Gamma)\times \mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)\times \widetilde{H}^1(\Omega)$ and define the Lagrangian functional

$$\begin{aligned}
&\mathcal{L}(\mathbf{u},p,\phi,g,\mu,\tau,s)\\
&= \mathcal{F}(\mathbf{u},p,\phi)+\frac{\delta}{2}\int_\Gamma |g|^2 d\Gamma - \tfrac{1}{M^2}\int_\Omega \nabla\mathbf{u}:\nabla\mu d\Omega + \int_\Omega [\nabla\phi-(\mathbf{u}\times\mathbf{B})]\cdot(\mu\times\mathbf{B})d\Omega\\
&\quad -\tfrac{1}{N}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mu d\Omega + \int_\Omega p\nabla\cdot\mu d\Omega + \int_\Omega \mathbf{f}\cdot\mu d\Omega - \int_\Omega [\nabla\phi-(\mathbf{u}\times\mathbf{B})]\cdot(\nabla s)d\Omega\\
&\quad +\int_\Omega k\, s d\Omega + \int_\Gamma g\, s d\Gamma + \int_\Omega \tau\nabla\cdot\mathbf{u} d\Omega \qquad \forall(\mathbf{u},p,\phi,g,\mu,\tau,s)\in\mathcal{X}.
\end{aligned} \tag{2.16}$$

An optimality system of equations that an optimum must satisfy is derived by taking variations with respect to every variable in the Lagrangian. By taking variations with respect to $\mathbf{u}$, p and ϕ, we obtain:

$$\begin{aligned}
&\tfrac{1}{M^2}\int_\Omega \nabla\mu:\nabla\mathbf{w}d\Omega + \int_\Omega (\mathbf{w}\times\mathbf{B})\cdot(\mu\times\mathbf{B})d\Omega + \tfrac{1}{N}\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mu d\Omega + \tfrac{1}{N}\int_\Omega(\mathbf{w}\cdot\nabla)\mathbf{u}\cdot\mu d\Omega\\
&\quad -\int_\Omega \tau\nabla\cdot\mathbf{w}d\Omega = \langle\mathcal{F}_\mathbf{u}(\mathbf{u},p,\phi),\mathbf{w}\rangle \quad \forall\mathbf{w}\in\mathbf{H}_0^1(\Omega),
\end{aligned} \tag{2.17}$$

$$\int_\Omega [\nabla s-(\mu\times\mathbf{B})]\cdot(\nabla r)d\Omega = \langle\mathcal{F}_\phi(\mathbf{u},p,\phi),r\rangle \quad \forall r\in\widetilde{H}^1(\Omega) \tag{2.18}$$

and

$$\int_\Omega \sigma\nabla\cdot\mu d\Omega = \langle\mathcal{F}_p(\mathbf{u},p,\phi),\sigma\rangle \quad \forall\sigma\in L_0^2(\Omega), \tag{2.19}$$

where $\mathcal{F}_\mathbf{u}$, $\mathcal{F}_\phi$ and $\mathcal{F}_p$ are the derivatives of the functional with respect to its three arguments, respectively. By taking variations with respect to μ, τ and ξ, we recover the constraint equations

(2.1)-(2.3). By taking variation with respect to g we obtain

$$\int_\Gamma (\delta\, g\, z + z\, s) d\Gamma = 0 \quad \forall z \in L^2(\Gamma) \qquad \text{i.e.,} \qquad g = -\frac{1}{\delta}\, s\,.$$

This last equation enables us to eliminate the control g in (2.13). Thus (2.12)-(2.14) can be replaced by

$$\frac{1}{M^2}\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} d\Omega - \int_\Omega [\nabla\phi - (\mathbf{u}\times\mathbf{B})]\cdot(\mathbf{v}\times\mathbf{B}) d\Omega + \frac{1}{N}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} d\Omega - \int_\Omega p\,\nabla\cdot\mathbf{v} d\Omega$$
$$= \int_\Omega \mathbf{f}\cdot\mathbf{v} d\Omega \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,, \qquad (2.20)$$

$$\int_\Omega [\nabla\phi - (\mathbf{u}\times\mathbf{B})]\cdot(\nabla\psi) d\Omega + \frac{1}{\delta}\int_\Gamma s\,\psi d\Omega = \int_\Omega k\,\psi d\Omega \quad \forall \psi \in \widetilde{H}^1(\Omega)\,, \qquad (2.21)$$

and

$$\int_\Omega q\,\nabla\cdot\mathbf{u} d\Omega = 0 \quad \forall q \in L_0^2(\Omega)\,, \qquad (2.22)$$

Equations (2.17)-(2.22) forms an optimality system of equations that an optimal solution must satisfy.

2.3 Finite Element Approximations

A finite element discretization of the optimality system (2.17)–(2.22) is defined in the usual manner. First one chooses families of finite dimensional subspaces $X^h \subset H^1(\Omega)$ and $S^h \subset L^2(\Omega)$. These families are parameterized by a parameter h that tends to zero; commonly, h is chosen to be some measure of the grid size. We set $\tilde{X}^h = X^h \cap L_0^2(\Omega)$, $\mathbf{X}^h = [X^h]^d$, $\mathbf{X}_0^h = \mathbf{X}^h \cap \mathbf{H}_0^1(\Omega)$ and $S_0^h = S^h \cap L_0^2(\Omega)$. We assume that as $h \to 0$,

$$\inf_{\mathbf{v}^h \in \mathbf{X}_0^h} \|\mathbf{v} - \mathbf{v}^h\|_1 \to 0\,, \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,,$$

$$\inf_{v^h \in \tilde{X}^h} \|v - v^h\|_1 \to 0\,, \quad \forall v \in \widetilde{H}^1(\Omega)$$

and

$$\inf_{q^h \in S_0^h} \|q - q^h\|_0 \to 0\,, \quad \forall q \in L_0^2(\Omega)\,.$$

Here we may choose any pair of subspaces X^h and S^h such that $\mathbf{X}_0^h$ and S_0^h can be used for finding finite element approximations of solutions of the Navier-Stokes equations. Thus, we make the following standard assumptions, which are exactly those employed in well-known finite element methods for the Navier-Stokes equations. First, we have the approximation properties: there exist an integer k and a constant C, independent of h, $\mathbf{v}$, v and q, such that

$$\inf_{\mathbf{v}^h \in \mathbf{X}_0^h} \|\mathbf{v} - \mathbf{v}^h\|_1 \le C h^m \|\mathbf{v}\|_{m+1} \quad \forall \mathbf{v} \in \mathbf{H}^{m+1} \cap \mathbf{H}_0^1(\Omega)\,,\ 1 \le m \le k \tag{2.23}$$

$$\inf_{v^h \in \tilde{X}^h} \|v - v^h\|_1 \le C h^m \|v\|_{m+1} \quad \forall v \in H^{m+1} \cap L_0^2(\Omega)\,,\ 1 \le m \le k \tag{2.24}$$

and

$$\inf_{q^h \in S_0^h} \|q - q^h\|_0 \le C h^m \|q\|_m \quad \forall q \in H^m \cap L_0^2(\Omega)\,,\ 1 \le m \le k\,. \tag{2.25}$$

Next, we assume the *inf-sup condition*, or *Ladyzhenskaya-Babuska-Brezzi condition*: there exists a constant C, independent of h, such that

$$\inf_{0 \ne q^h \in S_0^h} \ \sup_{0 \ne \mathbf{v}^h \in \mathbf{X}_0^h} \frac{-\int_\Omega q^h \nabla \cdot \mathbf{v}^h d\Omega}{\|\mathbf{v}^h\|_1 \|q^h\|_0} \ge C\,. \tag{2.26}$$

This condition assures the stability of finite element discretizations of the Navier-Stokes equations. It also assures the stability of the approximation of the constraint equations (2.21)–(2.22) and the optimality system (2.17)–(2.22). For thorough discussions of the approximation properties (2.23)–(2.25) and the stability condition (2.26), see, *e.g.*, [GR] or [MG]. These references may also be consulted for a catalogue of finite element subspaces that meet the requirements of (2.23)–(2.26).

Once the approximating subspaces have been chosen, we seek $\mathbf{u}^h \in \mathbf{X}_0^h$, $p^h \in S_0^h$, $\phi^h \in \tilde{X}^h$, $\mu^h \in \mathbf{X}_0^h$, $\tau^h \in S_0^h$ and $s^h \in \tilde{X}^h$ such that

$$\frac{1}{M^2}\int_\Omega \nabla \mathbf{u}^h : \nabla \mathbf{v}^h d\Omega - \int_\Omega [\nabla\phi^h - (\mathbf{u}^h \times \mathbf{B})]\cdot(\mathbf{v}^h \times \mathbf{B}) d\Omega + \frac{1}{N}\int_\Omega (\mathbf{u}^h\cdot\nabla)\mathbf{u}^h \cdot \mathbf{v}^h d\Omega$$
$$- \int_\Omega p^h \nabla\cdot\mathbf{v}^h d\Omega = \int_\Omega \mathbf{f}\cdot\mathbf{v}^h d\Omega \quad \forall \mathbf{v}^h \in \mathbf{X}_0^h\,, \tag{2.27}$$

$$\int_\Omega [\nabla\phi^h - (\mathbf{u}^h \times \mathbf{B})]\cdot(\nabla\psi^h) d\Omega + \frac{1}{\delta}\int_\Gamma s\,\psi^h d\Gamma = \int_\Omega k\,\psi^h d\Omega \quad \forall \psi^h \in \tilde{X}^h\,, \tag{2.28}$$

$$\int_\Omega q^h \nabla\cdot\mathbf{u}^h d\Omega = 0 \quad \forall q^h \in S_0^h\,, \tag{2.29}$$

$$\frac{1}{M^2}\int_\Omega \nabla\mu^h : \nabla\mathbf{w}^h d\Omega + \int_\Omega (\mathbf{w}^h \times \mathbf{B})\cdot(\mu^h \times \mathbf{B}) d\Omega + \frac{1}{N}\int_\Omega (\mathbf{u}^h\cdot\nabla)\mathbf{w}^h\cdot\mu^h d\Omega \tag{2.30}$$
$$+\frac{1}{N}\int_\Omega (\mathbf{w}^h\cdot\nabla)\mathbf{u}^h\cdot\mu^h d\Omega - \int_\Omega \tau^h \nabla\cdot\mathbf{w}^h d\Omega = (\mathcal{F}_{\mathbf{u}}(\mathbf{u}^h, p^h, \phi^h), \mathbf{w}^h) \quad \forall \mathbf{w}^h \in \mathbf{X}_0^h\,,$$

$$\int_\Omega [\nabla s^h - (\mu^h \times \mathbf{B})] \cdot (\nabla r^h) d\Omega = \langle \mathcal{F}_\phi(\mathbf{u}^h, p^h, \phi^h), r^h \rangle \quad \forall r^h \in \tilde{X}^h \tag{2.31}$$

and

$$\int_\Omega \sigma^h \nabla \cdot \mu^h d\Omega = \langle \mathcal{F}_p(\mathbf{u}^h, p^h, \phi^h), \sigma^h \rangle \quad \forall \sigma^h \in S_0^h . \tag{2.32}$$

From a computational standpoint, this is a formidable system. In three dimensions, we have a coupled system involving *ten* unknown discrete scalar fields. Therefore, how one solves this system is a rather important question.

2.4 Solution Methods for the Discrete Optimality System of Equations

We will present two methods for solving the discrete optimality system of equations (2.27)–(2.32). The first one is the Newton's method for the entire system; the second one is an iterative method which uncouples the computation of (2.30)–(2.32) and the computation of (2.27)–(2.29) at each iteration (the second method is in essence equivalent to a gradient method in minimization.)

2.4.1 Newton's method.

For notational convenience, we will use $(\mathbf{U}, P, \Phi, \mathbf{M}, T, S)$ to denote $(\mathbf{u}^h, p^h, \phi^h, \mu^h, r^h, s^h)$. We will only give the Newton's method for the special case $\mathcal{F}(\mathbf{u}, p, \phi) = \mathcal{F}(\mathbf{u})$. General cases can be treated similarly. Thus the Newton's method for solving the discrete optimality system (2.27)–(2.32) is given as follows:

1° Choose an initial guess $(\mathbf{U}^{(0)}, P^{(0)}, \Phi^{(0)}, \mathbf{M}^{(0)}, T^{(0)}, S^{(0)})$;

2° For $n = 1, 2, \cdots$, compute $(\mathbf{U}^{(n)}, P^{(n)}, \Phi^{(n)}, \mathbf{M}^{(n)}, T^{(n)}, S^{(n)})$ from the following discrete system of equations:

$$\begin{aligned}\frac{1}{M^2}\int_\Omega \nabla \mathbf{U}^{(n)} : \nabla \mathbf{v}^h d\Omega &- \int_\Omega [\nabla \Phi^{(n)} - (\mathbf{U}^{(n)} \times \mathbf{B})] \cdot (\mathbf{v}^h \times \mathbf{B}) d\Omega - \int_\Omega P^{(n)} \nabla \cdot \mathbf{v}^h d\Omega \\ &+ \frac{1}{N}\int_\Omega (\mathbf{U}^{(n)} \cdot \nabla)\mathbf{U}^{(n-1)} \cdot \mathbf{v}^h d\Omega + \frac{1}{N}\int_\Omega (\mathbf{U}^{(n-1)} \cdot \nabla)\mathbf{U}^{(n)} \cdot \mathbf{v}^h d\Omega \\ &= \int_\Omega \mathbf{f} \cdot \mathbf{v}^h d\Omega + \frac{1}{N}\int_\Omega (\mathbf{U}^{(n-1)} \cdot \nabla)\mathbf{U}^{(n-1)} \cdot \mathbf{v}^h d\Omega \quad \forall \mathbf{v}^h \in \mathbf{X}_0^h, \end{aligned} \tag{2.33}$$

$$\int_{\Omega}[\nabla\Phi^{(n)}-(\mathbf{U}^{(n)}\times\mathbf{B})]\cdot(\nabla\psi^h)d\Omega+\frac{1}{\delta}\int_{\Gamma}S^{(n)}\psi^h d\Omega=\int_{\Omega}k\,\psi^h d\Omega\quad\forall\psi^h\in\tilde{X}^h,\tag{2.34}$$

$$\int_{\Omega}q^h\,\nabla\cdot\mathbf{U}^{(n)}d\Omega=0\quad\forall q^h\in S_0^h,\tag{2.35}$$

$$\begin{aligned}&\tfrac{1}{M^2}\int_{\Omega}\nabla\mathbf{M}^{(n)}:\nabla\mathbf{w}^h d\Omega+\int_{\Omega}(\mathbf{w}^h\times\mathbf{B})\cdot(\mathbf{M}^{(n)}\times\mathbf{B})d\Omega+\tfrac{1}{N}\int_{\Omega}(\mathbf{U}^{(n)}\cdot\nabla)\mathbf{w}^h\cdot\mathbf{M}^{(n-1)}d\Omega\\&+\tfrac{1}{N}\int_{\Omega}(\mathbf{U}^{(n-1)}\cdot\nabla)\mathbf{w}^h\cdot\mathbf{M}^{(n)}d\Omega+\tfrac{1}{N}\int_{\Omega}(\mathbf{w}^h\cdot\nabla)\mathbf{U}^{(n-1)}\cdot\mathbf{M}^{(n)}d\Omega\\&+\tfrac{1}{N}\int_{\Omega}(\mathbf{w}^h\cdot\nabla)\mathbf{U}^{(n)}\cdot\mathbf{M}^{(n-1)}d\Omega-\int_{\Omega}T^{(n)}\nabla\cdot\mathbf{w}^h d\Omega-\langle\mathcal{F}_{\mathbf{uu}}(\mathbf{U}^{(n-1)})\mathbf{U}^{(n)},\mathbf{w}^h\rangle\\&=\langle\mathcal{F}_{\mathbf{u}}(\mathbf{U}^{(n-1)}),\mathbf{w}^h\rangle-\langle\mathcal{F}_{\mathbf{uu}}(\mathbf{U}^{(n-1)})\,\mathbf{U}^{(n-1)},\mathbf{w}^h\rangle+\tfrac{1}{N}\int_{\Omega}(\mathbf{U}^{(n-1)}\cdot\nabla)\mathbf{w}^h\cdot\mathbf{M}^{(n-1)}d\Omega\\&+\tfrac{1}{N}\int_{\Omega}(\mathbf{w}^h\cdot\nabla)\mathbf{U}^{(n-1)}\cdot\mathbf{M}^{(n-1)}d\Omega\quad\forall\mathbf{w}^h\in\mathbf{X}_0^h,\end{aligned}\tag{2.36}$$

$$\int_{\Omega}[\nabla S^{(n)}-(\mathbf{M}^{(n)}\times\mathbf{B})]\cdot(\nabla r^h)d\Omega=0\quad\forall r^h\in\tilde{X}^h\tag{2.37}$$

and

$$\int_{\Omega}\sigma^h\,\nabla\cdot\mathbf{M}^{(n)}d\Omega=0\quad\forall\sigma^h\in S_0^h.\tag{2.38}$$

Under suitable assumptions, the Newton's method converges at a quadratic rate to the finite element solution $(\mathbf{U},P,\Phi,\mathbf{M},T,S)$.

Quadratic convergence of Newton's method is valid within a contraction ball. In practice we normally first perform a few fixed point iteration and then switch to the Newton's method. For flow field matching problems, we use the desired field as initial data for the corresponding unknown field. This choice has proven to produce faster convergence compared with arbitrary choice of initial data.

2.4.2 An iterative method.

We now discuss an iterative method that uncouples the solution of the constraint equations (2.27)–(2.29) from the adjoint equations (2.30)–(2.32).

1° Choose $(\mathbf{U}^{(0)},P^{(0)},\Phi^{(0)})$;

2° For $n=0,1,2,\cdots$, compute $(\mathbf{M}^{(n)},T^{(n)},S^{(n)})$ from

$$\tfrac{1}{M^2}\int_{\Omega}\nabla\mathbf{M}^{(n)}:\nabla\mathbf{w}^h d\Omega+\int_{\Omega}(\mathbf{w}^h\times\mathbf{B})\cdot(\mathbf{M}^{(n)}\times\mathbf{B})d\Omega+\tfrac{1}{N}\int_{\Omega}(\mathbf{U}^{(n-1)}\cdot\nabla)\mathbf{w}^h\cdot\mathbf{M}^{(n)}d\Omega$$

$$+\frac{1}{N}\int_\Omega (\mathbf{w}^h\cdot\nabla)\mathbf{U}^{(n-1)}\cdot\mathbf{M}^{(n)}d\Omega - \int_\Omega T^{(n)}\nabla\cdot\mathbf{w}^h d\Omega$$
$$= \langle \mathcal{F}_{\mathbf{u}}(\mathbf{u}^{(n-1)}, p^{(n-1)}, \phi^{(n-1)}), \mathbf{w}^h\rangle \quad \forall \mathbf{w}^h \in \mathbf{X}_0^h, \tag{2.39}$$

$$\int_\Omega [\nabla S^{(n)} - (\mathbf{M}^{(n)}\times\mathbf{B})]\cdot(\nabla r^h)d\Omega = \langle \mathcal{F}_\phi(\mathbf{U}^{(n-1)}, P^{(n-1)}, \Phi^{(n-1)}), r^h\rangle \quad \forall r^h \in \tilde{X}^h \tag{2.40}$$

and

$$\int_\Omega \sigma^h \nabla\cdot\mathbf{M}^{(n)}d\Omega = \langle \mathcal{F}_p(\mathbf{U}^{(n-1)}, P^{(n-1)}, \Phi^{(n-1)}), \sigma^h\rangle \quad \forall \sigma^h \in S_0^h; \tag{2.41}$$

and compute $(\mathbf{U}^{(n)}, P^{(n)}, \Phi^{(n)})$ from

$$\frac{1}{M^2}\int_\Omega \nabla\mathbf{U}^{(n)} : \nabla\mathbf{v}^h d\Omega - \int_\Omega [\nabla\Phi^{(n)} - (\mathbf{U}^{(n)}\times\mathbf{B})]\cdot(\mathbf{v}^h\times\mathbf{B})d\Omega - \int_\Omega P^{(n)}\nabla\cdot\mathbf{v}^h d\Omega$$
$$+ \frac{1}{N}\int_\Omega (\mathbf{U}^{(n)}\cdot\nabla)\mathbf{U}^{(n)}\cdot\mathbf{v}^h d\Omega = \int_\Omega \mathbf{f}\cdot\mathbf{v}^h d\Omega \quad \forall \mathbf{v}^h \in \mathbf{X}_0^h, \tag{2.42}$$

$$\int_\Omega [\nabla\Phi^{(n)} - (\mathbf{U}^{(n)}\times\mathbf{B})]\cdot(\nabla\psi^h)d\Omega + \frac{1}{\delta}\int_\Gamma S^{(n)}\psi^h d\Omega = \int_\Omega k\,\psi^h d\Omega \quad \forall \psi^h \in \tilde{X}^h \tag{2.43}$$

$$\int_\Omega q^h \nabla\cdot\mathbf{U}^{(n)}d\Omega = 0 \quad \forall q^h \in S_0^h. \tag{2.44}$$

Formally this method is equivalent to a gradient method with unit step-length for minimizing the functional of g:

$$\tilde{\mathcal{J}}(g) \equiv \mathcal{J}(\mathbf{U}(g), P(g), \Phi(g), g).$$

Under suitable assumptions, most notably, for large parameters ϵ and δ in the functional, the sequence $(\mathbf{U}^{(n)}, P^{(n)}, \Phi^{(n)}, \mathbf{M}^{(n)}, T^{(n)}, S^{(n)})$ converges. We may modify this iterative method to a variable step-length gradient method which have better convergence properties.

The main advantage of this method over the Newton's method is that at each iteration we are dealing with a smaller size nonlinear system which requires less computer memory in computation. In our experience, the computing times for the Newton's method and this iterative method are comparable.

2.5 Some Mathematical Results Concerning the Optimality System of Equations and Finite Element Approximations

For completeness, we summarize without proof relevant mathematical results for the constrained minimization problem and error estimation results for the finite element approximation. Interested readers are referred to [HR1] for details and proofs. Specifically we will summarize results on the following: well-posedness of the constraint equations; existence of an optimal solution $(\mathbf{u}, p, \phi, g)$; the existence of Lagrange multipliers (μ, τ, s) such that the optimality system of equa' ¬ are satisfied; convergence and optimal error estimates for finite element approximations of optimal solutions.

It can be shown in a way similar to [JP] that there are constants $\alpha > 0$ and $\beta > 0$, independent of $(\mathbf{v}, \psi)$ and q, such that

$$\frac{1}{M^2}\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, d\Omega + \int_\Omega [\nabla\phi - (\mathbf{u}\times\mathbf{B})]\cdot[\nabla\psi - (\mathbf{v}\times\mathbf{B})]d\Omega$$
$$\geq \alpha(\|\mathbf{v}\|_1 + \|\psi\|_1) \qquad \forall(\mathbf{v}, \psi) \in \mathbf{H}^1_0(\Omega)\times \widetilde{H}^1(\Omega) \tag{2.45}$$

and

$$\inf_{p\in L^2_0(\Omega)} \sup_{(\mathbf{v},\psi)\in\mathbf{H}^1_0(\Omega)\times\widetilde{H}^1(\Omega)} \frac{-\int_\Omega p\nabla\cdot\mathbf{v}}{\|\mathbf{v}\|_1\,\|p\|_0} \geq \beta\,. \tag{2.46}$$

The ellipticity condition (2.45) and the inf-sup condition (2.46) are crucial in showing the existence of a solution to the constraint equations (2.1)–(2.3).

Theorem 2.5.1 *There exists a solution* $(\mathbf{u}, p, \phi) \in \mathbf{H}^1_0(\Omega)\times L^2_0(\Omega)\times \widetilde{H}^1(\Omega)$ *to the constraint equations (2.1)–(2.3). Furthermore, the following estimates hold for any solution* $(\mathbf{u}, p, \phi)$ *of (2.1)–(2.3):*

$$\|\mathbf{u}\|_1 + \|\phi\|_1 \leq C\left(\|\mathbf{f}\|_* + \|k\|_* + \|g\|_{-1/2,\Gamma}\right) \tag{2.47}$$

and

$$\|p\|_0 \leq C\left(\|\mathbf{f}\|_* + \|k\|_* + \|g\|_{-1/2,\Gamma} + \|\mathbf{u}\|_1\right)\,. \quad \square \tag{2.48}$$

Remark. Just as in the steady-state Navier-Stokes case, we do not in general have uniqueness for the solutions to (2.1)–(2.3).□

The precise mathematical statement of the optimal control problem was given in (2.9). The existence of an optimal solution can be proved based on the a priori estimates (2.47)–(2.48) and standard techniques.

Theorem 2.5.2 *There exists an optimal solution* $(\mathbf{u}, p, \phi, g) \in \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega) \times \tilde{H}^1(\Omega) \times L^2(\Gamma)$ *that solves the constrained minimization problem (2.9).* □

In Section 2, we formally applied Lagrange multiplier principles to turn the constrained minimization problem into an unconstrained one and derive an optimality system of equations. We did so without worrying about whether or not Lagrange multipliers exist. The following theorem guarantees the existence of a solution to the optimality system of equations and thereby justifies the use of Lagrange multiplier rules.

Theorem 2.5.3 *Assume* $(\mathbf{u}, p, \phi, g) \in \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega) \times \tilde{H}^1(\Omega) \times L^2(\Gamma)$ *is an optimal solution for (2.9). Then for almost all values of the interaction number* N*, there exist a Lagrange multiplier* $(\mu, \tau, s) \in \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega) \times \tilde{H}^1(\Omega)$ *such that the optimality system (2.17)–(2.22) is satisfied.* □

Remark. In fact, under suitable assumptions, e.g., for small Hartmann number M, we can show that the solution to the optimality system of equations (2.17)–(2.22) is unique. □

Concerning the finite element approximations (2.27)-(2.32), we may prove the existence of, the convergence of, and optimal error estimates for, finite element solutions.

Theorem 2.5.4 *Assume* $(\mathbf{u}, p, \phi, \mu, \tau, s) \in \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega) \times \tilde{H}^1(\Omega) \times \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega) \times \tilde{H}^1(\Omega)$ *is a nonsingular solution to the optimality system of equations (2.17)–(2.22). Then for each sufficiently small* $h > 0$*, there exists a unique finite element solution* $(\mathbf{u}^h, \phi^h, p^h, \mu^h, \tau^h, s^h) \in \mathbf{X}^h_0 \times S^h_0 \times \tilde{X}^h \times \mathbf{X}^h_0 \times S^h_0 \times \tilde{X}^h$ *for (2.27)–(2.32) such that*

$$\lim_{h \to 0+} \left\{ \|\mathbf{u} - \mathbf{u}^h\|_1 + \|p - p^h\|_0 + \|\phi - \phi^h\|_1 + \|\mu - \mu^h\|_1 + \|\tau - \tau^h\|_0 + \|s - s^h\|_1 \right\} = 0$$

and if the solution is smooth, i.e., $(\mathbf{u}, \phi, p, \mu, \tau, s) \in \mathbf{H}^{m+1} \times \tilde{H}^m \times H^{m+1} \times \mathbf{H}^{m+1} \times \tilde{H}^m \times H^{m+1}$*, then*

$$\begin{aligned} &\left\{ \|\mathbf{u} - \mathbf{u}^h\|_1 + \|p - p^h\|_0 + \|\phi - \phi^h\|_1 + \|\mu - \mu^h\|_1 + \|\tau - \tau^h\|_0 + \|s - s^h\|_1 \right\} \\ &\leq Ch^m (\|\mathbf{u}\|_{m+1} + \|p\|_m + \|\phi\|_{m+1} + \|\mu\|_{m+1} + \|\tau\|_m + \|s\|_{m+1} . \end{aligned} \quad (2.49)$$

Furthermore, the approximate control g^h converges to the exact control g and the estimate $\|g - g^h\|_{0,\Gamma} \leq Ch^m$ holds. □

2.6 Computational Examples

In this section, we report some computational examples that serve to illustrate the effectiveness and practicality of optimal control techniques in electrically conducting fluids. First, we treat the problem of steering the velocity field to a desired one. The second one deals with the minimization of potential gradient throughout the domain. Finally, we consider the problem of matching electric potential to a desired one.

All computations are done with the following choice of finite element spaces defined over the same triangulation of the domain $\Omega = \bigcup K$: continuous piecewise quadratic polynomials for both components of the velocity $\mathbf{u}^h$ and adjoint velocity μ^h; continuous piecewise quadratic polynomials for the potential ϕ^h and the adjoint potential s^h; continuous piecewise linear polynomials for the pressure p^h and adjoint pressure r^h. On each triangle, the degrees of freedom for quadratic elements are the function values at the vertices and midpoints of each edge; the degrees of freedom for linear elements are the function values at the vertices. Using standard finite element notations and those introduced in Section 2.3, we have that

$$X^h = \{v \in C^0(\overline{\Omega}) : v|_K \in P_2(K), \text{ on each element } K\},$$

$$\mathbf{X}^h = \{\mathbf{v} = (v_1, v_2)^T \in \mathbf{C}^0(\overline{\Omega}) : v_i \in X^h,\ i = 1,2\}$$

and

$$S^h = \{r \in C^0(\overline{\Omega}) : r|_K \in P_1(K), \text{ on each element } K\}.$$

Under these choices of finite element spaces, the velocity-pressure pair and the adjoint velocity-adjoint pressure pair are approximated by the Taylor-Hood element pair ([TH]) which has been shown to satisfy the div-stability condition (2.26). Approximation properties (2.23)-(2.25) hold with $k = 1$.

2.6.1 Velocity field matching

The first case we consider is the problem of minimizing (2.6) subject to (2.1)–(2.3), i.e., we attempt to match the velocity field with a desired one by finding an appropriate boundary current density g.

The optimality system of equations are given by (2.17)–(2.22) with

$$\langle \mathcal{F}_{\mathbf{u}}(\mathbf{u},p,\phi),\mathbf{w}\rangle = \frac{1}{\epsilon}\int_\Omega (\mathbf{u}-\mathbf{u}_d)\cdot \mathbf{w}\,d\Omega$$

and

$$\langle \mathcal{F}_p(\mathbf{u},p,\phi),\sigma\rangle = \langle \mathcal{F}_\phi(\mathbf{u},p,\phi),r\rangle = 0\,.$$

The corresponding system of partial differential equations for (2.17)–(2.22) is given by (2.1)–(2.4),

$$\frac{\partial\phi}{\partial n} = -\frac{1}{\delta}s \qquad \text{on } \Gamma\,,$$

$$-\tfrac{1}{M^2}\Delta\mu + \tfrac{1}{N}\mu\cdot(\nabla\mathbf{u})^T - \tfrac{1}{N}(\mathbf{u}\cdot\nabla)\mu + \nabla\tau - \mathbf{B}\times(\nabla s) - (\mu\times\mathbf{B})\times\mathbf{B} - \frac{1}{\epsilon}\mathbf{u} = -\frac{1}{\epsilon}\mathbf{u}_d \qquad \text{in } \Omega,$$

$$\nabla\cdot\mu = 0 \qquad \text{in } \Omega,$$

$$-\Delta s + \nabla\cdot(\mu\times\mathbf{B}) = 0 \qquad \text{in } \Omega,$$

$$\mu = 0 \qquad \text{on } \Gamma$$

and

$$\frac{\partial s}{\partial n} = 0 \qquad \text{on } \Gamma\,.$$

We now present some numerical results for the following choice of parameters and data:

The Hartmann number and interaction number: $M = N = 1$;

the domain Ω is the unit square $(0,1)\times(0,1)$.

body force $\mathbf{f} = (f_1, f_2)^T$ and electric source k:

$$\begin{aligned} f_1 &= \sin(2\pi y)[4\pi^2(2\cos(2\pi x)-1) - \frac{\pi y}{2}\cos(\pi x)\sec(\pi y)] \\ &\quad +(\cos(2\pi x)-1)[y^2\sin(2\pi y) + 2\pi\sin(2\pi x)(\cos(2\pi y)-1)]\,; \\ f_2 &= \sin(2\pi x)[\frac{\pi}{2}y\cos(\pi y)\sec(\pi x) - 4\pi^2(2\cos(2\pi y)-1)] \end{aligned}$$

$$
\begin{aligned}
& +(1-\cos(2\pi y))\,[y^2\sin(2\pi x)+2\pi\sin(2\pi y)\,(1-\cos(2\pi x))]\,; \\
k &= \sin(2\pi y)\,[1-\cos(2\pi x)]+2\pi^2\cos(\pi x)\cos(\pi y) \\
& +2\pi y[\cos(2\pi x)-2\cos(2\pi y)\cos(2\pi x)+\cos(2\pi y)]\,;
\end{aligned}
$$

applied magnetic field: $\mathbf{B}=(0,0,y)^T$;

desired velocity field:

$$
\mathbf{u}_d=\begin{pmatrix}\cos(2\pi y)\,[\cos(2\pi x)-1] \\ \sin(2\pi x)\sin(2\pi y)\end{pmatrix}.
$$

For these data, the exact solution of the uncontrolled problem, i.e., the solution for (2.1)–(2.5) with $g=0$, is given by

$$
\mathbf{u}_0=\begin{pmatrix}[\cos(2\pi x)-1]\sin(2\pi y) \\ \sin(2\pi x)\,[1-\cos(2\pi y)]\end{pmatrix},
$$

$$
p_0=C,\qquad \text{where } C \text{ is a constant}
$$

and

$$
\phi_0=\cos(\pi x)\cos(\pi y)\,.
$$

The proper choice of the constants ϵ and δ in the functional plays an important role in obtaining a best velocity matching. For the computational results shown in Figures 2.1 below, our choice of these two constants were $\epsilon=0.00001$ and $\delta=0.01$.

We give a brief description of the figures. Figure 2.1 a) and b) are the uncontrolled velocity field $\mathbf{u}_0$ and potential field ϕ_0, respectively (under zero electrical current on the boundary). Figure 2.1 c) is the desired velocity field $\mathbf{u}_d$. Figure 2.1 d), g), h) and i) are the optimal control solutions: velocity field $\mathbf{u}^h$, potential field ϕ^h, adjoint velocity field μ^h and adjoint potential field s^h; those were obtained by solving (2.17)–(2.22). The optimal control g^h can be gleaned from Figure 2.1 g) and the relation $g^h=-\frac{1}{\delta}s^h$. The function g^h along the top boundary is shown in Figure 2.1 j). All the computational results shown in figures 2.1 were obtained with a 10 by 10 triangulation of the unit square. A nonuniform grid with corner refinements was used. We see from the figures that optimal control does a very good job in matching the desired velocity field which has double circulations.

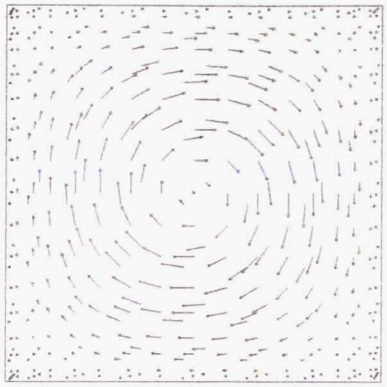

Figure 2.1 a) uncontrolled velocity field $\mathbf{u}_0$

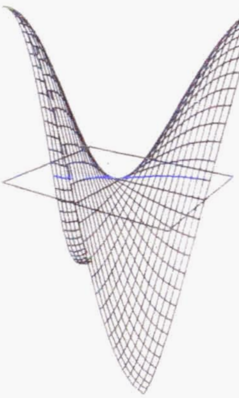

Figure 2.1 b) uncontrolled potential field ϕ_0

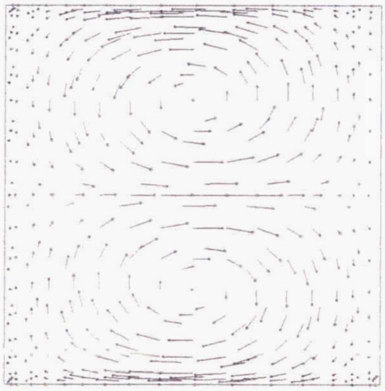

Figure 2.1 c) desired velocity field $\mathbf{u}_d$

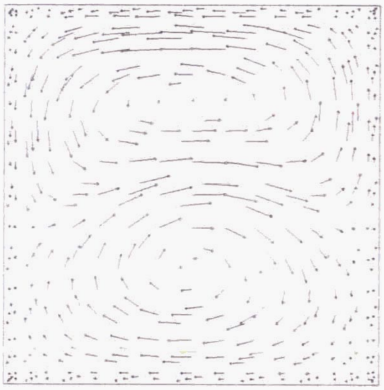

Figure 2.1 d) optimal control velocity $\mathbf{u}^h$

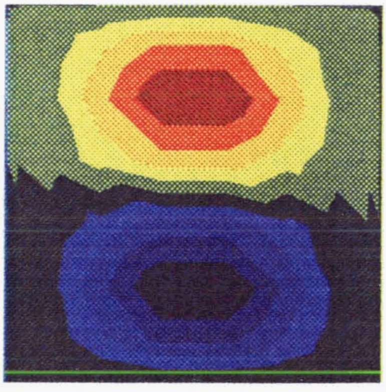

Figure 2.1 e) contours of desired velocity

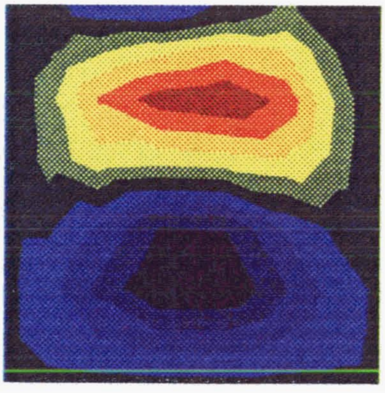

Figure 2.1 f) contours of optimal control velocity

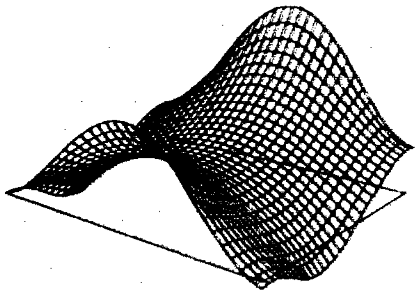

Figure 2.1 g) optimal control solution: potential field ϕ^h

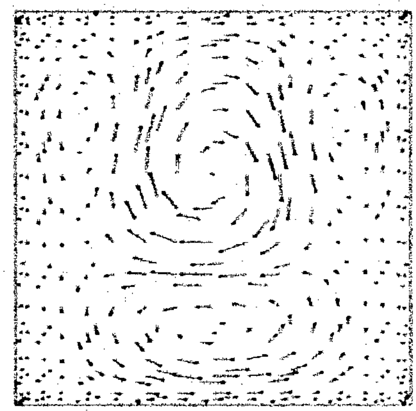

Figure 2.1 h) optimal control solution: adjoint velocity field μ^h

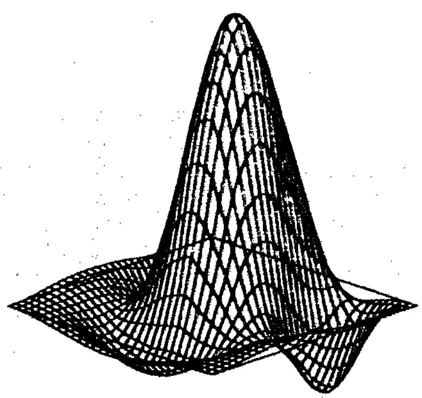

Figure 2.1 i) optimal control solution: adjoint potential field s^h

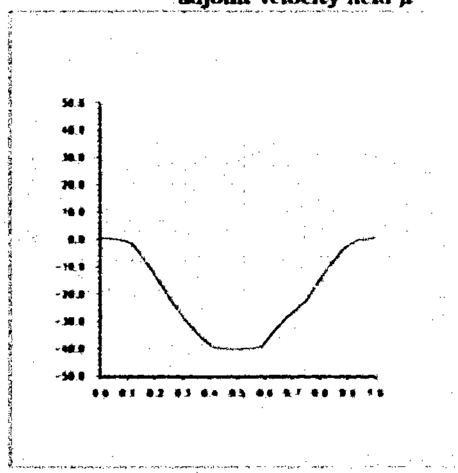

Figure 2.1 j) optimal control g^h along the top boundary

2.6.2 Potential field matching

The second case we consider is the problem of matching the electric potential ϕ to a desired distribution ϕ_d, i.e., we minimize functional (2.7) subject to (2.1)–(2.3).

The optimality system of equations are given by (2.17)–(2.22) with

$$\langle \mathcal{F}_\phi(\mathbf{u}, p, \phi), r \rangle = \frac{1}{\epsilon} \int_\Omega (\phi - \phi_d)\, r\, d\Omega$$

and

$$\langle \mathcal{F}_\mathbf{u}(\mathbf{u}, p, \phi), \mathbf{w} \rangle = \langle \mathcal{F}_p(\mathbf{u}, p, \phi), \sigma \rangle = 0\,.$$

The domain is chosen to be a curved quadrilateral as shown in Figures 2.2 a) to f). The length is 2 and the height is 1.125.

The Hartmann number M, interaction number N, body force $\mathbf{f}$, electric source k and applied magnetic field $\mathbf{B}$ are chosen to be the same as in Section 2.6.1

The desired potential field is a uniform one: $\phi_d = 1$.

The two parameters in the functional are chosen as $\epsilon = 0.002$ and $\delta = 0.1$.

For these data, the solution $(\mathbf{u}_0, p_0, \phi_0)$ of the uncontrolled problem, i.e., the solution for (2.1)–(2.5) with $g = 0$, can only be found through numerical approximations, the reason being that the uncontrolled solution given in Section 2.6.1 no longer satisfies the boundary conditions on the curved domain.

Some numerical results for this example is reported in Figures 2.2 a)-f). We give a brief description of the figures.

Figure 2.2 a) and b) are the uncontrolled velocity field $\mathbf{u}_0$ and potential field ϕ_0, respectively (under zero electrical current on the boundary.)

Figure 2.2 c), d), e) and f) are the optimal control solutions: velocity field $\mathbf{u}^h$, potential field ϕ^h, adjoint velocity field μ^h and adjoint potential field τ^h; those were obtained by solving (2.27)–(2.32). The optimal control g^h can be gleaned from Figure 2.2 f) and the relation $g^h = -\frac{1}{\delta}s^h$.

A look at Figure 2.2 d) reveals that the optimal potential matches well with the desired potential $\phi_d = 1$.

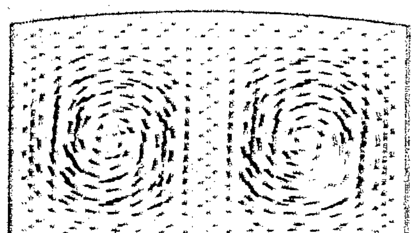

Figure 2.2 a) uncontrolled velocity field $\mathbf{u}_0$

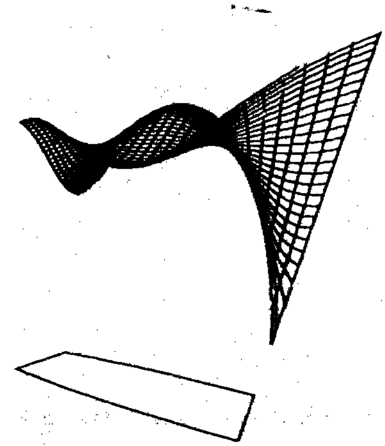

Figure 2.2 b) uncontrolled potential field ϕ_0

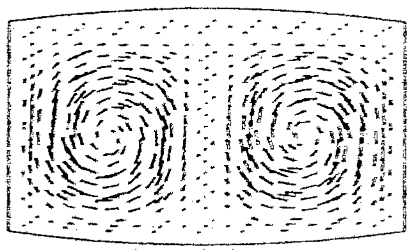

Figure 2.2 c) optimal control solution: velocity field $\mathbf{u}^h$

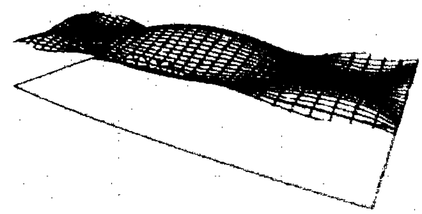

Figure 2.2 d) optimal control solution: potential field ϕ^h

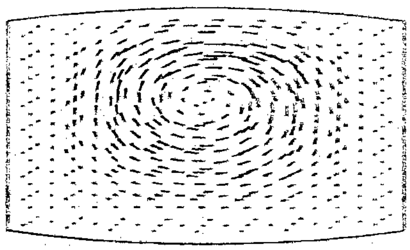

Figure 2.2 e) optimal control solution: adjoint velocity field μ^h

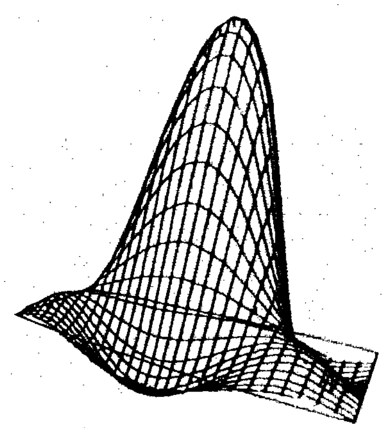

Figure 2.2 f) optimal control solution: adjoint potential field s^h

2.6.3 Potential gradient minimization

The third case we consider is the problem of minimizing the electric potential gradient, i.e., we minimize functional (2.8) subject to (2.1)–(2.3).

The optimality system of equations are given by (2.17)–(2.22) with

$$\langle \mathcal{F}_\phi(\mathbf{u}, p, \phi), r \rangle = \frac{1}{\epsilon} \int_\Omega \nabla\phi \cdot \nabla r \, d\Omega$$

and

$$\langle \mathcal{F}_\mathbf{u}(\mathbf{u}, p, \phi), \mathbf{w} \rangle = \langle \mathcal{F}_p(\mathbf{u}, p, \phi), \sigma \rangle = 0 \, .$$

The domain is chosen to be a rectangle with length 3 and height 1. The Hartmann number M, interaction number N, body force $\mathbf{f}$, electric source k and applied magnetic field $\mathbf{B}$ are chosen to be the same as in Section 2.6.1

The two parameters in the functional are chosen as $\epsilon = 0.0002$ and $\delta = 1$. For these data, the exact solution $(\mathbf{u}_0, p_0, \phi_0)$ of the uncontrolled problem, i.e., the solution for (2.1)–(2.5) with $g = 0$, is given by the same uncontrolled solution as in Section 2.6.1.

Some numerical results for this example is reported in Figures 2.3 a)-f). We give a brief description of the figures.

Figure 2.3 a) and b) are the uncontrolled velocity field $\mathbf{u}_0$ and potential field ϕ_0, respectively (under zero electrical current on the boundary.) Figure 2.3 c), d), e) and f) are the optimal control solutions: velocity field $\mathbf{u}^h$, potential field ϕ^h, adjoint velocity field μ^h and adjoint potential field τ^h; those were obtained by solving (2.27)–(2.32). The optimal control g^h can be gleaned from Figure 2.3 f) and the relation $g^h = -\frac{1}{\delta} s^h$.

By minimizing functional (2.8) we wish to obtain a quasi-uniform potential distribution. The numerical results (in particular, Figure 2.3 d)) demonstrate that the optimal control did a very good job in achieving the objective.

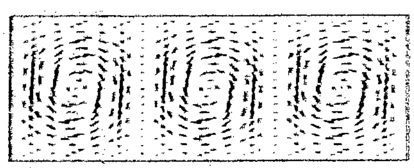

Figure 2.3 a) uncontrolled velocity field $\mathbf{u}_0$

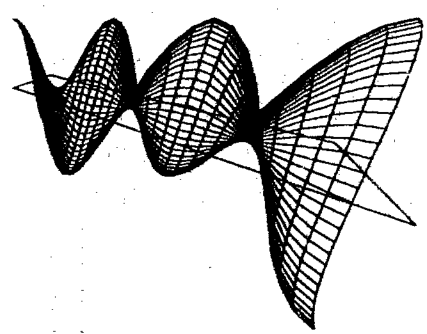

Figure 2.3 b) uncontrolled potential field ϕ_0

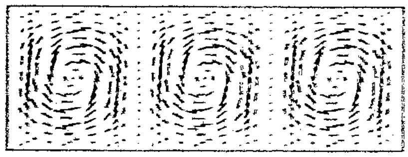

Figure 2.3 c) optimal control solution: velocity field $\mathbf{u}^h$

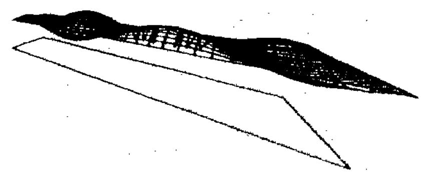

Figure 2.3 d) optimal control solution: potential field ϕ^h

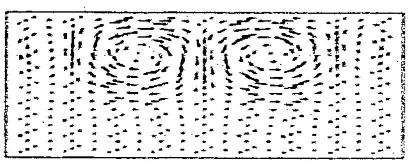

Figure 2.3 e) optimal control solution: adjoint velocity field μ^h

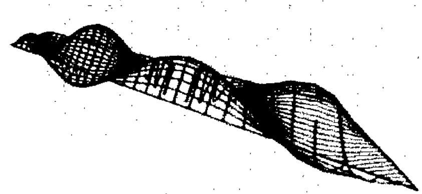

Figure 2.3 f) optimal control solution: adjoint potential field s^h

2.7 Concluding Remarks.

In this Chapter we studied numerical computation of boundary optimal control problems for an electrically conducting fluid using electric current control. We summarize the main points in this Chapter as follows:

- We converted the optimal control problem into a system of equations (i.e., the optimality system of equations) by using the Lagrange multiplier principles;

- We proposed some methods for solving the discrete optimality system of equations. Our discussions of these methods were made for finite element discretizations. Apparently, these methods are equally applicable to finite difference, collocation or pseudo-spectral discretizations;

- We successfully performed numerical computations for some prototype examples as presented in the Figures. Our work demonstrated the effectiveness of optimal control techniques in flow field matchings and in the minimization of some physical quantities.

- We finally remark that the Hartman number (Reynolds number in the non magnetic case) has to be moderately high or small for the algorithms to be robust and to keep the computational cost cheap. Our numerical experiments indicate that, with help of simple continuation methods, algorithms perform well for Hartman's number up to 300.

In principle, other types of boundary controls such as Dirichlet controls of the electrical potential or Dirichlet controls of the boundary velocity can all be treated by the techniques used in this chapter.

Chapter 3

Control of Steady Flows II: Heat Flux Control

In this chapter we study some control problems for a Boussinesq's model of heat transfer in a steady electrically conducting fluid. The control is the heat flux on the flow boundary. As in the previous chapter, we study these problems in detail. The structure of this chapter is that of the previous one.

3.1 Statement of the Optimal Control Problems

We denote by Ω the flow region which is a bounded open container in $\mathbf{R}^2$ or $\mathbf{R}^3$ with a boundary Γ. The dimensionless equations governing the steady Boussinesq's model of an electrically conducting fluid in the presence of a magnetic field are given by

$$-\frac{1}{M^2}\Delta \mathbf{u} + \frac{1}{N}(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - (\mathbf{B}\times\nabla\phi) - (\mathbf{u}\times\mathbf{B})\times\mathbf{B} - \frac{1}{L}\mathbf{g}T = \mathbf{f} \quad \text{in } \Omega, \tag{3.1}$$

$$\nabla\cdot\mathbf{u} = 0, \quad \text{in } \Omega \tag{3.2}$$

$$-\Delta\phi + \nabla(\mathbf{u}\times\mathbf{B}) = k_1 \quad \text{in } \Omega, \tag{3.3}$$

and

$$-\Delta T + \mathbf{u}\cdot\nabla T = k_2 \quad \text{in } \Omega. \tag{3.4}$$

In (3.1)–(3.4), **B**, **f**, k_1 and k_2 are given data and $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{L}$ are parameters. The system (3.1)–(3.4) is supplemented with boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma, \tag{3.5}$$

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on } \Gamma \tag{3.6}$$

and

$$\frac{\partial T}{\partial n} = g \quad \text{on } \Gamma, \tag{3.7}$$

where g denotes the only control variable, namely, the heating and cooling on Γ.

Our goal is to try to obtain a desired flow field by appropriately choosing the control – the normal temperature gradient on Γ. Specifically we will investigate the following cases: matching a desired velocity field, matching a desired temperature field, or minimizing the temperature gradient. Mathematically, these tasks can be described, respectively, by the following optimal control setting: minimize the cost functional

$$\mathcal{K}(\mathbf{u}, \phi, T, p, g) = \frac{1}{2\epsilon} \int_\Omega |\mathbf{u} - \mathbf{u}_d|^2 d\Omega + \frac{\delta}{2} \int_\Gamma |g|^2 d\Gamma, \tag{3.8}$$

or

$$\mathcal{M}(\mathbf{u}, \phi, T, p, g) = \frac{1}{2\epsilon} \int_\Omega |T - T_d|^2 d\Omega + \frac{\delta}{2} \int_\Gamma |g|^2 d\Gamma, \tag{3.9}$$

or

$$\mathcal{N}(\mathbf{u}, \phi, T, p, g) = \frac{1}{2\epsilon} \int_\Omega |\nabla T|^2 d\Omega + \frac{\delta}{2} \int_\Gamma |g|^2 d\Gamma, \tag{3.10}$$

subject to the constraints (3.1)–(3.7). Here $\epsilon > 0$ and $\delta > 0$ are positive parameters; $\mathbf{u}_d$ and T_d are, respectively, desired velocity field and temperature field.

The minimization of functional (3.8) or (3.9) or (3.10) subject to (3.1)–(3.7) is a special case of the following general optimal control setting:

minimize the cost functional

$$\mathcal{J}(\mathbf{u}, \phi, T, p, g) = \mathcal{F}(\mathbf{u}, \phi, T, p) + \frac{\delta}{2} \int_\Gamma |g|^2 d\Gamma \tag{3.11}$$

subject to the constraints (3.1)–(3.7),

where $\mathcal{F}(\mathbf{u}, \phi, T, p)$ is a functional of $(\mathbf{u}, \phi, T, p)$.

3.2 A Variational Formulation of the Constraints; An Optimality System of Equations

The variational formulation of the constraint equations is then given as follows: seek $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, $p \in L_0^2(\Omega)$, $\phi \in \tilde{H}^1(\Omega)$ and $T \in \tilde{H}^1(\Omega)$ such that

$$\begin{aligned}\frac{1}{M^2}\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\,d\Omega + \int_\Omega [\nabla\phi - (\mathbf{u}\times\mathbf{B})]\cdot[\nabla\psi - (\mathbf{v}\times\mathbf{B})]\,d\Omega + \frac{1}{N}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,d\Omega \\ - \int_\Omega p\nabla\cdot\mathbf{v}\,d\Omega - \frac{1}{L}\int_\Omega \mathbf{g}\cdot\mathbf{v}T\,d\Omega = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\Omega + \int_\Omega k_1\psi\,d\Omega \quad \forall(\mathbf{v},\psi)\in\mathbf{H}_0^1(\Omega)\times\tilde{H}^1(\Omega).\end{aligned} \tag{3.12}$$

$$\int_\Omega q\nabla\cdot\mathbf{u}\,d\Omega = 0 \quad \forall q\in L_0^2(\Omega). \tag{3.13}$$

and

$$\int_\Omega \nabla T\cdot\nabla\theta\,d\Omega + \int_\Omega \mathbf{u}\cdot\nabla T\theta\,d\Omega = \int_\Omega k_2\theta\,d\Omega + \int_\Gamma g\theta\,d\Gamma \quad \forall\theta\in\tilde{H}^1(\Omega). \tag{3.14}$$

Or, equivalently, seek $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, $p \in L_0^2(\Omega)$, $\phi \in \tilde{H}^1(\Omega)$ and $T \in \tilde{H}^1(\Omega)$ such that

$$\begin{aligned}\frac{1}{M^2}\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\,d\Omega - \int_\Omega [\nabla\phi - (\mathbf{u}\times\mathbf{B})]\cdot(\mathbf{v}\times\mathbf{B})\,d\Omega + \frac{1}{N}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,d\Omega - \int_\Omega p\nabla\cdot\mathbf{v}\,d\Omega \\ - \frac{1}{L}\int_\Omega \mathbf{g}\cdot\mathbf{v}T\,d\Omega = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\Omega \quad \forall\mathbf{v}\in\mathbf{H}_0^1(\Omega),\end{aligned} \tag{3.15}$$

$$\int_\Omega [\nabla\phi - (\mathbf{u}\times\mathbf{B})]\cdot(\nabla\psi)\,d\Omega = \int_\Omega k_1\psi\,d\Omega \quad \forall\psi\in\tilde{H}^1(\Omega), \tag{3.16}$$

$$\int_\Omega q\nabla\cdot\mathbf{u}\,d\Omega = 0 \quad \forall q\in L_0^2(\Omega), \tag{3.17}$$

and

$$\int_\Omega \nabla T\cdot\nabla\theta\,d\Omega + \int_\Omega \mathbf{u}\cdot\nabla T\theta\,d\Omega = \int_\Omega k_2\theta\,d\Omega + \int_\Gamma g\theta\,d\Gamma \quad \forall\theta\in\tilde{H}^1(\Omega). \tag{3.18}$$

Here the colon notation stands for the scalar product on $\mathbf{R}^{d\times d}$.

The precise mathematical statement of the optimal control problem (3.11) can now be given as follows:

$$\text{seek a } (\mathbf{u},p,\phi,T,g)\in\mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)\times\tilde{H}^1(\Omega)\times\tilde{H}^1(\Omega)\times L^2(\Gamma) \text{ such that the functional (3.11) is minimized subject to the constraints (3.15)-(3.18).} \tag{3.19}$$

We will turn the constrained optimization problem (3.11) into an unconstrained one by using Lagrange multiplier principles. (For mathematical theories of Lagrange multiplier principles, see, e.g., [9].) We set $\mathcal{X} = \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \widetilde{H}^1(\Omega) \times \widetilde{H}^1(\Omega) \times L^2(\Gamma) \times \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \widetilde{H}^1(\Omega) \times \widetilde{H}^1(\Omega)$ and define the Lagrangian functional

$$
\begin{aligned}
&\mathcal{L}(\mathbf{u}, p, \phi, T, g, \mu, \tau, s, t) \\
&= \mathcal{F}(\mathbf{u}, p, \phi, T) + \frac{\delta}{2}\int_\Gamma |g|^2 d\Gamma - \frac{1}{M^2}\int_\Omega \nabla \mathbf{u} : \nabla \mu d\Omega + \int_\Omega [\nabla \phi - (\mathbf{u} \times \mathbf{B})] \cdot (\mu \times \mathbf{B}) d\Omega \\
&\quad - \frac{1}{N}\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mu d\Omega + \int_\Omega p \nabla \cdot \mu d\Omega + \int_\Omega \mathbf{f} \cdot \mu d\Omega - \int_\Omega [\nabla \phi - (\mathbf{u} \times \mathbf{B})] \cdot (\nabla s) d\Omega \\
&\quad + \int_\Omega k_1 s d\Omega + \int_\Omega \tau \nabla \cdot \mathbf{u} d\Omega - \int_\Omega \nabla T \cdot \nabla t d\Omega - \int_\Omega \mathbf{u} \cdot \nabla T t d\Omega \\
&\quad + \int_\Omega k_2 \theta d\Omega + \int_\Gamma g \theta d\Gamma + \frac{1}{L}\int_\Omega \mathbf{g} \cdot \mu T d\Omega \qquad \forall (\mathbf{u}, p, \phi, T, g, \mu, \tau, s, t) \in \mathcal{X}. \qquad (3.20)
\end{aligned}
$$

An optimality system of equations that an optimum must satisfy is derived by taking variations with respect to every variable in the Lagrangian. By taking variations with respect to $\mathbf{u}$, p, T and ϕ we obtain:

$$
\begin{aligned}
&\frac{1}{M^2}\int_\Omega \nabla \mu : \nabla \mathbf{w} d\Omega + \int_\Omega (\mathbf{w} \times \mathbf{B}) \cdot (\mu \times \mathbf{B}) d\Omega + \frac{1}{N}\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{w} \cdot \mu d\Omega + \frac{1}{N}\int_\Omega \mathbf{w} \cdot \nabla)\mathbf{u} \cdot \mu d\Omega \\
&\quad - \int_\Omega \tau \nabla \cdot \mathbf{w} d\Omega + \int_\Omega \mathbf{w} \cdot \nabla T t d\Omega = \langle \mathcal{F}_\mathbf{u}(\mathbf{u}, p, \phi, T), \mathbf{w} \rangle \quad \forall \mathbf{w} \in \mathbf{H}_0^1(\Omega), \qquad (3.21)
\end{aligned}
$$

$$\int_\Omega [\nabla s - (\mu \times \mathbf{B})] \cdot (\nabla r) d\Omega = \langle \mathcal{F}_\phi(\mathbf{u}, p, \phi, T), r \rangle \quad \forall r \in \widetilde{H}^1(\Omega), \qquad (3.22)$$

$$\int_\Omega \sigma \nabla \cdot \mu d\Omega = \langle \mathcal{F}_p(\mathbf{u}, p, \phi, T), \sigma \rangle \quad \forall \sigma \in L_0^2(\Omega) \qquad (3.23)$$

and

$$\int_\Omega \nabla t \cdot \nabla l d\Omega + \int_\Omega \mathbf{u} \cdot \nabla l t d\Omega - \frac{1}{L}\int_\Omega \mathbf{g} \cdot \mu l d\Omega = \langle \mathcal{F}_T(\mathbf{u}, p, \phi, T), l \rangle \quad \forall l \in \widetilde{H}^1(\Omega), \qquad (3.24)$$

where $\mathcal{F}_\mathbf{u}$, $\mathcal{F}_\phi$, $\mathcal{F}_p$ and $\mathcal{F}_T$ are the derivatives of the functional with respect to its three arguments, respectively. By taking variations with respect to μ, τ, s and t, we recover the constraint equations (3.1)-(3.4). By taking variation with respect to g we obtain

$$\int_\Gamma (\delta g z + z t) d\Gamma = 0 \quad \forall z \in L^2(\Gamma) \qquad \text{i.e.,} \qquad g = -\frac{1}{\delta} t.$$

This last equation enables us to eliminate the control g in (3.18). Thus (3.15)-(3.18) can be replaced by

$$\frac{1}{M^2}\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} d\Omega - \int_\Omega [\nabla\phi - (\mathbf{u}\times\mathbf{B})]\cdot(\mathbf{v}\times\mathbf{B})d\Omega + \frac{1}{N}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}d\Omega$$
$$- \int_\Omega p\nabla\cdot\mathbf{v}d\Omega - \frac{1}{L}\int_\Omega \mathbf{g}\cdot\mathbf{v}Td\Omega = \int_\Omega \mathbf{f}\cdot\mathbf{v}d\Omega \quad \forall\mathbf{v}\in\mathbf{H}_0^1(\Omega), \quad (3.25)$$

$$\int_\Omega [\nabla\phi - (\mathbf{u}\times\mathbf{B})]\cdot(\nabla\psi)d\Omega = \int_\Omega k_1\,\psi d\Omega \quad \forall\psi\in\widetilde{H}^1(\Omega), \quad (3.26)$$

$$\int_\Omega q\nabla\cdot\mathbf{u}d\Omega = 0 \quad \forall q\in L_0^2(\Omega), \quad (3.27)$$

and

$$\int_\Omega \nabla T\cdot\nabla\theta d\Omega + \int_\Omega \mathbf{u}\cdot\nabla T d\Omega + \frac{1}{\delta}\int_\Gamma t\,\theta d\Omega = \int_\Omega k_2\,\theta d\Omega \quad \forall\theta\in\widetilde{H}^1(\Omega). \quad (3.28)$$

Equations (3.21)-(3.28) forms an optimality system of equations that an optimal solution must satisfy.

3.3 Finite Element Approximations

A finite element discretization of the optimality system (3.21)–(3.28) is defined in the usual manner. First one chooses families of finite dimensional subspaces $X^h \subset H^1(\Omega)$ and $S^h \subset L^2(\Omega)$. These families are parameterized by a parameter h that tends to zero; commonly, h is chosen to be some measure of the grid size. We set $\tilde{X}^h = X^h \cap L_0^2(\Omega)$, $\mathbf{X}^h = [X^h]^d$, $\mathbf{X}_0^h = \mathbf{X}^h \cap \mathbf{H}_0^1(\Omega)$ and $S_0^h = S^h \cap L_0^2(\Omega)$. We assume that as $h \to 0$,

$$\inf_{\mathbf{v}^h\in\mathbf{X}_0^h} \|\mathbf{v}-\mathbf{v}^h\|_1 \to 0\,, \quad \forall\mathbf{v}\in\mathbf{H}_0^1(\Omega),$$

$$\inf_{v^h\in\tilde{X}^h} \|v-v^h\|_1 \to 0\,, \quad \forall v\in\widetilde{H}^1(\Omega)$$

and

$$\inf_{q^h\in S_0^h} \|q-q^h\|_0 \to 0\,, \quad \forall q\in L_0^2(\Omega).$$

Here we may choose any pair of subspaces X^h and S^h such that $\mathbf{X}_0^h$ and S_0^h can be used for finding finite element approximations of solutions of the Navier-Stokes equations. We also assume the properties (2.23)–(2.26) hold.

Once the approximating subspaces have been chosen, we seek $\mathbf{u}^h \in \mathbf{X}_0^h$, $p^h \in S_0^h$, $\phi^h \in \tilde{X}^h$, $T^h \in \tilde{X}^h$, $\mu^h \in \mathbf{X}_0^h$, $r^h \in S_0^h$, $s^h \in \tilde{X}^h$, $t^h \in \tilde{X}^h$ such that

$$\frac{1}{M^2}\int_\Omega \nabla \mathbf{u}^h : \nabla \mathbf{v}^h d\Omega - \int_\Omega [\nabla\phi^h - (\mathbf{u}^h \times \mathbf{B})]\cdot(\mathbf{v}^h \times \mathbf{B})d\Omega + \frac{1}{N}\int_\Omega (\mathbf{u}^h\cdot\nabla)\mathbf{u}^h\cdot\mathbf{v}^h d\Omega$$
$$-\int_\Omega p^h \nabla\cdot\mathbf{v}^h d\Omega - \frac{1}{L}\int_\Omega \mathbf{g}\cdot\mathbf{v}^h T^h d\Omega = \int_\Omega \mathbf{f}\cdot\mathbf{v}^h d\Omega \quad \forall \mathbf{v}^h \in \mathbf{X}_0^h, \tag{3.29}$$

$$\int_\Omega [\nabla\phi^h - (\mathbf{u}^h \times \mathbf{B})]\cdot(\nabla\psi^h)d\Omega = \int_\Omega k_1\psi^h d\Omega \quad \forall\psi^h \in \tilde{X}^h, \tag{3.30}$$

$$\int_\Omega q^h \nabla\cdot\mathbf{u}^h d\Omega = 0 \quad \forall q^h \in S_0^h, \tag{3.31}$$

$$\int_\Omega \nabla T^h\cdot\nabla\theta^h d\Omega + \int_\Omega \mathbf{u}^h\cdot\nabla T^h\theta^h d\Omega + \frac{1}{\delta}\int_\Gamma t^h\theta^h d\Omega = \int_\Omega k_2\theta^h d\Omega \quad \forall\theta^h \in \tilde{X}^h. \tag{3.32}$$

$$\frac{1}{M^2}\int_\Omega \nabla\mu^h : \nabla\mathbf{w}^h d\Omega + \int_\Omega (\mathbf{w}^h\times\mathbf{B})\cdot(\mu^h\times\mathbf{B})d\Omega + \frac{1}{N}\int_\Omega (\mathbf{u}^h\cdot\nabla)\mathbf{w}^h\cdot\mu^h d\Omega$$
$$+\frac{1}{N}\int_\Omega (\mathbf{w}^h\cdot\nabla)\mathbf{u}^h\cdot\mu^h d\Omega - \int_\Omega r^h\nabla\cdot\mathbf{w}^h d\Omega + \int_\Omega t^h\mathbf{w}^h\cdot\nabla T^h d\Omega$$
$$= \langle \mathcal{F}_{\mathbf{u}}(\mathbf{u}^h, p^h, \phi^h, T^h), \mathbf{w}^h\rangle \quad \forall\mathbf{w}^h \in \mathbf{X}_0^h, \tag{3.33}$$

$$\int_\Omega [\nabla s^h - (\mu^h\times\mathbf{B})]\cdot(\nabla r^h)d\Omega = \langle \mathcal{F}_\phi(\mathbf{u}^h, p^h, \phi^h, T^h), r^h\rangle \quad \forall r^h \in \tilde{X}^h, \tag{3.34}$$

$$\int_\Omega \sigma^h\nabla\cdot\mu^h d\Omega = \langle \mathcal{F}_p(\mathbf{u}^h, p^h, \phi^h, T^h), \sigma^h\rangle \quad \forall\sigma^h \in S_0^h, \tag{3.35}$$

and

$$\int_\Omega \nabla t^h\cdot\nabla l^h d\Omega + \int_\Omega \mathbf{u}^h\cdot\nabla l^h t^h d\Omega - \frac{1}{L}\int_\Omega \mathbf{g}\cdot\mu^h l^h d\Omega = \langle \mathcal{F}_T(\mathbf{u}^h, p^h, \phi^h, T^h), l^h\rangle \quad \forall l^h \in \tilde{X}^h. \tag{3.36}$$

From a computational standpoint, this is a formidable system. In three dimensions, we have a coupled system involving *twelve* unknown discrete scalar fields. Therefore, how one solves this system is a rather important question.

3.4 Solution Methods for the Discrete Optimality System of Equations

We will present two methods for solving the discrete optimality system of equations (3.29)–(3.36). The first one is the Newton's method for the entire system; the second one is an iterative method which uncouples the computation of (3.29)–(3.32) and the computation of (3.33)–(3.36) at each iteration (the second method is in essence equivalent to a gradient method in minimization.)

3.4.1 Newton's method.

For notational convenience, we will use $(\mathbf{U}, P, \Phi, T, \mathbf{M}, N, S, R)$ to denote $(\mathbf{u}^h, p^h, \phi^h, T^h, \mu^h, \tau^h, s^h, t^h)$. We will only give the Newton's method for the special case $\mathcal{F}(\mathbf{u}, p, \phi, T) = \mathcal{F}(\mathbf{u})$. General cases can be treated similarly. Thus the Newton's method for solving the discrete optimality system (3.29)–(3.36) is given as follows:

1° Choose an initial guess $(\mathbf{U}^{(0)}, P^{(0)}, \Phi^{(0)}, T^{(0)}, \mathbf{M}^{(0)}, N^{(0)}, S^{(0)}, R^{(0)})$;

2° For $n = 1, 2, \cdots$, compute $(\mathbf{U}^{(n)}, P^{(n)}, \Phi^{(n)}, T^{(n)}, \mathbf{M}^{(n)}, N^{(n)}, S^{(n)}, R^{(n)})$ from the following discrete system of equations:

$$\begin{aligned}
&\tfrac{1}{M^2}\int_\Omega \nabla\mathbf{U}^{(n)} : \nabla\mathbf{v}^h d\Omega - \int_\Omega [\nabla\Phi^{(n)} - (\mathbf{U}^{(n)} \times \mathbf{B})] \cdot (\mathbf{v}^h \times \mathbf{B}) d\Omega \\
&\quad + \tfrac{1}{N}\int_\Omega (\mathbf{U}^{(n-1)} \cdot \nabla)\mathbf{U}^{(n)} \cdot \mathbf{v}^h d\Omega + \tfrac{1}{N}\int_\Omega (\mathbf{U}^{(n-1)} \cdot \nabla)\mathbf{U}^{(n)} \cdot \mathbf{v}^h d\Omega - \tfrac{1}{L}\int_\Omega \mathbf{g} \cdot \mathbf{v}^h T^{(n)} d\Omega \\
&\quad - \int_\Omega P^{(n)} \nabla \cdot \mathbf{v}^h d\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v}^h d\Omega + \int_\Omega (\mathbf{U}^{(n-1)} \cdot \nabla)\mathbf{U}^{(n-1)} \cdot \mathbf{v}^h d\Omega \quad \forall \mathbf{v}^h \in \mathbf{X}_0^h,
\end{aligned} \tag{3.37}$$

$$\int_\Omega [\nabla\Phi^{(n)} - (\mathbf{U}^{(n)} \times \mathbf{B})] \cdot (\nabla\psi^h) d\Omega = \int_\Omega k_1 \psi^h d\Omega \quad \forall \psi^h \in \tilde{X}^h, \tag{3.38}$$

$$\int_\Omega q^h \nabla \cdot \mathbf{U}^{(n)} d\Omega = 0 \quad \forall q^h \in S_0^h, \tag{3.39}$$

$$\begin{aligned}
&\int_\Omega \nabla T^{(n)} \cdot \nabla\theta^h d\Omega + \int_\Omega \mathbf{U}^{(n-1)} \cdot \nabla T^{(n)} \theta^h d\Omega + \int_\Omega \mathbf{U}^{(n)} \cdot \nabla T^{(n-1)} \theta^h d\Omega + \frac{1}{\delta}\int_\Gamma R^{(n)} \theta^h d\Gamma \\
&\quad = \int_\Omega \mathbf{U}^{(n-1)} \cdot \nabla T^{(n-1)} \theta^h d\Omega + \int_\Omega k_2 \theta^h d\Omega \quad \forall \theta^h \in \tilde{X}^h,
\end{aligned} \tag{3.40}$$

$$\begin{aligned}
&\tfrac{1}{M^2}\int_\Omega \nabla\mathbf{M}^{(n)} : \nabla\mathbf{w}^h d\Omega + \int_\Omega (\mathbf{w}^h \times \mathbf{B}) \cdot (\mathbf{M}^{(n)} \times \mathbf{B}) d\Omega + \tfrac{1}{N}\int_\Omega (\mathbf{U}^{(n)} \cdot \nabla)\mathbf{w}^h \cdot \mathbf{M}^{(n-1)} d\Omega \\
&\quad + \tfrac{1}{N}\int_\Omega (\mathbf{U}^{(n-1)} \cdot \nabla)\mathbf{w}^h \cdot \mathbf{M}^{(n)} d\Omega + \tfrac{1}{N}\int_\Omega (\mathbf{w}^h \cdot \nabla)\mathbf{U}^{(n-1)} \cdot \mathbf{M}^{(n)} d\Omega \\
&\quad + \tfrac{1}{N}\int_\Omega (\mathbf{w}^h \cdot \nabla)\mathbf{U}^{(n)} \cdot \mathbf{M}^{(n-1)} d\Omega - \int_\Omega N^{(n)} \nabla \cdot \mathbf{w}^h d\Omega + \int_\Omega R^{(n-1)} \mathbf{w}^h \cdot \nabla T^{(n)} d\Omega \\
&\quad - \langle \mathcal{F}_{\mathbf{uu}}(\mathbf{U}^{(n-1)})\mathbf{U}^{(n)}, \mathbf{w}^h \rangle + \int_\Omega R^{(n)} \mathbf{w}^h \cdot \nabla T^{(n-1)} d\Omega \\
&\quad = \langle \mathcal{F}_{\mathbf{u}}(\mathbf{U}^{(n-1)}), \mathbf{w}^h \rangle - \langle \mathcal{F}_{\mathbf{uu}}(\mathbf{U}^{(n-1)})\mathbf{U}^{(n-1)}, \mathbf{w}^h \rangle
\end{aligned}$$

$$+\frac{1}{N}\int_\Omega (\mathbf{U}^{(n-1)}\cdot\nabla)\mathbf{w}^h\cdot\mathbf{M}^{(n-1)}d\Omega + \int_\Omega R^{(n-1)}\mathbf{w}^h\cdot\nabla T^{(n-1)}d\Omega$$
$$+\frac{1}{N}\int_\Omega (\mathbf{w}^h\cdot\nabla)\mathbf{U}^{(n-1)}\cdot\mathbf{M}^{(n-1)}d\Omega \quad \forall \mathbf{w}^h\in\mathbf{X}_0^h, \tag{3.41}$$

$$\int_\Omega [\nabla S^{(n)} - (\mathbf{M}^{(n)}\times\mathbf{B})]\cdot(\nabla r^h)d\Omega = 0 \quad \forall r^h\in\tilde{X}^h \tag{3.42}$$

$$\int_\Omega \sigma^h\nabla\cdot\mathbf{M}^{(n)}d\Omega = 0 \quad \forall\sigma^h\in S_0^h \tag{3.43}$$

and

$$\int_\Omega \nabla R^{(n)}\cdot\nabla l^h d\Omega + \int_\Omega \mathbf{U}^{(n-1)}\cdot\nabla l^h R^{(n)}d\Omega + \int_\Omega \mathbf{U}^{(n)}\cdot\nabla l^h R^{(n-1)}d\Omega$$
$$-\frac{1}{L}\int_\Omega \mathbf{g}\cdot\mathbf{M}^{(n-1)}l^h d\Omega = \int_\Omega \mathbf{U}^{(n-1)}\cdot\nabla l^h R^{(n-1)}d\Omega \quad \forall l^h\in S_0^h \tag{3.44}$$

3.4.2 An iterative method.

We now discuss an iterative method that uncouples the solution of the constraint equations (3.29)–(3.32) from the adjoint equations (3.33)–(3.36).

1° Choose $(\mathbf{U}^{(0)}, P^{(0)}, \Phi^{(0)}, T^{(0)})$;

2° For $n = 0, 1, 2, \cdots$, compute $(\mathbf{M}^{(n)}, N^{(n)}, S^{(n)}, R^{(n)})$ from

$$\frac{1}{M^2}\int_\Omega \nabla\mathbf{M}^{(n)} : \nabla\mathbf{w}^h d\Omega + \int_\Omega (\mathbf{w}^h\times\mathbf{B})\cdot(\mathbf{M}^{(n)}\times\mathbf{B})d\Omega + \frac{1}{N}\int_\Omega (\mathbf{U}^{(n-1)}\cdot\nabla)\mathbf{w}^h\cdot\mathbf{M}^{(n)}d\Omega$$
$$+\frac{1}{N}\int_\Omega (\mathbf{w}^h\cdot\nabla)\mathbf{U}^{(n-1)}\cdot\mathbf{M}^{(n)}d\Omega - \int_\Omega N^{(n)}\nabla\cdot\mathbf{w}^h d\Omega$$
$$= \langle \mathcal{F}_\mathbf{u}(\mathbf{U}^{(n-1)}, P^{(n-1)}, \Phi^{(n-1)}, T^{(n-1)}), \mathbf{w}^h\rangle \quad \forall\mathbf{w}^h\in\mathbf{X}_0^h, \tag{3.45}$$

$$\int_\Omega [\nabla S^{(n)} - (\mathbf{M}^{(n)}\times\mathbf{B})]\cdot(\nabla r^h)d\Omega = \langle \mathcal{F}_\phi(\mathbf{U}^{(n-1)}, P^{(n-1)}, \Phi^{(n-1)}, T^{(n-1)}), r^h\rangle \quad \forall r^h\in\tilde{X}^h \tag{3.46}$$

$$\int_\Omega \sigma^h\nabla\cdot\mathbf{M}^{(n)}d\Omega = \langle \mathcal{F}_p(\mathbf{U}^{(n-1)}, P^{(n-1)}, \Phi^{(n-1)}, T^{(n-1)}), \sigma^h\rangle \quad \forall\sigma^h\in S_0^h; \tag{3.47}$$

and

$$\int_\Omega \nabla R^{(n)}\cdot\nabla l^h d\Omega + \int_\Omega \mathbf{U}^{(n-1)}\cdot\nabla l^h R^{(n)}d\Omega - \frac{1}{L}\int_\Omega \mathbf{g}\cdot\mathbf{M}^{(n)}l^h d\Omega$$
$$= \langle \mathcal{F}_T(\mathbf{U}^{(n-1)}, P^{(n-1)}, \Phi^{(n-1)}, T^{(n-1)}), l^h\rangle \quad \forall l^h\in\tilde{X}^h; \tag{3.48}$$

and compute $(\mathbf{U}^{(n)}, P^{(n)}, \Phi^{(n)}, T^{(n)})$ from

$$\frac{1}{M^2}\int_\Omega \nabla\mathbf{U}^{(n)} : \nabla\mathbf{v}^h d\Omega - \int_\Omega [\nabla\Phi^{(n)} - (\mathbf{U}^{(n)} \times \mathbf{B})]\cdot(\mathbf{v}^h \times \mathbf{B})d\Omega - \int_\Omega P^{(n)}\nabla\cdot\mathbf{v}^h d\Omega$$
$$+\frac{1}{N}\int_\Omega (\mathbf{U}^{(n)}\cdot\nabla)\mathbf{U}^{(n)}\cdot\mathbf{v}^h d\Omega - \frac{1}{L}\int_\Omega \mathbf{g}\cdot\mathbf{v}^h T^{(n)} d\Omega = \int_\Omega \mathbf{f}\cdot\mathbf{v}^h d\Omega \quad \forall \mathbf{v}^h \in \mathbf{X}_0^h, \qquad (3.49)$$

$$\int_\Omega [\nabla\Phi^{(n)} - (\mathbf{U}^{(n)} \times \mathbf{B})]\cdot(\nabla\psi^h)d\Omega + \frac{1}{\delta}\int_\Gamma S^{(n-1)}\psi^h d\Omega = \int_\Omega k_1\psi^h d\Omega \quad \forall\psi^h \in \tilde{X}^h, \qquad (3.50)$$

$$\int_\Omega q^h\nabla\cdot\mathbf{U}^{(n)} d\Omega = 0 \quad \forall q^h \in S_0^h. \qquad (3.51)$$

$$\int_\Omega \nabla T^{(n)}\cdot\nabla\theta^h d\Omega + \int_\Omega \mathbf{U}^{(n)}\cdot\nabla T^h\theta^h d\Omega + \frac{1}{\delta}\int_\Gamma R^{(n-1)}\theta^h d\Gamma = \int_\Omega k_2\theta^h d\Omega \quad \forall\theta^h \in \tilde{X}^h \qquad (3.52)$$

Formally this method is equivalent to a gradient method with unit step-length for minimizing the functional of g:

$$\tilde{\mathcal{J}}(g) \equiv \mathcal{J}(\mathbf{U}(g), P(g), \Phi(g), T(g), g).$$

3.5 Some Mathematical Results Concerning the Optimality System of Equations and Finite Element Approximations

For completeness, we summarize without proof relevant mathematical results for the constrained minimization problem and error estimation results for the finite element approximation. Specifically we will summarize results on the following: well-posedness of the constraint equations; existence of an optimal solution $(\mathbf{u}, p, \phi, T, g)$; the existence of Lagrange multipliers (μ, τ, s, t) such that the optimality system of equations are satisfied; convergence and optimal error estimates for finite element approximations of optimal solutions.

It can be shown in a way similar to [JP] that there are constants $\alpha > 0$ and $\beta > 0$, independent of $(\mathbf{v}, \psi)$ and q, such that

$$\frac{1}{M^2}\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}d\Omega + \int_\Omega [\nabla\phi - (\mathbf{u}\times\mathbf{B})]\cdot[\nabla\psi - (\mathbf{v}\times\mathbf{B})]d\Omega$$
$$\geq \alpha(\|\mathbf{v}\|_1 + \|\psi\|_1) \qquad \forall(\mathbf{v},\psi) \in \mathbf{H}_0^1(\Omega)\times\widetilde{H}^1(\Omega) \qquad (3.53)$$

and

$$\inf_{p\in L_0^2(\Omega)} \sup_{(\mathbf{v},\psi)\in\mathbf{H}_0^1(\Omega)\times\widetilde{H}^1(\Omega)} \frac{-\int_\Omega p\nabla\cdot\mathbf{v}d\Omega}{\|\mathbf{v}\|_1\|p\|_0} \geq \beta. \qquad (3.54)$$

The ellipticity condition (3.53) and the inf-sup condition (3.54) are crucial in showing the existence of a solution to the constraint equations (3.1)–(3.4).

Theorem 3.5.1 *There exists a solution* $(\mathbf{u}, p, \phi, T) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \widetilde{H}^1(\Omega) \times \widetilde{H}^1(\Omega)$ *to the constraint equations (3.1)–(3.4). Furthermore, the following estimates hold for any solution* $(\mathbf{u}, p, \phi, T)$ *of (3.1)–(3.4):*

$$\|\mathbf{u}\|_1 + \|\phi\|_1 + \|T\|_1 \leq C \left(\|\mathbf{f}\|_* + \|k_1\|_* + \|k_2\|_* + \|g\|_{-1/2,\Gamma} \right) \tag{3.55}$$

and

$$\|p\|_0 \leq C \left(\|\mathbf{f}\|_* + \|k_1\|_* + \|k_2\|_* + \|g\|_{-1/2,\Gamma} + \|\mathbf{u}\|_1 \right) .\square \tag{3.56}$$

Remark. Just as in the steady-state Navier-Stokes case, we do not in general have uniqueness for the solutions to (3.1)–(3.4).□

The precise mathematical statement of the optimal control problem was given in (3.11). The existence of an optimal solution can be proved based on the a priori estimates (3.55)–(3.56) and standard techniques.

Theorem 3.5.2 *There exists an optimal solution* $(\mathbf{u}, p, T, g) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \widetilde{H}^1(\Omega) \times \widetilde{H}^1(\Omega) \times L^2(\Gamma)$ *that solves the constrained minimization problem (3.11).* □

In Section 3.2, we formally applied Lagrange multiplier principles to turn the constrained minimization problem into an unconstrained one and derive an optimality system of equations. We did so without worrying about whether or not Lagrange multipliers exist. The following theorem guarantees the existence of a solution to the optimality system of equations and thereby justifies the use of Lagrange multiplier rules.

Theorem 3.5.3 *Assume* $(\mathbf{u}, p, \phi, T, g) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \widetilde{H}^1(\Omega) \times \widetilde{H}^1(\Omega) \times L^2(\Gamma)$ *is an optimal solution for (2.16). Then for almost all values of the interaction number* N, *there exist a Lagrange multiplier* $(\boldsymbol{\mu}, \tau, s, T) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \widetilde{H}^1(\Omega)$ *such that the optimality system (3.21)–(3.28) is satisfied.* □

Remark. In fact, under suitable assumptions, e.g., for small Hartmann number M, we can show that the solution to the optimality system of equations (3.21)–(3.28) is unique.□

Concerning the finite element approximations (3.21)-(3.28), we may prove the existence of, the convergence of, and optimal error estimates for, finite element solutions.

Theorem 3.5.4 *Assume* $(\mathbf{u}, p, \phi, T, \mu, \tau, s, t) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \widetilde{H}^1(\Omega) \times \widetilde{H}^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \widetilde{H}^1(\Omega) \times \widetilde{H}^1(\Omega)$ *is a nonsingular solution to the optimality system of equations (3.21)–(3.28). Then for each sufficiently small* $h > 0$, *there exists a unique finite element solution* $(\mathbf{u}^h, p^h, \phi^h, T^h, \mu^h, \tau^h, s^h, t^h) \in \mathbf{X}_0^h \times S_0^h \times \tilde{X}^h \times \tilde{X}^h \times \mathbf{X}_0^h \times S_0^h \times \tilde{X}^h \times \tilde{X}^h$ *for (3.29)–(3.35) such that*

$$\lim_{h\to 0+} \Big\{\|\mathbf{u} - \mathbf{u}^h\|_1 + \|p - p^h\|_0 + \|\phi - \phi^h\|_1 + \|T - T^h\|_1 + \|\mu - \mu^h\|_1 + \|\tau - \tau^h\|_0 + \|s - s^h\|_1 + \|t - t^h\|_1\Big\} = 0$$

and if the solution is smooth, i.e., $(\mathbf{u}, p, \phi, T, \mu, \tau, s, t) \in \mathbf{H}^{m+1} \times \widetilde{H}^m \times H^{m+1} \times H^{m+1} \times \mathbf{H}^{m+1} \times \widetilde{H}^m \times H^{m+1} \times H^{m+1}$, *then*

$$\Big\{\|\mathbf{u} - \mathbf{u}^h\|_1 + \|p - p^h\|_0 + \|\phi - \phi^h\|_1 + \|T - T^h\|_1 + \|\mu - \mu^h\|_1 + \|\tau - \tau^h\|_0 + \|s - s^h\|_1 + \|t - t^h\|_1\Big\} \le Ch^m(\|\mathbf{u}\|_{m+1} + \|p\|_m + \|\phi\|_{m+1} + \|T\|_{m+1} + \|\mu\|_{m+1} + \|\tau\|_m + \|s\|_{m+1} + \|t\|_{m+1}.$$

Furthermore, the approximate control g^h *converges to the exact control* g *and the estimate* $\|g - g^h\|_{0,\Gamma} \le Ch^m$ *holds.* □

3.6 Computational Examples

In this section, we report some computational examples that serve to illustrate the effectiveness and practicality of optimal control techniques for the control problems we studied in this chapter. First, we treat the problem of steering the velocity field to a desired one. The second one deals with the problem of matching the temperature field to a desired one. Finally, we consider the problem of minimizing the temperature gradient throughout the domain.

All computations are done with the following choice of finite element spaces defined over the same triangulation of the domain $\Omega = \bigcup K$: continuous piecewise quadratic polynomials for both components of the velocity $\mathbf{u}^h$ and adjoint velocity μ^h; continuous piecewise quadratic polynomials for the potential ϕ^h, the temperature T^h, the adjoint potential s^h and the adjoint temperature t^h; continuous piecewise linear polynomials for the pressure p^h and adjoint pressure τ^h. On each triangle, the degrees of freedom for quadratic elements are the function values at the vertices and midpoints of each edge; the degrees of freedom for linear elements are the function values at the vertices. Using standard finite element notations and those introduced in Section 3.3, we have that

$$X^h = \{v \in C^0(\overline{\Omega}) : v|_K \in P_2(K), \text{ on each element } K\},$$

$$\mathbf{X}^h = \{\mathbf{v} = (v_1, v_2)^T \in \mathbf{C}^0(\overline{\Omega}) : v_i \in X^h,\ i = 1,2\}$$

and

$$S^h = \{r \in C^0(\overline{\Omega}) : r|_K \in P_1(K), \text{ on each element } K\}.$$

3.6.1 Velocity field matching

The first case we consider is the problem of minimizing (3.8) subject to (3.1)–(3.7), i.e., we attempt to match the velocity field with a desired one by finding an appropriate boundary heat flux g.

The optimality system of equations are given by (3.21)–(3.28) with

$$\langle \mathcal{F}_{\mathbf{u}}(\mathbf{u}, p, \phi, T), \mathbf{w}\rangle = \frac{1}{\epsilon}\int_\Omega (\mathbf{u} - \mathbf{u}_d)\cdot \mathbf{w}\, d\Omega$$

and

$$\langle \mathcal{F}_p(\mathbf{u}, p, \phi, T), \sigma\rangle = \langle \mathcal{F}_\phi(\mathbf{u}, p, \phi, T), r\rangle = \langle \mathcal{F}_T(\mathbf{u}, p, \phi, T), l\rangle = 0.$$

The corresponding system of partial differential equations for (3.21)–(3.28) are given by (3.1)–(3.6),

$$\frac{\partial T}{\partial n} = -\frac{1}{\delta} t \qquad \text{on } \Gamma,$$

$$-\frac{1}{M^2}\Delta\mu + \frac{1}{N}\mu\cdot(\nabla\mathbf{u})^T - \frac{1}{N}(\mathbf{u}\cdot\nabla)\mu + \nabla\tau - \mathbf{B}\times(\nabla s) - (\mu\times\mathbf{B})\times\mathbf{B} + t\nabla T - \frac{1}{\epsilon}\mathbf{u} = -\frac{1}{\epsilon}\mathbf{u}_d \qquad \text{in } \Omega,$$

$$\nabla\cdot\mu = 0 \qquad \text{in } \Omega,$$

$$-\Delta s + \nabla \cdot (\mu \times \mathbf{B}) = 0 \qquad \text{in } \Omega,$$

$$-\Delta t - \mathbf{u} \cdot \nabla t - \tfrac{1}{L}\mathbf{g} \cdot \mu = 0 \qquad \text{in } \Omega,$$

$$\mu = \mathbf{0} \qquad \text{on } \Gamma,$$

$$\frac{\partial t}{\partial n} = 0 \qquad \text{on } \Gamma$$

and

$$\frac{\partial s}{\partial n} = 0 \qquad \text{on } \Gamma.$$

We now present some numerical results for the following choice of parameters and data:

The Hartmann number and interaction number: $M = N = 1$;

the domain Ω is the unit square $(0,1) \times (0,1)$;

body force $\mathbf{f} = (f_1, f_2)^T$, electric source k_1 and heat source k_2:

$$\begin{aligned}
f_1 &= \sin(2\pi y)\,[4\pi^2(2\cos(2\pi x) - 1) - \frac{\pi y}{2}\cos(\pi x)\sec(\pi y)] \\
&\quad +(\cos(2\pi x) - 1)\,[y^2 \sin(2\pi y) + 2\pi \sin(2\pi x)\,(\cos(2\pi y) - 1)]\,; \\
f_2 &= \sin(2\pi x)\,[\frac{\pi}{2} y \cos(\pi y)\sec(\pi x) - 4\pi^2(2\cos(2\pi y) - 1)] \\
&\quad +(1 - \cos(2\pi y))\,[y^2 \sin(2\pi x) + 2\pi \sin(2\pi y)\,(1 - \cos(2\pi x))] \\
&\quad - \cos(\pi x)\cos(\pi y)\,; \\
k_1 &= \sin(2\pi y)\,[1 - \cos(2\pi x)] + 2\pi^2 \cos(\pi x)\cos(\pi y) \\
&\quad +2\pi y[\cos(2\pi x) - 2\cos(2\pi y)\cos(2\pi x) + \cos(2\pi y)]\,; \\
k_2 &= \pi[\sin(2\pi x)\sin(\pi y)\cos(\pi x)(\cos(2\pi y) - 1) \\
&\quad +(1 - \cos(2\pi x))\sin(2\pi y)\sin(\pi x)\cos(\pi y)] \\
&\quad +2\pi^2 \cos(\pi x)\cos(\pi y)\,;
\end{aligned}$$

applied magnetic field: $\mathbf{B} = (0, 0, y)^T$;

desired velocity field:

$$\mathbf{u}_d = \begin{pmatrix} \cos(2\pi y)\,[\cos(2\pi x) - 1] \\ \sin(2\pi x)\sin(2\pi y) \end{pmatrix}.$$

For these data, the exact solution of the uncontrolled problem, i.e., the solution for (3.1)–(3.7) with $g = 0$, is given by

$$\mathbf{u}_0 = \begin{pmatrix} [\cos(2\pi x) - 1]\sin(2\pi y) \\ \sin(2\pi x)\,[1 - \cos(2\pi y)] \end{pmatrix},$$

$$p_0 = C, \qquad \text{where C is a constant},$$

$$T_0 = \cos(\pi x)\cos(\pi y)$$

and

$$\phi_0 = \cos(\pi x)\cos(\pi y).$$

The proper choice of the constants ϵ and δ in the functional plays an important role in obtaining a best velocity matching. For the computational results shown in Figures 3.1 below, our choice of these two constants were $\epsilon = 0.00001$ and $\delta = 0.001$.

We give a brief description of the figures. Figure 3.1 a) and c) are the uncontrolled velocity field $\mathbf{u}_0$ and temperature field T_0, respectively (under zero temperature flux on the boundary). Figure 3.1 c) is the desired velocity field $\mathbf{u}_d$. Figure 3.1 c) optimal control solution: adjoint temperature field t^h; those were obtained by solving (3.29)–(3.36). The optimal control g^h can be gleaned from Figure 3.1 c) and the relation $g^h = -\frac{1}{\delta}t^h$. All the computational results shown in figures 3.1 were obtained with a 10 by 10 triangulation of the unit square. A nonuniform grid with corner refinements was used. We see from the figures that optimal control does a very good job in matching the desired velocity field which has double circulations.

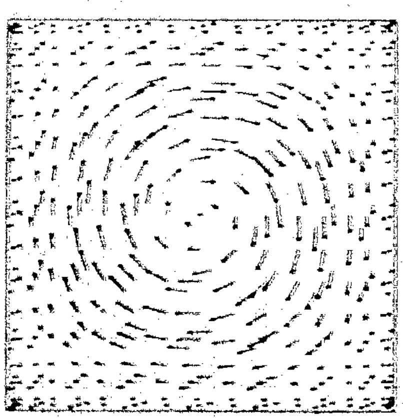

Figure 3.1 a) uncontrolled velocity field $\mathbf{u}_0$

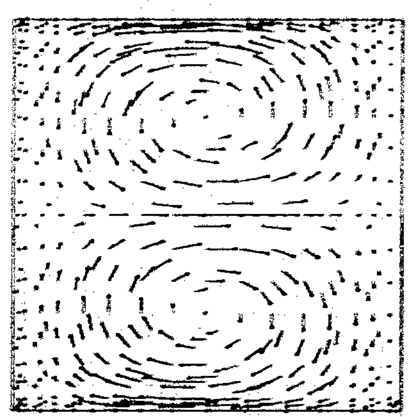

Figure 3.1 b) desired velocity field $\mathbf{u}_d$

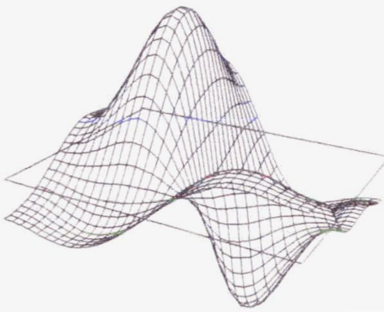

Figure 3.1 c) adjoint temperature field t^h

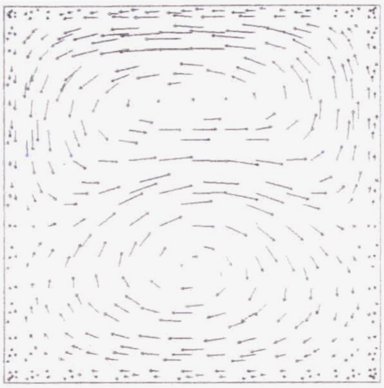

Figure 3.1 d) optimal control solution: velocity field $\mathbf{u}^h$

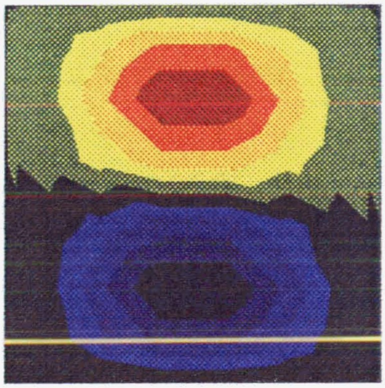

Figure 3.1 e) contours of desired velocity field

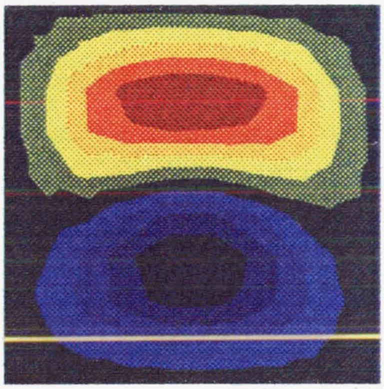

Figure 3.1 f) contours of optimal velocity field

3.6.2 Temperature field matching

The second case we consider is the problem of matching the temperature field T to a desired distribution T_d, i.e., we minimize functional (3.9) subject to (3.1)–(3.7).

The optimality system of equations are given by (3.21)–(3.28) with

$$\langle \mathcal{F}_T(\mathbf{u}, p, \phi, T), l \rangle = \frac{1}{\epsilon} \int_\Omega (T - T_d)\, l d\Omega$$

and

$$\langle \mathcal{F}_{\mathbf{u}}(\mathbf{u}, p, \phi, T), \mathbf{w} \rangle = \langle \mathcal{F}_p(\mathbf{u}, p, \phi, T), \sigma \rangle = \langle \mathcal{F}_\phi(\mathbf{u}, p, \phi, T), r \rangle = 0\,.$$

The domain is chosen to be a rectangle as shown in Figures 3.2 a) to c). The length is 2 and the height is 1. Control is applied partially on the boundary Γ as follows:

$$\frac{\partial T}{\partial n} = g \qquad \text{on} \quad \Gamma_1 = ([0,1]\times\{0\})\cup([0,1]\times\{1\})$$

and

$$\frac{\partial T}{\partial n} = 0 \qquad \text{on} \quad \Gamma_2 = (\{0\}\times[0,1])\cup(\{1\}\times[0,1]),$$

where $\Gamma_1 \cup \Gamma_2 = \Gamma$. The Hartmann number M, interaction number N, body force $\mathbf{f}$, electric source k_1, heat source k_2 and applied magnetic field $\mathbf{B}$ are chosen to be the same as in Section 3.6.1.

The desired temperature field is a uniform one: $\quad T_d = 1\,.$

The two parameters in the functional are chosen as $\epsilon = 0.001$ and $\delta = 1\,.$

For these data, the solution $(\mathbf{u}_0, p_0, \phi_0, T_0)$ of the uncontrolled problem, i.e., the solution for (3.1)–(3.7) with $g = 0$, are the same as in Section 3.6.1. Some numerical results for this example is reported in Figures 3.2 a)-c). We give a brief description of the figures. Figure 3.2 a) is the uncontrolled temperature field T_0 (under zero heat flux on the boundary). Figure 3.2 b) and c) are the optimal control solutions: temperature field T^h and adjoint temperature field t^h; those were obtained by solving (3.29)–(3.36). The optimal control g^h can be gleaned from Figure 3.2 c) and the relation $g^h = -\frac{1}{\delta}t^h$. A look at Figure 3.2 b) reveals that the optimal temperature matches well with the desired temperature field $T_d = 1$.

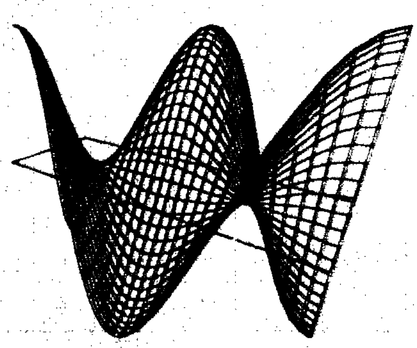

Figure 3.2 a) uncontrolled temperature field T_0

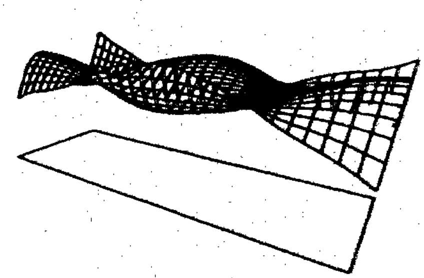

Figure 3.2 b) optimal control solution: temperature field T^h

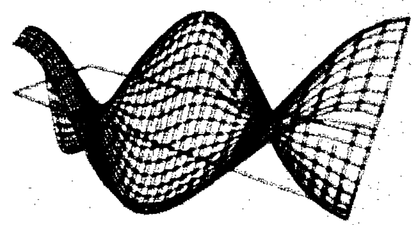

Figure 3.2 c) optimal control solution:
adjoint temperature field t^h

3.6.3 Temperature gradient minimization

The third case we consider is the problem of minimizing the temperature gradient, i.e., we minimize functional (3.10) subject to (3.1)–(3.7).

The optimality system of equations are given by (3.21)–(3.28) with

$$\langle \mathcal{F}_T(\mathbf{u}, p, \phi, T), l \rangle = \frac{1}{\epsilon} \int_\Omega \nabla T \cdot \nabla l d\Omega$$

and

$$\langle \mathcal{F}_{\mathbf{u}}(\mathbf{u}, p, \phi, T), \mathbf{w} \rangle = \langle \mathcal{F}_p(\mathbf{u}, p, \phi, T), \sigma \rangle = \langle \mathcal{F}_\phi(\mathbf{u}, p, \phi, T), r \rangle = 0 .$$

The domain is chosen to be a rectangle with length 2 and height 1. Control is applied partially on the boundary Γ as follows:

$$\frac{\partial T}{\partial n} = g \qquad \text{on} \quad \Gamma_1 = ([0,1] \times \{0\}) \cup ([0,1] \times \{1\})$$

and

$$\frac{\partial T}{\partial n} = 0 \qquad \text{on} \quad \Gamma_2 = (\{0\} \times [0,1]) \cup (\{1\} \times [0,1]),$$

where $\Gamma_1 \cup \Gamma_2 = \Gamma$. The Hartmann number M, interaction number N, body force $\mathbf{f}$, electric source k_1, heat source k_2 and applied magnetic field $\mathbf{B}$ are chosen to be the same as in Section 3.6.1. The two parameters in the functional are chosen as $\epsilon = 0.0002$ and $\delta = 1$.

For these data, the exact solution $(\mathbf{u}_0, p_0, c_0, T_0)$ of the uncontrolled problem, i.e., the solution for (3.1)–(3.7) with $g = 0$, is given by the same uncontrolled solution as in Section 3.6.1.

Some numerical results for this example is reported in Figures 3.3 a)-c). We give a brief description of the figures. Figure 3.3 a) is the uncontrolled temperature field T_0. Figure 3.3 b) and c) are the optimal control solutions: temperature field T^h and adjoint temperature field t^h; those were obtained by solving (3.29)–(3.36). The optimal control g^h can be gleaned from Figure 3.3 c) and the relation $g^h = -\frac{1}{\delta}t^h$.

By minimizing functional (3.10) we wish to obtain a quasi-uniform temperature distribution. The numerical results (in particular, Figure 3.3 b)) demonstrate that the optimal control did a very good job in achieving the objective.

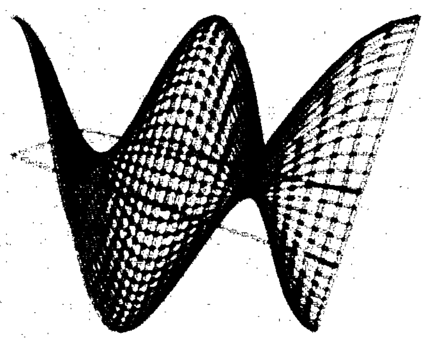

Figure 3.3 a) uncontrolled temperature field T_0

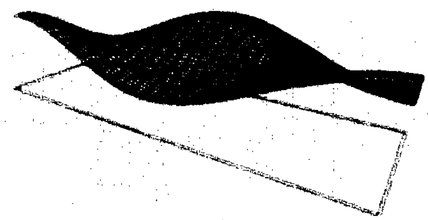

Figure 3.3 b) optimal temperature field T^h

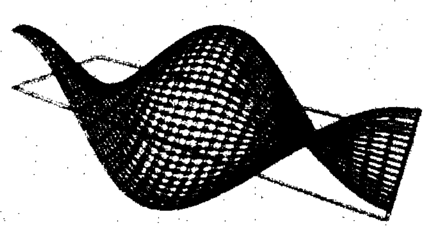

Figure 3.3 c) optimal control solution: adjoint temperature field t^h

3.7 Concluding Remarks.

In this chapter we studied numerical computation of boundary control problems for an electrically conducting fluid with heat flux controls. We summarize the main points in this chapter as follows:

- We converted the optimal control problem into a system of equations (i.e., the optimality system of equations) by using the Lagrange multiplier principles;
- We proposed some methods for solving the discrete optimality system of equations. Our discussions of these methods were made for finite element discretizations. Apparently, these methods are equally applicable to finite difference, collocation or pseudo-spectral discretizations;
- We successfully performed numerical computations for some prototype examples as presented in the Figures. Our work demonstrated the effectiveness of optimal control techniques in flow field matchings and in the minimization of some physical quantities. Our remark in Chapter 2 regarding the Hartman number (Reynolds number) restriction holds here as well.

Chapter 4

Control of Unsteady Flows I: Current Control

In this chapter we study some control problems for an unsteady electrically conducting flow model. The control is the normal electric current on the flow boundary. In Section 4.1, we present some optimal control problems and describe the computational methods used in this chapter. In Section 4.2, we derive the optimality system using the Lagrange multiplier techniques. In Section 4.3, we present some computational results and in Section 4.4 we make some concluding remarks.

4.1 Statement of the Optimal Control Problems

We denote by Ω the flow region which is a bounded container in $\mathbf{R}^2$ or $\mathbf{R}^3$ with the boundary Γ and by $[0,1]$ the time interval. The dimensionless equations governing the unsteady incompressible flow of an electrically conducting fluid in the presence of a magnetic field, when the Maxwell's displacement currents are neglected, are given by the following; see [TC]:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{N}(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + (\mathbf{j}\times\mathbf{B}) + \frac{1}{M^2}\Delta\mathbf{u} + \mathbf{f} \quad \text{in } [0,1]\times\Omega,$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } [0,1]\times\Omega,$$

$$\mathbf{j} = \mathbf{E} + (\mathbf{u} \times \mathbf{B}) \quad \text{in } [0,1] \times \Omega,$$

$$\nabla \cdot \mathbf{j} = 0 \quad \text{in } [0,1] \times \Omega,$$

$$\nabla \times \mathbf{B} = R_m \mathbf{j} \quad \text{in } [0,1] \times \Omega,$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{in } [0,1] \times \Omega,$$

and

$$\nabla \cdot \mathbf{B} = 0 \quad \text{in } [0,1] \times \Omega$$

where $\mathbf{u}$ denotes the velocity field, p the pressure field, $\mathbf{j}$ the electric current density, $\mathbf{B}$ the magnetic field, and $\mathbf{E}$ the electric field. Also, N is the interaction number, M is the Hartmann number, and R_m is the magnetic Reynolds number.

We will deal with a special case in which the externally applied magnetic field is steady and undisturbed by the flow; in particular, we assume that $\mathbf{B}$ is given. Then $\nabla \times \mathbf{E} = \mathbf{0}$ and the electric field can be expressed as $\mathbf{E} = -\nabla\phi$, where ϕ is the electric potential. We now arrive at the following simplified system by eliminating $\mathbf{j}$ as described in Chapter 2:

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{1}{M^2}\Delta \mathbf{u} + \frac{1}{N}(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - (\mathbf{B} \times \nabla\phi) - (\mathbf{u} \times \mathbf{B}) \times \mathbf{B} = \mathbf{f} \quad \text{in } [0,1] \times \Omega, \tag{4.1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } [0,1] \times \Omega \tag{4.2}$$

and

$$-\Delta\phi + \nabla \cdot (\mathbf{u} \times \mathbf{B}) = k \quad \text{in } [0,1] \times \Omega. \tag{4.3}$$

In (4.1)–(4.3), $\mathbf{B}$, $\mathbf{f}$ and k are given data and M and N are parameters.

The system (4.1)–(4.3) is supplemented with the following initial condition

$$\mathbf{u}(0, \mathbf{x}) = \hat{\mathbf{u}}(\mathbf{x}) \quad \text{in } \Omega, \tag{4.4}$$

and the boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on } [0,1] \times \Gamma \tag{4.5}$$

$$\frac{\partial \phi}{\partial n} = g \quad \text{on } [0,1] \times \Gamma, \tag{4.6}$$

where g denotes the only control variable, namely, the normal electric current on Γ, $\hat{\mathbf{u}}$ the initial velocity.

A cost function for this unsteady flow can be written as

$$\mathcal{J} = \frac{1}{2}\int_0^1 \mathcal{F}(\mathbf{u}, p, \phi)dt + \frac{1}{2}\int_0^1 \int_\Gamma |g|^2 d\Gamma dt,$$

where $\mathcal{F}(\mathbf{u}, \phi, p)$ is a functional of $(\mathbf{u}, \phi, p)$. The optimality system, when $\langle \mathcal{F}(\mathbf{u}, p, \mathbf{v}), l\rangle = \int_\Omega |\mathbf{u} - \mathbf{u}_d|^2 \mathbf{v} d\Omega$, is given by

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{1}{M^2}\Delta\mathbf{u} + \frac{1}{N}(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - (\mathbf{B}\times\nabla\phi) - (\mathbf{u}\times\mathbf{B})\times\mathbf{B} = \mathbf{f} \quad \text{in } [0,1]\times\Omega,$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } [0,1]\times\Omega$$

$$-\Delta\phi + \nabla\cdot(\mathbf{u}\times\mathbf{B}) = k \quad \text{in } [0,1]\times\Omega.$$

$$-\frac{\partial \mu}{\partial t} - \frac{1}{M^2}\Delta\mu + \frac{1}{N}\mu\cdot(\nabla\mathbf{u})^T - \frac{1}{N}(\mathbf{u}\cdot\nabla)\mu + \nabla\tau - \mathbf{B}\times(\nabla s) - (\mu\times\mathbf{B})\times\mathbf{B} = \mathbf{u} - \mathbf{u}_d \quad \text{in } [0,1]\times\Omega,$$

$$\nabla\cdot\mu = 0 \quad \text{in } [0,1]\times\Omega,$$

$$-\Delta s + \nabla\cdot(\mu\times\mathbf{B}) = 0 \quad \text{in } [0,1]\times\Omega,$$

and

$$\mu = 0 \quad \text{on } [0,1]\times\Gamma$$

$$\frac{\partial s}{\partial n} = 0 \quad \text{on } [0,1]\times\Gamma.$$

$$\mathbf{u} = 0 \quad \text{on } [0,1]\times\Gamma$$

$$\frac{\partial \phi}{\partial n} = -\frac{1}{\delta}s \quad \text{on } [0,1]\times\Gamma,$$

$$\mu(1,\mathbf{x}) = 0 \quad \text{in } \Omega,$$

$$\mathbf{u}(0,\mathbf{x}) = \hat{\mathbf{u}}(\mathbf{x}) \quad \text{in } \Omega.$$

This is a fully coupled nonlinear system with $\mathbf{u}$ forward in time and μ backward in time. This can not be solved by a simple time marching scheme. As a result, the solution of this system is computationally costly.

To overcome this difficulty, we instead study the following problem:

1° Select a sequence of time instances $0 = t_0 < t_1 < t_2 \ldots, t_{n-1}, t_n, \ldots t_{N-1} < t_N = 1$.

2° At each t_i we study the following optimal control problem:

minimize the cost functional

$$\mathcal{J}_{t_i}(\mathbf{u}(t_i,\cdot), \phi(t_i,\cdot), p(t_i,\cdot), g(t_i,\cdot)) = \frac{1}{2}\mathcal{F}(\mathbf{u}, \phi, p) + \frac{1}{2}\int_\Gamma |g(t_i,\cdot)|^2 d\Gamma$$

subject to the constraints

$$\frac{\partial \mathbf{u}}{\partial t}(t_i,\cdot) - \tfrac{1}{M^2}\Delta \mathbf{u}(t_i,\cdot) + \tfrac{1}{N}(\mathbf{u}(t_i,\cdot)\cdot\nabla)\mathbf{u}(t_i,\cdot) + \nabla p(t_i,\cdot) - (\mathbf{B}\times\nabla\phi(t_i,\cdot))$$
$$-(\mathbf{u}(t_i,\cdot)\times\mathbf{B})\times\mathbf{B} = \mathbf{f}(t_i,\cdot) \quad \text{in } \Omega,$$
$$\nabla\cdot\mathbf{u}(t_i,\cdot) = 0 \quad \text{in } \Omega$$
$$-\Delta\phi(t_i,\cdot) + \nabla\cdot(\mathbf{u}(t_i,\cdot)\times\mathbf{B}) = k \quad \text{in } \Omega.$$
$$\mathbf{u}(t_i,\cdot) = \mathbf{0} \quad \text{on } \Gamma$$
$$\frac{\partial\phi}{\partial n}(t_i,\cdot) = g(t_i,\cdot) \quad \text{on } \Gamma,$$

3° The time derivative $\mathbf{u}_t(t_i,\cdot)$ is replaced with $\frac{\mathbf{u}(t_i,\cdot)-\mathbf{u}(t_{i-1},\cdot)}{\Delta t}$ so that we can

minimize $\mathcal{J}_{t_i}$ subject to

$$\frac{1}{\Delta t}\mathbf{u}(t_i,\cdot) - \tfrac{1}{M^2}\Delta \mathbf{u}(t_i,\cdot) + \tfrac{1}{N}(\mathbf{u}(t_i,\cdot)\cdot\nabla)\mathbf{u}(t_i,\cdot) + \nabla p(t_i,\cdot) - (\mathbf{B}\times\nabla\phi(t_i,\cdot))$$
$$-(\mathbf{u}(t_i,\cdot)\times\mathbf{B})\times\mathbf{B} = \frac{1}{\Delta t}\mathbf{u}(t_{i-1},\cdot) + \mathbf{f}(t_i,\cdot) \quad \text{in } \Omega,$$
$$\nabla\cdot\mathbf{u}(t_i,\cdot) = 0 \quad \text{in } \Omega$$
$$-\Delta\phi(t_i,\cdot) + \nabla\cdot(\mathbf{u}(t_i,\cdot)\times\mathbf{B}) = k \quad \text{in } \Omega.$$
$$\mathbf{u}(t_i,\cdot) = \mathbf{0} \quad \text{on } \Gamma$$
$$\frac{\partial\phi}{\partial n}(t_i,\cdot) = g(t_i,\cdot) \quad \text{on } \Gamma.$$

4° Each minimization problem described in Step 2 is solved by the steady state method.

We are minimizing the functional at a number of time instances. If the time step length is small, this sequence of minimizers at $\{t_i\}_{i=1}^N$ will give us a quasiminimizer over the entire time interval (in $L^\infty(0,T)$ norm). There are further mathematical and numerical questions to be investigated. This is currently underway.

Remark: Efficient solution methods for the forward-backward problem are currently being developed and will be reported in a forthcoming article.□

This computational procedure is applied to the system (4.1) – (4.3) with the backward Euler method to discretize the time derivative. Let $\mathbf{u}(t_n, \mathbf{x}) = \mathbf{u}^n$, $\mathbf{u}(t_{n-1}, \mathbf{x}) = \mathbf{u}^{n-1}$, $\phi(t_n, \mathbf{x}) = \phi^n$ and $p(t_n, \mathbf{x}) = p^n$. Let $\Delta t = t_n - t_{n-1}$. Then the system (4.1) – (4.3) takes the form

$$\frac{1}{\Delta t}\mathbf{u}^n - \frac{1}{M^2}\Delta \mathbf{u}^n + \frac{1}{N}(\mathbf{u}^n \cdot \nabla)\mathbf{u}^n + \nabla p^n - (\mathbf{B} \times \nabla \phi^n) - (\mathbf{u}^n \times \mathbf{B}) \times \mathbf{B} = \mathbf{f}^n + \frac{1}{\Delta t}\mathbf{u}^{n-1} \quad \text{in } \Omega, \tag{4.7}$$

$$\nabla \cdot \mathbf{u}^n = 0 \quad \text{in } \Omega \tag{4.8}$$

and

$$-\Delta \phi^n + \nabla \cdot (\mathbf{u}^n \times \mathbf{B}) = k^n \quad \text{in } \Omega. \tag{4.9}$$

In (4.7)–(4.9), $\mathbf{B}$, $\mathbf{f}^n$ and k^n are given data. The system (4.7)–(4.9) is supplemented with the initial conditions

$$\mathbf{u}^0(0, \mathbf{x}) = \hat{\mathbf{u}}(\mathbf{x}) \quad \text{in } \Omega \tag{4.10}$$

and the boundary conditions

$$\mathbf{u}^n = \mathbf{0} \quad \text{on } \Gamma \tag{4.11}$$

$$\frac{\partial \phi^n}{\partial n} = g^n \quad \text{on } \Gamma \tag{4.12}$$

where g denotes the only control variable, namely, the normal electric current on Γ. We like to remark here that one could also use other implicit time discretization procedures such as Crank-Nicholson method in the above discretization.

Our goal is to try to obtain a desired flow field by appropriately choosing the control – the normal electric current on Γ. Specifically we will investigate the following cases: matching a desired velocity field, matching a desired potential field, or minimizing the potential gradient. Mathematically, these tasks can be described, respectively, by the following optimal control setting: minimize the cost functional at time levels $t_1, t_2, t_3, \ldots\ldots, t_N$

$$\mathcal{K}(\mathbf{u}^n, \phi^n, p^n, g^n) = \frac{1}{2\epsilon}\int_\Omega |\mathbf{u}^n - \mathbf{u}_d^n|^2 d\Omega + \frac{\delta}{2}\int_\Gamma |g^n|^2 d\Gamma, \tag{4.13}$$

or

$$\mathcal{M}(\mathbf{u}^n, \phi^n, p^n, g^n) = \frac{1}{2\epsilon}\int_\Omega |\phi^n - \phi_d^n|^2 d\Omega + \frac{\delta}{2}\int_\Gamma |g^n|^2 d\Gamma\,, \tag{4.14}$$

or

$$\mathcal{N}(\mathbf{u}^n, \phi^n, p^n, g^n) = \frac{1}{2\epsilon}\int_\Omega |\nabla\phi^n|^2 d\Omega + \frac{\delta}{2}\int_\Gamma |g^n|^2 d\Gamma\,, \tag{4.15}$$

subject to the constraints (4.7)–(4.12). Here $\epsilon > 0$ and $\delta > 0$ are positive parameters; $\mathbf{u}_d$ and ϕ_d are, respectively, desired velocity field and potential field.

The minimization of functional (4.13) or (4.14) or (4.15) subject to (4.7)–(4.12) is a special case of the following general optimal control setting:

minimize the cost functional

$$\mathcal{J}(\mathbf{u}^n, \phi^n, p^n, g^n) = \mathcal{F}(\mathbf{u}^n, \phi^n, p^n) + \frac{\delta}{2}\int_\Gamma |g^n|^2 d\Gamma \tag{4.16}$$

at $t_n = t_1, t_2, \ldots, t_N$ subject to the constraints (4.7)–(4.12),

where $\mathcal{F}(\mathbf{u}^n, \phi^n, p^n)$ is a functional of $(\mathbf{u}^n, \phi^n, p^n)$.

4.2 A Variational Formulation of the Constraints; An Optimality System of Equations

The variational formulation of the constraint equations is then given as follows: seek $\mathbf{u}^n \in \mathbf{H}_0^1(\Omega)$, $p^n \in L_0^2(\Omega)$ and $\phi^n \in \widetilde{H}^1(\Omega)$ such that

$$\frac{1}{\Delta t}\int_\Omega \mathbf{u}^n \cdot \mathbf{v}\, d\Omega + \frac{1}{M^2}\int_\Omega \nabla\mathbf{u}^n : \nabla\mathbf{v}\, d\Omega - \int_\Omega [\nabla\phi^n - (\mathbf{u}^n \times \mathbf{B})]\cdot(\mathbf{v}\times\mathbf{B})\, d\Omega - \int_\Omega p^n \nabla\cdot\mathbf{v}\, d\Omega$$
$$+\frac{1}{N}\int_\Omega (\mathbf{u}^n\cdot\nabla)\mathbf{u}^n\cdot\mathbf{v}\, d\Omega = \int_\Omega \mathbf{f}^n\cdot\mathbf{v}\, d\Omega + \frac{1}{\Delta t}\int_\Omega \mathbf{u}^{n-1}\cdot\mathbf{v}\, d\Omega \quad \forall \mathbf{v}\in\mathbf{H}_0^1(\Omega)\,, \tag{4.17}$$

$$\int_\Omega [\nabla\phi^n - (\mathbf{u}^n\times\mathbf{B})]\cdot(\nabla\psi)\, d\Omega = \int_\Omega k^n\,\psi\, d\Omega + \int_\Gamma g^n\,\psi\, d\Gamma \quad \forall\psi\in\widetilde{H}^1(\Omega) \tag{4.18}$$

and

$$\int_\Omega q\nabla\cdot\mathbf{u}^n d\Omega = 0 \quad \forall q \in L_0^2(\Omega)\,. \tag{4.19}$$

Here the colon notation stands for the scalar product on $\mathbb{R}^{d\times d}$.

The precise mathematical statement of the optimal control problem (4.16) can now be given as follows:

$$\text{seek a } (\mathbf{u}^n, p^n, \phi^n, g^n) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \widetilde{H}^1(\Omega) \times L^2(\Gamma) \text{ such that the functional (4.16) is minimized subject to the constraints (4.8)-(4.10).} \tag{4.20}$$

We will turn the constrained optimization problem (4.20) into an unconstrained one by using Lagrange multiplier principles. We set $\mathcal{X} = \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \widetilde{H}^1(\Omega) \times L^2(\Gamma) \times \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \widetilde{H}^1(\Omega)$ and define the Lagrangian functional

$$\begin{aligned}
&\mathcal{L}(\mathbf{u}^n, p^n, \phi^n, g^n, \mu^n, \tau^n, s^n) \\
&= \mathcal{F}(\mathbf{u}^n, p^n, \phi^n) + \frac{\delta}{2}\int_\Gamma |g^n|^2 d\Gamma - \tfrac{1}{\Delta t}\int_\Omega \mathbf{u}^n \cdot \mu^n d\Omega - \tfrac{1}{M^2}\int_\Omega \nabla \mathbf{u}^n : \nabla \mu^n d\Omega \\
&\quad + \int_\Omega [\nabla \phi^n - (\mathbf{u}^n \times \mathbf{B})] \cdot (\mu^n \times \mathbf{B}) d\Omega - \tfrac{1}{N}\int_\Omega (\mathbf{u}^n \cdot \nabla)\mathbf{u}^n \cdot \mu^n d\Omega \\
&\quad + \int_\Omega p^n \nabla \cdot \mu^n d\Omega + \int_\Omega \mathbf{f}^n \cdot \mu^n d\Omega + \tfrac{1}{\Delta t}\int_\Omega \mathbf{u}^{n-1} \cdot \mu^n d\Omega \\
&\quad - \int_\Omega [\nabla \phi^n - (\mathbf{u}^n \times \mathbf{B})] \cdot (\nabla s^n) d\Omega + \int_\Omega k^n s^n d\Omega + \int_\Gamma g^n s^n d\Gamma \\
&\quad + \int_\Omega \tau^n \nabla \cdot \mathbf{u}^n d\Omega \qquad \forall (\mathbf{u}^n, p^n, \phi^n, g^n, \mu^n, \tau^n, s^n) \in \mathcal{X}.
\end{aligned} \tag{4.21}$$

An optimality system of equations that an optimum must satisfy is derived by taking variations with respect to every variable in the Lagrangian. By taking variations with respect to $\mathbf{u}^n$, p^n and ϕ^n, we obtain:

$$\begin{aligned}
&\tfrac{1}{\Delta t}\int_\Omega \mu^n \cdot \mathbf{w} d\Omega + \tfrac{1}{M^2}\int_\Omega \nabla \mu^n : \nabla \mathbf{w} d\Omega + \int_\Omega (\mathbf{w} \times \mathbf{B}) \cdot (\mu^n \times \mathbf{B}) d\Omega \\
&\quad + \tfrac{1}{N}\int_\Omega (\mathbf{u}^n \cdot \nabla)\mathbf{w} \cdot \mu^n d\Omega + \tfrac{1}{N}\int_\Omega (\mathbf{w} \cdot \nabla)\mathbf{u}^n \cdot \mu^n d\Omega - \int_\Omega \tau^n \nabla \cdot \mathbf{w} d\Omega \\
&\quad = \langle \mathcal{F}_{\mathbf{u}^n}(\mathbf{u}^n, p^n, \phi^n), \mathbf{w} \rangle \quad \forall \mathbf{w} \in \mathbf{H}_0^1(\Omega),
\end{aligned} \tag{4.22}$$

$$\int_\Omega [\nabla s^n - (\mu^n \times \mathbf{B})] \cdot (\nabla r) d\Omega = \langle \mathcal{F}_{\phi^n}(\mathbf{u}^n, p^n, \phi^n), r \rangle \quad \forall r \in \widetilde{H}^1(\Omega) \tag{4.23}$$

and

$$\int_\Omega \sigma \nabla \cdot \mu^n d\Omega = \langle \mathcal{F}_{p^n}(\mathbf{u}^n, p^n, \phi^n), \sigma \rangle \quad \forall \sigma \in L_0^2(\Omega), \tag{4.24}$$

where $\mathcal{F}_{\mathbf{u}^n}$, $\mathcal{F}_{\phi^n}$ and $\mathcal{F}_{p^n}$ are the derivatives of the functional with respect to its three arguments, respectively. By taking variations with respect to μ^n, τ^n and ξ^n, we recover the constraint equations (4.17)-(4.19). By taking variation with respect to g^n we obtain

$$\int_\Gamma (\delta g^n z + z s^n) d\Gamma = 0 \quad \forall z \in L^2(\Gamma) \qquad \text{i.e.,} \qquad g^n = -\frac{1}{\delta} s^n .$$

This last equation enables us to eliminate the control g^n in (4.20). Thus (4.17)-(4.19) can be replaced by

$$\frac{1}{\Delta t}\int_\Omega \mathbf{u}^n \cdot \mathbf{v} d\Omega + \frac{1}{M^2}\int_\Omega \nabla \mathbf{u}^n : \nabla \mathbf{v} d\Omega - \int_\Omega [\nabla \phi^n - (\mathbf{u}^n \times \mathbf{B})] \cdot (\mathbf{v} \times \mathbf{B}) d\Omega + \frac{1}{N}\int_\Omega (\mathbf{u}^n \cdot \nabla)\mathbf{u}^n \cdot \mathbf{v} d\Omega$$
$$- \int_\Omega p^n \nabla \cdot \mathbf{v} d\Omega = \int_\Omega \mathbf{f}^n \cdot \mathbf{v} d\Omega + \frac{1}{\Delta t}\int_\Omega \mathbf{u}^{n-1} \cdot \mathbf{v} d\Omega \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \qquad (4.25)$$

$$\int_\Omega [\nabla \phi^n - (\mathbf{u}^n \times \mathbf{B})] \cdot (\nabla \psi) d\Omega + \frac{1}{\delta}\int_\Gamma s^n \psi d\Omega = \int_\Omega k^n \psi d\Omega \quad \forall \psi \in \widetilde{H}^1(\Omega), \qquad (4.26)$$

and

$$\int_\Omega q \nabla \cdot \mathbf{u}^n d\Omega = 0 \quad \forall q \in L_0^2(\Omega), \qquad (4.27)$$

Equations (4.22)-(4.27) forms an optimality system of equations that an optimal solution must satisfy. Finite element approximations and the computational methods, Newton's and iterative, presented in Chapter 2 for the steady state case can be extended here in a natural fashion.

4.3 Computational Examples

In this section, we report some computational examples that serve to illustrate the effectiveness and practicality of optimal control techniques in electrically conducting fluids. First, we treat the problem of steering the velocity field to a desired one. The second one deals with the minimization of potential gradient throughout the domain. Finally, we consider the problem of matching electric potential to a desired one.

All computations are done with the same choice of finite element spaces as in Chapter 2.

4.3.1 Velocity field matching

The first case we consider is the problem of minimizing (4.13) subject to (4.7)-(4.12), i.e., we attempt to match the velocity field with a desired one by finding an appropriate boundary current density g.

The optimality system of equations are given by (4.22)–(4.27) with

$$\langle \mathcal{F}_{\mathbf{u}^n}(\mathbf{u}^n, p^n, \phi^n), \mathbf{w}\rangle = \frac{1}{\epsilon}\int_\Omega (\mathbf{u}^n - \mathbf{u}_d^n)\cdot \mathbf{w}\,d\Omega$$

and

$$\langle \mathcal{F}_{p^n}(\mathbf{u}^n, p^n, \phi^n), \sigma\rangle = \langle \mathcal{F}_{\phi^n}(\mathbf{u}^n, p^n, \phi^n), r\rangle = 0\,.$$

The corresponding system of partial differential equations for (4.22)–(4.27) is given by (4.7)–(4.11),

$$\frac{\partial \phi^n}{\partial n} = -\frac{1}{\delta} s^n \qquad \text{on } \Gamma\,,$$

$$\frac{1}{\Delta t}\mu^n - \frac{1}{M^2}\Delta\mu^n + \frac{1}{N}\mu^n\cdot(\nabla \mathbf{u}^n)^T - \frac{1}{N}(\mathbf{u}^n\cdot\nabla)\mu^n + \nabla r^n - \mathbf{B}\times(\nabla s^n)$$
$$- (\mu^n\times\mathbf{B})\times\mathbf{B} - \frac{1}{\epsilon}\mathbf{u}^n = -\frac{1}{\epsilon}\mathbf{u}_d^n \qquad \text{in } \Omega,$$

$$\nabla\cdot\mu^n = 0 \qquad \text{in } \Omega,$$

$$-\Delta s^n + \nabla\cdot(\mu^n\times\mathbf{B}) = 0 \qquad \text{in } \Omega,$$

$$\mu^n = \mathbf{0} \qquad \text{on } \Gamma$$

and

$$\frac{\partial s^n}{\partial n} = 0 \qquad \text{on } \Gamma\,.$$

We now present some numerical results for the following choice of parameters and data:

The Hartmann number and interaction number: $N = 1$, $M = 1$;

the domain Ω is the unit square $(0,1)\times(0,1)$;

applied magnetic field: $\mathbf{B} = (0, 0, \sin(\pi y))^T$;

desired velocity field:

$$\mathbf{u}_d = \begin{pmatrix} [1-\cos(2\pi x t)]\cos(2\pi y t) \\ \sin(2\pi x t)\sin(2\pi y t)\end{pmatrix},$$

initial conditions:

$$\hat{\mathbf{u}} = \begin{pmatrix} [\cos(2\pi x) - 1]\sin(2\pi y) \\ \sin(2\pi x)[1-\cos(2\pi y)]\end{pmatrix}.$$

With these data, the body force $\mathbf{f} = (f_1, f_2)^T$ and electric source k are selected such that exact solution of the uncontrolled problem, i.e., the solution for (4.1)–(4.6) with $g = 0$, is given by

$$\mathbf{u}_0 = \begin{pmatrix} \exp(\frac{-t}{M})[\cos(2\pi x) - 1]\sin(2\pi y) \\ \exp(\frac{-t}{M})\sin(2\pi x)[1 - \cos(2\pi y)] \end{pmatrix},$$

$$p_0 = C, \qquad \text{where C is a constant}$$

and

$$\phi_0 = \exp(-t)\cos(\pi x)\cos(\pi y).$$

The proper choice of the constants ϵ and δ in the functional plays an important role in obtaining a best velocity matching. For the computational results shown in Figures 4.1 below, our choice of these two constants were $\epsilon = 0.00001$ and $\delta = 0.01$. The time step $\Delta t = 0.2$. We give a brief description of the figures. Figures 4.1 a), g), m), s), and x) are the uncontrolled velocity field $\mathbf{u}_0$ at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Figures 4.1 b) h), n), t) and y) are the desired velocity field $\mathbf{u}^h$ at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Figures 4.1 c), g), k), o) and s) are the optimal velocity fields at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Figures 4.1 d), h), l), p) and t) are the adjoint potential s^h at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Those were obtained by solving (4.22)-(4.27). The optimal control g^h can be gleaned from Figures 4.1 d), h), l), p) and t), and the relation $g^h = -\frac{1}{\delta}s^h$. All the computational results shown in figures 4.1 were obtained with a 10 by 10 triangulation of the unit square. A nonuniform grid with corner refinements was used. We see from the figures that optimal control does a very good job in matching the desired velocity field at t=1.

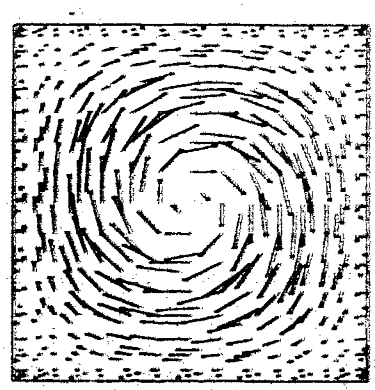

Figure 4.1 a) uncontrolled velocity $\mathbf{u}_0$ at $t = 0.2$

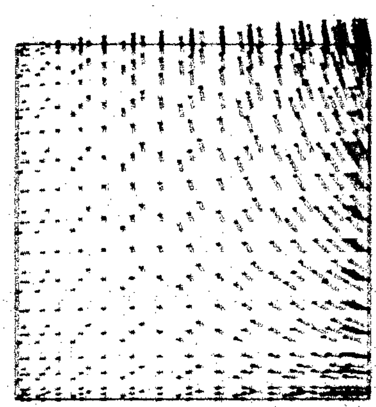

Figure 4.1 b) desired velocity $\mathbf{u}_d$ at $t = 0.2$

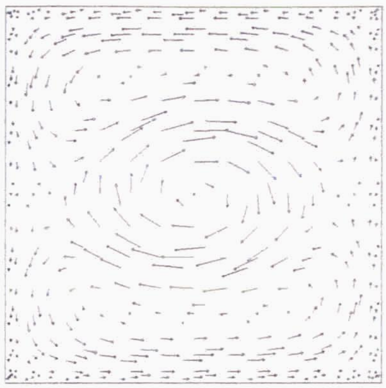

Figure 4.1 c) optimal velocity $\mathbf{u}^h$ at $t = 0.2$

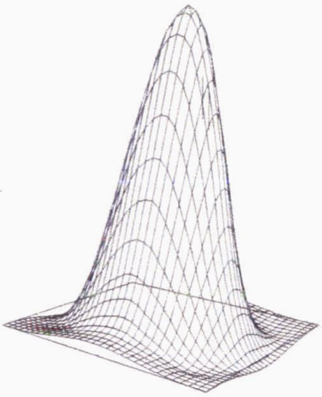

Figure 4.1 d) adjoint potential field s^h at $t = 0.2$

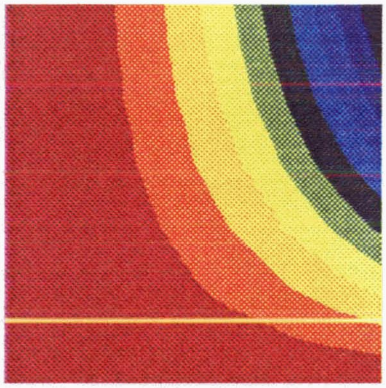

Figure 4.1 e) contours of desired velocity at $t = 0.2$

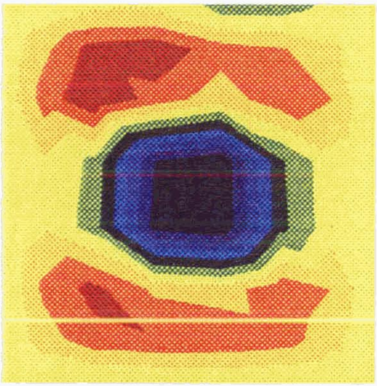

Figure 4.1 f) contours of optimal velocity field at $t = 0.2$

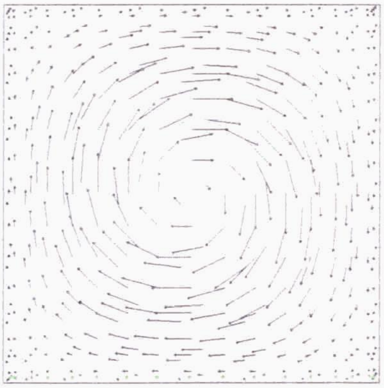

Figure 4.1 g) uncontrolled velocity $\mathbf{u}_0$ at $t = 0.4$

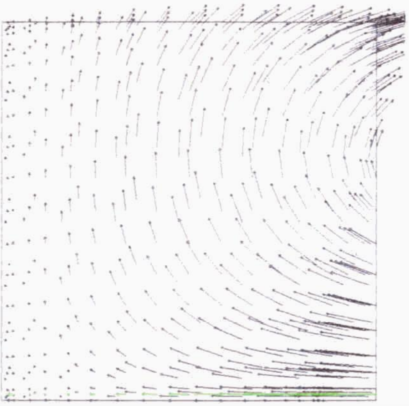

Figure 4.1 h) desired velocity $\mathbf{u}_d$ at $t = 0.4$

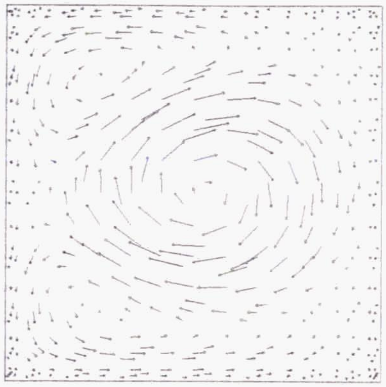

Figure 4.1 i) optimal velocity $\mathbf{u}^h$ at $t = 0.4$

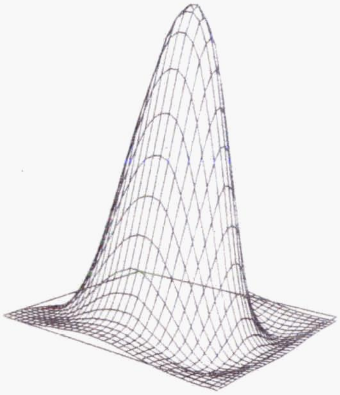

Figure 4.1 j) adjoint potential s^h at $t = 0.4$

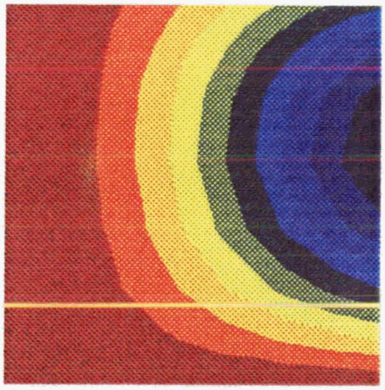

Figure 4.1 k) contours of desired velocity at $t = 0.4$

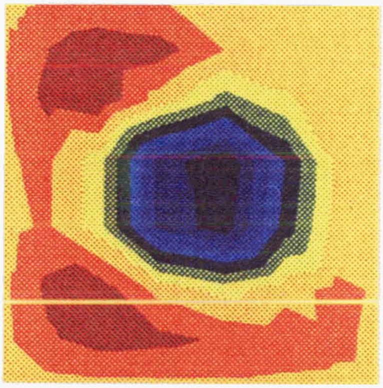

Figure 4.1 l) contours of optimal velocity field at $t = 0.4$

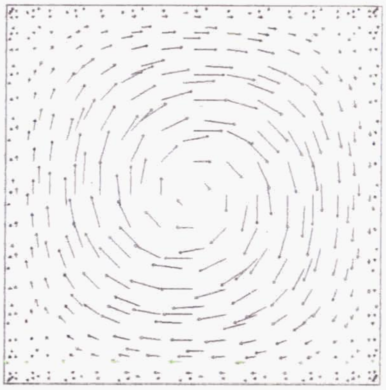

Figure 4.1 m) uncontrolled velocity $\mathbf{u}_0$ at $t = 0.6$

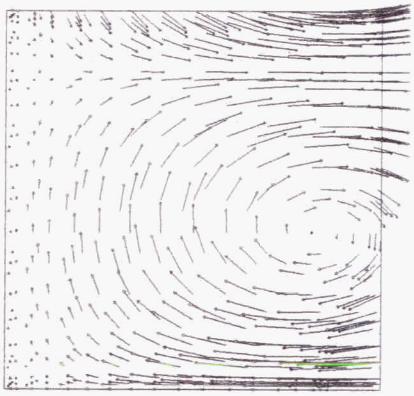

Figure 4.1 n) desired velocity $\mathbf{u}_d$ at $t = 0.6$

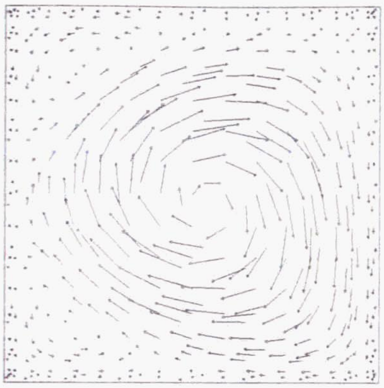

Figure 4.1 o) optimal velocity $\mathbf{u}^h$ at $t = 0.6$

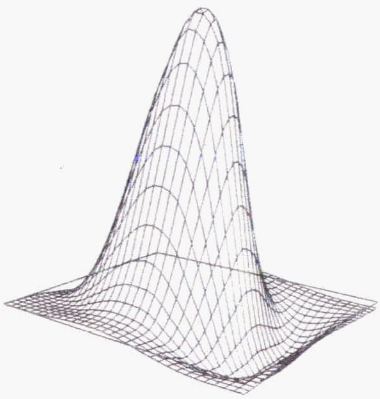

Figure 4.1 p) adjoint potential s^h at $t = 0.6$

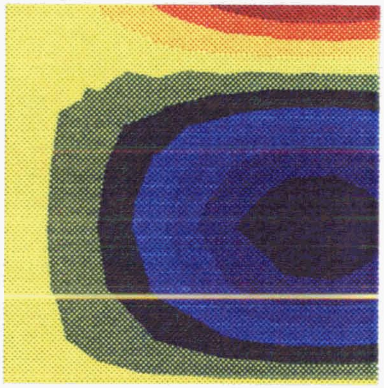

Figure 4.1 q) contours of desired velocity at $t = 0.6$

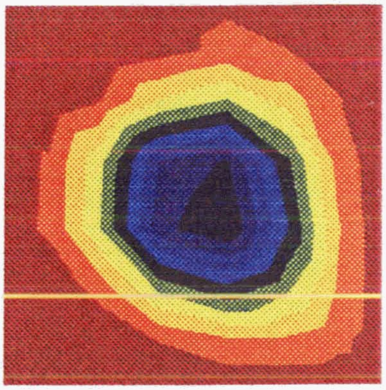

Figure 4.1 r) contours of optimal velocity field at $t = 0.6$

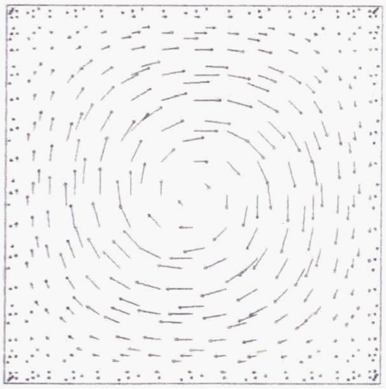

Figure 4.1 s) uncontrolled velocity $\mathbf{u}_0$ at $t = 0.8$

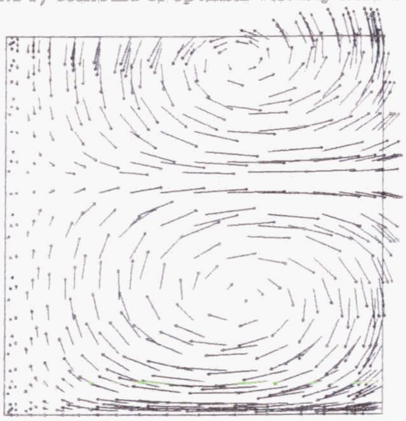

Figure 4.1 t) desired velocity $\mathbf{u}_d$ at $t = 0.8$

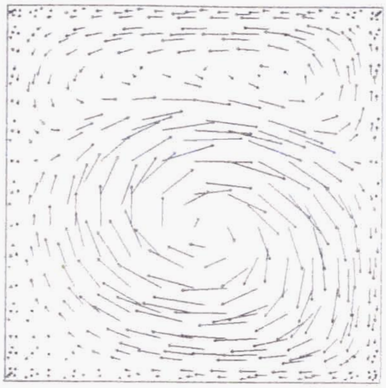

Figure 4.1 u) optimal velocity $\mathbf{u}^h$ at $t = 0.8$

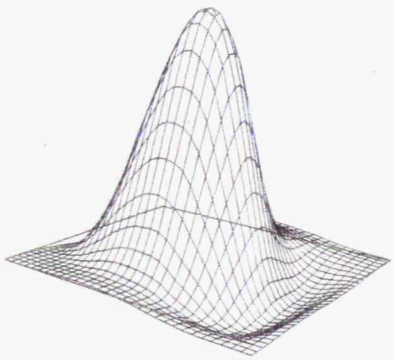

Figure 4.1 v) adjoint potential s^h at $t = 0.8$

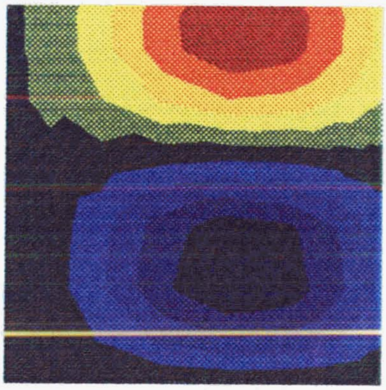

Figure 4.1 w) contours of desired velocity at $t = 0.8$

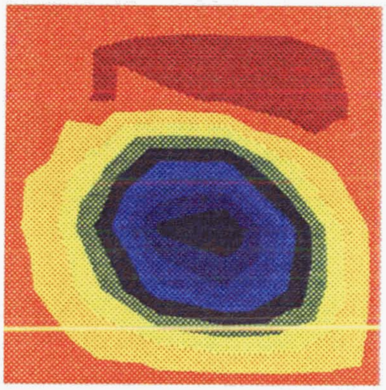

Figure 4.1 x) contours of optimal velocity field at $t = 0.8$

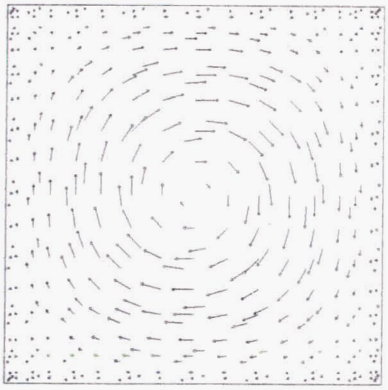

Figure 4.1 x) uncontrolled velocity $\mathbf{u}_0$ at $t = 1.0$

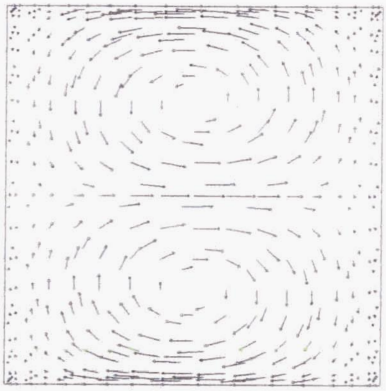

Figure 4.1 y) desired velocity $\mathbf{u}_d$ at $t = 1.0$

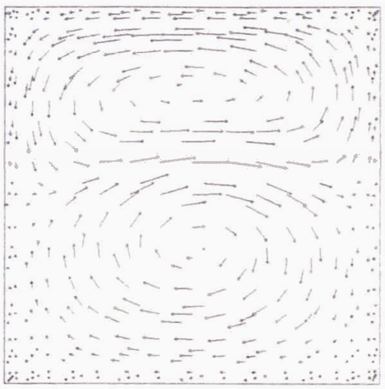

Figure 4.1 z) optimal velocity $\mathbf{u}^h$ at $t = 1.0$

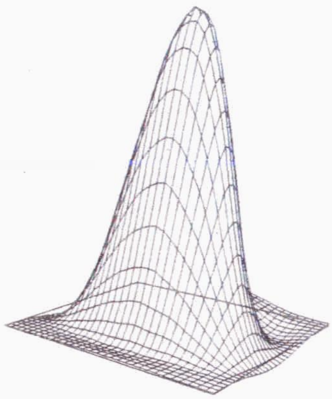

Figure 4.1 z1) adjoint potential s^h at $t = 1.0$

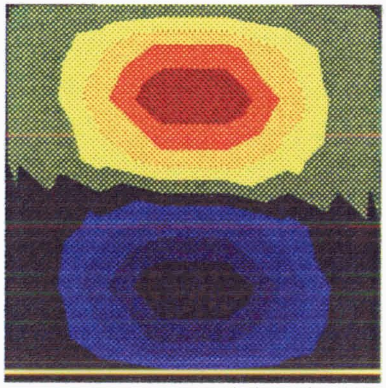

Figure 4.1 z2) contours of desired velocity at $t = 1.0$

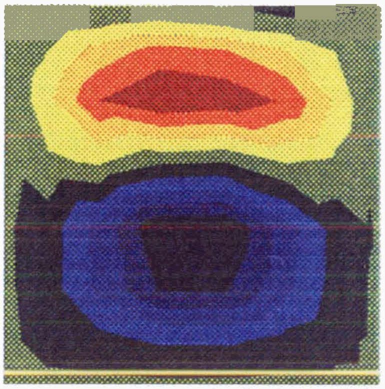

Figure 4.1 z3) contours of optimal velocity field at $t = 1.0$

4.3.2 Potential field matching

The second case we consider is the problem of matching the electric potential ϕ to a desired distribution ϕ_d, i.e., we minimize functional (4.14) subject to (4.7)–(4.12).

The optimality system of equations are given by (4.24)–(4.29) with

$$\langle \mathcal{F}_{\phi^n}(\mathbf{u}^n, p^n, \phi^n), r \rangle = \frac{1}{\epsilon} \int_\Omega (\phi^n - \phi_d^n)\, r\, d\Omega$$

and

$$\langle \mathcal{F}_{\mathbf{u}^n}(\mathbf{u}^n, p^n, \phi^n), \mathbf{w} \rangle = \langle \mathcal{F}_{p^n}(\mathbf{u}^n, p^n, \phi^n), \sigma \rangle = 0\,.$$

The domain is chosen to be a unit square. The interaction number N, body force $\mathbf{f}$, electric source k and applied magnetic field $\mathbf{B}$ are chosen to be the same as in Section 4.3.1. The Hartman's number $M = 10$.

The desired potential field is a function of time and space which is a uniform one at $t = 1$:

$$\phi_d = (1-t)\cos(\pi x)\cos(\pi y)\exp(-t) + t\,.$$

The two parameters in the functional are chosen as $\epsilon = 0.002$ and $\delta = 0.1$.

For these data, the solution $(\mathbf{u}_0, p_0, \phi_0)$ of the uncontrolled problem, i.e., the solution for (4.1)–(4.6) with $g = 0$, is the same as in Section 4.3.1. The time step $\Delta t = 0.2$.

Some numerical results for this example is reported in Figures 4.2 a)-t). We give a brief description of the figures. Figures 4.2 a), e), i), m), and q) are the uncontrolled potential fields ϕ_0 at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Figures 4.2 b) f), j), n) and r) are the desired potential fields ϕ^h at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Figures 4.2 c), g), k), o) and s) are the optimal potential fields at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Figures 4.2 d), h), l), p) and t) are the adjoint potential s^h at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Those were obtained by solving (4.22)-(4.27). The optimal control g^h can be gleaned from Figures 4.2 d), h), l), p) and t), and the relation $g^h = -\frac{1}{\delta}s^h$.

A look at Figures 4.2 c), g), k), o) and s) reveal that the optimal potential matches well with the desired potential. In fact we also have good matching at $t = 0.2, 0.4, 0.6$, and 0.8.

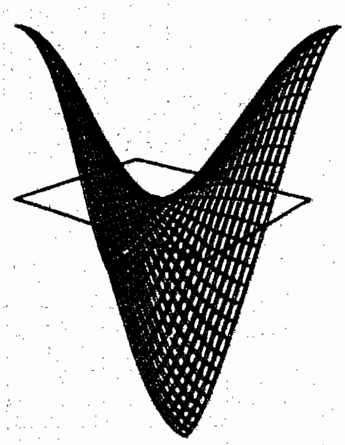

Figure 4.2 a) uncontrolled potential ϕ_0 at $t = 0.2$

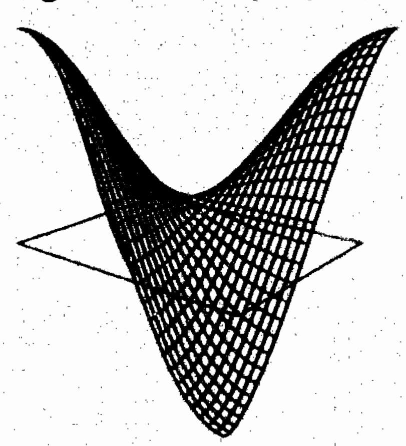

Figure 4.2 b) desired potential ϕ_d at $t = 0.2$

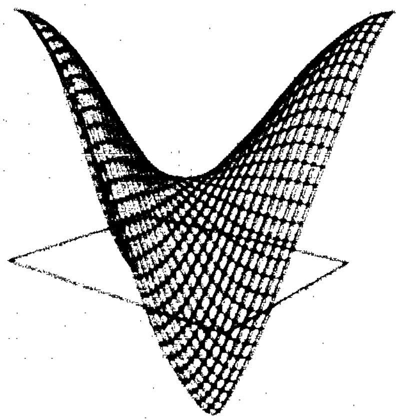

Figure 4.2 c) optimal potential ϕ^h at $t = 0.2$

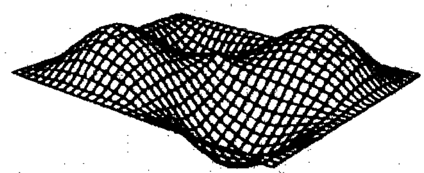

Figure 4.2 d) adjoint potential s^h at $t = 0.2$

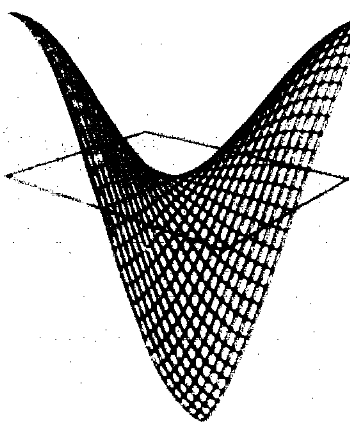

Figure 4.2 e) uncontrolled potential ϕ_0 at $t = 0.4$

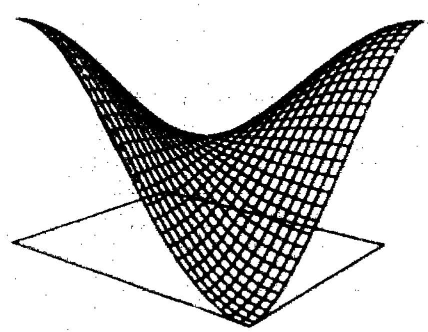

Figure 4.2 f) desired potential ϕ_d at $t = 0.4$

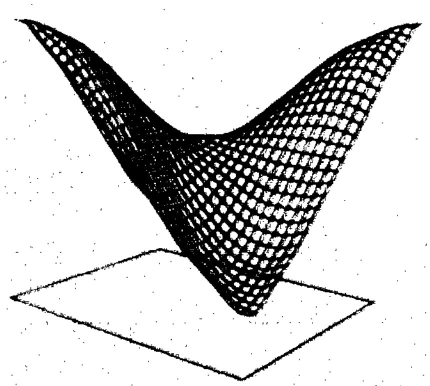

Figure 4.2 g) optimal potential ϕ^h at $t = 0.4$

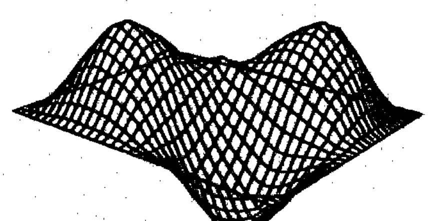

Figure 4.2 h) adjoint potential s^h at $t = 0.4$

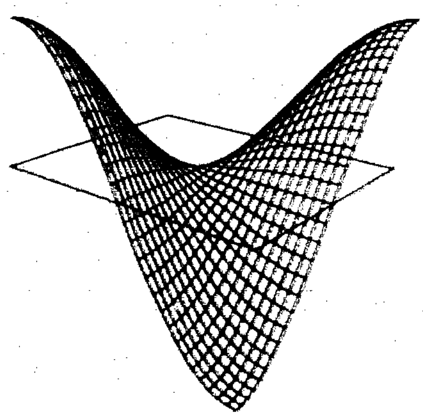

Figure 4.2 i) uncontrolled potential ϕ_0 at $t = 0.6$

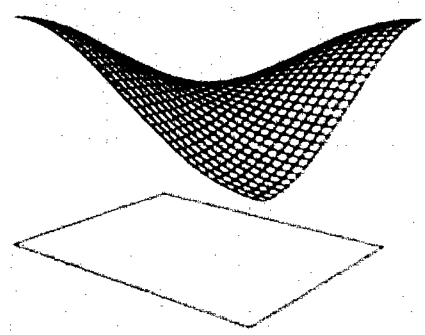

Figure 4.2 j) desired potential ϕ_d at $t = 0.6$

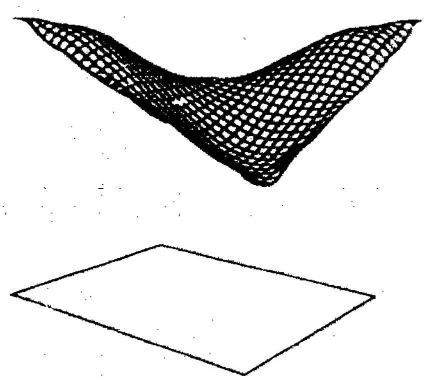

Figure 4.2 k) optimal potential ϕ^h at $t = 0.6$

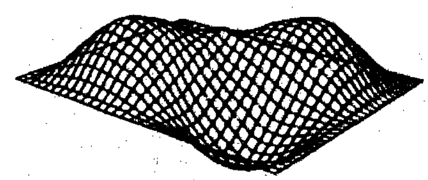

Figure 4.2 l) adjoint potential s^h at $t = 0.6$

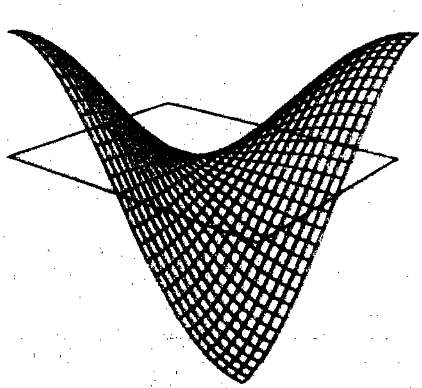

Figure 4.2 m) uncontrolled potential ϕ_0 at $t = 0.8$

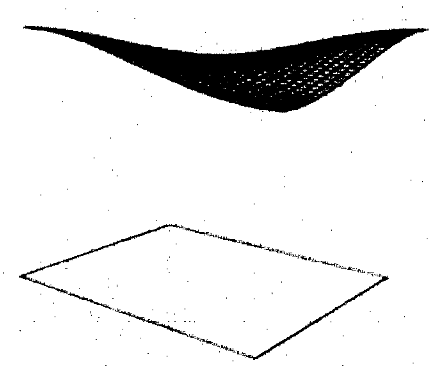

Figure 4.2 n) desired potential ϕ_d at $t = 0.8$

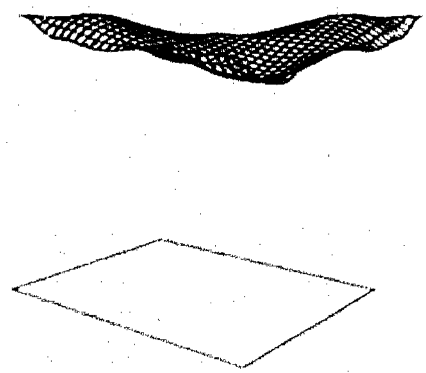

Figure 4.2 o) optimal potential ϕ^h at $t = 0.8$

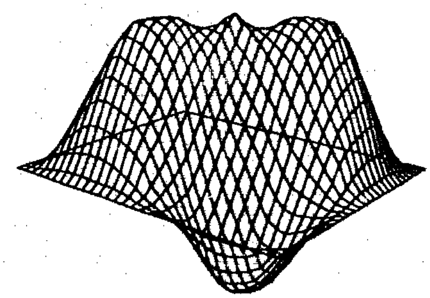

Figure 4.2 p) adjoint potential s^h at $t = 0.8$

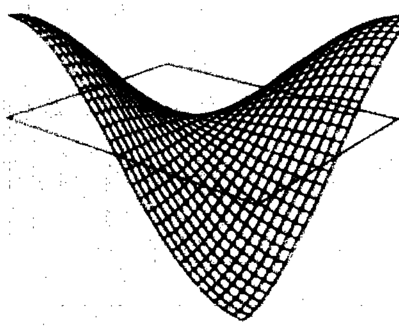

Figure 4.2 q) uncontrolled potential ϕ_0 at $t = 1.0$

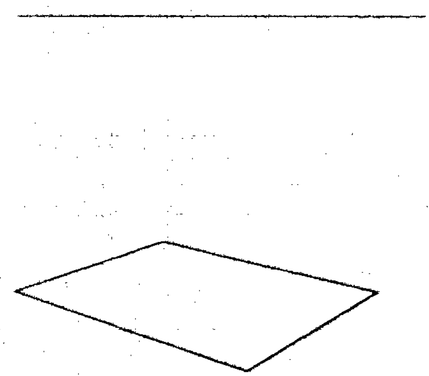

Figure 4.2 r) desired potential ϕ_d at $t = 1.0$

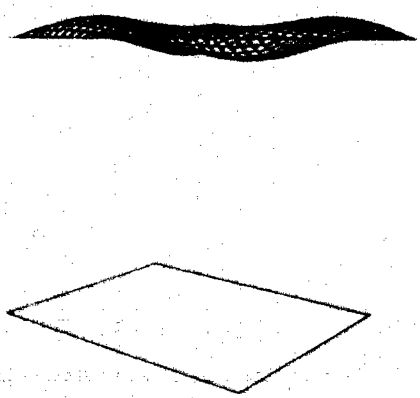

Figure 4.2 s) optimal potential ϕ^h at $t = 1.0$

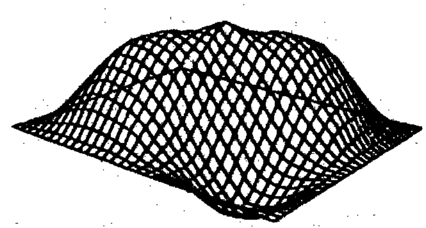

Figure 4.2 t) adjoint potential s^h at $t = 1.0$

4.3.3 Potential gradient minimization

The third case we consider is the problem of minimizing the electric potential gradient, i.e., we minimize functional (4.15) subject to (4.7)–(4.12).

The optimality system of equations are given by (4.22)–(4.27) with

$$\langle \mathcal{F}_{\phi^n}(\mathbf{u}^n, p^n, \phi^n), r\rangle = \frac{1}{\epsilon}\int_\Omega \nabla\phi^n \cdot \nabla r \, d\Omega$$

and

$$\langle \mathcal{F}_{\mathbf{u}^n}(\mathbf{u}^n, p^n, \phi^n), \mathbf{w}\rangle = \langle \mathcal{F}_{p^n}(\mathbf{u}^n, p^n, \phi^n), \sigma\rangle = 0\,.$$

The domain is chosen to be a unit square. The Hartmann number $M = 10$. The interaction number N, body force $\mathbf{f}$, electric source k and applied magnetic field $\mathbf{B}$ are chosen to be the same as in Section 4.3.1.

The two parameters in the functional are chosen as $\epsilon = 0.0002$ and $\delta = 1$. For these data, the exact solution $(\mathbf{u}_0, p_0, \phi_0)$ of the uncontrolled problem, i.e., the solution for (4.1)–(4.6) with $g = 0$, is given by the same uncontrolled solution as in Section 4.3.1. The time step $\Delta t = 0.2$.

Some numerical results for this example is reported in Figures 4.3 a)-o). We give a brief description of the figures. Figures 4.3 a), d), g), j), and m) are the uncontrolled potential field ϕ_0 at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Figures 4.3 b), e), h), k) and n) are the optimal potential field ϕ^h at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Figures 4.3 c), f), i), l) and o) are the adjoint potential s^h at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Those were obtained by solving (4.22)-(4.27). The optimal control g^h can be gleaned from Figures 4.3 c), f), i), l) and o), and the relation $g^h = -\frac{1}{\delta}s^h$.

By minimizing functional (4.15) we wish to obtain a quasi-uniform potential distribution at each time level. The numerical results (in particular, Figure 4.3 b), e), h), k) and n)) demonstrate that the optimal control did a very good job in achieving the objective.

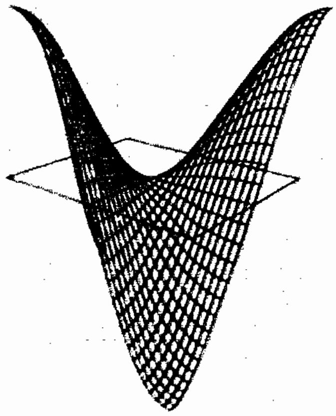

Figure 4.3 a) uncontrolled potential ϕ_0 at $t = 0.2$

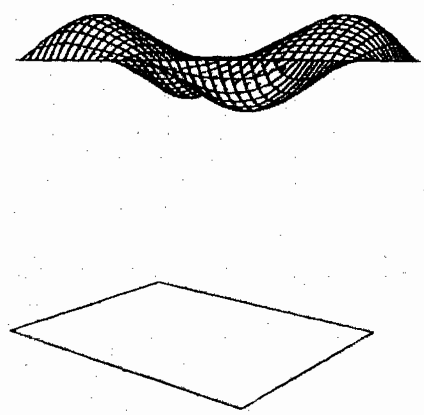

Figure 4.3 b) optimal potential ϕ^h at $t = 0.2$

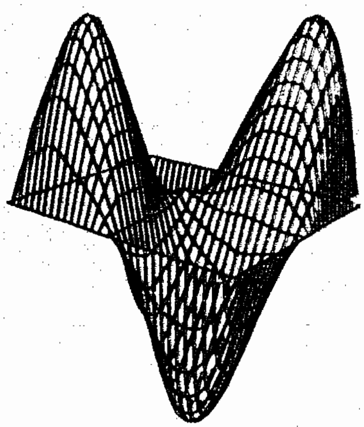

Figure 4.3 c) adjoint potential s^h at $t = 0.2$

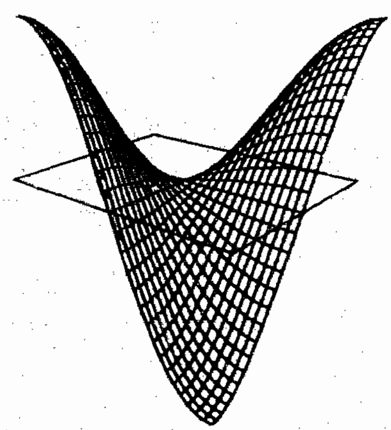

Figure 4.3 d) uncontrolled potential ϕ_0 at $t = 0.4$

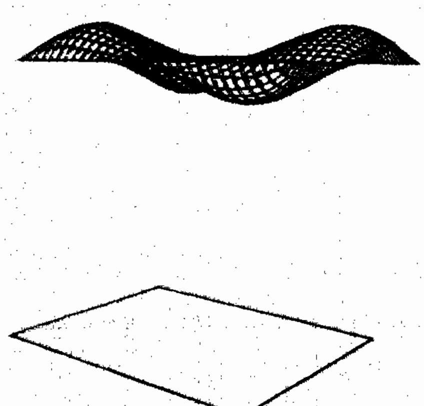

Figure 4.3 e) optimal potential ϕ^h at $t = 0.4$

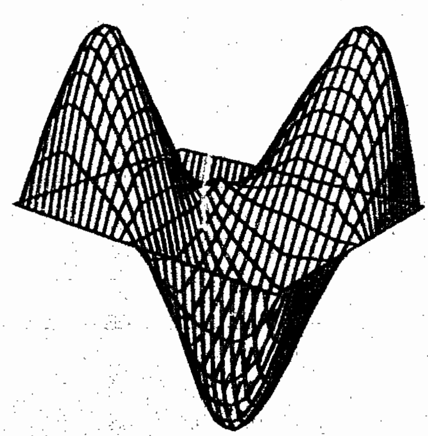

Figure 4.3 f) adjonit potential ϕ^h at $t = 0.4$

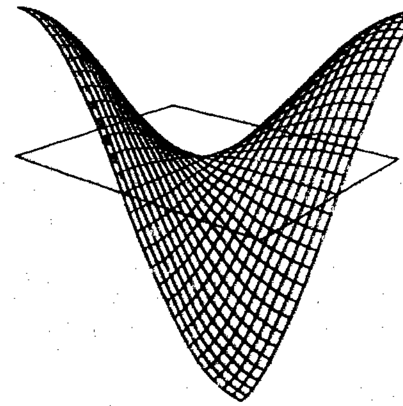

Figure 4.3 g) uncontrolled potential ϕ_0 at $t = 0.6$

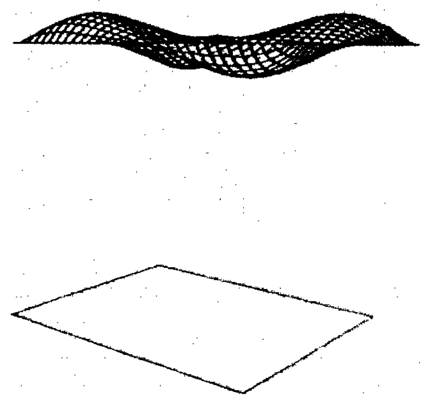

Figure 4.3 h) optimal potential ϕ^h at $t = 0.6$

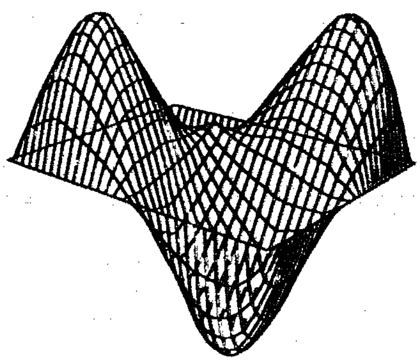

Figure 4.3 i) adjoint potential s^h at $t = 0.6$

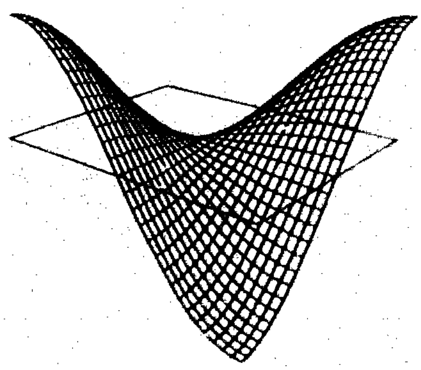

Figure 4.3 j) uncontroled potential ϕ_0 at $t = 0.8$

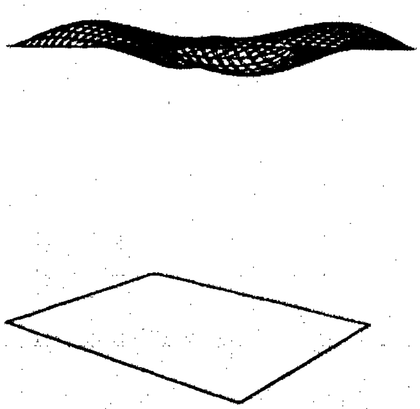

Figure 4.3 k) optimal potential ϕ^h at $t = 0.8$

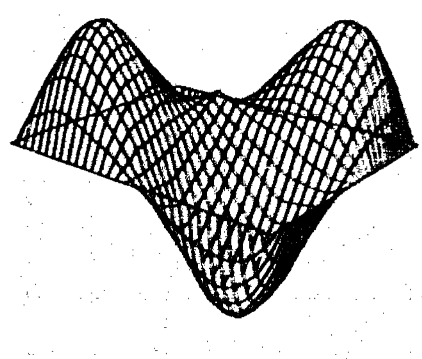

Figure 4.3 l) adjoint potential s^h at $t = 0.8$

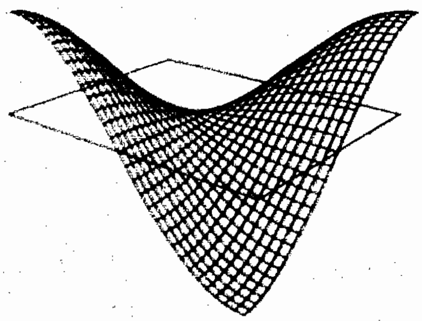

Figure 4.3 m) uncontrolled potential ϕ_0 at $t = 1.0$

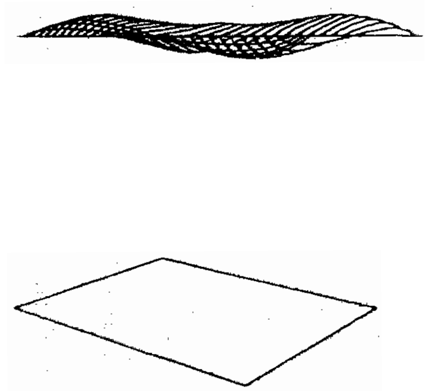

Figure 4.3 n) optimal potential ϕ^h at $t = 1.0$

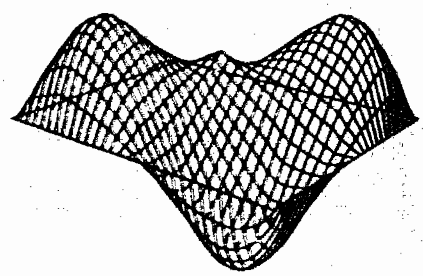

Figure 4.3 o) adjoint potential s^h at $t = 1.0$

4.4 Concluding Remarks.

In this chapter we studied numerical computation of unsteady boundary control problems for an electrically conducting fluid using electric current controls. We summarize the main points in this chapter as follows:

- We developed a computational procedure to solve unsteady optimal control problems.
- We converted the optimal control problem into a system of equations (i.e., the optimality of equations) by using the Lagrange multiplier principles;

- We proposed some methods for solving the discrete optimality system of equations. Our discussions of these methods were made for finite element discretizations. Apparently, these methods are equally applicable to finite difference, collocation or pseudo-spectral discretizations;

- We conducted numerical experiments for some prototype examples to show the performance of our computational procedure. Our work demonstrated the effectiveness of our computational techniques in flow field matchings and in the minimization of some physical quantities.

In principle, other types of boundary controls such as Dirichlet controls of the electrical potential or Dirichlet controls of the boundary velocity can all be treated by the techniques used in this chapter. Optimal control problems with distributed controls and Dirichlet can also be studied in a similar manner. See [HR2] for details and computational examples.

Chapter 5

Control of Unsteady Flows II: Heat Flux Control

In this chapter we study some control problems for a Boussinesq's model in an unsteady electrically conducting flow. The control is the heat flux on the flow boundary. The format of this chapter is the same as that of the previous one.

5.1 Statement of the Optimal Control Problems

We denote by Ω the flow region which is a bounded open container in $\mathbf{R}^2$ or $\mathbf{R}^3$ with boundary Γ and by $[0,1]$ the time interval. Then the dimensionless Boussinesq's model takes the form:

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{1}{M^2}\Delta \mathbf{u} + \frac{1}{N}(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - (\mathbf{B}\times\nabla\phi) - (\mathbf{u}\times\mathbf{B})\times\mathbf{B} - \frac{1}{L}\mathbf{g}T = \mathbf{f} \quad \text{in } [0,1]\times\Omega\,, \tag{5.1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } [0,1]\times\Omega\,, \tag{5.2}$$

$$-\Delta\phi + \nabla\cdot(\mathbf{u}\times\mathbf{B}) = k_1 \quad \text{in } [0,1]\times\Omega \tag{5.3}$$

and

$$T_t - \Delta T + \mathbf{u}\cdot\nabla T = k_2 \quad \text{in } [0,1]\times\Omega\,. \tag{5.4}$$

In (5.1)–(5.4), $\mathbf{B}$, $\mathbf{f}$, k_1 and k_2 are given data and M, N and L are parameters. The system (5.1)–(5.4) is supplemented with the following initial conditions

$$\mathbf{u}(0,\mathbf{x}) = \hat{\mathbf{u}}(\mathbf{x}) \quad \text{in } \Omega\,, \tag{5.5}$$

$$T(0,\mathbf{x}) = \hat{T}(\mathbf{x}) \quad \text{in } \Omega\,, \tag{5.6}$$

and the boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on } [0,1]\times\Gamma\,, \tag{5.7}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } [0,1]\times\Gamma\,, \tag{5.8}$$

$$\frac{\partial T}{\partial n} = g \quad \text{on } [0,1]\times\Gamma\,, \tag{5.9}$$

where g denotes the only control variable, namely, the heat flux on Γ, $\hat{\mathbf{u}}$ the initial velocity and $\hat{T}$ the initial temperature. We further test the computational procedure described in Chapter 4 on this model. For completeness we present the procedure here:

1° First the state equations are descretized using an implicit time descretization scheme in the time interval $[0,1]$.

2° Then in each time interval the state equation is steady and the cost function is taken instantaneous (no time averaging).

3° Lagrange Multiplier technique is applied at each time interval $[t_{n-1}, t_n]$ to derive the optimality system.

4° At each time level optimality systems are solved using the computational techniques we developed for the steady state control problems.

A fully implicit backward Euler time discretization is used here to discretize the system. Select $0 = t_1 < t_2 < t_3 ..., t_{n-1}, t_n, ...t_{N-1} < t_N = 1$. Let $\mathbf{u}^n = \mathbf{u}(t_n,\mathbf{x})$, $\mathbf{u}^{n-1} = \mathbf{u}(t_{n-1},\mathbf{x})$, $T^n = T(t_n,\mathbf{x})$, $T^{n-1} = T(t_{n-1},\mathbf{x})$ and $\phi^n = \phi(t_n,\mathbf{x})$. Let $\Delta t = t_n - t_{n-1}$. Then the time discrete state equations take the form

$$\begin{aligned}\frac{1}{\Delta t}\mathbf{u}^n - \frac{1}{M^2}\Delta\mathbf{u}^n + \frac{1}{N}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n + \nabla p^n \;-\; (\mathbf{B}\times\nabla\phi^n) - (\mathbf{u}^n\times\mathbf{B})\times\mathbf{B} - \frac{1}{L}\mathbf{g}T^n \\ = \mathbf{f}^n + \frac{1}{\Delta t}\mathbf{u}^{n-1} \quad \text{in } \Omega\,,\end{aligned} \tag{5.10}$$

$$\nabla \cdot \mathbf{u}^n = 0 \quad \text{in } \Omega, \tag{5.11}$$

$$-\Delta\phi^n + \nabla\cdot(\mathbf{u}^n \times \mathbf{B}) = k_1^n \quad \text{in } \Omega, \tag{5.12}$$

and

$$\frac{1}{\Delta t}T^n - \Delta T^n + \mathbf{u}^n \cdot \nabla T^n = k_2^n + \frac{1}{\Delta t}T^{n-1} \quad \text{in } \Omega. \tag{5.13}$$

The system (5.10)–(5.13) is supplemented with the initial conditions

$$\mathbf{u}(0,\mathbf{x}) = \widehat{\mathbf{u}}(\mathbf{x}) \quad \text{in } \Omega, \tag{5.14}$$

$$T(0,\mathbf{x}) = \widehat{T}(\mathbf{x}) \quad \text{in } \Omega, \tag{5.15}$$

and the boundary conditions

$$\mathbf{u}^n = \mathbf{0} \quad \text{on } \Gamma, \tag{5.16}$$

$$\frac{\partial\phi^n}{\partial n} = 0 \quad \text{on } \Gamma \tag{5.17}$$

and

$$\frac{\partial T^n}{\partial n} = g^n \quad \text{on } \Gamma \tag{5.18}$$

where g^n denotes the only control variable, namely, the heat flux on Γ.

Our goal is to try to obtain a desired flow field by appropriately choosing the control – the normal temperature gradient on Γ. Specifically we will investigate the following cases: matching a desired velocity field, matching a desired temperature field, or minimizing the temperature gradient. Mathematically, these tasks can be described, respectively, by the following optimal control setting: minimize the cost functional at time levels $t_1, t_2, t_3, \ldots\ldots, t_N$

$$\mathcal{K}(\mathbf{u}^n,\phi^n,T^n,p^n,g^n) = \frac{1}{2\epsilon}\int_\Omega |\mathbf{u}^n - \mathbf{u}_d^n|^2 d\Omega + \frac{\delta}{2}\int_\Gamma |g^n|^2 d\Gamma, \tag{5.19}$$

or

$$\mathcal{M}(\mathbf{u}^n,\phi^n,T^n,p^n,g^n) = \frac{1}{2\epsilon}\int_\Omega |T^n - T_d^n|^2 d\Omega + \frac{\delta}{2}\int_\Gamma |g^n|^2 d\Gamma, \tag{5.20}$$

or

$$\mathcal{N}(\mathbf{u}^n,\phi^n,T^n,p^n,g^n) = \frac{1}{2\epsilon}\int_\Omega |\nabla T^n|^2 d\Omega + \frac{\delta}{2}\int_\Gamma |g^n|^2 d\Gamma, \tag{5.21}$$

subject to the constraints (5.10)–(5.18). Here $\epsilon > 0$ and $\delta > 0$ are positive parameters; $\mathbf{u}_d$ and T_d are, respectively, desired velocity field and temperature field.

The minimization of functional (5.19) or (5.20) or (5.21) subject to (5.10)–(5.18) is a special case of the following general optimal control setting:

$$\begin{aligned}&\text{minimize the cost functional}\\ &\mathcal{J}(\mathbf{u}^n,\phi^n,T^n,p^n,g^n) = \mathcal{F}(\mathbf{u}^n,\phi^n,T^n,p^n) + \frac{\delta}{2}\int_\Gamma |g^n|^2 d\Gamma \\ &\text{at } t_n = t_1, t_2, \ldots\ldots, t_N \text{ subject to the constraints (5.10)–(5.18),}\end{aligned} \tag{5.22}$$

where $\mathcal{F}(\mathbf{u}^n,\phi^n,T^n,p^n)$ is a functional of $(\mathbf{u}^n,\phi^n,T^n,p^n)$.

5.2 A Variational Formulation of the Constraints; An Optimality System of Equations

The variational formulation of the constraint equations is then given as follows: seek $\mathbf{u}^n \in \mathbf{H}_0^1(\Omega)$, $p^n \in L_0^2(\Omega)$, $\phi^n \in \widetilde{H}^1(\Omega)$ and $T^n \in \widetilde{H}^1(\Omega)$ such that

$$\begin{aligned}&\tfrac{1}{\Delta t}\int_\Omega \mathbf{u}^n\cdot\mathbf{v}\,d\Omega + \tfrac{1}{M^2}\int_\Omega \nabla\mathbf{u}^n : \nabla\mathbf{v}\,d\Omega - \int_\Omega[\nabla\phi^n - (\mathbf{u}^n\times\mathbf{B})]\cdot(\mathbf{v}\times\mathbf{B})d\Omega - \int_\Omega p^n\,\nabla\cdot\mathbf{v}\,d\Omega\\ &\quad+\tfrac{1}{N}\int_\Omega(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n\cdot\mathbf{v}\,d\Omega - \tfrac{1}{L}\int_\Omega \mathbf{g}\cdot\mathbf{v}T^n d\Omega\\ &\quad= \int_\Omega \mathbf{f}^n\cdot\mathbf{v}\,d\Omega + \tfrac{1}{\Delta t}\int_\Omega \mathbf{u}^{n-1}\cdot\mathbf{v}\,d\Omega \quad \forall\mathbf{v}\in\mathbf{H}_0^1(\Omega),\end{aligned} \tag{5.23}$$

$$\int_\Omega[\nabla\phi^n - (\mathbf{u}^n\times\mathbf{B})]\cdot(\nabla\psi)d\Omega = \int_\Omega k^n\,\psi\,d\Omega \quad \forall\psi\in\widetilde{H}^1(\Omega), \tag{5.24}$$

$$\begin{aligned}&\tfrac{1}{\Delta t}\int_\Omega T^n\cdot\theta\,d\Omega + \int_\Omega \nabla T^n\cdot\nabla\theta\,d\Omega + \int_\Omega \mathbf{u}^n\cdot\nabla T^n\theta\,d\Omega\\ &\qquad= \int_\Omega k_2\theta\,d\Omega + \int_\Gamma g^n\theta\,d\Gamma + \tfrac{1}{\Delta t}\int_\Omega T^{n-1}\cdot\theta\,d\Omega \quad \forall\theta\in\widetilde{H}^1(\Omega),\end{aligned} \tag{5.25}$$

and

$$\int_\Omega q\,\nabla\cdot\mathbf{u}^n d\Omega = 0 \quad \forall q\in L_0^2(\Omega). \tag{5.26}$$

Here the colon notation stands for the scalar product on $\mathbf{R}^{d\times d}$.

The precise mathematical statement of the optimal control problem (5.22) can now be given as follows:

$$\text{seek a } (\mathbf{u}^n, p^n, \phi^n, T^n, g^n) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \widetilde{H}^1(\Omega) \times \widetilde{H}^1(\Omega) \times L^2(\Gamma) \text{ such that the functional (5.22) is minimized subject to the constraints (5.23)-(5.26).} \tag{5.27}$$

We will turn the constrained optimization problem (5.27) into an unconstrained one by using Lagrange multiplier principles. We set $\mathcal{X} = \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \widetilde{H}^1(\Omega) \times L^2(\Gamma) \times \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \widetilde{H}^1(\Omega)$ and define the Lagrangian functional

$$\begin{aligned}
&\mathcal{L}(\mathbf{u}^n, p^n, \phi^n, T^n, g^n, \mu^n, \tau^n, s^n) \\
&= \mathcal{F}(\mathbf{u}^n, p^n, \phi^n, T^n) + \frac{\delta}{2}\int_\Gamma |g^n|^2 d\Gamma - \frac{1}{\Delta t}\int_\Omega \mathbf{u}^n \cdot \mu^n d\Omega - \frac{1}{M^2}\int_\Omega \nabla \mathbf{u}^n : \nabla \mu^n d\Omega \\
&\quad + \int_\Omega [\nabla \phi^n - (\mathbf{u}^n \times \mathbf{B})] \cdot (\mu^n \times \mathbf{B}) d\Omega - \frac{1}{N}\int_\Omega (\mathbf{u}^n \cdot \nabla)\mathbf{u}^n \cdot \mu^n d\Omega \\
&\quad + \int_\Omega p^n \nabla \cdot \mu^n d\Omega + \int_\Omega \mathbf{f}^n \cdot \mu^n d\Omega + \frac{1}{\Delta t}\int_\Omega \mathbf{u}^{n-1} \cdot \mu^n d\Omega \\
&\quad - \int_\Omega [\nabla \phi^n - (\mathbf{u}^n \times \mathbf{B})] \cdot (\nabla s^n) d\Omega + \int_\Omega k^n s^n d\Omega + \int_\Gamma g^n s^n d\Gamma - \frac{1}{\Delta t}\int_\Omega T^n t^n d\Omega \\
&\quad + \int_\Omega \tau^n \nabla \cdot \mathbf{u}^n d\Omega - \int_\Omega \nabla T^n \cdot \nabla t^n d\Omega - \int_\Omega \mathbf{u}^n \cdot \nabla T^n t d\Omega + \frac{1}{\Delta t}\int_\Omega T^{n-1} t^n d\Omega \\
&\quad + \int_\Omega k_2^n t^n d\Omega + \int_\Gamma g^n t^n d\Gamma + \frac{1}{L}\int_\Omega \mathbf{g} \cdot \mu^n T^n d\Omega \\
&\qquad \forall (\mathbf{u}^n, p^n, \phi^n, T^n, g^n, \mu^n, \tau^n, s^n, t^n) \in \mathcal{X}.
\end{aligned} \tag{5.28}$$

An optimality system of equations that an optimum must satisfy is derived by taking variations with respect to every variable in the Lagrangian. By taking variations with respect to $\mathbf{u}^n$, p^n, T^n and ϕ^n, we obtain:

$$\begin{aligned}
&\frac{1}{\Delta t}\int_\Omega \mu^n \cdot \mathbf{w} d\Omega + \frac{1}{M^2}\int_\Omega \nabla \mu^n : \nabla \mathbf{w} d\Omega + \int_\Omega (\mathbf{w} \times \mathbf{B}) \cdot (\mu^n \times \mathbf{B}) d\Omega \\
&\quad + \frac{1}{N}\int_\Omega (\mathbf{u}^n \cdot \nabla)\mathbf{w} \cdot \mu^n d\Omega + \frac{1}{N}\int_\Omega (\mathbf{w} \cdot \nabla)\mathbf{u}^n \cdot \mu^n d\Omega - \int_\Omega \tau^n \nabla \cdot \mathbf{w} d\Omega \\
&= \langle \mathcal{F}_{\mathbf{u}^n}(\mathbf{u}^n, p^n, \phi^n), T^n, \mathbf{w} \rangle \quad \forall \mathbf{w} \in \mathbf{H}_0^1(\Omega),
\end{aligned} \tag{5.29}$$

$$\int_\Omega [\nabla s^n - (\mu^n \times \mathbf{B})] \cdot (\nabla r) d\Omega = \langle \mathcal{F}_{\phi^n}(\mathbf{u}^n, p^n, \phi^n), T^n, r \rangle \quad \forall r \in \widetilde{H}^1(\Omega), \tag{5.30}$$

$$\int_\Omega \sigma \nabla \cdot \mu^n d\Omega = \langle \mathcal{F}_{p^n}(\mathbf{u}^n, p^n, \phi^n, T^n), \sigma \rangle \quad \forall \sigma \in L_0^2(\Omega) \tag{5.31}$$

and

$$\frac{1}{\Delta t}\int_\Omega t^n l d\Omega + \int_\Omega \nabla t^n \cdot \nabla l d\Omega + \int_\Omega \mathbf{u}^n \cdot \nabla l t^n d\Omega - \frac{1}{L}\int_\Omega \mathbf{g}\cdot \mu^n l d\Omega$$
$$= \langle \mathcal{F}_{T^n}(\mathbf{u}^n, p^n, \phi^n, T^n), l\rangle \quad \forall l \in \widetilde{H}^1(\Omega), \qquad (5.32)$$

where $\mathcal{F}_{\mathbf{u}^n}$, $\mathcal{F}_{\phi^n}$, $\mathcal{F}_{T^n}$ and $\mathcal{F}_{p^n}$ are the derivatives of the functional with respect to its three arguments, respectively. By taking variations with respect to μ^n, τ^n, t^n and ξ^n, we recover the constraint equations (5.23)-(5.26). By taking variation with respect to g^n we obtain

$$\int_\Gamma (\delta g^n z + z t^n) d\Gamma = 0 \quad \forall z \in L^2(\Gamma) \qquad \text{i.e.,} \qquad g^n = -\frac{1}{\delta} s^n .$$

This last equation enables us to eliminate the control g^n in (5.25). Thus (5.23)-(5.26) can be replaced by

$$\frac{1}{\Delta t}\int_\Omega \mathbf{u}^n \cdot \mathbf{v} d\Omega + \frac{1}{M^2}\int_\Omega \nabla \mathbf{u}^n : \nabla \mathbf{v} d\Omega - \int_\Omega [\nabla \phi^n - (\mathbf{u}^n \times \mathbf{B})]\cdot(\mathbf{v}\times\mathbf{B})d\Omega - \int_\Omega p^n \nabla\cdot \mathbf{v} d\Omega$$
$$+\frac{1}{N}\int_\Omega (\mathbf{u}^n\cdot\nabla)\mathbf{u}^n \cdot \mathbf{v} d\Omega - \frac{1}{L}\int_\Omega \mathbf{g}\cdot \mathbf{v} T^n d\Omega$$
$$= \int_\Omega \mathbf{f}^n \cdot \mathbf{v} d\Omega + \frac{1}{\Delta t}\int_\Omega \mathbf{u}^{n-1}\cdot \mathbf{v} d\Omega \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \qquad (5.33)$$

$$\int_\Omega [\nabla\phi^n - (\mathbf{u}^n \times \mathbf{B})]\cdot(\nabla\psi)d\Omega + \frac{1}{\delta}\int_\Gamma s^n \psi d\Omega = \int_\Omega k_1^n \psi d\Omega \quad \forall \psi \in \widetilde{H}^1(\Omega), \qquad (5.34)$$

$$\int_\Omega q \nabla\cdot \mathbf{u}^n d\Omega = 0 \quad \forall q \in L_0^2(\Omega) \qquad (5.35)$$

and

$$\frac{1}{\Delta t}\int_\Omega T^n \cdot \theta d\Omega + \int_\Omega \nabla T^n \cdot \nabla\theta d\Omega + \int_\Omega \mathbf{u}^n \cdot \nabla T^n \theta d\Omega + \frac{1}{\delta}\int_\Omega s^n \theta d\Gamma$$
$$= \int_\Omega k_2 \theta d\Omega + \frac{1}{\Delta t}\int_\Omega T^{n-1}\cdot\theta d\Omega \quad \forall \theta \in \widetilde{H}^1(\Omega), \qquad (5.36)$$

Equations (5.29)-(5.36) forms an optimality system of equations that an optimal solution must satisfy.

Finite element approximations and the computational methods, Newton's and iterative, presented in Chapter 2 for the steady state case can be extended here in a natural fashion.

5.3 Computational Examples

In this section, we report some computational examples that serve to illustrate the effectiveness and practicality of optimal control techniques in electrically conducting fluids with heat flux controls. First, we treat the problem of steering the velocity field to a desired one. The second one deals with the matching the temperature field to a desired one. Finally, we consider the problem of minimizing the temperature gradient throughout the domain.

All computations are done with the same choice of finite element spaces as in Chapter 2.

5.3.1 Velocity field matching

The first case we consider is the problem of minimizing (5.19) subject to (5.23)–(5.26), i.e., we attempt to match the velocity field with a desired one by finding an appropriate boundary current density g.

The optimality system of equations are given by (5.31)–(5.38) with

$$\langle \mathcal{F}_{\mathbf{u}^n}(\mathbf{u}^n, p^n, \phi^n, T^n), \mathbf{w}\rangle = \frac{1}{\epsilon}\int_\Omega (\mathbf{u}^n - \mathbf{u}_d^n)\cdot \mathbf{w}\, d\Omega\,,$$

$$\langle \mathcal{F}_{p^n}(\mathbf{u}^n, p^n, \phi^n, T^n), \sigma\rangle = \langle \mathcal{F}_{\phi^n}(\mathbf{u}^n, p^n, \phi^n, T^n), r\rangle = \langle \mathcal{F}_{T^n}(\mathbf{u}^n, p^n, \phi^n, T^n), l\rangle = 0\,.$$

The corresponding system of partial differential equations for (5.29)–(5.36) is given by (5.1)–(5.9),

$$\frac{\partial T^n}{\partial n} = -\frac{1}{\delta} t^n \qquad \text{on } \Gamma\,,$$

$$\frac{1}{\Delta t}\mu^n - \frac{1}{M^2}\Delta\mu^n + \frac{1}{N}\mu^n\cdot(\nabla\mathbf{u}^n)^T - \frac{1}{N}(\mathbf{u}^n\cdot\nabla)\mu^n + \nabla\tau^n$$
$$- \mathbf{B}\times(\nabla s^n) - (\mu^n\times\mathbf{B})\times\mathbf{B} - \frac{1}{\epsilon}\mathbf{u}^n = -\frac{1}{\epsilon}\mathbf{u}_d^n \qquad \text{in } \Omega,$$

$$\nabla\cdot\mu^n = 0 \qquad \text{in } \Omega,$$

$$-\Delta s^n + \nabla\cdot(\mu^n\times\mathbf{B}) = 0 \qquad \text{in } \Omega,$$

$$\frac{1}{\Delta t}t^n - \Delta t^n - \mathbf{u}^n\cdot\nabla t^n - \frac{1}{L}\mathbf{g}\cdot\mu^n = 0 \qquad \text{in } \Omega,$$

$$\mu^n = \mathbf{0} \qquad \text{on } \Gamma\,,$$

$$\frac{\partial t^n}{\partial n} = 0 \qquad \text{on } \Gamma$$

and

$$\frac{\partial s^n}{\partial n} = 0 \qquad \text{on } \Gamma$$

We now present some numerical results for the following choice of parameters and data:
The Hartmann number and interaction number: $N = 1$, $M = 100$;
the domain Ω is the unit square $(0,1) \times (0,1)$;
applied magnetic field: $\mathbf{B} = (0, 0, \sin(\pi y))^T$;
desired velocity field:

$$\mathbf{u}_d = \begin{pmatrix} [1 - \cos(2\pi xt)] \cos(2\pi yt) \\ \sin(2\pi xt) \sin(2\pi yt) \end{pmatrix},$$

initial conditions:

$$\hat{\mathbf{u}} = \begin{pmatrix} [\cos(2\pi x) - 1] \sin(2\pi y) \\ \sin(2\pi x) [1 - \cos(2\pi y)] \end{pmatrix},$$

and

$$\hat{T} = \cos(2\pi x) \cos(2\pi y)$$

With these data, the body force $\mathbf{f} = (f_1, f_2)^T$, electric source k_1 and the heat source k_2 are selected such that exact solution of the uncontrolled problem, i.e., the solution for (5.1)-(5.9) with $g = 0$, is given by

$$\mathbf{u}_0 = \begin{pmatrix} \exp(\frac{-t}{M})[\cos(2\pi x) - 1] \sin(2\pi y) \\ \exp(\frac{-t}{M}) \sin(2\pi x) [1 - \cos(2\pi y)] \end{pmatrix},$$

$$p_0 = C, \qquad \text{where C is a constant},$$

$$T_0 = \exp(-t) \cos(2\pi x) \cos(2\pi y)$$

and

$$\phi_0 = \exp(-t) \cos(\pi x) \cos(\pi y),$$

The proper choice of the constants ϵ and δ in the functional plays an important role in obtaining a best velocity matching. For the computational results shown in Figures 5.1 below, our choice of these two constants were $\epsilon = 0.00001$ and $\delta = 0.01$. The time step $\Delta t = 0.2$.

We give a brief description of the figures. Figures 5.1 a), g), m), s), and y) are the uncontrolled velocity field $\mathbf{u}_0$ at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Figures 5.1 b), h), n), t) and z) are the desired velocity field $\mathbf{u}_d$ at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Figures 5.1 c), i), o), u) and z1) are the optimal velocity field u^h at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Figures 5.1 d), j), p), v) and z2) are the adjoint temperature s^h at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively; those were obtained by solving (5.31)–(5.38). The optimal control g^h can be gleaned from Figure 5.1 d), j), p), v) and z2) and the relation $g^h = -\frac{1}{\delta}t^h$.

All the computational results shown in Figures 5.1 were obtained with a 10 by 10 triangulation of the unit square. A nonuniform grid with corner refinements was used. We see from the figures that optimal control does a very good job in matching the desired velocity field at $t = 1$.

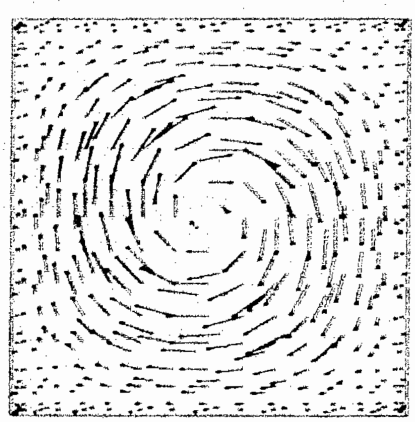

Figure 5.1 a) uncontrolled velocity $\mathbf{u}_0$ at $t = 0.2$

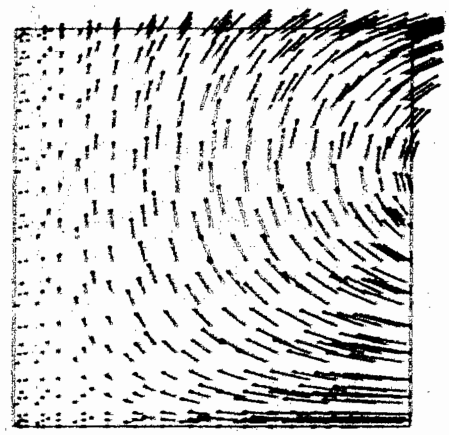

Figure 5.1 b) desired velocity $\mathbf{u}_d$ at $t = 0.2$

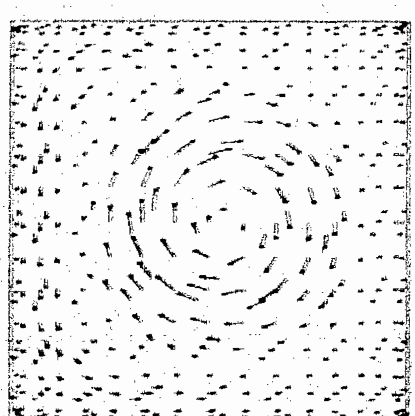

Figure 5.1 c) optimal velocity $\mathbf{u}^h$ at $t = 0.2$

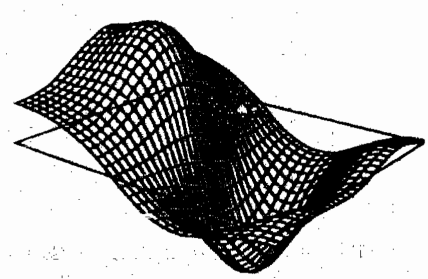

Figure 5.1 d) adjoint temperature field t^h at $t = 0.2$

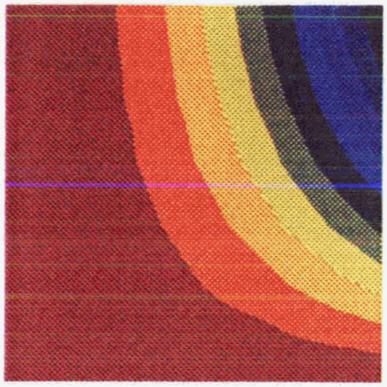

Figure 4.1 e) contours of desired velocity at $t = 0.2$

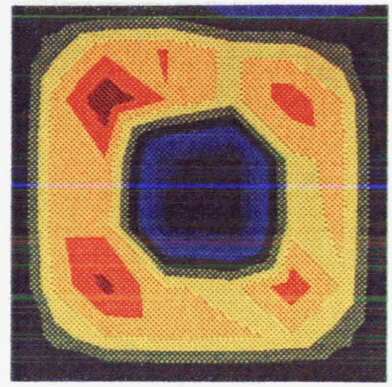

Figure 4.1 f) contours of optimal velocity field at $t = 0.2$

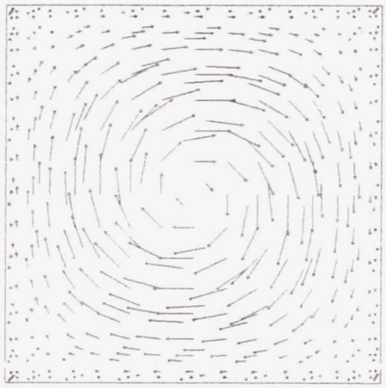

Figure 5.1 g) uncontrolled velocity $\mathbf{u}_0$ at $t = 0.4$

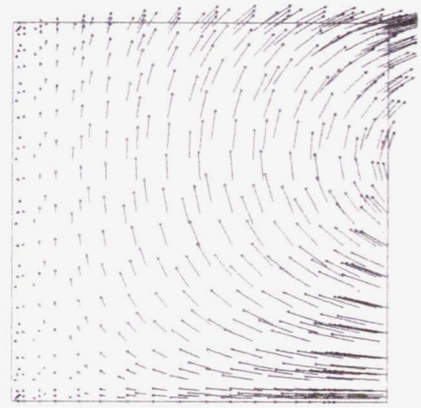

Figure 5.1 h) desired velocity $\mathbf{u}_d$ at $t = 0.4$

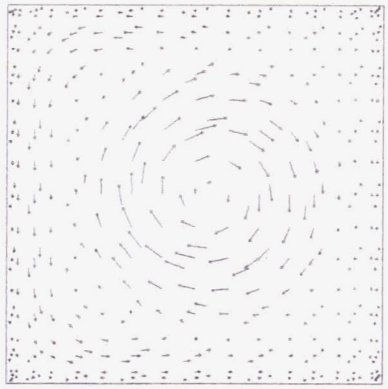

Figure 5.1 i) optimal velocity $\mathbf{u}^h$ at $t = 0.4$

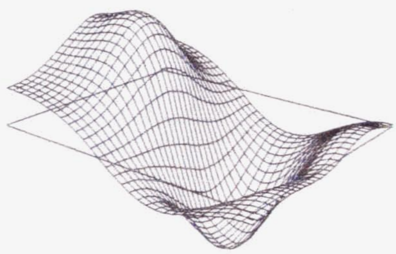

Figure 5.1 j) adjoint temperature t^h at $t = 0.4$

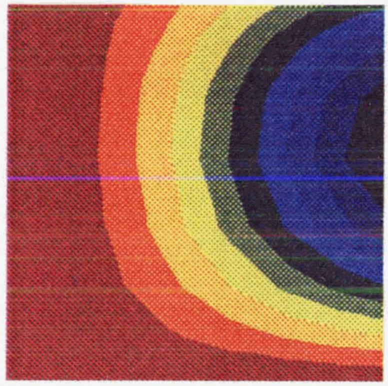

Figure 4.1 k) contours of desired velocity at $t = 0.4$

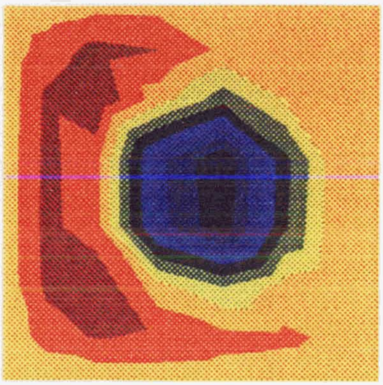

Figure 4.1 l) contours of optimal velocity field at $t = 0.4$

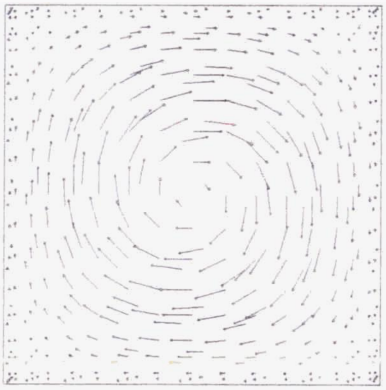

Figure 5.1 m) uncontrolled velocity $\mathbf{u}_0$ at $t = 0.6$

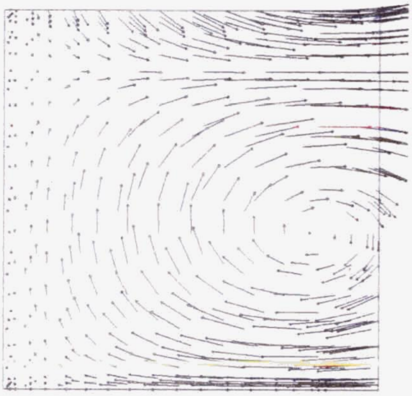

Figure 5.1 n) desired velocity $\mathbf{u}_d$ at $t = 0.6$

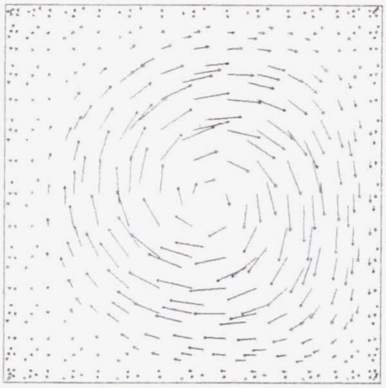

Figure 5.1 o) optimal velocity $\mathbf{u}^h$ at $t = 0.6$

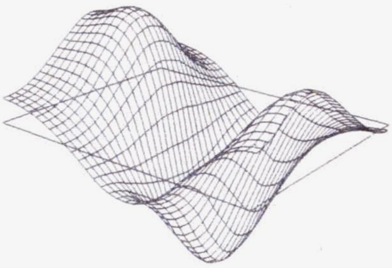

Figure 5.1 p) adjoint temperature t^h at $t = 0.6$

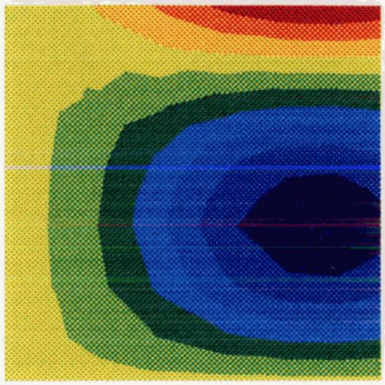

Figure 4.1 q) contours of desired velocity at $t = 0.6$

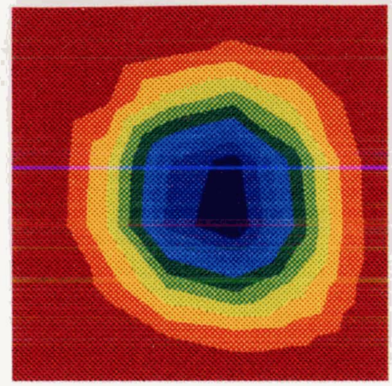

Figure 4.1 r) contours of optimal velocity field at $t = 0.6$

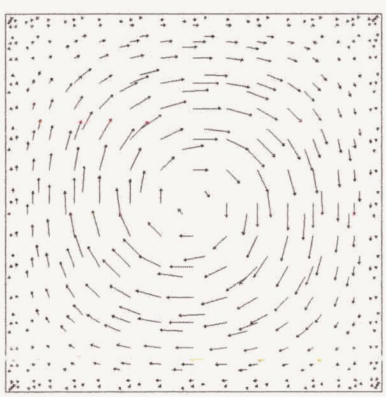

Figure 5.1 s) uncontrolled velocity $\mathbf{u}_0$ at $t = 0.8$

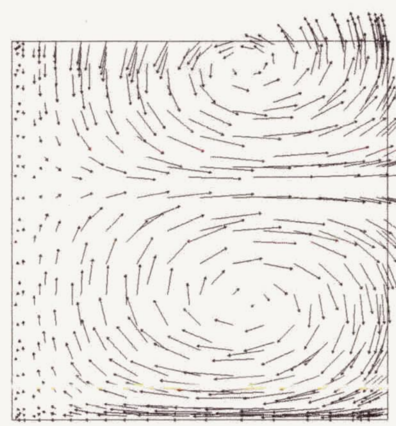

Figure 5.1 t) desired velocity $\mathbf{u}_d$ at $t = 0.8$

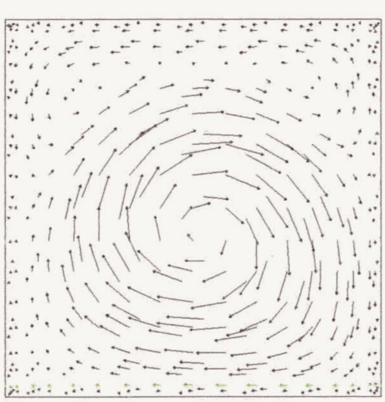

Figure 5.1 u) optimal velocity $\mathbf{u}^h$ at $t = 0.8$

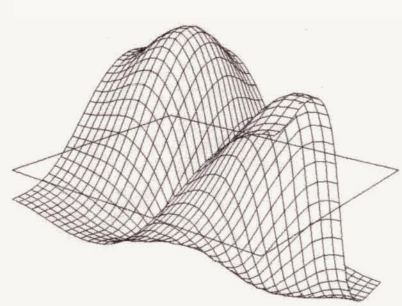

Figure 5.1 v) adjoint temperature t^h at $t = 0.8$

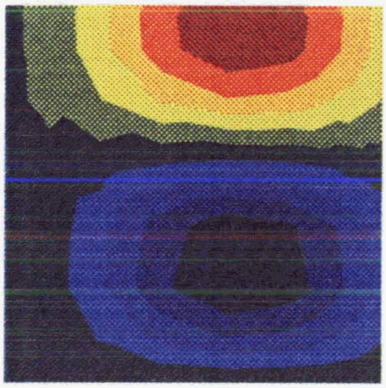

Figure 4.1 w) contours of desired velocity at $t = 0.8$

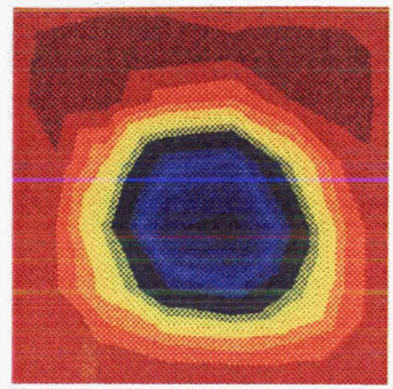

Figure 4.1 x) contours of optimal velocity field at $t = 0.8$

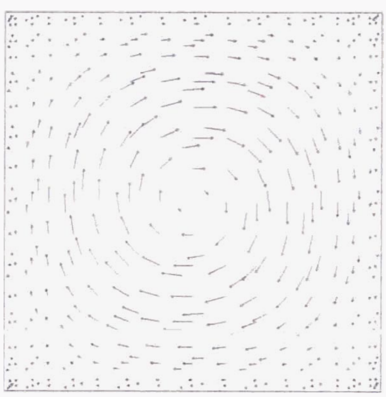

Figure 5.1 y) uncontrolled velocity $\mathbf{u}_0$ at $t = 1.0$

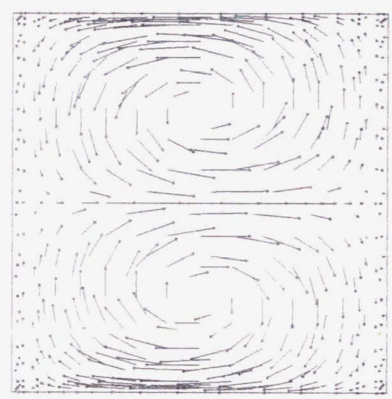

Figure 5.1 z) desired velocity $\mathbf{u}_d$ at $t = 1.0$

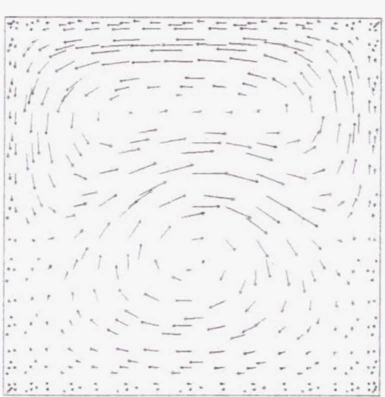

Figure 5.1 z1) optimal velocity $\mathbf{u}^h$ at $t = 1.0$

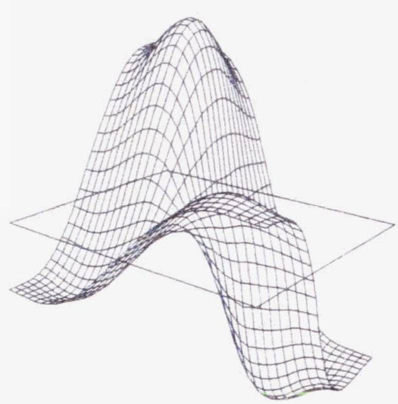

Figure 5.1 z2) adjoint temperature t^h at $t = 1.0$

Figure 4.1 z3) contours of desired velocity at $t = 1.0$

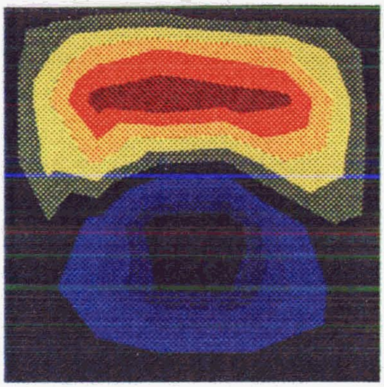

Figure 4.1 z4) contours of optimal velocity field at $t = 1.0$

5.3.2 Temperature field matching

The second case we consider is the problem of matching the temperature T to a desired distribution T_d, i.e., we minimize functional (5.20) subject to (5.23)–(5.26).

The optimality system of equations are given by (5.29)–(5.36) with

$$\langle \mathcal{F}_{T^n}(\mathbf{u}^n, p^n, \phi^n, T^n), l \rangle = \frac{1}{\epsilon} \int_\Omega (T^n - T_d^n)\, l d\Omega$$

and

$$\langle \mathcal{F}_{\mathbf{u}^n}(\mathbf{u}^n, p^n, \phi^n, T^n), \mathbf{w} \rangle = \langle \mathcal{F}_{p^n}(\mathbf{u}^n, p^n, \phi^n, T^n), \sigma \rangle = \langle \mathcal{F}_{\phi^n}(\mathbf{u}^n, p^n, \phi^n, T^n), \sigma \rangle = 0\,.$$

The domain is chosen to be a unit square. The Hartmann number M, interaction number N, body force $\mathbf{f}$, electric source k_1, heat source k_2 and applied magnetic field $\mathbf{B}$ are chosen to be the same as in Section 5.3.1.

The desired temperature field is taken as a function in space and time which approaches a uniform value one in unit time :

$$T_d = (1-t)\cos(2\pi x)\cos(2\pi y)\exp(-t) + t\,.$$

The two parameters in the functional are chosen as $\epsilon = 0.002$ and $\delta = 0.1$. The time step $\Delta t = 0.2$ For these data, the solution $(\mathbf{u}_0, p_0, \phi_0, T_0)$ of the uncontrolled problem, i.e., the solution for (5.1)–(5.9) with $g = 0$, is the same as in Section 5.3.1.

Some numerical results for this example is reported in Figures 5.2 a)-t). We give a brief description of the figures. Figures 5.2 a), e), i), m), and q) are the uncontrolled velocity field $\mathbf{u}_0$ at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Figures 5.2 b) f), j), n) and r) are the desired velocity field $\mathbf{u}^h$ at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. figures 5.2 c), g), k), o) and s) are the optimal velocity fields at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Figures 5.2 d), h), l), p) and t) are the adjoint temperature s^h at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Those were obtained by solving (5.29)-(5.36). The optimal control g^h can be gleaned from figures 5.2 d), h), l), p) and t) and the relation $g^h = -\frac{1}{\delta}t^h$. A look at Figure 5.2 r) and s) reveals that the optimal temperature matches well with the desired temperature T_d.

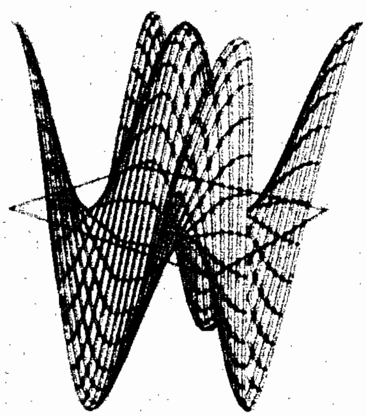

Figure 5.2 a) uncontrolled temperature T_0 at $t = 0.2$

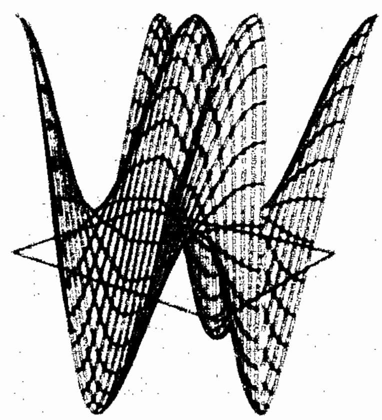

Figure 5.2 b) desired temperature T_d at $t = 0.2$

Figure 5.2 c) optimal temperature T^h at $t = 0.2$

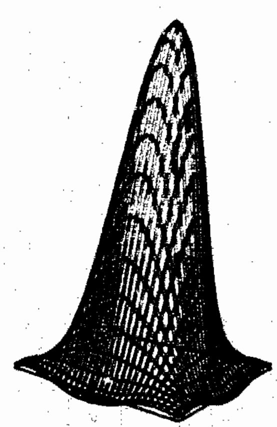

Figure 5.2 d) adjoint temperature t^h at $t = 0.2$

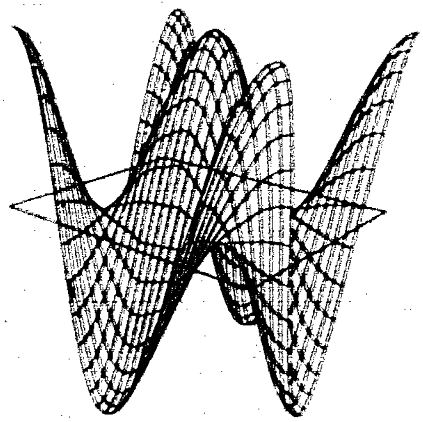

Figure 5.2 e) uncontrolled temperature T_0 at $t = 0.4$

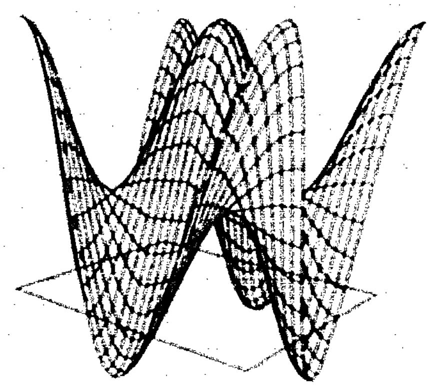

Figure 5.2 f) desired temperature T_d at $t = 0.4$

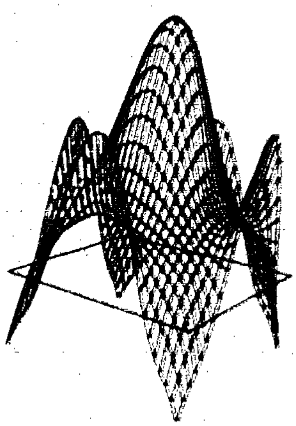

Figure 5.2 g) optimal temperature T^h at $t = 0.4$

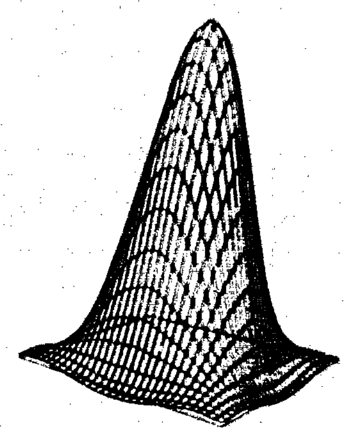

Figure 5.2 h) adjoint temperature t^h at $t = 0.4$

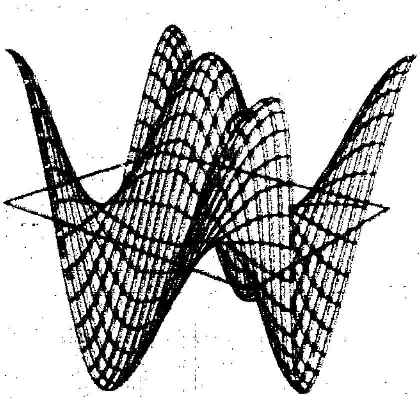

Figure 5.2 i) uncontrolled temperature T_0 at $t = 0.6$

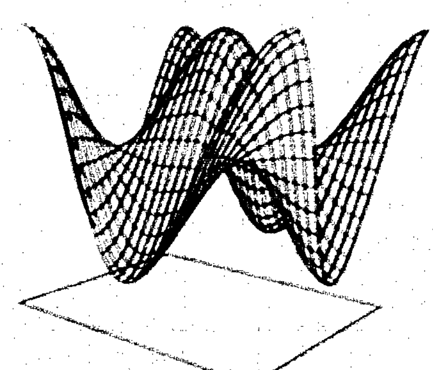

Figure 5.2 j) desired temperature T_d at $t = 0.6$

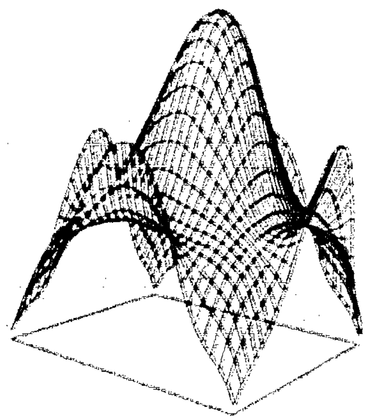

Figure 5.2 k) optimal temperature T^h at $t = 0.6$

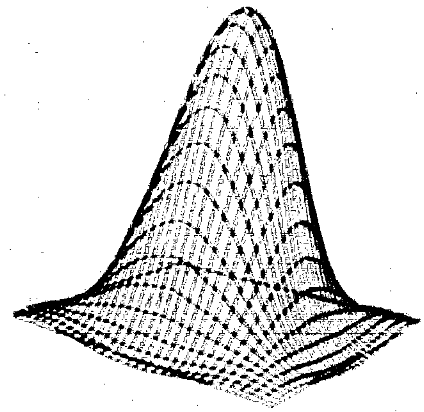

Figure 5.2 l) adjoint temperature t^h at $t = 0.6$

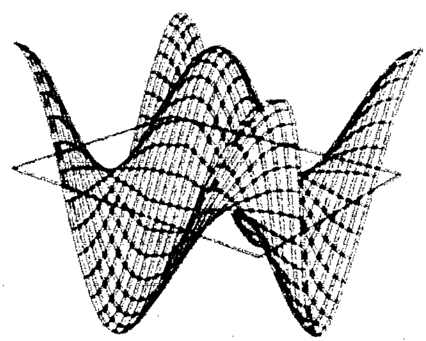

Figure 5.2 m) uncontrolled temperature T_0 at $t = 0.8$

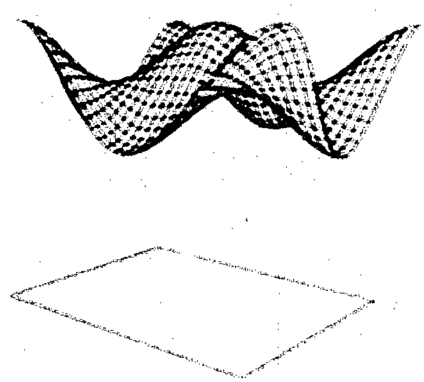

Figure 5.2 n) desired temperature T_d at $t = 0.8$

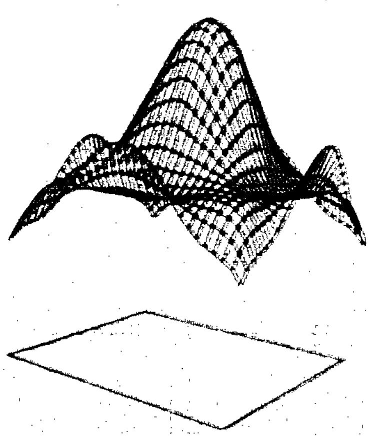

Figure 5.2 o) optimal temperature T^h at $t = 0.8$

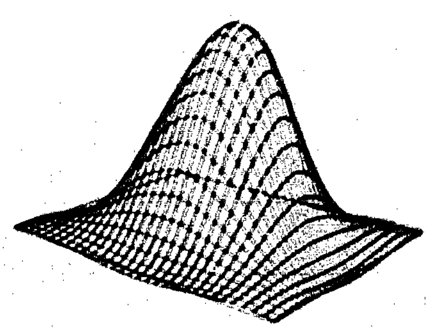

Figure 5.2 p) adjoint temperature t^h at $t = 0.8$

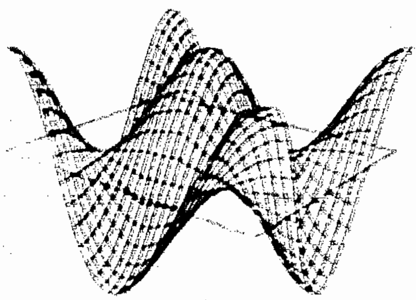

Figure 5.2 q) uncontrolled temperature T_0 at $t = 1.0$

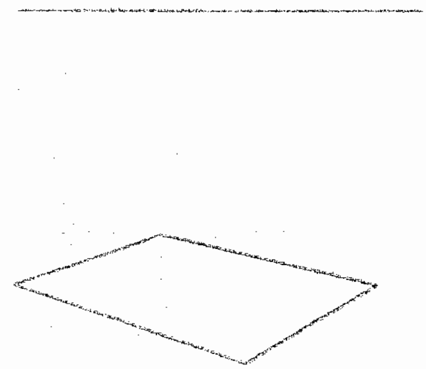

Figure 5.2 r) desired temperature T_d at $t = 1.0$

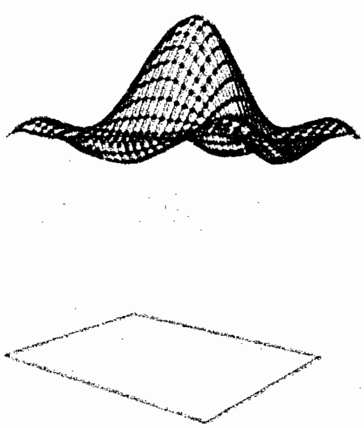

Figure 5.2 s) optimal temperature T^h at $t = 1.0$

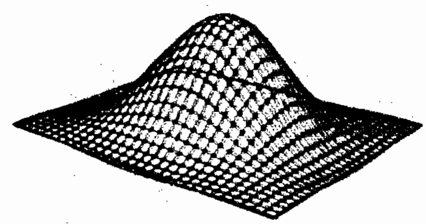

Figure 5.2 t) adjoint temperature t^h at $t = 1.0$

5.3.3 Temperature gradient minimization

The third case we consider is the problem of minimizing the temperature gradient, i.e., we minimize functional (5.21) subject to (5.23)–(5.26).

The optimality system of equations are given by (5.31)–(5.38) with

$$\langle \mathcal{F}_{T^n}(\mathbf{u}^n, p^n, \phi^n, T^n), l \rangle = \frac{1}{\epsilon} \int_\Omega \nabla T^n \cdot \nabla l d\Omega$$

and

$$\langle \mathcal{F}_{\mathbf{u}^n}(\mathbf{u}^n, p^n, \phi^n, T^n), \mathbf{w} \rangle = \langle \mathcal{F}_{p^n}(\mathbf{u}^n, p^n, \phi^n, T^n), \sigma \rangle = 0\,.$$

The domain is chosen to be a unit square. The Hartmann number M, interaction number N, body force $\mathbf{f}$, electric source k and applied magnetic field $\mathbf{B}$ are chosen to be the same as in Section 5.3.1.

The two parameters in the functional are chosen as $\epsilon = 0.0002$ and $\delta = 1$. The time step $\Delta = 0.2$. For these data, the exact solution $(\mathbf{u}_0, p_0, \phi_0)$ of the uncontrolled problem, i.e., the solution for (5.1)-(5.9) with $g = 0$, is given by the same uncontrolled solution as in Section 5.3.1.

Some numerical results for this example is reported in figures 5.3 a)-o). We give a brief description of the figures. Figures 5.3 a), d), g), j), and m) are the uncontrolled temperature field T_0 at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Figures 5.3 b), e), h), k) and n) are the optimal temperature field T^h at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Figures 5.3 c), f), i), l) and o) are the adjoint temperature field t^h at t=0.2, 0.4, 0.6, 0.8, 1.0, respectively. Those were obtained by solving (5.29)-(5.36). The optimal control g^h can be gleaned from figures 5.3 c), f), i), l) and o) and the relation $g^h = -\frac{1}{\delta}t^h$.

By minimizing functional (5.21) we wish to obtain a quasi-uniform temperature distribution. The numerical results (in particular, figures 5.3 b), e), h), k) and n) demonstrate that the optimal control did a very good job in achieving the objective.

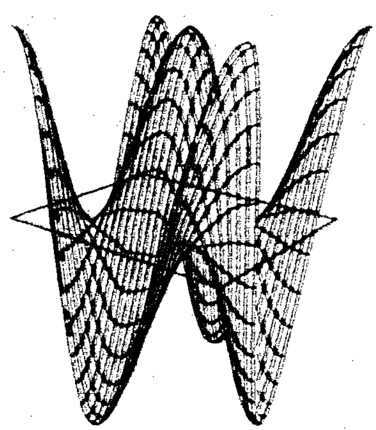

Figure 5.3 a) uncontrolled temperature T_0 at $t = 0.2$

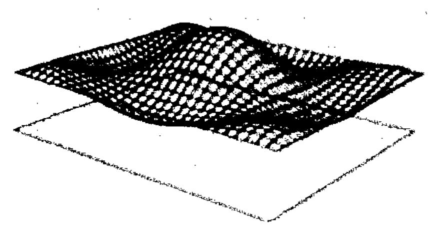

Figure 5.3 b) optimal temperature T^h at $t = 0.2$

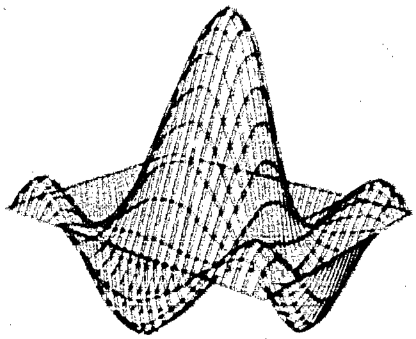

Figure 5.3 c) adjoint temperature t^h at $t = 0.2$

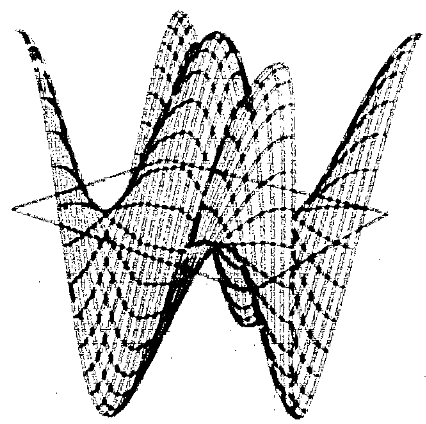

Figure 5.3 d) uncontrolled temperature T_0 at $t = 0.4$

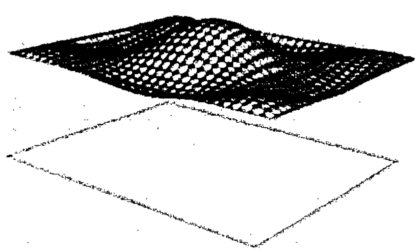

Figure 5.3 e) optimal temperature T^h at $t = 0.4$

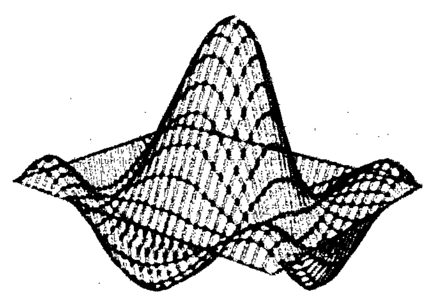

Figure 5.3 f) adjoint temperature t^h at $t = 0.4$

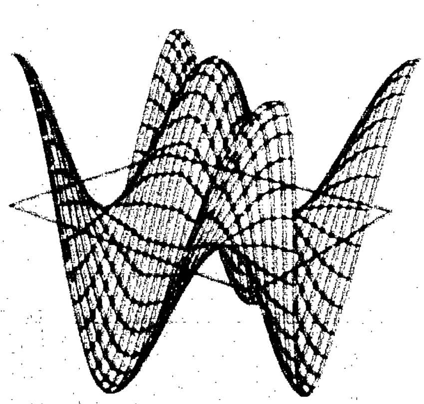

Figure 5.3 g) uncontrolled temperature T_0 at $t = 0.6$

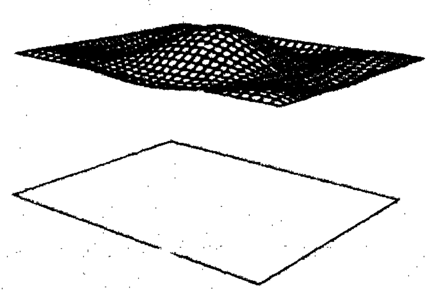

Figure 5.3 h) optimal temperature T^h at $t = 0.6$

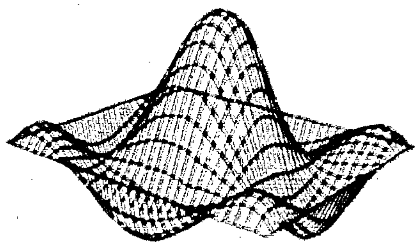

Figure 5.3 i) adjoint temperature t^h at $t = 0.6$

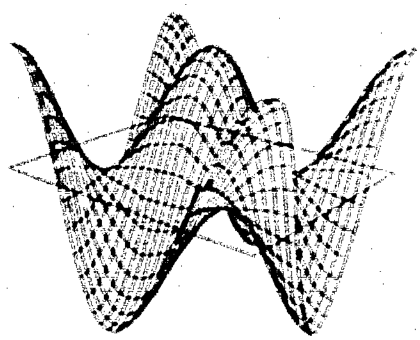

Figure 5.3 j) uncontrolled temperature T_0 at $t = 0.8$

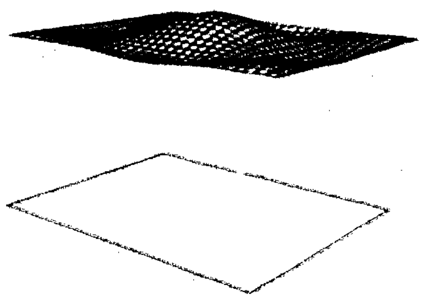

Figure 5.3 k) optimal temperature T^h at $t = 0.8$

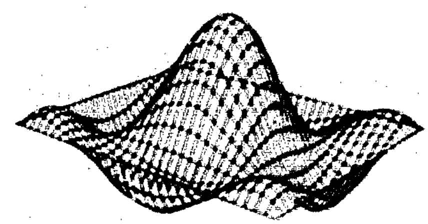

Figure 5.3 l) adjoint temperature t^h at $t = 0.8$

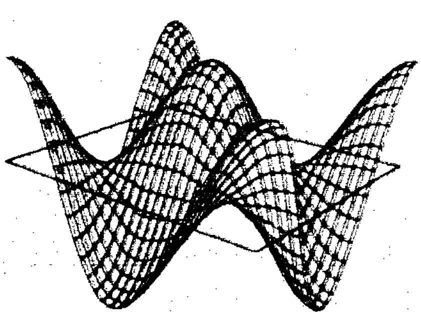

Figure 5.3 m) uncontrolled temperature T_0 at $t = 1.0$

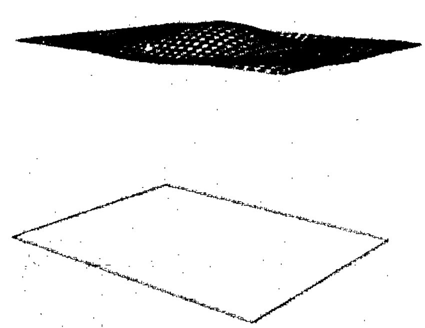

Figure 5.3 n) optimal temperature T^h at $t = 1.0$

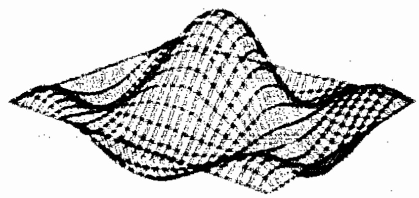

Figure 5.3 c) adjoint temperature t^* at $t = 1.0$

5.4 Concluding Remarks.

In this chapter we studied numerical computation of boundary control problems for an unsteady electrically conducting fluid using heat flux controls. We summarize the main points in this Chapter as follows:

- We further tested the computational procedure described in Chapter 4 for unsteady control problems considered here.
- We converted the optimal control problem into a system of equations (i.e., the optimality system of equations) by using the Lagrange multiplier principles;
- We proposed some methods for solving the discrete optimality system of equations. Our discussions of these methods were made for finite element discretizations. Apparently, these methods are equally applicable to finite difference, collocation or pseudo-spectral discretizations;
- We numerical experiments further confirms the effectiveness of our techniques in flow field matchings and in the minimization of some physical quantities.

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