

CONNECTIVITY IN VERTEX-TRANSITIVE GRAPHS

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CHEN, Tai-Yu

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APPROVAL

Name: CHEN, Tai-Yu
Degree: Master of Science
Title of thesis: CONNECTIVITY IN VERTEX-TRANSITIVE GRAPHS

Examining Committee: Dr. S.K. Thomason
Chair

Dr. B.R. Alspach, Senior Supervisor

Dr. K. Heinrich

Dr. L. Goddyn

Dr. J. Peters, External Examiner
Department of Computing Science
Simon Fraser University

Date Approved:

July 22, 1994

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Connectivity in Vertex-Transitive Graphs

Author:

(signature)

CHEN, Tai-Yu
(name)

July 20, 1994
(date)

Abstract

We are interested in determining the vertex-connectivity and the edge-connectivity of vertex-transitive graphs. We show easily that the edge-connectivity is simply the degree of the graph. Hence, the vertex-connectivity of vertex-transitive graphs is our main interest. In 1968, M. Watkins characterized vertex-transitive graphs by means of atomic parts which play a crucial role in determining the vertex-connectivity of these graphs. We study the structure of vertex-transitive graphs, in particular, Cayley graphs. The vertex-connectivity of Cayley graphs with minimal generating sets was easily characterized in a paper written by C. Godsil. Later, B. Alspach generalized Godsil's result to Cayley graphs with quasi-minimal generating set. Finally, we look at an application of atomic parts for an algorithm to determine the vertex-transitivity of circulant graphs — a special subfamily of Cayley graphs. F. Boesch and R. Tindell in 1984, and independently Watkins in 1985, gave algorithms for finding the vertex-connectivity of circulant graphs.

Dedication

To my wonderful parents, Tsung-Yao Chen and Ming-Chu Pong, and my wife Jennifer I-Chun Chen.

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Chapter 1

Introduction

Let G be a graph. The vertex-connectivity $\kappa(G)$ is the minimum number of vertices whose deletion will disconnect G or leave a single vertex, and the edge-connectivity $\kappa_1(G)$ is the minimum number of edges whose deletion will disconnect G . One can easily see that $\kappa(G) \leq \kappa_1(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree in G . However, the difference between $\kappa(G)$ and $\kappa_1(G)$ can be made arbitrarily large. To show this, consider the graph G which is formed by taking two copies of K_n and identifying one vertex from each copy. The identified vertex is a cut-vertex which implies that $\kappa(G) = 1$. However, it is obvious that $\kappa_1(G) = n - 1$.

Given any graph, a natural question to ask is how well connected it is. Connectivity is an extensively studied topic in graph theory, and many important results in graph theory are related to it. One well-known result about connectivity is Menger's Theorem which relates connectivity with vertex-disjoint or edge-disjoint paths. Many other people also proved a lot of theorems about vertex-connectivity and edge-connectivity. However, it is still not easy to determine them for an arbitrary graph. This problem has been widely studied because of its many applications.

The notion of connectivity plays a major role in the reliability of networks modeled by graphs. A concept related to networks is the notion of the fault tolerance of a network. The fault tolerance of a network is the maximum number k such that if any k stations (or nodes) fail simultaneously, the transmission of information between the remaining stations would **not be interrupted**. Hence, in graph theoretic terms,

the fault tolerance of a graph is the maximum number of vertices whose deletion will not disconnect the graph. It is obvious that in any graph, k must be less than the vertex-connectivity unless the graph is complete, in which case they are the same. We say that a network is more reliable the larger its fault tolerance. So to find reliable networks means we want to find graphs with large κ or κ_1 . One family of graphs which deserves special attention is the family of regular graphs.

A graph is *regular* if the degrees of its vertices are all the same. We hope that this family of graphs will yield large vertex-connectivity and large edge-connectivity compared to the degrees of the graphs. The following example will show that the difference between the two connectivities can also be made arbitrarily large. Take two copies of K_n , where n is even. Let u and v be two new vertices. We form a new graph G by connecting u to half of the vertices in each copy of K_n and connecting v to the other vertices in the two copies of K_n . It is easy to see that $\kappa(G) = 2$ and $\kappa_1(G) = n$. Hence we consider a smaller set of regular graphs — vertex-transitive graphs.

Vertex-transitive graphs (sometimes called point-symmetric graphs) are often considered in designing networks because of their symmetry. When designing networks, one area of the design can be readily translated to another area if the network is modeled by vertex-transitive graphs. In addition, as shown in the next section, the edge-connectivity of any vertex-transitive graph is nothing but the degree of the graph. As for the vertex-connectivity of any vertex-transitive graph G , M. Watkins [7] showed that $\kappa(G)/\delta(G) > 2/3$ which means that the vertex-connectivity of any given vertex-transitive graph will not be too small. All of these theorems reaffirm the idea of using vertex-transitive graphs when designing networks. Furthermore, Watkins introduced the notion of atomic parts. He showed that we can partition a vertex-transitive graph into isomorphic subgraphs, called atomic parts, such that the vertex-transitivity is preserved in each subgraph and in the quotient graph obtained by contracting each subgraph to a vertex. However, it is not easy to determine an atomic part of an arbitrary vertex-transitive graph. Therefore, the subfamily of vertex-transitive graphs was being investigated with respect to this problem.

A special family of vertex-transitive graphs of particular interest is Cayley graphs.

Because there are some interactions between groups and graphs, some problems concerning groups are best attacked by using graphs. Cayley graphs can be viewed as pictorial representations of groups.

Several authors have proposed using Cayley graphs as possible candidates for networks. One reason is that the fault tolerances of some Cayley graphs are easier to calculate. C.D. Godsil [3] showed that for any Cayley graph $X(G; H)$, $\kappa(X)$ is easy to find if H is a minimal Cayley generating set. B. Alspach [1] generalized Godsil's result to quasiminimal Cayley generating set. Furthermore, Alspach showed in the same paper that the atomic parts of any Cayley graph with quasiminimal generating set will be isomorphic to a single vertex unless the Cayley graph is in a special family, in which case the atomic parts are all isomorphic to K_2 .

However, in general it is still not easy to determine an atomic part for an arbitrary Cayley graph. Several authors began to study how the idea of atomic parts can be applied in determining the vertex-connectivity of any vertex-transitive graph. In 1984, F. Boesch and R. Tindell [2] found a way to determine the vertex-connectivity of any circulant graph using the idea of atomic parts. Independently in 1985, Watkins developed an algorithm which can determine the vertex-connectivity of any finite or infinite, but locally finite, circulant graph G . Furthermore, this algorithm also generates the atomic part in G which contains the identity.

Chapter 2

Preliminaries

Let $G(V, E)$ be a graph with the vertex set V and the edge set E . Let $Aut(G)$ denote the group of automorphisms of the graph G .

Definition 2.1 A graph G is *vertex-transitive* if $\forall u, v \in V(G), \exists \phi \in Aut(G)$ such that $\phi(u) = v$.

Definition 2.2 A graph G is *edge-transitive* if $\forall (u, v), (x, y) \in E(G), \exists \phi \in Aut(G)$ such that $(\phi(u), \phi(v)) = (x, y)$ or $(\phi(u), \phi(v)) = (y, x)$.

Throughout the thesis, we are interested only in connected graphs because otherwise, the vertex connectivity and the edge connectivity are simply zero. Furthermore, in vertex-transitive graphs, we are interested only in the vertex-connectivity because of the following theorem which shows that the edge-connectivity is simply the degree of the graph. However, before proving the theorem, some notations need to be introduced. Suppose X is a subset of $V(G)$ and $G[X]$ is a connected subgraph of G , then $\nabla(X)$ is the minimum edge-cut which will disconnect X from the rest of G . In this case, X is called a *shore* with respect to $\nabla(X)$. In addition, $\kappa(G)$ denotes the vertex-connectivity of the graph G , and $\kappa_1(G)$ denotes the edge-connectivity of G . Also $\delta(G)$ and $\delta(x)$ represent the minimum degree of the graph G and the degree of the vertex x , respectively.

Theorem 2.3 *In any vertex-transitive graph G , the edge-connectivity always equals $\delta(G)$.*

PROOF: Let G be a vertex-transitive graph, and let X be a connected subgraph of G so that $|\nabla(X)| = \kappa_1(G)$. Without loss of generality, assume X is a minimum shore among all the shores admitted by minimum edge cuts. We will show that $\nabla(X)$ is a star-cut, that is, $|X| = 1$. Suppose $|X| \geq 2$. Then $\exists u, v \in X$ such that $u \neq v$. Since G is vertex-transitive, $\exists \phi \in \text{Aut}(G)$ such that $\phi(u) = v$. There are two cases that an automorphism will do to the shore X .

Case I: Assume $\phi(X) \neq X$. The partition of the vertex sets is as shown in Figure 2.1.

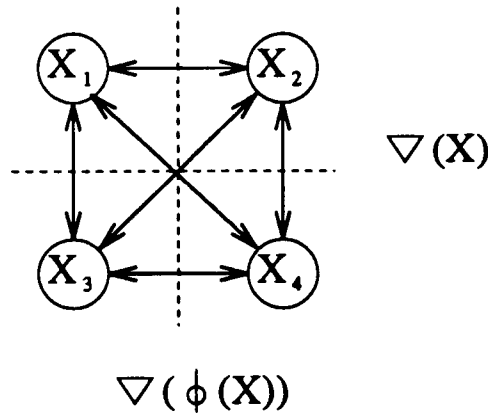


Figure 2.1: Different shores

Since $\phi(X) \neq X$,

$$X_1 = X \cap \phi(X) \neq \emptyset,$$

$$X_2 = X \cap \overline{\phi(X)} \neq \emptyset,$$

$$X_3 = \overline{X} \cap \phi(X) \neq \emptyset, \text{ and}$$

$$X_4 = \overline{X} \cap \overline{\phi(X)} \neq \emptyset.$$

Since $\nabla(X)$ is a minimum edge cut,

$$|\nabla(X_1)| \geq |\nabla(X)|,$$

$$\begin{aligned}
|\nabla(X_2)| &\geq |\nabla(X)|, \\
|\nabla(X_3)| &\geq |\nabla(X)|, \quad \text{and} \\
|\nabla(X_4)| &\geq |\nabla(X)|.
\end{aligned}$$

Hence, $4|\nabla(X)| \leq |\nabla(X_1)| + |\nabla(X_2)| + |\nabla(X_3)| + |\nabla(X_4)| \leq 2|\nabla(X)| + 2|\nabla(\phi(X))| = 4|\nabla(X)|$. Therefore, $|\nabla(X \cap \phi(X))| = |\nabla(X)|$. But this is a contradiction because we chose X to be a minimum shore and now we have a smaller shore with a minimum edge cutset.

Case II : Suppose $\phi(X) = X$. By case I, we may assume that if $\phi \in \text{Aut}(G)$ and $\phi(X) \cap X \neq \emptyset$, then $\phi(X) = X$. Hence, we know that the image of the shore X is preserved under automorphisms of this kind. Assume there are $t \geq 1$ edges of $\nabla(X)$ incident with u . Since $\phi(X) = X$ and G is vertex-transitive, every vertex in X will contribute t edges to $\nabla(X)$. Assume $|X| = n$, $\delta(G) = d$, and $|\nabla(X)| = m$. Then $n \geq d - t + 1$ and $nt = m$. Furthermore, $n > 2$ implies that $t < d$ and $m < d$. Hence, by substitution, we get $d > t(t - d + 1)$. After a little algebraic simplification, we get $(t - 1)(t - d) > 0$ which is equivalent to $t < 1$ or $t > d$. This is again a contradiction.

Therefore $|X| = 1$ which implies that $\kappa_1(G) = d = \delta(G)$. ■

Definition 2.4 Given two graphs H_1 and H_2 , the *lexicographic product* $H_1 \wr H_2$ of H_1 and H_2 is defined as follows : $V(H_1 \wr H_2) = V(H_1) \times V(H_2)$ and $((x_1, x_2), (y_1, y_2)) \in E(H_1 \wr H_2)$ if and only if either $(x_1, y_1) \in E(H_1)$ or $x_1 = y_1$ and $(x_2, y_2) \in E(H_2)$

Notice that the lexicographic product of H_1 and H_2 is obtained by taking $|V(H_1)|$ vertex-disjoint copies of H_2 corresponding to the vertices of H_1 and then taking all the edges between two copies of H_2 if and only if there is a corresponding edge in H_1 .

Definition 2.5 Let S be a subset of the vertex set of G . Then $G[S]$ denotes the subgraph induced by the vertices in S .

Definition 2.6 Given a connected graph G and $C \subseteq V(G)$, C is a *cutset* if $G[V(G) \setminus C]$ is disconnected or a trivial graph.

Let $\mathcal{C}(G) = \{ C \mid C \text{ is a cutset of } G \text{ and } |C| = \kappa(G) \}$, that is, $\mathcal{C}(G)$ is the set of all minimum cutsets of G .

Definition 2.7 P is a *part* of G with respect to $C \in \mathcal{C}(G)$ if P is a component in $G[V(G) \setminus C]$.

Example 2.8 Let G be the graph shown in Figure 2.3. Then P , Q and R are parts with respect to the cutset C .

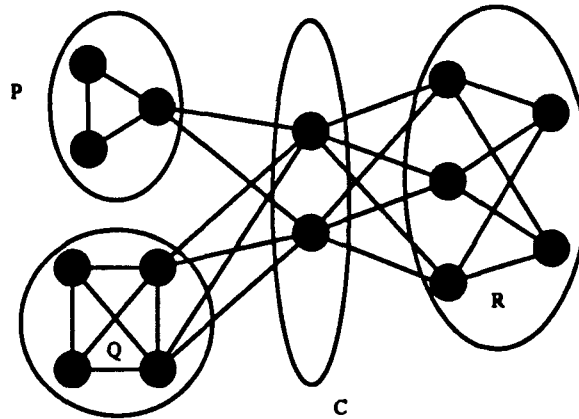


Figure 2.2: Cutset and parts

Proposition 2.9 For every part P , there is a unique cutset in $\mathcal{C}(G)$ corresponding to it.

PROOF: By the definition of part, $\exists C \in \mathcal{C}(G)$ such that P is a component of the disconnected graph $G[V(G) \setminus C]$. Assume $\exists C_1, C_2 \in \mathcal{C}(G)$ such that P is a component associated with both cutsets and $C_1 \neq C_2$. Since $|C_1| = |C_2|$, $\exists u \in C_1 \setminus C_2$ and $\exists v \in C_2 \setminus C_1$. The vertex u must be adjacent to some vertices in P , otherwise $C_1 \setminus u$ is a smaller cutset which is a contradiction. Now we know that u is not in C_2 , and so u is still adjacent to some vertices in P in $G[V(G) \setminus C_2]$. This implies that P is not a component of $G[V(G) \setminus C_2]$ which is a contradiction. Hence, the cutset must be unique. ■

We use the notation $x \sim y$ to denote that x and y are adjacent. If C be a minimum cutset in G and P is a part of G with respect to C , then $\forall x \in C, x \sim w$ for some

$w \in P$ and $x \sim v$ for some $v \in V(G) \setminus (C \cup P)$. Let $p(G) = \min\{\min\{|V(P)| : P \text{ is a part with respect to } C\} : C \in \mathcal{C}(G)\}$. In another word, $p(G)$ is the cardinality of a minimum component over all the minimum cutsets.

Definition 2.10 P is an *atomic part* if P is a part and $|V(P)| = p(G)$

Lemma 2.11 *Given a connected graph G , the following statements are equivalent :*

- (i) $p(G) \geq 2$,
- (ii) $\kappa(G) < \delta(G)$, and
- (iii) $\forall x \in V(G)$, $\{v | v \sim x\} \notin \mathcal{C}(G)$.

PROOF: (i) \Rightarrow (iii).

Assume $\exists x \in V(G)$ such that $C = \{v | v \sim x\} \in \mathcal{C}(G)$. Hence $G[V(G) \setminus C]$ will have a component consisting of the vertex x which implies that $p(G) \leq 1$. This is a contradiction.

(iii) \Rightarrow (ii).

If $\kappa(G) = \delta(G)$, then $\exists C \in \mathcal{C}(G)$ such that C consists of all the neighbours of some vertex $x \in V(G)$. This is a contradiction.

(ii) \Rightarrow (i).

If $p(G) = 1$, then $\kappa(G) = \delta(x)$ for some $x \in V(G)$. Again this is a contradiction.

■

Let P_1 and P_2 denote two distinct atomic parts. Let $U_1 = V(P_1)$ and $U_2 = V(P_2)$. Let C_1 and C_2 be the cutsets with respect to P_1 and P_2 , respectively, and let $R_1 = V(G) \setminus (U_1 \cup C_1)$ and $R_2 = V(G) \setminus (U_2 \cup C_2)$. Hence $|U_1| = |U_2| = p(G)$, $|C_1| = |C_2| = \kappa(G)$, and $|R_1| = |R_2|$.

Let G be partitioned as in Figure 2.4, that is, let

$$S_1 = U_1 \cap U_2,$$

$$S_2 = U_1 \cap C_2,$$

$$S_3 = U_1 \cap R_2,$$

$$S_4 = C_1 \cap U_2,$$

$$\begin{aligned}
S_5 &= C_1 \cap C_2, \\
S_6 &= C_1 \cap R_2, \\
S_7 &= R_1 \cap U_2, \\
S_8 &= R_1 \cap C_2, \quad \text{and} \\
S_9 &= R_1 \cap R_2.
\end{aligned}$$

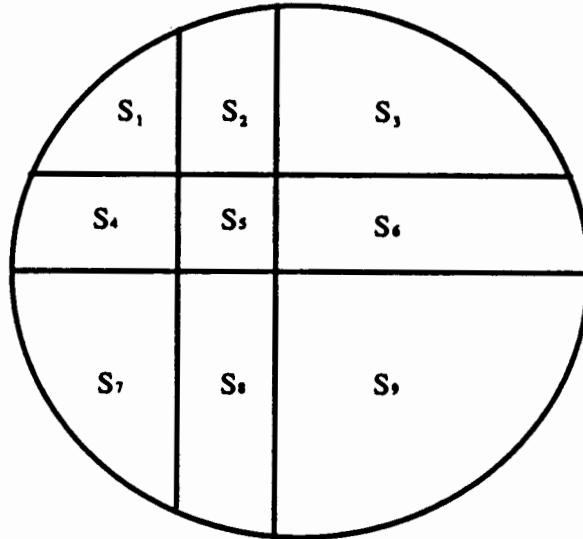


Figure 2.3: Partition of the vertex set

Lemma 2.12 *If $S_1 \cup S_2 \cup S_4 \neq \emptyset$, then either $S_3 = \emptyset$ or $S_7 = \emptyset$.*

PROOF: Suppose $S_3 \neq \emptyset$ and $S_7 \neq \emptyset$. Let $D_1 = S_2 \cup S_5 \cup S_6$ and $D_2 = S_4 \cup S_5 \cup S_8$. Notice that $\forall v \in S_3$, v can only be adjacent to vertices in $S_2 \cup S_3 \cup S_5 \cup S_6$. Otherwise, either C_1 or C_2 is not a cutset. Similarly, $\forall v \in S_7$, v can only be adjacent to vertices in $S_4 \cup S_5 \cup S_7 \cup S_8$. Therefore, D_1 and D_2 are cutsets which implies that $|D_1| \geq \kappa(G)$ and $|D_2| \geq \kappa(G)$. However, $|D_1| + |D_2| = |C_1| + |C_2| = 2\kappa(G)$. Hence, $|D_1| = |D_2| = \kappa(G)$. Therefore, $D_1, D_2 \in \mathcal{C}(G)$. But $0 < |S_3| < p(G)$ or $0 < |S_7| < p(G)$ since

$S_1 \cup S_2 \cup S_4 \neq \emptyset$. This implies that either S_3 or S_7 contains a part with respect to D_1 or D_2 having fewer than $p(G)$ vertices, a contradiction. Hence $S_3 = \emptyset$ or $S_7 = \emptyset$ ■

Corollary 2.13 *If $U_1 \cap U_2 \neq \emptyset$, then $S_3 = \emptyset$ or $S_7 = \emptyset$.*

Lemma 2.14 *If $U_2 \cap C_1 \neq \emptyset$ and $U_2 \cap R_1 = \emptyset$, then $U_2 \subset C_1$.*

PROOF: Suppose $U_2 \not\subset C_1$. By the hypothesis, $S_4 \neq \emptyset$ and $S_7 = \emptyset$. Therefore, $S_1 \neq \emptyset$. Let $Q_1 = S_1$ and $Q_2 = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_7$. Let $D_1 = S_2 \cup S_4 \cup S_5$ and $D_2 = S_5 \cup S_6 \cup S_8$. Notice that any vertex in Q_1 or Q_2 can only be adjacent to vertices in either $Q_1 \cup D_1$ or $Q_2 \cup D_2$, respectively. Hence D_i is a cutset for Q_i if $D_i \cup Q_i \neq V(G)$, $i = 1, 2$. First, D_1 is a cutset since $R_1 \neq \emptyset$ which implies $D_1 \cup Q_1 \neq V(G)$. We need that $R_1 \cap R_2 \neq \emptyset$ in order to conclude D_2 is a cutset.

If $R_1 \cap R_2 = \emptyset$, that is, $S_9 = \emptyset$, then $|S_8| = |R_1|$ and $|R_2| = |S_3 \cup S_6|$ since $S_7 = \emptyset$. Hence $|S_8| = |S_3 \cup S_6|$ implying that $2|R_1| = |S_8| + |S_3 \cup S_6| = |S_8| + |S_3| + |S_6| = |S_3| + |S_5| + |S_6| + |S_8| - |S_5| = |S_3| + |D_2| - |S_5|$. However, since $|D_1| + |D_2| = 2\kappa(G)$ and $|D_1| \geq \kappa(G)$, $|D_2| \leq \kappa(G) \leq |D_1|$. Therefore, $|S_3| + |D_2| - |S_5| \leq |D_1| + |S_3| - |S_5| = |S_2| + |S_3| + |S_4| < 2p(G)$. That is, $2|R_1| < 2p(G)$. This is a contradiction because $p(G) \leq |R_1|$. Hence, $R_1 \cap R_2 \neq \emptyset$. So D_2 is also a cutset.

Since $|D_1| \geq \kappa(G)$ and $|D_2| \geq \kappa(G)$, and $|D_1| + |D_2| = 2\kappa(G)$, $|D_1| = |D_2| = \kappa(G)$ which implies that $D_1, D_2 \in \mathcal{C}(G)$. Hence Q_1 contains a part whose size is smaller than $p(G)$. Again, this gives a contradiction. Hence, $U_2 \subset C_1$. ■

Notice that the above lemma still holds if the indices 1 and 2 are interchanged.

Lemma 2.15 *If $U_1 \cap U_2 \neq \emptyset$, then $U_2 \cap R_1 = \emptyset$.*

PROOF: Suppose $U_2 \cap R_1 \neq \emptyset$. By Corollary 2.14 we have that $S_3 = \emptyset$ or $S_7 = \emptyset$. Since $U_1 \cap U_2 \neq \emptyset$, we have that U_1 is not a subset of C_2 . Hence, by Lemma 2.15, $U_1 \cap C_2 = \emptyset$ must hold. Therefore, $U_1 \subset U_2$ implying $U_1 = U_2$. This contradicts the choice of two distinct atomic parts. ■

Theorem 2.16 *In any connected graph, two distinct atomic parts are disjoint.*

PROOF: Let G be a connected graph, and let P_1 and P_2 be two distinct atomic parts. Let $U_1 = V(P_1)$ and $U_2 = V(P_2)$. Let C_1 and C_2 be the cutsets with respect to P_1 and P_2 , respectively, and let $R_1 = V(G) \setminus (U_1 \cup C_1)$ and $R_2 = V(G) \setminus (U_2 \cup C_2)$. Hence $|U_1| = |U_2| = p(G)$, $|C_1| = |C_2| = \kappa(G)$, and $|R_1| = |R_2|$. Let $S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8$, and S_9 be defined as before. Suppose $U_1 \cap U_2 \neq \emptyset$. Then $U_2 \cap R_1 = \emptyset$ by the preceding lemma. Since P_1 and P_2 are distinct atomic parts, $U_2 \cap C_1 \neq \emptyset$ and $U_2 \subset C_1$ by Lemma 2.15. This implies that $U_1 \cap U_2 = \emptyset$ contradicting $U_1 \cap U_2 \neq \emptyset$. Therefore, $V(P_1) \cap V(P_2) = \emptyset$. ■

Lemma 2.17 *Let G be a graph. If P is an atomic part of G and $\phi \in \text{Aut}(G)$, then $\phi(P)$ is an atomic part of G .*

PROOF: Let C be the minimum cutset corresponding to the atomic part P . Then any vertex $u \in V(P)$ can only be adjacent to vertices in $V(P) \cup C$. Since $\phi \in \text{Aut}(G)$, $\phi(u)$ can only be adjacent to vertices in $V(\phi(P)) \cup \phi(C)$. As a result, the deletion of $\phi(C)$ will disconnect $\phi(P)$ from the rest of the graph. Hence $\phi(C)$ is a minimum cutset because $|\phi(C)| = |C|$. Since $|V(\phi(P))| = |V(P)|$, $\phi(P)$ is an atomic part of G . ■

Now, we can easily characterize the vertex-connectivity of edge-transitive graphs.

Corollary 2.18 *If a connected graph G is edge-transitive, then*

$$\kappa(G) = \delta(G).$$

PROOF: Assume $\kappa(G) < \delta(G)$. By Lemma 2.12(ii), $p(G) \geq 2$. Let P be an atomic part. Since G is connected, $\exists z \in V(G) \setminus V(P)$ such that z is adjacent to some vertex in P . Without loss of generality, assume $x, y \in V(P)$, $x \sim y$ and $z \sim x$. Since G is edge-transitive, $\exists \phi \in \text{Aut}(G)$ such that either $\phi(x) = x$ and $\phi(y) = z$ or $\phi(x) = z$ and $\phi(y) = x$. Hence $\phi(V(P)) \cap V(P) \neq \emptyset$ and $\phi(V(P)) \setminus V(P) \neq \emptyset$ which contradicts Theorem 2.17 and Lemma 2.18. Therefore, $\kappa(G) = \delta(G)$. ■

Lemma 2.19 *Let P_1 and P_2 be two distinct atomic parts with respect to cutsets C_1 and C_2 , respectively. Then either $S_4 = \emptyset$ or $S_7 = \emptyset$.*

PROOF: Assume $P_1 \neq P_2$. Then we have $S_1 = \emptyset$ by Theorem 2.17. Suppose $S_4 \neq \emptyset$ and $S_7 \neq \emptyset$. $S_4 \neq \emptyset$ implies $S_3 = \emptyset$ or $S_7 = \emptyset$ by Lemma 2.13. Since $S_7 \neq \emptyset$, $S_3 = \emptyset$. Let $x \in S_7$. The vertex x can only be adjacent to vertices in $S_4 \cup S_5 \cup S_7 \cup S_8$, and therefore, $\delta(x) \leq p(G) - 1 + \kappa(G) - p(G) = \kappa(G) - 1$. This is a contradiction. Hence $S_4 = \emptyset$ or $S_7 = \emptyset$. ■

The above lemma actually states that given an atomic part and a minimum cutset, they are either disjoint or the atomic part is contained in the cutset.

Chapter 3

Connectivity of vertex-transitive graphs

Atomic parts will give us a way to look at the structure of vertex-transitive graphs as developed in this section. Before we begin to study the structure of vertex-transitive graphs, some definitions and terminology about automorphism groups of graphs must be given.

Definition 3.1 Let $B \subset V(G)$. We say B is a *block* of a graph G with respect to $Aut(G)$ if $\forall \phi \in Aut(G)$ either $\phi(B) = B$ or $\phi(B) \cap B = \emptyset$.

Note that \emptyset , $V(G)$, and all singleton subsets of the vertex set are blocks. We call them *trivial blocks*.

Example 3.2 Let G be as in Figure 3.1. Then, in addition to the trivial blocks, $\{P, Q, R\}$ is a block.

Definition 3.3 A transitive permutation group is *primitive* if it has only trivial blocks. Otherwise, it is called *imprimitive*.

Definition 3.4 If B_1 and B_2 are blocks and $\exists \phi \in Aut(G)$ such that $\phi(B_1) = B_2$, then B_1 and B_2 are said to be *conjugate*.

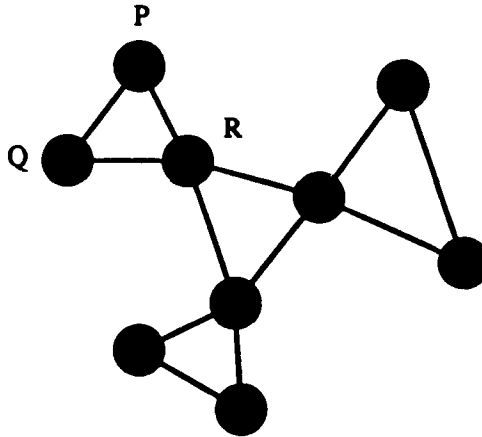


Figure 3.1: Blocks

Definition 3.5 The set of all blocks of G which are conjugate to some block B is called a *complete block system*.

Lemma 3.6 *The vertex sets of atomic parts of a vertex-transitive graph G form a complete block system of $\text{Aut}(G)$.*

PROOF: Let P be an atomic part of G . Let $x \in V(P)$ and let $y \in V(G \setminus P)$. Since G is vertex-transitive, $\exists \phi \in \text{Aut}(G)$ such that $\phi(x) = y$. By Lemma 2.18, $\phi(P)$ is also an atomic part, and by Theorem 2.17, $V(P) \cap V(\phi(P)) = \emptyset$. Hence, $V(\phi(P))$ is conjugate to $V(P)$. By repeating the process, the vertex set of G is partitioned into blocks which are all conjugate to $V(P)$. ■

Corollary 3.7 *If $p(G) > 1$, then $\text{Aut}(G)$ is imprimitive.*

PROOF: If $p(G) > 1$, then the vertex sets of the atomic parts will be a complete block system of non-trivial blocks of $\text{Aut}(G)$. Hence, $\text{Aut}(G)$ is imprimitive. ■

For any graph G , we know that $\kappa(G) \leq \kappa_1(G) \leq \delta(G)$. Consider the case when $\kappa(G) < \delta(G)$.

Corollary 3.8 *Let G be vertex-transitive and suppose $0 < \kappa(G) < \delta(G)$. Let P be an atomic part of G . Then*

(i) P is a vertex-transitive graph;

(ii) G is isomorphic to a disjoint union of two or more copies of P together with some edges joining them.

PROOF: Since $\kappa(G) < \delta(G)$, we know $|V(P)| \geq 2$ by Lemma 2.12. Hence by Corollary 3.7, $\text{Aut}(G)$ is imprimitive. Since $V(P)$ is a block of $\text{Aut}(G)$, the automorphisms of G fixing $V(P)$ setwise act transitively on $V(P)$. Therefore, P is vertex-transitive. Furthermore, the blocks of $\text{Aut}(G)$ are conjugate since $\text{Aut}(G)$ is imprimitive. Hence, (ii) holds. ■

Lemma 3.9 *Let G be vertex-transitive and suppose $0 < \kappa(G) < \delta(G)$. Then $\kappa(G) = np(G)$ for some integer $n \geq 2$.*

PROOF: Let P be an atomic part of G . Let C be the cutset determined by P . By (ii) of the previous theorem, $V(G)$ can be partitioned into copies of P , and by Lemma 2.20, every atomic part must be either contained in C or disjoint from C . Hence, $\kappa(G) = np(G)$ for some integer n . Assume $n = 1$. Since G is vertex-transitive, $V(P)$ is a minimum cutset and $G[C]$ is an atomic part. Since $V(P)$ is a minimum cutset, $G \setminus V(P)$ has at least two parts. Let L_1 and L_2 be two parts of $G \setminus V(P)$. Since G is connected and C is the cutset for P , $L_1 \cap C \neq \emptyset$ and $L_2 \cap C \neq \emptyset$. Because there is no path between L_1 and L_2 without using vertices in P , there is no path between $L_1 \cap C$ and $L_2 \cap C$ without using vertices in P . This implies $G[C]$ is not connected which is a contradiction. Hence, $n \geq 2$. ■

Theorem 3.10 *Let H_1 and H_2 be graphs which are connected and vertex-transitive. If H_1 is not complete and $|V(H_2)| \geq 2$, then $G = H_1 \wr H_2$ is vertex-transitive, $0 < \kappa(G) < \delta(G)$, and $\kappa(G) = \kappa_1(G)|V(H_2)|$.*

PROOF: Let $((x_1, x_2), (y_1, y_2)) \in E(G)$. Since x_1 and y_1 are in $V(H_1)$ and H_1 is vertex-transitive, $\exists \rho \in \text{Aut}(H_1)$ such that $\rho(x_1) = y_1$. Since x_2 and y_2 are in $V(H_2)$ and H_2

is vertex-transitive, $\exists \psi \in \text{Aut}(H_2)$ such that $\psi(x_2) = y_2$. Define $\phi : E(G) \mapsto E(G)$ as $\phi(x, y) = (\rho(x), \psi(y))$. We first show that ϕ is one-to-one:

$$\begin{aligned} \phi(x_1, x_2) &= \phi(x_3, x_4) \\ \Leftrightarrow \rho(x_1) &= \rho(x_3), \psi(x_2) = \psi(x_4) \\ \Leftrightarrow x_1 &= x_3, x_2 = x_4. \end{aligned}$$

It remains to show $(\phi(x_1, x_2), \phi(y_1, y_2))$ is an edge in G . Because of the definition of lexicographic product, either $(x_1, y_1) \in E(H_1)$ or $x_1 = y_1$ and $(x_2, y_2) \in E(H_2)$. If $(x_1, y_1) \in E(H_1)$, then since $\rho \in \text{Aut}(H_1)$, $(\rho(x_1), \rho(y_1)) \in E(H_1)$. Now, if the second case holds, then $\rho(x_1) = \rho(y_1)$ and $(\psi(x_2), \psi(y_2)) \in E(H_2)$ implying $\phi((x_1, x_2), (y_1, y_2))$ is still an edge in G . Therefore, $\phi \in \text{Aut}(G)$.

Now we want to show that $\kappa(G) = \kappa(H_1)|V(H_2)|$. Since H_1 is connected, $\kappa(H_1) > 0$. Let C_1 be a minimum cutset of H_1 . There are precisely $\kappa(H_1)|V(H_2)|$ vertices of G of the form (x_1, x_2) where $x_1 \in C_1$ and $x_2 \in V(H_2)$ and they form a cutset. So $\kappa(G) \leq \kappa(H_1)|V(H_2)|$. Now let $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$ be two vertices from distinct parts of G . It is sufficient to show that there are at least $\kappa(H_1)|V(H_2)|$ internally disjoint paths between the two vertices.

Case 1: If $x_1 = y_1$. Then \bar{x} and \bar{y} are in the same copy of H_2 . Since $\delta(x_1) = \delta(H_1) \geq \kappa(H_1)$, \bar{x} is adjacent to at least $\kappa(H_1)|V(H_2)|$ vertices outside of this copy of H_2 . However, these vertices are all adjacent to \bar{y} , and so, there are at least $\kappa(H_1)|V(H_2)|$ internally disjoint $\bar{x}\bar{y}$ paths in G .

Case 2: If $x_1 \neq y_1$. Then by Menger's theorem, there are $\kappa(H_1)$ internally disjoint paths A_i in H_1 joining x_1 and y_1 . Let $A_i = x_1, w_{i1}, w_{i2}, \dots, w_{im_i}, y_1$ for $i = 1, 2, \dots, \kappa(H_1)$. Now for every $z \in V(H_2)$, we can determine $A_{i,z}$ in G by $(x_1, x_2), (w_{i1}, z), (w_{i2}, z), \dots, (w_{im_i}, z), (y_1, y_2)$. Hence, there are $\kappa(H_1)|V(H_2)|$ internally disjoint $\bar{x}\bar{y}$ paths in G .

Therefore, $\kappa(G) \geq \kappa(H_1)|V(H_2)|$. So, $\kappa(G) = \kappa(H_1)|V(H_2)|$. ■

When H_1 is complete, it is easy to verify that $\kappa(H_1 \wr H_2) = \kappa(H_2) + \kappa(H_1)|V(H_1)|$, and the atomic parts of $H_1 \wr H_2$ are all isomorphic to atomic parts of H_2 .

The above theorem gives a characterization to the structure of the lexicographic product of two vertex-transitive graphs. The connectivity of a vertex-transitive graph obtained from the lexicographic product of two vertex-transitive graphs is related to the connectivities and the orders of the smaller vertex-transitive graphs. Furthermore, the structure of an atomic part of the new graph is obtainable from the smaller graphs.

For any graph G , we know that $\kappa(G) \leq \kappa_1(G) \leq \delta(G)$ and by choosing an appropriate graph G the ratio $\delta(G)/\kappa(G)$ can be made as large as desired. However, for the case when the graph is vertex-transitive, the ratio cannot be too big as is shown by the next result.

Theorem 3.11 *Let G be a vertex-transitive graph and assume that $\delta(G) \neq \kappa(G)$. Then $\text{l.u.b.}\{ \delta(G)/\kappa(G) \mid G \text{ is vertex-transitive and connected} \} = 3/2$ and the bound is never met.*

PROOF: Let P be an atomic part of G and let $N = \{Q \mid Q \neq P \text{ is an atomic part such that } \exists y \in V(Q) \text{ and } \exists x \in V(P), \text{ so that } (x, y) \in E(G)\}$. Let $C = \cup\{V(Q) \mid Q \in N\}$. Then C is the minimum cutset determined by P . Hence $\kappa(G) = |N|p(G) \geq 2p(G)$. But considering the degree of a vertex in P will yield the following : $\delta(G) \leq |C| + \delta(P) \leq \kappa(G) + p(G) - 1$. Hence $\delta(G)/\kappa(G) \leq 1 + (p(G) - 1)/\kappa(G) \leq 1 + (p(G) - 1)/(2p(G)) = 3/2 - 1/(2p(G))$, and $3/2$ is an upper bound. Now we need to present a graph such that $\delta(G)/\kappa(G) = 3/2 - 1/(2p(G))$. Let $G = H \wr K$ where H is a cycle of length at least 4 and K is the complete graph with $m = p(G)$ vertices. By Theorem 3.10 we have $\delta(G) = 3m - 1$ and $\kappa(G) = 2m$. Hence $\delta(G)/\kappa(G) = 3/2 - 1/m$. ■

Corollary 3.12 *Any vertex-transitive graph G having $\kappa(G) = 2$ is a cycle.*

PROOF: From the previous result we know that $\delta(G)/\kappa(G) = \delta(G)/2 < 3/2$ implying that $\delta(G) < 3$. Hence $\delta(G) = 1$ or 2 . But the only connected vertex-transitive graph with $\delta(G) = 1$ is an edge which doesn't have vertex-connectivity 2. So G is a cycle. ■

Corollary 3.13 *If G is vertex-transitive with $\delta(G) = 2$, $\delta(G) = 3$, $\delta(G) = 4$, or $\delta(G) = 6$, then $\kappa(G) = \delta(G)$.*

PROOF: When $\delta(G) = 2$, the graph is a cycle, and therefore, $\kappa(G) = 2$.

When $\delta(G) = 3$, $\delta(G)/\kappa(G) = 3/\kappa(G) < 3/2$ implies that $\kappa(G) > 2$. Hence $\kappa(G) = 3$ because $\kappa(G) \leq \delta(G) = 3$.

When $\delta(G) = 4$, $\delta(G)/\kappa(G) = 4/\kappa(G) < 3/2$ implies that $\kappa(G) = 3$, or $\kappa(G) = 4$. Consider the case when $\kappa(G) = 3$. By Lemma 2.12 we know that $p(G) \geq 2$. However, $\kappa(G) \geq 2p(G) \geq 4$ by Lemma 3.9. Therefore, $\kappa(G)$ can only be 4.

When $\delta(G) = 6$, $\delta(G)/\kappa(G) = 6/\kappa(G) < 3/2$ implies that $\kappa(G) = 5$ or $\kappa(G) = 6$. In the case when $\kappa(G) = 5$, $p(G) \geq 2$ by Lemma 2.12. By Lemma 3.9, $p(G)$ is a factor of $\kappa(G)$, and so, $p(G) = 5$. However, $\kappa(G) \geq 2p(G) \geq 10$ is impossible. Therefore, $\kappa(G) = 6$. ■

Chapter 4

A Generalization

The atomic parts give us a way to look at the structure of vertex-transitive graphs. A.C. Green [4] generalized the idea of atomic parts to the blocks obtained from the atomic parts with respect to a cutset.

The next lemma is a generalization of Lemma 3.6.

Lemma 4.1 *Let G be a graph and suppose $S \in \mathcal{C}(G)$. Assume there are exactly $k \geq 1$ atomic parts P_1, P_2, \dots, P_k with respect to S . Let $B = \cup\{V(P_i) | i = 1, 2, \dots, k\}$. Then B is a block of $\text{Aut}(G)$.*

PROOF: Suppose $\exists \phi \in \text{Aut}(G)$ such that $\phi(P_i) \cap P_j \neq \emptyset$ for some $i, j \in \{1, 2, 3, \dots, k\}$. Since atomic parts are blocks with respect to $\text{Aut}(G)$, $\phi(P_i) = P_j$. Since $\phi(P_i)$ is uniquely determined by $\phi(S)$ and P_j is uniquely determined by S , the atomic parts determined by S are exactly the atomic parts determined by $\phi(S)$. Hence $\phi(S) = S$ and $\phi(B) = B$. ■

Given a set S of vertices, we say that S is *independent* if there is no edge between any two vertices in S .

Theorem 4.2 *Let G be vertex-transitive. Suppose $S \in \mathcal{C}(G)$ admits exactly $k \geq 1$ atomic parts. Let B denote the union of the vertices of these atomic parts. Then*

(i) *G is isomorphic to a disjoint union of $n \geq 2$ copies of $G[B]$ together with some edges joining them;*

(ii) $\kappa(G) = tkp(G)$ for some integer $t \geq 1$; and

(iii) every minimum cutset contains an independent k -subset of vertices.

PROOF:

(i) Since B is a block of $Aut(G)$, $\phi(B) = B$ or $\phi(B) \cap B = \emptyset$ for every $\phi \in Aut(G)$. Hence, G is a disjoint union of at least 2 copies of $G[B]$ together with some edges joining them because G is connected.

(ii) We want to show that the size of a minimum cutset is just some multiple of the size of B . Suppose $S \in \mathcal{C}(G)$ admits atomic parts P_1, P_2, \dots, P_k . Let $C \in \mathcal{C}(G)$ be another minimum cutset which admits parts Q_1, Q_2, \dots, Q_j . First, we want to show that either $\phi(B) \subset C$ or $\phi(B) \cap C = \emptyset$ for all $\phi \in Aut(G)$. Assume $\phi(B) \cap C \neq \emptyset$ and $\phi(B) \not\subset C$. Hence, $\exists P \in \{P_1, P_2, \dots, P_k\}$ such that $P \cap C \neq \emptyset$. Since P is an atomic part, $P \subset C$. Since $C \in \mathcal{C}(G)$ and by vertex-transitivity of the graph, every vertex in P is adjacent to some vertex in Q_i for $i = 1, 2, \dots, j$. Since the vertices in $\phi(P)$ can only be adjacent to vertices in $\phi(P) \cup \phi(S)$, $\phi(S) \cap Q_i \neq \emptyset$ for all i . Since P_1, P_2, \dots, P_k are atomic parts with respect to S , then $\phi(P_1), \phi(P_2), \dots, \phi(P_k)$ are atomic parts with respect to $\phi(S)$, and every vertex in $\phi(S)$ is adjacent to some vertex in $\phi(P_i)$ for all i . Choose $P_m \in \{P_1, P_2, \dots, P_k\}$. If $\phi(P_m) \cap Q_t \neq \emptyset$ for some $t \in \{1, 2, \dots, j\}$, then $\phi(P_m) \subset Q_t$. So if $x_i \in \phi(S) \cap V(Q_i)$, then x_i is adjacent to some vertex in $\phi(P_m)$ and hence in Q_t . But this is impossible since Q_1, Q_2, \dots, Q_j are parts with respect to C . Hence $\phi(V(P_m)) \subset C$ for $m = 1, 2, \dots, k$. This shows that B is either contained in a minimum cutset or is totally disjoint from it. Now, if there is another vertex z in C , then since G is vertex-transitive, there is a block B_z containing z . Hence, B_z is again a subset of C . Therefore, $|C| = \kappa(G) = tkp(G)$ for some $t \geq 1$.

(iii) From (ii) we know $\phi(V(P_m)) \subset C, \forall m = 1, 2, \dots, k$. Hence, C contains these k copies of the atomic parts, and we can choose one vertex from each atomic part to form an independent set of cardinality k . ■

It is natural to ask whether or not the n and the t in the previous theorem can be improved. Let's consider the graph $G = K_{q,t} \wr K_t$, where $t \geq 1$ and $q \geq 2$. It is easy to see that $\kappa(G) = qt$ and a minimum cutset admits q atomic parts which are isomorphic to K_t . In this case, $n = 2$ and $m = 1$. Hence the numbers are best possible.

Here is a theorem which gives some characteristics of atomic parts of a vertex-transitive graph.

Theorem 4.3 *Let G be vertex-transitive and suppose $S \in \mathcal{C}(G)$ admits exactly $k \geq 1$ atomic parts. Then*

- (i) *every $C \in \mathcal{C}(G)$ admits exactly 0 or k atomic parts, and*
- (ii) *S admits at most one non-atomic part together with the k atomic parts.*

PROOF: Let $C_0 \in \mathcal{C}(G)$ be a minimum cutset. Either $G \setminus C_0$ has an atomic part P_0 or not. If P_0 is an atomic part with respect to C_0 , then $\exists \phi \in \text{Aut}(G)$ such that ϕ will map an atomic part of S to P_0 . Because of the vertex-transitivity and the proposition 2.10, the cutset C_0 must also produce exactly k atomic parts. Hence (i) is proven.

Suppose P is an atomic part with respect to S and suppose S admits $t \geq 0$ non-atomic parts Q_1, Q_2, \dots, Q_t . Without loss of generality we may assume Q_1 is chosen such that $|V(Q_1)|$ is maximum over all the parts admitted by all the minimum cutsets that admit atomic parts. Assume $t \geq 2$. Let $y \in V(Q_2)$. Then y belongs to a unique atomic part P^* with respect to a unique minimum cutset S^* . Furthermore, $V(P^*) \subset V(Q_2)$. So $S^* \subset (S \cup V(Q_2))$. Since $|S| = |S^*| = \kappa(G)$ and $S \neq S^*$, $\exists z \in S \setminus S^*$ and z is adjacent to some vertex in Q_1 because $k \geq 1$ and S is a minimum cutset. Hence $V(Q_1) \cup \{z\}$ is contained in some non-atomic part with respect to S^* . This is a contradiction. Hence $t \leq 1$. ■

Now we know that the structure of a vertex-transitive graph is heavily dependent on the number of atomic parts admitted by a minimum cutset. We do not expect that any minimum cutset always generates an atomic part for any given graph. Instead of considering the atomic parts as Watkins did in [7], Green [4] considered a minimal non-atomic part together with some other non-atomic parts admitted by a minimum cutset which admits no atomic part. If three or more non-atomic parts occur together, some results similar to the results about atomic parts can be proved. However the results are not true for only two non-atomic parts. Before showing Green's results, we need the following definition.

Definition 4.4 Suppose $S \in \mathcal{C}(G)$ and P is a part with respect to S . The part P is said to be S -minimal if $|V(P)| \leq |V(Q)|$ for every part Q admitted by S .

Example 4.5 Consider the graph in Figure 4.1. The parts P and R are S -minimal.

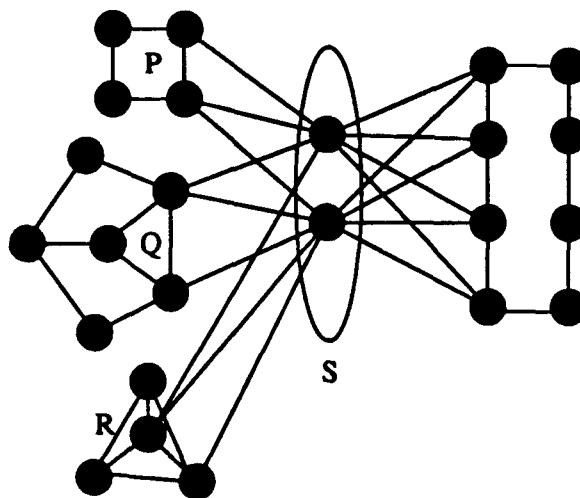


Figure 4.1: P and R are S -minimal parts of G

Theorem 4.6 Let G be a vertex-transitive graph such that $S \in \mathcal{C}(G)$ admits $k \geq 3$ non-atomic parts Q_1, Q_2, \dots, Q_k . Let P be S -minimal and $\phi \in \text{Aut}(G)$. Then either $\phi(V(Q)) \subset S$ or $\phi(V(Q)) \cap S = \emptyset$.

PROOF: Without loss of generality, we may assume that $P = Q_j$ for some $j \in \{1, 2, \dots, k\}$. Suppose the result fails for some $\phi \in \text{Aut}(G)$. Hence there is an $i \in \{1, 2, \dots, k\}$ such that $\phi(Q_j) \cap P_i \neq \emptyset$. Let $\phi(Q_j) \cap Q_i = Q''$ and $G[S \cap \phi(V(Q_j))] = Q'$. Note that both $V(Q')$ and $V(Q'')$ are non-empty.

Let $\phi(S) = K$. Since G is vertex-transitive, K is in $\mathcal{C}(G)$ and K admits k non-atomic parts which are pairwise isomorphic to the parts admitted by S . Let $D_\ell = V(Q_\ell) \cap \phi(S)$, $\ell = 1, 2, \dots, k$. Let $D_S = S \cap \phi(S)$ and $S' = S \setminus (D_S \cup V(Q'))$. Notice that these sets are all mutually disjoint from each other. Let $x \in Q'$. Since $S \in \mathcal{C}(G)$, x is adjacent to some vertex y_ℓ in Q_ℓ for all $\ell = 1, 2, \dots, k$. Since y_ℓ is adjacent to $x \in V(Q') \subset \phi(V(Q_j))$ and both Q_j and $\phi(Q_j)$ are connected, either $y_\ell \in \phi(S)$ or

$y_\ell \in \phi(V(Q_j))$. Since Q_j is S -minimal, $|\phi(V(Q_j))| = |V(Q_j)| \leq |V(Q_\ell)|$. Since $Q' \neq \emptyset$, there is at least one vertex in $V(Q_\ell)$ which is not in $\phi(V(Q_j))$. Hence there exists a $z \in V(Q_\ell) \setminus \phi(V(Q_j))$ such that z is adjacent to some $x \in \phi(V(Q_j))$ or P_ℓ is not a part. Since Q_ℓ is connected, $z \in \phi(S)$. Therefore, $D_\ell \neq \emptyset$ for all $\ell = 1, 2, \dots, k$.

For notational simplicity, we let $i = 1$ for the remainder of the proof. Since Q'' is a part with respect to the cutset $C = D_1 \cup D_S \cup V(Q')$, $|D_1| + |D_S| + |V(Q')| \geq \kappa(G) = |\phi(S)| = |D_S| + |D_1| + |D_2| + \dots + |D_k|$. Therefore, $|V(Q')| \geq |D_2| + |D_3| + \dots + |D_k|$. Suppose $\exists t \in \{2, 3, \dots, k\}$ such that $Q = V(Q_t) \setminus \phi(S \cup V(Q_j)) \neq \emptyset$. Then elements of Q may only be adjacent to vertices in $Q \cup D_t \cup D_S \cup S'$. Hence $G[Q]$ is a part admitted by $D_t \cup D_S \cup S'$. Hence, $|D_t| + |D_S| + |S'| \geq \kappa(G) = |S| = |D_S| + |S'| + |V(Q')|$. So, $|D_t| \geq |V(Q')|$. Hence $|D_2| + |D_3| + \dots + |D_k| > |D_t| \geq |V(Q')| \geq |D_2| + |D_3| + \dots + |D_k|$. This is a contradiction. Therefore for all t in $\{2, 3, \dots, k\}$, $V(Q_t) \subset \phi(S \cup V(Q_j))$. Because Q_j is S -minimal, $|V(Q_2) \cup V(Q_3)| \geq 2|V(Q_j)|$. Since $Q' \neq \emptyset$, $|V(\phi(Q_j) \cap (Q_2 \cup Q_3))| < |V(Q_j)| \leq 1/2(|V(Q_2) \cup V(Q_3)|)$. Since $V(Q_t) \subset \phi(S \cup V(Q_j))$ for all $t = 2, 3, \dots, k$ and Q_j is S -minimal, $|\phi(S) \cap (V(Q_2) \cup V(Q_3))| > 1/2|V(Q_2) \cup V(Q_3)|$. Therefore $|D_2| + |D_3| = |\phi(S) \cap (V(Q_2) \cup V(Q_3))| > 1/2|V(Q_2) \cup V(Q_3)| \geq |V(Q_j)|$. Since Q' is a proper subgraph of $\phi(Q_j)$, $|\phi(V(Q_j))| > |V(Q')| \geq |D_2| + |D_3| + \dots + |D_k| \geq |D_2| + |D_3| > |V(Q_j)|$. This is a contradiction implying our assumption is wrong. Therefore the result is true. ■

Corollary 4.7 *Let G be vertex-transitive. Let S be in $\mathcal{C}(G)$ such that S admits $k \geq 3$ non-atomic parts. Let P be an S -minimal part and Q be any other part with respect to S . Suppose there is a $\phi \in \text{Aut}(G)$ such that $\phi(V(P)) \cap V(Q) \neq \emptyset$. Then $\phi(V(P)) \subset V(Q)$.*

PROOF: Since $\phi(V(P)) \cap V(Q) \neq \emptyset$, then by the previous theorem and since $\phi(P)$ is connected, $\phi(P) \subset Q$. ■

Corollary 4.8 *Let G be vertex-transitive. Let S be in $\mathcal{C}(G)$ such that S admits $k \geq 3$ non-atomic parts. Suppose P is an S -minimal part of G . Then P is vertex-transitive.*

PROOF: For any two vertices x and y in P , there is an automorphism ϕ in $\text{Aut}(G)$ such that $\phi(x) = y$. However $\phi(P) = P$ by the previous corollary. By restricting such

automorphisms of G to the vertices of P , we obtain automorphisms of P . Hence P is vertex-transitive. ■

Corollary 4.9 *Let G be vertex-transitive. Let S be in $\mathcal{C}(G)$ such that S admits $k \geq 3$ non-atomic parts. Let P_1 and P_2 be S -minimal. Then P_1 and P_2 are isomorphic.*

PROOF: Let $x \in V(P_1)$ and $y \in V(P_2)$. Since G is vertex-transitive, $\exists \phi \in \text{Aut}(G)$ such that $\phi(x) = y$. Therefore, $\phi(P_1) \cap P_2 \neq \emptyset$. Hence $\phi(P_1) \subset P_2$. Because $|V(\phi(P_1))| = |V(P_1)| = |V(P_2)|$ and by applying corollary 4.7, we can conclude that P_1 and P_2 are isomorphic. ■

Corollary 4.10 *Let G be vertex-transitive. Let S be in $\mathcal{C}(G)$ such that S admits $k \geq 3$ non-atomic parts. Suppose S admits exactly $d \geq 2$ isomorphic S -minimal parts P_1, P_2, \dots, P_d . Let $B = \cup\{V(P_i) | i = 1, 2, \dots, d\}$. Then B is a block of $\text{Aut}(G)$.*

PROOF: Suppose there is a $\phi \in \text{Aut}(G)$ such that $\phi(P_i) \cap P_j \neq \emptyset$ for some $i, j \in \{1, 2, \dots, d\}$. From the proof of the previous corollary we know $\phi(P_i) = P_j$. Hence $\phi(S) = S$. Since S admits only these d isomorphic S -minimal parts, for all $m = 1, 2, \dots, d$, $\phi(P_m)$ must be P_n for some $n \in \{1, 2, \dots, d\}$. As a result, $\phi(B) = B$. Therefore, B is a block of G with respect to $\text{Aut}(G)$. ■

Now based on these theorems and corollaries, we can characterize the vertex-transitive graphs which have some minimum cutset admitting at least three non-atomic parts.

Theorem 4.11 *Let G be vertex-transitive. Let S be in $\mathcal{C}(G)$ such that S admits $k \geq 3$ non-atomic parts. Suppose S admits exactly $d \geq 2$ isomorphic S -minimal parts P_1, P_2, \dots, P_d . Let $B = \cup\{V(P_i) | i = 1, 2, \dots, d\}$. Then*

(i) G is isomorphic to a disjoint union of $n \geq 2$ copies of $G[B]$ together with some additional edges joining them;

(ii) $\kappa(G) = m|B|$, for some $m \geq 1$.

The proof is an exact analogue of the proof of Theorem 4.2.

By considering the following example, we will be able to see that $m = 1$ and $n = 2$ are best possible.

Example 4.12 Let G be the graph in Figure 4.2. It is easy to see that the graph G is vertex-transitive and the degree of the graph is 6. The vertex-connectivity is 6 and an atomic part is a single vertex. However, if we remove the cutset S , the resulting graph will consist of three S -minimal parts which are all isomorphic to K_2 .

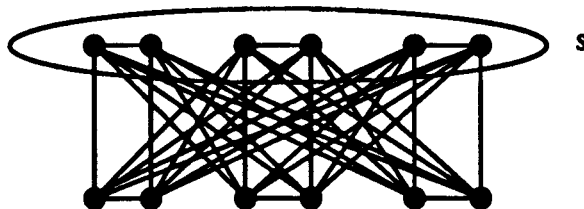


Figure 4.2: A graph which shows Theorem 4.11 is best possible

However, in the case when $k = 2$ and $d = 1$ or 2, the following example shows that the theorem is false.

Example 4.13 Let $V(C_8) = \{a_0, a_1, \dots, a_7\}$ and $V(C_4) = \{x_0, x_1, x_2, x_3\}$. Let H be the vertex-transitive graph $C_8 \wr C_4$. We know that $|V(H)| = 32$. Consider the subgraph H' of H , with all the edges $\{(a_i, x_k), (a_j, x_p)\}$ in H except the edges satisfying the following condition:

$$j \equiv i+1 \pmod{8} \text{ and } p \equiv k+2 \pmod{4} \text{ for } i, j = 1, 2, \dots, 7 \text{ and } k, p = 0, 1, 2, 3.$$

Since the deletion of the edges is symmetric and these edges form four disjoint isomorphic cycles, the new graph H' is also vertex-transitive. An atomic part is a single vertex and $\kappa(H') = 8$. However, $S = \{(a_i, x_k) | i = 1, 5 \text{ and } k = 0, 1, 2, 3\}$ is a minimum cutset of H' which admits two non-atomic isomorphic parts which contain 12 vertices in each part. But 12 divide neither 32 nor 8. Hence the theorem fails for the case when $k = 2$ and $d = 2$. For the other case, consider the graph $G = C_{11} \wr C_4$ with similar edges being deleted. We can see that $|V(G)| = 44$, $\kappa(G) = 8$ and for the same choice of indices for S , we obtain a minimum cutset which admits two non-atomic parts with 12 and 24 vertices, respectively. Hence the theorem still fails.

Here is another result similar to Theorem 4.3 part (ii) which gives a characteristic of S -minimal parts.

Theorem 4.14 *Let G be vertex-transitive, and let $S \in \mathcal{C}(G)$ admit $k \geq 3$ non-atomic parts. Suppose P is an S -minimal part. Then S admits at most one part P_1 which is not isomorphic to P .*

PROOF: Suppose S admits $k \geq 3$ non-atomic parts. Without loss of generality, we may number them such that $|V(P_1)| \geq |V(P_2)| \geq \dots \geq |V(P_k)|$. Notice that P_k is an S -minimal part.

If $|V(P_i)| = |V(P_k)|$ for all $i \in \{1, 2, \dots, k-1\}$, then P_i and P_k are isomorphic, and therefore, S admits only parts isomorphic to P_k . Hence assume that there is an $i \in \{1, 2, \dots, k-1\}$ such that $|V(P_i)| > |V(P_k)|$. Then $|V(P_1)| > |V(P_k)|$. By vertex-transitivity, there exists a $\phi \in \text{Aut}(G)$ such that $\phi(P_k) \subset P_1$. As a result, $\phi(S) \subset (S \cup P_1)$. Since P_1 is connected, $\phi(S) \cap V(P_1) \neq \emptyset$. Therefore, there is a vertex $z \in S$ that is not in $\phi(S)$, and $G[\{z\} \cup V(P_2) \cup V(P_3) \cup \dots \cup V(P_{k-1})]$ is connected and it is a subgraph of a part Q with respect to the cutset $\phi(S)$. Since $|V(Q)| > |V(\phi(P_i))|$ for all $i \in \{2, 3, \dots, k-1\}$, Q must be $\phi(P_1)$. Hence $|V(P_1)| > |V(P_i)|$ for all $i \in \{2, 3, \dots, k\}$. But if $|V(P_2)| > |V(P_k)|$, then the same arguments hold which imply that $|V(P_2)| > |V(P_1)|$. This is impossible. Therefore, $|V(P_2)| = |V(P_k)|$ which implies that S admits at most two non-isomorphic parts. ■

Chapter 5

Connectivity of Cayley Graphs

Now we are interested in a specific type of vertex-transitive graph, namely, a Cayley graph. First, we have to introduce some definitions.

Definition 5.1 Let G be a finite group and let H be a subset of G . H is said to be a *Cayley set* if $1 \notin H$, where 1 denotes the identity element of G , and $h \in H$ implies $h^{-1} \in H$.

Definition 5.2 The *Cayley graph* $X = X(G; H)$ is the graph with $V(X) = G$ and $E(X) = \{(g_1, g_2) | g_2 = g_1h \text{ where } h \in H\}$ where H is a Cayley set.

Example 5.3 Let G be Z_6 and let $H = \{2, 4\}$. Then, the Cayley graph $X = X(G; H)$ is as in Figure 5.1.

The Cayley set is so defined because loops in a graph will not change its vertex-connectivity, and hence we exclude the identity from the Cayley set. Furthermore, we are interested in undirected graphs, and that is the reason why $h \in H$ implies $h^{-1} \in H$ is in the conditions for a Cayley set. In the previous example, the Cayley graph is not connected which means that the vertex-connectivity is equal to zero.

Definition 5.4 Let G be a finite group, and let $H \subset G$. Then, $\langle H \rangle$ denotes the subgroup generated by H .

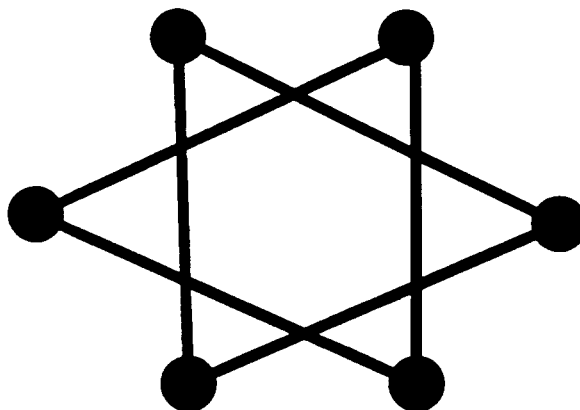


Figure 5.1: The Cayley graph $X = X(G;H)$

Definition 5.5 Let G be a finite group, and let $H \subset G$ be a Cayley set. H is said to be a *generating set* of G if $\langle H \rangle = G$. In addition, H is said to be *minimal* if $\langle H \setminus \{h, h^{-1}\} \rangle$ is a proper subset of G for all $h \in H$. In the case when H is minimal, the Cayley graph is said to be a *minimal* Cayley graph.

The condition $\langle H \rangle = G$ will guarantee that $X(G; H)$ is connected because otherwise, $\langle H \rangle$ is a proper subgroup of G and there is no path from the identity 0 which is in H to any vertex in $G \setminus \langle H \rangle$.

Cayley graphs are vertex-transitive. To show this, consider a connected Cayley graph X with a Cayley set H . Let u and v be two vertices in $V(X)$. Since X is connected, there is a path from u to v . Hence, there is a sequence to elements $\{h_1, h_2, \dots, h_r\}$ from H such that $u, h_1u, h_2h_1u, \dots, h_r \dots h_2h_1u = v$ is the path. Hence $h_r \dots h_2h_1$ is the automorphism in $Aut(G)$ which maps u to v .

Example 5.6 Let G be Z_{12} , the additive group of integers modulo 12, and let $H = \{2, 3, 9, 10\}$. The Cayley graph $X = X(G; H)$ is as in Figure 5.2. It is obvious that H is a minimal generating set for Z_{12} .

In 1981, C.D. Godsil [3] proved that the connectivity of a minimal Cayley graph is always equal to the degree of the graph which is the cardinality of the minimal Cayley set. In 1992, a paper by B. Alspach [1] generalized Godsil's result to Cayley graphs

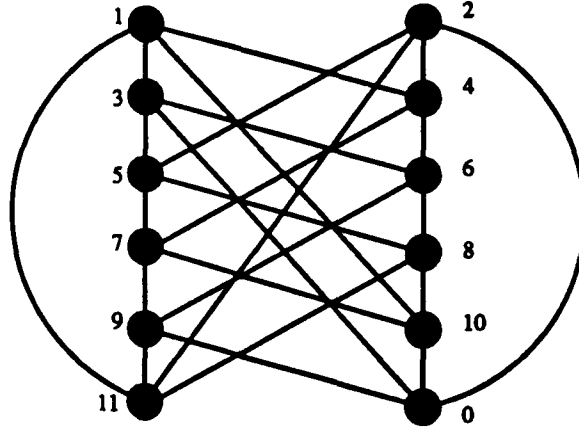


Figure 5.2: The minimal Cayley graph $X = X(G;H)$

with quasiminimal generating Cayley sets. Here is the definition of a quasiminimal generating Cayley set.

Definition 5.7 A Cayley set H is said to be *quasiminimal* if the elements of H can be arranged in the order h_1, h_2, \dots, h_r such that

(i) if the order of h_i is greater than 2, then h_i^{-1} is either h_{i-1} or h_{i+1} , and

(ii) if H_i denotes $\langle \{h_1, h_2, \dots, h_i\} \rangle$, then

(1) for each i such that h_i has order 2, $\langle H_i \rangle$ is a strict supergroup of $\langle H_{i-1} \rangle$,

and

(2) for each i such that h_i has order greater than 2 and $h_i^{-1} = h_{i-1}$, $\langle H_i \rangle$ is a strict supergroup of $\langle H_{i-2} \rangle$.

Notice that any minimal generating Cayley set is a quasiminimal generating Cayley set because the conditions hold trivially. However the converse is not true.

Example 5.8 Let $G = Z_{12}$, $H_1 = \{2, 3, 4, 8, 9, 10\}$ and $H_2 = \{3, 4, 8, 9\}$. We know that $\langle H_1 \rangle = \langle H_2 \rangle = Z_{12}$. H_2 is obviously a minimal generating set which implies that H_1 is not a minimal generating set. However, if we rearrange the elements of H_1 in the order 4, 8, 2, 10, 3, 9, then we see that H_1 is a quasiminimal generating Cayley set.

In the rest of this section, we will assume that G is a finite group and H is a quasiminimal generating Cayley set. Let $H' = H \setminus \{h, h^{-1}\}$ for some $h \in H$ and

$G' = \langle H' \rangle$. Let $X = (G; H)$ and $X' = (G'; H')$ be two Cayley graphs. Since G is the union of the left cosets of G' in G , X consists of vertex-disjoint copies of X' , called the X' -constituents of X , together with some edges joining them which are caused by the elements h and h^{-1} .

Before showing the main result from the paper by B. Alspach, we need the following definition and lemmas.

Definition 5.9 The *quotient graph* X/X' for a Cayley graph X is the graph obtained from X by replacing each X' -constituent of X with a single vertex, and letting two such vertices be adjacent if and only if the corresponding X' -constituents have an edge joining them.

Lemma 5.10 *Let $C \subset V(X)$. If C lies in a single X' -constituent, then $X \setminus C$ is connected.*

PROOF: Let X/X' denote the quotient graph. It is clear that the quotient graph is vertex-transitive since each X' -constituent can be mapped to another by left-multiplication. As a result, X/X' cannot have a cut-vertex. Let Y be the X' -constituent which contains C . Then, the removal of the vertex corresponding to Y will not disconnect X/X' . Hence, $X \setminus Y$ is still connected. Let v be a vertex in $Y \setminus C$. Then, v is connected to at least one vertex in $X \setminus Y$ by an h edge. Therefore, $X \setminus C$ is still connected. ■

Lemma 5.11 *If $s = |H| \geq 4$, then $|G| \geq 3 \cdot 2^a$, where $a = \lfloor (s - 2)/2 \rfloor$ and $s = |H|$. If $|H| = 3$, then either $X = K_4$ or $|G| \geq 6$.*

PROOF: When $|H| = 3$, one possibility is that one of the elements in H has order 2 and the other two are the inverses of one other, that is, $H = \{h_1, h_2, h_2^{-1}\}$. Hence the order of G must be even. One possibility is that $H = \{h_1 = h_2^2, h_2, h_2^{-1} = h_2^3\}$. In this case, G has order 4, and X is K_4 . Otherwise, $G \geq 6$ since h_2 will have to generate at least six elements. The other possibility is that all three of them are of order 2. In this case, G has $8 > 6$ elements. Thus, assume $|H| \geq 4$ and let $H = \{h_1, h_2, \dots, h_s\}$

be in quasiminimal order. If h_1 has order larger than 2, then $h_1^{-1} = h_2$ and $\{h_1, h_2\}$ generates a group of at least 3 elements. Now we add elements one or two at a time and check the order of the group generated by the new elements. If the next element has order greater than 2, we add the next two elements, and the order of the group is at least double the size of the preceding group. If the next element is of order 2, then we only add that element, and the order of the new group is twice the order of the preceding one. In this case, we must repeat this process at least $\lfloor (s-2)/2 \rfloor$ times and the result follows. If h_1 has order 2, then we start with it and continue the preceding process. We notice that the process must be repeated at least $b = \lceil (s-1)/2 \rceil$ times which means that the order of the group generated by H is at least $2 \cdot 2^b$ which is greater than or equal to $3 \cdot 2^a$ where $a = \lfloor (s-2)/2 \rfloor$. ■

One simple observation of the previous lemma is that the number of the vertices in a Cayley graph is always greater than $|H|$.

Lemma 5.12 *Let $|H| > 5$. Let C be a minimum cutset of $X = X(G; H)$ which admits an atomic part and $|C| < |H|$. If $Y \setminus C$ is connected for every X' -constituent Y of X , then some $Y \setminus C$ is an isolated component in $X \setminus C$.*

PROOF: If for every X' -constituent Y , $Y \setminus C$ is connected and joined by an edge to some other $Z \setminus C$ in $X \setminus C$, then every component of $X \setminus C$ has at least $2|X'| - |C|$ vertices. Hence, $p(X) \geq 2|X'| - |C|$. By the previous lemma, $|X'| > |H'|$. Since $|C| \leq |H'| + 1 \leq |X'|$, $p(G) \geq |X'| \geq |C|$ which contradicts Lemma 3.9. Hence, some $Y \setminus C$ must be a part in $X \setminus C$. ■

Lemma 5.13 *A subset of r vertices, $0 \leq r \leq |X'|$, of any X' -constituent Y has at least r neighbors amongst the other X' -constituents.*

PROOF: Let U be a subset of r vertices in some X' -constituent Y . In the case when h has order 2, each $u \in U$ has a neighbor uh in another X' -constituent. The neighbors must be distinct otherwise there are two distinct vertices, x and y , in U such that $xh = yh$ which is a nonsense. If h has order greater than 2, then every $u \in U$ has two

neighbors in another X' -constituent. Hence, U should have $2r$ neighbors. However, some of them might be the same. We see that U must have at least r neighbors because otherwise, there are x and y in U and h_1 and h_2 in H such that $xh_1 = yh_2$ which means that H is not minimal. ■

We intend to show that for any Cayley graph with quasiminimal generating Cayley set, the vertex-connectivity is always equal to the cardinality of the quasiminimal generating Cayley set unless it belongs to a special family. Let \mathcal{F} denote the special family of Cayley graphs. An arbitrary member of \mathcal{F} is a Cayley graph $X = (G; H)$ on a group G with quasiminimal generating Cayley set $H = \{h_1, h_2, \dots, h_{2k}, h_{2k+1}\}$ (with the order in which the elements appear in H being the order satisfying the definition of quasiminimality), where h_1 has order 2, and all others have order 4, commute with h_1 , and have their squares equal to h_1 .

Theorem 5.14 *Let H be a quasiminimal generating Cayley set of the finite group G , and let $X = (G; H)$ be the Cayley graph. Then $\kappa(X) = |H|$ unless X is a member of \mathcal{F} in which case $\kappa(X) = |H| - 1$, $p(X) = 2$ and edges of the form (y, yh_1) are atoms.*

PROOF: We will use induction on $|H|$. For $|H| = 2, 3$, and 4 , $\kappa(X) = |H|$ by Corollary 3.13. Consider the case when $|H| = 5$. Suppose the vertex connectivity of the Cayley graph $X(G, H)$ is not 5. By Theorem 3.11 we know that its vertex connectivity is 4. Hence, the atomic parts of X are all isomorphic to two vertices connected by an edge. Let $A = \{u, v\}$ be an atomic part of X and let C be the minimum cutset of X whose deletion isolates A . By Lemma 3.9 we know that a minimum cutset consists of the atomic parts. As a result, we may assume that $C = \{w, x, y, z\}$ and $x \sim y$ and $w \sim z$. Therefore, the local subgraph of the atomic part is as described in Figure 5.3. Now it is easy to see that the Cayley graph is the lexicographical product (the wreath product) $C_n \wr K_2$ of C_n and K_2 where $n \geq 4$. Since $|H| = 5$ and H is quasiminimal, we can see that n must be even. This graph is in \mathcal{F} because it is the Cayley graph on the group $G = \langle h_1, h_2, h_3 \rangle$, where $1 = h_1^2 = h_2^4 = h_3^4$, $h_1 = h_2^2 = h_3^2$, $h_1h_2 = h_2h_1$, $h_1h_3 = h_3h_1$, and $(h_2h_3)^{n/2} = 1$, with the quasiminimal generating set $\{h_1, h_2, h_2^{-1}, h_3, h_3^{-1}\}$. Hence, the base case is established. Let H be a

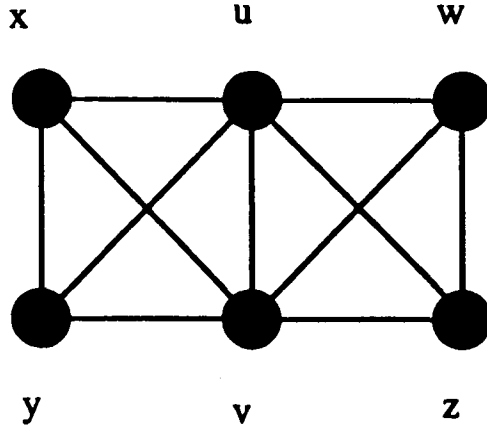


Figure 5.3: The local subgraph of the atomic part A

quasiminimal Cayley generating set and assume $|H| > 5$. Let h be the last element in a quasiminimal ordering of H . First suppose that h has order 2. By induction, each X' -constituent has vertex-connectivity $|H'|$ or $|H'| - 1$, and if it has vertex-connectivity $|H'| - 1$, its atomic parts are isomorphic to an edge. Assume $\kappa(X) < |H|$ and let C be a minimum cutset which produces an atomic part. Since h has order 2, $|H'| = |H| - 1$. Hence, $\kappa(X) = |C| \leq |H'| < |X'|$ by Lemma 5.11. By Lemma 5.10, C cannot belong to a single X' -constituent. Suppose C has non-empty intersections with at least 3 X' -constituents. Then $Y \setminus C$ is connected for any X' -constituent Y because $|Y \cap C| \leq |C| - 2 \leq |H'| - 2$. By Lemma 5.13, $Y \setminus C$ has at least $|Y \setminus C|$ neighbors in each of the other X' -constituents. Not all these neighbors can belong to C otherwise $|C| \geq |Y| = |X'|$ which is a contradiction. Hence there is an edge joining $Y \setminus C$ to a vertex in $Z \setminus C$ for some other X' -constituent Z . This contradicts Lemma 5.12. Hence, there are exactly two X' -constituents, say Y and Z , which have nonempty intersection with C . However, the above arguments still work as long as $Y \setminus C$ and $Z \setminus C$ are connected. It can fail only if one of them is disconnected, say $Y \setminus C$. But in this case, we can conclude, by induction, that $|C \cap Y| = |H'| - 1$ and $X' \in \mathcal{F}$. Hence, every component of $Y \setminus C$ has at least two vertices, one of which must be adjacent to a vertex in $K \setminus C$ where K is another X' -constituent. This contradicts Lemma 3.9 for the same reason as in Lemma 5.12. Now consider the case

when the order of h is greater than 2. By induction, the X' -constituent has vertex-connectivity $|H'| = |H| - 2$ or $|H'| - 1 = |H| - 3$. First, let's assume that $X' \notin \mathcal{F}$, that is, $\kappa(X') = |H'|$. Let C be a minimum cutset which produces an atomic part of X . Suppose that $\kappa(X) < |H|$. By Lemma 5.10, C has non-empty intersections with at least two X' -constituents. The same arguments as before will lead to the conclusion that C has non-empty intersections with exactly two X' -constituents, say Y and Z , and the removal of C will disconnect one of them, say Y . But then C intersects Y in $|H'|$ vertices, and C intersects Z at one vertex. However, $Y \setminus C$ has at least two vertices, and by Lemma 5.13, they must be adjacent to at least two other vertices in another X' -constituent. This again contradicts Lemma 3.9. Hence, we may assume that $X' \in \mathcal{F}$ and C is a minimum cutset which produces an atomic part of X . Suppose that $\kappa(X) < |H|$ which implies that $\kappa(X) \leq |H'| + 1$. Again, by Lemma 5.10, C intersects at least two X' -constituents. If C intersects at least three X' -constituents, say Y , Z , and W , then the same arguments as before will lead to the same contradiction unless C intersects one of them, say Y , in $|H'| - 1$ vertices and the other two, W and Z , at one vertex each and there are no others. Furthermore, since $X' \in \mathcal{F}$, $Y \setminus C$ is disconnected. By induction, every part in $Y \setminus C$ has at least two vertices. For any part of $Y \setminus C$ with more than two vertices, there must be an edge joining this part to another X' -constituent which means that it cannot be an atomic part of X . Therefore, the only possible atomic part of X is a part of $Y \setminus C$ with exactly two vertices and all the neighbors of these two vertices have been deleted. Let $D = \{x, y\}$ be an atomic part of X . However, the only vertices that have been deleted are one vertex u from W and one vertex v from Z . By induction we know that $y = xh_1$. Furthermore, we may assume that $u = xh$ and $v = xh^{-1}$. As a result, $uh = y = xh_1$ which implies that $h^2 = h_1$. This shows that h has order 4 and $hh_1 = h_1h$. Therefore, $X \in \mathcal{F}$. ■

Now, the main theorem in [3] is an immediate consequence of the previous theorem which is stated in the next corollary.

Corollary 5.15 *Let $H = H^{-1}$ be a minimal generating set of G . Then, $\kappa(X(G; H)) = |H|$.*

PROOF: Clearly, X is not in \mathcal{F} . Therefore, by Theorem 5.14 we have the result. ■

Definition 5.16 A vertex-transitive graph G is called *hypoconnected* if $\kappa(G) < \delta(G)$.

The graphs in the family \mathcal{F} are all hypoconnected. A natural question to ask is whether or not there are any other graphs which are also hypoconnected. A result from M.E. Watkins [8] presented in the following theorem shows another class of hypoconnected graphs.

Theorem 5.17 *Let \mathcal{H} denote the set of graphs of the form $G_1 \wr G_2$, where G_1 and G_2 are vertex-transitive graphs, and G_1 is not complete and $V(G_2) > 2$. Then the graphs in \mathcal{H} are all hypoconnected.*

PROOF: Let $G \in \mathcal{H}$. By the proof of Theorem 3.10 we know that $\kappa(G) = \kappa(G_1)|V(G_2)|$. Furthermore, $\delta(G) = \delta(G_2) + \kappa(G_1)|V(G_2)|$. Since $\delta(G_2) \geq 1$, $\kappa(G) < \delta(G)$. ■

It is obvious that \mathcal{H} is non-empty. As for the set \mathcal{F} , the following example will show it is not empty.

Example 5.18 Let G be the group $Z_4 \times Z_2 \times Z_2 \times \dots \times Z_2$ with $k-2$ Z_2 factors. Then $|G| = 2^k$. Let H contain the following elements: $h_1 = (2, 0, \dots, 0)$, $h_2 = (1, 0, \dots, 0)$, $h_4 = (1, 1, 0, \dots, 0)$, $h_{2i} = (1, 0, \dots, 0, 1, 0, \dots, 0)$, where the second 1 is in the i^{th} coordinate for $i = 2, 3, \dots, k-1$. Then $X(G; H)$ has degree $2k+1$, but the deletion of vertices $\{h_2, h_2^{-1}, h_4, h_4^{-1}, \dots, h_{2k}, h_{2k}^{-1}\}$ will disconnect the vertices $\{1, h_1\}$ from the rest of the graph. Therefore, $X(G; H)$ is in \mathcal{F} .

Note that in the proof of Theorem 5.14, the only subgraphs of degree 5 in \mathcal{F} are of the form $C_{2k} \wr K_2$. A generalized result regarding odd degree graphs in \mathcal{F} was obtained by J. Morris [6] and is stated in the following theorem.

Theorem 5.19 *If $X \in \mathcal{F}$ has degree $2n+1$, $n \geq 2$, then X is isomorphic to $Y \wr K_2$ where Y is vertex-transitive and has degree n .*

PROOF: Since $X \in \mathcal{F}$, $p(X) = 2$. Let $\{u, v\}$ be an atomic part of X . Since the degree of X is $2n+1$, $\delta(u) = \delta(v) = 2n+1$. Because $X \in \mathcal{F}$, $\kappa(X) = 2n$ implying that u and

v must be adjacent to the same $2n$ vertices. Furthermore, a minimum cutset consists of copies of the atomic part. Hence $G = Y \wr K_2$. Since atomic parts are disjoint and X is vertex-transitive, Y must be vertex-transitive. ■

By the previous theorem we know that any graph X in \mathcal{F} is of the form $Y \wr K_2$, where Y is some vertex-transitive graph. Then by definition 5.9, Y is the quotient graph X/K_2 . J. Morris [6] gave the following theorem showing that there are some restrictions on a vertex-transitive graph G such that $G \wr K_2$ is in \mathcal{F} .

Theorem 5.20 *Given a Cayley graph $Y = X(G', H')$ of degree d , where H' is a minimal or quasiminimal generating set for G' , and $h^2 = 1$ for all $h \in H'$, Y is the quotient graph for some graph $X(G, H) \in \mathcal{F}$. That is, $Y \wr K_2 \in \mathcal{F}$.*

PROOF: Let $H' = \{h_2, h_3, \dots, h_{d+1}\}$. Take $h_1 \notin G'$, and define $h_1^2 = 1$. Let $H = \{h_1, h_2, h_1 h_2, h_3, h_1 h_3, \dots, h_{d+1}, h_1 h_{d+1}\}$, where $h_i^2 = h_1$ for all $i = 2, 3, \dots, d + 1$. Let $G = \langle H \rangle$. We can see that G is just the union of G' and $h_1 G'$. Since h_1 has order 2, h_i must have order 4 in G , and since $h_i^2 = h_1$, $h_1^{-1} = h_1 h_i$. Therefore, H is already in quasiminimal order. Since H is of the form required, $X(G, H) \in \mathcal{F}$. Because $\{1, h_1\}$ is normal in G , $G/\{1, h_1\}$ is isomorphic to G' . Hence, Y is the quotient graph of $X(G, H)$. ■

Note that the conditions in Theorem 5.20 are both necessary and sufficient to find a graph in \mathcal{F} . Similarly, the conditions in Theorem 5.17 are both necessary and sufficient to find a graph in \mathcal{H} . Furthermore, Theorem 5.20 allows us to generate Cayley graphs in \mathcal{F} . But Theorem 5.17 can be used to generate vertex-transitive graphs with $\delta(G) = \kappa(G) + k$ for all positive integers k . However in the case of vertex-transitive graphs, it is still no easy task in determining the vertex-connectivity. The reason is mainly due to the difficulty of finding an atomic part of a vertex-transitive graph. In 1984, F. Boesch and R. Tindell [2] gave an algorithm to determine the vertex-connectivity of any finite circulant graph. Independently in 1985, M.E. Watkins [9] gave an algorithm to determine the vertex-connectivity of any finite or infinite but locally finite, circulant graph together with an atomic part. We are going to present a simplified version of the algorithm done by Watkins in the next section.

Chapter 6

Algorithm for Circulant Graphs

Watkins applied knowledge on atomic parts of vertex-transitive graphs to a special family, circulant graphs, and got an easy algorithm for determining the vertex-connectivity of circulant graphs. It can be shown that the complexity of his algorithm is of order $O(n^{3/2})$ where n is the number of vertices in the graph. Before we get into the details of his algorithm, let's become familiar with some terminology and develop the theorems upon which the algorithm is based.

Definition 6.1 Let $n \geq 2$ be any finite integer, and let $S = \{i_1, i_2, \dots, i_r\}$ be a finite subset of $Z_{\lfloor n/2 \rfloor}$ where $1 \leq i_1 \leq i_2 \leq \dots \leq i_r \leq n/2$. Then a *finite circulant graph* is defined to be $G = G(n, S)$ where $V(G) = Z_n$ and $(x, y) \in E(G)$ iff there exists a $j \in \{1, 2, \dots, r\}$ such that $x + i_j \equiv y \pmod{n}$ or $y + i_j \equiv x \pmod{n}$.

Example 6.2 For $n = 8$ and $S = \{1, 3, 4\}$, the circulant graph $G(8, S)$ is as in Figure 6.1.

It is obvious that $G(n, S)$ is just $X(Z_n, S \cup S^{-1})$, where S^{-1} consists the additive inverse elements of the elements in S . As a result, circulant graphs are also vertex-transitive. Since we are interested in vertex-connectivity, it is natural to consider only connected circulant graphs. The following theorem gives necessary and sufficient conditions for circulant graphs to be connected.

Theorem 6.3 *The circulant graph $G(n, S)$, where $S = \{i_1, i_2, \dots, i_r\}$, is connected iff $\gcd(i_1, i_2, \dots, i_r, n) = 1$.*

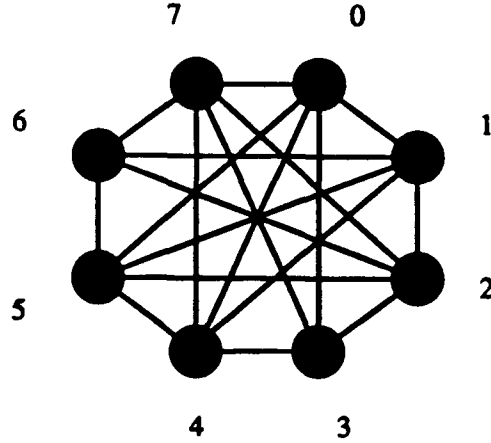


Figure 6.1: The circulant graph $G(8, \{1, 3, 4\})$

PROOF: Let $G(n, S)$ be a connected circulant graph where $S = \{i_1, i_2, \dots, i_r\}$. Since G is connected, there is a path from the vertex 0 to any other vertex x . Therefore, $x \equiv \sum_{m=1}^r a_m i_m \pmod{n}$ where the a_m 's are integers for $m = 1, 2, \dots, r$. Especially, $1 \equiv \sum_{m=1}^r a_m i_m \pmod{n}$. Hence there is an integer $k \geq 1$ such that $kn + 1 = a_1 i_1 + a_2 i_2 + \dots + a_r i_r$, or $1 = a_1 i_1 + a_2 i_2 + \dots + a_r i_r + (-k)n$. This means that the greatest common divisor of the elements in $S \cup \{n\}$ is 1. Assume that $\gcd(i_1, i_2, \dots, i_r, n) = 1$. Therefore, $\exists a_1, a_2, \dots, a_r, a_n \in \mathbb{Z}$ such that $1 = a_1 i_1 + a_2 i_2 + \dots + a_r i_r + a_n n$ or $1 \equiv a_1 i_1 + a_2 i_2 + \dots + a_r i_r \pmod{n}$. Hence we can use the last equation to express any integer x between 0 and $n - 1$ as an integral linear combination of i_1, i_2, \dots, i_r which means that we have a walk from the vertex 0 to any vertex x . Therefore, G is connected. ■

Definition 6.4 Let G and H be finite circulant graphs with n_1 and n_2 vertices respectively. Let Y be a connected spanning subgraph of $G \wr H$. For each $x \in V(G)$, define H_x to be the subgraph of Y with vertex-set $V(H_x) = \{(x, y) | y \in V(H)\}$. We call Y a *circulant product* or *c-product* of G by H , denoted by $G \circ H$, if $\text{Aut}(Y)$ contains an automorphism σ satisfying the following: (a) the sets $V(H_x)$ for $x \in V(G)$ are orbits of σ^{n_1} , and (b) $\text{Aut}(G)$ contains an n_1 -cycle τ such that $\sigma[H_x] = H_{\tau(x)}$.

Example 6.5 Let G be C_4 and H be K_5 . Then the graph in Figure 6.2 is a circulant product of G by H .

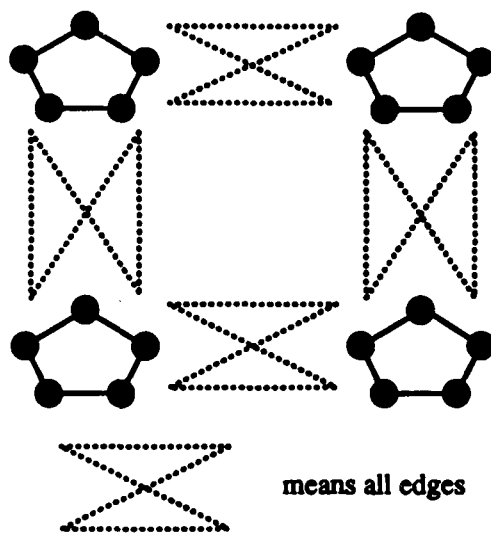


Figure 6.2: A circulant product of C_4 by K_5

Recall that if A is an atomic part of a vertex-transitive graph G , then by Definition 5.9 and Lemma 3.6, the quotient graph $H = G/A$ is vertex-transitive. It is clear that if G is not complete, then neither is H , and G is just a spanning subgraph of $H \wr A$. Furthermore, if G and H are vertex-transitive graphs, then $G \wr H$ is also vertex-transitive. We would like to conclude that the circulant product of two circulant graphs is still circulant. The following lemma will enable us to conclude that.

Lemma 6.6 *Let G and H be circulant graphs. Then $G \circ H$ is also a circulant graph.*

PROOF: Let $(u, v), (x, y) \in V(G \circ H)$. First get σ . By the second condition of the c-product definition, we know there exists an n -cycle $\tau \in \text{Aut}(G)$ so that $\tau^r(u) = x$ for some $r \in \mathbb{Z}_{n_1}$, where $n_1 = |V(G)|$. Hence, $\sigma^r(H_u) = H_x$. As a result, $\sigma^r((u, v)) = (x, z)$ for some $z \in V(H)$. Since the vertices in H_x are in the same orbit of σ^{n_1} , $\exists q \in \mathbb{Z}$ such that $\sigma^{qn_1}((x, z)) = (x, y)$. Hence, $\sigma^{(qn_1+r)}((u, v)) = (x, y)$. Hence $\text{Aut}(G \circ H)$ is transitive implying that $G \circ H$ is circulant. ■

We also want to conclude that any circulant graph is a circulant product of some circulant graphs. For this case, we first need a theorem from Y.O. Hamidoune [5] about Cayley generating sets.

Theorem 6.7 *Let G be a finite abelian group and let A be an atomic part of $X(G; H)$ containing the identity element e of G . Then $V(A)$ is the subgroup of G generated by $V(A) \cap H$.*

PROOF: Let $x \in V(A)$. Because A is an atomic part, $xV(A) = V(A)$ which implies that $V(A)$ is closed. Hence, $V(A)$ is a subgroup. We now only have to show that $V(A)$ is generated by $V(A) \cap H$. Let $x \in V(A)$. Since e and x are both in $V(A)$, there is a path, say $e = x_0, x_1, x_2, \dots, x_k = x$, from e to x contained in $V(A)$. Since there is an edge from x_i to x_{i+1} , $\exists h_{i+1} \in H$ such that $x_{i+1} = x_i h_{i+1}$. Therefore, $x = \prod_{i=1}^k h_i$ and $h_i = x_i^{-1} x_{i+1}$. Therefore, x is the product of some elements in $V(A) \cap H$. ■

Lemma 6.8 *Let Y be a circulant graph and let H be an atomic part of Y . Then there exists a unique circulant graph G (up to isomorphism) such that $Y = G \circ H$.*

PROOF: Without loss of generality, choose an arbitrary vertex v . Since Y is circulant, $\exists \sigma \in \text{Aut}(Y)$ such that σ is an n -cycle where $n = |V(Y)|$. Now relabel a vertex with i if it is $\sigma^i(v)$. Let H be the atomic part which contains 0. By the previous theorem we know that the vertices of H form a subgroup of Z_n . Hence, $V(H) = \{0, m, 2m, \dots, (a-1)m\}$, where $am = n$. Let U_i denote the atomic parts of Y , where $U_i = \sigma^i(H)$, $i = 1, 2, \dots, m-1$. Let G be the quotient graph Y/H . Define $\tau : V(G) \rightarrow V(G)$ by $\tau(U_i) = U_{\sigma(i)} = U_{i+1}$ where the indices are all taken modulo m . Therefore, G is circulant and G is unique. ■

The following corollary is an immediate consequence of the previous two lemmas.

Corollary 6.9 *Let G be a vertex-transitive graph, and let A be an atomic part of G . Then G is circulant iff $G = H \circ A$ for some circulant graph H .*

The basic idea of Watkins' algorithm is to run through all possible c-products $H \circ A$ of a given circulant graph $G(n, S)$ seeking a c-product in which A is isomorphic to an

atomic part of G . Since $p(G)$ must divide n , for any divisor d of n , an n/d -element subgroup $A(d) = \{0, d, 2d, \dots, n - d\}$ is a good candidate to be an atomic part. For each candidate, there are five tests to ensure that it can be a non-trivial atomic part. If no candidate can pass all five tests, then G has only trivial atomic parts which means that $\kappa(G) = \delta(G)$. By Theorem 6.7 and the fact that $0 \notin S$, we can see that if G has a non-trivial atomic part, then a candidate which has empty intersection with S cannot be an atomic part. This is Test 1. Since a candidate $A(d)$ has n/d elements and is a subgroup of Z_n , the greatest common divisor of $(A(d) \cap S) \cup \{n\}$ must be d . This constitutes Test 2. Furthermore, if $A(d)$ is an atomic part, then by the previous corollary we know that $G = H(d) \circ A(d)$ for some circulant graph $H(d)$. Since $A(d)$ is an atomic part, $H(d)$ can not be complete. Hence we know that $d \geq 4$. This is Test 3. If G is complete, then $\kappa(G) = \delta(G)$. Hence we may assume that G is not complete. As a result, $H(d)$ is not complete. Since $H(d)$ is not complete, $\delta(H(d)) \leq d - 2$. This is Test 4. Now all candidates which pass all four tests can be used in producing c-products of G . Since $\kappa(G) \leq \delta(G)$, we can eliminate those candidates $A(d)$ whose neighborhoods $N(A(d))$ have cardinality greater than $\delta(G)$. This is Test 5. After the last test, we have a collection of candidates which have $|N(A(d))| < \delta(G)$. The minimum of all such $|N(A(d))|$'s is the vertex-connectivity of the circulant graph G . Now eliminate all the candidates for which $|N(A(d))|$ is not minimum. At this stage, we are left with a collection of candidates which can be either an atomic part of G or a non-atomic part of G and all we have to do is choose d such that $|A(d)|$ will be minimum. Therefore, the vertex-connectivity of the circulant graph G is $|N(A(d))|$ and the atomic part of G containing 0 is $A(d)$. An immediate consequence of the first three tests is the following:

Proposition 6.10 *Let $G(n, S)$ be a circulant graph where $S = \{a_1, a_2, \dots, a_k\}$. If a_j is relatively prime to n whenever $a_j \geq 4$, then $\kappa(G) = \delta(G)$.*

PROOF: If $a_j \in A(d) \cap S$, where $a_j \geq 4$, then $\gcd((A(d) \cap S) \cup \{n\}) = 1$. Hence a_j cannot be in $A(d) \cap S$ whenever $a_j \geq 4$. Therefore, $A(d) \cap S$ can only have elements which are less than 4. However, none of the elements in $A(d) \cap S$ can be less than 4 by Test 3. Hence, the intersection must be empty. Therefore, the atomic parts of G

are all trivial, and $\kappa(G) = \delta(G)$. ■

Here is a pseudo-version of simplified Watkins' algorithm. Comments in this algorithm will have a leading #.

THE ALGORITHM

Step I Input circulant graph $G(n, S)$ where $S = \{a_1, a_2, \dots, a_k\}$.

Step II Initialization

- Let $S^{-1} = \{n - a_k, n - a_{k-1}, \dots, n - a_1\}$.
- $\Delta = 2k$ if $a_k \neq n/2$, otherwise $\Delta = 2k - 1$.
- Let $\kappa = \Delta$.
- Let $NFactors$ be the set of all proper divisor of n .
- $i = 0$.
- $Dset = \emptyset$.

Step III • $i = i + 1$.

- If $i > NFactors$, then go to Step X.
- Let d be the i^{th} elements in $NFactors$.

Step IV • If $d < 4$, then go to Step III.

This is Test 3.

Step V • Let $S_d = \{s \in S \mid s = md \text{ for some } m \in \mathbb{Z}^+\}$.

S_d is $A(d) \cap S$.

- If $S_d = \emptyset$, then go to Step III.

This is Test 1.

Step VI • If $\gcd(S_d \cup \{n\}) \neq d$, then go to Step III.

This is Test 2.

Step VII • Let $NBHD = \{s \pmod{d} \mid s \in S \cup S^{-1}, s \not\equiv 0 \pmod{d}\}$.

• If $|NBHD| > d - 2$, then go to Step III.

This is Test 4.

Step VIII • Let $New = |NBHD| \cdot (n/d)$.

• If $New \geq Delta$, then go to Step III.

This is Test 5.

Step IX • If $New < Kappa$, then $kappa = New$ and $Dset = \{d\}$.

• If $New = Kappa$, then $Dset = Dset \cup \{d\}$.

• Go to Step III.

Step X • Output: $Kappa$ is the vertex-connectivity of G and the atomic parts of G are all isomorphic to $A(d)$.

• Terminate.

Now let's apply the algorithm to an example.

Example 6.11 Let $S = \{1, 4, 8, 9, 12, 15, 16, 20, 23, 24, 25, 28, 32, 33\}$. Consider the circulant graph $G(72, S)$. We observe that $\delta(G) = 28$. The proper divisors of 72 are $\{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36\}$. Test 3 eliminates 1, 2, and 3 as choices of d , and Test 1 eliminates 18. Furthermore, Test 2 eliminates 6 since $A(d) \cap S = \{12, 24\}$ and the greatest common divisor of $\{12, 24, 72\}$ is $12 \neq 6$. Now for $d = 9$, $\delta(H(d)) = |\{1, 4, 8, 3, 6, 7, 2, 5, 6, 7, 1, 5, 6\}| = |\{1, 2, 3, 4, 5, 6, 7, 8\}| = 8 > 9 - 2$. Hence 9 is eliminated. Similarly we can calculate the following: $\delta(H(4)) = 2$, $\delta(H(8)) = 3$, $\delta(H(12)) = 6$, $\delta(H(24)) = 9$, and $\delta(H(36)) = 14$. Furthermore, $|N(A(d))| < \delta(G)$ only holds for $d = 8$ or 24 and $|N(A(8))| = |N(A(24))| = 27$. Therefore, $\kappa(G) = 27$. Since we are left with only two candidates $d = 8$ and $d = 24$. However, $|A(8)| = 9$ and $|A(24)| = 3$. Hence $A(24)$ is the graph which is isomorphic to the atomic parts of G .

Bibliography

- [1] B. Alspach. *Cayley Graphs with Optimal Fault Tolerance*, IEEE Transactions on Computers, Vol. 41, No 10, pp 1337-1339, 1992
- [2] F. Boesch and R. Tindell. *Circulants and Their Connectivities*, J. of Graph Theory, Vol. 8, pp 487 - 499, 1984
- [3] C.D. Godsil. *Connectivity of Minimal Cayley Graphs*, Arch. Math Vol. 37, pp 473 - 476, 1981
- [4] A.C. Green. *Structure of Vertex-Transitive graphs*, J. Comb. Theory B. Vol. 18, pp 1 - 18, 1975
- [5] Y.O. Hamidoune *On the Connectivity of Cayley Digraphs*, Europ. J. Combinatorics, 5, pp 309 - 312, 1984
- [6] J. Morris. *Connectivity of Cayley Graphs : A Special Family*, Preprint
- [7] M.E. Watkins. *Connectivity of Transitive Graphs*, J. Comb. Theory Vol. 8, pp 23 - 29, 1970
- [8] M.E. Watkins. *Some Classes of Hypoconnected Vertex-Transitive Graphs*, Recent Progress in Combinatorics; Proceedings of the Third Waterloo Conference on Combinatorics, Editors : W.T. Tutte and C. St. J. A. Nash-Williams, Academic Press, N.Y., pp 323-328, 1969
- [9] M.E. Watkins. *Computing the Connectivity of Circulant Graphs*, Congressus Numerantium, 49, pp 247 - 258, 1985