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IS THERE A CONNECTION BETWEEN PLANCK'S CONSTANT, BOLTZMANN'S CONSTANT AND THE SPEED OF LIGHT?

by

Peter Danenhower

B.Sc. (Hons. First Class), Simon Fraser University, 1977

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
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of
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Examining Committee:

Chairman: C. Villegas

E. Pechlaner
Senior Supervisor

A. Das

S. Kloster

C. Y. Shen
External Examiner
Department of Mathematics and Statistics
Simon Fraser University

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IS THERE A CONNECTION BETWEEN PLANKK'S CONSTANT, BOLTZMANN'S CONSTANT AND THE SPEED OF LIGHT?

Author:

(signature)

PETER DANENHOWER

(name)

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(date)

ABSTRACT

An attempt is made to find a connection between Planck's constant, h, Boltzmann's constant, k, and the speed of light, c. The method used is to study blackbody radiation without quantum mechanics. Classical thermodynamics and statistical mechanics are reviewed. The problem of finding a satisfactory relativistic generalization of these theories is then discussed and the canonical approach due to Balescu is presented. A discussion of the standard treatment of blackbody radiation (including quantum results) follows. The quantum result (Planck law with zero point energy) for the spectral density of a blackbody radiator is then derived, as first done by Einstein, Hopf and Boyer, using the techniques of stochastic electrodynamics (non-quantum derivation). Unfortunately, Planck's constant must be introduced as a scale factor in this treatment. Hence, there are still too many free choices for a relationship between h, c, and k to be a necessity. Accordingly, two attempts are made to study this problem in more detail: 1. A "classical Fermi-Dirac" statistics is developed, to treat the walls of the blackbody cavity as a Fermi gas. 2. Adjustments to thermodynamics, required by the non-quantum derivation of the spectral density, are subjected to the relativistic thermodynamics previously developed. These two problems are very difficult and little progress is made on either. Hence, we are left with no conclusion about definite independence or dependence of h, k and c.

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CHAPTER I

INTRODUCTION

An ongoing problem in the development of modern mathematical physics is deciding just how many degrees of freedom are available for choosing units and dimensions of basic physical parameters. While the literature and many texts include brief overviews of this topic, detailed discussions are not too common. For an in depth review, see Lévy-LeBlond [1]. We can illustrate the problem with a simple example: If we choose a time scale arbitrarily, we then appear to be able to choose a length scale arbitrarily, thereby fixing the speed of light. We can introduce a mass scale and determine the dimension and scale of force (if we choose dimensions for the constant in Newton's second law - normally set equal to 1). We seem to be able to continue this chain of defining parameters and constants (subject to physical relevance) as we please. For example, if we choose electric charge and temperature as independent then we get two more fundamental constants; the permitivity constant (of free space) and Boltzmann's constant. The question is: Do we really have all this freedom? Is it possible that by developing mathematical physics in this manner, we have made arbitrary choices which are in fact incompatible? If so, then unification of diverse fields of physics would be impossible.

In this thesis we attempt to gain more insight into this situation by searching for a physical connection between three fundamental physical constants: Planck's constant, h, Boltzmann's constant, k, and the speed of light, c. We wish to make it clear at the on-set that we are not referring to "mathematical relationships". A search for mathematical relationships involves looking for clusters of constants that have the same dimension. For example, if the unit of length is defined as $\frac{h^2}{4\pi^2 me^2}$ where m and e are respectively the mass and charge on an electron, and the unit of time is defined as $\frac{h^3}{8\pi^3 me^4}$, then that leaves c proportional to $\frac{2\pi e^2}{h}$. This sort of approach seems unhelpful, since we can take an extreme (but very convenient) case and choose a system of units and dimensions such that $h = k = c = 1$. Then there are all manner of relationships between h, k and c. For example, $k^2 = h^3\sqrt{c}$.

Accordingly, to make progress on the question we have posed, we need to search for a <u>physical</u> relationship between h, k and c. By this we mean starting with any two of the three disciplines of physics associated with these constants (quantum mechanics, thermodynamics, and electrodynamics-relativity) and deriving results thought to require theory from the third. The particular trio of constants, h, k, c, is considered in this paper because a relationship between them was suggested from considerations of another project. That work is an attempt to extend special relativity by incorporating a generalized position function $Y(\chi,\tau)$ where χ and τ are surface parameters (we are suppressing two dimensions). The idea is that, in general, a particle is 'smeared' over a world hypersurface rather than restricted to a world line. The idea of the generalization is to connect $\Upsilon(\chi,\tau)$ with the 'rest mass' of the particle via some conformally invariant second order differential equation (not yet discovered). Of course, the symmetry group might be some other extension of the Poincarè group. In any case, given that there exists a physically relevant function $\Upsilon(\chi,\tau)$, by analogy with relativity the parameters, $\frac{\partial \Upsilon}{\partial x}$ and $\frac{\partial \Upsilon}{\partial t}$, where x and t are space-time coordinates, should have fundamental constants associated with them. We would like to associate $\frac{\partial \Upsilon}{\partial x}$ and $\frac{\partial \Upsilon}{\partial t}$, with h and k respectively. Then in the standard relativistic limit in which the particle follows a trajectory, $\frac{dx}{dt}$ is non-zero so that $\frac{\partial \Upsilon}{\partial x}\frac{dx}{dt} = \frac{\partial \Upsilon}{\partial t}$. This suggest that hc = k in this limit. Of course, k must be redefined dimensionally for this relationship to be acceptable. In general, since we are searching for a relationship between h, k and c, we must leave one constant undefined dimensionally until we have discovered the relationship between all three (assuming such a relationship exists). We will nevertheless, continue to refer to the 'thermodynamic constant' as Boltzmann's constant. As the above project has not been worked-out in detail, it will be discussed no further.

We would like to suggest some more concrete physical reasons why a relationship might hold between h, k, and c. Before doing this, however, we will briefly discuss a common perception amongst theoreticians: It is that Boltzmann's constant is not as fundamental as Planck's constant and the speed of light, being as it is, merely a scale factor between energy and temperature (private conversations with M. Plischke and L. Ballentine, S.F.U. physics dept. [this may or may not be their personal view]). However, several of the important constants of physics are scale factors between various parameters and energy, for example, $E = h\nu$ and $E = m_0c^2$. Thus, what is really meant here, is that there does not seem to be some physical realm in which k is taken to be either infinite or zero. (Classically, h->0 and c->∞ ; these limits change the physics in a fundamental way, eg., momentum and position operators commute, and Lorentz symmetry reduces to Galilean symmetry, respectively.) However, as a test for 'fundamentalness' the above criterium will not do: It could be that k is <u>more</u> fundamental than h and c, having the same relationship to all of physics that h and c have in their respective realms, i.e., it could be that letting k->0 (or ∞) leads to the whole universe being in a different realm, whereas letting h-> 0 or c->∞ leads to a different realm within the present universe. After all, in the usual formulations of physics, temperature is treated as a fundamental dimension along with length, time, mass and electric charge (possibly among others, such as, color or baryon number). Since, temperature seems to be definable (statistically at least) for even very simple systems, such as a point particle subject to a central force, the status of k in relation to h and c is still unclear.

With this question in mind we can ask another, namely, exactly how does the concept of temperature arise? The answer seems to be that temperature arises from the fact that specifying the energy of a system is not sufficient, in general, to specify the microsopic details of the system. For example, for a particle subject to a central force with energy E, there are many possible trajectories. Temperature is essentially a reflection of how the amount of degeneracy (number of possible states) depends on E (see equation (2.20) in chapter six). Saying that k, h, and c are all independent is tantamount to claiming that relativity and quantum mechanics do not restrict degeneracy any further than classical physics, i.e., every system displaying degeneracy according to classical physics also displays degeneracy according to quantum mechanics and relativity and vice versa. (This statement needs to be distinguished from the actual calculation of the amount of degeneracy which, in general, depends on whether quantum mechanics, relativistic or classical theories are used for the calculation.) So, once more we are stuck with considerations which are not very tractable.

In any case, from these rather vague considerations, we have put together a concrete plan of study (which, however, may only be less obviously as untractable as the previous considerations!): Find a phenomenon that involves all three of the above disciplines for a standard explanation and try to explain it using theory from at most two of the above disciplines. The phenomenon we have chosen is blackbody radiation, since it has been extensively studied, is well understood, and has several properties which are independent of the details of the particular system used to model it. The problem of deriving the spectral density (the central problem, see chapter six) using only relativity (electromagnetic fields) and thermodynamics has already been done by Timothy Boyer [2-7]. Unfortunately, Planck's constant must be introduced into this work as a proportionality constant, thereby leaving h, k, and c independent. Actually, this is not surprising, since by using electromagnetic field theory we have introduced a fourth discipline and a fourth constant, namely the permittivity constant, ϵ_0 (or equivalently ~~the per~~meability, μ_0. Recall that $c^2 = \frac{1}{\mu_0 \epsilon_0}$).

Since this development is apparently unavoidable, (essentially, a force law must be introduced somewhere, if we are to explain a dynamical system) in this thesis we have concentrated on two possible refinements of Boyer's derivation: We have developed a "classical Fermi-Dirac" statistics, assuming the Pauli - Exclusion principle is independent of quantum mechanics. (This has surely been done by others as well, but we have not found a reference to such.) Then the walls of the blackbody cavity can be treated as a Fermi-gas subject to local forced oscillations from the radiation in the cavity. Of course, the phase average of the electric field components tangent to the walls is zero as usual, but $\langle E_T^2 \rangle$ may not be. Essentially, the problem is to modify the Rayleigh - Jeans approach (see chapter three) to include the Fermi energy of the electrons. We have not been able to do this calculation. The other approach we have taken is to see what relativity has to say about adjustments, required by Boyer's analysis, to the thermodynamic and statistical mechanical notions of entropy. In particular, these two concepts of entropy are no longer equal (see chapter 4, section 2). For this work we have made a thorough study of relativistic statistical mechanics (for systems in equilibrium), but, alas, to no avail - we have not solved this problem either.

Before beginning detailed discussions, we give a brief overview. Chapter two is a brief review of classical thermodynamics and statistical mechanics. Topics covered are restricted to those used in later chapters. In chapter three we present the various generalizations of equilibrium thermodynamics to relativistic thermodynamics. The development of relativistic statistical mechanics due to Balescu is also discussed in detail here. Chapter four is a summary of the usual treatment of blackbody radiation, beginning with the Stefan-Boltzmann law and concluding with the modern quantum treatment (included for later comparison). In chapter five, the rather long derivation of the blackbody spectral density due to Boyer is explained. Also given here are the adjustments to thermodynamics and statistical mechanics required by Boyer's approach. Finally, chapter six includes the development of "classical Fermi-Dirac statistics", some comments on relativizing Boyer's adjustments to thermodynamics and statistical mechanics, and some concluding remarks.

The reader should be forwarned that there is a serious problem with notation arising from bringing together several diverse fields of study which employ coinciding symbols to mean different things. For example, traditionally, "P" is used to denote pressure in thermodynamics, the generator of space translations in canonical mechanics, and the radiation drag function in the equation, $F = -P'v$, from stochastic electrodynamics. We have chosen not to break with tradition in our notation, and have attempted to insure that new usages of symbols already introduced in a previous context are not confusing. Some other overworked symbols to be careful with are: "p", invariably used to denote momentum (or the canonical momentum variable), in the earlier chapters, but also used for the dipole moment in chapter 5. "V" is used to denote volume throughout, whereas "v" is used to express velocity. "K" with various subscripts and superscripts is used to denote frames of reference, while "k" is used variously, as the wave vector, unit vector in the z-direction, Boltzmann's constant, and the magnitude of the wave vector (vector quantities are denoted by boldface type).

REVIEW OF THERMODYNAMICS AND STATISTICAL MECHANICS

Classical Thermodynamics

We begin with a concise overview of thermodynamics and statistical mechanics. Quantum statistics will not be necessary since the analysis of blackbody radiation presented in Chapter 4 is specifically intended to use relativity and thermodynamics alone. The classical case is developed here and extended to relativity in the next chapter. The development presented here follows Reif [8].

Classical Thermodynamics is concerned with making macroscopic statements about the properties of macroscopic systems. No attempt is made to understand the microscopic picture of the system at hand. This approach is based on four empirical laws, described as follows:

1. Zeroth law - If two systems are in thermal equilibrium with a third system then they are in equilibrium with each other. Experimentally, "thermal equilibrium" means there is no net heat flow between the two systems. This law establishes temperature as a useful parameter with which to measure thermal equilibrium (The third system acts as a thermometer.).
2. First law - A system in equilibrium, in a specific macrostate, can be described by a parameter E called the internal energy which has the following properties:
For an isolated system, E = constant. If the system is

interacting with another system, then

$$\Delta E = -W + Q \tag{2.1}$$

where W is the work done by the first system on the second and Q is the heat absorbed by the first system.

An isolated system is defined to be a system in which no heat is absorbed or lost and no work is done on or by the system. There is a certain circularity in the definitions of E, W, and Q, since Q is usually defined as $Q = \Delta E + W$, while the first law describes a property of E in terms of Q. This dilemma is not a problem in the classical case, since the difference between heat and work is usually obvious. However, as will be discussed in the next chapter, this ambiguity is really troublesome in the relativistic generalization of thermodynamics.

3. Second law - A system in a specific equilibrium macrostate can be described by a parameter S, (called the entropy) which has the following properties:

In any process in which a thermally isolated system goes from one macrostate to another,

$$S_f - S_i \geq 0 ,$$

where S_f and S_i are the entropies of the final and initial states respectively.

For a non-isolated system undergoing a quasi-static process at constant volume, V, in which an infinitesimal amount of heat đQ is absorbed then,

$$dS = \frac{đQ}{T} , \tag{2.2}$$

where T is the temperature of the system (measured on an absolute scale). A quasi-static process is a process carried out slowly enough to keep the system in approximate equilibrium throughout the process. Of course, "slowly enough" depends on the particular system at hand.

To calculate changes to the macroscopic parameters of a system due to an arbitrary process is usually very difficult in classical thermodynamics, since, in general, the expression

$$đQ = dE + đW ,$$

is not an exact differential, and hence a detailed knowledge of the process is required to integrate. That is, a knowledge of the initial and final states is not sufficient since $\int_i^f đQ$ is path dependent. To get around this problem, generally only quasi-static processes are considered. Henceforth, all processes will be assumed to be quasi-static unless otherwise stated. In this case, the first law can be written as,

$$TdS = dE + PdV , \qquad (2.3)$$

where P is the pressure and V the volume of the system.

If the equation of state of the system is known, then all the macroscopic parameters of the system can be determined. The two mathematical techniques involved are Legendre transformations and properties of exact differentials. Since this kind of analysis will be used in the next chapter, a brief illustration is given here for the case of an ideal gas.

The equation of state for an ideal gas in a container is:

$$PV = \nu RT, \qquad (2.4)$$

where P is the pressure, V is the volume, ν is the number of moles of gas, R is the gas constant, and T is the temperature of the gas ($R = N_0 k$, where N_0 is Avogadro's number and k is Boltzmann's constant.). The equation of state and the first law can be used to show that the internal energy of the gas is a function of T alone as follows: Assume $E = E(V,T)$, then,

$$dE = \left(\frac{\partial E}{\partial V}\right)_T dV + \left(\frac{\partial E}{\partial T}\right)_V dT \; .$$

Substituting into the first law (2.3), we get,

$$TdS = \left(\frac{\partial E}{\partial V}\right)_T dV + \left(\frac{\partial E}{\partial T}\right)_V dT + PdV \; .$$

Dividing by T and substituting for P from (2.4),

$$dS = \frac{1}{T}\left(\frac{\partial E}{\partial V}\right)_T dV + \frac{1}{T}\left(\frac{\partial E}{\partial T}\right)_V dT + \frac{\nu R}{V} dV \; .$$

Comparing this with the mathematical relation,

$$dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV \; , \tag{2.5}$$

yields the two equations:

$$\left(\frac{\partial S}{\partial V}\right)_T = \frac{1}{T}\left(\frac{\partial E}{\partial V}\right)_T + \frac{\nu R}{V} \; ; \tag{2.6}$$

$$\left(\frac{\partial S}{\partial T}\right)_V = \frac{1}{T}\left(\frac{\partial E}{\partial T}\right)_V \; . \tag{2.7}$$

Taking the partial derivative of (2.6) and (2.7) with respect to T and V respectively, and equating,

$$\frac{-1}{T^2}\left(\frac{\partial E}{\partial V}\right)_T + \frac{1}{T}\left(\frac{\partial^2 E}{\partial T \partial V}\right) = \frac{1}{T}\left(\frac{\partial^2 E}{\partial V \partial T}\right) \; .$$

Thus, $\frac{-1}{T^2}\left(\frac{\partial E}{\partial V}\right)_T = 0$, giving the desired result.

To calculate the change in entropy of the gas, define the molar specific heat at constant volume,

$$C_V \equiv \frac{1}{\nu}\left(\frac{\partial E}{\partial T}\right)_V . \qquad (2.8)$$

C_V may be a function of T but not V. Then by the previous result, $dE = \left(\frac{\partial E}{\partial T}\right)_V dT$. Substituting into (2.3) yields,

$$TdS = \left(\frac{\partial E}{\partial T}\right)_V dT + PdV .$$

Using (2.4) and (2.8) to substitute for P and $\left(\frac{\partial E}{\partial T}\right)_V$ respectively we get,

$$dS = \frac{1}{T}\nu C_V(T)dT + \frac{\nu R}{V}dV . \qquad (2.9)$$

If we define arbitrarily the value of the molar entropy, S_0, of some standard state, then this equation can be integrated along any convenient quasi-static path to get:

$$S(V,T,\nu) - \nu S_0 = \nu\int_{T_0}^{T} \frac{C_V(T)}{T}dT + \nu\int_{V_0'}^{V} \frac{R}{V}dV ,$$

where ν_0, V_0, and T_0 are the number of moles, the volume and temperature that define the standard state respectively, $S(V,T,\nu)$ is the entropy of ν moles of gas in the final state, S_0 the molar entropy of the standard state, and $V_0' \equiv \frac{V_0\nu}{\nu_0}$, is the volume of ν moles of gas in the standard state. The number of moles of gas, ν, remains fixed throughout the process. For example, start with volume V_0', raise the temperature from T_0 to T at constant volume, then change the volume from $V_0\nu/\nu_0$ to V at constant temperature.

For a monatomic ideal gas $C_V = \frac{3}{2}R$ so,

$$S(V,T,\nu) - \nu S_0 = \frac{3\nu R}{2}\ln\frac{T}{T_0} + \nu R\ln\frac{V\nu_0}{V_0\nu}$$

$$= \frac{3\nu R}{2}\ln\frac{T}{T_0} + \nu R\ln\frac{V}{\nu} - \nu R\ln\frac{V_0}{\nu_0}\ .\ \text{So,}$$

$$S(T,V,\nu) = \frac{3\nu R}{2}\ln T + \nu R\ln\frac{V}{\nu} - \frac{3\nu R}{2}\ln T_0 - \nu R\ln\frac{V_0}{\nu_0} + \nu S_0\ ,\ \text{or}$$

$$S(V,T,\nu) = \nu\{\frac{3}{2}R\ln T + R\ln\frac{V}{\nu} + \text{constant}\} \qquad (2.10)$$

Of course, to derive the result, $C_V = \frac{3}{2}R$ for a monatomic ideal gas, microscopic considerations need to be taken into account.

Central to any discussion of macroscopic thermodynamics are the Maxwell relations and various Legendre transforms of the internal energy, E. These are presented below followed by a sample derivation.

Maxwell's relations are:

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V\ ;\ \left(\frac{\partial T}{\partial P}\right)_S = \left(\frac{\partial V}{\partial S}\right)_P\ ;$$

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V\ ;\ \left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P \qquad (2.11)$$

In each case the parameter in the numerator is being thought of as a function of two variables; the variable of differentiation and the variable being held constant as indicated by the subscript to each bracket.

The four thermodynamic energy functions with independent parameters as indicated are:

1. Internal Energy, $E \equiv E(S,V)$; $dE = TdS - PdV$ (2.12)
2. Enthalpy, $H \equiv H(S,P)$; $H \equiv E + PV$; $dH = TdS + VdP$ (2.13)

3. Helmholtz Free Energy, $F \equiv F(T,V)$; $F \equiv E - TS$;

$$dF = -SdT - PdV \qquad (2.14)$$

4. Gibb's Free Energy, $G \equiv G(T,P)$; $G \equiv E - TS + PV$;

$$dG = -SdT + VdP \; . \qquad (2.15)$$

In each case the differential expresses the first law in terms of the new energy function. Each function is useful for particular types of processes. For example, for processes at constant pressure, H is most useful. If volume is held constant during the process then E is most useful. For an adiabatic (thermally isolated) process at constant volume, the Helmholtz free energy, F, is most useful. Finally, G is used to describe adiabatic processes at constant pressure. To see how these relations are developed from the first law (2.3), we proceed as follows: (The Helmholtz free energy is used to illustrate, as it will be used in chapter three.)

Starting with $dE = TdS - PdV$, substitute $TdS = d(TS) - SdT$, to get $dE = d(TS) - SdT - PdV$, or $d(E - ST) = -SdT - PdV \equiv dF$, since $F \equiv E - ST$.
The Maxwell relations (2.11) follow immediately from the fact that dE, dF, dH, and dG are all exact differentials. For example, for the case of dF above,

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V \; .$$

The other equations follow similarly. Naturally, these relations only hold for quasi-static processes.

To illustrate an application of the Maxwell relations consider the problem of finding a relationship between the molar specific heats at constant volume and constant pressure, C_V and C_P, respectively. They are defined as,

$$C_V \equiv \left(\frac{dQ}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V ; \qquad (2.16)$$

$$C_P \equiv \left(\frac{dQ}{dT}\right)_P = T\left(\frac{\partial S}{\partial T}\right)_P . \qquad (2.17)$$

Taking $S \equiv S(P,T)$, then

$$dS = \left(\frac{\partial S}{\partial T}\right)_P dT + \left(\frac{\partial S}{\partial P}\right)_T dP . \qquad (2.18)$$

If the pressure is assumed to be a function of temperature and volume, i.e., $P \equiv P(V,T)$, dP can be expressed as,

$$dP = \left(\frac{\partial P}{\partial T}\right)_V dT + \left(\frac{\partial P}{\partial V}\right)_T dV . \qquad (2.19)$$

Substituting (2.19) into (2.18),

$$dS = \left(\frac{\partial S}{\partial T}\right)_P dT + \left(\frac{\partial S}{\partial P}\right)_T \left\{\left(\frac{\partial P}{\partial T}\right)_V dT + \left(\frac{\partial P}{\partial V}\right)_T dV\right\} .$$

Setting $dV = 0$, for a constant volume process, dividing by dT, and multiplying by T,

$$T\left(\frac{dS}{dT}\right) = T\left(\frac{\partial S}{\partial T}\right)_P + T\left(\frac{\partial S}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)_V .$$

It is easily shown that for constant volume processes,

$$T\left(\frac{dS}{dT}\right) = T\left(\frac{\partial S}{\partial T}\right)_V = C_V,$$

and substituting (2.17), we obtain,

$$C_V = C_P + T\left(\frac{\partial S}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)_V .$$

Since $(\frac{\partial S}{\partial P})_T$ is hard to determine experimentally, the Maxwell relation $(\frac{\partial S}{\partial P})_T = -(\frac{\partial V}{\partial T})_P$ can be used to yield, $C_V = C_P - T(\frac{\partial V}{\partial T})_P(\frac{\partial P}{\partial T})_V$. The quantity, $\alpha \equiv \frac{1}{V}(\frac{\partial V}{\partial T})_P$, is the volume coefficient of expansion, and is readily measured. It is easy to show that $(\frac{\partial P}{\partial T})_V = \frac{\alpha}{\kappa}$, where $\kappa \equiv -\frac{1}{V}(\frac{\partial V}{\partial P})_T$ is the isothermal compressibility. Hence, $C_V = C_P - TV\frac{\alpha^2}{\kappa}$. Thus, the difference, $C_P - C_V$, has been expressed in terms which are readily measurable.

Before discussing the statistical approach, we will briefly discuss Carnot cycles as they will be used in the analysis of blackbody radiation in chapter four.

A Carnot cycle is the simplest process for converting heat from a reservoir to external work. The usual picture is illustrated in Figure 2.1. Each figure represents the system at the end of the previous step in the cycle (all processes are quasi-static).

a) An isothermal expansion is performed in this step. An amount of heat, Q, is absorbed from the reservoir and work is done by the system. On completion of the expansion the system has pressure and volume P_2 and V_2 respectively, but the temperature is still T_1.

b) The reservoir is now sealed-off and the system allowed to expand adiabatically (by relaxing the external pressure). The

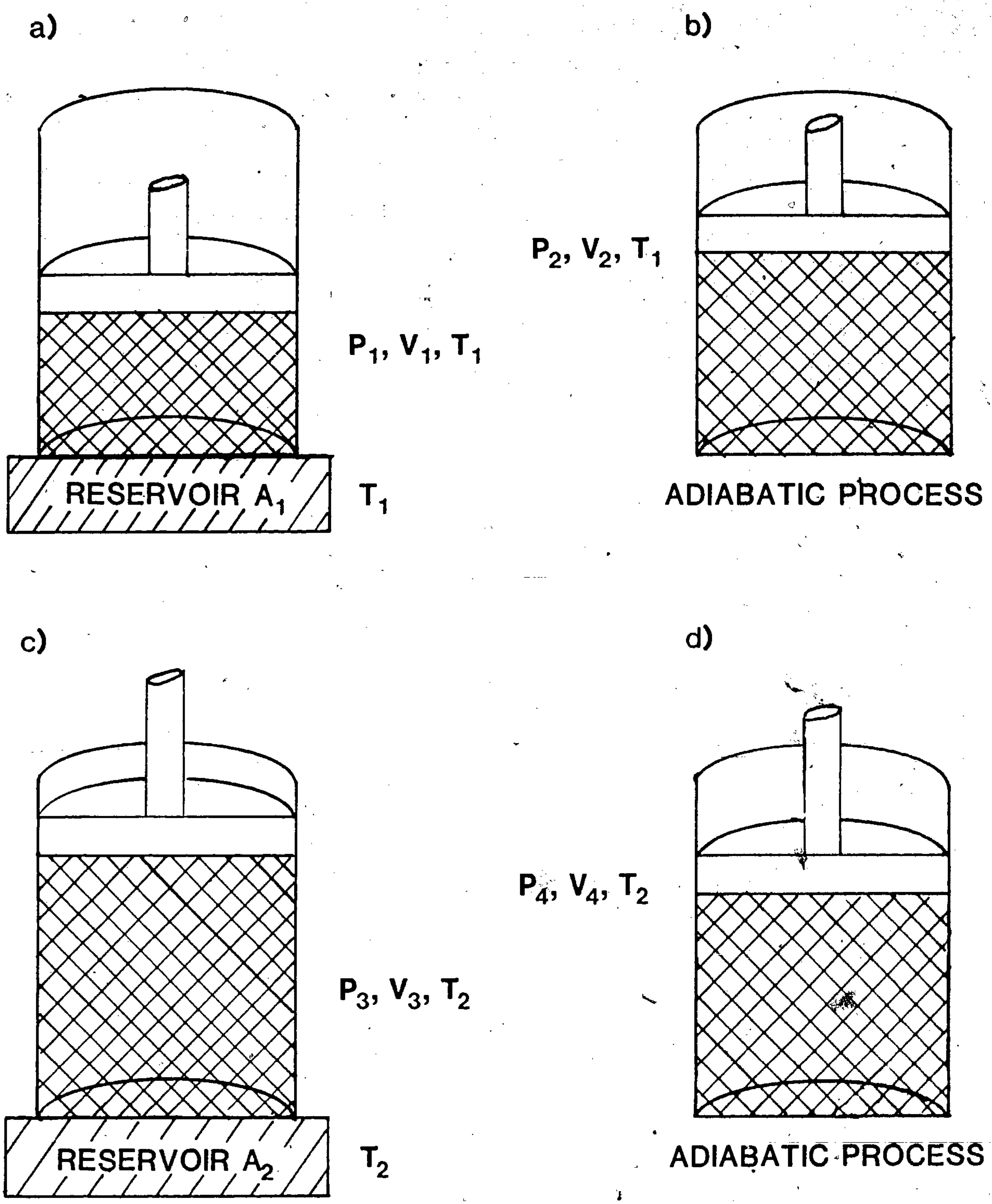

Figure 2.1 – Carnot Cycle

temperature and pressure drop to T_2 and P_3 respectively, while the volume increases to V_3.

c) The system is connected to a reservoir at temperature T_2 ($<T_1$) and an isothermal contraction is performed. This can be done, for example, by increasing the pressure to P_4. The volume is reduced to V_4 and the temperature remains constant at T_2.

d) The last stage is an adiabatic compression to the original state in stage (a), i.e., pressure P_1, volume V_1, and temperature T_1.

The process can be illustrated on a pressure-volume diagram, Figure 2.2. The work done by the engine is just, $\oint PdV$, where the integral is taken counterclockwise around the path bordering the shaded region in the figure.

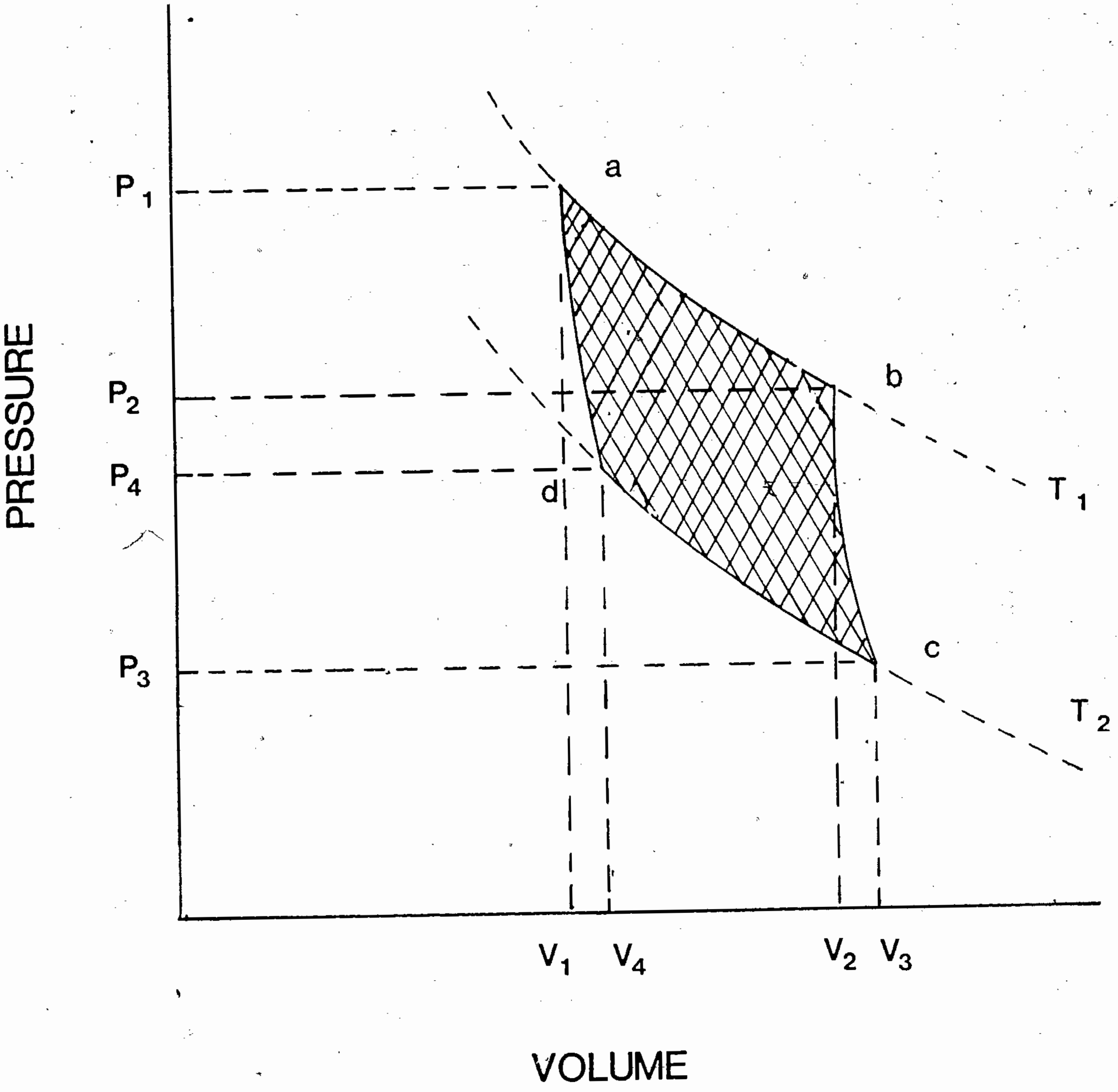

Figure 2.2 – Pressure–Volume Diagram for Carnot Cycle

Classical Statistical Mechanics

In classical statistical mechanics a system is described in terms of its microstates. Specification of these microstates together with the main postulate of statistical mechanics allows calculation of all the macroscopic parameters of the system. A brief description of this approach is presented here.

To specify the microstates of the system one must first choose some microscopic standard of the system. This could be, for example, atoms or molecules or electrons. Next, a suitable number of generalized coordinates must be chosen to completely specify each microscopic unit. For the case of an ideal gas in a box with N particles, each particle can be taken to be a microunit, and the position and momemtum of each particle specify it completely. In three dimensions this gives a total of 6N degrees of freedom. In order to count the number of states an approximation has to be made as follows: The phase space (consisting of 3 momentum and 3 position coordinates for each particle for a total of 6N dimensions) is divided into cells of volume h_0^{3N}. The cells, of total volume V, are then enumerated, 1, 2,...r,... $\frac{V}{h_0^{3N}}$, in some convenient fashion and the system considered to be in state r if the coordinates and momenta describing the system lie anywhere inside the r^{th} cell.

Classically, h_0 can be made arbitrarily small. In quantum mechanics h_0 must be greater than h (Planck's constant). In addition, the system is normally specified by the quantum states of the wave function. Once a procedure for specifying the microstates of the system has been determined, an ensemble of similar systems is imagined, so that the macroscopic parameters of the system can be deduced on the basis of probability calculations.

To complete the foundation of the statistical approach some connection must be found between the microstates of the system and the probabilities of the system being in those states. The fundamental postulate of statistical mechanics serves this purpose: An isolated system in equilibrium is equally likely to be in any of its accessible states. An accessible state is a state that the system can actually occupy. For example, if the total energy of the system is E, it cannot occupy a macrostate with energy E' ≠ E. Only states with total energy E are accessible.

In the usual approach to statistical mechanics the existence of a function, $\Omega(E)$, which denotes the number of states accessible to the system in the energy range E to E + dE is assumed. Then the parameter β is defined by,

$$\beta \equiv \frac{\partial \ln\Omega(E)}{\partial E}; \quad \beta \equiv \frac{1}{kT} , \qquad (2.20)$$

where k is Boltzmann's constant and T the absolute temperature. So (2.20) gives the statistical mechanics definition of temperature. The motivation for this definition comes from a short analysis of two systems in thermal equilibrium, however, in chapter four these definitions will be considered further.

In practice the function $\Omega(E)$ is not very useful since, in general, it is difficult to calculate. Instead of actually counting states it is much easier to calculate the probability of a state being occupied. The sum of the probabilities over all states is then equal to one. There are several ways to calculate probabilities depending on how the statistical ensemble is chosen, but the most common distribution is called the canonical distribution. By considering a system in equilibrium with a heat reservoir, to excellent approximation the probability, P_r, of the system being in a microstate r is,

$$P_r = \frac{e^{-\beta E_r}}{\sum_n e^{-\beta E_n}} ,$$

where the sum is taken over all accessible states. The sum,

$$Z \equiv \sum_n e^{-\beta E_n} , \tag{2.21}$$

is termed the partition function and is usually much more convenient to work with than $\Omega(E)$.

An ensemble of such systems can be imagined as consisting of a large number of identical systems, each in contact with a heat reservoir. If the energy, E_s, of the system is very much less than the energy, E, of the resevoir, then the probability of the system having energy in the range E_s to $E_s + dE_s$ is, approximately, $P(E_s) = C\Omega(E_s)\exp\{-\beta E_s\}$, where C is a constant. Since $\Omega(E)$ is rapidly increasing and $\exp\{-\beta E\}$ rapidly decreasing, $\Omega(E)\exp\{-\beta E\}$ has a sharp maximum around the average energy, $\bar{E}$. Since $\bar{E}$ is the macroscopic parameter that is measured, measurement will agree very closely with the probabilistic calculation, with the agreement improving for $E>>E_S$.

A simple example of the calculation of Z is furnished by a monatomic ideal gas. For an ideal gas in a container of volume V, the total energy of the gas is,

$$E = \sum_{i=1}^{N} \frac{p_i^2}{2m} + U(r_1, r_2, \ldots r_N) \quad ,$$

where N is the number of gas particles in the box, m is the mass of each particle, p_i is the momentum of each particle and $U(r_1, r_2, \ldots r_N)$ is the potential energy of the gas. Since the states are continuous, the sum $\sum_n e^{-\beta E_n}$ can be calculated using an integral as follows: The states are specified by the N position and momentum vectors, r_1, $r_2, \ldots r_N$, p_1, $p_2, \ldots p_N$, respectively. The number of states with momenta and coordinates in the range p_1 to $p_1 + dp_1, \ldots,$ r_N to $r_N + dr_N$ per unit cell of phase space is, $\frac{d^3p_1 d^3p_2 \ldots d^3p_N d^3r_1 d^3r_2 \ldots d^3r_N}{h_0^{3N}}$. Multiplying by the Boltzmann factor, $\exp\{-\beta[\sum_{i=1}^{N} \frac{p_i^2}{2m} + U(r_1, r_2, \ldots r_N)]\}$,

dividing by N!, and summing over all phase space (as an integral), we have,

$$Z = \frac{1}{N!}\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \exp\{-\beta[\sum_{i=1}^{N} \frac{p_i^2}{2m} + U(r_1, r_2, \ldots r_N)]\} \frac{d^3p_1 d^3p_2 \ldots d^3r_N}{h_0^{3N}} \quad .$$

The factor of $\frac{1}{N!}$ arises because permuting two gas particles does not lead to a new state of the gas. Thus, the total number of states is reduced by a factor of one over N!. For an ideal gas $U(r_1, r_2, \ldots r_N) = 0$, so the integral works out to,

$$Z = \frac{V^N}{N!}\{\frac{2\pi m}{h_0^2\beta}\}^{\frac{3}{2}N} . \tag{2.22}$$

Several macroscopic parameters can now be calculated. The average energy of the system is given by,

$$\bar{E} = \frac{\sum_n E_n e^{-\beta E_n}}{Z} = -\frac{\partial \ln Z}{\partial \beta} , \tag{2.23}$$

where as always, the sum is calculated over all possible states of the gas. Substituting (2.22) into (2.23) we have,

$$\bar{E} = -\frac{\partial}{\partial\beta}\{-\ln N! + N\ln V + \frac{3N}{2}\ln\frac{2\pi m}{h_0^2} - \frac{3N}{2}\ln\beta\}$$

$$= \frac{3N}{2\beta} = \frac{3}{2}NkT .$$

If X is the generalized force corresponding to the parameter x, then $\bar{X} \equiv \frac{1}{\beta}\frac{\partial \ln Z}{\partial x}$. If x = V, then $\bar{X} = \bar{P}$, the average pressure, and $\bar{P} = \frac{NkT}{V}$, or $\bar{P}V = NkT$, which is the equation of state of the gas. To calculate the entropy, set $Z = Z(\beta,x)$. Then,

$$d\ln Z = \frac{\partial \ln Z}{\partial \beta}d\beta + \frac{\partial \ln Z}{\partial x}dx ,$$

$$= -\bar{E}d\beta + \beta\bar{X}dx = -d(\bar{E}\beta) + \beta d\bar{E} + \beta P dV ,$$

$$= -d(\bar{E}\beta) + \beta T dS ,$$

for a quasi-static process in which the parameter x undergoes a change dx. The last step follows by the first law (2.3), with $dW = PdV$. Thus, it follows that

$$S = k(\ln Z + \beta\bar{E}) \tag{2.24}$$

Substituting the expressions for Z and $\bar{E}$ already calculated we get,

$$S = k\{-\ln N! + N\ln V + \frac{3N}{2}\ln\frac{2\pi m}{h_0^2} - \frac{3N}{2}\ln\beta + \frac{3N}{2}\}$$

$$= k\{-\ln N! + N\ln V + \frac{3N}{2}\ln\frac{2\pi m}{h_0^2} + \frac{3N}{2}\ln T + \frac{3N}{2}\} .$$

Since $N >> 1$, we can use Stirling's formula to approximate $\ln N! \simeq N\ln N - N$, so we have finally,

$$S = kN\{\ln\frac{V}{N} + \frac{3}{2}\ln T + \frac{3}{2}\ln\frac{2\pi mk}{h_0^2} + \frac{5}{2}\} .$$

The Helmholtz free energy has a simple relationship to Z that makes it particularly useful. Using (2.24) and rearranging,

$$F = \bar{E} - TS = -kT\ln Z . \tag{2.25}$$

Having surveyed classical thermodynamics and statistical mechanics, we will next discuss attempts to generalize these ideas to relativity. But as will be seen the problem is very complicated.

CHAPTER III

RELATIVISTIC THERMODYNAMICS AND STATISTICAL MECHANICS

General Considerations

This chapter is divided into three sections. In section one the many problems of generalizing thermodynamics to relativity are discussed in a general way. In section two several systems for generalizing thermodynamics are described. In section three statistical mechanics is generalized using the scheme of Balescu and Kotera [9,10].

A major problem in relativistic thermodynamics arises from the problems associated with simultaneity and extended bodies. Other problems include ambiguities in defining thermodynamic parameters, questions about exactly how to incorporate into the first law the systematic energy resulting from the motion of the center of mass of the system, and a lack of experimental evidence with which to resolve disagreement between theories.

Before further discussion of these problems it is perhaps useful to mention several things that all authors agree on. Even here there is room for discussion. All authors implicitly assume that Boltzmann's constant (and where relevant Planck's constant) is a relativistic invariant. Such an assumption is esthetically pleasing, but has no experimental basis. Most authors agree that entropy and pressure are relativistic invariants. In the case of entropy there are two arguments used:

1. Since the entropy is proportional to the number of states accessible to the system at some energy E, this is just a number and so must be a relativistic invariant.

2. The overall motion of the center of mass of the system can not affect the microscopic distribution of states so the entropy must be invariant.

Both these arguments are, in general, unclear. In relativity it is true that a scalar is invariant, but not all scalars in Newtonian physics are Lorentz scalars. A simple example to illustrate that argument 1 is incorrect, in general, is provided by energy. In Newtonian mechanics, energy is a scalar, so by argument 1, energy must be a scalar in relativity as well. As is well known, this is not the case; energy is the time component of the momentum 4-vector. The difficulty arising here can be traced to the fact that entropy is a measurement of a distribution in three dimensional space and so it is not obvious that this distribution will retain the same "shape" under a Lorentz transformation.

Some authors, for example Dixon [11], have tried to formulate an entropy 4-vector, but there is an obvious circularity in reasoning here. (In fairness to Dixon and others who have adopted this method, we point out that they are well aware of the difficulty with this approach. They circumvent the logical problem by not expecting thermodynamic parameters which have been generalized to 4-vectors to have the same physical interpretation as their classical counterparts.)

The second of the above arguments is unclear because of peculiarities of the relativistic velocity transformation. For example, Landesberg [12], has discussed the velocity distribution of an ideal gas in relative motion. He points out that if the box has velocity v with respect to an observer, K, then for K, more particles travel in the direction of motion, and at slower speeds than in the direction opposite to the motion of the box. In addition, for K, the interparticle spacing appears to be less in the direction of motion than in other directions, so the position distribution appears anistropic. It is not a all obvious if the number of states accessible to the system is preserved. Of course, the incorrectness of these arguments does not disprove the claim that $S' = S_0$ for the system. After all, the velocity transformation is one to one, so one expects the number of velocity states to be preserved. It would be nice to have a rigorous proof.

The argument that pressure is invariant can be understood as follows: Consider a square box of side L with one edge moving along the x-axis, with speed v. Then for gas particles of constant rest mass, the relativistic force transformation laws for the 3-force are (see Ritchmeyer [13]):

$$F_x' = F_x - \frac{u_y v F_y}{(1-u_x v)} - \frac{u_z v F_z}{(1-u_x v)} \;;$$

$$F_y' = \frac{\sqrt{1-v^2}}{(1-u_x v)} F_y \;;\; F_z' = \frac{\sqrt{1-v^2}}{(1-u_x v)} F_z \;, \qquad (3.1)$$

where the primed frame moves with speed v along the x-axis with respect to the unprimed frame, the speed of light is taken to be

1, and u_x, u_y, and u_z are the components of the velocity u, of a particle, as observed in the unprimed frame. For the case of the above box, containing an ideal fluid (so that the force against the walls is static), we can set $u_x = u_y = u_z = 0$. Then the pressure against the y-z and x-y faces of the box is respectively,

$$P_{yz} = \frac{F'_x}{L^2} = \frac{F_x}{L^2}, \quad \text{and} \quad P_{xy} = \frac{F'_z}{L^2}\cdot\frac{1}{\sqrt{1-v^2}} = \frac{\sqrt{1-v^2}}{\sqrt{1-v^2}}\cdot\frac{F_z}{L^2} .$$

And similarly for the pressure against the x-z face. Alternatively, for an ideal fluid, T_{ij} has the pressure for spatial eigenvalues, so these must be preserved under a Lorentz transformation.

Finally, most authors have assumed (generally, implicitly) that the notion of equilibrium is a relativistic invariant. In fact, a system in equilibrium may look very different to a moving observer. For example, the ideal gas as already mentioned, does not appear to be in equilibrium to a moving observer. Dixon [11] has tackled this problem by generalizing the notion of equilibrium. He suggests that if the microscopic distribution is static, then the system should be ruled to be in equilibrium. Balescu [10] also uses this definition. Having covered areas of general consensus, some contentious problems will now be discussed.

The attempt to generalize thermodynamics by expressing laws in terms of tensor equations does not yield a unique theory because there are many ways to express thermodynamic quantities as 4-vectors that reduce to the usual quantities in the rest frame. In addition, as already pointed out, thermodynamic parameters lose their usual physical interpretations in these theories. There are several consistent tensor theories (for perfect fluids). Only experiment can resolve the different theories and these are considered no further here, as they seem no more fundamental ~~than~~ the non-tensor theories. The interested reader should consult, for example, Dixon [11], Israel [14], or Synge [15]. (The tensor formulations have been more widely accepted than the non-tensor theories.)

Since thermodynamic quantities such as temperature, volume and pressure are averages over an extended body, simultaneity is a problem. Thus, if the rest frame observer and moving observer measure a thermodynamic quantity using a notion of simultaneity in their own frame, they are not measuring the same thing, in general. For quasi-static processes or systems in static equilibrium this should not be a problem, since simultaneous measurement is not crucial in this case.

The final topic in this section is the problem of how to distribute the increase in internal energy, due to the motion of the center of mass of the system, between the heat and work increments in the first law (2.1). Heat is generally agreed to be energy flow of a random nature, whereas work is ordered energy flow. An attempt to calculate the kinetic energy of the moving system and call this the work done on the system fails because of the peculiarity of the relativistic velocity transformation law. For example, for an ideal gas in a box moving with velocity v in the x direction, the observed velocity components of a given gas particle are,

$$u_x' = \frac{u_x+v}{1+vu_x} \; ; \; u_y' = \frac{u_y}{\gamma(1+vu_x)} \; ; \; u_z' = \frac{u_z}{\gamma(1+vu_x)} \; ,$$

where $\gamma = \frac{1}{\sqrt{1 - v^2}}$. The kinetic energy associated with the velocity, v, of the center of mass of the box can be taken as the work that has been done on the box, but if this energy is simply subtracted from each particle's total energy what is left is not equal to the rest frame heat and work. For example, consider a particle moving with velocity $u=u_x$ in the x-direction with respect to the box. Then if the box is moving with velocity v in the x-direction with respect to the observer, this particle appears to have kinetic energy:

$m_0\left(\frac{u_x + v}{1+vu_x}\right)\{1 - \left(\frac{u_x + v}{1+vu_x}\right)^2\}^{-\frac{1}{2}}$. This particle's contribution to the kinetic energy of the box with respect to the observer is $\gamma m_0 v$. Then the difference of these two expressions is:

$$m_0\left(\frac{u_x + v}{1+vu_x}\right)\{1 - \left(\frac{u_x + v}{1+vu_x}\right)^2\}^{-\frac{1}{2}} - \gamma m_0 v \neq \frac{m_0 u_x}{\sqrt{1 - u_x^2}} \; .$$

The expression on the right is the kinetic energy of the particle in the box frame. So it is not clear what to do with the systematic energy of the motion of the box. (Evidently, the problem arising here has its origins in the indiscriminant comparison of quantities in different frames.)

We conclude this section with a discussion of the difficulty of obtaining experimental data about the temperature of moving bodies. The chief obstacle of measuring the temperature of a moving body, A, arises because the thermometer, B, must come to equilibrium with A as A moves by B. If A is moving rapidly enough then A and B do not have time to come into equilibrium, so the temperature measurement is meaningless. Landesberg, and others [12,16-19], have discussed this problem extensively. He argues that $T=T_0$, i.e., temperature invariance, is the only physically reasonable transformation as follows: Suppose $T = T_0/\gamma$. Then if A and B are two identical objects both of rest temperature T_0, A moving with speed v with respect to B, then according to B, A appears cold and heat should flow from B to A. The reverse occurs in A's rest frame. Since, Landesberg considers that this is evidently contradictory, he concludes that $T=T_0$. This problem is similar to the twin (paradox) problem as seen in the following version: An object is quasi-statically accelerated along a very long, straight, frictionless, conducting rail, which is in thermal equilibrium with a very long heat reservoir. The object is then insulated, slowed down and reversed to return to its original starting point. An observer at rest with respect to the rail concludes that the object is now hotter since it appeared cooler during the acceleration stage of its trip, and so absorbed heat from the reservoir. The observer on the object says the object is now colder since it lost heat to the, according to him, colder reservoir. So we have the same problem as the two twins (each claims the other is younger). Perhaps the thermal problem can be resolved in a manner similar to the twin paradox. That is, one felt accelerations and the other did not, etc. The general response to Landesberg's problem has been that it cannot be discussed within the realm of equilibrium thermodynamics. Kellerman [19] has suggested an alternative way out, via modification of the definition of entropy. We will discuss this briefly later, as it ties-in with the modifications to entropy required by the stochastic electrodynamics of chapter six.

Relativistic Thermodynamics

Since the original Planck - Einstein formulation, and especially since 1963, a large number of formulations of relativistic thermodynamics have been proposed. Several good classification schemes have been proposed, but we will defer classifying the various formulations until the last section. An overview of the various prescriptions is presented here. As usual $\gamma = (1 - v^2)^{-\frac{1}{2}}$, c=1, the subcript "0" denotes rest frame values, and K and K_0 denote the moving and rest frame, respectively.

Planck, Einstein and others [20-22] worked out the following scheme:

$$U = \gamma(U_0 + v^2 P_0 V_0) \ ; \ T = T_0/\gamma \ ; \ S = S_0 \ ;$$

$$P = P_0 \ ; \ p = \gamma(U_0 + P_0 V_0)v \ , \qquad (3.2)$$

where p is the momentum of the whole system, P the pressure, V the volume, S the entropy, and U the total energy. The second law, TdS = đQ , is retained by these authors so that

$$đQ = đQ_0/\gamma \ . \qquad (3.3)$$

The work done by the system is defined,

$$dW \equiv dW_0/\gamma - v^2\gamma dH_0 \ , \qquad (3.4)$$

where we recall that, $H_0 \equiv U_0 + P_0V_0$, is the enthalpy, and the last term represents the work done in accelerating the system to velocity v and momentum p. This definition for dW keeps the first law (2.3) form invariant. Using (3.2, 3.3, and 3.4), the heat transfer works-out to, $dQ = dU + dW$. Evidently, in this formulation energy-momentum is not conserved with a change in V_0, because of the term P_0V_0 . This is a consequence of the fact that in this formulation, U and p are obtained by simply performing a Lorentz transformation on T_{ij} for a perfect fluid. Thus, according to this definition, K makes measurements at constant time in his frame and K_0 does likewise. Because of the relativity of simultaneity they are not measuring the same thing (see, for example, Yuen[23]). The usual explanation for the P_0V_0 term is that stresses are induced in the walls of the container. This explanation is made more apparent in the next section.

In 1963 Ott [24], and later Møller and Kibble [25,26], challenged the Planck - Einstein formulation and proposed instead that,

$$U = \gamma(U_0 + v^2P_0V_0) \ ; \ T = \gamma T_0 \ ; \ P = P_0 \ ;$$
$$S = S_0 \ ; \ dQ = dU + dW \ ; \ dS \geq dQ/T \ . \qquad (3.5)$$

Thus, they retain the first and second laws, but reject the Planck - Einstein temperature transformation law.

Their primary objection to the classical fomulation arises out of the ambiguity of defining thermodynamic quantities such as heat and work in relativity. Their argument runs as follows: Suppose thermodynamics variables have been chosen so that,

$$dQ = dU + dW \ ; \ dS = dQ/T \ , \ T = T_0/\gamma \ , \ dQ = dQ_0/\gamma \ .$$

Then define new variables by $T' \equiv Tg(\gamma)$, and $dQ' \equiv dQg(\gamma)$, where $g(\gamma)$ is an arbitrary smooth function of γ. Then to keep the first law invariant in the new variables we require that dW' be defined as, $dW' \equiv dW - dQ'\{[g(\gamma)]^{-1} - 1\}$. Thus, to have a clear theory it is important to distinguish carefully between heat and work. For the classical treatment with $dW = dW_0/\gamma - v^2\gamma dH_0$, dW need not be 0 even if dW_0 is, so that it is possible for a purely radiative system to do work of amount $dW = - dQ_0 v^2\gamma$ (where for $dW_0 = 0$, $dE_0 = dQ_0$). The above three authors find this result physically unreasonable, and propose (3.5) as a remedy. It may be that in relativity the distinction between work and heat is a frame dependent concept. This is not too farfetched, for, as Kibble [26] has pointed out, fundamentally there is not much distinction between heat and work - only a notion of randomness separates them. Since such a notion is a space-time relation we should not expect it to be Lorentz invariant.

In the mid sixties Arzelies and Gamba [27,28] suggested the scheme:

$$U = \gamma U_0 \ ; \ T = \gamma T_0 \ ; \ dQ = \gamma dQ_0 \ ; \ S = S_0 \ ;$$
$$P = P_0 \ ; \ dW = \gamma dW_0 \ . \qquad (3.6)$$

These authors argue that $U = \gamma U_0$ so that the two observers are measuring the same quantity. If one uses $U = \gamma(U_0 + v^2P_0V_0)$, derived by performing a Lorentz transformation on T_{ij} (we are using T_{ij} to indicate the stress-energy tensor, not its components) and integrating over a constant time surface in K, this is not the same quantity as T_{ij} integrated over a constant time surface in K_0. They argue that the only quantities of interest are those that transform by a Lorentz transformation.

As previously mentioned, Landesberg, John, and Van Kampen among others [12,16-18,29,30] have argued for temperature invariance:

$$T = T_0 \ ; \ P = P_0 \ ; \ S = S_0 \ ; \ dQ = dQ_0/\gamma \ ;$$
$$U = \gamma U_0 \ ; \ dW = dW_0/\gamma - v^2\gamma dE_0 \ . \tag{3.7}$$

These authors distinguish between confined systems (includes walls) and inclusive systems (free). For confined systems $U = \gamma(U_0 + v^2P_0V_0)$. In this formulation the second law is not invariant. Also the transformation law for temperature is assumed to be an independent physical law, that can not be determined from relativity and equilibrium thermodynamics alone.

Landesberg [12] has found a classification scheme for the various formulations, but instead of going through his synopsis, we will present, in the next section, the scheme worked out by Balescu.

Relativistic Statistical Mechanics

There have been two basic approaches to relativistic statistical mechanics. Both approaches start with scalar distribution functions. The argument for this assumption is that if n particles of the system have the same 4-momentum **P**, then those particles will have the same 4-momentum, **P**, for any Lorentz observer, so that n(**P**), the distribution function of the particles, is invariant, i.e., $n(\mathbf{P}) = \bar{n}(\mathbf{P})$. This argument would be obvious if we were using the Galilean transformation, in which relative motion of the system would simply be added to all the particles. However, as already pointed out, this argument is not so clear in relativity. In any case, we will only briefly describe the 4-tensor approach worked out by several authors [11,14,15,31], but will discuss in detail the canonical approach due to Balescu and Kotera [9,10,32].

As we have seen, (see equation (2.21) *et seq*) in the rest frame the Boltzmann distribution depends on the quantity $E\beta = \frac{E}{kT_0}$. The tensor approach is to generalize $E\beta$ to the product of two 4-vectors, one of which is obviously the 4-momentum **P**. β_μ can be defined as, $\beta_\mu \equiv \frac{u_\mu}{kT_0}$, where u_μ are the components of the 4-velocity (μ=0,1,2,3). If $\beta_0 = \frac{\gamma}{kT_0}$, is chosen so that $\beta_0 = \frac{1}{kT}$, then we recover the Planck - Einstein law, $T = T_0/\gamma$. Of course, β_μ can be defined in a variety of ways. The ambiguity problem has thus, not been solved because, as already pointed out, no matter how β_μ is defined we must discover what the physical connection is between the new parameter β and the standard notion of temperature.

The canonical approach is attractive since it makes it clear (once a scalar distribution is accepted) that all of the various generalizations of thermodynamics are self-consistent and amount to different choices of two arbitrary functions of γ. In this development, the Poincarè group is represented as a subgroup of the group of canonical transformations of the dynamical system. No attempt is made to retain duality between time and space coordinates. The subgroup of canonical transformations satisfy the Lorentz group axioms, but the specific coordinates q_r and p_r need not be components of 4-vectors. The key result needed to make progress is that under appropriate conditions a canonical distribution goes over into a canonical distribution under a Lorentz transformation. Then since any thermodynamic system can be represented by a canonical distribution (see for example, Tolman [22], Landau and Lifshitz [33], or Huang [34]), we will have solved the problem of the transformation of thermodynamic quantities in relativity.

Before showing the above results in detail, we will briefly discuss the canonical approach to mechanics and show that the Poincarè group can be expressed as a subgroup of the group of canonical transformations.

Classically, in the canonical approach a system is described in terms of three space coordinates, q_r, and three momentum coordinates, p_r, for each particle, yielding a configuration space of 6N dimensions (r=1,2,3). Following the prescription of

Dirac [35], the coordinates are specified at some specific observer time, say t=0 . The macroscopic parameters of the system are described by "dynamic functions" of the canonical variables, (q_r, p_r). An example of such a function is the Hamiltonian. Of special importance are the canonical transformations, defined as follows: $Q = Q(q,p)$ and $P = P(q,p)$ is a canonical transformation provided,

$$\begin{aligned}
&1)\ \frac{\partial Q_r}{\partial q_i}\frac{\partial Q_s}{\partial p_i} - \frac{\partial Q_r}{\partial p_i}\frac{\partial Q_s}{\partial q_i} \equiv [Q_r, Q_s] = 0 \ , \\
&2)\ \frac{\partial Q_r}{\partial q_i}\frac{\partial P_s}{\partial p_i} - \frac{\partial Q_r}{\partial p_i}\frac{\partial P_s}{\partial q_i} \equiv [Q_r, P_s] = \delta_{rs} \ , \\
&3)\ \frac{\partial P_r}{\partial q_i}\frac{\partial P_s}{\partial p_i} - \frac{\partial P_r}{\partial p_i}\frac{\partial P_s}{\partial q_i} \equiv [P_r, P_s] = 0 \ ,
\end{aligned} \tag{3.8}$$

where we have used the Einstein summing convention, and $(i = 1, 2, 3)$.

It is then not difficult to show (see, e.g., Desloge [36]) that if $F(q,p)$ and $G(q,p)$ are dynamic functions of the system, then the Poisson bracket,

$$[F,G] \equiv \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \ , \tag{3.9}$$

is invariant under a canonical transformation.

A canonical distribution function is any function, $f(q,p)$, sufficiently smooth, (usually $f(q,p) \in C^1$ is enough, but we will assume that $f(q,p)$ is C^∞ or at least that $f(q,p)$ has a sufficiently accurate C^∞ approximation) such that $f(q,p) \geq 0$ and $\iint f(q,p)dqdp = 1$, where the integral is taken over all of phase space. Once such a distribution has been specified for the system, a dynamic observable can be calculated at some specific time say t=0 by,

$$\langle A\rangle(X) \equiv \iint A(q,p;X)f(q,p)dqdp \ . \tag{3.10}$$

The evolution of the system is then dependent only on the Hamiltonian. That is, for an infinitesimal time translation,

$$A(q,p;\delta t) = A(q,p) + [A(q,p),H(q,p)]\delta t \quad \text{or equivalently,}$$

$$\frac{\partial A(q,p;t)}{\partial t} = [A(q,p;t),H(q,p)] \ . \tag{3.11}$$

This equation can be solved to yield,

$$A(q,p;t) = e^{[H]t}A(q,p) \ , \tag{3.12}$$

where A(q,p) = A(q,p;0), and the righthandside is defined:

$$e^{[H]t}A(q,p) \equiv A(q,p) + t[A,H]|_{t=0} + \frac{t^2}{2!}[[A,H],H]|_{t=0} + \ldots \ . \tag{3.13}$$

The procedure for solving (3.11) is sketched as follows: Treating A(q,p;t) as a function of several variables with only t explicitly changing,

$$A(q,p;t) = A(q,p) + t\{\frac{\partial A(q,p;t)}{\partial t}|_{t=0}\} + \frac{t^2}{2!}\{\frac{\partial^2 A(q,p;t)}{\partial t^2}|_{t=0}\} + \ldots \ .$$

The coefficients can be evaluated using the differential equation. For example, $\{\frac{\partial A(q,p;t)}{\partial t}|_{t=0}\} = [A(q,p;t),H(q,p)]|_{t=0}$;

$$\left\{\frac{\partial^2 A(q,p;t)}{\partial t^2}\Big|_{t=0}\right\} = \left\{\frac{\partial[A,H]}{\partial t}\Big|_{t=0}\right\} = \left\{\left[\frac{\partial A}{\partial t},H\right]\Big|_{t=0}\right\} = [[A,H],H]\big|_{t=0} \ ,$$

where we have used Hamilton's equations,

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \ ; \ \frac{\partial H}{\partial q_i} = -\dot{p}_i \ ; \ \frac{\partial H}{\partial t} = 0 \ , \qquad (3.14)$$

and (3.11) twice. The distribution function itself evolves in a similar manner except for a sign change, i.e.,

$$\frac{\partial f(q,p;t)}{\partial t} = -[f(q,p;t),H(q,p)] \ , \text{ and}$$

$$f(q,p;t) = e^{-[H]t}f(q,p;0) \ . \qquad (3.15)$$

The change in sign is necessary to preserve the equivalence of the "Schrödinger" and "Heisenberg" pictures, i.e.,

$$\langle A\rangle_t = \iint A(q,p)e^{-[H]t}f(q,p)dqdp \ ; \text{ "Schrödinger"}$$

$$\langle A\rangle_t = \iint f(q,p)e^{[H]t}A(q,p)dqdp \ ; \text{ "Heisenberg"} \qquad (3.16)$$

(Of course, quantum mechanics is not involved here, we are just pointing out that we can view the distribution as evolving or view the dynamic variable as evolving.) The two expressions in (3.16) can be shown to be equal using (3.13), and the fact that f(q,p) and all its derivatives go to zero as p -> ∞ or q -> ∞ , but we will omit this.

Our next task in the canonical development is to represent the Poincarè group as subgroup of canonical transformations. The infinitesimal Poincarè group can be expressed as:

$$\acute{x}^{\mu} = x^{\mu} + a^{\mu} + b^{\mu}{}_{\nu}x^{\nu} \ , \qquad (3.17)$$

where $b_{\mu\nu} = -b_{\nu\mu}$. (Greek indices take values in the range 0,1,2,3, and Latin indices are in the range 1,2,3.) We next construct a dynamical function, $F = F(q,p,a_\mu,b_{\mu\nu})$, such that under an infinitesimal Poincarè transformation, an arbitrary dynamic parameter, $A(q,p)$, transforms as,

$$A'(q,p) = A(q,p) + [A(q,p),F(q,p,a_\mu,b_{\mu\nu})] . \qquad (3.18)$$

Since the transformation is infinitesimal we require F to be linear in a_μ and $b_{\mu\nu}$, so that

$$F = -P^\mu(q,p)a_\mu + \frac{1}{2}M^{\mu\nu}(q,p)b_{\mu\nu} . \qquad (3.19)$$

The ten functions, $P^\mu(q,p)$ and $M^{\mu\nu}(q,p)$,are the generators of the representation (the minus sign and $\frac{1}{2}$ are convention). The exact choice of $P^\mu(q,p)$ and $M^{\mu\nu}(q,p)$ depends on the nature of the system at hand, however, they must satisfy the commutator relations of the Lie Algebra for the Poincarè group:

$$[P_\mu,P_\nu] = 0 \; ; \; [M_{\mu\nu},P_\sigma] = -g_{\mu\sigma}P_\nu + g_{\nu\sigma}P_\mu \; ;$$

$$[M_{\mu\nu},M_{\rho\sigma}] = -g_{\mu\rho}M_{\nu\sigma} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\sigma}M_{\rho\nu} + g_{\nu\sigma}M_{\rho\mu} . \qquad (3.20)$$

It is not hard to show that, as well, the equations (3.20) hold with all the indices raised. At this stage it is helpful to identify the generators, P^μ and $M_{\mu\nu}$, in the usual manner:

$P_0 = H$ is the Hamiltonian and the generator of time translations;

P_r is the momentum and the generator of space translations;

$M_{rs} = \epsilon_{rsi}J^i$ is related to the angular momentum and is the generator of space rotations;

$M_{SO} = K_S$ is the generator of space-time rotations. (3.21)

As an illustration, consider a Lorentz transformation consisting of constant velocity $v = \tanh s$, along the x^1-axis. Then the finite and infinitesimal transformations are, respectively:

$$\acute{x}^0 = x^0 \cosh s - x^1 \sinh s \;;$$

$$\acute{x}^1 = x^1 \cosh s - x^0 \sinh s \;;$$

$$\acute{x}^2 = x^2 \;;\; \acute{x}^3 = x^3 \;, \tag{3.22}$$

and,

$$\acute{x}^0 = x^0 - (\delta s)x^1 \;;\; \acute{x}^2 = x^2$$

$$\acute{x}^1 = x^1 - (\delta s)x^0 \;;\; \acute{x}^3 = x^3 \;. \tag{3.23}$$

A dynamic variable, $A(q,p)$, is then transformed under (3.23) as,

$$A(q,p,\delta s) = A(q,p) + [A(q,p),K_1]\delta s \;, \tag{3.24}$$

which yields the differential equation:

$$\frac{\partial A(q,p;s)}{\partial s} = [A(q,p;s),K_1] \;. \tag{3.25}$$

This equation can be solved as previously described to get,

$$A(q,p;s) = e^{[K_1]s} A(q,p) \;, \tag{3.26}$$

for the finite transformation (3.22). Similarly, the distribution function transforms as,

$$f(q,p;s) = e^{-[K_1]s} f(q,p) \;. \tag{3.27}$$

Finally, the Lie Algebra conditions (3.20) reduce to,

$$[H,P] = 0 \ , \ [K,P] = H \ , \text{ and } [K,H] = P \ , \qquad (3.28)$$

where we have set $P_0 \equiv H$, $K_1 \equiv K$, $P_1 \equiv P$ (not to be confused with the pressure), and used the metric with signature -2.

We now wish to show that under appropriate circumstances, a canonical distribution function is transformed into a canonical distribution under the subgroup of canonical transformations satisfying (3.20) or in this (special) case (3.28). Let $f(q,p,T,V,s)$ be the distribution function for a system in thermal equilibrium at temperature T, volume V, and as seen by an observer moving in the x-direction with respect to the center of mass with velocity v = tanhs. Bors [37,38] has shown that $f(q,p,T,V,s)$ has the explicit form,

$$f(q,p,T,V,s) = \exp\{\frac{1}{kT}[F(T,V,s) - H(q,p) - (\tanh s)P(q,p)]\} \ . \qquad (3.29)$$

The factor $\exp\{\frac{F(T,V,s)}{kT}\}$ is a normalizing factor in which $F(T,V,s)$ is the Helmholtz free energy of a system with temperature T, volume V, and relative velocity tanhs (with respect to the observer). Then under a canonical transformation,

$$\begin{aligned} f(q,p,\acute{s}) &= e^{-[K](\acute{s}-s)} f(q,p,T,V,s) \\ &= \exp\{\frac{1}{kT}[F(T,V,s) - e^{-[K](\acute{s}-s)} H(q,p) \\ &\qquad - (\tanh s) e^{-[K](\acute{s}-s)} P(q,p)]\} \ . \qquad (3.30) \end{aligned}$$

Using the relations (3.28); and the definition of

$\exp\{-[K](\acute{s}-s)\}G(q,p)$, (3.13), the expressions in the brackets in (3.30) can be simplified:

$$
\begin{aligned}
e^{-[K](\acute{s}-s)}H(q,p) &= H(q,p) - (\acute{s}-s)[H(q,p),K(q,p)] \\
&\qquad + \frac{(\acute{s}-s)^2}{2!}[[H,K],K] - \ldots \\
&= H(q,p) + (\acute{s}-s)P + \frac{(\acute{s}-s)^2}{2!}H + \frac{(\acute{s}-s)^3}{3!}P + \ldots \\
&= H(q,p)\cosh(\acute{s}-s) + P(q,p)\sinh(\acute{s}-s) \ ,
\end{aligned}
$$

and,

$$
\begin{aligned}
e^{-[K](\acute{s}-s)}P(q,p) &= P(q,p) - (\acute{s}-s)[P(q,p),K(q,p)] \\
&\qquad + \frac{(\acute{s}-s)^2}{2!}[[P,K],K] - \ldots \\
&= P(q,p) + (\acute{s}-s)H + \frac{(\acute{s}-s)^2}{2!}P + \frac{(\acute{s}-s)^3}{3!}H + \ldots \\
&= P(q,p)\cosh(\acute{s}-s) + H(q,p)\sinh(\acute{s}-s) \ .
\end{aligned}
$$

Substituting these results into (3.30),

$$
\begin{aligned}
f(q,p,\acute{s}) = \exp\{\tfrac{1}{kT}[&F(T,V,s) - H(q,p)\cosh(\acute{s}-s) - P(q,p)\sinh(\acute{s}-s) \\
&- (\tanh s)[P(q,p)\cosh(\acute{s}-s) + H(q,p)\sinh(\acute{s}-s)]]\} \ .
\end{aligned}
$$

Using the hyperbolic trig identities:

$$\cosh(\acute{s}-s) = (\cosh\acute{s})\cosh s - (\sinh\acute{s})\sinh s \ ,$$

$$\sinh(\acute{s}-s) = (\sinh\acute{s})\cosh s - (\cosh\acute{s})\sinh s \ ,$$

we get after some algebra,

$$f(q,p,\acute{s}) = \exp\{\frac{1}{kT}[F(T,V,s) - \frac{\mathrm{ch}\acute{s}}{\mathrm{ch}s}H(q,p) - \frac{\mathrm{sh}\acute{s}}{\mathrm{ch}s}P(q,p)]\}$$

$$= \exp\{\frac{1}{kT}[F(T,V,s) - \frac{ch\acute{s}}{chs}H(q,p) - (\frac{ch\acute{s}}{chs})(th\acute{s})P(q,p)]\}$$

$$= \exp\{\frac{1}{kT}\cdot\frac{ch\acute{s}}{chs}[\frac{chs}{ch\acute{s}}F(T,V,s) - H(q,p) - (th\acute{s})P(q,p)]\} \ . \tag{3.31}$$

In the primed frame the distribution is normalized over the primed volume of the system, $V' = \frac{chs}{ch\acute{s}}V$. Thus, if we require,

$$T\,\frac{chs}{ch\acute{s}} = T' \ ; \ \frac{chs}{ch\acute{s}}\,F(T,V,s) = F(T',V',\acute{s}) \ ,$$

the distribution function is form invariant under the canonical Lorentz transformation. This is precisely the Planck - Einstein formulation, i.e.,

$$T = T_0/\gamma \ ; \ V = V_0/\gamma \ ; \ F(T\gamma,V\gamma,\acute{s}) = \gamma F(T,V,s) \ ,$$

where $\gamma = \frac{chs}{ch\acute{s}}$.

For the rest of the chapter we will set $\acute{s} = 0$ to simplify the calculations. Then,

$$F(T_0,V_0,0) = (chs)F(T,V,s) \ , \text{ or}$$

$$F(T,V,s) = \frac{1}{chs}F(Tchs,Vchs,0) \ . \tag{3.32}$$

We are now in a position to discuss the disagreements over tensor and thermodynamic transformation laws mentioned previously. To do this we need to derive the relation between the internal energy, $E(T,V,s)$, of the system and $F(T,V,s)$. Differentiating (3.32) and substituting for $F(Tchs,Vchs,0)$ we have,

$$(\frac{\partial F}{\partial s})_{TV} = ths\{-F + T(\frac{\partial F}{\partial T})_{Vs} + V(\frac{\partial F}{\partial V})_{Ts}\} \ . \tag{3.33}$$

We define the partition function,

$$Z(T,V,s) \equiv \int_{\Omega} \exp\{-\frac{1}{kT}[H + (\text{th}s)P]\}d\Gamma \ , \qquad (3.34)$$

where $d\Gamma$ is the number of states in the volume $d^{3N}p\,d^{3N}q$, of phase space. From the normalization condition (3.29), on the distribution, $f(q,p,V,T,s)$, it follows that,

$$F(T,V,s) = -kT\ln Z(T,V,s) \ . \qquad (3.35)$$

We can then use (3.32) to conclude that $Z(T\text{ch}s,V\text{ch}s,0) = Z(T,V,s)$, so that $Z(T,V,s)$ must be a scalar. This result then yields the differential equation,

$$\left(\frac{\partial Z}{\partial s}\right)_{TV} = \text{th}s\{T\left(\frac{\partial Z}{\partial T}\right)_{Vs} + V\left(\frac{\partial Z}{\partial V}\right)_{Ts}\} \ . \qquad (3.36)$$

Since $P(V,T,s)$ and $S(V,T,s)$ (the pressure and entropy respectively) are assumed to be invariants they also satisfy this equation (3.36a, 3.36b, 3.36c) respectively.

We define the internal energy of the system (actually, the average internal energy, but time is not a parameter in this treatment, so we can ignore the distinction):

$$E(s) \equiv \int_{\Omega} Hf(s)d\Gamma \equiv \int_{\Omega} H\exp\{-[K]s\}f(0)d\Gamma \ . \qquad (3.37)$$

Then using (3.25) and (3.27) to transform the energy and the definition (3.13) we have:

$\frac{\partial E(s)}{\partial s} = \frac{\partial}{\partial s}\int_{\Omega} Hf(s)d\Gamma = \int_{\Omega} H[K,f(s)]d\Gamma$. Integrating by parts, and using the Lie Algebra conditions (3.28), we have,

$$\frac{\partial E(s)}{\partial s} = -\int_{\Omega} f(s)[K,H]d\Gamma = -\int_{\Omega} f(s)Pd\Gamma \ . \qquad (3.38)$$

The last integral is the x-component of the average momentum, G(s), so we have,

$$\frac{\partial E(s)}{\partial s} = -G(s) \ . \tag{3.39}$$

This calculation is actually only valid for an unconfined system. The definition (3.37) has been adopted from standard canonical work (see [9] and Balescu, *et al* [39]) where f(q,p) is not dependent on V and T. The partial, $\frac{\partial E(s)}{\partial s}$, is holding q_i and p_i constant. With the introduction of V and T as independent variables, we need to calculate a total derivative with respect to s, V and T, instead of $\frac{\partial E(s)}{\partial s}$. That is, we have to take into account the implicit dependence of V and T on s. This arises both in f(q,p,V,T,s) and the limits of integration for confined systems. We will discuss this problem further, shortly. In any case, if we ignore V and T for the moment, (3.39) is the transformation equation of the energy component of the momentum 4-vector. That is:

$$\frac{dp_0(s)}{ds} = [p_0(s),K_1] = p_0 shs - p_1 chs = -p_1(s) \ .$$

See [39] for more details.

We now wish to treat the internal energy of the system as a state variable, $E \equiv E(V,T,s)$. We wish to calculate $\left(\frac{\partial E(s)}{\partial s}\right)_{TV}$ to compare this with the vector law (3.39). To do this we calculate E and G from their definitions, (3.37), (3.38) and (3.29):

$$E \equiv \int_\Omega Hf(s)d\Gamma = \int_\Omega H\exp\{\frac{1}{kT}[F(T,V,s) - H(q,p) - (\tanh s)P(q,p)]\}d\Gamma$$

$$= \exp\{\frac{F(T,V,s)}{kT}\}\int_\Omega H\exp\{\frac{-1}{kT}[H(q,p) + (\tanh s)P(q,p)]\}d\Gamma$$

Using the relations (3.34) and (3.35) we get,

$$E = \frac{\int_\Omega H\exp\{\frac{-1}{kT}[H + (\mathrm{ths})P]\}d\Gamma}{\int_\Omega \exp\{\frac{-1}{kT}[H + (\mathrm{ths})P]\}d\Gamma}$$

$$= \frac{\int_\Omega [H + (\mathrm{ths})P]\exp\{\frac{-1}{kT}[H + (\mathrm{ths})P]\}d\Gamma}{\int_\Omega \exp\{\frac{-1}{kT}[H + (\mathrm{ths})P]\}d\Gamma} + \frac{-\int_\Omega (\mathrm{ths})P\exp\{\frac{-1}{kT}[H + (\mathrm{ths})P]\}d\Gamma}{\int_\Omega \exp\{\frac{-1}{kT}[H + (\mathrm{ths})P]\}d\Gamma}$$

$$= -\frac{\partial}{\partial\beta}\ln\{\int_\Omega \exp\{\frac{-1}{kT}[H + (\mathrm{ths})P]\}d\Gamma\} + \frac{kT\mathrm{ths}}{\mathrm{sech}^2 s}\cdot\frac{\partial}{\partial s}\ln\{\int_\Omega \exp\{\frac{-1}{kT}[H + (\mathrm{ths})P]\}d\Gamma\},$$

where we have used (2.20) to replace kT. The terms in brackets are just Z(T,V,s), so using (3.35), we have,

$$E(V,T,s) = -\frac{\partial}{\partial\beta}\{-\frac{F(T,V,s)}{kT}\} + kT(\mathrm{chs})(\mathrm{shs})\frac{\partial}{\partial s}\{-\frac{F(T,V,s)}{kT}\}$$

$$= F(V,T,s) - T(\frac{\partial F}{\partial T})_{Vs} - (\mathrm{chs})(\mathrm{shs})(\frac{\partial F}{\partial s})_{VT}\ . \qquad (3.40)$$

Similarly,

$$G(s) \equiv \int_\Omega f(s)P(q,p)d\Gamma = \frac{\int_\Omega P\exp\{\frac{-1}{kT}[H + (\mathrm{ths})P]\}d\Gamma}{\int_\Omega \exp\{\frac{-1}{kT}[H + (\mathrm{ths})P]\}d\Gamma}$$

$$= \frac{-\ kT}{\mathrm{sech}^2 s}\cdot\frac{\partial}{\partial s}\ln Z(V,T,s) = (\mathrm{ch}^2 s)\left(\frac{\partial F}{\partial s}\right)_{VT} \tag{3.41}$$

Substituting (3.40) and (3.41) into (3.33) we get,

$$\frac{G}{\mathrm{ch}^2 s} = \mathrm{ths}\{-E(V,T,s) - \frac{(\mathrm{shs})(\mathrm{chs})}{\mathrm{ch}^2 s}G + V\left(\frac{\partial F}{\partial V}\right)_{Ts}\}\ .$$

We define the pressure, $P \equiv -\left(\frac{\partial F}{\partial V}\right)_{Ts}$, so we get,

$$G = -(\mathrm{ths})(E + PV)\ . \tag{3.42}$$

(From here until the discussion following (3.52), "P" is used exclusively to denote pressure and is unrelated to the generator of space translations also denoted "P".) Differentiating both sides of (3.40) and (3.41) with respect to s, we have

$$\left(\frac{\partial E}{\partial s}\right)_{TV} = \left(\frac{\partial F}{\partial s}\right)_{VT} - T\{\frac{\partial}{\partial T}\left(\frac{\partial F}{\partial s}\right)\}$$
$$- (\mathrm{ch}^2 s + \mathrm{sh}^2 s)\left(\frac{\partial F}{\partial s}\right)_{TV} - (\mathrm{chs})(\mathrm{shs})\left(\frac{\partial^2 F}{\partial s^2}\right)_{TV}\ , \tag{3.43}$$

and,

$$\left(\frac{\partial G}{\partial s}\right)_{TV} = 2(\mathrm{chs})(\mathrm{shs})\left(\frac{\partial F}{\partial s}\right)_{TV} + (\mathrm{ch}^2 s)\left(\frac{\partial^2 F}{\partial s^2}\right)_{TV}\ , \tag{3.44}$$
$$= 2(\mathrm{ths})G + (\mathrm{ch}^2 s)\left(\frac{\partial^2 F}{\partial s^2}\right)_{TV}\ ,\ \text{using (3.41).}$$

Further applications of (3.41) yield,

$$\left(\frac{\partial E}{\partial s}\right)_{TV} = \frac{G}{\mathrm{ch}^2 s} - \frac{T}{\mathrm{ch}^2 s}\left(\frac{\partial G}{\partial T}\right)_{Vs} - (1 + \mathrm{th}^2 s)G + \{2(\mathrm{ths})G - \left(\frac{\partial G}{\partial s}\right)_{TV}\}\mathrm{ths}$$
$$= -(\mathrm{ths})\left(\frac{\partial G}{\partial s}\right)_{TV} - \frac{T}{\mathrm{ch}^2 s}\left(\frac{\partial G}{\partial T}\right)_{Vs}\ .$$

Substituting (3.42) for G we have,

$$(\frac{\partial E}{\partial s})_{TV} = -\frac{G}{ch^2 s} + (th^2 s)\{(\frac{\partial E}{\partial s})_{VT} + V(\frac{\partial P}{\partial s})_{TV}\} + \frac{ths}{ch^2 s}T\{(\frac{\partial E}{\partial T})_{Vs} + V(\frac{\partial P}{\partial T})_{Vs}\} .$$

Using (3.36b) to eliminate $(\frac{\partial P}{\partial s})_{TV}$,

$$(\frac{\partial P}{\partial s})_{TV} = ths\{T(\frac{\partial P}{\partial T})_{Vs} + V(\frac{\partial P}{\partial V})_{Ts}\} , \tag{3.45}$$

$$\frac{1}{ch^2 s}(\frac{\partial E}{\partial s})_{VT} = V(th^3 s)\{T(\frac{\partial P}{\partial T})_{Vs} + V(\frac{\partial P}{\partial V})_{Ts}\} - \frac{G}{ch^2 s} + \frac{ths}{ch^2 s}T\{(\frac{\partial E}{\partial T})_{Vs} + V(\frac{\partial P}{\partial T})_{Vs}\}$$

$$= \frac{-G}{ch^2 s} + \frac{ths}{ch^2 s}T\{(\frac{\partial E}{\partial T})_{Vs} + VT(ths)(\frac{\partial P}{\partial T})_{Vs}\} + (th^3 s)V^2(\frac{\partial P}{\partial V})_{Ts}$$

or,

$$(\frac{\partial E}{\partial s})_{VT} = -G + T(ths)(\frac{\partial E}{\partial T})_{Vs} + (chs)(shs)VT(\frac{\partial P}{\partial T})_{Vs} + (ths)(sh^2 s)V^2(\frac{\partial P}{\partial V})_{Ts} \tag{3.46}$$

This equation is rather different from the vector law, (3.39). Balescu suggests that the three additional terms can be explained by thinking of a Lorentz transformation as a thermodynamic process. The view that a Lorentz transformation can be looked at as a dynamical process was first proposed by Fock [40]. Then the volume and temperature are not constant and to compare (3.39) with the statistical law, (3.46), we need to conduct further expansion and heating processes. The three additional terms reflect this, being clearly related to the specific heat, coefficients of thermal expansion and compressibility.

Evidently, we need to calculate a total derivative, $(\frac{dE}{ds})_L$, to take into account the dependence of T and V on s. The subscript L indicates that a total derivative is to be taken with respect to V, T, and s, but that the canonical variables (q_i,p_i) are held constant. This is actually what is required in calculating $\frac{\partial E(s)}{\partial s}$ in (3.39), since we ignored any dependence f(s) or Ω may have had on V and T. To calculate $(\frac{dE}{ds})_L$, we proceed as follows:

$$(\frac{dE(V',T';s)}{ds})_L \equiv (\frac{\partial E}{\partial T'})(\frac{\partial T'}{\partial s}) + (\frac{\partial E}{\partial V'})(\frac{\partial V'}{\partial s}) + (\frac{\partial E}{\partial s}) , \quad (3.47)$$

where $T' = \frac{T}{chs}$, and, $V' = \frac{V}{chs}$. Then,

$$(\frac{\partial T'}{\partial s}) = -T\frac{ths}{chs} = -T'ths , \text{ and } (\frac{\partial V'}{\partial s}) = -V'ths ,$$

so we have (dropping the primes),

$$(\frac{dE(V,T,s)}{ds})_L \equiv -\{(\frac{\partial E}{\partial T})_{Vs}T + (\frac{\partial E}{\partial V})_{Ts}V\}ths + (\frac{\partial E}{\partial s})_{VT} . \quad (3.48)$$

We have already calculated $(\frac{\partial E}{\partial s})_{VT}$, in (3.46), so we need only determine $(\frac{\partial E}{\partial V})_{Ts}$. Differentiating (3.40) with respect to V, we get,

$$(\frac{\partial E}{\partial V})_{Ts} = (\frac{\partial F}{\partial V})_{Ts} - T\{\frac{\partial}{\partial T}(\frac{\partial F}{\partial V})_{Ts}\}_{Vs} - (chs)(shs)\{\frac{\partial}{\partial s}(\frac{\partial F}{\partial V})_{Ts}\}_{VT} .$$

Replacing $(\frac{\partial F}{\partial V})_{Ts}$ with -P and using (3.36b) to eliminate $(\frac{\partial P}{\partial s})_{TV}$, we have:

$$(\frac{\partial E}{\partial V})_{Ts} = -P + T(\frac{\partial P}{\partial T})_{Vs} + (chs)(shs)(ths)\{T(\frac{\partial P}{\partial T})_{Vs} + V(\frac{\partial P}{\partial V})_{Ts}\}$$

$$= -P + T(\mathrm{ch}^2 s)\left(\frac{\partial P}{\partial T}\right)_{Vs} + V(\mathrm{sh}^2 s)\left(\frac{\partial P}{\partial V}\right)_{Ts} . \tag{3.49}$$

Combining (3.46) and (3.49) with (3.48) we get:

$$\left(\frac{dE(V,T,s)}{ds}\right)_L = PV\mathrm{th}s - G(s) = \mathrm{th}s(E + 2PV) . \tag{3.50}$$

If we now do the statistical calculation correctly, taking into account a finite system, of volume V, and temperature T, we get the same result. To show this we proceed as follows: Using the relations (3.35) and (3.40), expressing F in terms of Z, and E in terms of F respectively, we can derive a relation between E and Z:

$$E = \frac{kT}{Z}\left\{T\left(\frac{\partial Z}{\partial T}\right)_{Vs} + (\mathrm{ch}s)(\mathrm{sh}s)\left(\frac{\partial Z}{\partial s}\right)_{TV}\right\} . \tag{3.51}$$

Then,

$$\left(\frac{dE(V,T,s)}{ds}\right)_L = \left\{\left(\frac{\partial}{\partial s}\right)_{VT} - \mathrm{th}s\left[T\left(\frac{\partial}{\partial T}\right)_{Vs} + V\left(\frac{\partial}{\partial V}\right)_{Ts}\right]\right\} \cdot\left\{\frac{kT}{Z}\left[T\left(\frac{\partial Z}{\partial T}\right)_{Vs} + (\mathrm{ch}s)(\mathrm{sh}s)\left(\frac{\partial Z}{\partial s}\right)_{TV}\right]\right\} , \tag{3.52}$$

where we have used (3.48). We can now calculate the various derivatives of Z using (3.34). Since the domain of integration does not depend on s and T explicitly, these partials are not difficult. However, the volume derivatives are best avoided by using the relation (3.36) to express $\left(\frac{\partial Z}{\partial V}\right)_{Ts}$ in terms of the other two derivatives:

$$V\left(\frac{\partial Z}{\partial V}\right)_{Ts} = -T\left(\frac{\partial Z}{\partial T}\right)_{Vs} + \frac{1}{\mathrm{th}s}\left(\frac{\partial Z}{\partial s}\right)_{TV}$$

The complete calculation is rather long and will be omitted. The result is (3.50). However, we will give an example of how the calculation proceeds. In (3.52), the first (of six) terms on the right is,

$$\frac{\partial}{\partial s}\{\frac{kT^2}{Z}(\frac{\partial Z}{\partial T})_{Vs}\} = -\frac{\partial}{\partial s}\{\frac{\partial}{\partial \beta}\ln Z\} = -\frac{\partial}{\partial s}\{\frac{\partial}{\partial \beta}[\frac{-F}{kT}]\}$$

$$= \frac{\partial}{\partial s}\{F - T(\frac{\partial F}{\partial T})_{Vs}\} .$$

Balescu evidently calculates derivatives of Z directly, using (3.34), getting various expressions involving averages such as <HP> and <P²>, but there is no need to do this, because of the fundamental relation, (3.35), which normalizes the distribution. The calculations are essentially equivalent, and yield the same result, (3.50).

Thus, we conclude that thermodynamic quantities such as energy do not transform like components of vectors except for unconfined systems. The loss of vector character is clearly an effect of the confinement of the system. This supports the often made suggestion that the non-vector character of the transformation equation for energy is due to stresses in the walls of the system.

To conclude this section we wish to generalize the derivation of the invariance of the canonical distribution. We start with a generalized equilibrium distribution,

$$\bar{f}(T,V,s,q,p) = \exp\{\frac{\alpha(s)}{kT}[\frac{\bar{F}(T,V,s)}{\gamma(s)}$$

$$- H(q,p) - (\text{ths})P(q,p)]\} . \qquad (3.53)$$

Here $\alpha(s)$ and $\gamma(s)$ are even functions, satisfying the condition, $\alpha(0)=\gamma(0)=1$, but are otherwise arbitrary, sufficiently smooth functions of s. We note that $\bar{f}(T,V,0) = f(T,V,0)$, and that $\alpha(s)$ and $\gamma(s)$ do not interfere with the canonical formulism, i.e., $\bar{f}(T,V,s,q,p) = \exp\{-[K]s\}\bar{f}(T,V,0,q,p)$. Thus, we can proceed as previously shown, and arrive at,

$$\bar{F}\left(T\frac{\mathrm{ch}s\,\alpha(\acute{s})}{\mathrm{ch}\acute{s}\,\alpha(s)}, V\frac{\mathrm{ch}s}{\mathrm{ch}\acute{s}}, \acute{s}\right) = \frac{\mathrm{ch}s\,\gamma(\acute{s})}{\mathrm{ch}\acute{s}\,\gamma(s)}\bar{F}(T,V,s) \ . \tag{3.54}$$

This shows that the distribution function (3.53), while unique when s=0, has a doubly infinite class of Lorentz invariant generalizations. If we set $\acute{s}=0$, to compare with the rest frame values, we have,

$$T = T_0\frac{\alpha(s)}{\mathrm{ch}s} \ ; \ V = V_0/\mathrm{ch}s \ ;$$

$$\bar{F}\left(T_0\frac{\alpha(s)}{\mathrm{ch}s}, \frac{V_0}{\mathrm{ch}s}, s\right) = \frac{\gamma(s)}{\mathrm{ch}s}\bar{F}(T_0, V_0, 0) \ . \tag{3.55}$$

In the rest frame we have the fundamental equations:

$$\left(\frac{\partial\bar{F}}{\partial V_0}\right)_{T_0} = -P \text{ and } \left(\frac{\partial\bar{F}}{\partial T_0}\right)_{V_0} = -S \ , \tag{3.56}$$

where P and S are the pressure and entropy respectively. If we differentiate both sides of (3.55) we get,

$$\left(\frac{\partial\bar{F}}{\partial T}\right)_{Vs} = \frac{\gamma(s)}{\mathrm{ch}s}\left(\frac{\partial\bar{F}}{\partial T_0}\right)_{V_0}\left(\frac{\partial T_0}{\partial T}\right) = \frac{\gamma(s)}{\mathrm{ch}s}\left(\frac{\partial\bar{F}}{\partial T_0}\right)_{V_0}\frac{\mathrm{ch}s}{\alpha(s)} = -\frac{\gamma(s)}{\alpha(s)}S \ ;$$

and,

$$\left(\frac{\partial\bar{F}}{\partial V}\right)_{Ts} = \frac{\gamma(s)}{\mathrm{ch}s}\left(\frac{\partial\bar{F}}{\partial V_0}\right)_{T_0}\left(\frac{\partial V_0}{\partial V}\right) = \gamma(s)\left(\frac{\partial\bar{F}}{\partial V_0}\right)_{T_0} = -\gamma(s)P \ . \tag{3.57}$$

If we assume P and S are invariant and that thermodynamics is to be form invariant, then we must set $\gamma(s) = \alpha(s) = 1$. This is the Planck-Einstein formulation (moving bodies appear cooler). The choice $\gamma(s) = 1$ and $\alpha(s) = \mathrm{ch}^2 s$ is the Ott-Arzelies formulation (moving bodies appear hotter). It is not form invariant unless (3.56) is abandoned. If we set $\gamma(s) = 1$ and $\alpha(s) = \text{ch}s$ we get Landesberg's scheme (invariance of temperature).

These results pretty well summarize all that can be said about relativistic equilibrium thermodynamics. We wish only to mention a paper by Kellerman [19] in which he suggests an addendum to Balescu's work to help clear up the thought experiment devised by Landesberg and already described. He suggests that we can consider the two moving bodies as an isolated system in which the relative velocity is treated as a thermodynamic parameter, as Balescu has done. Then the exchange of energy between the two bodies will be governed by maximization of the total entropy. He shows that entropy can be generalized to include a dependence on the relative difference $s_A - s_B$, where $v_A = \text{th}s_A$ and $v_B = \text{th}s_B$ are the velocities of system A and B respectively, as seen by an observer in his own rest frame. Kellerman then gets a system of equations and contraints that is covariant and independent of any temperature transformation laws. Thus, under these assumptions Landesberg's claim that $T = T_0$ is the only consistent transformation law is incorrect. Unfortunately, Kellerman was not able to actually solve the equations.

Our next task is to review blackbody radiation in preparation for attempting to apply the work of this chapter to the problem of finding a general relation between the thermal and statistical notions of entropy that is compatible with the stochastic electrodynamics worked out by Boyer and discussed in chapter 4.

CHAPTER IV

BLACKBODY RADIATION

Since this thesis is intimately concerned with blackbody radiation, we will give a rather complete and systematic treatment of the standard approach to this subject. We do this in spite of the thorough treatment given in most texts on Modern Physics or Statistical Mechanics (see, for example,[8,13], Eisberg [41], or Wieder [42]), because comparison of the quantum mechanical derivation (given in section 3) and the "classical fluctuation" method covered in chapter 4, makes it hard to believe that the complete coincidence of the results from these two methods is fortuitous.

We begin with a description of the problem, and the derivation of some early results. Wien's displacement law is derived, and Wien's radiation law is briefly discussed. We conclude with the derivations of the Rayleigh - Jeans law and the modern quantum mechanical law for the spectral density.

Early Results

The blackbody problem is essentially the problem of determining the spectral density $\rho(\nu,T)$; that is, the energy density radiated in the frequency range ν to $\nu + d\nu$, of a body at temperature T. It is assumed that the radiative process has reached a steady state so that T and $\rho(\nu,T)$ are time independent. In early studies of this problem $\rho(\nu,T)$ was found to be independent of the shape of the radiating object. $\rho(\nu,T)$ does depend on a parameter, $0 \leq e \leq 1$, which is a measure of the emitivity of the object. For a perfect blackbody $e = 1$. Such is also a perfect absorber of radiation. The theoretical significance of this problem is that a great deal of progress can be made towards solving it with very general thermodynamic arguments.

In order to keep these analyses as simple as possible we need to consider a model that is in thermal equilibrium. The model usually used is that of a cavity with perfectly reflecting walls. (In the stochastic electrodynamic approach used in chapter 5, we will modify this model by including an ideal gas inside the cavity.) It is not hard to see that the radiation in the cavity must be isotropic, in all frequencies, as otherwise the second law would be violated.

The first precise experimental results about blackbody radiation were determined in 1879 by Stefan. He found that the total power emitted per unit area of a perfect blackbody was proportional to the fourth power of the temperature. In 1884 Boltzmann managed to derive this result using a Carnot cycle as follows (see figure 4.1): The cylinder and piston are assumed to be perfectly reflecting and frictionless. The working medium in this case is radiation. In step (a) the piston is moved from O_1 to O_2 by isothermal expansion in contact with the reservoir, A_1, at temperature T_1. The work done in this step is,

$$W_1 = P_1(V_2 - V_1) = \frac{1}{3}U_1(V_2 - V_1) \quad . \tag{4.1}$$

The last expression follows from the classical result connecting pressure and radiation density (see Richtmyer [13]):

$$P_1 = \frac{1}{3}U_1 \; . \tag{4.2}$$

Since the volume of the cylinder has expanded from V_1 to V_2, to keep the energy density, U_1, constant, the cavity must absorb an amount of energy equal to $U_1(V_2 - V_1)$. Thus, the total energy absorbed from the reservoir in this step is,

$$\frac{4}{3}U_1(V_2 - V_1) \equiv H_1 \; . \tag{4.3}$$

In step (b) the piston is moved from O_2 to O_3 adiabatically. If the change in volume in this step, $V_3 - V_2$, is made small enough, then $T_2 - T_1 = dT$, $U_2 - U_1 = dU$, and $P_2 - P_1 = dP = \frac{1}{3}dU$, since $P = \frac{1}{3}U$. In step (c) the piston is compressed from O_3 to O_4, isothermally, while in contact with the reservoir A_2 at temperature T_2. In this step, heat H_2, is exhausted to the reservoir, A_2. The last step returns the piston to its original position by adiabatic compression. Since the pressure was constant in steps (a) and (c), if the change in pressure is kept small in step (b) (and therefor in step (d) as well), the net external work done is just,

$$dW = (V_2 - V_1)dP = \frac{1}{3}(V_2 - V_1)dU \; . \tag{4.4}$$

For any Carnot cycle we have,

$$\frac{dW}{H_1} = \frac{T_1 - T_2}{T_1} \; . \tag{4.5}$$

This last expression is equal to $\frac{dT}{T_1}$ if $P_2 - P_1$ is taken to be infinitesimal. Substituting for dW and H_1 from (4.3) and (4.4) into (4.5) we have, with $T_1 \equiv T$ and $U_1 \equiv U$,

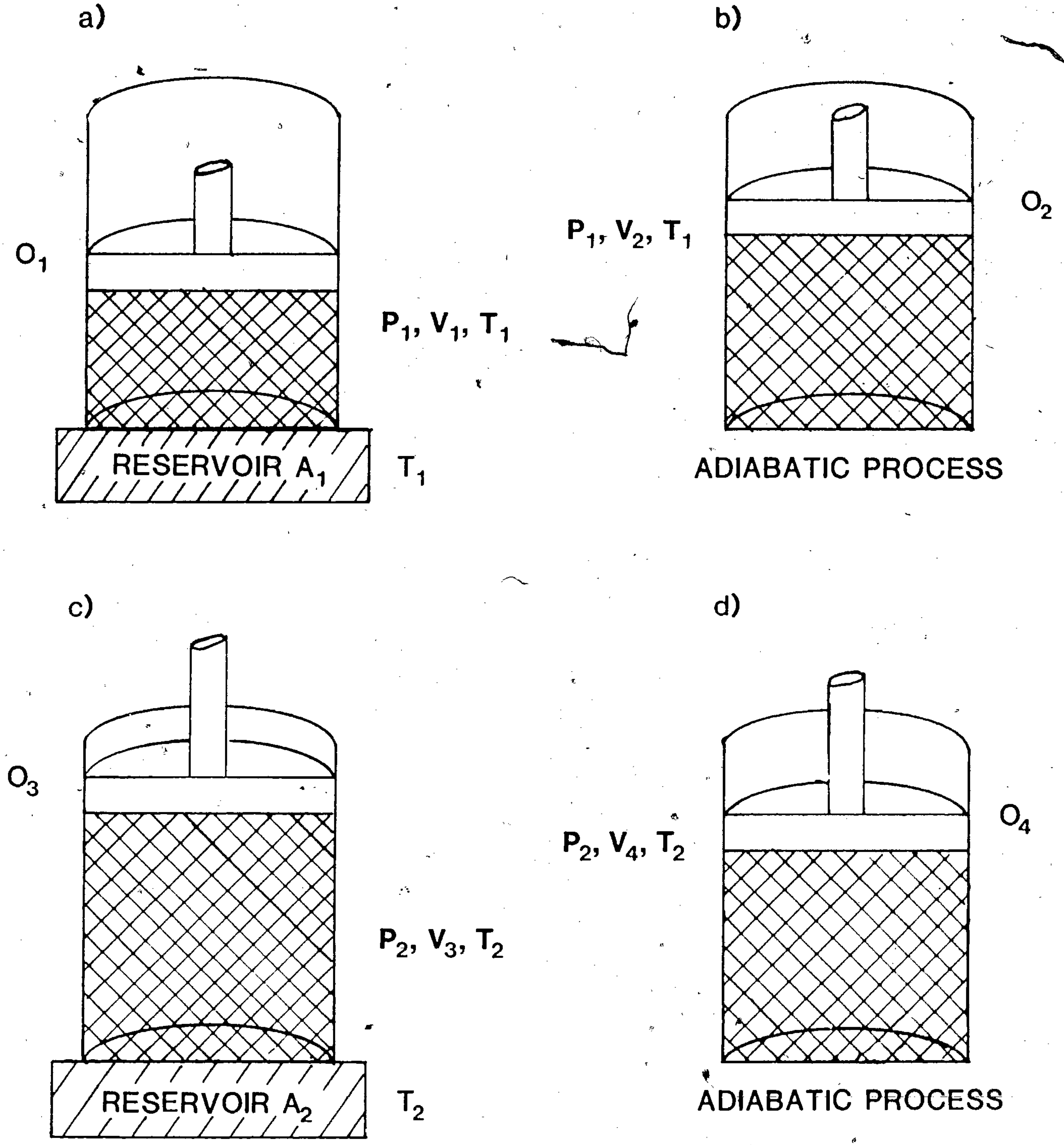

Figure 4.1–Carnot Cycle for Radiation

$$\frac{\frac{1}{3}(V_2 - V_1)dU}{\frac{4}{3}(V_2 - V_1)U} = \frac{dT}{T} , \text{ or } \frac{dU}{U} = \frac{4dT}{T} .$$

Integrating this last expression yields the Stefan - Boltzmann law,

$$U = aT^4 . \tag{4.6}$$

Wien's Displacement Law

The next important development came in 1893 with Wien's displacement law. To derive Wien's law we consider an adiabatic expansion of the cavity, i.e., step 2 of the Carnot cycle, figure 2.1 (or step (b) of figure 4.1). We assume that the walls have been lined with some perfectly, diffusely reflecting material, such as magnesium oxide, so that the radiation remains isotropic throughout the (slow) expansion. Then the radiation pressure is given by (4.2). The work done by the radiation during the expansion, if the cylinder has length l and cross section A, is $PdV = \frac{1}{3}UAdl$. The total energy of the radiation in the cavity is lAU, so,

$$-\frac{1}{3}UAdl = d(lAU) = AUdl + AldU , \text{ or}$$

$$-\frac{4}{3}UAdl = AldU , \text{ so } \frac{dU}{U} = -\frac{4}{3}\frac{dl}{l} ,$$

and integrating we get,

$$U \propto l^{-\frac{4}{3}} . \tag{4.7}$$

Since the radiation in the cavity must remain black to conform with the second law, we conclude that,

$$T \propto l^{-\frac{1}{3}} . \qquad (4.8)$$

We next consider the change in a particular spectral element arising from the expansion. For a ray of light reflecting off the walls of the cavity it is evident that (ideally) the angle of incidence remains constant, so the speed along the axis of the cylinder is $c(\cos\theta)$. In one second the ray strikes the piston $\frac{c(\cos\theta)}{2l}$ times. The Doppler shift of a ray of light reflected from a moving mirror, with speed v, is, $\Delta\lambda = 2\frac{v}{c}\lambda\cos\theta$. Then the rate of shifting of λ, $\frac{d\lambda}{dt}$, is,

$$\frac{d\lambda}{dt} = (2\frac{v}{c}\lambda\cos\theta)\frac{c(\cos\theta)}{2l} , \text{ where } v = \frac{dl}{dt} .$$

Thus,

$$\frac{d\lambda}{dt} = \frac{dl}{dt}\cdot\frac{\lambda}{l}\cos^2\theta . \qquad (4.9)$$

The average value of $\cos^2\theta$ (for isotropic radiation) over all possible solid angles is, $\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \cos^2\theta \sin\theta \, d\theta d\phi = \frac{1}{3}$, so the average value of $\frac{d\lambda}{dt} = \frac{1}{3}\cdot\frac{\lambda}{l}\cdot\frac{dl}{dt}$. This implies $\frac{d\lambda}{\lambda} = \frac{1}{3}\cdot\frac{dl}{l}$, or $\lambda \propto l^{\frac{1}{3}} \propto \frac{1}{T}$, using (4.8) . Evidently, each component of the radiation must change in such a way that

$$\lambda \propto \frac{1}{T} , \text{ or } \nu \propto T , \qquad (4.10)$$

where ν is the frequency.

Now we consider once again an adiabatic expansion, and the spectral range ν_1 to $\nu_1 + d\nu_1$, with total energy density $\rho(\nu_1,T_1)d\nu_1$. If the expansion lowers the temperature to T_2 then the frequency range is shifted to the range ν_2 to $\nu_2 + d\nu_2$. By

(4.10) we have,

$$\frac{\nu_2}{\nu_1} = \frac{\nu_2 + d\nu_2}{\nu_1 + d\nu_1} = \frac{T_2}{T_1} , \text{ or } \frac{d\nu_2}{d\nu_1} = \frac{T_2}{T_1} . \qquad (4.11)$$

The energy density in the range ν_1 to $\nu_1 + d\nu_1$ is decreased by the same ratio as the total energy. Thus, we have,

$\frac{\rho(\nu_2,T_2)d\nu_2}{\rho(\nu_1,T_1)d\nu_1} = \frac{U_2}{U_1} = \frac{T_2^4}{T_1^4}$, and using (4.11), $\frac{\rho(\nu_2,T_2)}{\rho(\nu_1,T_1)} = \frac{T_2^3}{T_1^3}$. Then $\frac{d\rho(\nu,T)}{\rho(\nu,T)} = \frac{3dT}{T}$. Using the mathematical relation,

$$d\rho(\nu,T) = \frac{\partial\rho(\nu,T)}{\partial T}dT + \frac{\partial\rho(\nu,T)}{\partial\nu}d\nu ,$$

we have $3\rho(\nu,T)\frac{dT}{T} = \frac{\partial\rho(\nu,T)}{\partial T}dT + \frac{\partial\rho(\nu,T)}{\partial\nu}d\nu$, from which we get, with the help of (4.10), $\rho(\nu,T) = \frac{\partial\rho(\nu,T)}{\partial T}\cdot\frac{T}{3} + \frac{\partial\rho(\nu,T)}{\partial\nu}\cdot\frac{\nu}{3}$. This differential equation is readily solved to yield Wien's Displacement law:

$$\rho(\nu,T) = \nu^3 g\left(\frac{\nu}{T}\right) , \qquad (4.12)$$

where g is an arbitrary $\epsilon\ C^1[R^2\text{->}R]$. Some authors refer to Wien's Displacement law as the special case of (4.10) arising when ν and T are chosen so that $\frac{\partial\rho(\nu,T)}{\partial\nu} = 0$; i.e., along the curve $\frac{\partial\rho(\nu,T)}{\partial\nu} = 0$, $\nu = CT$, where C is a fixed constant. Equation (4.12) contains this statement for "physically reasonable" choices of $g(\frac{\nu}{T})$, where "physically reasonable" is defined once close agreement between theory and experiment is reached. Since this has been done, (4.12) is now often referred to as Wien's Displacement law.

Unfortunately, thermodynamics alone seems to be able to make no further headway on this problem. Hence, at the end of the last century, attention turned to the details of absorbtion and emission of radiation by the walls. Using this approach, Wien derived the following law: $\rho(\nu,T) = C_1\frac{\nu^3}{c^4}\exp\{\frac{-C_2\nu}{cT}\}$, where c is the speed of light and C_1 and C_2 are constants. By proper choice of C_1 and C_2, this law can be made to fit experimental results except at low frequencies: $\rho(\nu,T) = \frac{8\pi h\nu^3}{c^3}\exp\{\frac{-h\nu}{kT}\}$. The inaccuracy of this law did not become evident until 1899, when improved experimental results by Lummer and Pringsheim made the low frequency discrepancy apparent.

The Rayleigh - Jeans and Planck Laws

In 1900 Rayleigh and Jeans worked out a different approach using equipartition of energy. They counted the number of modes of electromagnetic radiation in a given frequency range and then multiplied by kT, i.e., $\frac{1}{2}kT$ for each of the 2 degrees of freedom per mode. Their calculation goes as follows:

The radiation in the cavity must obey Maxwell's (wave) equations in a vacuum: $\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0$, and $\nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = 0$, subject to the boundary conditions that the tangential components of **E** and the normal components of **B** must be zero at the walls of the cavity (see Wangsness or Jackson [43,44], for more details on electromagnetic waves in a cavity). Assuming a cubic cavity of side L, and imagining the coordinate planes to coincide with three walls of the cavity, we have,

$$\mathbf{E}_l(\mathbf{x},t) = f_l(\omega,T)e^{-i\omega_n t}[\sin\frac{n_x\pi}{L}x][\sin\frac{n_y\pi}{L}y][\sin\frac{n_z\pi}{L}z] ,$$

where $l = x,y,z$; and n_x,n_y,n_z are positive integers, such that

$$\omega_n{}^2 = \frac{n^2\pi^2c^2}{L^2} ;\; n^2 = n_x{}^2 + n_y{}^2 + n_z{}^2 . \qquad (4.13)$$

The number of modes with this frequency, is then just equal to the number of ways to choose the integers n_x, n_y, and n_z subject to the given constraints, (4.13). For temperatures and frequencies of interest, n is very large so that the number of modes between frequencies ω and $\omega + d\omega$ can be estimated by the volume of one eighth of a spherical shell of radius n. That is,

$dN = 2\cdot\frac{4\pi}{8}\{\frac{L\omega_n}{\pi c}\}^2 d(\frac{\omega L}{\pi c})$. The factor of $\frac{1}{8}$ arises because we require n_x, n_y, n_z to be positive. The factor of 2 arises because for each mode there are two possible polarizations. Thus, the mode density (number of modes per unit volume) is

$$dN = \frac{\omega^2 d\omega}{\pi^2c^3} . \qquad (4.14)$$

In this approach the spectral density is then just

$$\rho(\omega,T)d\omega = \bar{\epsilon}(\omega,T)dN = \frac{\bar{\epsilon}(\omega,T)\omega^2 d\omega}{\pi^2c^3} ,$$

where $\bar{\epsilon}(\omega,T)$ is the average energy per mode.

Rayleigh and Jeans assumed that the oscillators in the walls absorbed and reradiated constantly and independently so that they could apply the classical equipartition theorem to the oscillators to get $\overline{\epsilon}(\omega,T) = kT$, i.e., $\frac{1}{2}kT$ for each of the kinetic and potential energy degrees of freedom of the oscillators. Then we have for the Rayleigh - Jeans spectral density,

$$\rho_{RJ}(\omega,T) = \frac{\omega^2 kT}{\pi^2 c^3} \text{ , or, } \rho_{RJ}(\nu,T) = \frac{8\pi\nu^2 kT}{c^3} . \tag{4.15}$$

This result fits experiment for low frequencies better than Wien's law does. However, at higher frequencies it is completely incorrect. (The failure of this law at higher frequencies is often referred to as "the ultraviolet catastrophe". There is hardly a "catastrophe" involved here; just another theory that is inadequate.) The graph of ν vs $\rho_{RJ}(\nu,T)$ (at fixed T) does not exhibit the charateristic maximum evident from experiment. It also gives an infinite thermal energy for the cavity:

$$\int_0^\infty \rho_{RJ}(\nu,T)d\nu \to \infty .$$

Planck arrived at the correct solution by guessing the answer, then seeking to modify the classical procedure for calculating $\overline{\epsilon}(\omega,T)$ of the oscillators. He was eventually led to the idea of setting the possible energies of the oscillators at $\epsilon_n = nh\nu$ $(\omega = 2\pi\nu)$. To calculate $\overline{\epsilon}(\nu,T)$, for these values of ϵ_n, we have by (2.23),

$$\overline{\epsilon}(\nu,T) = \frac{\sum_{n=0}^{\infty} \epsilon_n e^{-\epsilon_n \beta}}{\sum_{n=0}^{\infty} e^{-\epsilon_n \beta}} .$$

To save work we can use a standard trick:

$$\overline{\epsilon}(\nu,T) = -\frac{\partial}{\partial\beta}\ln\{\sum_{n=0}^{\infty} e^{-\epsilon_n\beta}\}$$

$$= -\frac{\partial}{\partial\beta}\ln\{\frac{1}{1 - \exp(-\frac{h\nu}{kT})}\} = \frac{\partial}{\partial\beta}\ln\{1 - \exp(-\frac{h\nu}{kT})\}$$

$$= \frac{h\nu\exp\{-\frac{h\nu}{kT}\}}{1 - \exp(-\frac{h\nu}{kT})} = \frac{h\nu}{\exp(\frac{h\nu}{kT}) - 1} ,$$

where we have used (2.20). Combining this last result with the mode density, (4.14), we have,

$$\rho_P(\nu,T) = \frac{8\pi h\nu^3}{c^3}\cdot\{\frac{1}{\exp(\frac{h\nu}{kT}) - 1}\} . \qquad (4.16)$$

This is the correct result as determined by experiment. The value of the constant h can be determined by fitting data to the experimental curve, however, better ways of finding h have been devised. The accepted value is $h = 6.6256 \times 10^{-34}$ Joule sec..

To conclude this chapter we wish to point out that in a modern quantum mechanical treatment of this subject the wave equation is solved using a harmonic potential $(\frac{1}{2}KX^2)$, yielding energy eigenvalues for the oscillators: $\epsilon_n = (\frac{1}{2} + n)h\nu$. If we use this result in the calculation for $\overline{\epsilon}(\nu,T)$, (2.23), we get,

$$\overline{\epsilon}(\nu,T) = \frac{h\nu}{\exp(\frac{h\nu}{kT}) - 1} + \frac{h\nu}{2} .$$

And the spectral density becomes,

$$\rho_{QM}(\nu,T) = \frac{8\pi h\nu^3}{c^3}\{\frac{1}{\exp(\frac{h\nu}{kT}) - 1} + \frac{1}{2}\} . \qquad (4.17)$$

Thus, we have a non-thermal background spectrum that persist even at T = 0 .

In the next chapter we will derive this equation without explicit quantum assumptions. We assume a zero-point background energy density. Such an assumption is a possible solution to Maxwell's vacuum equations, under the usual boundary conditions.

CHAPTER V

BLACKBODY SPECTRUM USING STOCHASTIC ELECTRODYNAMICS

In this chapter we derive the blackbody spectral density using techniques which have come to be known as stochastic electrodynamics. The idea of this procedure is based on the fact that classical electrodynamics allows unspecified radiation in coming from the far past. As long as the phase averages of **E** and **B** are zero, this radiation will not affect the problem at hand. Maxwell's formulation of electromagnetism sets this radiation equal to 0, i.e., $<\mathbf{E}> = <\mathbf{B}> = 0$, and $<\mathbf{E}^2> = <\mathbf{B}^2> = 0$. If instead, we do not require $<\mathbf{E}^2> = <\mathbf{B}^2> = 0$, then we get stochastic electrodynamics. The primary characteristic of this procedure is the assumed existence of a non-zero energy density even when the temperature of the blackbody cavity is zero.

Before we begin the calculation of $\rho(\omega,T)$, we give an overview. First we calculate the zero point spectrum, $\rho(\omega)\equiv\rho(\omega,T=0)$, under the assumption of Lorentz invariance of the spectrum (equation 5.8). Lorentz invariance is necessary as otherwise we should be able to find a preferred inertial frame with respect to the assumed universal zero point spectrum. In the next step, using a model of an ideal gas in equilibrium with radiation in a cavity we calculate the momentum balance equation (5.16). Equation (5.16) involves averages of two statistical parameters, Δ and P, related to the impulse received by a gas particle from the thermal bath and the radiation damping, respectively. Thus, we proceed to calculate these two parameters

(5.30 and 5.23, respectively). Substituting these results into (5.16) yields a differential equation for $\rho(\omega,T)$, (5.32). This can then be solved to give (4.17). In section two of this chapter, we discuss several theoretical points about the derivation and some adjustments to statistical mechanics alluded to previously.

Derivation of the Zero Point Spectrum

We derive, under the assumption of Lorentz invariance, the spectral density of the zero point radiation, $\rho(\omega,T=0) \equiv \rho(\omega)$. The derivation follows Boyer [2,3].

We begin with the expression of the Electric and Magnetic fields for plane waves of arbitrary phase:

$$\mathbf{E}(\mathbf{x},t) = \mathrm{Re} \sum_{\sigma=1}^{2} \int \epsilon(\mathbf{k},\sigma)h(\omega_k)\exp\{i\omega_k t - i\mathbf{k}\cdot\mathbf{x} - i\theta(\mathbf{k},\sigma)\}d^3\mathbf{k} \ ;$$

$$\mathbf{B}(\mathbf{x},t) = \mathrm{Re} \sum_{\sigma=1}^{2} \int \frac{\mathbf{k} \times \epsilon(\mathbf{k},\sigma)}{k} h(\omega_k)\exp\{i\omega_k t - i\mathbf{k}\cdot\mathbf{x} - i\theta(\mathbf{k},\sigma)\}d^3\mathbf{k} \ , \tag{5.1}$$

where the equations have been expressed in Gaussian units. The integrals are taken over all $\mathbf{k}$ space, i.e., $-\infty < k_r < \infty$, for each component (r=x,y,z). Strictly speaking, for plane waves in a cubic box, the wave vectors are discrete, i.e., $k_r = \frac{2\pi n_r}{L}$, where n_r is an integer and L is the length of a side of of the box. But for large L we can treat the wave vector as continuous and replace the usual series, summed over $\mathbf{k}$, with an integral over **k** . The fluctuating phase, $\theta(\mathbf{k},\sigma)$, has been introduced in accordance with the procedures of Einstein and Hopf [45,46]. The Fourier amplitudes, $h(\omega_k)$, are assumed to depend on, $\omega_k = ck = c|\mathbf{k}|$, alone, since space is assumed to be isotropic. The subscript "k" on ω_k serves as a reminder of the relationship between ω and k, and is needed to indicate the different frequency variables associated with different **k** spaces (see below). When there is no ambiguity we will often drop the k subscript from ω. The polarization vectors, $\epsilon(\mathbf{k},\sigma)$, and the wave vector, **k**, satisfy the following orthogonality conditions:

$$\epsilon(\mathbf{k},\sigma)\cdot\mathbf{k} = 0 \; ; \; \epsilon(\mathbf{k},\sigma)\cdot\epsilon(\mathbf{k},\sigma') = \delta_{\sigma\sigma'} \; . \qquad (5.2)$$

The vectors $\epsilon(\mathbf{k},1)$, $\epsilon(\mathbf{k},2)$ and $\frac{\mathbf{k}}{k}$ form an orthonormal righthanded triple in that order. We have used "X" to denote the cross product. (See Jackson [44] p. 269 *et seq* for details.)

Next we calculate the average energy density of the field, where the average is calculated over all phase angles, $\theta(\mathbf{k},\sigma)$:

$$\frac{1}{8\pi}\langle\mathbf{E}^2 + \mathbf{B}^2\rangle$$

$$= \frac{2}{8\pi} \sum_{\sigma_1=1}^{2} \sum_{\sigma_2=1}^{2} \int\int \{\epsilon(\mathbf{k}_1,\sigma_1)\cdot\epsilon(\mathbf{k}_2,\sigma_2)\; h(\omega_{k_1})\; h(\omega_{k_2})\}$$

$$\cdot\{\langle\cos[\omega_{k_1}t - \mathbf{k}_1\cdot\mathbf{x} - \theta(\mathbf{k}_1,\sigma_1)]$$

$$\cdot\cos[\omega_{k_2}t - \mathbf{k}_2\cdot\mathbf{x} - \theta(\mathbf{k}_2,\sigma_2)]\rangle\}\; d^3\mathbf{k}_1 d^3\mathbf{k}_2 \; .$$

The phase average inside the integral, indicated by the <> symbols, is just $\frac{1}{2}\,\delta^3(\mathbf{k}_1 - \mathbf{k}_2)\delta_{\sigma_1\sigma_2}$, since the average value of $\cos^2\theta$ over one period is $\frac{1}{2}$. The factor of $\delta^3(\mathbf{k}_1 - \mathbf{k}_2)$ is the Dirac delta function, which has the property that integration with respect to $\mathbf{k}_2$ of the Dirac delta function simply changes all remaining $\mathbf{k}_2$ variables (inside the integral) to $\mathbf{k}_1$. The factor $\delta_{\sigma_1\sigma_2}$, is the Kronecker delta. The factor of 2 in front of the summation sign arises since $\langle\mathbf{E}^2 + \mathbf{B}^2\rangle = 2\langle\mathbf{E}^2\rangle$. Thus,

$$\frac{1}{8\pi}\langle\mathbf{E}^2 + \mathbf{B}^2\rangle = \frac{2}{8\pi} \sum_{\sigma_1=1}^{2} \sum_{\sigma_2=1}^{2} \iint \{\epsilon(\mathbf{k}_1,\sigma_1)\cdot\epsilon(\mathbf{k}_2,\sigma_2)$$

$$\cdot h(\omega_{k_1})\, h(\omega_{k_2})\, \{\tfrac{1}{2}\, \delta^3(\mathbf{k}_1 - \mathbf{k}_2)\delta_{\sigma_1\sigma_2}\} d^3\mathbf{k}_1 d^3\mathbf{k}_2 \,.$$

Performing the first integration and the sums, we get,

$$\frac{1}{8\pi}\langle\mathbf{E}^2 + \mathbf{B}^2\rangle = \frac{1}{4\pi} \int h^2(\omega_k)\, d^3\mathbf{k} \,,$$

where the subscript 1 has been dropped. Switching to spherical k space, and dropping the primes,

$$\frac{1}{8\pi}\langle\mathbf{E}^2 + \mathbf{B}^2\rangle = \frac{1}{4\pi} \int_0^\infty 4\pi k^2 h(\omega_k) h(\omega_k) dk = \int_0^\infty \frac{\omega^2}{c^3} h^2(\omega) d\omega \,,$$

where we have used the relation $\omega^2 = c^2k^2$ in the last step. Comparing with the spectral density, defined by,

$$\frac{1}{8\pi}\langle\mathbf{E}^2 + \mathbf{B}^2\rangle \equiv \int_0^\infty \rho(\omega) d\omega = \int_0^\infty \frac{\omega^2}{c^3}\, h^2(\omega) d\omega \,, \qquad (5.3)$$

we arrive at

$$\rho(\omega) = \frac{\omega^2}{c^3}\, h^2(\omega) \,, \qquad (5.4)$$

as the relationship between $\rho(\omega)$ and $h(\omega)$. Of course, equivalence of the integrals in (5.3) does not imply (5.4). However, we can rectify this objection, by doing the calculation on an arbitrary frequency range.

Next we perform a Lorentz transformation on $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ for uniform motion, v, along the x-axis. Starting with (5.1) we used the Lorentz transformation for the electric and magnetic fields (see [44]),

$$\begin{aligned} E_x' &= E_x & B_x' &= B_x \\ E_y' &= \gamma(E_y - \tfrac{v}{c}B_z) & B_y' &= \gamma(B_y + \tfrac{v}{c}E_z) \qquad (5.5) \\ E_z' &= \gamma(E_z + \tfrac{v}{c}B_y) & B_z' &= \gamma(B_z - \tfrac{v}{c}E_y) \ , \end{aligned}$$

to calculate $\frac{1}{8\pi}\langle \mathbf{E}'^2 + \mathbf{B}'^2 \rangle$ in the K frame (we will deviate slightly from our previous convention and use K for the rest frame, and K' for a moving frame with respect to K) as follows:

$$\begin{aligned} \mathbf{E}'(\acute{\mathbf{x}},\acute{t}) = \mathrm{Re} \sum_{\sigma=1}^{2} \int \Big\{ \mathbf{i}\,\epsilon_x(\mathbf{k},\sigma) + \mathbf{j}\gamma\Big[\epsilon_y(\mathbf{k},\sigma) - \frac{v}{c}\frac{(\mathbf{k}\times\epsilon(\mathbf{k},\sigma))_z}{k}\Big] \\ + \mathbf{k}\gamma\Big[\epsilon_z(\mathbf{k},\sigma) + \frac{v}{c}\frac{(\mathbf{k}\times\epsilon(\mathbf{k},\sigma))_y}{k}\Big]\Big\} h(\omega_k) \\ \cdot \exp[i\acute{\omega}\acute{t} - i\acute{\mathbf{k}}\cdot\acute{\mathbf{x}} - i\theta(\mathbf{k},\sigma)]\, d^3\mathbf{k} \ , \end{aligned}$$

and,

$$B'(\acute{x},\acute{t}) = \mathrm{Re}\sum_{\sigma=1}^{2}\int \{\mathbf{i}\epsilon_x(\mathbf{k},\sigma) + j\gamma[\frac{v}{c}\epsilon_z(\mathbf{k},\sigma) + \frac{(\mathbf{k}X\epsilon(\mathbf{k},\sigma))_y}{k}]$$

$$+ \mathbf{k}\gamma[\frac{(\mathbf{k}X\epsilon(\mathbf{k},\sigma))_z}{k} - \frac{v}{c}\epsilon_y(\mathbf{k},\sigma)]\}h(\omega_k)$$

$$\cdot\exp[i\acute{\omega}\acute{t} - i\acute{\mathbf{k}}\cdot\acute{\mathbf{x}} - i\theta(\mathbf{k},\sigma)]\,d^3\mathbf{k}\ ,$$

where we have used the fact that plane waves transform into plane waves with phase invariance (see [44], p. 521). In addition, we note that the '**k**' appearing in front of the third term in each integral is the unit vector in the z-direction, and has nothing to do with the wave vector also denoted '**k**'. We also point out that here and elsewhere we have abused notation by denoting the components of the polarization and wave vector with x,y,z. We have done this to avoid later confusion that would arise if numerical subscripts had been used. **E**' and **B**' are functions of $\acute{x}$ and $\acute{t}$ because $\acute{x}$ and $\acute{t}$ appear in the phase of the exponential. As well, the relationship between $(\acute{\omega},\acute{k})$ and (ω,k) is,

$$\acute{k}_x = \gamma(k_x - \frac{v\omega}{c^2}) \ ; \ \acute{\omega}_{\acute{k}} = \gamma(\omega_k - vk_x)$$

$$\acute{k}_y = k_y \ ; \ \acute{k}_z = k_z \ ; \ d^3\acute{\mathbf{k}} = d^3\mathbf{k}\gamma(1 - \frac{vk_x}{\omega}) \ . \qquad (5.6)$$

As usual, $\gamma \equiv (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$. Calculating the mean squares, $<\mathbf{E}'^2>$ and $<B'^2>$, we have:

$$\frac{1}{8\pi}\langle \mathbf{E}'^2 + \mathbf{B}'^2\rangle = \frac{2}{8\pi}\sum_{\sigma_2=1}^{2}\sum_{\sigma_1=1}^{2}\iint \{\epsilon_x(\mathbf{k}_1,\sigma_1)\cdot\epsilon_x(\mathbf{k}_2,\sigma_2)$$

$$+ \gamma^2[\epsilon_y(\mathbf{k}_1,\sigma_1) - \frac{v}{c}\frac{(\mathbf{k}_1 X\epsilon(\mathbf{k}_1,\sigma_1))_z}{k_1}][\epsilon_y(\mathbf{k}_2,\sigma_2) - \frac{v}{c}\frac{(\mathbf{k}_2 X\epsilon(\mathbf{k}_2,\sigma_2))_z}{k_2}]$$

$$+ \gamma^2[\epsilon_z(\mathbf{k}_1,\sigma_1) + \frac{v}{c}\frac{(\mathbf{k}_1 X\epsilon(\mathbf{k}_1,\sigma_1))_y}{k_1}][\epsilon_z(\mathbf{k}_2,\sigma_2)$$

$$+ \frac{v}{c}\frac{(\mathbf{k}_2 X\epsilon(\mathbf{k}_2,\sigma_2))_y}{k_2}]\}\; h(\omega_{k_1})\; h(\omega_{k_2})\; \frac{1}{2}\delta^3(\mathbf{k}_1 - \mathbf{k}_2)\delta_{\sigma_1\sigma_2}d^3\mathbf{k}_1 d^3\mathbf{k}_2 ,$$

where we have expressed the phase average in terms of Dirac and Kronecker delta functions. Calculating the first integral, the first sum, and dropping the remaining subscript "2", we have,

$$= \frac{1}{8\pi}\sum_{\sigma=1}^{2}\int\{\epsilon_x^2 + \gamma^2[\epsilon_y - \frac{v}{c}\frac{(k_x\epsilon_y - k_y\epsilon_x)}{k}]^2$$

$$+ \gamma^2[\epsilon_z + \frac{v}{c}\frac{(k_z\epsilon_x - k_x\epsilon_z)}{k}]^2\}h^2(\omega_k)d^3\mathbf{k} .$$

$$= \frac{2}{8\pi}\int\{\epsilon_x^2 + \gamma^2[\epsilon_y^2 + \frac{v^2}{c^2k^2}(k_x^2\epsilon_y^2 + k_y^2\epsilon_x^2 - 2k_x\epsilon_y k_y\epsilon_x)$$

$$- \frac{2v\epsilon_y}{kc}(k_x\epsilon_y - k_y\epsilon_x) + \epsilon_z^2 + \frac{v^2}{c^2k^2}(k_z^2\epsilon_x^2 + k_x^2\epsilon_z^2 - 2k_z\epsilon_x k_x\epsilon_z)$$

$$+ \frac{2v\epsilon_z}{kc}(k_z\epsilon_x - k_x\epsilon_z)]\}h^2(\omega_k)d^3\mathbf{k}$$

$$= \frac{1}{4\pi} \int \{ \epsilon_x^2 + \gamma^2 [\epsilon_y^2 + \frac{v^2}{c^2 k^2} (\epsilon_x^2 k^2 + k_x^2 - 2 k_x \epsilon_x \mathbf{k} \cdot \epsilon)$$

$$+ \epsilon_z^2 - \frac{2v}{ck} (k_x)] \} \, h^2 (\omega_k) d^3 \mathbf{k}$$

$$= \frac{1}{4\pi} \int (1 - \frac{k_x v}{\omega})^2 \gamma^2 \, h^2 (\omega) \, d^3 \mathbf{k} = \frac{1}{4\pi} \int h^2 (\acute{\omega}) \, d^3 \acute{\mathbf{k}} \, ,$$

where the last equation follows by form invariance, and we have dropped the subscript, "k", on ω. Lorentz invariance of the zero point spectrum is defined to mean that the energy density contained between two arbitrary frequencies $\omega_k = a$ and $\omega_k = b$ is the same for both the "at rest" spectrum and the transformed spectrum, where the measurement is done in the K frame. The numbers a and b are to be held fixed for both measurements. Thus, if we use (5.6) to transform the $d^3 \acute{\mathbf{k}}$ on the right, we get:

$$\int_{\omega_k = a}^{\omega_k = b} (1 - \frac{k_x v}{\omega})^2 \gamma^2 h^2 (\omega) d^3 \mathbf{k} = \int_{\omega_k = a}^{\omega_k = b} (1 - \frac{k_x v}{\omega}) \gamma h^2 (\acute{\omega}) d^3 \mathbf{k} \, .$$

Since a and b are arbitrary, this equation implies that

$$(1 - \frac{k_x v}{\omega}) \gamma \, h^2 (\omega) = h^2 (\acute{\omega}) \, .$$

Thus, $h^2(\omega) \equiv 0$, or $h^2(\omega) = \text{constant} \cdot \omega$. The first possibility is the usual zero point energy, $\rho(\omega) \equiv 0$. In order that the present work should give the usual answers, i.e. agrees with experiment, the constant needs to be chosen so that the zero point energy per normal mode is $\frac{h\omega}{4\pi}$, i.e., equal to the ground state energy of a quantum mechanical oscillator. Then using

(4.14) and (5.4),

$$\rho(\omega) = \frac{\omega^2}{\pi^2 c^3} \pi^2 h^2(\omega) \quad \Rightarrow \quad h^2(\omega)\pi^2 = \frac{h\omega}{4\pi} . \qquad (5.7)$$

So the constant of proportionality between $h^2(\omega)$ and ω is $\frac{h}{4\pi^3}$ and the zero point spectrum is:

$$\rho(\omega) = \frac{h\omega^3}{4\pi^3 c^3} . \qquad (5.8)$$

Our next task is to derive the differential equation governing the spectral density $\rho(\omega,T)$ at equilibrium for any temperature, T. To do this we follow the classical procedure worked out by Einstein and Hopf [45,46] and modified by Boyer [2,3] . They consider particles confined to a box in presence of a radiation field. (In many treatments, the cavity contains radiation only. The idea of including both particles and radiation is to take care of the objection, raised by these authors, to the radiation only treatments, that the radiation and the particles in the walls of the container can not come to equilibrium.) They next consider a particular particle, of mass m, assumed to be traveling in the x-direction, $v=v_x$. To model the interaction of the particle with the radiation, the particle is assumed to consist of a sperical shell containing a non-relativistic dipole oscillator with oscillations of frequency, ω_0, in the z direction. It is convenient to assume further, that one pole of the dipole is much more massive than the other, so that the massive pole is essentially fixed. This model allows both classical and statistical calculations (free translation of entire particles) and dipole interactions with the electromagnetic field (radiation).

Assuming the particles to be non-interacting (with each other), and that collisions with the walls and other particles are elastic, (ideal gas), there are two forces acting on the particles from the radiation:

1) In a short time interval, τ, there is an impulse associated with the random fluctuations, Δ, of the fields in the time τ.

2) The velocity dependent resistive force F_x . If the velocity of the particle is small compared to c, then F_x can be assumed to be linear in v, so that,

$$F_x = -Pv \; . \tag{5.9}$$

where P is a function of the spectral density that needs to be determined. (Note that we are introducing a new meaning for the symbol "P"; the "P" in (5.9) is unrelated to any of the parameters previously denoted by "P", e.g., the pressure or the generator of space translations.) Then if τ is small, the resistive impulse can be taken to be $F_x\tau = -Pv\tau$. In the analysis that follows, no attempt is made to explain in detail exactly how the particle and radiation field exchange energy. We assume that force 1 causes absorption of energy by the particle and that force 2 causes the particle to lose energy. If the particle's velocity at time t is v(t) (<<c, where c is the speed of light), then (by conservation of momentum),

$$mv(t+\tau) = mv(t) + \Delta - Pv\tau \; . \tag{5.10}$$

For the purpose of the following calculations it is not necessary to assume $v=v_x$. The particle can have arbitrary direction since the radiation is assumed to be isotropic and the gas in equilibrium. But if the gas is in equilibrium, $<(mv(t))^2> = <(mv(t+\tau))^2>$, where the average is taken over all particles, so that, with the help of (5.10),

$$<(mv(t+\tau))^2> = <(mv(t) + \Delta - Pv\tau)^2>$$

$$= <(mv(t))^2> + <\Delta^2> + P^2\tau^2<v^2> - 2P\tau<\Delta v> - 2mP\tau<v^2> + 2m<v\Delta> ,$$

which implies that,

$$0 = <\Delta^2> + P^2\tau^2<v^2> - 2P\tau<\Delta v> - 2mP\tau<v^2> + 2m<v\Delta> .$$

Since $v(t)$ is fixed while Δ is random during the time interval t to $t+\tau$, $<v\Delta> = 0$. We can choose $m >> P\tau$ by making τ small enough and m large enough so that $P^2\tau^2<v^2>$ can be omitted in comparison to $mP\tau<v^2>$. Thus we remain with,

$$<\Delta^2> = 2mP\tau<v^2> , \qquad (5.11)$$

which is the Einstein - Hopf equation.

When the classical values for $<\Delta^2>$ and P (calculated below) are substituted into this equation together with the equipartition relation, $\frac{1}{2}m<v^2> = \frac{1}{2}kT$, (for one dimension) then we get, $\rho(\omega,T) = \frac{\omega^2}{c^3\pi^2}kT$, the Rayleigh - Jeans law, (4.15). (Recall that $\omega = 2\pi\nu$). Up to this point we have assumed that the particles collide elastically with the walls. In fact this cannot possibly be the case, since on striking the wall a particle undergoes an acceleration and therefore the charges forming the dipole in each particle must radiate. (Recall that we are assuming the particles interact with the radiation field.) The radiative loss is quickly restored, however, as the particle moves toward the middle of the cavity. As the temperature, T->0, something quite different happens. In this limit the particles gradually lose energy (through collisions with the walls) to the zero point radiation field. In this limit there can be no velocity dependent forces, i.e. $P(T=0) = 0$, since the zero point spectrum (5.8) is Lorentz invariant. To take into account the interaction between the particles and the walls as T->0 we now modify the derivation of (5.11) following Boyer [2,3]. As T->0 ,

$$mv(t+\tau) = mv(t) + \Delta + J \,, \tag{5.12}$$

where J is the impulse received in time τ from a collision with the wall, if one occurs for this particle. Averaging the square over all particles gives:

$$<(mv(t+\tau))^2> = <(mv(t) + \Delta + J)^2> = <(mv(t))^2> \,,$$

if the system is in a steady state. Then,

$$0 = <\Delta^2> + <J^2> + 2m<v\Delta> + 2m<vJ> + 2<\Delta J> \,.$$

As usual $<\Delta J>$ and $<\Delta v>$ are 0 , but $<vJ> \neq 0$, since vJ is always negative for collisions with the wall. Boyer argues that $<J^2>$ can be neglected since $\Delta << mv_t$ so that J has to be small. This is only one possibility, i.e., $J \simeq -\frac{1}{2}\frac{\Delta^2}{mv}$, or $J \simeq \{2mv - \frac{1}{2}\frac{\Delta^2}{mv}\}$. In the second case J is much larger than Δ

and should not be neglected. However, this second possibility corresponds to almost every particle having a collision with a wall in the time τ . This cannot happen because we are explicitly assuming that τ is so small that most particles do not strike the walls. Thus, we can ignore the second possibility. Ignoring $\langle J^2\rangle$, we have,

$$\langle\Delta^2\rangle_{T=0} + 2m\langle vJ\rangle_{T=0} = 0 \, . \tag{5.13}$$

Repeating the calculation for $T \neq 0$, we include the velocity dependent resistive force:

$$mv(t+\tau) = mv(t) + \Delta - Pv(t)\tau + J \, .$$

Squaring, averaging over all particles, and using the equilibrium condition, $\langle(mv(t+\tau))^2\rangle = \langle(mv(t))^2\rangle$, we get:

$$0 = \langle\Delta^2\rangle + P^2\tau^2\langle v^2\rangle + \langle J^2\rangle + 2\langle\Delta J\rangle + 2m\langle Jv\rangle$$

$$- 2P\tau\langle vJ\rangle - 2P\tau\langle\Delta v\rangle - 2P\tau m\langle v^2\rangle + 2m\langle v\Delta\rangle \, .$$

As usual, $\langle\Delta J\rangle = \langle\Delta v\rangle = 0$, and we can neglect $\langle J^2\rangle$ and $P\tau$ compared to m and and mv respectively. Hence,

$$0 = \langle\Delta^2\rangle + 2m\langle Jv\rangle_T - 2P\tau m\langle v^2\rangle \, . \tag{5.14}$$

The term $2m\langle Jv\rangle_T$ is just equal to $2m\langle vJ\rangle_{T=0}$, since at $T\neq 0$, the walls are in thermal equilibrium with the gas, so that only the zero point energy contributes to this term. Substituting (5.13) into (5.14);

$$\langle\Delta^2\rangle - \langle\Delta^2\rangle_{T=0} = 2P\tau m\langle v^2\rangle \, . \tag{5.15}$$

Equation (5.15) is the zero-point modification of the Einstein - Hopf equation, (5.11).

If we assume otherwise elastic collisions at the walls, the average power, Q, radiated at the walls during the time interval τ, is given by,

$$\tau Q = \frac{1}{2}m<(v(t+\tau))^2> - \frac{1}{2}m<(v(t))^2> ,$$

where the average is taken over all particles. Using (5.10) we have,

$$\tau Q = \frac{1}{2}m<(v(t) - \frac{Pv\tau}{m} + \frac{\Delta}{m})^2> - \frac{1}{2}m<(v(t))^2> .$$
$$= \frac{1}{2}m\{- \frac{2P\tau}{m}<(v(t))^2> + \frac{<\Delta^2>}{m^2}\} ,$$

where we have neglected terms of order τ^2 . As we will show presently, (equations 5.23 and 5.30), P and $<\Delta^2>$ are inversely proportional to m (because of the dependence on Γ). Thus, Q will be negligible only in the limit $m \to \infty$. Therefore in this limit we expect the equipartition theorem to be valid. This is incidently, the same limit in which the Rayleigh - Jeans law holds, since for a harmonic oscillator, $\omega^2 = \frac{k}{m}$, and as $m \to \infty$, $\omega \to 0$.

Thus if we assume the particles are massive enough, we can replace $<v^2>$ in (5.15): $\frac{1}{2}m<v^2> = \frac{1}{2}kT$, where $v=v_x$. So (5.15) becomes,

$$<\Delta^2> - <\Delta^2>_{T=0} = 2P\tau kT . \qquad (5.16)$$

To make further progress we need to calculate $<\Delta^2>$ and P in (5.16). This was done by Einstein and Hopf [45,46], but we follow Boyer [2], who uses the same procedure in modern notation. We begin by calculating P in (5.16).

The equation of motion for a non-relativistic dipole oscillator of frequency ω_0, aligned in the z-direction is given by the Abrahms - Lorentz equation, (see [44], chapter 17) ,

$$\frac{d^2p}{dt^2} - \Gamma\frac{d^3p}{dt^3} + \omega_0^2 p = \frac{3}{2}\Gamma c^3 E_z \ , \tag{5.17}$$

where p is the dipole moment, p = ez , for a dipole consisting of two point charges, and $\Gamma = \frac{2}{3}\frac{e^2}{mc^3}$, is the radiation damping constant. (e is the charge on an electron; m is the mass of the particle, c is the speed of light, z is the displacement between the charges of the dipole, and E_z is the z component of the electric field of the radiation in the cavity.) Recalling that our model of a particle consist of a massive outer shell traveling in the x direction and an oscillator inside aligned in the z direction, we treat the rest frame of the particle as the K' frame and the rest frame of the box as the K frame. Then although (5.17) is not Lorentz invariant it will be valid in the K' frame, in primed coordinates, i.e.,

$$\frac{d^2p'}{dt'^2} - \Gamma\frac{d^3p'}{dt'^3} + \acute{\omega}_0^2 p' = \frac{3}{2}\Gamma c^3 E_z' \ , \tag{5.18}$$

Using (5.5) and (5.6), (5.18) can be solved to yield,

$$p' = \sum_{\sigma=1}^{2} \int \{\frac{3c^3}{2\acute{\omega}^3}\gamma[\epsilon_z(\mathbf{k},\sigma) + \frac{v}{c}\frac{(\mathbf{k}\mathbf{X}\epsilon)_y}{k}]h(\omega_k,T)\ \sin\alpha(\acute{\omega})$$

$$\cdot\cos[\acute{\omega}\acute{t} - \acute{\mathbf{k}}\cdot\acute{\mathbf{x}} - \alpha(\acute{\omega}) - \theta(\mathbf{k},\sigma)]\}d^3\mathbf{k} \ , \tag{5.19}$$

where $\cot\alpha(\acute{\omega}) \equiv \frac{(\acute{\omega}_0^2 - \acute{\omega}^2)}{\Gamma\acute{\omega}^3}$, and the amplitudes $h(\omega_k,T) = h(\omega_k)$ at T=0 : Equation (5.19) expresses the dipole moment, p', as observed from the K frame. Some of the variables have been left primed for conveneince. Note that we would not expect p' observed in the K frame, to satisfy (5.18) with the primes removed, since (5.18) is not Lorentz invariant.

The force on the particle arising from the action of the radiation on the dipole in the K' frame is (see [44]),

$$F'_x = \frac{\partial E'_x}{\partial z'} p' - \frac{1}{c} \frac{\partial p'}{\partial t'} B'_y \; . \qquad (5.20)$$

Substituting (5.19) and (5.5) into (5.20) we get:

$$
\begin{aligned}
\langle F'_x\rangle = \sum_{\sigma_1=1}^{2}\sum_{\sigma_2=1}^{2}\iint &\{k'_{1z}\epsilon_x(\mathbf{k}_1,\sigma_1)h(\omega_{k_1},T)\,\frac{3c^3}{2\omega'^3_{k_2}}\gamma \\
&\cdot[\epsilon_z(\mathbf{k}_2,\sigma_2)+\frac{v}{c}\,\frac{(\mathbf{k}X\epsilon)_y}{k}]\}\ \{\sin\alpha(\omega'_{k_2})h(\omega_{k_2},T) \\
&\cdot\langle\cos[\omega'_{k_2}t'-\mathbf{k}'_2\cdot\mathbf{x}'-\alpha(\omega'_{k_2})-\theta(\mathbf{k}_2,\sigma_2)] \\
&\qquad\cdot\sin[\omega'_{k_1}t'-\mathbf{k}'_1\cdot\mathbf{x}'-\theta(\mathbf{k}_1,\sigma_1)]\rangle d^3\mathbf{k}_1d^3\mathbf{k}_2 \\
+\frac{1}{c}\sum_{\sigma_1=1}^{2}\sum_{\sigma_2=1}^{2}\iint &\{[\frac{(\mathbf{k}_1X\epsilon(\mathbf{k}_1,\sigma_1))_y}{k_1}+\frac{v}{c}\epsilon_z(\mathbf{k}_1,\sigma_1)]\gamma h(\omega_{k_1},T)\frac{3c^3}{2\omega'^2_{k_2}}\gamma \\
&\cdot[\epsilon_z(\mathbf{k}_2,\sigma_2)+\frac{v}{c}\,\frac{(\mathbf{k}_2X\epsilon(\mathbf{k}_2,\sigma_2))_y}{k_2}]\}h(\omega_{k_2},T)\sin\alpha(\omega'_{k_2}) \\
&\cdot\langle\sin[\omega'_{k_2}t'-\mathbf{k}'_2\cdot\mathbf{x}'-\alpha(\omega'_{k_2})-\theta(\mathbf{k}_2,\sigma_2)] \\
&\qquad\cdot\cos[\omega'_{k_1}t'-\mathbf{k}'_1\cdot\mathbf{x}'-\theta(\mathbf{k}_1,\sigma_1)]\rangle d^3\mathbf{k}_1d^3\mathbf{k}_2\ .
\end{aligned}
$$

We can drop the primes in the calculation of phase averages, by (5.6), and since the phases are invariant. Then the average over phases in the first term is $\frac{1}{2}\sin\alpha(\omega'_{k_2})\delta^3(\mathbf{k}_1-\mathbf{k}_2)\delta_{\sigma_1\sigma_2}$, while the average in the second term is the negative of this. Performing the first integration, the sums (in both terms), and dropping the subscript "2", we get:

$$\langle F'_x\rangle = \frac{1}{2}\sum_{\sigma=1}^{2}\int k_z\epsilon_x h^2(\omega_k,T)\frac{3c^3}{2\hat{\omega}_k^3}\gamma\,[\epsilon_z + \frac{v}{c}\frac{(\mathbf{kX}\epsilon(\mathbf{k},\sigma))_y}{k}]\sin^2\alpha(\hat{\omega}_k)d^3\mathbf{k}$$

$$-\frac{1}{2}\sum_{\sigma=1}^{2}\int \gamma^2h^2(\omega_k,T)\frac{3c^2}{2\hat{\omega}_k^2}[\frac{(\mathbf{kX}\epsilon(\mathbf{k},\sigma))_y}{k} + \frac{v}{c}\epsilon_z(\mathbf{k},\sigma)]$$

$$\cdot[\epsilon_z(\mathbf{k},\sigma) + \frac{v}{c}\frac{(\mathbf{kX}\epsilon(\mathbf{k},\sigma))_y}{k}]\sin^2\alpha(\hat{\omega}_k)d^3\mathbf{k}\;.$$

Next, we sum over the polarizations recalling that $\epsilon(\mathbf{k},1)$, $\epsilon(\mathbf{k},2)$, and $\mathbf{k}$ form a right handed triple in that order so that the components of $\epsilon(\mathbf{k},2)$ can be computed in terms of the components of $\epsilon(\mathbf{k},1)$ and $\mathbf{k}$, i.e., $\epsilon(\mathbf{k},2)_i = \frac{(\mathbf{kX}\epsilon(\mathbf{k},1))_i}{k}$. This yields:

$$\langle F'_x\rangle = \frac{1}{2}\int h^2(\omega_k,T)\frac{3c^2}{2\hat{\omega}_k^2}\cdot[A]\sin^2\alpha(\hat{\omega}_k)d^3\mathbf{k}\;, \tag{5.21}$$

where $A = \frac{c}{\hat{\omega}_k}k_z\gamma\{\epsilon_x[\epsilon_z + \frac{v}{c}\frac{(k_z\epsilon_x - k_x\epsilon_z)}{k}]$

$$+ [\frac{(k_y\epsilon_z - k_z\epsilon_y)}{k}][\frac{(k_x\epsilon_y - k_y\epsilon_x)}{k} - \frac{v}{c}\epsilon_y]\}$$

$$-\gamma^2\{[\frac{(k_z\epsilon_x - k_x\epsilon_z)}{k} + \frac{v}{c}\epsilon_z][\epsilon_z + \frac{v}{c}\frac{(k_z\epsilon_x - k_x\epsilon_z)}{k}]$$

$$+ [\frac{(k_x\epsilon_y - k_y\epsilon_x)}{k} - \frac{v}{c}\epsilon_y][-\epsilon_y + \frac{v}{c}\frac{(k_x\epsilon_y - k_y\epsilon_x)}{k}]\}\;.$$

This expression can be simplified using the conditions (5.2) to give:

$$A = \frac{c}{\acute{\omega}_k}\acute{k}_z\gamma[\frac{v}{c}\frac{k_z}{k} - \frac{k_z k_x}{k^2}] - \gamma^2[\frac{v}{c} - \frac{k_x}{k} - \frac{v^2}{c^2}\frac{k_x}{k} + \frac{v}{c}\frac{k_x^2}{k^2}] .$$

Substituting this last expression for A into (5.21), we have:

$$<F'_x> = \frac{1}{2}\int h^2(\omega_k,T)\frac{3c^2}{2\acute{\omega}_k^2}\sin^2\alpha(\acute{\omega}_k)$$

$$\cdot\{\frac{c}{\acute{\omega}}\acute{k}_z\gamma[\frac{v}{c}\frac{k_z}{k} - \frac{k_z k_x}{k^2}] - \gamma^2[\frac{v}{c} - \frac{k_x}{k} - \frac{v^2}{c^2}\frac{k_x}{k} + \frac{v}{c}\frac{k_x^2}{k^2}]\}d^3\mathbf{k} .$$

It remains to transform all the variables into primed coordinates using (5.6). Since the form of $\rho(\omega,T) = \frac{\omega^2}{c^3}h^2(\omega,T)$ is not yet known, we express it as a power series in $\frac{v}{c}$ (to first order) as follows:

$$\rho(\omega,T) \simeq \rho(\acute{\omega},T) + \frac{\partial\rho(\acute{\omega},T)}{\partial\acute{\omega}}(\omega-\acute{\omega}) + \ldots$$

$$\simeq \rho(\acute{\omega},T) + \frac{v}{c}\frac{\acute{k}_x}{\acute{k}}\acute{\omega}\frac{\partial\rho(\acute{\omega},T)}{\partial\acute{\omega}} + \ldots$$

Then, dropping the subscript k on ω_k,

$$<F'_x> = \frac{3}{4}\int\frac{c^2}{\acute{\omega}^2}[\rho(\acute{\omega},T) + \acute{\omega}\frac{v}{c}\frac{\acute{k}_x}{\acute{k}}\frac{\partial\rho(\acute{\omega},T)}{\partial\acute{\omega}}][\frac{\Gamma^2\acute{\omega}^6}{\Gamma^2\acute{\omega}^6 + (\acute{\omega}_0^2-\acute{\omega}^2)^2}]$$

$$\cdot[\acute{k}_x - \frac{\acute{k}_z^2\acute{k}_x}{\acute{k}^2} - \frac{3v}{c}\frac{\acute{k}_x^2}{\acute{k}} + \frac{3v}{c}\frac{\acute{k}_z^2\acute{k}_x^2}{\acute{k}^3}]\frac{c}{\acute{k}^3}d^3\acute{k} ,$$

where we get a factor of $\frac{c}{k^2}$ from replacing $h^2(\omega_k,T)$ with $\rho(\acute{\omega}_k,T)$, and each factor in square brackets has been approximated to first order in $\frac{v}{c}$. We now change to spherical coordinates and change over to frequency space using the relation $\omega = ck$:

$$\langle F_x' \rangle = \frac{3c^2}{4} \int_0^{\pi} \int_0^{2\pi} \int_0^{\infty} [\rho(\acute{\omega},T) + \frac{v}{c}\acute{\omega}(\sin\theta\cos\phi)\frac{\partial\rho(\acute{\omega},T)}{\partial\acute{\omega}}] \cdot [\frac{\Gamma^2\acute{\omega}^4}{\Gamma^2\acute{\omega}^6 + (\acute{\omega}_0^2-\acute{\omega}^2)^2}] \{\sin\theta\cos\phi - \sin\theta\cos^2\theta\cos\phi - \frac{3v}{c}\sin^2\theta\cos^2\phi + \frac{3v}{c}\cos^2\theta\sin^2\theta\cos^2\phi\}\sin\theta d\theta d\phi d\acute{\omega} .$$

Computing the angular part to first order in $\frac{v}{c}$, we have,

$$\langle F_x' \rangle = \frac{3c^2}{4}\int_0^{\infty} - \frac{3v}{c}\cdot\frac{16\pi}{15}\cdot[\rho(\acute{\omega},T) - \frac{1}{3}\acute{\omega}\frac{\partial\rho(\acute{\omega},T)}{\partial\acute{\omega}}][\frac{\Gamma^2\acute{\omega}^4}{\Gamma^2\acute{\omega}^6 + (\acute{\omega}_0^2-\acute{\omega}^2)^2}]d\acute{\omega} .$$

To make further progress, we assume $\Gamma\omega_0 << 1$ so that the integral is sharply peaked at $\acute{\omega} = \acute{\omega}_0$. Since $\Gamma \simeq 6.3 \times 10^{-24}$ (statcoul^{2}sec/erg cm), this assumption is reasonable for a non-relativistic dipole oscillator. Changing the variable $\acute{\omega}$ to $x = \acute{\omega} - \acute{\omega}_0$, and substituting $\acute{\omega}_0$ for $\acute{\omega}$ in terms not involving $\acute{\omega} - \acute{\omega}_0$, we get:

$$\langle F_x' \rangle = \frac{-12}{5}vc\pi[\rho(\acute{\omega}_0,T) - \frac{1}{3}\acute{\omega}_0\frac{\partial\rho(\acute{\omega}_0,T)}{\partial\acute{\omega}_0}] \int_{-\acute{\omega}_0}^{\infty} \frac{\Gamma^2\acute{\omega}_0^4 \, dx}{\Gamma^2\acute{\omega}_0^6 + 4\acute{\omega}_0^2 x^2} .$$

Since the integral is sharply peaked at $x=0$, and $\omega_0 >> 0$, we can extend the lower limit of integration to $-\infty$, without significantly affecting the results. Then using the transformation, $x = \frac{1}{2}\Gamma\acute{\omega}_0^2\tan\theta$, the integral works out to $\frac{\Gamma\pi}{2}$. So we have,

$$\langle F_x' \rangle = -\frac{6}{5}vc\pi^2\Gamma[\rho(\acute{\omega}_0,T) - \frac{1}{3}\acute{\omega}_0\frac{\partial\rho(\acute{\omega}_0,T)}{\partial\acute{\omega}_0}] . \qquad (5.22)$$

Finally, we need to transform to the rest frame of the box. This involves negligible terms of order $\frac{v^2}{c^2}$, so the primes can simply be dropped, and using (5.9) we get (dropping the subscript "0"):

$$P' = \frac{6}{5}c\pi^2\Gamma[\rho(\omega,T) - \frac{1}{3}\omega\frac{\partial\rho(\omega,T)}{\partial\omega}] . \tag{5.23}$$

Our next task is to calculate $<\Delta^2>$ following Einstein and Hopf (according to Boyer). Recall that Δ is the impulse, due to the fluctuating radiation, given to the particle in a time interval τ. Then for the oscillator in the rest frame of the particle (we now let the unprimed frame be the rest frame of the particle):

$$\Delta = \int_0^\tau F_x dt = \int_0^\tau [\frac{\partial E_x}{\partial z}p - \frac{1}{c}B_y \frac{dp}{dt}]dt , \tag{5.24}$$

where we have use (5.20). Integrating by parts in the second term,

$$\int_0^\tau B_y \frac{dp}{dt}dt = [B_y p]_0^\tau - \int_0^\tau \frac{\partial B_y}{\partial t} pdt . \tag{5.25}$$

And using Maxwell's equation,

$$\nabla X E = -\frac{1}{c}\frac{\partial B}{\partial t} , \text{ or } \frac{1}{c}\frac{\partial B_y}{\partial t} = \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} ,$$

so (5.24) becomes,

$$\Delta = -\frac{1}{c}[B_y p]_0^\tau + \int_0^\tau \frac{\partial E_z}{\partial x} pdt .$$

The first term on the right can be ignored since by inspection of (5.1) and (5.19) (without primes) it is of order τ^2 . As will become clear later in the calculation, the integral is of order τ. Thus, for $\tau << 1$, we can ignore the boundary term. So we are left with,

$$\Delta = \int_0^\tau \frac{\partial E_z}{\partial x} p dt \ . \tag{5.26}$$

Since we are interested in finding averages of quantities integrated over time, $\frac{\partial E_z}{\partial x}$ and E_z must be regarded as having independent phases. This is a theorem from probability theory, proved by Einstein and Hopf [46]. Thus, rather than just differentiating E_z we have,

$$\frac{\partial E_z(\mathbf{x},t)}{\partial x} = \sum_{\sigma=1}^{2} \int k_x \epsilon_z(\mathbf{k},\sigma) h(\omega_k,T) \cos[\omega t - \mathbf{k}\cdot\mathbf{x} - \eta(\mathbf{k},\sigma)] d^3\mathbf{k}$$

Then using the solution of the dipole equation (5.17) in the rest frame of the particle,

$$p = \sum_{\sigma=1}^{2} \int \frac{3c^3}{2\omega^3} \epsilon_z(\mathbf{k},\sigma) h(\omega_k,T) \sin\alpha(\omega) \cdot \cos[\omega t - \mathbf{k}\cdot\mathbf{x} - \alpha(\omega) - \eta(\mathbf{k},\sigma)] d^3\mathbf{k} \ , \tag{5.27}$$

where $\cot\alpha(\omega) \equiv \frac{\omega_0^2-\omega^2}{\Gamma\omega^3}$, we have,

$$\Delta = \int_0^\tau \sum_{\sigma_1=1}^{2} \sum_{\sigma_2=1}^{2} \iint \epsilon_z(\mathbf{k}_1,\sigma_1) \epsilon_z(\mathbf{k}_2,\sigma_2) h(\omega_{k_1},T) h(\omega_{k_2},T) \cdot k_{1x} \sin\alpha(\omega_2) \frac{3c^3}{2\omega_2^3} \cos[\omega_1 t - \mathbf{k}_1\cdot\mathbf{x} - \eta(\mathbf{k}_1,\sigma_1)] \cdot \cos[\omega_2 t - \mathbf{k}_2\cdot\mathbf{x} - \alpha(\omega_2) - \theta(\mathbf{k}_2,\sigma_2)] d^3\mathbf{k}_1 d^3\mathbf{k}_2 dt \ . \tag{5.28}$$

If the origin of the coordinates is chosen at the location of the particle, suitably oriented, then $\mathbf{k}_1\cdot\mathbf{x} = \mathbf{k}_2\cdot\mathbf{x} = 0$. Then the time integral becomes:

$$\int_0^\tau \cos\{\omega_1 t - \eta_1\}\cos\{\omega_2 t - \alpha(\omega_2) - \theta_2\}dt$$

$$= \frac{1}{2}\int_0^\tau \cos\{\omega_1 t - \eta_1 + \omega_2 t - \alpha(\omega_2) - \theta_2\}$$

$$+ \cos\{\omega_1 t - \eta_1 - \omega_2 t + \alpha(\omega_2) + \theta_2\}dt \; .$$

Calculating the integral and doing some algebra, the righthandside is:

$$= \frac{1}{\omega_1+\omega_2}\sin\{\frac{(\omega_1+\omega_2)}{2}\tau\}\cos\{\frac{(\omega_1+\omega_2)}{2}\tau - \eta_1 - \alpha(\omega_2) - \theta_2\}$$

$$+ \frac{1}{\omega_2-\omega_1}\sin\{\frac{(\omega_2-\omega_1)}{2}\tau\}\cos\{\frac{(\omega_2-\omega_1)}{2}\tau + \eta_1 - \alpha(\omega_2) - \theta_2\} \; .$$

To compute the average $<\Delta^2>$, we need to square the above result, but we note that since the phases, $\eta(\mathbf{k}_1,\sigma_1) + \theta(\mathbf{k}_2,\sigma_2)$ and $\eta(\mathbf{k}_1,\sigma_1) - \theta(\mathbf{k}_2,\sigma_2)$, are independently correlated the cross terms will average out to 0, so we are left with the squared terms:

$<\Delta^2>$(time part)

$$= \frac{1}{(\omega_1+\omega_2)^2}\sin^2\{\frac{(\omega_1+\omega_2)}{2}\tau\}<\cos\{\frac{(\omega_1+\omega_2)}{2}\tau - \eta_1 - \alpha(\omega_2) - \theta_2\}$$

$$\cdot\cos\{\frac{(\acute{\omega}_1+\acute{\omega}_2)}{2}\tau - \acute{\eta}_1 - \alpha(\acute{\omega}_2) - \theta_2'\}>$$

$$+ \frac{1}{(\omega_1-\omega_2)^2}\sin^2\{\frac{(\omega_2-\omega_1)}{2}\tau\}<\cos\{\frac{(\omega_2-\omega_1)}{2}\tau + \eta_1 - \alpha(\omega_2) - \theta_2\}$$

$$\cdot\cos\{\frac{(\acute{\omega}_2-\acute{\omega}_1)}{2}\tau + \acute{\eta}_1 - \alpha(\acute{\omega}_2) - \theta_2'\}>$$

$$= [\frac{1}{(\omega_1+\omega_2)^2}\sin^2\{\frac{(\omega_1+\omega_2)}{2}\tau\} + \frac{1}{(\omega_1-\omega_2)^2}\sin^2\{\frac{(\omega_2-\omega_1)}{2}\tau\}] \cdot \frac{1}{2}\delta^3(\mathbf{k}_1-\acute{\mathbf{k}}_1)\delta_{\sigma_1\acute{\sigma}_1}\delta^3(\mathbf{k}_2-\acute{\mathbf{k}}_2)\delta_{\sigma_2\acute{\sigma}_2} . \quad (5.29)$$

Strictly speaking the sine terms in (5.29) should be the product of two factors: The factor given in the equation (without the square) and a similar factor with primes in it. However, the sine factors do not appear in the phase averages and integration over the primed variables will change primed variables to unprimed variables (on account of the delta functions). Thus, we can safely ignore the primes in the sine terms in (5.29). Squaring (5.28) and substituting (5.29) for the square of the time integral, we have,

$$\begin{aligned}<\Delta^2> = & \sum_{\sigma_1=1}^{2}\sum_{\sigma_2=1}^{2}\sum_{\acute{\sigma}_1=1}^{2}\sum_{\acute{\sigma}_2=1}^{2} \iiiint \epsilon_z(\mathbf{k}_1,\sigma_1)\epsilon_z(\acute{\mathbf{k}}_1,\acute{\sigma}_1)h(\omega_{k_1},T) \\ & \cdot h(\acute{\omega}_{k_1},T)k_{1x}\acute{k}_{1x}\frac{3c^3}{2\omega_2^3}\frac{3c^3}{2\acute{\omega}_2^3}\epsilon_z(\mathbf{k}_2,\sigma_2)\epsilon_z(\acute{\mathbf{k}}_2,\acute{\sigma}_2)h(\omega_{k_2},T)h(\acute{\omega}_{k_2},T) \\ & \cdot[\frac{1}{(\omega_1+\omega_2)^2}\sin^2\{\frac{(\omega_1+\omega_2)}{2}\tau\} + \frac{1}{(\omega_1-\omega_2)^2}\sin^2\{\frac{(\omega_2-\omega_1)}{2}\tau\}]\sin\alpha(\omega_2) \\ & \cdot\sin\alpha(\acute{\omega}_2)\ \frac{1}{2}\delta^3(\mathbf{k}_1-\acute{\mathbf{k}}_1)\Delta_{\sigma_1\acute{\sigma}_1}\delta^3(\mathbf{k}_2-\acute{\mathbf{k}}_2)\Delta_{\sigma_2\acute{\sigma}_2}d^3\mathbf{k}_1d^3\mathbf{k}_2d^3\acute{\mathbf{k}}_1d^3\acute{\mathbf{k}}_2 .\end{aligned}$$

Carrying out the integrations over primed variables,

$$<\Delta^2> \; = \; \frac{1}{2} \sum_{\sigma_1=1}^{2} \sum_{\sigma_2=1}^{2} \int\!\!\int \frac{9c^6}{4\omega_2^6} \, k_{1x}^2 h^2(\omega_{k_1},T) h^2(\omega_{k_2},T) \epsilon_z^2(\mathbf{k}_1,\sigma_1)$$

$$\cdot \epsilon_z^2(\mathbf{k}_2,\sigma_2) \; \sin^2\alpha(\omega_2) [\frac{1}{(\omega_1+\omega_2)^2} \sin^2\{\frac{(\omega_1+\omega_2)}{2}\tau\}$$

$$+ \; \frac{1}{(\omega_1-\omega_2)^2} \sin^2\{\frac{(\omega_2-\omega_1)}{2}\tau\}] d^3\mathbf{k}_1 d^3\mathbf{k}_2 \; .$$

Summing over polarizations and using (5.2) we have,

$$<\Delta^2> \; = \; \frac{1}{2} \int\!\!\int \frac{9c^6}{4\omega_2^6} k_{1x}^2 h^2(\omega_{k_1},T) h^2(\omega_{k_2},T) \{1 - \frac{k_{1z}^2}{k_1^2}\}\{1 - \frac{k_{2z}^2}{k_2^2}\} \sin^2\alpha(\omega_2)$$

$$\cdot [\frac{1}{(\omega_1+\omega_2)^2} \sin^2\{\frac{(\omega_1+\omega_2)}{2}\tau\} \; + \; \frac{1}{(\omega_1-\omega_2)^2} \sin^2\{\frac{(\omega_2-\omega_1)}{2}\tau\}] d^3\mathbf{k}_1 d^3\mathbf{k}_2 \; .$$

We now switch to spherical coordinates in each k_i variable:

$$<\Delta^2> \; = \; \frac{1}{2} \int_0^\infty \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \int_0^\pi \int_0^\pi \; (1 - \cos^2\theta_1)(\sin^2\theta_1 \cos^2\phi_1) \frac{\omega_1^2}{c^2}$$

$$\cdot (1 - \cos^2\theta_2) \frac{\omega_1^2}{c^2} \; \sin\theta_1 \frac{\omega_2^2}{c^2} \; \sin\theta_2 \; h^2(\omega_{k_1},T) h^2(\omega_{k_2},T) \; \frac{9c^6}{4\omega_2^6}$$

$$\cdot \sin^2\alpha(\omega_2) [\frac{1}{(\omega_1+\omega_2)^2} \sin^2\{\frac{(\omega_1+\omega_2)}{2}\tau\}$$

$$+ \; \frac{1}{(\omega_1-\omega_2)^2} \sin^2\{\frac{(\omega_2-\omega_1)}{2}\tau\}] \frac{1}{c^2} d\theta_1 d\theta_2 d\phi_1 d\phi_2 d\omega_1 d\omega_2 \; .$$

Calculating the angular integrals,

$$<\Delta^2> = \frac{9}{4}\cdot\frac{64}{45}\cdot\frac{\pi^2}{c^2}\int_0^\infty\int_0^\infty h^2(\omega_{k_1},T)h^2(\omega_{k_2},T)\frac{\omega_1^4}{\omega_2^4}\cdot\frac{\Gamma^2\omega_2^6}{(\omega_0^2-\omega_2^2)^2 + \Gamma^2\omega_2^6}$$

$$\cdot\left[\frac{1}{(\omega_1+\omega_2)^2}\sin^2\left\{\frac{(\omega_1+\omega_2)}{2}\tau\right\} + \frac{1}{(\omega_1-\omega_2)^2}\sin^2\left\{\frac{(\omega_2-\omega_1)}{2}\tau\right\}\right]d\omega_2 d\omega_1 \quad ,$$

where we have used (5.27) to replace $\sin^2\alpha(\omega_2)$. To make further progress, we need to make an approximation: Since the amplitudes, $h(\omega_{k_1},T)$ and $h(\omega_{k_2},T)$, will be completely negligible at temperatures of interest unless ω_1 and ω_2 are much larger than zero, we can neglect the term in $\frac{1}{(\omega_1+\omega_2)^2}$ compared with the term in $\frac{1}{(\omega_2-\omega_1)^2}$. Then the integral has a sharp maximum at $\omega_2 = \omega_1$. Changing variables to $x = \omega_2-\omega_1$ (in the first integral), replacing factors of ω_2 not involved in differences, $\omega_2-\omega_1$, with ω_1, and letting the lower limit of integration go to $-\infty$ (as part of the approximation), we have,

$$<\Delta^2> = \frac{16\pi^2}{5c^2}\int_0^\infty\int_{-\infty}^\infty h^4(\omega_1,T)\frac{\Gamma^2\omega_1^6}{(\omega_0^2-\omega_1^2)^2 + \Gamma^2\omega_1^6}\,\frac{\sin^2\frac{x\tau}{2}}{x^2}\,dx\,d\omega_1$$

$$= \frac{8\pi^3\tau}{5c^2}\int_0^\infty h^4(\omega_1,T)\frac{\Gamma^2\omega_1^6}{(\omega_0^2-\omega_1^2)^2 + \Gamma^2\omega_1^6}d\omega_1 \quad .$$

Substituting (5.4) for $h(\omega_{k_1},T)$ we have:

$$<\Delta^2> = \frac{8\pi^3\tau}{5c^2}\int_0^\infty \frac{c^6}{\omega_1^4}\rho^2(\omega_1,T)\frac{\Gamma^2\omega_1^6}{(\omega_0^2-\omega_1^2)^2 + \Gamma^2\omega_1^6}d\omega_1 \quad .$$

As in the calculation of P (in 5.23), we can treat the integral as sharply peaked at $\omega_1 = \omega_0$ to arrive at:

$$<\Delta^2> = \frac{8\pi^3\tau}{5c^2}\,\frac{c^6}{\omega_0^4}\,\frac{\Gamma\pi}{2}\omega_0^2\rho^2(\omega_0,T) = \frac{4\pi^4\tau\Gamma c^4}{5\omega_0^2}\,\rho^2(\omega_0,T)\ . \qquad (5.30)$$

We can now combine (5.16), (5.23) and (5.30) to get the differential equation governing $\rho(\omega,T)$:

$$\frac{4\pi^4\tau\Gamma c^4}{5\omega^2}\,\{\rho^2(\omega,T) - \rho^2(\omega)\}$$

$$= 2\cdot\frac{6}{5}kTc\pi^2\tau\Gamma\ \{\rho(\omega,T) - \frac{\omega}{3}\,\frac{\partial\rho(\omega,T)}{\partial\omega}\}\ , \qquad (5.31)$$

where we have dropped the subscript 'o'. Substituting the zero point spectrum, (5.8), and simplifying,

$$\frac{\pi^2c^3}{3\omega^2}\,\{\rho^2(\omega,T) - \frac{h^2\omega^6}{16\pi^6c^6}\} = kT\{\rho(\omega,T) - \frac{\omega}{3}\,\frac{\partial\rho(\omega,T)}{\partial\omega}\}\ . \qquad (5.32)$$

Boyer [2] claims to have solved this equation using power series methods (which he does not elucidate). Instead, we have used the general procedure for solving a Riccati equation, $y' = p(x)y^2 + q(x)y + r(x)$, which for (5.32) turns out to be not difficult (see e.g., Spiegel [47], p.60). The result is:

$$\rho(\omega,T) = \frac{\omega^3 h}{2\pi^3c^3}\,\{\frac{1}{2} - [C_1(T)\exp(\frac{h\omega}{2\pi kT}) + 1]^{-1}\}\ , \qquad (5.33)$$

where $C_1(T)$ is an arbitrary, once continuously differentiable function of T alone ($C_1(T) \in C^1([0,\infty) \to R)$). However, by Wien's law, (4.12), C_1 = constant ($\neq 0$). We also expect $\{\rho(\omega,T) - \rho(\omega)\}$ to approach the Rayleigh - Jeans limit, (4.15), as $\omega \to 0$.

Thus, we require $C_1 = -1$. So the final result is:

$$\rho(\omega,T) = \frac{\omega^3 h}{2\pi^3 c^3}\left\{\frac{1}{2} + \left[\exp\left(\frac{h\omega}{2\pi kT}\right) - 1\right]^{-1}\right\} . \tag{5.34}$$

This is exactly the modern quantum result for the spectral density of blackbody radiation, (4.17).

At this point we need to make two consistency checks: That P=0 for the zero point spectrum in (5.23) and that we get the Rayleigh - Jeans spectrum if we set $\rho(\omega) = 0$ in (5.32). For the zero point spectrum (5.8), $\frac{\partial\rho(\omega)}{\partial\omega} = \frac{3\omega^2 h}{4\pi^3 c^3}$. So we have, with the help of (5.23),

$$P(T=0) = \frac{6}{5}c\pi^2\Gamma\left[\frac{\omega^3 h}{4\pi^3 c^3} - \frac{\omega}{3}\cdot\frac{3\omega^2 h}{4\pi^3 c^3}\right] \equiv 0 ,$$

as required by the Lorentz invariance of $\rho(\omega)$, i.e., the zero point spectrum cannot give rise to velocity dependent forces. Substituting $\rho(\omega) = 0$ into (5.31), we get:

$$\frac{\partial\rho(\omega,T)}{\partial\omega} = -\frac{\pi^2 c^3}{\omega^3 kT}\rho^2(\omega,T) + \frac{3}{\omega}\rho(\omega,T) . \tag{5.35}$$

Using the Riccati procedure, we have:

$$\rho(\omega,T) = \frac{kT\omega^3}{\pi^2 c^3}\,\frac{1}{\omega + C_1(T)} , \tag{5.36}$$

where $C_1(T) \in C^1\{[0,\infty) \to R\}$. To satisfy Wien's law we require $C_1(T) = C_2 T$, where C_2 is an arbitrary constant. Thus,

$$\rho(\omega,T) = \frac{kT\omega^3}{\pi^2 c^3}\,\frac{1}{\omega + C_2 T} . \tag{5.37}$$

This spectrum is incorrect except in the limit $\omega \to 0$, where it will match experimental results just in case $C_2 \equiv 0$. This choice of C_2 yields the Rayleigh - Jeans law, (4.15).

Some Theoretical Considerations in More Detail

In this section we wish to discuss in more detail some claims made in the previous section. In particular, we wish to show that the breakdown of the equipartition theorem, mentioned previously, suggests that "thermodynamic" entropy and "statistical" entropy cannot be equal, as is usually assumed.

For our purposes, the equipartition theorem says that for a gas of non-interacting particles, in equilibrium at temperature T, the average energy per particle is $\frac{3}{2}kT$, i.e., independent of the mass of the particles. Stated another way, this theorem says that while the particles in the gas will, in general, exhibit an energy distribution, the likelihood of any given particle being in an infinitesimal energy range, E to $E + dE$, is independent of the mass of the particle (see Reif [8] or Feynman, *et al* [48] for more discussion of the equipartition theorem). Recall that in the previous section, we showed that this theorem is correct only in the limit $m \to \infty$.

As has already been mentioned, classical thermodynamics claims the equivalence of two distinct notions of entropy: The thermodynamic notion is defined (for quasi-static processes) by (2.2), $dS_{cal} = \frac{dQ}{T}$, where S_{cal} is found to be a state function of the system (see chapter two). The statistical notion of entropy is defined by $S_{prob} \equiv S_0 + k\ln\Omega(E)$, where S_0 is a constant and, as explained in chapter two, $\Omega(E)$ is the density of states. A major result of thermodynamics, due to Boltzmann, is that these two notions of entropy are coincident. We wish next to give an example to show that this result cannot be correct in the presence of zero point radiation.

It has been shown by Marshall [49] that the entropy of a harmonic oscillator (in one dimension) is given by $S = S_0 + k\ln E$, where E is the average energy of the oscillator. This result has long been known from early statistical calculations, but Marshall gets this result using the procedures of Einstein and Hopf (without a zero point spectrum) given in the previous section. Then we have the equation, $\frac{\partial S_{prob}}{\partial E} = \frac{k}{E}$. From the first law (2.3), we have (for a system in a box, where $dV \equiv 0$), $\frac{\partial S_{cal}}{\partial E} = \frac{1}{T}$. Thus, if $S_{prob} = S_{cal}$, it follows that $\frac{k}{E} = \frac{1}{T}$, or $E = kT$, as the average energy of an oscillator at temperature T. Yet we have just shown (in the previous section, c.f. (5.34)) that $E = \frac{h\omega}{2\pi}\{\frac{1}{2} + [\exp(\frac{h\omega}{2\pi kT}) - 1]^{-1}\}$. Thus we conclude that $S_{prob} \neq S_{cal}$. (Of course, the usual way out of this dilemma is to assume that the radiation is quantized. Then equivalence between the different notions of entropy can be retained. This point is covered in more detail below.)

To see what the relationship between S_{prob} and S_{cal} should be, we follow an analysis originally worked out by Einstein [50]. (It is a further tribute to his genius that such simple considerations are so fruitful!) We consider a box of volume V, containing radiation in equilibrium. A volume $V_1 << V$ is now considered. The energy density U_1 of V_1, of the radiation in the frequency range ω to $\omega + d\omega$, undergoes fluctuations in accordance with the random phase approach. (See the previous section; recall that in chapter three U was the total energy density of the cavity over all frequencies. We are using U in a different sense here, for lack of enough symbols. As the radiation fluctuates, so does the entropy of the cavity. If S_T is the total entropy of the cavity due to the radiation in the frequency range ω to $\omega + d\omega$, then $S_T = S_1 + S$, where S_1 is the entropy contribution from the volume V_1 and S is the entropy contribution from the volume $V - V_1$, due to the radiation in the range ω to $\omega + d\omega$. We can then express S_T as a power series in ϵ, expanded about the equilibrium entropies, S_{10} and S_0, where ϵ is the fluctuation in U, i.e., $\epsilon = U - U_0$, and U_0 is the average equilibrium value of U:

$$S_T(\epsilon) = S_{10} + S_0 + \frac{\partial}{\partial \epsilon}(S_1 + S)\epsilon + \frac{\partial^2}{\partial \epsilon^2}(S_1 + S)\frac{\epsilon^2}{2} + \ldots .$$

In this expression all derivatives are evaluated at $\epsilon = 0$. At equilibrium S_T will be maximum so the first derivative is 0 at $\epsilon = 0$. Since $S >> S_1$, $\frac{\partial^2 S}{\partial \epsilon^2}$ will be negligible (ϵ is just the fluctuation of the small volume, V_1). Hence, to second order,

$$S_T(\epsilon) \simeq S_{10} + S_0 + \frac{1}{2}\frac{\partial^2 S_1}{\partial \epsilon^2}\epsilon^2 . \qquad (5.38)$$

We can now use the definition of the probabilistic notion of entropy,

$$S_T \equiv k\ln\Omega(\epsilon) + S_{To} + \text{constant}$$

to get (In this definition $S_T(\epsilon)$ is the entropy in excess of S_{To} due to the fluctuations, ϵ, and the constant term reflects the arbitrary nature of the scale chosen in the function $\Omega(\epsilon)$; see below):

$$\Omega(\epsilon) = C\exp\{\frac{S_T - S_{To}}{k}\} ,$$

where C is a constant that can be determined from the normalization condition, $\int_{-\infty}^{\infty} \Omega(\epsilon)d\epsilon = 1$. Since $\Omega(\epsilon)$ will have a sharp maximum at $\epsilon = 0$, we can use our approximation of $S_T - S_{To}$, (5.38), to yield:

$$\Omega(\epsilon)d\epsilon = C\exp\{\frac{1}{2k} \frac{\partial^2 S_1}{\partial \epsilon^2} \epsilon^2\}d\epsilon .$$

If we replace $\frac{\partial^2 S_1}{\partial \epsilon^2}$ with $\frac{\partial^2 S_1}{\partial U^2}$, using the relation $\epsilon = U - U_0$, we have,

$$\int_{-\infty}^{\infty} \Omega(\epsilon)d\epsilon = 1 \simeq C\int_{-\infty}^{\infty} \exp\{\frac{1}{2k} \frac{\partial^2 S_1}{\partial U^2} \epsilon^2\}d\epsilon = C\sqrt{2k\pi} \{- \frac{\partial^2 S_1}{\partial U^2}\}^{-\frac{1}{2}} .$$

Hence, $C = \frac{1}{\sqrt{2k\pi}} \{- \frac{\partial^2 S_1}{\partial U^2}\}^{\frac{1}{2}}$. Furthermore, we now calculate $<\epsilon^2>$ using the relation,

$$<\epsilon^2> \equiv \int_{-\infty}^{\infty} \epsilon^2 \Omega(\epsilon)d\epsilon \simeq C\int_{-\infty}^{\infty} \epsilon^2 \exp\{\frac{1}{2k} \frac{\partial^2 S_1}{\partial U^2} \epsilon^2\}d\epsilon .$$

Computing the integral we get,

$$\frac{\partial^2 S_1}{\partial U^2} = \frac{-k}{<\epsilon^2>} . \qquad (5.39)$$

In the calculation of the integral we use the fact that S_1 has a sharp maximum at $\epsilon = 0$ to approximate the integral with $\frac{\partial^2 S_1}{\partial U^2}|_{U=U_0}$. To eliminate U and $<\epsilon^2>$, we use the following three results:

1. The average energy, E, per oscillator is given by (see (4.14) *et seq*),

$$E = \frac{\pi^2 c^3}{\omega^2} \rho(\omega,T) \ . \qquad (5.40)$$

2. From the definition of U and $\rho(\omega,T)$,

$$U = \rho(\omega,T) d\omega \ . \qquad (5.41)$$

3. From the purely electromagnetic theory of waves (see Tomonaga [51]), the magnitude of the fluctuations per unit mode is,

$$<\acute{\epsilon}^2> = \frac{\pi^2 c^3}{\omega^2} \rho^2(\omega,T) d\omega \ . \qquad (5.42)$$

Then the relationship between $<\epsilon^2>$ and $<\acute{\epsilon}^2>$ is,

$$<\epsilon^2> = <\acute{\epsilon}^2> \frac{\omega^2 d\omega}{c^3 \pi^2} \ . \qquad (5.43)$$

Using the purely mathematical relation (and dropping the subscript "1" on S),

$$\frac{\partial^2 S}{\partial U^2} = \frac{\partial^2 S}{\partial E^2}\left\{\frac{\partial E}{\partial U}\right\}^2 + \frac{\partial S}{\partial E}\,\frac{\partial^2 E}{\partial U^2} \ ,$$

together with (5.39) we have,

$$\frac{\partial^2 S}{\partial E^2}\left\{\frac{\partial E}{\partial U}\right\}^2 = \frac{-k}{<\epsilon^2>} \ , \qquad (5.44)$$

where E is the average energy per oscillator. We have also used the fact that $\frac{\partial S}{\partial E} = 0$ at the equilibrium energy. Using (5.40), (5.41) and (5.42), we get $E = \frac{\pi^2 c^3}{\omega^2 d\omega} U$, so that $\frac{\partial E}{\partial U}$ works out to $\frac{\pi^2 c^3}{\omega^2 d\omega}$. Substituting this result into (5.44) we get, with the help of (5.40), (5.42) and (5.43),

$$\frac{\partial^2 S}{\partial E^2} = - \left\{\frac{\omega^2 d\omega}{\pi^2 c^3}\right\}^2 \frac{k}{(\rho d\omega)^2} = \frac{-k}{E^2} \ . \qquad (5.45)$$

Equation (5.45) relates the probabilistic entropy of the region V_1 to the average energy per mode of an oscillator in this region. It was first derived by Einstein. At this stage Einstein assumes $S_{prob} = S_{cal}$, from which it follows immediately (as shown above) that $E = kT$. Einstein proposed that the wave theory of light be replaced with a quantum theory. Then $<\hat{\epsilon}^2>$ works out to (see [51], chapter 2),

$$<\hat{\epsilon}^2> = [\frac{h\omega}{2\pi}\rho + \frac{\pi^2 c^3}{\omega^{\text{[illegible]}}}\rho^2]d\omega ,$$

which leads to the Planck value for E,

$$E = \frac{h\omega}{2\pi}\{\exp[\frac{h\omega}{2\pi kT} - 1\}^{-1} .$$

Instead we can retain (5.42) as follows: Since the zero point radiation and thermal radiation are independent we have,

$$<\epsilon^2>_{total} = <\epsilon^2>_{zero\ point} + <\epsilon^2>_{cal} .$$

We can repeat the derivation of (5.39) using $S_{cal} = S_{prob} - S_{zero\ point}$ to get,

$$\frac{\partial^2 S_{cal}}{\partial U^2} = \frac{-k}{<\epsilon^2>_{cal}} = \frac{-k}{<\epsilon^2>_{total} - <\epsilon^2>_{zero\ point}} . \qquad (5.46)$$

Then using, $<\hat{\epsilon}^2>_{zero\ point} = \frac{\pi^2 c^3}{\omega^2}\rho^2(\omega)d\omega$, (5.43) and (5.8), we can repeat the procedure for deriving (5.45) to arrive at

$$\frac{\partial^2 S_{cal}}{\partial E^2} = \frac{-k}{E^2 - (\frac{h\omega}{4\pi})^2} . \qquad (5.47)$$

Recalling that S is being treated as a function of V and E with V held fixed, we can integrate (5.47) once with respect to E without consideration of the arbitrary function of V arising in this process. So we have,

$$\frac{\partial S_{cal}}{\partial E} = \frac{2k\pi}{h\omega} \ln\{\frac{E + \frac{h\omega}{4\pi}}{E - \frac{h\omega}{4\pi}}\} = \frac{1}{T} , \tag{5.48}$$

where the last equality follows from the first law, (2.3). Solving (5.48) for E, we have, finally,

$$E = \frac{h\omega}{2\pi} \{\frac{1}{2} + \exp[\frac{h\omega}{2\pi kT} - 1]^{-1}\} ,$$

which is the modern quantum result (compare with (4.17)). Thus, abandoning the classical equivalence between probabilistic entropy and caloric entropy is an acceptable way to resolve the difficulty posed by the Einstein equation, (5.45).

CHAPTER VI

EXTENSIONS OF BOYER'S ANALYSIS

Classical Fermi - Dirac Statistics

We begin this section with the derivation of the Fermi-Dirac (FD) distribution, using the Pauli exclusion principle, but otherwise from a classical point of view. We are treating the walls of the blackbody cavity as a free electron gas subject to the condition that at most one electron can occupy any given cell in phase space. First we enumerate the cells of phase space, 1, 2, 3, ...,r ... . As usual, each cell is of volume h_0^3. Then each cell has energy, ϵ_r, associated with it. Next we calculate the partition function, (2.21), given by, $Z = \sum_r e^{-\beta\epsilon_r n_r}$, where n_r is the number of particles in the r^{th} cell. In this case $n_r = 0$ or 1 , and $\sum_r n_r = N$, the total number of particles. As we have seen in several examples, the partition function is easy to calculate for unrestricted sums. To calculate Z with restricted sums we use a rather ingenious mathematical trick: We note that for different values of N, Z(N) will, in general, change. In fact, Z(N), is a very rapidly increasing function of N. So, we can substitute the dummy variable N' for N (so that N will be arbitrary but fixed) and define,

$$\Xi(N') \equiv \sum_{N'} Z(N')e^{-\alpha N'} , \qquad (6.1)$$

where the sum is taken over all possible choices of N', that is, 1, 2, 3, ... ($\Xi(N')$ is called a "grand partition function"). Then $\Xi(N')$ will have a sharp maximum which can be arranged to occur at $N' = N$ with a proper choice of α . That $\Xi(N')$ will be useful can be seen as follows:

$$\Xi(N') \equiv \sum_{N'} Z(N')e^{-\alpha N'} \simeq Z(N)e^{-\alpha N}\Delta N' \ , \tag{6.2}$$

where $\Delta N'$ is the width of the maximum which has been arranged to occur at $N' = N$. Then,

$$\ln\Xi(N') \simeq \ln Z - \alpha N + \ln\Delta N' \simeq \ln Z - \alpha N \ , \tag{6.3}$$

since the term $\ln\Delta N'$ is completely negligible compared to $\ln Z - \alpha N$ (we will show this after we have demonstrated the mathematics required to calculate $\Delta N'$). The point is, that $\Xi(N')$ is easy to calculate since it involves unrestricted sums. Thus, using (6.2), we have,

$$\begin{aligned}\Xi(N') &\simeq \sum_{n_1,n_2,\ldots} e^{-\beta n_1\epsilon_1-\beta n_2\epsilon_2-\ldots}\ e^{-\alpha(n_1+n_2+\ldots)} \\ &= \sum_{n_1,n_2,\ldots} e^{-(\beta\epsilon_1+\alpha)n_1-(\beta\epsilon_1+\alpha)n_2-\ldots} \ ,\end{aligned} \tag{6.4}$$

where the sum is taken over all possible choices of each n_k for all $k = 1, 2, \ldots$. For FD statistics each n_k has only 2 values, 0 or 1, and since the sums over each n_k are independent, (6.4) becomes,

$$\Xi(N') = (1 + e^{-\beta\epsilon_1-\alpha})(1 + e^{-\beta\epsilon_2-\alpha})\cdot\ldots \ ,$$

and,

$$\ln\Xi(N') = \sum_r \ln(1 + e^{-\alpha-\beta\epsilon_r}) \ . \tag{6.5}$$

Using (6.3) and (6.5) we have,

$$\ln Z(N) \simeq \sum_r \ln(1 + e^{-\alpha-\beta\epsilon_r}) + \alpha N , \qquad (6.6)$$

where we have dropped the prime on N', since α has been chosen to make $\Xi(N')$ have a maximum at $N' = N$. To find α explicitly, we set the derivative of $\ln\Xi(N')$ equal to 0:

$$\frac{\partial}{\partial N'}\ln\Xi(N')\Big|_{N'=N} = \frac{\partial}{\partial N'}[\ln Z(N') - \alpha N']\Big|_{N'=N} = 0 \text{ , or,}$$

$$\alpha \doteq \frac{\partial \ln Z(N')}{\partial N'}\Big|_{N'=N} . \qquad (6.7)$$

Thus, we expect α to depend on N. Differentiating (6.6),

$$\frac{\partial \ln Z(N)}{\partial N} = \frac{\partial}{\partial N}\sum_r \ln(1 + e^{-\alpha-\beta\epsilon_r}) + \frac{\partial(\alpha N)}{\partial N}$$

$$= \{-\sum_r \frac{\exp[-(\beta\epsilon_r+\alpha)]}{1 + \exp[+(\beta\epsilon_r-\alpha)]} + N\}\frac{\partial\alpha}{\partial N} + \alpha .$$

Substituting this expression into (6.7) and simplifying, we get,

$$\sum_r \frac{1}{\exp(\beta\epsilon_r+\alpha) + 1} = N . \qquad (6.8)$$

Equation (6.8) is the normalization condition on the partition function Z.

The FD distribution is the average number of particles, $\bar{n}_r$, in the r^th^ cell. For any statistics,

$$\bar{n}_r = \frac{\sum_r n_r e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+\dots)}}{\sum_r e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+\dots)}} = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \epsilon_r} , \qquad (6.9)$$

where we have used (2.21). If we apply (6.9) to (6.6) we have:

$$\bar{n}_r = -\frac{1}{\beta}\{\frac{\partial\alpha N}{\partial\epsilon_r} + \frac{\partial}{\partial\epsilon_r}\sum_s \ln(1 + e^{-\alpha-\beta\epsilon_s})\} = \frac{1}{\exp(\beta\epsilon_r+\alpha) + 1} \ . \qquad (6.10)$$

If N is very large (N is typically of order of Avogadro's number) we can change the sum in (6.8) to an integral over all phase space as follows (see chapter two):

$$N = \sum_r \frac{1}{\exp(\beta\epsilon_r+\alpha) + 1} \Rightarrow N = \frac{1}{h_0^3}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{dp_1\ldots dx_3}{\exp(\beta\epsilon_r+\alpha) + 1} \ . \qquad (6.11)$$

Next we wish to calculate the Fermi energy ϵ_F of a free electron gas. If we define the chemical potential, μ, by $\mu \equiv -\frac{\alpha}{\beta}$, then, $\epsilon_F = \mu$ at $T = 0$. In this case all the lowest energy cells are filled as can be seen by writing (6.10) in the form,

$$\bar{n}_r = \frac{1}{\exp[\beta(\epsilon_r-\mu)] + 1} \ .$$

Then if $\epsilon_r > \epsilon_F$, $\bar{n}_r = 0$ (for $\beta \to \infty$), and if $\epsilon_r < \epsilon_F$, $\bar{n}_r = 1$. For free electrons,

$$\epsilon_F = \frac{p_F^2}{2m} \ , \qquad (6.12)$$

where m is the mass of an electron and p_F the momentum at the Fermi energy. The total number of phase space cells in a sphere of radius p_F and volume V is,

$$\frac{4\pi p_F^3 V}{3h_0^3} = N \ . \qquad (6.13)$$

Solving for p_F and substituting into (6.12) we get,

$$\epsilon_F = \frac{h_0^2}{2m}\{\frac{3N}{4\pi V}\}^{\frac{2}{3}} \ . \qquad (6.14)$$

If we set $h_0 = h$ and replace N with $\frac{N'}{2}$ to take into account the two spin states of each electron, then (6.14) is exactly the quantum result.

In chapter five we introduced Planck's constant as a scale factor in the derivation of the zero-point spectrum. In the following section we attempt to connect the "scale factor" notion of Planck's constant to the size of a cell in phase space, h_0, which classically must be taken to zero at the end of the calculation. The idea is that there must be a balance (at very low temperatures) between the emission and absorption of radiation by electrons in the walls of the cavity. We would expect this balance to depend on the spectral density of the radiation in the cavity. If we find an equation representing this balance that involves h_0, then we will have found the connection that we seek.

Modification of the Rayleigh - Jeans Method

We wish to calculate the average energy of an oscillator in the wall of the cavity at very low temperature, ≈ 1°K or less. We are treating the walls as a free electron gas subject to forced oscillations from the radiation. Recall that the walls are being treated as a perfect, or at least very good conductor, so that charges respond to the radiation, essentially instantaneously, in such a way as to cancel the impinging electric (tangental component) and magnetic (normal component) fields. The calculations are done at very low temperature since we are interested in the connection, if any, between the zero-point spectrum and the Fermi energy. Furthermore, at higher temperatures the metallic walls of the cavity absorb energy primarily by mechanical waves (phonons) which obey Maxwell-Boltzmann statistics. We need not be concerned with superconductivity as this effect has not been observed in good conductors such as the alkali metals or noble metals (see Ashcroft and Mermin [53]).

The procedure we have followed is as follows:

1. Calculate the chemical potential, μ, of a free electron gas.
2. Calculate the average energy, $\bar{\epsilon}$, per electron of a free electron gas (in terms of μ).
3. Assign a "perturbed" energy, $\epsilon = \bar{\epsilon} + \frac{\mathbf{p}\cdot\mathbf{p}}{2m} + \frac{1}{2}m\omega^2\mathbf{x}\cdot\mathbf{x}$, per electron, where $\bar{\epsilon}$ is the average translational energy calculated in part 2, and $\frac{\mathbf{p}\cdot\mathbf{p}}{2m} + \frac{1}{2}m\omega^2\mathbf{x}\cdot\mathbf{x}$, is the small oscillator energy induced by radiation absorption. In this expression $\mathbf{p}$ is the additional momentum the particle acquires from the oscillations.
4. Recalculate μ, and find the new average energy, $\bar{\epsilon}'$.
5. Calculate the density of modes in terms of the carrier density, n (number of free electrons per unit volume in the walls).
6. Using the Rayleigh - Jeans prescription, the spectral density, is given by $\rho(\omega,T) = \bar{\epsilon}'\cdot$mode density .

We have not been able to calculate the mode density. Comparison of the radiation mode density, $\frac{\omega^2}{\pi^2 c^3}d\omega$, with a typical value of the effective carrier density reveals a discrepancy of several orders of magnitude. To make this comparison, we integrate the radiation mode density over all frequencies up to the minimum frequency, ω_F, that results in the integral being equal to the effective carrier density, that is,

$$\int_0^{\omega_F} \frac{\omega^2}{\pi^2 c^3} d\omega = \frac{\omega_F^3}{3\pi^2 c^3} = n_{eff} \simeq 10^{23} \text{charges/m}^3 \text{ at } 1^\circ\text{K} .$$

(In this equation the integral is the total number of modes per unit volume with frequencies between 0 and ω_F, and n_{eff} is the total number of charges per unit volume not in their ground state; that is, $n_{eff} \simeq \{\frac{T}{T_F}\}n$, where T_F is the Fermi temperature. For our considerations, $\frac{T}{T_F} \simeq 10^{-5}$.) Then using 3 x 10^8m/s for c, ω_F works out to $\simeq 10^{16}$hertz, which is well above the plasma frequency (the frequency at which the metal becomes transparent to radiation). We require some physical criteria for determining how many carriers are actually involved in the process of "neutralizing" the electric and magnetic fields at the surface, as required by the classical model of a good conductor. In any case, we include the results from the first four steps. We have omitted most of the calculations, since they are straightforward adaptations of standard calculations in FD statistics (see [8, 52, or 53]).

1. Calculation of μ for a free electron gas:

We start with the normalization condition, (6.11),

$$N = \frac{V}{h_0^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d^3p}{\exp\beta\{\frac{p_1^2+p_2^2+p_3^2}{2m} - \mu\} + 1} .$$

We next change to spherical coordinates with $r^2 = \frac{p_1^2+p_2^2+p_3^2}{2m}$. Then,

$$N = \frac{4\pi V(2m)^{\frac{3}{2}}}{h_0^3}\int_0^\infty \frac{r^2 dr}{\exp\beta(r^2-\mu) + 1} ,$$

where the factor of 4π comes from the angular part of the integration and $(2m)^{\frac{3}{2}}$ comes from the Jacobian of the transformation. It is convenient to change variables again; $r^2 \to z$, so we have,

$$N = \frac{2\pi V(2m)^{\frac{3}{2}}}{h_0^3}\int_0^\infty \frac{z^{\frac{1}{2}} dz}{\exp\beta(z-\mu) + 1} .$$

Integrating by parts yields,

$$N = \frac{2\pi V(2m)^{\frac{3}{2}}}{h_0^3}\{\frac{2}{3}\beta\int_0^\infty \frac{z^{\frac{3}{2}}\exp\beta(z-\mu)}{[\exp\beta(z-\mu) + 1]^2}dz\} .$$

If T is near 0, then $\beta = \frac{1}{kT}$ is very large and the integrand is sharply peaked. We expand $z^{\frac{3}{2}}$ into a power series about μ, retaining only the first few terms as we need only calculate μ to order $\frac{1}{\beta^2}$ for T near 0:

$$N = \frac{2\pi V(2m)^{\frac{3}{2}}}{h_0^3}\cdot\frac{2}{3}\beta \int_0^\infty\{\mu^{\frac{3}{2}} + \frac{3}{2}\mu^{\frac{1}{2}}(z-\mu) + \frac{3}{8}\mu^{-\frac{1}{2}}(z-\mu)^2 + \ldots\} \frac{\exp\beta(z-\mu)}{[\exp\beta(z-\mu) + 1]^2}dz .$$

$$= \frac{2\pi V(2m)^{\frac{3}{2}}}{h_0^3}\cdot\frac{2}{3}\beta\{\mu^{\frac{3}{2}}\int_0^\infty \frac{\exp\beta(z-\mu)}{[\exp\beta(z-\mu) + 1]^2}dz + \frac{3}{2}\mu^{\frac{1}{2}}\int_0^\infty \frac{(z-\mu)\exp\beta(z-\mu)}{[\exp\beta(z-\mu) + 1]^2}dz + \frac{3}{8}\mu^{-\frac{1}{2}}\int_0^\infty \frac{(z-\mu)^2\exp\beta(z-\mu)}{[\exp\beta(z-\mu) + 1]^2}dz + \ldots\} .$$

Changing variables, $x = (z-\mu)\beta$, and using the by now familiar approximation of letting the lower limit of integration (after the change of variables) go from $-\mu\beta$ to $-\infty$, (see, for example, the discussion preceeding (5.22)) we have,

$$N = \frac{2\pi V(2m)^{\frac{3}{2}}}{h_0^3}\cdot\frac{2}{3}\beta\ \{\frac{\mu^{\frac{3}{2}}}{\beta}\int_{-\infty}^{\infty}\frac{e^x}{(e^x+1)^2}dx + \frac{3}{2}\cdot\frac{\mu^{\frac{1}{2}}}{\beta^2}\int_{-\infty}^{\infty}\frac{xe^x}{(e^x+1)^2}dx$$

$$+ \frac{3}{8}\cdot\frac{\mu^{-\frac{1}{2}}}{\beta^3}\int_{-\infty}^{\infty}\frac{x^2e^x}{(e^x+1)^2}dx + \ldots\} .$$

The first integral on the righthandside can be evaluated directly and is 1. The second integral is 0 since the integrand is odd. The third integral can be calculated by integration by parts and expressing, $\frac{1}{e^x+1} = \frac{e^{-x}}{e^{-x}+1}$ as a geometric series in e^{-x} . Term by term integration leads to the result, $\frac{\pi^2}{3}$. Thus,

$$N \simeq \frac{4\pi V(2m)^{\frac{3}{2}}}{3h_0^3}\ \{\mu^{\frac{3}{2}} + \frac{\pi^2}{8}\cdot\frac{\mu^{-\frac{1}{2}}}{\beta^2} + \ldots\} .$$

Using (6.14), we have,

$$\epsilon_F^{\frac{3}{2}} \simeq \mu^{\frac{3}{2}} + \frac{\pi^2}{8}\cdot\frac{\mu^{-\frac{1}{2}}}{\beta^2} .$$

Next we expand μ about ϵ_F to first order since $\mu - \epsilon_F \simeq O(\frac{1}{\beta^2})$. The result is,

$$\mu \simeq \epsilon_F\{1 - \frac{\pi^2}{12\epsilon_F^2\beta^2}\} . \qquad (6.15)$$

Before continuing with the calculation of the average energy, $\overline{\epsilon}$, (step 2) we will return to the calculation of $\Delta N'$ to show that it is indeed negligible compared to N. We define the dispersion, $\Delta N'$ by the equation,

$$(\Delta N')^2 \equiv \overline{(\overline{N'} - N')^2} = \overline{N'^2} - (\overline{N'})^2$$

With the help of (6.2) and some algebra we have,

$$(\Delta N')^2 = \frac{\partial}{\partial\alpha}\{\frac{\partial}{\partial\alpha}\ln\Xi(N')\} = \frac{\partial^2}{\partial\alpha^2}\sum_r \ln\{1 + \exp(-\alpha-\beta\epsilon_r)\} ,$$

where we have used (6.5) in the last step. Calculation of the derivatives gives,

$$(\Delta N')^2 = \sum_r \frac{\exp(\alpha+\beta\epsilon_r)}{[\exp(\alpha+\beta\epsilon_r) + 1]^2} .$$

If N is very large then we can approximate the sum with an integral,

$$(\Delta N')^2 = \frac{V}{h_0^3} \int_{-\infty}^{\infty} \frac{\exp\{\frac{p^2}{2m} - \mu\}\beta}{[\exp\{\frac{p^2}{2m} - \mu\}\beta + 1]^2} d^3p .$$

This integral is straightforward to calculate to second order in $\frac{1}{\beta}$ by the methods just outlined. The result is,

$$(\Delta N')^2 = \frac{2\pi V(2m)^{\frac{3}{2}}}{h_0^3 N} . \frac{\mu^{\frac{1}{2}}}{\beta} N .$$

We can use (6.14) and (6.15) to clarify the order of magnitude of $\Delta N'$:

$$(\Delta N')^2 = \frac{3N}{2\beta\epsilon_F}\{1 - \frac{\pi^2}{12\epsilon_F^2\beta^2}\}^{\frac{1}{2}} \simeq \frac{3}{2} . N . \frac{T}{T_F} ,$$

where we have use the Fermi temperature, T_F, defined as, $T_F \equiv \frac{\epsilon_F}{k}$. The Fermi temperature is typically of order 10^4 to 10^5°K, so that $\Delta N'$ is of $O(\sqrt{N}) << N$, for physically reasonable temperatures. Of course, to determine the order of magnitude of T_F, we need to set $h_0 \simeq h$ in (6.14). This is acceptable even though the classical approach requires that

$h_0 \to 0$ because we do not expect to actually take this limit, that is, we are trying to show that if h (from the stochastic analysis of chapter 4) is not zero, then h_0 (from phase space) cannot be zero either.

2. We define the average energy per particle, $\bar{\epsilon}$, by the equation:

$$\bar{\epsilon} = \frac{\frac{V}{h_0^3}\int_{-\infty}^{\infty} \frac{\frac{\mathbf{p}^2}{2m}}{\exp\beta\{\frac{\mathbf{p}^2}{2m}-\mu\} + 1} d^3p}{\frac{V}{h_0^3}\int_{-\infty}^{\infty} \frac{d^3p}{\exp\beta\{\frac{\mathbf{p}^2}{2m}-\mu\} + 1}} . \tag{6.16}$$

Both the integrals in (6.16) can be done in the manner already outlined for calculating μ. Actually, we have just done the denominator. Alternatively, the denominator could be replaced with N. The result, to $O(\frac{1}{\beta^2})$ is,

$$\bar{\epsilon} = \frac{3}{5}\epsilon_F\{1 + \frac{5\pi^2}{12\epsilon_F^2\beta^2}\} . \tag{6.17}$$

3. - 4. We now repeat the calculations using $\epsilon = \bar{\epsilon} + \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2 x^2$, for the energy of each particle. We are assuming with this expression that the free translational kinetic energy of the electron gas particles has a negligible interaction with the radiation at low temperatures. If we accept this assumption we have,

$$N = \frac{1}{h_0^3}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{d^3p\, d^3x}{\exp\{\frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \bar{\epsilon} - \mu\}\beta + 1} .$$

Changing to six dimensional spherical coordinates, $r^2 = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{x}^2$, we have,

$$N = \frac{8\pi^3}{\omega^3 h_0^3} \int_0^\infty \frac{r^5 dr}{\exp\{r^2 + \bar{\epsilon} - \mu\}\beta + 1} ,$$

where the factor of $\frac{8}{\omega^3}$ comes from changing variables and the factor of π^3 comes from the angular part of the integral. The integral can be evaluated using the procedure already explained in part 1. The result is (to second order in $\frac{1}{\beta}$):

$$N \simeq \frac{4\pi^3}{3\omega^3 h_0^3}\{(\mu - \bar{\epsilon})^3 + \frac{(\mu - \bar{\epsilon})\pi^2}{\beta^2}\} . \tag{6.18}$$

Unfortunately, we cannot approximate further since μ is now dependent on the frequency, ω. (Recall, however, from chapter three that the frequency is proportional to the temperature, i.e., inversely proportional to β.) The cubic equation (6.18) can be solved exactly (one real root):

$$\begin{aligned}(\mu - \bar{\epsilon}) = {} & \{\frac{3N\omega^3 h_0^3}{8\pi^3} + [\frac{\pi^6}{27\beta^6} + \frac{9N^2\omega^6 h_0^6}{64\pi^6}]^{\frac{1}{2}}\}^{\frac{1}{3}} \\ & + \{\frac{3N\omega^3 h_0^3}{8\pi^3} - [\frac{\pi^6}{27\beta^6} + \frac{9N^2\omega^6 h_0^6}{64\pi^6}]^{\frac{1}{2}}\}^{\frac{1}{3}} .\end{aligned} \tag{6.19}$$

We cannot do much with (6.19) since for some frequencies the term involving ω inside the square root is much larger than the term in $\frac{1}{\beta}$, but for other frequencies it is much smaller. Finally we have calculated the new average energy, $\bar{\epsilon}'$:

$$\bar{\epsilon}' = \bar{\epsilon} + \frac{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \dfrac{\{\frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{x}^2\}d^3p\,d^3x}{\exp\{\frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{x}^2 + \bar{\epsilon} - \mu\}\beta + 1}}{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \dfrac{d^3p\,d^3x}{\exp\{\frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{x}^2 + \bar{\epsilon} - \mu\}\beta + 1}} .$$

The result is,

$$\overline{\epsilon}' = \overline{\epsilon} + (\mu - \overline{\epsilon}) + \frac{\pi^2}{\beta^2(\mu - \overline{\epsilon})} + \ldots, \qquad (6.20)$$

to second order in $\frac{1}{\beta}$, where $(\mu - \overline{\epsilon})$ is now given by (6.19).

At this stage we need to know how many charges are actually being oscillated, since as we have already pointed out, the effective carrier density is much larger than the mode density of the radiation. We have not discovered any way to determined the density of oscillations required by the Rayleigh - Jeans method.

Another approach altogether, which we have not worked out, is to repeat all the calculations of chapter five for a free electron gas, using FD statistics. We could start with (5.11) and would then get the average energy, $m\langle v^2\rangle$, in terms of temperature using some equation such as (6.20), since the equipartition theorem does not hold (at low temperatures) for a free electron gas. Next we could attempt to calculate the radiation drag and impulse, P and Δ, respectively, as in chapter five, using the equation of motion for a free charge (see [44], chapter 12). A question that is hard to answer *a priori* is whether or not the calculation should be done in two or three dimensions: For a good conductor the charges are confined to the surface, so a two dimensional calculation seems appropriate. In any case, we have left this calculation for a future project.

Relativistic Thermodynamics

In chapter three we presented a canonical formulation of relativistic thermodynamics. We have not been able to apply this machinery to the blackbody problem. The chief difficulty is that the entropy of the zero-point field does not lend itself to ready calculation. For example, it cannot be dependent on the energy of the system (otherwise $\frac{\partial S_0}{\partial E} \neq 0$, so that there would be a thermal contribution from the zero-point entropy). As a matter of fact, calculation of the entropy of a classical radiation field has not met with a lot of success: Attempts to do this calculation require many *ad hoc* assumptions (see Marshall [54]).

Conclusion

We set out to show that h, k, and c are physically interdependent. We chose to do this by studying blackbody radiation without quantum mechanics. Since this leaves relativity, thermodynamics and electromagnetism, we made a study of relativistic thermodynamics as a matter of preliminary ground work. We chose the canonical approach over the tensor approach since the former is more physically accessible than the latter. Next we studied the standard derivation of blackbody radiation. We then went through a classical derivation of the blackbody spectrum using stochastic electrodynamics. Again, we chose Boyer's original 1969 derivation, since this calculation uses statistical mechanics, and avoids physical complications (in later derivations, Boyer studies acceleration of charged particles in the zero-point field, and magnetic moments, see [4-7]). Finally, we have tried to discern a connection between h, k, and c by deriving an explicit relation between the constant of proportionality (Planck's constant) arising in stochastic electrodynamics and h_0, the size of a cell in classical phase space. We have not suceeded in deriving such an equation. Thus, we are left with no conclusion about the possible interdependence of h, k, and c.

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