SEMIGROUP EXPANSIONS

by

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SEMIGROUP EXPANSIONS

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ABSTRACT

This thesis consists of a study of expansions in some subcategories of the category $\mathscr{S}_X$ of semigroups . In particular , we consider expansions in the category $\mathscr{S}_X{}^*$ of quotients of the free semigroup on X.

The first chapter includes an introduction to the subject of the thesis and a brief resume of the results.

The second chapter contains some background and preliminaries for the succeeding chapters.

In the third chapter we first develop the concept of contractions on the lattice $\Gamma(X)$ of congruences on the free semigroup on X and then we show that there exist mappings φ and ψ between the set of contractions in $\Gamma(X)$ and the set of expansions in $\mathscr{S}_X{}^*$ which are inverse order anti-isomorphisms , and we give these mappings explicitly. We also give some basic properties of these lattices.

The fourth chapter consists of the characterization of some special expansions in terms of contractions and the explicit definitions of the joins and the meets of some known expansions.

In the final chapter we characterize the expansions in the category of monogenic semigroups and we also give results related to the lattice of these expansions such as the order in this lattice and the compatibility of these lattice operations with multiplication.

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CHAPTER 1

Introduction.

From the point of view of the theory of semigroups, it is natural that one should want to consider for a given semigroup S those semigroups $\bar{S}$ for which there is a natural epimorphism $\eta_S : \bar{S} \to S$; that is, S is a homomorphic image of $\bar{S}$, and such that $\bar{S}$ is *close* to the particular semigroup S. More precisely, we are interested in functors F, from special categories of semigroups and morphisms into special categories of semigroups and morphisms. Such a functor F, is called an expansion if in addition there exists a natural transformation η from the functor F to the identity functor such that each η_S is surjective.

So, given a semigroup S, we are interested in the situation where there exists an expanded semigroup F(S) and an epimorphism η_S: F(S) $\to$ S ; given a morphism $\varphi: S \to T$, there exists a morphism $F(\varphi): F(S) \to F(T)$; if φ is surjective, $F(\varphi)$ should also be surjective ; and (functorially) if ι is the identity function of S, then $F(\iota)$ is the identity of F(S), and if $S \xrightarrow{\varphi} T \xrightarrow{\psi} U$ then $F(\Phi \circ \Psi) = F(\Phi) \circ F(\Psi)$; morover the following diagram commutes :

$$\begin{tikzcd} F(S) \arrow[r, "F(\varphi)"] \arrow[d, "\eta_S"'] & F(T) \arrow[d, "\eta_T"] \\ S \arrow[r, "\varphi"] & T \end{tikzcd}$$

This thesis is devoted to the investigation of expansions in the category $\mathscr{S}_X$ and its subcategories. In order to do that, the concept of a contraction in $\mathrm{Con}X^+$ is introduced.

We begin in Chapter 2 by giving the appropriate background for the Chapters 3, 4 and 5. For a general introduction to the theory of semigroups, the reader is referred to [4] , [6] or [9] .

In the third chapter we start with the definition of an expansion in $\mathscr{S}_X$ and we introduce the subcategory $\mathscr{S}_X{}^*$ of $\mathscr{S}_X$, and we prove that every expansion in $\mathscr{S}_X$ is congruent to an expansion in $\mathscr{S}_X{}^*$ in order to work only in $\mathscr{S}_X{}^*$. We then describe the concept of a contraction in $\mathrm{Con}X^+$ and we give the order on the set of contractions in $\mathrm{Con}X^+$, $\mathscr{C}$, which is shown to be a lattice. We also introduce the expansions in $\mathscr{S}_X{}^*$ based on contractions in $\mathrm{Con}X^+$ and conversely the contractions in $\mathrm{Con}X^+$ based on expansions in $\mathscr{S}_X{}^*$. Finally, we give two mappings , Ψ and Φ, from the set of expansions in $\mathscr{S}_X{}^*$, $\mathscr{E}$, onto $\mathscr{C}$, and from $\mathscr{C}$ onto $\mathscr{E}$ respectively. The main result of this chapter is the fact that these mappings are order anti-isomorphisms and consequently , we define a partial order on $\mathscr{E}$ which is then viewed as a lattice. We also show that $\mathscr{C}$ and $\mathscr{E}$ form as well semigroups with composition as multiplication and that Ψ and Φ are also semigroup isomorphisms. The last result of this chapter concerns the compatibility of the product and the join of two expansions, and is illustrated by an example.

In Chapter 4 we consider some known expansions. The first section is devoted to the machine expansion $\overline{S}^{\mathscr{L}}$. We begin by introducing a congruence $\overline{\rho}^{\mathscr{L}}$

and a contraction $f : \rho \to \bar{\rho}^{\mathcal{L}}$. We also give an isomorphism φ from the free semigroup on X, X^+, modulo ρ to the cutdown to generators A of the left machine expansion on X^+/ρ. We close this section by proving that $\Phi(f)$ is congruent to the left machine expansion cutdown to generators. In the second section we turn our attention to an expansion based on the machine expansion, $\tilde{S}^{\mathcal{L}}$, and we introduce a congruence $\tilde{\rho}^{\mathcal{L}}$ and a contraction $l : \rho \to \tilde{\rho}^{\mathcal{L}}$. We give an isomorphism φ from $X^+/\tilde{\rho}^{\mathcal{L}}$ to the cutdown to generators A of the expansion $(\widetilde{X^+/\rho})^{\mathcal{L}}$. We end by proving that $\Phi(l)$ is congruent to the expansion L which maps X^+/ρ to the cutdown to generators A of the expansion $(\widetilde{X^+/\rho})^{\mathcal{L}}$. In the third section we are concerned with the Henckell's expansion $\hat{S}^{(2)}$ and as before we give the congruence $\hat{\rho}^{(2)}$, the contraction $h : \rho \to \hat{\rho}^{(2)}$, the isomorphism φ from $X^+/\hat{\rho}^{(2)}$ onto $(\widehat{X^+/\rho})^{(2)}_A$ and we end by proving that $\Phi(h)$ is congruent to the expansion H which maps X^+/ρ to the cutdown to generators A of the expansion $(\widehat{X^+/\rho})^{(2)}$. The fourth section contains some lattice theoretical results about the congruences and the expansions stated in sections 1, 2 and 3. We introduce an expansion P, which is shown to be the join of the expansions $E_{\mathcal{L}} : S \to \tilde{S}^{\mathcal{L}}$ and $E_{\mathcal{R}} : S \to \tilde{S}^{\mathcal{R}}$. We define the congruence ρ^P corresponding to P, we prove that the contraction $p : \rho \to \rho^P$ is congruent to the expansion P_A which maps S to the cutdown to generators A of P(S). The last result of this section is that ρ^P is the meet of $\tilde{\rho}^{\mathcal{L}}$ and $\tilde{\rho}^{\mathcal{R}}$.

In the last chapter we are concerned with the expansions in category of monogenic semigroup $\mathcal{M}$, and the contractions in the congruences on the free monogenic semigroup F. In the first section we give a characterization of the congruences on F and the order in ConF. We remark that ConF can be identified with $\mathscr{M}$. This section is closed by a characterization of contractions in ConF and an example. The second section is devoted to the expansions in the category of monogenic semigroups. We first remark that the set of expansions in $\mathscr{M}$, $\mathscr{E}_{\mathscr{M}}$, is equal to the set of contractions in $\mathscr{C}_{\mathscr{M}}$. We then give a partial order in $\mathscr{E}_{\mathscr{M}}$, and we define the lattice operations. Finally, we investigate the compatibility conditions of these lattice operations with multiplication.

CHAPTER 2

Preliminaries.

The purpose of this chapter is to introduce basic concepts used throughout this thesis and to establish a few properties of these to be used in the succeeding chapters.

2.1. Basic concepts concerning semigroups.

Definition 2.1.1. Let S be a set and $*$ be a binary operation on S. Then $(S,*)$ is called a *semigroup* if and only if $(a * b) * c = a * (b * c)$ for any $a,b,c \in S$.

It is customary to say " semigroup S " rather than " semigroup $(S,*)$ ".

Definition 2.1.2. A semigroup S is *generated by* a subset G if every element of S can be written as a product of some elements of G, and it is denoted by $S = \langle G \rangle$.

Definition 2.1.3. An equivalence relation ρ on a semigroup S is a *left congruence* if for all $a,b,c \in S$, $a \ \rho \ b$ implies $ca \ \rho \ cb$, a *right congruence* if $a \ \rho \ b$ implies $ac \ \rho \ bc$; ρ is a *congruence* if it is both a left and a right congruence.

Lemma 2.1.4. An equivalence relation ρ on a semigroup S is a congruence if and only if for all $a,b,c,d \in S$ $a \ \rho \ b$ and $c \ \rho \ d$ implies $ac \ \rho \ bd$.

Proof. Let ρ be a congruence on S, let $a,b,c,d \in S$ be such that $a \ \rho \ b$ and $c \ \rho \ d$. Then since ρ is a right congruence we have $ac \ \rho \ bc$ and since ρ is also a left congruence we have $bc \ \rho \ bd$. Hence $ac \ \rho \ bd$. Conversely, if for any $a,b,c,d \in S$, $a \ \rho \ b$ and $c \ \rho \ d$ implies $ac \ \rho \ bd$ then ρ is clearly a congruence .•

Definition 2.1.5. For a semigroup S let *ConS* denote the set of all congruences on S.

Let X be a non-empty finite set. A non-empty finite sequence $x_1,x_2,\ldots,x_n$ usually written by juxtaposition , $x_1x_2\ldots x_n$, of elements of X is called a *word over the alphabet* X . Let 1 denote the empty word .

Definition 2.1.6. The set X^+ of all non-empty words with operation of juxtaposition

$$(x_1x_2\ldots x_n)\cdot(y_1y_2\ldots y_n) = x_1x_2\ldots x_ny_1y_2\ldots y_n$$

is a semigroup called *the free semigroup on the set* X . Let $X^* = X^+ \cup \{1\}$.

For $u = x_1x_2\ldots x_n \in X^+$, let $|u|$ denote *the length of* u , which is equal to n in this case .

Definition 2.1.7. For any $i \geq 0$, let e^i and s^i be functions from X^* into X^* defined by : Let $u = x_1x_2\ldots x_n \in X^+$, then $e^i(u) = x_{i+1}x_{i+2}\ldots x_n$ and $s^i(u) = x_1x_2\ldots x_{n-i}$.

2.2. Basic concepts concerning expansions and $\mathcal{S}_X$.

Definition 2.2.1. Let S,T be semigroups and φ be a function from S into T. Then φ is called a *morphism of semigroup* if and only if $(s\varphi)\cdot(s'\varphi) = (s\cdot s')\varphi$ for any $s,s' \in S$.

For a formal definition of a *functor*, a *category* and a *subcategory* the reader is referred to [5] or [8].

Let $\mathcal{S}$ denote the category of all semigroups and morphisms.

Definition 2.2.2. A functor f from a category $\mathcal{A}$ of semigroups into $\mathcal{A}$ is called an *expansion* if there is a natural transformation η from the functor f to the identity functor, such that each η_S is surjective . This concept was introduced and studied in the papers by Birget and Rhodes [1] and [2] .

Definition 2.2.3. A semigroup S is "generated" by a set X if and only if there exists a function $\alpha : X \to S$ such that $S = \langle (X)\alpha \rangle$; i.e., S is generated -in the classical sense- by the range $(X)\alpha$ of α ; we do not assume that $X \subseteq S$, nor even $|X| \leq |S|$.

Definition 2.2.4. For a given set X , we consider the category $\mathcal{S}_X$ of all semigroups "generated" by X : The objects of the category $\mathcal{S}_X$ are of the form (S,α) where S is a semigroup and $\alpha : X \to S$ is a function such that $S = \langle (X)\alpha \rangle$.

The morphisms from (S,α) into (T,β) are those semigroup morphisms $\varphi : S \to T$ such that the following diagram commutes :

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ S \arrow[rr, "\varphi"] & & T \end{tikzcd}$$

The elements of $\mathscr{S}_X$ are (S,α) but when convenient we will simply write S for (S,α).

Remark 2.2.5. In $\mathscr{S}_X$, due to the commutativity of the above diagram, any morphism φ is necessarily surjective. Moreover, since any morphism is completely determined by its action on a set of generators, the morphisms in $\mathscr{S}_X$ are uniquely determined.

Some other basic properties of $\mathscr{S}_X$ are given in [1].

CHAPTER 3

In this chapter we develop the concept of *contraction* and we explicitly give two mappings which are inverse order anti-isomorphisms between the lattice of expansions in $\mathscr{S}_X$ and the lattice of contractions in $\mathrm{Con}X^+$.

In the context of $\mathscr{S}_X$, due to the Remark 2.2.5, the definition of an expansion simplifies to the following :

Definition 3.1. Let $\mathscr{A}$ be a subcategory of $\mathscr{S}_X$. A functor $F: \mathscr{A} \to \mathscr{A}$ is called an *expansion* if the following condition is satisfied:

E(i) For any $(A, \alpha) \in \mathscr{A}$, there exists an epimorphism $\eta_A: F(A) \to A$.

Definition 3.2. Let $\mathscr{A}$ be a subcategory of $\mathscr{S}_X$. Let F and G be expansions within $\mathscr{S}_X$ and $\mathscr{A}$, respectively. We say that F is *congruent* to G if, for any $(S, \alpha) \in \mathscr{S}_X$, there exist $(T, \beta) \in \mathscr{A}$ and isomorphisms φ and ψ such that (S, α) is isomorphic to (T, β) via φ and $F(S, \alpha)$ is isomorphic to $G(T, \beta)$ via ψ.

Note that, by the remark following Definition 2.2.2, the following diagram is automatically commutative:

$$\begin{array}{ccc} F(S,\alpha) & \xrightarrow{\ \psi\ } & G(T,\beta) \\ {\scriptstyle \eta_S}\downarrow & & \downarrow{\scriptstyle \eta_T} \\ (S,\alpha) & \xrightarrow[\ \varphi\ ]{} & (T,\beta) \end{array}$$

For any congruence $\rho \in \mathrm{Con}X^+$, let $\iota_\rho : X \to X^+/\rho$ be the morphism defined by $(x)\iota_\rho = x\rho$.

Let $\mathscr{S}_X{}^* = \{(X^+/\rho, \iota_\rho) : \rho \in \mathrm{Con}X^+\}$. Clearly $\mathscr{S}_X{}^*$ is a subcategory of $\mathscr{S}_X$.

<u>Remark 3.3.</u> In $\mathscr{S}_X{}^*$ there exists a homomorphism $\varphi: (X^+/\rho, \iota_\rho) \to (X^+/\tau, \iota_\tau)$ if and only if $\rho \subseteq \tau$ and when this is the case φ is unique.

That being understood, we can identify $(X^+/\rho, \iota_\rho)$ with X^+/ρ.

<u>Proposition 3.4</u>. Let F be an expansion in $\mathscr{S}_X$. Then F is congruent to an expansion in $\mathscr{S}_X{}^*$.

<u>Proof</u>. Let F be an expansion in $\mathscr{S}_X$. By the universal property of X^+, for any $(S,\alpha) \in \mathscr{S}_X$, there exists a unique congruence $\rho \in \mathrm{Con}X^+$ such that $(X^+/\rho, \iota_\rho) \overset{\varphi}{\cong} (S,\alpha)$, namely $\rho = \alpha \circ \alpha^{-1}$.

$$\begin{tikzcd} & X \arrow[dl, "\iota_\rho"'] \arrow[dr, "\alpha"] & \\ X^+/\rho \arrow[rr, "\varphi"] & & S \end{tikzcd}$$

Now, define

$$G: \mathscr{S}_X{}^* \to \mathscr{S}_X{}^* \quad \text{by}$$

$$G:\ X^+/\rho \to X^+/\rho^F \quad (\rho \in \mathrm{Con}X^+)$$

where ρ^F is the unique congruence on X^+ such that $(X^+/\rho^F, \iota_{\rho^F}) \cong F(X^+/\rho, \iota_\rho)$. For any $X^+/\rho \in \mathscr{S}_X{}^*$, since there exists an epimorphism η from $F(X^+/\rho)$ onto X^+/ρ and an isomorphism φ_ρ from $G(X^+/\rho) = X^+/\rho^F$ onto $F(X^+/\rho)$, $\eta' = \eta \circ \varphi_\rho$ is an epimorphism from $X^+/\rho^F = G(X^+/\rho)$ onto X^+/ρ.

$$\begin{tikzcd} G(X^+/\rho) = X^+/\rho^F \arrow[r, "\varphi_\rho"] \arrow[dr, "\eta'"'] & F(X^+/\rho) \arrow[d, "\eta"] \\ & X^+/\rho \end{tikzcd}$$

Now, let X^+/ρ, $X^+/\tau \in \mathscr{S}_X{}^*$ and $\theta: X^+/\rho \to X^+/\tau$ be a morphism. Then there exists an epimorphism $F(\theta): F(X^+/\rho) \to F(X^+/\tau)$, and isomorphisms

$$\Phi_\rho: X^+/\rho^F \to F(X^+/\rho) \quad \text{and} \quad \Psi_\tau: F(X^+/\tau) \to X^+/\tau^F.$$

Therefore $G(\theta) = \Psi_\tau \circ F(\theta) \circ \Phi_\rho$ is an epimorphism from $X^+/\rho^F = G(X^+/\rho)$ onto $X^+/\tau^F = G(X^+/\tau)$.

$$\begin{tikzcd} X^+/\rho^F \arrow[r, "\Phi_\rho"'] & F(X^+/\rho) \arrow[r, "F(\theta)"] \arrow[d] & F(X^+/\tau) \arrow[r, "\Psi_\tau"'] \arrow[d] & X^+/\tau^F \\ & X^+/\rho \arrow[r, "\theta"] & X^+/\tau & \end{tikzcd}$$

Hence, G is a functor consequently, G is an expansion in $\mathscr{S}_X{}^*$.

Next, let $(S, \alpha) \in \mathscr{S}_X$. Again by the universal property of X^+ there exists a unique homomorphism $\varphi: X^+ \to S$ such that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "\iota"'] \arrow[dr, "\alpha"] & \\ X^+ \arrow[rr, "\varphi"] & & S \end{tikzcd}$$

Since $(S, \alpha) \in \mathscr{S}_X$, φ is an epimorphism and therefore $S \cong X^+/\rho$, where $\rho = \varphi \circ \varphi^{-1}$. By the definition of G, F(S) is isomorphic to $G(X^+/\rho)$. Thus, F is congruent to G.•

For the remainder of this chapter, we restrict our attention to $\mathscr{S}_X{}^*$.

Definition 3.5. A function $f : \mathrm{Con}X^+ \to \mathrm{Con}X^+$ is called a *contraction* if the following conditions are satisfied:

C(i) $f(\rho) \subseteq \rho$ for any $\rho \in \mathrm{Con}X^+$.

C(ii) If $\rho, \tau \in \mathrm{Con}X^+$ and $\rho \subseteq \tau$ then $f(\rho) \subseteq f(\tau)$.

Definition 3.6. Let

$$\mathscr{E} = \{ F: \mathscr{S}_X^* \to \mathscr{S}_X^*, F \text{ is an expansion} \},$$

$$\mathscr{C} = \{ f : \mathrm{Con}X^+ \to \mathrm{Con}X^+, f \text{ is a contraction} \}.$$

We now introduce order relations on $\mathscr{E}$ and $\mathscr{C}$:

Definition 3.7. For $F, G \in \mathscr{E}$, let $F \geq G$ if for any $X^+/\rho \in \mathscr{S}_X^*$ there exists an epimorphism θ_ρ from $F(X^+/\rho)$ onto $G(X^+/\rho)$. For $f, g \in \mathscr{C}$, let $f \leq g$ if $f(\rho) \subseteq g(\rho)$ for all $\rho \in \mathrm{Con}X^+$.

Clearly these are partial orders on $\mathscr{E}$ and $\mathscr{C}$, respectively.

Lemma 3.8. $\mathscr{C}$ is a lattice where

$$(f \wedge g)(\rho) = f(\rho) \wedge g(\rho) \quad \text{and} \quad (f \vee g)(\rho) = f(\rho) \vee g(\rho).$$

Proof. Let f and $g \in \mathscr{C}$ and define $h: \mathrm{Con}X^+ \to \mathrm{Con}X^+$ by :

$$h(\rho) = f(\rho) \wedge g(\rho) \qquad (\rho \in \mathrm{Con}X^+).$$

We first show that $h \in \mathscr{C}$.

Since $f(\rho) \subseteq \rho$ and $g(\rho) \subseteq \rho$ we have $h(\rho) = f(\rho) \wedge g(\rho) \subseteq \rho$ so that h satisfies C(i). Let $\tau \subseteq \rho$. Then $f(\tau) \subseteq f(\rho)$ and $g(\tau) \subseteq g(\rho)$ since $f, g \in \mathscr{C}$ and therefore, we have that $h(\tau) = f(\tau) \wedge g(\tau) \subseteq f(\rho) \wedge g(\rho) = h(\rho)$. Thus h satisfies C(ii) and $h \in \mathscr{C}$.

Now, let $t \in \mathscr{C}$ be such that $t \leq f$ and $t \leq g$. Then $t(\rho) \subseteq f(\rho)$ and $t(\rho) \subseteq g(\rho)$ for any $\rho \in \mathrm{Con}X^+$. Therefore we have

$$t(\rho) \subseteq f(\rho) \wedge g(\rho) = h(\rho) \quad \text{for any } \rho \in \mathrm{Con}X^+.$$

Thus $t \leq h$. Clearly, h is a lower bound of f and g , consequently h is the greatest lower bound of f and g .

Next , define $k\colon \mathrm{Con}X^+ \to \mathrm{Con}X^+$ by

$$k(\rho) = f(\rho) \vee g(\rho) \quad (\rho \in \mathrm{Con}X^+).$$

Let us see that $k \in \mathscr{C}$:

Since $f(\rho) \subseteq \rho$ and $g(\rho) \subseteq \rho$ we have $k(\rho) = f(\rho) \vee g(\rho) \subseteq \rho$ and k satisfies C(i) . Let $\tau \subseteq \rho$ then $f(\tau) \subseteq f(\rho)$ and $g(\tau) \subseteq g(\rho)$ and therefore we have $k(\tau) = f(\tau) \vee g(\tau) \subseteq f(\rho) \vee g(\rho) = k(\rho)$. Thus k satisfies C(ii) and $k \in \mathscr{C}$. Clearly , k is an upper bound of f and g .

Now, let $t \in \mathscr{C}$ be such that $t \geq f$ and $t \geq g$. Then $f(\rho) \subseteq t(\rho)$ and $g(\rho) \subseteq t(\rho)$ for any $\rho \in \mathrm{Con}X^+$. Therefore we have:

$$k(\rho) = f(\rho) \vee g(\rho) \subseteq t(\rho) \quad \text{for any } \rho \in \mathrm{Con}X^+$$

Thus $t \geq k$, and k is the least upper bound of f and g .•

<u>Definition 3.9</u>. For any $f \in \mathscr{C}$ let $E_f : \mathscr{S}_X{}^* \to \mathscr{S}_X{}^*$ be defined by :

$$E_f(X^+/\rho) = X^+/f(\rho) \quad (\rho \in \mathrm{Con}X^+)$$

and similarly for any $F \in \mathscr{E}$ let $c_F\colon \mathrm{Con}X^+ \to \mathrm{Con}X^+$ be defined by:

$$c_F(\rho) = \rho^F \quad \text{where} \quad F(X^+/\rho) = X^+/\rho^F.$$

<u>Lemma 3.10.</u> (i) For any $f \in \mathscr{C}$, E_f is an expansion in $\mathscr{S}_X{}^*$.

(ii) For any $F \in \mathscr{E}$, c_F is a contraction .

<u>Proof</u>. (i) Let $X^+/\rho \in \mathscr{S}_X{}^*$. Then $E_f(X^+/\rho) = X^+/f(\rho)$, by the definition of E_f.

Define η, from $X^+/f(\rho)$ into X^+/ρ, by

$$(wf(\rho))\eta = w\rho \qquad (w \in X^+)$$

Then η is a well-defined epimorphism since $f(\rho) \subseteq \rho$ and $\rho \in \mathrm{Con}X^+$. This proves that E_f satisfies E(i) .

Now let X^+/ρ , $X^+/\tau \in \mathscr{S}_X{}^*$ and let $\theta: X^+/\rho \to X^+/\tau$ be a morphism . For this morphism to exist ρ must be contained in τ and therefore θ is necessarily surjective. Since $\rho \subseteq \tau$ and f satisfies C(ii) , we have $f(\rho) \subseteq f(\tau)$. Next , define $E_f(\theta) : E_f(X^+/\rho) \to E_f(X^+/\tau)$; i.e. , $E_f(\theta)$ maps $X^+/f(\rho)$ into $X^+/f(\tau)$, by

$$(wf(\rho))E_f(\theta) = wf(\tau) \qquad (w \in X^+)$$

Then $E_f(\theta)$ is a well-defined epimorphism since $f(\rho) \subseteq f(\tau)$ and $f(\rho)$, $f(\tau)$ are from $\mathrm{Con}X^+$. Thus, E_f is a functor . Hence , $E_f \in \mathscr{E}$.

(ii) Let $X^+/\rho \in \mathscr{S}_X{}^*$. Since $F \in \mathscr{E}$, there exists an epimorphism η from $F(X^+/\rho) = X^+/\rho^F$ onto X^+/ρ. Since η is a morphism in $\mathscr{S}_X{}^*$, we must have $\rho^F \subseteq \rho$. Thus E_f satisfies C(i).

Next, let $\rho \subseteq \tau$. The only morphism $\theta: X^+/\rho \to X^+/\tau$ is given by $(x\rho)\theta = x\tau$. Then since F is a functor there exists an epimorphism $F(\theta)$ from $F(X^+/\rho)$ onto $F(X^+/\tau)$; i.e., $F(\theta)$ maps X^+/ρ^F onto X^+/τ^F . On the other hand $c_F(\rho) = \rho^F$ and $c_F(\tau) = \tau^F$. Hence , because we are dealing with morphisms in $\mathscr{S}_X{}^*$, we must have $c_F(\rho) = \rho^F \subseteq \tau^F = c_F(\tau)$. This verifies that c_F satisfies C(ii). Therefore, $c_F \in \mathscr{C}$.•

<u>Definition 3.11</u>. Let $\Phi: \mathscr{C} \to \mathscr{E}$ and $\Psi: \mathscr{E} \to \mathscr{C}$ be defined by:

$$\Phi: f \to E_f \quad (f \in \mathscr{C}) \qquad \text{and} \qquad \Psi: F \to c_F \quad (F \in \mathscr{E}).$$

<u>Theorem 3.12</u> . The mappings Φ and Ψ are inverse order anti-isomorphisms.

Proof. By definition it is clear that for all $f \in \mathscr{C}$, $F \in \mathscr{E}$,

$$\Psi \circ \Phi(f) = \Psi(E_f) = c_{E_f} = f \quad \text{and} \quad \Phi \circ \Psi(F) = \Phi(c_F) = E_{c_F} = F,$$

and so that Φ and Ψ are inverse bijections.

Now, let $f,g \in \mathscr{C}$ be such that $f \leq g$. Then $f(\rho) \subseteq g(\rho)$ for any ρ in $\mathrm{Con}X^+$. Therefore, $\theta_\rho: X^+/f(\rho) \to X^+/g(\rho)$ defined by

$$(wf(\rho))\theta_\rho = wg(\rho) \qquad (w \in X^+)$$

is an epimorphism. Since $E_f(X^+/\rho) = X^+/f(\rho)$ and $E_g(X^+/\rho) = X^+/g(\rho)$ and since there exists an epimorphism $\theta_\rho: E_f(X^+/\rho) \to E_g(X^+/\rho)$ for any X^+/ρ in $\mathscr{S}_X{}^*$, we have that $\Phi(f) = E_f \geq E_g = \Phi(g)$.

Finally, let $F, G \in \mathscr{E}$ with $F \geq G$. Then for any $X^+/\rho \in \mathscr{S}_X{}^*$ there exists an epimorphism θ_ρ from $F(X^+/\rho)$ onto $G(X^+/\rho)$; that is

$$\theta_\rho: X^+/\rho^F \to X^+/\rho^G.$$

Then since θ_ρ is a morphism in $\mathscr{S}_X{}^*$, we must have that $\rho^F \subseteq \rho^G$. Hence, $c_F(\rho) = \rho^F \subseteq \rho^G = c_G(\rho)$ for any $\rho \in \mathrm{Con}X^+$, and therefore

$$\Psi(F) = c_F \leq c_G = \Psi(G)$$

as required. Therefore, Φ and Ψ are inverse order anti-isomorphisms.•

One important consequence of Lemma 3.8 and Theorem 3.12 is that we can now consider $\mathscr{E}$ as a lattice with respect to the operations

$$F \wedge G = E_{(c_F \vee c_G)} \quad \text{and} \quad F \vee G = E_{(c_F \wedge c_G)}.$$

Definition 3.13. Let $F, G \in \mathscr{E}$. Define $F{\cdot}G$ by

$$F{\cdot}G(X^+/\rho) = F(G(X^+/\rho)) \qquad (X^+/\rho \in \mathscr{S}_X{}^*).$$

Proposition 3.14. $\mathscr{E}$ is a semigroup with the multiplication defined above.

Proof. First let $X^+/\rho \in \mathcal{S}_X{}^*$. By E(i) there exist epimorphisms η_G from $G(X^+/\rho)$ onto X^+/ρ and η_F from $F(G(X^+/\rho))$ onto $G(X^+/\rho)$. Let $\eta = \eta_G \circ \eta_F$. Then η is an epimorphism which maps $F{\cdot}G(X^+/\rho) = F(G(X^+/\rho))$ onto X^+/ρ. Next let X^+/ρ, $X^+/\tau \in \mathcal{S}_X{}^*$ and let $\theta: X^+/\rho \to X^+/\tau$ be a morphism. Then there exist epimorphisms θ_G and θ_F, where

$$\theta_G: G(X^+/\rho) \to G(X^+/\tau) \quad \text{and} \quad \theta_F: F(G(X^+/\rho)) \to F(G(X^+/\tau)).$$

Hence $F{\cdot}G$ is a functor and so, $F{\cdot}G \in \mathcal{E}$. Since the composition of functions is associative, we have that $F{\cdot}(G{\cdot}H) = F{\cdot}(G{\cdot}H) = F{\cdot}G{\cdot}H$ for any F,G and $H \in \mathcal{E}$.•

Definition 3.15. Let f and $g \in \mathcal{C}$. Define $f{\cdot}g$ by

$$f{\cdot}g(\rho) = f(g(\rho)) \qquad (\rho \in \mathrm{Con}X^+).$$

Proposition 3.16. $\mathcal{C}$ is a semigroup with the multiplication defined above.

Proof. If $\rho \in \mathrm{Con}X^+$ then $f{\cdot}g(\rho) = f(g(\rho)) \subseteq g(\rho) \subseteq \rho$, and if $\rho, \tau \in \mathrm{Con}X^+$ with $\rho \subseteq \tau$, then $g(\rho) \subseteq g(\tau)$. Therefore, $f(g(\rho)) \subseteq f(g(\tau))$; that is, $f{\cdot}g(\rho) \subseteq f{\cdot}g(\tau)$. Hence, if $f,g \in \mathcal{C}$ then $f{\cdot}g \in \mathcal{C}$. Since the composition of functions is associative, we have that $f{\cdot}(g{\cdot}h) = (f{\cdot}g){\cdot}h = f{\cdot}g{\cdot}h$, for any f,g and h in $\mathcal{C}$. Consequently, $\mathcal{C}$ is a semigroup .•

Let $F, G \in \mathcal{E}$. Since

$$X^+/\rho^{F\cdot G} = F{\cdot}G(X^+/\rho) = F(G(X^+/\rho)) = F(X^+/\rho^G) = X^+/(\rho^G)^F,$$

we have that $\rho^{F\cdot G} = (\rho^G)^F$. Consequently,

$$c_{F\cdot G}(\rho) = \rho^{F\cdot G} = (\rho^G)^F = c_F(\rho^G) = c_F(c_G(\rho)).$$

That is $c_{F\cdot G} = c_F{\cdot}c_G$. Therefore,

$$\Psi(F{\cdot}G) = c_{F\cdot G} = c_F{\cdot}c_G = \Psi(F){\cdot}\Psi(G).$$

Theorem 3.17. Φ and Ψ are semigroup isomorphisims.

Proof. That Ψ is a homomorphism was established prior to the theorem. That Φ is a semigroup homomorphism follows from the fact that the inverse mapping of a semigroup homomorphism is a semigroup homomorphism .•

Proposition 3.18. For any $F, G \in \mathscr{E}$, $F{\cdot}G \geq F \vee G$.

Proof. Let $X^+/\rho \in \mathscr{S}_X{}^*$. Then

$$(F \vee G)(X^+/\rho) = E_{(c_F \wedge c_G)}(X^+/\rho) = X^+/((c_F \wedge c_G)(\rho)) = X^+/(c_F(\rho) \wedge c_G(\rho))$$
$$= X^+/(\rho^F \wedge \rho^G).$$

On the other hand, $F{\cdot}G(X^+/\rho) = F(G(X^+/\rho)) = F(X^+/\rho^G) = X^+/(\rho^G)^F$. Also $(\rho^G)^F \subseteq \rho^F$ by C(ii) since $\rho^G \subseteq \rho$, and $(\rho^G)^F \subseteq \rho^G$ by C(i). Thus we have that $(\rho^G)^F \subseteq \rho^F \wedge \rho^G$.

Now define $\theta_\rho: F{\cdot}G(X^+/\rho) \to (F \vee G)(X^+/\rho)$ by

$$(\,w(\rho^G)^F\,)\theta_\rho = w(\rho^F \wedge \rho^G) \qquad (w \in X^+).$$

Then θ_ρ is an epimorpism, and so $F{\cdot}G \geq F \vee G$.•

Example 3.19. The inequality in Proposition 3.18. may be strict. Define the Henckell's expansion H , [1] , on a semigroup S by

$$H(S) = \hat{S}^{(2)} = \left\{ \{ (\prod_{i}^{m} s_i \,,\, \prod_{m+1}^{k} s_i) \mid 0 \leq m \leq k \} \mid (s_1, s_2, \ldots\ldots, s_k) \in S^+ \right\}$$

with multiplication

$$\{ (\prod_{i=1}^{m} s_i \,,\, \prod_{m+1}^{k} s_i) \mid 0 \leq m \leq k \} \cdot \{ (\prod_{i=k+1}^{r} s_i \,,\, \prod_{r+1}^{k+h} s_i) \mid k \leq r \leq k+h \}$$

$$=\{(\prod_{i=1}^{n} s_i , \prod_{i=n+1}^{k+h} s_i) \mid 0 \leq n \leq k+h \}.$$

By [1] we know that $H(S)$ is a homomorphic image of $H^2(S)$, but $H(S)$ is not isomorphic to $H^2(S)$. So, we have that

$$(H \vee H)(S) = H(S) \vee H(S) = H(S)$$

however,

$$H \cdot H(S) = H(H(S)) = H^2(S).$$

Thus, $H \cdot H \neq H \vee H$ in this case.

CHAPTER 4

In this chapter we construct the contractions corresponding to some known expansions and we give some lattice theoretical results concerning these expansions and contractions.

4.1. The contraction corresponding to the machine expansion.

We begin this section with the definition of the *left machine expansion* . We then introduce a congruence $\bar{\rho}^{\mathscr{L}}$ on X^+ for any given congruence ρ on X^+, and a contraction $f : \rho \to \bar{\rho}^{\mathscr{L}}$ which is shown to correspond to the *left machine expansion.*

For $(S,\alpha) \in \mathscr{S}_X$, let

$$\bar{S}^{\mathscr{L}} = \{\ (s_1s_2s_3\ldots s_n, s_2s_3\ldots s_n, \ldots, s_n) \mid s_1, s_2, \ldots, s_n \in S\ \}$$

with multiplication given by

$$(s_1s_2s_3\ldots s_n, s_2s_3\ldots s_n, \ldots, s_n)\cdot(t_1t_2t_3\ldots t_m, t_2t_3\ldots t_m, \ldots, t_m) =$$

$$(s_1s_2\ldots s_nt_1t_2\ldots t_m, s_2s_3\ldots s_nt_1t_2\ldots t_m, \ldots, s_nt_1t_2\ldots t_m, t_1t_2t_3\ldots t_m, t_2t_3\ldots t_m, \ldots, t_m\ ),$$

and let $\bar{\alpha}^{\mathscr{L}} : x \to (x\alpha)$. Then $(\bar{S}^{\mathscr{L}}, \bar{\alpha}^{\mathscr{L}})$ is called the *machine expansion* and was introduced by Birget and Rhodes in [1].

Let $A \subseteq S$ such that $S = \langle A \rangle$. Then $\bar{S}^{\mathscr{L}}_A \subseteq \bar{S}^{\mathscr{L}}$ defined by

$$\bar{S}^{\mathscr{L}}_A = \langle\, (a) \mid a \in A \,\rangle$$

is called *the cutdown of* $\bar{S}^{\mathscr{L}}$ *to generators* A.

For any $\rho \in \mathrm{Con}X^+$ define $\bar{\rho}^{\mathscr{L}}$ on X^+ as follows :

For $u, v \in X^+$,

$u \ \bar{\rho}^{\mathscr{L}} \ v$ if and only if $e^i(u) \ \rho \ e^i(v)$ for any $i \geq 0$ and $|u| = |v|$.

Since $u \ \bar{\rho}^{\mathscr{L}} \ v$ implies , in particular, that $u = e^0(u) \ \rho \ e^0(v) = v$, we have $\bar{\rho}^{\mathscr{L}} \subseteq \rho$.

Lemma 4.1.1. $\bar{\rho}^{\mathscr{L}}$ is a congruence on X^+.

Proof. Clearly $\bar{\rho}^{\mathscr{L}}$ is an equivalence relation. In order to see that it is a congruence, let u,v,s and $t \in X^+$ be such that $u \ \bar{\rho}^{\mathscr{L}} \ v$ and $s \ \bar{\rho}^{\mathscr{L}} \ t$. It suffices to show that $us \ \bar{\rho}^{\mathscr{L}} \ vt$.

Since $|u| = |v|$ and $|s| = |t|$, $|us| = |u| + |s| = |v| + |t| = |vt|$.

Next, let $i \geq 0$.

If $i \leq |u| = |v|$ then $e^i(us) = e^i(u)s$ and $e^i(vt)= e^i(v)t$. Since $e^i(u) \ \rho \ e^i(v)$ and $s \ \rho \ t$ we have $e^i(us) = e^i(u)s \ \rho \ e^i(v)t = e^i(vt)$.

If $i > |u| = |v|$ then $e^i(us) = e^k(s)$ and $e^i(vt) = e^k(t)$ where $k = i - |u| = i - |v|$. Since $e^k(s) \ \rho \ e^k(t)$ we have that $e^i(us) \ \rho \ e^i(vt)$. Hence , $us \ \bar{\rho}^{\mathscr{L}} \ vt$. Consequently, $\bar{\rho}^{\mathscr{L}}$ is a congruence on X^+, as required .•

Let $\rho \in \mathrm{Con}X^+$ be such that $X^+/\rho \cong S$ and let $A = \{x\rho : x \in X\}$. Then $A \subseteq S$ and $S = \langle A \rangle$. We also have that

$$\bar{S}^{\mathscr{L}}_A \cong (\overline{X^+/\rho})^{\mathscr{L}}_A = \langle\, (x\rho) \mid x \in X \,\rangle$$

$= \{ ((x_1x_2....x_n)\rho,(x_2....x_n)\rho,.......,x_n\rho) \mid x_i \in X \text{ and } n \geq 0 \}.$

Now define $\varphi : X^+/\bar{\rho}^{\mathscr{S}} \to \overline{(X^+/\rho)}^{\mathscr{S}}_A \cong \bar{S}^{\mathscr{S}}_A$ by

$$\varphi : (x_1x_2\ldots x_n)\bar{\rho}^{\mathscr{S}} \to ((x_1x_2\ldots x_n)\rho,(x_2\ldots x_n)\rho,\ldots,x_n\rho)$$

Proposition 4.1.2. φ is an isomorphism.

Proof. We first show that φ is well-defined.

Let $u, v \in X^+$ be such that

$$u = x_1x_2\ldots x_n \quad \bar{\rho}^{\mathscr{S}} \quad y_1y_2\ldots y_m = v\,.$$

Then $e^i(u)\ \rho\ e^i(v)$ for any $i \geq 0$ and $n = m$. Therefore we have

$$\varphi(u\bar{\rho}^{\mathscr{S}}) = (e^0(u)\rho,e^1(u)\rho,......) = (e^0(v)\rho,e^1(v)\rho,......) = \varphi(v\bar{\rho}^{\mathscr{S}}).$$

Thus, φ is well-defined and clearly maps $X^+/\bar{\rho}^{\mathscr{S}}$ into $\overline{(X^+/\rho)}^{\mathscr{S}}_A$.

We first show that φ is injective. Let $u = x_1x_2.....x_n$ and $v = y_1y_2.....y_m$ be elements of X^+ such that $(u\bar{\rho}^{\mathscr{S}})\varphi = (v\bar{\rho}^{\mathscr{S}})\varphi$. Then $e^i(u)\rho = e^i(v)\rho$ for any $i \geq 0$, and $m = n$. That is, $e^i(u)\ \rho\ e^i(v)$ and $|u| = |v|$. It follows from the definition of $\bar{\rho}^{\mathscr{S}}$ that $u\ \bar{\rho}^{\mathscr{S}}\ v$. Hence φ is one-to-one.

We now show that φ is a homomorphism.

Let $u = (x_1x_2.....x_n)$ and $v = (y_1y_2.......y_m) \in X^+$. Then

$$((x_1x_2.....x_n)\bar{\rho}^{\mathscr{S}})\varphi = ((x_1x_2.....x_n)\rho,(x_2.....x_n)\rho,...........,x_n\rho)$$

and

$$((y_1y_2.....y_m)\bar{\rho}^{\mathscr{S}})\varphi= ((y_1y_2......y_m)\rho,(y_2....y_m)\rho,..........,y_m\rho).$$

Therefore,

$$(u\bar{\rho}^{\mathscr{S}})\varphi\cdot(v\bar{\rho}^{\mathscr{S}})\varphi$$

$$= ((x_1x_2\ldots x_n)\rho(y_1y_2\cdots y_m)\rho,\ldots,x_n\rho(y_1y_2\cdots y_m)\rho,(y_1y_2\cdots y_m)\rho,\ldots,y_m\rho)$$

$$= ((x_1x_2\ldots x_ny_1y_2\cdots y_m)\rho,\ldots,(x_ny_1y_2\cdots y_m)\rho,(y_1y_2\cdots y_m)\rho,\ldots,y_m\rho)$$

and

$$((uv)\bar{\rho}^{\mathscr{S}})\varphi = ((x_1x_2\ldots x_ny_1y_2\cdots y_m)\bar{\rho}^{\mathscr{S}})\varphi$$

$$= ((x_1x_2\ldots x_ny_1y_2\cdots y_m)\rho,\ldots,(x_ny_1y_2\cdots y_m)\rho,(y_1y_2\cdots y_m)\rho,\ldots,y_m\rho).$$

Therefore,

$$(u\bar{\rho}^{\mathscr{S}})\varphi\cdot(v\bar{\rho}^{\mathscr{S}})\varphi = ((uv)\bar{\rho}^{\mathscr{S}})\varphi.$$

We finally show that φ is surjective. For any $x \in X$, we have that $x\bar{\rho}^{\mathscr{S}} \in X^+/\bar{\rho}^{\mathscr{S}}$ and $(x\bar{\rho}^{\mathscr{S}})\varphi = (x\rho)$. Since $\overline{(X^+/\rho)}{}_A^{\mathscr{S}} = \langle\, (x\rho) \mid x \in X \,\rangle$, φ maps the generators of $X^+/\bar{\rho}^{\mathscr{S}}$ onto those of $\overline{(X^+/\rho)}{}_A^{\mathscr{S}}$.

Thus, φ is an isomorphism .•

Now define $f:\ \mathrm{Con}\, X^+ \to \mathrm{Con}\, X^+$ by

$$f:\ \rho \to \bar{\rho}^{\mathscr{S}} \qquad (\rho \in \mathrm{Con}\, X^+).$$

<u>Lemma 4.1.3</u>. $f \in \mathscr{C}$.

<u>Proof</u>. That $\bar{\rho}^{\mathscr{S}} \subseteq \rho$ was discussed before. For $\rho, \tau \in \mathrm{Con}\, X^+$ such that $\rho \subseteq \tau$ we clearly have $\bar{\rho}^{\mathscr{S}} \subseteq \bar{\tau}^{\mathscr{S}}$.•

Now let F be defined by

$$F:\ (S,\alpha) \to (\bar{S}_A^{\mathscr{S}}, \bar{\alpha}^{\mathscr{S}}) \quad ((S,\alpha) \in \mathscr{S}_X).$$

It follows easily from [1] that F is indeed an expansion in $\mathscr{S}_X$.

<u>Theorem 4.1.4</u>. F is congruent to E_f.

<u>Proof</u>. As before , for any $(S,\alpha) \in \mathscr{S}_X$ we know that there exists a unique

congruence $\rho \in \mathrm{Con}\, X^+$ such that $(S,\alpha) \cong (X^+/\rho, \iota_\rho)$. Then

$$F(S) = \bar{S}_A^{\mathscr{L}} \cong X^+/\bar{\rho}^{\mathscr{L}} \qquad \text{(by Proposition 4.1.2)}$$

$$= X^+/f(\rho) = E_f(X^+/\rho),$$

so that F is congruent to E_f .•

4.2. The contraction corresponding to the expansion $\tilde{S}^{\mathscr{L}}$.

In this section we turn our attention to an expansion based on the machine expansion of a semigroup. We introduce a congruence $\tilde{\rho}^{\mathscr{L}}$ and a contraction $f : \rho \to \tilde{\rho}^{\mathscr{L}}$. We close the section by showing that this contraction corresponds to that expansion .

For $(S,\alpha) \in \mathscr{S}_X$ let

$$\tilde{S}^{\mathscr{L}} = \{(\prod_1^k s_i, \{\prod_{m+1}^k s_i \mid 0 \le m \le k\}) \mid s_i \in S\}$$

and

$$\tilde{S}^{\mathscr{R}} = \{(\{\prod_1^n s_i \mid 0 \le n \le k\}, \prod_1^k s_i) \mid s_i \in S\}.$$

Equivalently,

$$\tilde{S}^{\mathscr{L}} = \{ (s_1 \ldots s_n, \{s_1 s_2 \ldots s_n, s_2 \ldots s_n, \ldots\ldots\ldots, s_n, 1\}) \mid s_1,\ldots,s_n \in S \}$$

and

$$\tilde{S}^{\mathscr{R}} = \{(\{s_1 s_2 \ldots s_n , s_1 s_2 \ldots s_{n-1}, \ldots, s_1 s_2, s_1\}, s_1 s_2 \ldots s_n) \mid s_1, s_2, \ldots, s_n \in S \}$$

Define a multiplication on $\tilde{S}^{\mathscr{L}}$ by :

$$(s, A)\cdot(t, B) = (st, A\cdot t \cup B)$$

and on $\widetilde{S}^{\mathscr{R}}$ by :

$$(A, s)\cdot(B, t) = (A \cup s\cdot B, st)$$

where

$$A\cdot t = \{ xt \mid x \in A \} \quad \text{and} \quad s\cdot B = \{ sy \mid y \in B \}.$$

and let $\tilde{\alpha}^{\mathscr{L}} : x \to (x\alpha, \{x\alpha, 1\})$ and $\tilde{\alpha}^{\mathscr{R}} : x \to (\{1, x\alpha\}, x\alpha)$.

The expansions $(\widetilde{S}^{\mathscr{L}}, \tilde{\alpha}^{\mathscr{L}})$ and $(\widetilde{S}^{\mathscr{R}}, \tilde{\alpha}^{\mathscr{R}})$ were introduced in [1] and [3] .

Let $A \subseteq S$ such that $S = \langle A \rangle$. Then $\widetilde{S}^{\mathscr{L}}_A \subseteq \widetilde{S}^{\mathscr{L}}$ defined by

$$\widetilde{S}^{\mathscr{L}}_A = \langle \ (a, \{a, 1\}) \mid a \in A \ \rangle$$

is called *the cutdown of* $\widetilde{S}^{\mathscr{L}}$ *to generators* A.

For any $\rho \in \mathrm{Con}X^+$ define $\tilde{\rho}^{\mathscr{L}}$ on X^+ as follows :

For $u, v \in X^+$ with $|u| = m$ and $|v| = n$,

$u \ \tilde{\rho}^{\mathscr{L}} \ v$ if and only if

(i) $u \ \rho \ v$;

(ii) for any $1 \le k < m$, there exists $0 \le l < n$ such that

$e^k(u) \ \rho \ e^l(v)$;

(iii) for any $1 \le r < n$, there exists $0 \le s < m$ such that

$e^r(v) \ \rho \ e^s(u)$.

<u>Lemma 4.2.1</u>. $\tilde{\rho}^{\mathscr{L}}$ is a congruence on X^+.

<u>Proof</u>. Clearly $\tilde{\rho}^{\mathscr{L}}$ is an equivalence relation . We wish to show that $\tilde{\rho}^{\mathscr{L}}$ is also a congruence .

Clearly $\tilde{\rho}^{\mathcal{L}} \subseteq \rho$. Let u,v,w and z $\in X^+$ be such that $u \, \tilde{\rho}^{\mathcal{L}} \, v$ and $w \, \tilde{\rho}^{\mathcal{L}} \, z$ where $|u| = m$, $|v| = n$, $|w| = p$ and $|z| = t$. Then $u \rho v$ and $w \rho z$ implies $uw \ \rho \ vz$. Let $1 \le k < m+p$.

If $1 \le k < m$ then there exists $0 \le l \le n < n+t$ with $e^k(u) \ \rho \ e^l(v)$. Since $e^k(u)w = e^k(uw)$ and $e^l(v) = e^l(vz)$ we have that $e^k(uw) \ \rho \ e^l(vz)$.

If $k = m$, let $l = n$. Then $e^k(uw) = w \ \rho \ z = e^l(vz)$ and thus $e^k(uw) \ \rho \ e^l(vz)$.

If $m < k < m+p$, let $k' = k-m$. Then $1 \le k' < p$ and so there exists $0 \le l' < t$ with $e^{k'}(w) \ \rho \ e^{l'}(z)$. Also, $e^{k'}(w) = e^k(uw)$ and, if we let $l = l'+n$, then $0 \le n \le l < t+n$ and $e^{l'}(z) = e^l(vz)$.

Hence in all cases, $e^k(uw) \ \rho \ e^l(vz)$ for some $0 \le l < t+n$.

Similarly for any $1 \le r < n+t$, there exists $0 \le s < m+p$ such that $e^r(vz) \ \rho \ e^s(uw)$. Therefore, $uw \ \tilde{\rho}^{\mathcal{L}} \ vz$ and so $\tilde{\rho}^{\mathcal{L}}$ is a congruence .•

Let $\rho \in \mathrm{Con}X^+$ be such that $X^+/\rho \cong S$ and $A = \{x\rho : x \in X\}$. Then $A \subseteq S$ and $S = \langle A \rangle$. Also, we have that

$$\tilde{S}_A^{\mathcal{L}} \cong \widetilde{(X^+/\rho)}_A^{\mathcal{L}} = \langle \, (x\rho , \{x\rho, 1\}) \mid x \in X \, \rangle$$

$$= \{ \, (x_1 \ldots x_n)\rho , \{ \, (x_1 \ldots x_n)\rho, \ldots\ldots, (x_n)\rho, 1 \} \,) \mid x_i \in X \}.$$

Now, define $\varphi : X^+/\tilde{\rho}^{\mathcal{L}} \to \widetilde{(X^+/\rho)}_A^{\mathcal{L}} \cong \tilde{S}_A^{\mathcal{L}}$ by :

$$\varphi\colon (x_1 \ldots x_n)\tilde{\rho}^{\mathcal{L}} \to (\, (x_1 \ldots x_n)\rho, \ \{(x_1 \ldots x_n)\rho, \ldots\ldots, (x_n)\rho, 1 \, \} \,).$$

Proposition 4.2.2. φ is an isomorphism.

Proof. We first show that φ is well-defined. Let $u,v \in X^+$ such that

$$u = x_1 \ldots x_n \ \ \tilde{\rho}^{\mathcal{L}} \ \ y_1 \ldots y_m = v.$$

Then, by the definition of $\tilde{\rho}^{\mathcal{L}}$, we have that $(x_1....x_n)\rho = (y_1...y_m)\rho$ and for any $1 \le i < n$, there exists $0 \le j < m$ such that $e^i(u)\rho = e^j(v)\rho$. We also have that for any $1 \le i' < m$, there exists $0 \le j' < n$ such that $e^{i'}(v)\rho = e^{j'}(u)\rho$. Therefore,

$$\{(x_1...x_n)\rho,..., (x_n)\rho,1\} = \{(y_1...y_m)\rho,...,(y_m)\rho,1\}$$

Hence,

$$((x)\tilde{\rho}^{\mathcal{L}})\varphi = ((y)\tilde{\rho}^{\mathcal{L}})\varphi .$$

Thus, φ is well-defined.

To see that φ is injective, let $u = x_1...x_n$, $v= y_1...y_m \in X^+$ be such that $((u)\tilde{\rho}^{\mathcal{L}})\varphi = ((v)\tilde{\rho}^{\mathcal{L}})\varphi$. Then

$$(x_1...x_n)\rho = (y_1...y_m)\rho \quad (*)$$

and

$$\{ (x_1...x_n)\rho,...,(x_n)\rho,1 \} = \{(y_1...y_m)\rho,...,(y_m)\rho,1 \}.$$

Hence, for any $1 \le i < n$ there exists $0 \le j < m$ such that $e^j(u)\ \rho\ e^j(v)$ $(**)$ and conversely, for any $1 \le i' < m$, there exists $0 \le j' < n$ such that $e^{i'}(v)\ \rho\ e^{j'}(u)$ $(***)$.

Whence, by $(*)$, $(**)$ and $(***)$, $u\ \tilde{\rho}^{\mathcal{L}}\ v$; that is, $(u)\tilde{\rho}^{\mathcal{L}} = (v)\tilde{\rho}^{\mathcal{L}}$. Thus, φ is injective.

To see that φ is also a homomorphism let $u = x_1....x_n$, $v = y_1....y_m \in X^+$. Then

$$((u)\tilde{\rho}^{\mathcal{L}}))\varphi\cdot((v)\tilde{\rho}^{\mathcal{L}})\varphi =$$

$$= ((x_1...x_n)\rho, \{(x_1...x_n)\rho,...,(x_n)\rho,1\})\cdot((y_1...y_m)\rho, \{(y_1...y_m)\rho,...,(y_m)\rho,1\})$$

$$= ((x_1...x_ny_1...y_m)\rho,\{(x_1...x_ny_1...y_m)\rho,...,(x_ny_1...y_m)\rho,(y_1...y_m)\rho,...,(y_m)\rho,1\})$$

$$= ((x_1...x_ny_1...y_m)\tilde{\rho}^{\mathcal{L}})\ \varphi = ((uv)\tilde{\rho}^{\mathcal{L}})\varphi$$

Finally, we show that φ is surjective. Let $x \in X$ and

$(x\rho,\{x\rho,1\}) \in \widetilde{(X^+/\rho)}_A^{\mathscr{S}}$. Then $x\tilde{\rho}^{\mathscr{S}} \in X^+/\tilde{\rho}^{\mathscr{S}}$ and $(x\tilde{\rho}^{\mathscr{S}})\varphi = (x\rho,\{x\rho,1\})$. Since $\widetilde{(X^+/\rho)}_A^{\mathscr{S}}$ is generated by $\{(x\rho,\{x\rho,1\}) \mid x \in X\}$ and φ is onto on generators, φ is surjective .•

Now define l: Con $X^+ \rightarrow$ Con X^+ by:

$$l: \rho \rightarrow \tilde{\rho}^{\mathscr{S}}.$$

<u>Lemma 4.2.3</u>. $l \in \mathscr{C}$.

<u>Proof</u>. That $\tilde{\rho}^{\mathscr{S}} \subseteq \rho$ is clear from the definition of $\tilde{\rho}^{\mathscr{S}}$, part (i). For $\rho,\tau \in$ Con X^+ such that $\rho \subseteq \tau$, we clearly have $\tilde{\rho}^{\mathscr{S}} \subseteq \tilde{\tau}^{\mathscr{S}}$.•

Now let L be defined by

$$L: (S,\alpha) \rightarrow (\tilde{S}_A^{\mathscr{S}}, \tilde{\alpha}^{\mathscr{S}}) \qquad (S,\alpha) \in \mathscr{S}_X.$$

It follows from [1] or [7] that L is indeed an expansion in $\mathscr{S}_X$.

<u>Proposition 4.2.4</u>. L is congruent to E_l .

<u>Proof</u>. As before, for any $(S,\alpha) \in \mathscr{S}_X$,we know that there exists a unique $\rho \in$ ConX^+ such that $X^+/\rho \cong S$. Then ,by Proposition 4.2.2 ,

$$L(S) = \tilde{S}_A^{\mathscr{S}} \cong X^+/\tilde{\rho}^{\mathscr{S}} = X^+/l(\rho) = E_l(X^+/\rho),$$

so that L is congruent to E_l .•

<u>4.3. The contraction corresponding to the Henckell's expansion.</u>

This section is devoted to *Henckell's expansion.* We first recall this expansion and then introduce a congruence $\hat{\rho}^{(2)}$ for any given $\rho \in \mathrm{Con}\, X^+$. This is followed by the definition of a contraction $h : \rho \to \hat{\rho}^{(2)}$. We then show that this contraction corresponds to the Henckell's expansion.

For $(S,\alpha) \in \mathscr{S}_X$ let

$$\hat{S}^{(2)} = \{ \{(\prod_{1}^{m} s_i , \prod_{m+1}^{k} s_i) : 0 \le m \le k \} \mid s_1,\dots,s_k \in S \}$$

with multiplication

$$\{(\prod_{1}^{m} s_i , \prod_{m+1}^{k} s_i) : 0 \le m \le k \}\cdot\{(\prod_{k+1}^{l} s_i , \prod_{l+1}^{n+k} s_i): k \le l \le n+k \}$$

$$= \{(\prod_{1}^{r} s_i , \prod_{r+1}^{n+k} s_i) : 0 \le r \le n+k \}$$

and let $\hat{\alpha} : x \to \{(x\alpha,1), (1,x\alpha)\}$.

Then $(\hat{S}^{(2)},\hat{\alpha})$ is called the *Henckell's expansion* and was introduced in [1].

Let $A \subseteq S$ such that $S = \langle A \rangle$. Then $\hat{S}^{(2)}_A \subseteq \hat{S}^{(2)}$ defined by

$$\hat{S}^{(2)}_A = \langle \{(1,a),(a,1)\} \mid a \in A \rangle$$

is called the *cutdown of* $\hat{S}^{(2)}$ *to generators* A.

For $\rho \in \mathrm{Con}X^+$ define $\hat{\rho}^{(2)}$ on X^+ as follows :

For $u, v \in X^+$, where $|u| = m$ and $|v| = n$,

$u \ \hat{\rho}^{(2)} \ v$ if and only if

(i) for any $0 \le i < m$, there exists $0 \le j < n$ such that

$s^i(u) \ \rho \ s^j(v)$ and $e^{m-i}(u) \ \rho \ e^{n-j}(v)$;

(ii) for any $0 \leq k \leq n$ there exists $0 \leq l \leq n$ such that

$$s^k(v) \ \rho \ s^l(u) \quad \text{and} \quad e^{n-k}(v) \ \rho \ e^{m-l}(u).$$

Lemma 4.3.1. $\hat{\beta}^{(2)}$ is a congruence on X^+.

Proof. Clearly $\hat{\beta}^{(2)}$ is an equivalence relation. To see that it is also a congruence we first show that $\hat{\beta}^{(2)} \subseteq \rho$. Let $u, v \in X^+$ be such that $u \ \hat{\beta}^{(2)} \ v$. Then $s^0(u) \ \rho \ s^j(v)$ for some $0 \leq j < n$ and $e^m(u) \ \rho \ e^{m-j}(v)$. Therefore, $u = s^0(u)e^m(u) \ \rho \ s^j(v)e^{m-j}(v) = v$. Hence, $u \ \rho \ v$, as required.

Now let u, v, w and $z \in X^+$ be such that $u \ \hat{\beta}^{(2)} \ v$ and $w \ \hat{\beta}^{(2)} \ z$, where $|u| = m$, $|v| = n$, $|w| = p$ and $|z| = r$. Then $|uw| = m+p$ and $|vz| = n+r$.

Next, let $0 \leq i < m+p$.

Case (i) $0 \leq i < p$. Since $w \ \hat{\beta}^{(2)} \ z$ there exists $0 \leq j < r$ such that $s^i(w) \ \rho \ s^j(z)$ and $e^{p-i}(w) \ \rho \ e^{r-j}(z)$. Since $u \ \hat{\beta}^{(2)} \ v$ and therefore $u \ \rho \ v$, by the above, we have that $us^i(w) \ \rho \ vs^j(z)$. But since $0 \leq i < p = |w|$ and $0 \leq j < r = |z|$, we have $us^i(w) = s^i(uw)$ and $vs^j(z) = s^j(vz)$. Hence, $s^i(uv) \ \rho \ s^j(vz)$. Also,since $0 \leq i < p = |w|$ and $0 \leq j < r = |z|$, we have that $e^{p-i}(w) = e^{m+p-i}(uw)$ and $e^{r-j}(z) = e^{n+r-j}(vz)$. Thus, $e^{(m+p)-i}(uw) \ \rho \ e^{(n+r)-j}(vz)$.

Case (ii) $i = p$. Let $j = r$. Then we have $s^i(uw) = u \ \rho \ v = s^j(vz)$ and $e^{(m+p)-i}(uw) = e^{m+p-p}(uw) = e^m(uw) = w \ \rho \ z = e^n(vz) = e^{n+r-r}(vz) = e^{(n+r)-j}(vz)$.

Case (iii) $p < i < p+m$. Then $0 < i-p < m$ and so there exists $0 \leq j' < n$ such that $s^{i-p}(u) \ \rho \ s^{j'}(v)$ and $e^{m-(i-p)}(u) \ \rho \ e^{n-j'}(v)$. Let $j = j'+r$. Then $r \leq j < n+r$ and $s^{j'}(v) = s^j(vz)$. Since $s^{i-p}(u) = s^i(uw)$, we have that $s^i(uw) \ \rho \ s^j(vz)$. Also, since $w \ \rho \ z$, $e^{m+p-i}(u)w = e^{(m+p)-i}(uw)$ and $e^{n-j'}(v)z = e^{n-(j-r)}(vz) = e^{(n+r)-j}(vz)$, we have that $e^{(m+p)-i}(uw) \ \rho \ e^{(n+r)-j}(vz)$.

In cases (i) ,(ii) and (iii) we have shown that uw and vz satisfy condition (i) in the definition of $\hat{\rho}^{(2)}$. That they also satisfy condition (ii) is similarly shown. Hence, uw $\hat{\rho}^{(2)}$ vz and therefore, $\hat{\rho}^{(2)}$ is a congruence .•

Let $\rho \in \mathrm{Con}X^{+}$ be such that $X^{+}/\rho \cong S$ and $A = \{x\rho \mid x \in X\}$. Then $A \subseteq S$ and $S = \langle A \rangle$. Also, we have that

$$\hat{S}_A^{(2)} \cong \widehat{(X^{+}/\rho)}_A^{(2)} = \langle\ \{(1,x\rho),(x\rho,1)\} \mid x \in X\ \rangle$$

$$= \{\{((\prod_1^m x_i)\rho,(\prod_{m+1}^n x_i)\rho) \mid 0 \le m \le n\ \} \mid\ n \ge 1 \text{ and } x_i \in X\}.$$

Now define $\varphi : X^{+}/\hat{\rho}^{(2)} \ \rightarrow\ \widehat{(X^{+}/\rho)}_A^{(2)} \cong \hat{S}_A^{(2)}$ by

$$\varphi : (x_1,\ldots,x_n)\,\hat{\rho}^{(2)} \ \rightarrow\ \{((\prod_1^m x_i)\rho,(\prod_{m+1}^n x_i)\rho) \mid 0 \le m \le n\}.$$

<u>Proposition 4.3.2</u>. φ is an isomorphism.

<u>Proof.</u> We first show that φ is well defined.

Let $u, v \in X^{+}$ be such that $u = x_1\ldots\ldots x_n\ \hat{\rho}^{(2)}\ y_1\ldots\ldots y_k = v$. Let $1 \le m \le n$ and $i = n - m$. Then $0 \le i < n$ and hence there exists $0 \le j < k$ such that

$$\prod_1^m x_r = s^i(x_1\ldots..x_n) \quad \rho \quad s^j(y_1\ldots..y_k) = \prod_1^{k-j} y_r$$

and

$$\prod_{m+1}^n x_r = e^{n-i}(x_1\ldots..x_n) \quad \rho \quad e^{k-j}(y_1\ldots..y_k) = \prod_{k-j+1}^k y_r.$$

For $m = 0$, since $u\ \rho\ v$, $(1, u\rho) = (1, v\rho)$.

Hence, for any $0 \le m \le n$ there exist $0 \le k\text{-}j \le k$ such that :

$$((\prod_1^m x_r)\rho,(\prod_{m+1}^n r_i)\rho) = ((\prod_1^{k-j} y_r)\rho,(\prod_{k-j+1}^k y_r)\rho)$$

Similarly any pair

$$((\prod_1^l y_i)\rho,(\prod_{l+1}^k y_i)\rho)$$

is equal to a pair of the form

$$((\prod_1^{p_l} x_i)\rho,(\prod_{p_l+1}^n x_i)\rho).$$

Hence, $u\varphi = v\varphi$. Consequently, φ is well-defined.

To see that φ is injective, let $u = x_1.....x_n$, $v = y_1.....y_m \in X^+$ be such that

$$((u)\hat{\rho}^{(2)})\varphi = ((v)\hat{\rho}^{(2)})\varphi .$$

Then for any $0 \le r < n$ there exists $0 \le t < m$ such that

$$(\prod_1^r x_i)\rho = (\prod_1^t y_i)\rho \quad \text{and} \quad (\prod_{r+1}^n x_i)\rho = (\prod_{t+1}^m x_i)\rho;$$

$$\text{i. e., } (\prod_1^r x_i) \ \rho \ (\prod_1^t y_i) \quad \text{and} \quad (\prod_{r+1}^n x_i) \ \rho \ (\prod_{t+1}^m y_i).$$

That is, for any $0 \le k < n$, if we let $r = n - k$ then there exists $0 \le t < m$ such that

$$s^k(u) = \prod_1^r x_i \quad \rho \quad \prod_1^t y_i = s^{m-t}(v)$$

and

$$e^{n-k}(u) = \prod_{r+1}^n x_i \quad \rho \quad \prod_{t+1}^m y_i = e^{t-j}(v).$$

Similarly, for any $0 \le p < m$, there exists $0 \le l < n$ such that

$$s^p(v) \ \rho \ s^l(u) \quad \text{and} \quad e^{m-p}(v) \ \rho \ s^{n-l}(u).$$

Thus, $u \ \hat{\rho}^{(2)} \ v$ and so, φ is injective.

To see that φ is a homomorphism, let $u = x_1....x_n$, $v = x_{n+1}...x_{n+k} \in X^+$. Then we have that :

$$((u)\,\hat{\rho}^{(2)})\varphi.((v)\,\hat{\rho}^{(2)})\varphi = \{(\prod_{1}^{m} x_i)\rho\,,\,(\prod_{m+1}^{n+k} x_i)\rho) \mid 0 \le m \le n+k\,\}$$

$$= ((xx')\hat{\rho}^{(2)})\varphi\,.$$

Finally, we show that φ is surjective. Let $x \in X$ and

$$\{(x\rho,1),\,(1,\,x\rho)\} \in \widehat{(X^{+}/\rho)}^{(2)}_{A}\,.$$

Then $(x)\,\hat{\rho}^{(2)} \in X^{+}/\hat{\rho}^{(2)}$ and

$$[(x)\hat{\rho}^{(2)}]\varphi = \{(1,x\rho),(x\rho,1)\}.$$

Since $\widehat{(X^{+}/\rho)}^{(2)}_{A}$ is generated by $\{\{(1,x\rho),(x\rho,1)\} \mid x \in X\}$, φ is surjective .•

Now define $h : \mathrm{Con}X^{+} \to \mathrm{Con}X^{+}$ by

$$h : \rho \to \hat{\rho}^{(2)}.$$

Lemma 4.3.3. $h \in \mathscr{C}$.

Proof. That $\hat{\rho}^{(2)} \subseteq \rho$ was discussed in the proof of Lemma 4.3.1. For $\rho,\tau \in \mathrm{Con}X^{+}$ such that $\rho \subseteq \tau$ we clearly have $\hat{\rho}^{(2)} \subseteq \hat{\tau}^{(2)}$.•

Now let H be defined by

$$H :\ (S,\alpha) \ \to (\,\hat{S}^{(2)}_{A}\,,\,\hat{\alpha}\,) \qquad (\,S,\alpha) \in \mathscr{S}_{X}.$$

It follows easily from [1] that H is indeed an expansion.

Proposition 4.3.4. H is congruent to E_h .

Proof. As before, for any $(S,\alpha) \in \mathscr{S}_X$,we know that there exists a unique $\rho \in \mathrm{Con}X^{+}$ such that $X^{+}/\rho \cong S$. Then

$$H(S) = \hat{S}^{(2)}_{A} \cong X^{+}/\hat{\rho}^{(2)} \qquad \text{(by Proposition 4.3.2)}$$

$$= X^{+}/h(\rho) = E_h(\,X^{+}/\rho\,),$$

so that H is congruent to E_h .•

4.4. Some lattice theoretical results.

In this section we introduce the congruence $\tilde{\rho}^{\mathscr{R}}$ which is the dual of $\tilde{\rho}^{\mathscr{L}}$ and we show , by an example , that $\hat{\rho}^{(2)} \lneqq \tilde{\rho}^{\mathscr{L}} \wedge \tilde{\rho}^{\mathscr{R}}$. We then turn our attention to expansions; in particular to $\tilde{S}^{\mathscr{L}}$ and $\tilde{S}^{\mathscr{R}}$. We define an expansion $P: S \to P(S)$ and we prove that $P(S) = \tilde{S}^{\mathscr{L}} \vee \tilde{S}^{\mathscr{R}}$. We then introduce the contraction $p : \rho \to \rho^P$ corresponding to the expansion $P: S \to P(S)$. We conclude this section by showing that $\rho^P = \tilde{\rho}^{\mathscr{L}} \vee \tilde{\rho}^{\mathscr{R}}$.

Definition 4.4.1. Let $\rho \in \mathrm{Con}X^+$. Define $\tilde{\rho}^{\mathscr{R}}$ on X^+ as follows :

For $u , v \in X^+$, where $|u| = m$ and $|v| = n$,

$u \ \tilde{\rho}^{\mathscr{R}} \ v$ if and only if

(i) $u \ \rho \ v$;

(ii) For any $1 \le k < m$ there exists $0 \le l < n$ such that $s^k(u) \ \rho \ s^l(v)$;

and

(iii) For any $1 \le r < n$ there exists $0 \le s < m$ such that $s^r(v) \ \rho \ s^s(u)$.

Proposition 4.4.2.

(i) $\tilde{\rho}^{\mathscr{R}}$ is a congruence on X^+.

(ii) Let $r : \mathrm{Con}X^+ \to \mathrm{Con}X^+$ be defined by

$$r : \rho \to \tilde{\rho}^{\mathscr{R}} \qquad (\rho \in \mathrm{Con}X^+).$$

Then $r \in \mathscr{C}$.

(iii) Let $R : (S,\alpha) \to (\tilde{S}^{\mathscr{R}}_A, \tilde{\alpha}^{\mathscr{R}})$ $\quad (S,\alpha) \in \mathscr{S}_X$ (it follows from [1] or [7] that R is indeed an expansion in $\mathscr{S}_X$). Then E_r is congruent to R.

Proof. All statements are the duals of results in 4.2 and may be proved similarly.•

Clearly $\hat{\rho}^{(2)} \leq \tilde{\rho}^{\mathcal{L}}$ and $\hat{\rho}^{(2)} \leq \tilde{\rho}^{\mathcal{R}}$. Thus, $\hat{\rho}^{(2)} \leq (\tilde{\rho}^{\mathcal{L}} \wedge \tilde{\rho}^{\mathcal{R}})$.

We now prove, by an example, that $\hat{\rho}^{(2)} \lneqq (\tilde{\rho}^{\mathcal{L}} \wedge \tilde{\rho}^{\mathcal{R}})$.

Example 4.4.3. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$. Define ρ on X^+ by

$$u \ \rho \ v \quad \text{if and only if} \quad |u| = |v| = 1 \quad \text{or} \quad |u| \text{ and } |v| > 1 \qquad (u, v \in X^+)$$

Clearly ρ is a congruence on X^+ and $X^+/\rho \cong S = \{0,1\}$ where the multiplication on S is defined by $1 \cdot 1 = 0 \cdot 1 = 1 \cdot 0 = 0 \cdot 0 = 0$.

Let $u = x_1x_2x_3x_4$ and $v = x_1x_2$, then $|u| = 4$ and $|v| = 2$. We now will show that $(u,v) \in \tilde{\rho}^{\mathcal{L}}$, $(u,v) \in \tilde{\rho}^{\mathcal{R}}$ but $(u,v) \notin \hat{\rho}^{(2)}$.

To see that $u \ \tilde{\rho}^{\mathcal{L}} \ v$, first note that $u \ \rho \ v$. Next let $1 \leq k < 4$. If $k \leq 2$ then with $l = 0$ we have $e^k(u) \ \rho \ e^l(v)$, while if $k = 2$ then with $l = 1$ we have $e^k(u) \ \rho \ e^l(v)$. Conversely, since $|v| = 2$ and $1 \leq r \leq |v|$, r must be 1, in which case taking $s = 3$ we have $e^r(v) \ \rho \ e^s(u)$. Hence, $(u,v) \in \tilde{\rho}^{\mathcal{L}}$.

Dually, $(u,v) \in \tilde{\rho}^{\mathcal{R}}$.

However, for $i = 1$, $s^i(u) = x_1x_2x_3$ and $|s^i(u)| = 3 > 1$. Therefore, $s^i(u) \ \rho \ s^j(v)$ only if $|s^j(v)| > 1$. But $2 = |v| \geq s^j(v)$ for any $j \geq 0$ and so $s^i(u) \ \rho \ s^j(v)$ only if $j = 0$; that is, $s^j(v) = s^0(v) = v$. But then we have

$$e^{m-i}(u) = e^{4-1}(u) = x_4 \qquad \text{and} \qquad e^{n-j}(v) = e^{2-0}(v) = e^2(v) = \emptyset$$

so that $(e^{m-i}(u), e^{n-j}(v)) \notin \rho$. Thus, u and v do not satisfy the condition (i) in the definition of $\hat{\rho}^{(2)}$, and as a result $(u,v) \notin \hat{\rho}^{(2)}$.

It follows that $\hat{\rho}^{(2)} \lneqq \tilde{\rho}^{\mathcal{L}} \wedge \tilde{\rho}^{\mathcal{R}}$.•

We now turn our attention to expansions .

For $(S,\alpha) \in \mathscr{S}_X$, let

$$P(S) = \{(\{ \prod_1^n s_i \mid 0 \le n \le k \} , \prod_1^k s_i , \{ \prod_{m+1}^k s_i \mid 0 \le m \le k \}) \ s_i \in S \}.$$

Define the multiplication on $P(S)$ by

$$(A_1 ,s, A_2)\cdot(B_1 ,t, B_2) = (A_1 \cup s\cdot B_1 ,st, A_2\cdot t \cup B_2)$$

where $s\cdot B_1 = \{ sx \mid x \in B_1 \}$ and $A_2\cdot t = \{ yt \mid y \in A_2 \}$

and let $p(\alpha) : x \to (\{1,x\alpha\},x\alpha,\{x\alpha,1\})$.

It is easily seen that $P(S)$ is a semigroup.

<u>Lemma 4.4.4</u>. Define $P : \mathscr{S}_X \to \mathscr{S}_X$ by $P : (S,\alpha) \to (P(S),p(\alpha))$. Then P is an expansion in $\mathscr{S}_X$.

<u>Proof</u>. Let $(S,\alpha) \in \mathscr{S}_X$. Define $\eta : P(S) \to S$ as follows :

$$\eta : (\{\prod_1^n s_i \mid 0 \le n \le k\}, \prod_1^k s_i ,\{ \prod_{m+1}^k s_i \mid 0 \le m \le k \}) \to \prod_1^k s_i .$$

Clearly η is an epimorphism. Next , let $S,T \in \mathscr{S}_X$ and $\theta : S \to T$ an epimorphism. Define $P(\theta) : P(S) \to P(T)$ by

$$P(\theta) : (\{\prod_1^n s_i \mid 0 \le n \le k\}, \prod_1^k s_i ,\{ \prod_{m+1}^k s_i \mid 0 \le m \le k \}) \to$$
$$(\{(\prod_1^n s_i)\theta \mid 0 \le n \le k\}, (\prod_1^k s_i)\theta, \{(\prod_{m+1}^k s_i)\theta \mid 0 \le m \le k\}).$$

Then $P(\theta)$ is an epimorphism , thus P is a functor. Hence P is an expansion in $\mathscr{S}_X$.•

Let $E_{\mathscr{L}}, E_{\mathscr{R}} \in \mathscr{E}$ be defined by

$$E_{\mathscr{L}} : S \to \tilde{S}^{\mathscr{L}} \quad \text{and} \quad E_{\mathscr{R}} : S \to \tilde{S}^{\mathscr{R}} \qquad (S \in \mathscr{S}_X).$$

For expansions E, F in $\mathscr{S}_X$, let $E \leq F$ if and only if there exists an epimorphism $\theta(S) : F(S) \to E(S)$ for any $(S,\alpha) \in \mathscr{S}_X$. The relation $\leq$ is not a partial order. However, although we are aware of this, for the next part of the discussion we will proceed in the understanding that we are only working within isomorphism of expansions.

Proposition 4.4.5. $P = E_{\mathscr{L}} \vee E_{\mathscr{R}}$.

Proof. For any $S \in \mathscr{S}_X{}^*$, clearly

$$\theta_{\mathscr{L}} : P(S) \to \widetilde{S}^{\mathscr{L}} \quad \text{and} \quad \theta_{\mathscr{R}} : P(S) \to \widetilde{S}^{\mathscr{R}}$$

defined by:

$$\theta_{\mathscr{L}} : (\{\prod_1^n s_i \mid 0 \leq n \leq k\}, \prod_1^k s_i, \{\prod_{m+1}^k s_i \mid 0 \leq m \leq k\}) \to (\prod_1^k s_i, \prod_{m+1}^k s_i \mid 0 \leq m \leq k\})$$

and

$$\theta_{\mathscr{R}} : (\{\prod_1^n s_i \mid 0 \leq n \leq k\}, \prod_1^k s_i, \{\prod_{m+1}^k s_i \mid 0 \leq m \leq k\}) \to (\{\prod_1^n s_i \mid 0 \leq n \leq k\}, \prod_1^k s_i)$$

are epimorphisms. Hence $P \geq E_{\mathscr{L}}$ and $P \geq E_{\mathscr{R}}$.

To see that $P = E_{\mathscr{L}} \vee E_{\mathscr{R}}$, let $F \in \mathscr{E}$ be such that $F \geq E_{\mathscr{L}}$ and $F \geq E_{\mathscr{R}}$. Let $S \in \mathscr{S}_X{}^*$. Then there exist epimorphisms

$\varphi_{\mathscr{L}} : F(S) \to \widetilde{S}^{\mathscr{L}}$ and $\varphi_{\mathscr{R}} : F(S) \to \widetilde{S}^{\mathscr{R}}$, such that the following diagram commutes :

$$\begin{tikzcd}
 & F(S) \arrow[dl, "\varphi_{\mathscr{L}}"'] \arrow[dr, "\varphi_{\mathscr{R}}"] \arrow[dd, "\eta_F"] & \\
\widetilde{S}^{\mathscr{L}} \arrow[dr, "\eta_{\mathscr{L}}"'] & & \widetilde{S}^{\mathscr{R}} \arrow[dl, "\eta_{\mathscr{R}}"] \\
 & S &
\end{tikzcd}$$

Now, let

$$I = \{(s\varphi_{\mathscr{R}}, s\varphi_{\mathscr{L}}) \mid s \in F(S)\}.$$

Since the above diagram is commutative we have $(s)\varphi_{\mathscr{L}} o \eta_{\mathscr{L}} = (s)\varphi_{\mathscr{R}} o \eta_{\mathscr{R}}$.
Hence the pairs $(s\varphi_{\mathscr{R}}, s\varphi_{\mathscr{L}})$ are of the form :

$$((\{\prod_1^n s_i \mid 0 \le n \le k\}, \prod_1^k s_i), (\prod_1^k s_i, \{\prod_{m+1}^k s_i \mid 0 \le m \le k\}))$$

(Recall that $\eta_{\mathscr{L}}$ and $\eta_{\mathscr{R}}$ are the projection mappings on the first and second coordinates, respectively). We define the multiplication on I by :

$$((A_1,s),(s,A_2))\cdot((B_1,t),(t,B_2)) = ((A_1 \cup s\cdot B_1, st),(st, A_2\cdot t \cup B_2))$$

where $s\cdot B_1 = \{ sx \mid x \in B_1\}$ and $A_2\cdot t = \{ yt \mid y \in A_2\}$.

Let $\Psi : F(S) \to I$ be defined by

$$\Psi : s \to (s\varphi_{\mathscr{R}}, s\varphi_{\mathscr{L}}).$$

Since

$$(s\varphi_{\mathscr{R}}, s\varphi_{\mathscr{L}})\cdot(t\varphi_{\mathscr{R}}, t\varphi_{\mathscr{L}}) = (s\varphi_{\mathscr{R}}\cdot t\varphi_{\mathscr{R}}, s\varphi_{\mathscr{L}}\cdot t\varphi_{\mathscr{L}})$$
$$= ((st)\varphi_{\mathscr{R}}, (st)\varphi_{\mathscr{L}})$$

and $\varphi_{\mathscr{R}}$ and $\varphi_{\mathscr{L}}$ are epimorphisms, Ψ is an epimorphism.

Next, define $\chi : I \to P(S)$ by

$$\chi : ((\{\prod_1^n s_i \mid 0 \le n \le k\}, \prod_1^k s_i), (\prod_1^k s_i, \{\prod_{m+1}^k s_i \mid 0 \le m \le k\})) \to$$
$$(\{\prod_1^n s_i \mid 0 \le n \le k\}, \prod_1^k s_i, \{\prod_{m+1}^k s_i \mid 0 \le m \le k\}).$$

It is not difficult to verify that χ is an isomorphism. Let $\theta = \chi o \Psi$, then we have that $\theta : F(S) \to P(S)$ is an epimorphism. Thus, $F \ge P$ and this proves that $P = E_{\mathscr{L}} \vee E_{\mathscr{R}}$.•

For $\rho \in \mathrm{Con}X^+$, define ρ^P on X^+ as follows :

For $u,v \in X^+$, where $|u| = m$ and $|v| = n$,

$u \ \rho^P \ v$ if and only if

(i) $u \ \rho \ v$;

(ii) for any $1 \leq i < m$ there exist $0 \leq j, \ j' < n$ such that

$s^i(u) \ \rho \ s^j(v)$ and $e^i(u) \ \rho \ e^{j'}(v)$;

(iii) for any $1 \leq k < n$ there exist $0 \leq l, l' < m$ such

that $s^k(v) \ \rho \ s^l(u)$ and $e^k(v) \ \rho \ e^{l'}(u)$.

Lemma 4.4.6. $\rho^P \in \mathrm{Con}X^+$.

Proof. Clearly ρ^P is an equivalence relation. To see that it is also a congruence, let u,v,w and $z \in X^+$ be such that $u \ \rho^P v$ and $w \ \rho^P z$, where $|u| = m$, $|v| = n$, $|w| = p$ and $|z| = t$.

Let $1 \leq i < m+p$.

If $1 \leq i < m$ then there exists $0 \leq j' < n \leq n+t$ such that $e^i(u) \ \rho \ e^{j'}(v)$. Since $e^i(u)w = e^i(uw)$ and $e^{j'}(v)z = e^{j'}(vz)$, we have $e^i(uw) \ \rho \ e^{j'}(vz)$.

If $m < i < m+p$ then setting $i' = i-m$, we have that $1 \leq i' < p$ and so there exists $0 \leq j < t$ such that $e^{i'}(w) \ \rho \ e^j(z)$. Let $j' = j+n$. Then $0 \leq j' < n+t$. We also have $e^j(z) = e^{j'}(vz)$ and $e^{i'}(w) = e^i(uw)$, whence $e^i(uw) \ \rho \ e^{j'}(vz)$.

If $i = m$ then setting $j' = n$ we have that $e^i(uw) = w \ \rho \ z = e^{j'}(vz)$. Hence, in all cases, there exists $0 \leq j' < n+t$ such that $e^i(uw) \ \rho \ e^{j'}(vz)$. We now look at the following cases :

If $1 \leq i < p$ then there exists $0 \leq j < t$ such that $s^i(w) \ \rho \ s^j(z)$. Since $s^i(uw) = us^i(w)$ and $s^j(vz) = vs^j(z)$ and $u \ \rho \ v$ we have $s^i(uw) \ \rho \ s^j(vz)$.

If $p < i \leq m+p$ then setting $i' = i-p$, we have that $1 \leq i' < m$ and so there exists $0 \leq k < n$ such that $s^{i'}(u) \ \rho \ s^k(v)$. Let $j = k+t$. Then we have

$s^k(v) = s^j(vz)$ and $s^{i'}(u) = s^i(uw)$. Thus $s^i(uw)\ \rho\ s^j(vz)$.

If $i = p$ then setting $j = t$ we have that $s^i(uw) = u\ \rho\ v = s^j(vz)$. Thus in all cases there also exists $0 \le j < n+t$ such that $s^i(uw)\ \rho\ s^j(vz)$.

Dually for any $1 \le k < n+t$ there exist $0 \le l, l' < m+p$ such that $e^k(vz)\ \rho\ e^{l'}(uw)$ and $s^k(vz)\ \rho\ s^l(uw)$. Therefore, $uv\ \rho^P\ vz$. Hence ρ^P is a congruence .•

Let $A \subseteq S$ such that $S = \langle A \rangle$. Then $P(S)_A \subseteq P(S)$ defined by

$$P(S)_A = \langle\ (\{1,a\},a,\{a,1\})\ |\ a \in A\ \rangle$$

is called *the cutdown of* $P(S)$ *to generators* A.

Let $\rho \in \mathrm{Con}X^+$ be such that $X^+/\rho \cong S$. Let $A = \{\ x\rho\ |\ x \in X\ \}$. Then $A \subseteq S$, $S = \langle A \rangle$ and we have $P(S)_A \cong P(X^+/\rho)_A$ where

$$\begin{aligned} P(X^+/\rho)_A &= \langle\ (\{1,x\rho\},x\rho,\{x\rho,1\})\ |\ x \in X\ \rangle \\ &= \{(\{1,(x_1)\rho,\ldots,(x_1\ldots x_n)\rho\},(x_1\ldots x_n)\rho,\{(x_1\ldots x_n)\rho,\ldots,(x_n)\rho,1\})|\ x_i \in X\ \}. \end{aligned}$$

Define $\varphi\colon X^+/\rho^P \to P(X^+/\rho)_A \cong P(S)_A$ by :

$$\varphi\colon (x_1\ldots x_n)\rho^P \to (\{1,(x_1)\rho,\ldots,(x_1\ldots x_n)\rho\},(x_1\ldots x_n)\rho,\{(x_1\ldots x_n)\rho,\ldots,(x_n)\rho,1\}).$$

Proposition 4.4.7. φ is an isomorphism.

Proof. We first show that φ is well defined. Let $u = x_1\ldots x_n\ \rho^P\ y_1\ldots y_m = v$. Then from the definition of ρ^P we have :

$$\{1,(x_1)\rho,\ldots,(x_1\ldots x_n)\rho\} = \{1,(y_1)\rho,\ldots,(y_1\ldots y_m)\rho\}$$

$$\{(x_1\ldots x_n)\rho,\ldots,(x_n)\rho,1\} = \{(y_1\ldots y_m)\rho,\ldots,(y_m)\rho,1\}$$

and

$$(x_1\ldots x_n)\rho = (y_1\ldots y_m)\rho.$$

Thus, $((u)\rho^P)\varphi = ((v)\rho^P)\varphi$ and φ is well defined.

To see that φ is injective, let $u = x_1\ldots x_n$, $v = y_1\ldots y_m \in X^+$ be such that $((u)\rho^P)\varphi = ((v)\rho^P)\varphi$. Then

$$(x_1\ldots x_n)\rho = (y_1\ldots y_m)\rho\ ;$$

$$\{1,(x_1)\rho,\ldots,(x_1\ldots x_n)\rho\} = \{1,(y_1)\rho,\ldots(y_1\ldots y_m)\rho\}\ ;$$

and

$$\{(x_1\ldots x_n)\rho,\ldots,(x_n)\rho\} = \{(y_1\ldots y_m)\rho,\ldots,(y_m)\rho,1\}.$$

Hence $u\ \rho^P\ v$, i.e., $(u)\rho^P = (v)\rho^P$ and φ is injective.

To see that φ is also a homomorphism let $u = x_1\ldots x_n$, $v = y_1\ldots y_m \in X^+$. Then

$$((u)\rho^P)\varphi \cdot ((v)\rho^P)\varphi$$

$$=(\{1,(x_1)\rho,\ldots,(x_1\ldots x_n)\rho\}\ ,(x_1\ldots x_n)\rho,\{(x_1\ldots x_n)\rho,\ldots,(x_n)\rho\})\cdot$$

$$(\{1,(y_1)\rho,\ldots(y_1\ldots y_m)\rho\},(y_1\ldots y_m)\rho,\{(y_1\ldots y_m)\rho,\ldots,(y_m)\rho,1\})$$

$$=(\{1,(x_1)\rho,\ldots,(x_1\ldots x_n)\rho,(x_1\ldots x_ny_1)\rho,\ldots,(x_1\ldots x_ny_1\ldots y_m)\rho\},(x_1\ldots x_ny_1\ldots y_m)\rho,$$

$$\{x_1\ldots x_ny_1\ldots y_m)\rho,\ldots,(x_ny_1\ldots y_m)\rho,(y_1\ldots y_m)\rho,\ldots(y_m)\rho,1\})$$

$$= ((x_1\ldots x_ny_1\ldots y_m)\rho^P)\varphi\ .$$

Finally we show that φ is surjective. Let $x \in X$, and $(\{1,x\rho\},x\rho,\{x\rho,1\})$ be an element of $P(X^+/\rho)_A$. Then $x\rho^P \in X^+/\rho^P$ and $(x\rho^P)\varphi = (\{1,x\rho\},x\rho,\{x\rho,1\})$. Since $P(X^+/\rho)_A$ is generated by

$$\{(\{1,x\rho\},x\rho,\{x\rho,1\}) \mid x \in X\ \}$$

and φ is onto on generators, φ is surjective .•

Now define $p : \mathrm{Con}X^+ \to \mathrm{Con}X^+$ by

$$p : \rho \to \rho^P\ .$$

Lemma 4.4.8. $p \in \mathscr{C}$.

Proof. That $p(\rho) = \rho^P \subseteq \rho$ is clear from the definition of ρ^P. For $\rho,\tau \in \mathrm{Con}X^+$ such that $\rho \subseteq \tau$ we clearly have $p(\rho) = \rho^P \subseteq \tau^P = p(\tau)$.•

Now , let P_A be defined by

$$P_A : (S, \alpha) \rightarrow (P(S)_A, p(\alpha)) \qquad (S,\alpha) \in \mathscr{S}_X .$$

It is easy to verify that P_A is an expansion in $\mathscr{S}_X$.

Proposition 4.4.9. P_A is congruent to E_p .

Proof. As before , for any $(S,\alpha\cdot) \in \mathscr{S}_X$ we know that there exists a unique congruence $\rho \in \mathrm{Con}\,X^+$ such that $(S,\alpha) \cong (X^+/\rho, \iota_\rho)$. Then

$$P_A(S) = P(S)_A \cong X^+/p(\rho) \qquad \text{(by Proposition 4.4.7)}$$
$$= E_p(X^+/\rho),$$

so that P_A is congruent to E_p .•

Proposition 4.4.10. $\rho^P = \tilde{\rho}^{\mathscr{L}} \wedge \tilde{\rho}^{\mathscr{R}}$.

Proof. Let L, R be defined , as before, by

$$L : S \rightarrow \tilde{S}_A^{\mathscr{L}} \qquad \text{and} \qquad R : S \rightarrow \tilde{S}_A^{\mathscr{R}} .$$

Then ,by 4.4.5 , $P_A = L \vee R$. Let $\rho \in \mathrm{Con}X^+$. Then ,by 4.4.9,

$$\rho^P = p(\rho) = c_{P_A}(\rho) = (\Psi(P_A))(\rho) = (\Psi(L \vee R))(\rho)$$
$$= (\Psi(L) \wedge \Psi(R))(\rho)$$
$$= (\Psi(L)(\rho) \wedge (\Psi(R))(\rho)$$
$$= \tilde{\rho}^{\mathscr{L}} \wedge \tilde{\rho}^{\mathscr{R}} .•$$

CHAPTER 5

In this chapter we turn our attentions to the category of monogenic semigroups. First we are going to characterize the contractions in ConX^+, where $X = \{x\}$. Then we look at the expansions in the category of monogenic semigroups and we give some results concerning the lattice of the expansions in this category.

5.1. The contractions in the congruences on the free monogenic semigroup.

We start this section with a characterization of the congruences on the free monogenic semigroup, F. These results are well-known, although not perhaps in this form, and follow easily from the description of monogenic semigroups to be found in Howie [6] and Clifford and Preston [4]. Then, we give the order on ConF. Finally, we define the contractions in ConF and we present an example.

Let $X = \{x\}$. The free semigroup F on X is $F = \{x^m : m = 1,2,\ldots\}$ with the usual multiplication $x^m \cdot x^n = x^{m+n}$. Let $[i,p]$ denote the monogenic semigroup of index i and period p.

Let $\mathcal{M} = \{[i,p] \mid i,p \text{ are integers}, i \geq 0, p \geq 1\}$.

Definition 5.1.1. For any $[i,p] \in \mathcal{M}$, define $\rho_{[i,p]}$ on F by

$$x^m \ \rho_{[i,p]} \ x^n \quad \text{if and only if} \quad m = n < i+1 \quad \text{or} \quad m,n \geq i+1 \ \text{ and } \ p \mid m-n .$$

Lemma 5.1.2. $\rho_{[i,p]}$ is a congruence on F.

Proof. That $\rho_{[i,p]}$ is an equivalence relation is clear. To see that it is a congruence, let m,n,r and s $\in \{1,2,3,\ldots\}$ be such that $x^m \ \rho_{[i,p]} \ x^n$ and $x^r \ \rho_{[i,p]} \ x^s$. We want to show that $x^m.x^r = x^{m+r} \ \rho_{[i,p]} \ x^{n+s} = x^n.x^s$.

If $m = n < i+1$ and $r = s < i+1$ then $m+r = n+s$ and either $m+r = n+s < i+1$, in which case we have $x^{m+r} \ \rho_{[i,p]} \ x^{n+s}$, or $m+r = n+s \geq i+1$, and if this is the case we have $p \mid (m+r)-(n+s) = 0$ hence, $x^{m+r} \ \rho_{[i,p]} \ x^{n+s}$.

If $m = n < i+1$, $r,s \geq i+1$ and $p \mid r-s$ or $m,n \geq i+1$, $p \mid m-n$ and $r = s < i+1$, then we have $m+r$, $n+s \geq i+1$ and $p \mid (m+r)-(n+s)$ hence, $x^{m+r} \ \rho_{[i,p]} \ x^{n+s}$.

If $m,n \geq i+1$, $p \mid m-n$, $r,s \geq i+1$ and $p \mid r-s$ then we have $m+r$, $n+s \geq i+1$ and $p \mid (m+r)-(n+s)$ hence, $x^{m+r} \ \rho_{[i,p]} \ x^{n+s}$. Thus, in all cases $x^{m+r} \ \rho_{[i,p]} \ x^{n+s}$. Therefore $\rho_{[i,p]}$ is a congruence on F.•

Lemma 5.1.3. $\mathrm{Con}F = \{\iota\} \cup \{\rho_{[i,p]} : [i,p] \in M\}$.

Proof. Let $\rho \in \mathrm{Con}F$, $\rho \neq \iota$. Then the set

$$\{m \in \{1,2,\ldots\} \mid \text{there exists } n \in \{1,2,\ldots\} \text{ such that } x^m \rho x^n, m \neq n\}$$

is non empty and so has a least element k. Then the set

$$\{r \in \{1,2,\ldots\} : x^k \rho x^{k+r}\}$$

is non empty and so it too has a least element p.

First of all let $r \in \{1,2,\ldots\}$ be such that $x^k \rho x^{k+r}$. Then $r \geq p$ and so $r = ap+b$ for some integers a and b such that $a \geq 1$, $0 \leq b < p$. Hence, $k+r = k+ap+b$ and $x^k \rho x^{k+r} = x^{k+ap+b} = x^{k+ap}.x^b \rho x^k.x^b = x^{k+b}$. But then b must be 0 since p is the minimum of such elements, hence $p \mid r$.

Now let $0 \leq a < p$, and $x^a \rho x^b$ for some $b \in \{1,2,\ldots\}$. Then $x^{k+a} \rho x^{k+b}$. And, since $p-a > 0$, $x^k \rho x^{k+p} = x^{k+a+(p-a)} \rho x^{k+b+(p-a)}$. **Then, by the above,** $p \mid b+p-a = p+(b-a)$. Hence $p \mid b-a$.

Finally, let $s,t \in \{1,2,\ldots\}$ be such that $s,t \geq k$ and $x^s \rho x^t$. Then $k+ap \leq s < k+(a+1)p$ and $k+bp \leq t < k+(b+1)p$. Hence $0 \leq s-(k+ap) < p$. Since $s = k+ap+s-(k+ap)$ and $t = k+bp+t-(k+bp)$ it follows that:

$$x^{s-ap} = x^k x^{s-(k+ap)} \ \rho\ x^{k+ap} x^{s-(k+ap)} = x^s \ \rho\ x^t = x^{k+bp} x^{t-(k+bp)} \ \rho\ x^k x^{t-(k+bp)} = x^{t-bp},$$

and so, $x^{s-ap} \rho\ x^{t-bp}$. Then, by the above, we have that

$$p \mid (s-ap)-(t-bp) = s-t+p(b-a).$$

Hence $p \mid s-t$. Now, by setting $i = k-1$, we have $\rho = \rho_{[i,p]}$.•

Lemma5.1.4. $\rho_{[i,p]} \leq \rho_{[k,r]}$ if, and only if $i \geq k$ and $r \mid p$.

Proof. First, let $\rho_{[i,p]} \leq \rho_{[k,r]}$. Then, by definition of $\rho_{[i,p]}$, $x^{i+1}\ \rho_{[i,p]}\ x^{i+p+1}$ and so $x^{i+1}\ \rho_{[k,r]}\ x^{i+p+1}$. Then, since $i+1 \neq i+p+1$, we must have $i+1$ and $i+p+1 \geq k+1$, in particular $i+1 \geq k+1$ and so $i \geq k$, and $r \mid (i+p+1)-(i+1) = p$.

Conversely, let $i \geq k$ and $r \mid p$. If $m,n, \in \{1,2,\ldots\}$ are such that $x^m\ \rho_{[i,p]}\ x^n$ then either $m = n < i+1$ in which case $x^m\ \rho_{[k,r]}\ x^m = x^n$ or, $m,n \geq i+1$ and $p \mid m-n$ then, since $i+1 \geq k+1$ and $r \mid p$, we have $x^m\ \rho_{[k,r]}\ x^n$. Hence $\rho_{[i,p]} \leq \rho_{[k,r]}$.•

Let $\mathscr{C}_{\mathscr{M}} = \{ c : c$ is a contraction in ConF $\}$. Let $c \in \mathscr{C}_{\mathscr{M}}$ and $\rho \in$ ConF. Then, by Lemma 5.1.3, either $\rho = \iota$ or $\rho = \rho_{[i,p]}$ for some $[i,p] \in \mathscr{M}$. Note that, by C(i) in the definition of contraction, $f(\iota) = \iota$ for any contraction f. Therefore, it suffices to define the contraction on ConF $\setminus \{\iota\}$ and so, we identify $\rho_{[i,p]}$ by $[i,p]$. Hence c can be regarded as a function from $\mathscr{M}$ into $\mathscr{M}$; i.e.,

$$c : [i,p] \to [k,r] = [c_1(i,p), c_2(i,p)].$$

We recognize that $[i,p]$ has two meanings, one being a congruence $\rho_{[i,p]}$ and the other being the monogenic semigroup with index i and period p, because of the close relation between the sets ConF and $\mathscr{M}$. However, we believe that there will be no confusion due to the context.

Proposition 5.1.5. $c \in \mathscr{C}_{\mathscr{M}}$ if and only if c is of the form :

$$c : [i,p] \to [c_1(i,p), c_2(i,p)]$$

where c_1 and c_2 are functions of two variables satisfying the following conditions:

(i) $c_1(i,p) \geq i$ for any $i,p \in \{1,2,\ldots\}$;

(ii) $p \mid c_2(i,p)$ for any $i,p \in \{1,2,\ldots\}$;

(iii) if $i \geq k$ and $r \mid p$ then $c_1(i,p) \geq c_1(k,r)$ and $c_2(k,r) \mid c_2(i,p)$.

Proof. Let $c \in \mathscr{C}_{\mathscr{M}}$ be such that $c : [i,p] \to [c_1(i,p), c_2(i,p)]$. We will now show that c_1 and c_2 satisfy the above three conditions.

By C(i) in the definition of contraction $c(\rho) \subseteq \rho$ for any $\rho \in$ ConF, that is, $\rho_{[c_1(i,p),\, c_2(i,p)]} \subseteq \rho_{[i,p]}$. Then, by Lemma 5.1.4, $c_1(i,p) \geq i$ and $p \mid c_2(i,p)$. Also, by C(ii) in the definition of contraction, if $\rho \subseteq \tau$ then $c(\rho) \subseteq c(\tau)$; i.e., if $\rho_{[i,p]} \subseteq \rho_{[k,r]}$ then $\rho_{[c_1(i,p),\, c_2(i,p)]} \subseteq \rho_{[c_1(k,r),\, c_2(k,r)]}$; that is, by Lemma 5.1.4, if $i \geq k$ and $r \mid p$ then $c_1(i,p) \geq c_1(k,r)$ and $c_2(k,r) \mid c_2(i,p)$.

Next, let c be a function as defined in the proposition. Let $\rho \in$ ConF, say $\rho = \rho_{[i,p]}$. Let $c(\rho) = \rho_{c([i,p])} = \rho_{[c_1(i,p),c_2(i,p)]}$. Then, since $c_1(i,p) \geq i$, $p \geq c_2(i,p)$ and by Lemma 5.1.4, $c(\rho) \subseteq \rho$.

Let $\rho,\tau \in$ ConF be such that $\rho \subseteq \tau$, say $\rho = \rho_{[i,p]}$ and $\tau = \rho_{[k,r]}$,

$[i,p],[k,r] \in \mathscr{M}$. Then, by **Lemma 5.1.4, $i \geq k$ and $r \mid p$ and so, $c_1(i,p) \geq c_1(k,r)$** and $c_2(k,r) \mid c_2(i,p)$. Hence, $c(\rho) = \rho_{[c_1(i,p),c_2(i,p)]} \subseteq \rho_{[c_1(k,r),c_2(k,r)]} = c(\tau)$ **by** Lemma 5.1.4. Thus $c \in \mathscr{C}_{\mathscr{M}}$.•

Example 5.1.6. Define $c : \mathscr{M} \rightarrow \mathscr{M}$ by

$$c : [i,p] \rightarrow [i+p, 2^i \cdot p]$$

Then for $[i,p] \in \mathscr{M}$ we have that $c_1(i,p) = i+p \geq i$ and $p \mid c_2(i,p) = 2^i \cdot p$ and for $i \geq k$ and $r \mid p$ we have $c_1(i,p) = i+p \geq k+r = c_1(k,r)$ and $c_2(k,r) = 2^k \cdot r \mid 2^i \cdot p = c_2(i,p)$. Hence $c \in \mathscr{C}_{\mathscr{M}}$.

5.2. The expansions in $\mathscr{M}$.

In this section we turn our attention to the expansions in the category of monogenic semigroups $\mathscr{M}$. We define these expansions and we give the order in the lattice of expansions in this category.

Let $\mathscr{E}_{\mathscr{M}} = \{ E : E$ is an expansion in $\mathscr{M} \}$. Since $F/\rho_{[i,p]} \cong [i,p]$ and we have the anti-isomorphisms Φ and Ψ we have that $\mathscr{E}_{\mathscr{M}} = \mathscr{C}_{\mathscr{M}}$. We remark that the relation $\leq$ in $\mathscr{E}_{\mathscr{M}}$ is the reverse of the relation $\leq$ in $\mathscr{C}_{\mathscr{M}}$.

We have the relation $\leq$ in $\mathscr{E}_{\mathscr{M}}$,as defined previously in $\mathscr{E}$, as follows :

For $E, F \in \mathscr{E}_{\mathscr{M}}$, where $E = (e_1,e_2)$ and $F = (f_1,f_2)$, $E \leq F$ if and only if there exists an epimorphism $\varphi_{[i,p]}$ from $F([i,p])$ onto $E([i,p])$ for any $[i,p] \in \mathscr{M}$; that is $E \leq F$ if and only if $e_1(i,p) \geq f_1(i,p)$ and $f_2(i,p) \mid e_2(i,p)$.

In contrast to the situation in general for expansions this is clearly a partial order on $\mathscr{E}_{\mathscr{M}}$. In fact, $\mathscr{E}_{\mathscr{M}}$ is a lattice where the join and the meet of two expansions are given as in the proceeding proposition.

Proposition 5.2.1. For $E, F \in \mathscr{E}_{\mathscr{M}}$, where $E = (e_1,e_2)$ and $F = (f_1,f_2)$

$$E \vee F = J = (j_1,j_2) \quad \text{where} \quad j_1(i,p) = \max\{e_1(i,p),f_1(i,p)\}$$
$$j_2(i,p) = \text{l.c.m.}\{e_2(i,p),f_2(i,p)\},$$

and

$$E \wedge F = M = (m_1,m_2) \quad \text{where} \quad m_1(i,p) = \min\{e_1(i,p),f_1(i,p)\}$$
$$m_2(i,p) = \text{g.c.d.}\{e_2(i,p),f_2(i,p)\}.$$

Proof. It is clear that $J \in \mathscr{E}_{\mathscr{M}}$ and $J \geq E, F$. Let $G = (g_1,g_2) \in \mathscr{E}_{\mathscr{M}}$ be such that $G \geq E$ and $G \geq F$. Then $g_1(i,p) \geq e_1(i,p)$ and $g_1(i,p) \geq f_1(i,p)$ and so $g_1(i,p) \geq \max\{e_1(i,p),f_1(i,p)\} = j_1(i,p)$. Also, since $e_2(i,p) \mid g_2(i,p)$ and $f_2(i,p) \mid g_2(i,p)$ $j_2(i,p) = \text{l.c.m.}\{e_2(i,p),f_2(i,p)\} \mid g_2(i,p)$. Hence, $G \geq J$ and consequently $J = E \vee F$.

It is also clear that $M \in \mathscr{E}_{\mathscr{M}}$ and $M \leq E, F$. Let $H = (h_1,h_2) \in \mathscr{E}_{\mathscr{M}}$ be such that $H \leq E$ and $H \leq F$. Then $e_1(i,p) \geq h_1(i,p)$ and $f_1(i,p) \geq h_1(i,p)$ and so $h_1(i,p) \leq \min\{e_1(i,p),f_1(i,p)\} = m_1(i,p)$. Also, since $h_2(i,p) \mid e_2(i,p)$ and $h_2(i,p) \mid e_2(i,p)$, $h_2(i,p) \mid = \text{g.c.d.}\{e_2(i,p),f_2(i,p)\} = m_2(i,p)$. Hence, $H \leq M$ and consequently $M = E \wedge F$.•

For expansions in $\mathscr{M}$ we have the following compatibility conditions.

Proposition 5.2.2. Let E, F and $G \in \mathscr{E}_{\mathscr{M}}$, $E = (e_1,e_2)$, $F = (f_1,f_2)$ and $G = (g_1,g_2)$. Then $(E \wedge F)\cdot G = E\cdot G \wedge F\cdot G$ however, $G(E \wedge F) \leq G\cdot E \wedge G\cdot F$

but $G(E \wedge F)$ is not necessarily equal to $G \cdot E \wedge G \cdot F$.

Proof. Let $[i,p] \in \mathscr{M}$. Then,

$$[(E \wedge F) \cdot G]([i,p]) = (E \wedge F)(G([i,p])) = (E \wedge F)([g_1(i,p),g_2(i,p)])$$
$$= [m_1(i,p),m_2(i,p)]$$

where $m_1(i,p) = \min\{e_1([g_1(i,p),g_2[i,p]),f_1([g_1(i,p),g_2(i,p)]\}$ and

$$m_2(i,p) = \text{g.c.d.}\{e_2([g_1(i,p),g_2[i,p]),f_2([g_1(i,p),g_2(i,p)]\}.$$

On the other hand,

$$(E \cdot G \wedge F \cdot G)([i,p]) = [\min\{e_1([g_1(i,p),g_2[i,p]),f_1([g_1(i,p),g_2[i,p])\},$$
$$\text{g.c.d.}\{e_2([g_1(i,p),g_2[i,p]),f_2([g_1(i,p),g_2[i,p])\}].$$

Therefore, $(E \wedge F) \cdot G = E \cdot G \wedge F \cdot G$.

First of all since $E \wedge F \leq E, F$ we have that $G \cdot (E \wedge F) \leq G \cdot E$, $G \cdot F$. Hence, $G \cdot (E \wedge F) \leq G \cdot E \wedge G \cdot F$. Now we will give an example for which $G \cdot (E \wedge F)$ is not equal to $G \cdot E \wedge G \cdot F$. Define E,F and G by

$$e_1(i,p) = e_1(i) = f_1(i,p) = f_1(i) = g_1(i,p) = g_1(i) = i$$

$$e_2(i,p) = e_2(p) = \begin{cases} p & \text{if } p \neq 2 \\ 4 & \text{if } p = 2 \end{cases}$$

$$f_2(i,p) = f_2(p) = \begin{cases} p & \text{if 2 does not divide } p \\ 6k & \text{if } p = 2k \end{cases}$$

and

$$g_2(i,p) = g_2(p) = \begin{cases} 10k & \text{if } p=2k \text{ and } k \in N \setminus \{1\} \\ p & \text{otherwise} \end{cases}$$

It is easy to verify that E ,F and $G \in \mathscr{E}_{\mathscr{M}}$. Then for $[i,p] = [2,2]$ we have that

$$(G \cdot (E \wedge F))([2,2]) = G([\min\{e_1(2),f_1(2)\}, \text{g.c.d.}\{e_2(2),f_2(2)\}])$$
$$= G([2, \text{g.c.d.}\{4,6\}]) = G([2,2]) = [2,2].$$

However,

$$(G \cdot E \wedge G \cdot F)([2,2]) = [\min\{g_1(e_1(2)),g_1(f_1(2))\},\text{g.c.d.}\{g_2(e_2(2)),g_2(f_2(2))\}]$$

$$= [\min\{2,2\}, \text{g.c.d.}\{g_2(4), g_2(6)\}]$$

$$= [2, \text{g.c.d.}\{20,30\}] = [2,10].$$

Thus, we have $G\cdot(E \wedge F) \neq G\cdot E \wedge G\cdot F$ in this case.

Remark5.2.3. The dual result holds for the joins; that is, $(E \vee F)\cdot G = E\cdot G \vee F\cdot G$ and $G\cdot(E \vee F) \geq G\cdot E \vee G\cdot F$ but $G\cdot(E \vee F)$ is not necessarily equal to $G\cdot E \vee G\cdot F$.

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