A SURVEY OF RESULTS OF KELLY'S CONJECTURE ON GRAPH ISOMORPHISMS

by

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ABSTRACT

This is an expository paper surveying the results of Kelly's Conjecture (Reconstruction Problem or Vertex Problem) and related results in Graph Theory. Chapter 1 summarizes the present state of the conjecture, i.e., a graph G is reconstructed uniquely from its subgraphs G_v (obtained by deleting one vertex at a time from G), for graphs with no multiple edges and loops. That is, we present the classes of graphs for which the conjecture is true. In particular, trees, disconnected graphs, separable graphs without pendant vertices and some special graphs, e.g., regular graphs, cacti, and Eulerian graphs.

It is shown in Chapter 2 that the problem of reconstructing a graph G given the subgraphs G^e (obtained by deleting one edge at a time from G) is equivalent to that of reconstructing its line graph L(G) from its subgraphs $(L(G))_e$. That is, the Edge Problem is a special case of the Vertex Problem.

Chapter 3 summarizes the results of the conjecture for Tournaments.

In Chapter 4 we simplify the conjecture by labeling some of the vertices in the graph and present these results.

Throughout the paper we present some new conjectures related to Kelly's Conjecture, but as yet unsolved.

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INTRODUCTION

The following very interesting conjecture was proposed by P. Kelly in 1942 and presented in S. Ulam's problem book [18].

Suppose that in two sets A and B, each of n elements, there is defined a distance fuction ρ such that for two elements x,y, $\rho(x,y)$ equals either 1 or 2, and $\rho(x,x) = 0$. Assume that for every subset of n - 1 elements of A, there exists an isometric system of n - 1 elements of B, and that the number of distinct subsets isometric to any given subset of n - 1 elements is the same in A as in B. Then for $n \geq 3$, A and B are isometric.

This conjecture can be restated in graph theory terms but first we give some definitions.

Definition 0.1 - A graph is a triple $G = (V,E,f)$ where $f: E \to V^{(2)}$ and $V^{(2)} = \{[u,v]: u,v \in V\}$, the set of unordered pairs on V and $f^{-1}[u,v]$ contains 1 or 0 elements, $f^{-1}[u,u]$ contains 0 elements and V is a nonempty finite set.

Definition 0.2 - The elements of V are called vertices (nodes, points) of G and V is called the vertex set of G and we denote it by V(G).

Definition 0.3 - An element e of E is called an edge (line, join) of G. If $f(e) = [u,v] \in V^{(2)}$ we say e is incident with u and v, and two vertices (edges) are said to be adjacent if they are incident with a common edge (vertex). We denote such an edge by $<u,v> \in G$. We denote an edge set of G by E(G).

Definition 0.4 - The order of a graph G is the number of vertices in V(G).

Definition 0.5 - If $G = (V,E,f)$ and $H = (V',E',f')$ are graphs then H is an induced subgraph of G if $V' \subseteq V$, $f' = f|E'$ and $f'^{-1}[u,v] = f^{-1}[u,v]$.

Definition 0.6 - Two graphs G and G' are said to be isomorphic if there exists a one-to-one mapping σ from the vertices of G onto the vertices of G' such that $<u,v> \in G$ if and only if $<\sigma(u),\sigma(v)> \in G'$.

For any vertex $v \in V$ let G_v denote the induced subgraph of G with vertex set $V(G) - \{v\} = V(G_v)$. Thus the conjecture proposed by Kelly which we shall call the vertex problem is as follows in graph theory terms.

Conjecture A - (Kelly's Conjecture) - The Vertex Problem
If G and H are graphs, $|V(G)| > 2$, and $\sigma: V(G) \to V(H)$ is a one-to-one onto function such that $G_v \cong H_{\sigma(v)}$ for all $v \in V(G)$, then $G \cong H$.

An equivalent formulation of this problem is as follows:

<u>Conjecture A'</u> - An arbitrary graph G, $|V(G)| = n$, $n > 2$, can be uniquely reconstructed, up to isomorphism, from its n subgraphs G_{v_i}, $i = 1,\cdots,n$.

If $n = 2$, say $V(G) = \{u,v\}$, then $G_u = v$ and $G_v = u$. G cannot be reconstructed uniquely from G_u and G_v as G may contain an edge or may not as in Figure 1. However, there are no other known counterexamples.

G_u: •v

G_v: •u

G: v• •u or v•——•u

<u>Figure 1</u>

It is the purpose of this paper to summarize most of the known results about this conjecture and any other results related to this conjecture in graph theory.

Kelly [13] has verified by exhaustion that Conjecture A holds for all graphs with at most six vertices. Harary and Palmer [9] found the same to be true of the seven vertex graphs. The conjecture has been verified for disconnected graphs. P. Kelly [13] has also verified the conjecture true for trees. There have been a number of improvements of

Kelly's result for trees. Others have tried to prove the vertex problem without success, although some convinced themselves that they had produced a valid proof. Chapter 1 will discuss all the above results including the latest partial results on the vertex problem. The definitions of tree and disconnected graph will be stated in Chapter 1.

For any element e of E(G) let G^e denote the induced subgraph of G with $V(G^e) = V(G)$ and $E(G^e) = E(G) - \{e\}$.

F. Harary [6] has suggested the following intuitively simpler but also as yet unsolved conjecture related to Conjecture A.

<u>Conjecture B</u> - <u>The Edge Problem</u> - If G and H are graphs, $|E(G)| > 3$, and $\sigma: E(G) \to E(H)$ is a one-to-one onto function such that $G^e \cong H^{\sigma(e)}$ for all e in E(G), then $G \cong H$.

Or equivalently:

<u>Conjecture B'</u> - An arbitrary graph G, $|E(G)| = m$, $m > 3$, can be uniquely reconstructed, up to isomorphism, from its m subgraphs G^e.

If $m = 3$, say $E(G) = \{c,d,e\}$ and G^c, G^d and G^e are the three subgraphs in Figure 2, then G cannot be uniquely reconstructed, as G can be the two graphs in Figure 3.

If $m = 2$, say $E(G) = \{d,e\}$ and G^d and G^e are the two graphs in Figure 4, then G cannot be uniquely reconstructed, as G can be the two graphs in Figure 5.

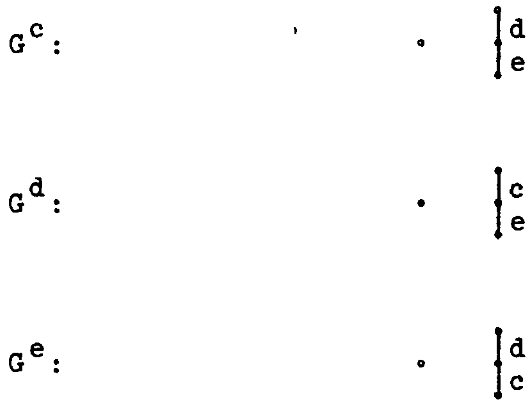


Figure 2

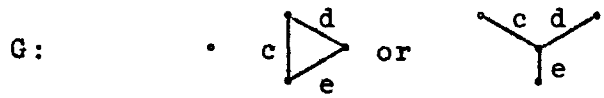


Figure 3

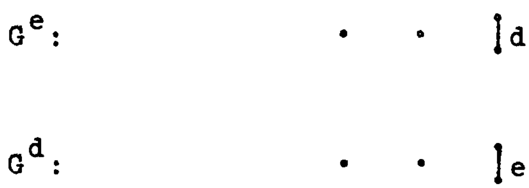


Figure 4

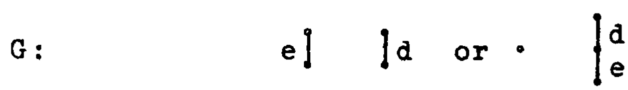


Figure 5

However, as yet, for any graph G such that $|E(G)| > 3$ there are no known counterexamples.

Greenwell and Hemminger in [5] show that the edge problem is a special case of the vertex problem. Chapter 2 will present this result.

Chapter 3 will diverge slighly and consider the vertex problem and edge problem for a special subclass of directed graphs.

Definition 0.7 - A directed graph is a graph G such that $V^{(2)} = \{[u,v]: u,v \varepsilon V\}$ in Definition 0.1 is the set of ordered pairs on V.

In particular we will present Harary and Palmer's [11] results that the edge problem holds for all tournaments, a special subclass of directed graphs, and the vertex problem holds for some tournaments. We also show that the vertex problem does not hold for all tournaments. We defer the definition of tournament to Chapter 3.

Harary and Manvel [8] simplified the vertex problem by labeling some of the vertices in a graph G. Using the labels, they tried to find exactly how many maximal subgraphs may be needed to determine partially labeled graphs. Thus in Chapter 4 we conclude this paper by considering the vertex problem for labeled graphs.

CHAPTER 1 - THE VERTEX PROBLEM

The current approach to the vertex problem is to solve it for a class of graphs, hoping to eventually include every graph in one of the solved classes.

Definition 1.1 - G is a graph. A walk is a sequence of vertices and edges of the form $v_1, \langle v_1, v_2 \rangle, v_2, \langle v_2, v_3 \rangle, \cdots, v_{k-1}, \langle v_{k-1}, v_k \rangle, v_k$ where $v_i \varepsilon V(G)$ and $\langle v_i, v_{i+1} \rangle \varepsilon E(G)$ and is called a walk of length k-1 from v_1 to v_k. A path of length k from u to v, $u,v \varepsilon V(G)$, is a walk of length k from u to v in which all vertices are distinct. A cycle of length k is a walk of length k in which all the vertices are distinct except $v_1 = v_{k+1}$. A spanning path is a path which contains all the vertices of G.

Definition 1.2 - Two vertices u,v of a graph G are said to be connected if there is a path from u to v. A graph G is connected if every pair of distinct vertices in G is joined by a path.

We present the results on the vertex problem in three parts. First, we look at the solution of the vertex problem for disconnected graphs and the class of graphs for which the vertex problem holds due to the result. Part two will present the results of the vertex problem for graphs that are connected but without cycles. At the time of writing of this paper there is no known solution of the vertex problem for connected graphs with cycles. The third part will give some partial results for this class of graphs.

§1.1 Disconnected Graphs

The first question we answer is how we know if a graph G is connected or disconnected from its subgraphs G_v.

Definition 1.3 - A vertex v is said to be a cut-vertex of a connected graph G if G_v is not connected.

Definition 1.4 - A graph G with no cycles is acyclic. A tree is a connected acyclic graph. A spanning tree of a graph G is a tree which contains all the vertices of G.

Theorem 1.1 - The graph G, $|V(G)| \geq 3$, is connected if and only if at least two of the subgraphs G_v are connected.

Proof - Every connected graph G has a spanning tree none of whose end-vertices can be cut-vertices of G. Since a nontrivial tree has at least two end-vertices, G has at least two vertices v for which G_v is connected.

To prove sufficiency, we assume without loss of generality that G_{v_1} and G_{v_2} are connected. In G_{v_1} and thus in G, there is a path joining v_2 and v_i, for each $i \geq 3$. Similarily, in G_{v_2} there is a path joining v_1 and v_3. Therefore G has every pair of distinct vertices joined by a path.

In [13] Kelly has stated that the vertex problem holds for disconnected graphs. A proof of this result using Lemma

1.6 [Kelly's Lemma], which is presented in the next chapter, is given by Greenwell and Hemminger [5]. F. Harary [6] and [7] has presented another proof of the vertex problem for disconnected graphs, but we have found a counterexample for his method of proof.

We now give a proof of a statement Harary used in his proof, present Harary's proof and give the counterexample to his proof.

Definition 1.5 - The degree $d(v)$ of a vertex v is the number of edges incident with it. A vertex of degree zero is said to be isolated.

Definition 1.6 - A component of a graph G is a maximal connected induced subgraph.

Lemma 1.2 - The graph G has exactly s $(s \geq 1)$ isolated vertices if and only if exactly s of the subgraphs G_v have exactly $s-1$ isolated vertices and the remaining subgraphs G_v each have at least s isolated vertices.

Proof - For a graph G with exactly s isolated vertices we have the following: If v is an isolated vertex then G_v will have exactly $s-1$ isolated vertices; if v is not an isolated vertex then G_v will have at least s isolated vertices. Thus exactly s of the subgraphs G_v have exactly $s-1$ isolated vertices and the remaining G_v have at least s isolated vertices each.

To prove sufficiency we assume G has more than s isolated vertices. Then each of the subgraphs G_v will have more than s-1 isolated vertices, thus contradicting that we have exactly s subgraphs G_v with s-1 isolated vertices.

Assume G has less than s isolated vertices and has exactly zero isolated vertices. Now any component of G has a spanning tree. If the component has 3 or more vertices, then we obtain at least two subgraphs G_v for the end-vertices of the spanning tree such that they have zero isolated vertices. For $s \geq 1$, this contradicts that we have 1 or less subgraphs with zero isolated vertices. Thus every component of G must contain exactly two vertices and all subgraphs G_v have one isolated vertex each. If $s=1$, this contradicts that we have exactly one subgraph with zero isolated vertices. If $s \geq 2$, this contradicts that we have two or less subgraphs with one isolated vertex. Thus G cannot have zero isolated vertices.

Assume G has less than s isolated vertices, but more than zero. Then there exists a subgraph that will contain s-2 or less isolated vertices. This contradicts the fact that all subgraphs G_v have s-1 or more isolated vertices.

Therefore G has exactly s isolated vertices.

From Lemma 1.2 we obtain the following corollary.

Corollary 1.3 - The graph G has exactly zero isolated vertices if and only if either

(i) each G_v has exactly one isolated vertex.

or (ii) at least two G_v have no isolated vertices.

<u>Proof</u> - By considering the components of G as in the proof of Lemma 1.2 we immediately see that either (i) or (ii) must hold.

If (ii), then by Theorem 1.1 G is connected and thus has no isolated vertices. If (i), then by Lemma 1.2 G must contain zero isolated vertices.

<u>Theorem 1.4</u> - (Kelly [13]) - If a graph G is disconnected, then it is reconstructible.

<u>Harary's Proof of Theorem 1.4 [7]</u> - In proving this, one can exploit either the components with the fewest or the most vertices. We choose the former, and first determine if G has any isolated vertices by Lemma 1.2. If G has $s \geq 1$ isolated vertices, it is recontructed by adding such a vertex to one of the subgraphs G_v having s-1 isolated vertices.

We assume G has no isolated vertices and k components. Since each component has a spanning tree, G will contain at least two subgraphs G_v with k components each. The remaining subgraphs G_v will contain k or more components depending on whether or not v is a cut-vertex. Thus the number k of components of G is the minimum number of components in any of the subgraphs G_v.

Consider these subgraphs G_v with k components and choose one, say G_{v_j}, in which the number of vertices in the smallest component is minimal. Now all the components of G_{v_j} will be the same as the components of G except the component with the minimal number of vertices. Thus we determine k-1 components of G. Now, among these k-1 components, we select one and remove a vertex which is not a cut-vertex of the component, thus forming a new component say H. Among the subgraphs G_v with k components, we find one, say G_{v_i}, having k-1 components isomorphic to the k-2 known components and H. The k^{th} component in this subgraph must be the heretofore unknown one, thereby reconstructing G.

The counterexample to this proof is illustrated in Figure 6. If we choose G_{v_1} as G_{v_j} and G_{v_3} as G_{v_i} we obtain a triangle as our k^{th} component, while we actually want a path of length two.

This is not the only counterexample to Harary's proof. It is just one of many.

We now present a proof of Theorem 1.4 following Harary's approach in his attempt to prove the theorem.

Proof of Theorem 1.4 - If G has isolated vertices, then Harary's proof holds. We assume G has no isolated vertices. As we saw in Harary's proof, the number k of components of G is the minimum number in any of the subgraphs G_v.

Consider these subgraphs G_v with k components and choose one, say G_{v_j}, in which the number of vertices, say n, in the smallest component is minimal. Thus we determine k-1 components of G and must determine the unique k^{th} unknown component. We consider three cases.

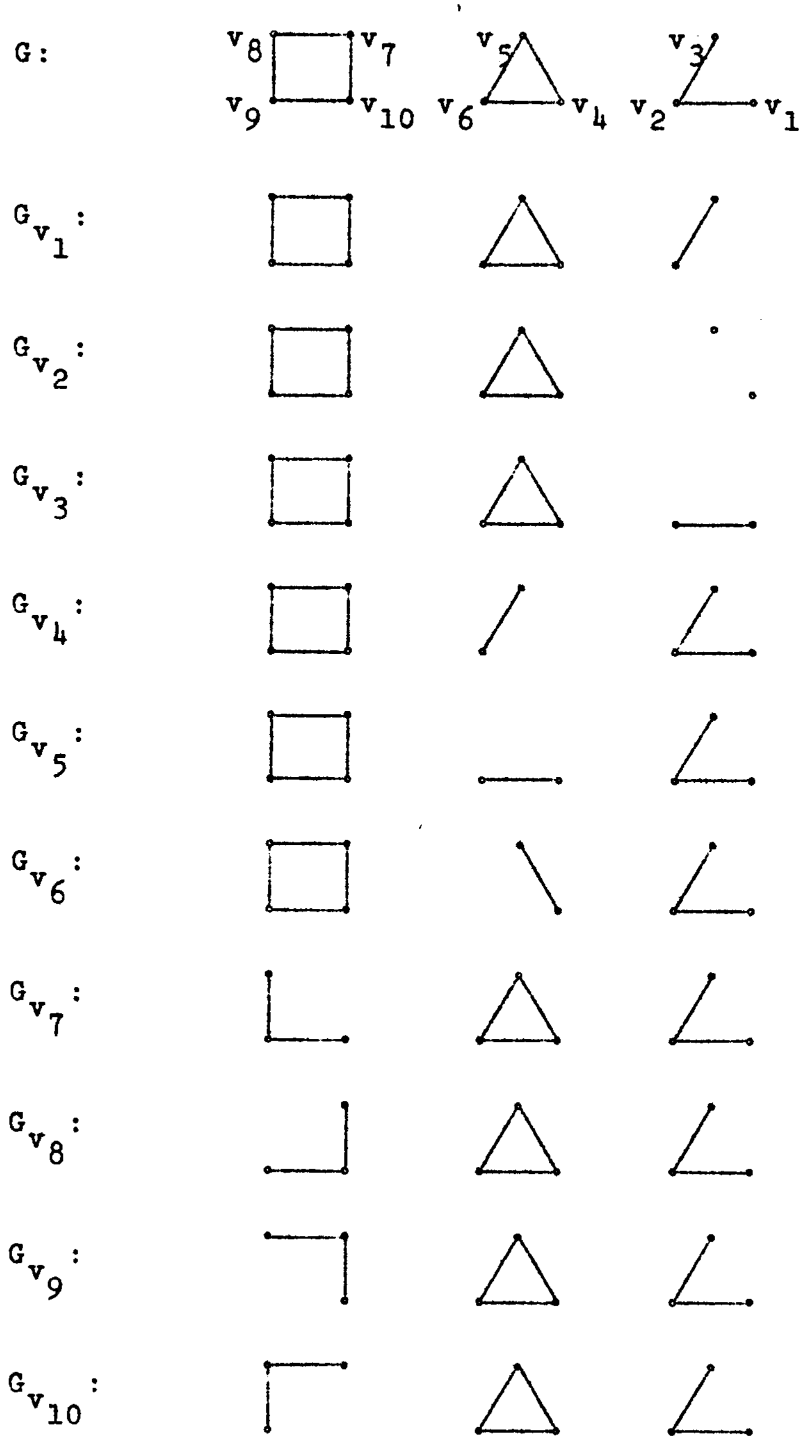

G:
v8
v7
v5
v3
v9
v10
v6
v4
v2
v1
G_{v_1}:
G_{v_2}:
G_{v_3}:
G_{v_4}:
G_{v_5}:
G_{v_6}:
G_{v_7}:
G_{v_8}:
G_{v_9}:
$G_{v_{10}}$:

Figure 6

Case (a) - Among the k components of G_{v_j}, k-1 components have greater than n+2 vertices each and one component has n vertices. Harary's proof holds for this case.

Case (b) - Among the k components of G_{v_j}, one component has n vertices, at least one component has n+2 vertices and the remaining components each have greater than n+2 vertices.

Consider all the subgraphs G_v with k components such that two components in each subgraph contain n+1 vertices. If in one of these subgraphs G_v, the two components with n+1 vertices are isomorphic then the k^{th} unknown component is one of the components with n+1 vertices. If no such subgraph exists, then choose one of these subgraphs, and check which component of the two components with n+1 vertices is not a subgraph of the components with n+2 vertices in G_{v_j}. This component then will be the k^{th} unknown component.

Case (c) - Among the k components of G_{v_j}, one component has n vertices, at least one component has n+1 vertices and the remaining components each have greater than n+1 vertices.

Consider all the subgraphs G_v with k components such that one component in each subgraph contains n vertices. If G_{v_j} contains different types of components, say k_1 of type A, k_2 of type B, etc., each with n+1 vertices and if the number of one type of component with n+1 vertices increases in any one of the subgraphs G_v considered here, then this type of component is the k^{th} unknown component; or if a new type of component with n+1 vertices appears in the considered subgraphs G_v, then this component is the $k^{\underline{th}}$ unknown component. Otherwise if the above does not happen, then all the components with n+1 vertices in G_{v_j} are isomorphic, i.e., all of one type, and the $k^{\underline{th}}$ unknown component is any one of these components with n+1 vertices.

Thus we have reconstructed G.

Definition 1.7 - The complement C(G) of a graph G has the same vertices as G, and two vertices are adjacent in C(G) if and only if they are not adjacent in G.

We conclude this section with the following corollary to Theorem 1.4.

Corollary 1.5 - If the complement C(G) of a graph G is disconnected, then the vertex problem is true for G.

Proof - Suppose there is a one-to-one onto function $\sigma:V(G)\to V(H)$ such that $G_v \simeq H_{\sigma(v)}$ for all $v\epsilon V(G)$ where G and H are n order graphs. Then $C(G_v)\simeq C(H_{\sigma(v)})$ for all $v\epsilon V(G)=V(C(G))$. But $(C(G))_v=C(G_v)$ and $(C(H))_{\sigma(v)}=C(H_{\sigma(v)})$ and therefore $(C(G))_v\simeq(C(H))_{\sigma(v)}$ for all $v\epsilon V(C(G))$. Since C(G) is disconnected by Theorem 1.4 $C(G)\simeq C(H)$. Therefore $G\simeq H$.

§1.2 Connected Graphs Without Cycles

In Definition 1.4 we define a tree as a connected graph without cycles. Thus we will be concerned with the vertex problem for trees in this section. Kelly [13] was the first person to prove the vertex problem for trees by proving the following theorem.

Kelly's Theorem [13] 1.6 - A tree T with at least three vertices is uniquely reconstructible by the subgraphs T_v.

The proof of Kelly's Theorem used two very important lemmas. The conditions for the vertex problem are satisfied for both lemmas.

Kelly's Lemma [13] 1.7 - Every type of vertex proper induced subgraph which occurs in G or H occurs the same number of times in both, and v and $\sigma(v)$ are vertices in the same number of these induced subgraphs for all v in V(G).

Proof - Let T denote a certain type of graph on j vertices, where $2 \leq j < |V(G)|$, which occurs as an induced subgraph α times in G and β times in H. Also let $V(G)=\{v_i : i=1,\cdots,n\}$ and α_i be the number of T-type induced subgraphs which have v_i as a vertex. Then

$$\alpha = \sum_{i=1}^{n} \alpha_i / j \qquad \text{and} \qquad \beta = \sum_{i=1}^{n} \beta_i / j \tag{1}$$

where β_i is the number of T-type induced subgraphs having $\sigma(v_i)$ as a vertex. Since $G_{v_i} \cong H_{\sigma(v_i)}$, the number of T-type induced subgraphs which do not have v_i as a vertex is the same as the number which do not have $\sigma(v_i)$ as a vertex. Thus

$\alpha-\alpha_i=\beta-\beta_i$, $i=1,\cdots,n$. Therefore,

$\sum_{i=1}^{n}(\alpha-\beta) = \sum_{i=1}^{n}(\alpha_i-\beta_i)$ and hence

$$n(\alpha-\beta)=\sum_{i=1}^{n}\alpha_i-\sum_{i=1}^{n}\beta_i=j\alpha-j\beta=j(\alpha-\beta)$$

from (1). But since $n\neq j$ we have $\alpha-\beta=0$ and hence $\alpha=\beta$. Since $\alpha=\beta$ we also know $\alpha_i=\beta_i$, $i=1,\cdots,n$ and the lemma is proved.

Lemma 1.8 - (Kelly [13]) - The vertices v_i and $\sigma(v_i)$ have the same degree, i.e., $d(v_i) = d(\sigma(v_i))$ for $i=1,\cdots,n$.

Proof - In Lemma 1.7 let $j = 2$ and we have $d(v_i)=\alpha_i=\beta_i=d(\sigma(v_i))$.

Harary and Palmer [10] showed that not all of the T_v are needed in Kelly's Theorem by proving the following theorem.

Theorem 1.9 - (Harary and Palmer [G]) - A tree T with at least three vertices is uniquely reconstructible by the subgraphs $\{T_v\}_{v\in S}$ where $S=\{v\in V(T):d(v)=1\}$.

J.A. Bondy [2] has improved this result even further, the proof of which we present.

Definition 1.8 - We shall denote the length of the shortest path between vertices u,v by d(u,v). The radius of the graph G is

$$r = \min_{u\in G}.(\max_{v\in G}.d(u,v)).$$

Any vertex c for which $\max_{v\in G}.d(c,v) = r$ is called a center of G. A peripheral vertex is a vertex p for which

$$\max_{v \in G} d(p,v) = \max_{u \in G}(\max_{v \in G} d(u,v))$$

and for convience we shall call it a p-vertex from now on.

Theorem 1.10 - (Bondy [2]) - Let G and H be two trees, each containing N p-vertices. If the p-vertices $\{u_i\}_{i=1}^N$ of G and $\{v_i\}_{i=1}^N$ of H may be paired so that $G_{u_i} \simeq H_{v_i}$, $1 \le i \le N$, then $G \simeq H$.

Before presenting the proof of Theorem 1.10 we present some preliminary results.

Theorem 1.11 - A tree has either one center or two centers.

A proof of Theorem 1.11 may be found in [14].

Definition 1.9 - We call a tree central or bicentral according as it has one or two centers.

We let the pairs (u_i, v_i) with the same index denote the corresponding p-vertices in the sense of Theorem 1.10. We also assume, until the completion of the proof of Theorem 1.10, that G and H satisfy the hypothesis of Theorem 1.10.

Theorem 1.12 - G and H have the same radius and are either both central or both bicentral.

Proof - If T is a central tree of radius r, longest paths in T have length 2r; if T is a bicentral tree of radius r, longest paths in T have length 2r-1. Hence it suffices to show that longest paths in G and H have the same length. Now let d_G, d_H be the lengths of the longest paths in G,H respectively and let C_G, C_H be two such paths.

(a) N=2. If u is a p-vertex in G, longest paths in G_u have length d_G-1 and longest paths in H_v (where v corresponds to u) have length d_H-1. Since $G_u \simeq H_v$ we must have $d_G-1 = d_H-1$. Therefore $d_G = d_H$.

(b) N>2. There exists a p-vertex $u \varepsilon G$ such that $u \notin C_G$. Then $C_G \subset G_u \simeq H_v \subset H$ and therefore $d_G \leq d_H$. Similiarly there exists a p-vertex $v' \varepsilon H$ such that $v' \notin C_H$. Then $C_H \subset H_{v'} \simeq G_{u'} \subset G$ and so $d_H \leq d_G$. Therefore $d_G = d_H$.

We now assume that G and H are central trees of radius r. Proofs for the case when G and H are bicentral trees will follow later.

Lemma 1.13 - Let T and U be central trees of radius r. Then, whenever T is isomorphic to a subgraph of U, p-vertices in T correspond under the isomorphism to p-vertices in U.

Proof - Suppose $T \simeq T' \subset U$. We must show that every p-vertex of T' is a p-vertex of U. Let u be the center of U. Then $d(u,v) \leq r$ for all $v \varepsilon V$, thus $d(u,v) \leq r$ for all $v \varepsilon T'$ and therefore u is the center of T'. If v' is a p-vertex of T' then $d(u,v') = r$ and hence v' is a p-vertex of U.

Theorem 1.14 - Let T be any central tree of radius r with n p-vertices ($n < N$). Suppose there are α distinct subgraphs of G which are isomorphic to T, and that the p-vertex

u_i of G is in α_i of these subgraphs; that there are β distinct subgraphs of H which are isomorphic to T and that the p-vertex v_i of H is in β_i of these subgraphs. Then $\alpha = \beta$ and $\alpha_i = \beta_i$ $(1 \leq i \leq N)$.

Proof - From Lemma 1.13 and by an argument analogus to the proof of Lemma 1.7 the theorem follows.

Definition 1.10 - A pendant vertex is a vertex of degree one. A limb of a central tree T is a subtree which contains the center of T as a pendant vertex and is joined to the rest of T at the center. A limb which contains a p-vertex of T is called a radial limb.

We shall denote the sets of radial limbs $\{\bar{G}_i\},\{\bar{H}_i\}$ of G,H by Γ,T respectively; the sets of non-radial limbs $\{\bar{K}_i\}$, $\{\bar{L}_i\}$ of G,H by Ω,Λ respectively. In addition, we write $K = \bigcup_i \bar{K}_i$, $L = \bigcup_i \bar{L}_i$

Theorem 1.15 - Let N> 2. Then if the p-vertex u_i is on a limb containing γ_i p-vertices, v_i is also on a limb containing γ_i p-vertices.

Proof - Since a tree is connected and contains no cycles, there is one and only one path between any pair u,v of its vertices. We denote this path by C(u,v). If u_i is on a limb containing γ_i p-vertices then u_i lies on exactly $N-\gamma_i$ paths of length 2r in G, namely, $\{C(u_i,g) \cup C(g,u_j)\}_{u_j \in I}$ where I is the set of p-vertices of G not the limb containing u_i and g is the center of G. Now a path of length 2r is a central tree with two p-vertices, and therefore by Theorem 1.14, the p-vertex v_i of H lies on $N - \gamma_i$ paths of length 2r in H. Hence v_i is on a limb containing γ_i p-vertices.

Corollary 1.16 - If G has M radial limbs $\{\tilde{G}_i\}_1^M$ where $\tilde{G}_i$ contains γ_i p-vertices, then H also has M radial limbs $\{\tilde{H}_i\}_1^M$ and these may be ordered so that $\tilde{H}_i$ contains γ_i p-vertices.

Proof - If N = 2, then both G and H have just two radial limbs with one p-vertex on each limb.

If N > 2, denote by u_{ij}, $1 \leq j \leq \gamma_i$, the p-vertices on $\tilde{G}_i$. Then u_{ij} lies on a limb containing γ_i p-vertices. Therefore, by Theorem 1.15, v_{ij} lies on a limb containing γ_i p-vertices, $1 \leq j \leq \gamma_i$, $1 \leq i \leq M$, and this implies the truth of the corollary.

Proof of Theorem 1.10 - Central Case - By Corollary 1.16, G and H have the same number of radial limbs; say M. For $1 \leq i \leq M$ we may suppose that $\tilde{G}_i$ and $\tilde{H}_i$ each contain γ_i p-vertices, where $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_M$.

θ, ϕ, and ϕ' respectively will denote isomorphisms of G_{u_1} onto H_{v_1}, of G_{u_2} onto H_{v_2}, and of $G_{u_2'}$ onto $H_{v_2'}$, where $v_2' = \theta(u_2)$.

The notation $(U,u) \cong (W,v)$ will mean that the graphs U,W are isomorphic under an isomorphism mapping vertex $u \in U$ into vertex $v \in W$. If α is such an isomorphism we shall write

$\alpha(U,u) = (W,v)$. Two sets $\Xi = \{S_i\}$, $\P = \{T_i\}$ of graphs are isomorphic ($\Xi \simeq \P$) if there is a one-to-one function from Ξ onto $\P$ which maps each graph in Ξ onto an isomorphic graph in $\P$. $(\Xi,u) \simeq (\P,v)$ (or $\alpha(\Xi,u) = (\P,v)$) will mean that $(\Xi \simeq \P)$, that $u \in S_i$, $v \in T_i$ for all i, and that for each pair of isomorphic graphs there is an isomorphism (α) mapping vertex u into vertex v.

The proof will be achieved by showing $(\Gamma,g) \simeq (T,h)$ and $(\Omega,g) \simeq (\Lambda,h)$.

Case 1. $M > 2$. If u_i is a p-vertex in G, then $M > 2$ implies G_{u_i} and H_{v_i} are central trees of radius r. In any isomorphism of G_{u_i} onto H_{v_i} the centers must correspond. Hence

$$\theta(g) = h, \ \phi(g) = h, \ \phi'(g) = h \qquad (1)$$

where g and h are, respectively, the centers of G and H. It is also clear that these isomorphisms map radial limbs onto radial limbs. Let u_1, u_2 be p-vertices in $\tilde{G}_1, \tilde{G}_2$, respectively. We may assume $v_1 \in \tilde{H}_1$. Let $(\tilde{G}_i)_u$ denote the induced subgraph of $\tilde{G}_i$ with the vertex set of $\tilde{G}_i$ minus u. Since $(\tilde{G}_1)_{u_1}, (\tilde{H}_1)_{v_1}$ each contain $\gamma_1 - 1$ p-vertices, $\theta(\tilde{G}_{u_1}) = \tilde{H}_{v_1}$ implies

$$\theta(\{\tilde{G}_i\}_2^M, g) = (\{\tilde{H}_i\}_2^M, h)$$

and there is no loss of generality in assuming that

$$\theta(\tilde{G}_i, g) = (\tilde{H}_i, h) \qquad (2 \leq i \leq M). \qquad (2)$$

We now consider three subcases.

(a) $\gamma_2 > \gamma_1 + 1$. Then $\phi(G_{u_2}) = H_{v_2}$ implies

$$\phi(\tilde{G}_1) = \tilde{H}_1$$

since $(\tilde{G}_i)_{u_2}, (\tilde{H}_i)_{v_2}$, for $i \geq 2$, each contain more than γ_1 p-vertices.

(b) $\gamma_2 = \gamma_1 + 1$. From (2) $\theta(u_2) = v_2' \in \tilde{H}_2$, and hence

$$\theta((\tilde{G}_2)_{u_2}) = (\tilde{H}_2)_{v_2'}.$$

Also $\phi(G_{u_2}) = H_{v_2}$ implies either $\phi(\tilde{G}_1) = \tilde{H}_1$ or $\phi((\tilde{G}_2)_{u_2}) = \tilde{H}_1$ and $\phi'(G_{u_2'}) = H_{v_2'}$ implies either

$$\phi'(\tilde{G}_1) = \tilde{H}_1 \text{ or } \phi'(\tilde{G}_1) = (\tilde{H}_2)_{v_2'}.$$

Therefore, at least one of ϕ, ϕ', $\phi\theta^{-1}\phi'$ (product of isomorphisms) maps $\tilde{G}_1$ onto $\tilde{H}_1$.

(c) $\gamma_2 = \gamma_1$. Suppose $v_2 \in \tilde{H}_k$. Then $\phi(G_{u_2}) = H_{v_2}$ implies

$$\phi(\{\tilde{G}_i\}_{i\neq 2}, g) = (\{\tilde{H}_i\}_{i\neq k}, h).$$

Put $\phi_r = \phi(\theta^{-1}\phi)^r$. We shall show that, for some r, either

$$\phi_r(\tilde{G}_1) = \tilde{H}_1 \text{ or } \phi_r(\tilde{G}_1) = \tilde{H}_2. \qquad (3)$$

For assume otherwise. Then $\phi_r(\tilde{G}_1)$ is well defined for all r as the cases $\phi(\tilde{G}_2)$ and $\theta^{-1}(\tilde{H}_1)$ do not occur by our assumption. Thus we may put $\phi_r(\tilde{G}_1) = \tilde{H}_{j_r}$ where $j_r > 2$. Thus

$$\theta^{-1}\phi_r(\tilde{G}_1) = \theta^{-1}(\tilde{H}_{j_r}) = \tilde{G}_{j_r}. \qquad (4)$$

Since M is finite there are integers m,n ($m < n$) such that $\tilde{H}_{j_m} = \tilde{H}_{j_n}$. Then

$$\phi_n(\tilde{G}_1) = \phi_m(\tilde{G}_1)$$

$$\theta_m^{-1}\phi_n(\tilde{G}_1) = \tilde{G}_1$$

$$(\phi(\theta^{-1}\phi)^m)^{-1}(\phi(\theta^{-1}\phi)^n)(\tilde{G}_1) = \tilde{G}_1$$

$$(\theta^{-1}\phi)^{-m}\phi^{-1}\phi(\theta^{-1}\phi)^n(\tilde{G}_1) = \tilde{G}_1$$

$$(\theta^{-1}\phi)^{-m}(\theta^{-1}\phi)^n(\tilde{G}_1) = \tilde{G}_1$$

$$(\theta^{-1}\phi)^{n-m}(\tilde{G}_1) = \tilde{G}_1$$

$$\theta^{-1}\theta(\theta^{-1}\phi)^{n-m}(\tilde{G}_1) = \tilde{G}_1$$

$$\theta^{-1}\phi\phi^{-1}\theta(\theta^{-1}\phi)^{n-m}(\tilde{G}_1) = \tilde{G}_1$$

$$\theta^{-1}\phi(\theta^{-1}\phi)^{-1}(\theta^{-1}\phi)^{n-m}(\tilde{G}_1) = \tilde{G}_1$$

$$\theta^{-1}\phi(\theta^{-1}\phi)^{n-m-1}(\tilde{G}_1) = \tilde{G}_1$$

$$\theta^{-1}\phi_{n-m-1}(\tilde{G}_1) = \tilde{G}_1.$$

Thus contradicting (4).

Hence (3) follows.

From (2) $v_2' \varepsilon \tilde{H}_2$. Then $\phi'(G_{u_2'}) = H_{v_2'}$ implies

$$\phi'(\{\tilde{G}_i\}_{i\neq k}, g) = (\{\tilde{H}_i\}_{i\neq 2}, h).$$

Put $\phi_s' = \phi'(\theta^{-1}\phi')^s$. It is clear that

$$\phi_s'^{-1} = \phi'^{-1}(\theta\phi'^{-1})^s = \psi_s$$

and by an argument analogous to the above that, for some s, either $\psi_s(\tilde{H}_1) = \tilde{G}_1$ or $\psi_s(\tilde{H}_1) = \tilde{G}_2$. Thus for some s, either

$$\phi_s'(\tilde{G}_1) = \tilde{H}_1 \text{ or } \phi_s'(\tilde{G}_2) = \tilde{H}_1.$$

Therefore, for some r,s at least one of ϕ_r, ϕ_s', $\phi_s'\theta^{-1}\phi_r$ maps $\tilde{G}_1$ onto $\tilde{H}_1$.

(a), (b), (c) and (1) together imply that an isomorphism θ_1 exists such that $\theta_1(\tilde{G}_1, g) = (\tilde{H}_1, h)$

Thus $(\Gamma,g) \approx (T,h)$.

We now show that $(\Omega,g) \approx (\Lambda,h)$. If $\gamma_1 > 1$ then G_{u_1} has the same non-radial limbs as G and H_{v_1} has the same non-radial limbs as H. Hence $\theta(G_{u_1}) = H_{v_1}$ implies $\theta(\Omega,g) = (\Lambda,h)$.

If $\gamma_1 = 1$, then $\theta_1(\tilde{G}_1,g) = (\tilde{H}_1,h)$ implies $\theta_1(u_1) = v_1$ and hence

$$\theta_1((\tilde{G}_1)_{u_1},g) = ((\tilde{H}_1)_{v_1},h). \qquad (5)$$

Now G_{u_1} has non-radial limbs $(\tilde{G}_1)_{u_1} \cup \Omega$ and H_{v_1} has non-radial limbs $(\tilde{H}_1)_{v_1} \cup \Lambda$. Therefore

$$\theta((\tilde{G}_1)_{u_1} \cup \Omega,g) = ((\tilde{H}_1)_{v_1} \cup \Lambda,h).$$

If $\theta((\tilde{G}_1)_{u_1},g) = ((\tilde{H}_1)_{v_1},h)$ then $\theta(\Omega,g) = (\Lambda,h)$. Otherwise $\theta((\tilde{G}_1)_{u_1},g) = (\tilde{L}_j,h)$ and $\theta(\tilde{K}_k,g) = ((\tilde{H}_1)_{v_1},h)$ for some j,k. Then $\theta(\{\tilde{K}_i\}_{i\neq k},g) = (\{\tilde{L}_i\}_{i\neq j},h)$. But from (5)

$$\theta\theta_1^{-1}\theta(\tilde{K}_k,g) = (\tilde{L}_j,h).$$

Hence $(\Omega,g) \approx (\Lambda,h)$.

Therefore $G \approx H$.

<u>Case 2</u>. M = 2. For $\gamma_1 > 1$ the proof of Case 1 suffices. We shall therefore assume that $\gamma_1 = 1$. For i = 1,2, denote by g_i the vertex adjacent to g on $\tilde{G}_i$ and by h_i the vertex adjacent to h of $\tilde{H}_i$. Let u_1 be the p-vertex in $\tilde{G}_1$. We may

assume that $v_1 \in \tilde{H}_1$. Then G_{u_1} and H_{v_1} are bicentral trees of radius r. In the isomorphism θ: $G_{u_1} \to H_{v_1}$ the bicenters g, g_2 of G_{u_1} and h, h_2 of H_{v_1} must correspond in some order. Hence either

$$(\alpha_1): \theta(g) = h \text{ and } \theta(g_2) = h_2$$

or

$$(\alpha_2): \theta(g) = h_2 \text{ and } \theta(g_2) = h$$

$$\begin{aligned} &(\alpha_1) \text{ implies } \theta((\tilde{G}_1)_{u_1} \cup \Omega, g) = ((\tilde{H}_1)_{v_1} \cup \Lambda, h) \\ &\qquad \text{and } \theta(\tilde{G}_2, g) = (\tilde{H}_2, h) \end{aligned} \tag{6}$$

$$\begin{aligned} &(\alpha_2) \text{ implies } \theta((\tilde{G}_1)_{u_1} \cup K, g) = ((\tilde{H}_2)_h, h_2) \\ &\qquad \text{and } \theta((\tilde{G}_2)_g, g_2) = ((\tilde{H}_1)_{v_1} \cup L, h) \end{aligned} \tag{7}$$

We first make two points.

$$(\tilde{G}_1, g) \simeq (\tilde{H}_1, h) \text{ implies } ((\tilde{G}_1)_{u_1}, g) \simeq ((\tilde{H}_1)_{v_1}, h) \tag{8}$$

since $\gamma_1 = 1$, u_1 and v_1 must correspond under any such isomorphism.

$$\begin{aligned} &((\tilde{G}_i)_g, g_i) \simeq ((\tilde{H}_i)_h, h_i) \text{ implies} \\ &\qquad (\tilde{G}_i, g) \simeq (\tilde{H}_i, h) \qquad (i = 1,2) \end{aligned} \tag{9}$$

since g is joined to g_i and h is joined to h_i.

We now consider three subcases.

(a) $\gamma_2 > 2$. Let u_2 be a p-vertex in $\tilde{G}_2$. Then $v_2 \in \tilde{H}_2$ since v_1 is the p-vertex in $\tilde{H}_1$. G_{u_2} and H_{v_2} are central trees of radius r and $\phi(G_{u_2}) = H_{v_2}$ implies

$$\phi(\tilde{G}_1,g) = (\tilde{H}_1,h) \text{ and } \phi(\Omega,g) = (\Lambda,h). \tag{10}$$

If (α_1) holds, then from (6) $\theta(\tilde{G}_2,g) = (\tilde{H}_2,h)$.

If (α_2) holds, then from (8) and (10)

$$\phi((\tilde{G}_1)_{u_1} \cup K,g) = ((\tilde{H}_1)_{v_1} \cup L,h).$$

Hence by (7), $\theta\phi^{-1}\theta$ maps $(\tilde{G}_2)_g$ onto $(\tilde{H}_2)_h$. Thus

$$((G_2)_g,g_2) \simeq ((\tilde{H}_2)_h,h_2)$$

and by (9) $(\tilde{G}_2,g) \simeq (\tilde{H}_2,h)$.

Hence in both cases $(\tilde{G}_2,g) \simeq (\tilde{H}_2,h)$ and this with (10) shows

$$G \simeq H.$$

(b) $\gamma_2 = 2$. Let u_2 be a p-vertex in $\tilde{G}_2$. Then

$$\phi(G_{u_2}) = H_{v_2} \text{ implies } \phi(\Omega,g) = (\Lambda,h) \tag{11}$$

and either

$$(\beta_1)\colon \phi(g_1) = h_1 \text{ and } \phi(g_2) = h_2$$

or

$$(\beta_2)\colon \phi(g_1) = h_2 \text{ and } \phi(g_2) = h_1.$$

(β_1) implies $\phi(\tilde{G}_1,g) = (\tilde{H}_1,h)$ and

$$\phi((\tilde{G}_2)_{u_2},g) = ((\tilde{H}_2)_{v_2},h). \tag{12}$$

(β_2) implies $\phi(\tilde{G}_1,g) = ((\tilde{H}_2)_{v_2},h)$ and

$$\phi((\tilde{G}_2)_{u_2},g) = (\tilde{H}_1,h). \tag{13}$$

Also $\phi'(G_{u_2'}) = H_{v_2'}$ implies either

(δ_1): $\phi'(g_1) = h_1$ and $\phi'(g_2) = h_2$

or

(δ_2): $\phi'(g_1) = h_2$ and $\phi'(g_2) = h_1$.

(δ_1) implies $\phi'(\tilde{G}_1,h) = (\tilde{H}_1,h)$ and

$$\phi'((\tilde{G}_2)_{u_2'},g) = ((\tilde{H}_2)_{v_2'},h). \tag{14}$$

(δ_2) implies $\phi'(\tilde{G}_1,h) = ((\tilde{H}_2)_{v_2'},h)$ and

$$\phi'((\tilde{G}_2)_{u_2'},g) = (\tilde{H}_1,h). \tag{15}$$

If (α_1) holds, then from (6) $\theta(\tilde{G}_2,g) = (\tilde{H}_2,h)$.

Since $\theta(u_2) = v_2'$

$$\theta((\tilde{G}_2)_{u_2},g) = ((\tilde{H}_2)_{v_2'},h). \tag{16}$$

Now (11) through (16) imply that at least one of ϕ, ϕ', and $\phi\theta^{-1}\phi'$ maps $\tilde{G}_1$ onto $\tilde{H}_1$. Hence $(\tilde{G}_1,g) \simeq (\tilde{H}_1,h)$ and with (6) and (11) this shows that $G \simeq H$.

If (α_2) and (β_1) hold, then from (12) $\phi(\tilde{G}_1,g) = (\tilde{H}_1,h)$ and by (7), (8), (11) and (12), $\theta\phi^{-1}\theta$ maps $(\tilde{G}_2)_g$ onto $(\tilde{H}_2)_h$. Therefore $((\tilde{G}_2)_g,g_2) \simeq ((\tilde{H}_2)_h,h_2)$ and by (9)

$$(\tilde{G}_2,g) \simeq (\tilde{H}_2,h).$$

Hence $G \simeq H$.

If (α_2) and (β_2) hold, then from (7)

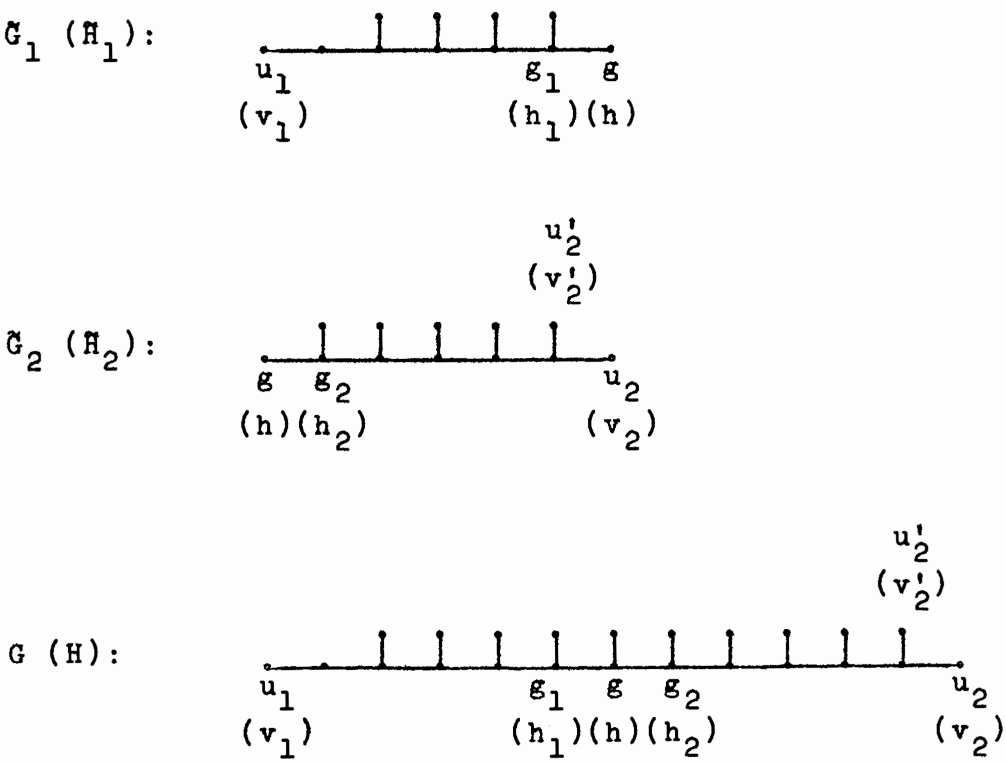
$$|E(\tilde{H}_2)| = |E(\tilde{G}_1)| + |E(K)|$$

and from (13)

$$|E(\tilde{H}_2)| = |E(\tilde{G}_1)| + 1.$$

Therefore K has just one edge. By means of (7) and (13) one can construct $\tilde{G}_1$ and $\tilde{H}_2$ and show that they must be the following graphs.

$\tilde{G}_1$ is a path of length r with an edge attached to each vertex except the vertex at the central end and two vertices at the other end. $\tilde{H}_2$ is a path of length r with an edge attached to each vertex except the vertices at either end. An example of the two graphs is presented in Figure 7.



<u>Figure 7</u>

By a similiar argument one may prove that L has just one edge, that $\tilde{H}_1$ is a graph isomorphic to $\tilde{G}_1$ and that $\tilde{G}_2$ is a graph isomorphic to $\tilde{H}_2$. Examples of these two graphs again are presented in Figure 7. Thus $(\tilde{G}_1,g) \simeq (\tilde{H}_1,h)$ and $(\tilde{G}_2,g) \simeq (\tilde{H}_2,h)$.

Hence $G \simeq H$.

(c) $\gamma_2 = 1$. Under the isomorphism ϕ: $G_{u_2} \to H_{v_2}$ either

(β_1): $\phi(g) = h$ and $\phi(g_1) = h_1$

or

(β_2): $\phi(g) = h_1$ and $\phi(g_1) = h$.

(β_1) implies $\phi(\tilde{G}_1,g) = (\tilde{H}_1,h)$ and

$$\phi((\tilde{G}_2)_{u_2} \cup \Omega,g) = ((\tilde{H}_2)_{v_2} \cup \Lambda,h). \qquad (17)$$

(β_2) implies $\phi((\tilde{G}_1)_g,g_1) = ((\tilde{H}_2)_{v_2} \cup L,h)$ and

$$\phi((\tilde{G}_2)_{u_2} \cup K,g) = ((\tilde{H}_1)_h,h_1). \qquad (18)$$

We have four cases to consider (α_1,β_1), (α_1,β_2), (α_2,β_1) and (α_2,β_2). As (α_1,β_2) and (α_2,β_1) are symmetric cases, it suffices to consider just one of these two.

(α_1,β_1) holds: From (6) $\theta(\tilde{G}_2,g) = (\tilde{H}_2,h)$.

From (17) $\phi(\tilde{G}_1,g) = (\tilde{H}_1,h)$.

Now (6), (8), and (17) imply that $(\Omega,g) \simeq (\Lambda,h)$ by an argument similar to that used in Case 1(c) for $\gamma_1 = 1$.

Therefore $G \simeq H$.

(α_1,β_2) holds: From (6), $\theta(\tilde{G}_2,g) = (\tilde{H}_2,h)$ and, hence, since $\gamma_2 = 1$,

$$\theta((\tilde{G}_2)_{u_2},g) = ((\tilde{H}_2)_{v_2},h). \tag{19}$$

If $(\Omega,g) \simeq (\Lambda,h)$ then $(K,g) \simeq (L,h)$ and thus from (19)

$$\theta((\tilde{G}_2)_{u_2} \cup K,g) = ((\tilde{H}_2)_{v_2} \cup L,h). \tag{20}$$

From (18) and (20) $\phi\theta^{-1}\phi((\tilde{G}_1)_g,g_1) = ((\tilde{H}_1)_h,h_1)$.

From (9) $(\tilde{G}_1,g) \simeq (\tilde{H}_1,h)$.

Thus $G \simeq H$.

If $(\Omega,g) \not\simeq (\Lambda,h)$, then by (6)

$$((\tilde{G}_1)_{u_1},g) \not\simeq ((\tilde{H}_1)_{v_1},h).$$

Therefore, $(\tilde{G}_1)_{u_1}$ is isomorphic to a non-radial limb in H, say $((\tilde{G}_1)_{u_1},g) \simeq (\tilde{L}_1,h)$. Similarly we may put

$$((\tilde{H}_1)_{v_1},h) \simeq (\tilde{K}_1,g) \tag{21}$$

From (18) $|E(\tilde{H}_1)| = |E(K)| + |E(\tilde{G}_2)|$

and from (21) $|E(\tilde{H}_1)| = |E(\tilde{K}_1)| + 1 \leqq |E(K)| + 1$.

Therefore $|E(\tilde{G}_2)| \leqq 1$. Since $M = 2$, Ω and Λ are both empty, contradicting the supposition $(\Omega,g) \not\simeq (\Lambda,h)$.

Hence $G \simeq H$.

(α_2,β_2) holds: From (7) $|E(\tilde{G}_1)| + |E(K)| = |E(\tilde{H}_2)|$ and from (18) $|E(\tilde{H}_2)| + |E(L)| = |E(\tilde{G}_1)|$.

Therefore $|E(K)| + |E(L)| = 0$ and so Ω and Λ are both empty.

Hence by (7) and (18)

$$((G_1)_{u_1},g) \simeq ((H_2)_h,h_2) \text{ and } ((G_1)_g,g_1) \simeq ((H_2)_{v_2},h).$$

Using these two equations one can construct G_1 and H_2 and show that they must both be paths of length r. Similarly G_2 and H_1 are paths of length r. Therefore G and H are both paths of length 2r and hence $G \simeq H$.

This completes the proof of the central case of Theorem 1.10. Before presenting the proof of the bicentral case, we again present some preliminary results.

The following is the analogue of Theorem 1.14 for bicentral trees, and may be proved in a similar way.

Theorem 1.14A - Let T be any bicentral tree of radius r with n p-vertices ($n < N$). Suppose there are α distinct subgraphs of G which are isomorphic to T, and that the p-vertex u_i of G is in α_i of these subgraphs; that there are β distinct subgraphs of H which are isomorphic to T, and that the p-vertex v_i of H is in β_i of these subgraphs. Then

$$\alpha = \beta \quad \text{and} \quad \alpha_i = \beta_i \qquad (1 \leqq i \leqq N).$$

Definition 1.11 - The central edge of a bicentral tree is the edge joining its bicenters. A limb of a bicentral tree T is a subtree which contains the central edge and both bicenters of T, one center as a pendant vertex and the other as a vertex of degree two and which is joined to the rest of

T at the bicenters.

Let G have bicenters g and $\bar{g}$ and radius r; let H have bicenters h and $\bar{h}$ and radius r. Denote by $\Gamma = \{G_i\}$ and $\bar{\Gamma} = \{\bar{G}_i\}$, respectively, the sets of radial limbs of G having $\bar{g}$ and g as pendant vertices and by $T = \{H_i\}$ and $\bar{T} = \{\bar{H}_i\}$, respectively, the sets of radial limbs of H having $\bar{h}$ and h as pendant vertices; by $\Omega = \{K_i\}$ and $\bar{\Omega} = \{\bar{K}_i\}$, respectively, the sets of non-radial limbs of G having $\bar{g}$ and g as pendant vertices and by $\Lambda = \{L_i\}$, $\bar{\Lambda} = \{\bar{L}_i\}$, respectively, the sets of non-radial limbs of H having $\bar{h}$ and h as pendant vertices. As before we write $K = \bigcup_i K_i$, $\bar{K} = \bigcup_i \bar{K}_i$, $L = \bigcup_i L_i$, and $\bar{L} = \bigcup_i \bar{L}_i$.

θ, ϕ, and ψ, respectively, will denote isomorphisms of G_{u_1} onto H_{v_1}, of G_{u_2} onto H_{v_2} and of G_{u_3} onto H_{v_3}. We extend the notation of the central case by writing

$$(\Xi, u, u') \simeq (\P, v, v') \quad (\text{or } \alpha(\Xi, u, u') = (\P, v, v')$$

if there is an isomorphism α such that $\alpha(\Xi, u) = (\P, v)$ and $\alpha(\Xi, u') = (\P, v')$.

<u>Theorem 1.15A</u> - Suppose there are γ_G and $\bar{\gamma}_G$ p-vertices of G in Γ and $\bar{\Gamma}$, respectively, and γ_H and $\bar{\gamma}_H$ p-vertices of H in T and $\bar{T}$, respectively. We may assume that g and $\bar{g}$, and h and $\bar{h}$ are ordered so that $\gamma_G \leqq \bar{\gamma}_G$ and $\gamma_H \leqq \bar{\gamma}_H$. Then

$$\gamma_G = \gamma_H \quad \text{and} \quad \bar{\gamma}_G = \bar{\gamma}_H.$$

<u>Proof</u> - G and H each have N p-vertices.

If $N = 2$, then $\gamma_G = \bar{\gamma}_G = \gamma_H = \bar{\gamma}_H = 1$.

If $N > 2$, let u_i be a p-vertex in Γ then u_i lies on $\bar{\gamma}_G$ paths of length $2r - 1$ in G. Hence by Theorem 1.13A, v_i lies on $\bar{\gamma}_G$ paths of length $2r - 1$ in H. If $v_i \in T$ then $\bar{\gamma}_G = \bar{\gamma}_H$ and $\gamma_G = N - \bar{\gamma}_G = N - \bar{\gamma}_H = \gamma_H$. If $v_i \in \bar{T}$ then $\bar{\gamma}_G = \gamma_H$ and $\bar{\gamma}_H = \gamma_G$. Therefore $\bar{\gamma}_G = \gamma_H \leq \bar{\gamma}_H = \gamma_G$ and since $\gamma_G \leq \bar{\gamma}_G$ by assumption we have $\gamma_G = \gamma_H = \bar{\gamma}_G = \bar{\gamma}_H$.

In view of Theorem 1.15A we write $\gamma_G = \gamma_H = \gamma$ and $\bar{\gamma}_G = \bar{\gamma}_H = \bar{\gamma}$ where $\gamma \leq \bar{\gamma}$.

<u>Proof of Theorem 1.10</u> - <u>Bicentral Case</u> -

<u>Case 1</u> - $\gamma > 1$. For any p-vertex u in G, G_{u_i} and H_{v_i} are bicentral trees of radius r. If X is a bicentral tree, denote by X' the central tree obtained from X by adding a vertex on its central edge. $G_{u_i} \simeq H_{v_i}$ $(1 \leq i \leq N)$ implies

$$(G_{u_i})' \simeq (H_{v_i})' \qquad (1 \leq i \leq N) \qquad \text{implies}$$

$$(G')_{u_i} \simeq (H')_{v_i} \qquad (1 \leq i \leq N)$$

since $(G_{u_i})' = (G')_{u_i}$ and $(H_{v_i})' = (H')_{v_i}$. This implies

$$G' \simeq H'$$

since Theorem 1.10 has been proved for central trees. Thus

$$G \simeq H.$$

<u>Case 2</u> - $\gamma = 1$. The proof will be based on showing that $(\Gamma,\bar{g}) \simeq (T,\bar{h})$, $(\bar{\Gamma},g) \simeq (\bar{T},g)$, $(\Omega,g,\bar{g}) \simeq (\Lambda,h,\bar{h})$ and $(\bar{\Omega},g,\bar{g}) \simeq (\bar{\Lambda},h,\bar{h})$.

Let u_1 be a p-vertex in G_1 and let u_2 be a p-vertex in $\bar{G}_1$. We may assume that $v_1 \in H_1$; then v_2 is in a limb of $\bar{T}$, which we may take to be $\bar{H}_1$. Since G_{u_1} and H_{v_1} are central trees with centers $\bar{g}$ and $\bar{h}$ respectively $\theta(G_{u_1}) = H_{v_1}$ implies

$$\theta(\bar{\Omega},g,\bar{g}) = (\bar{\Lambda},h,\bar{h})$$
$$\text{and}\quad \theta(\{(G_1)_{u_1} \cup K\} \cup \{(\bar{G}_i)_g\},\bar{g}) = (\{(H_1)_{v_1} \cup L\} \cup \{(\bar{H}_i)_h\},\bar{h}). \tag{22}$$

Now we consider three subcases.

(a) $\bar{\gamma} > 2$. G_{u_2} and H_{v_2} are bicentral trees and therefore

$$\phi(\bar{G}_1,\bar{g}) = (\bar{H}_1,\bar{h}) \quad \text{and} \quad \phi(\Omega,g,\bar{g}) = (\Lambda,h,\bar{h}). \tag{23}$$

Since $\gamma = 1$, $\phi((G_1)_{u_1} \cup K,\bar{g}) = ((H_1)_{v_1} \cup L,\bar{h})$ and so from (22) $(\{(\bar{G}_i)_g\},\bar{g}) \simeq (\{(\bar{H}_i)_g\},\bar{h})$. Therefore, $(\bar{\Gamma},g) \simeq (\bar{T},h)$ since g is joined to $\bar{g}$ and h is joined to $\bar{h}$. (This is an argument analogus to that used in (9)). Hence, with (22) and (23) this shows that $G \simeq H$.

(b) $\bar{\gamma} = 2$. Let u_3 be the other p-vertex in $\bar{\Gamma}$, say $u_3 \in \bar{G}_j$; then v_3 is in a limb $\bar{H}_k$ of $\bar{T}$. $\phi(G_{u_2}) = H_{v_2}$ implies either

(α_1): $\phi(g) = h$ and $\phi(\bar{g}) = \bar{h}$

or

(α_2): $\phi(g) = \bar{h}$ and $\phi(\bar{g}) = h$.

(α_1) implies $\phi(\bar{G}_1,\bar{g}) = (\bar{H}_1,\bar{h})$ and

$\phi(\Omega,g,\bar{g}) = (\Lambda,h,\bar{h})$. (24)

(α_2) implies $\phi(\bar{G}_1 \cup \Omega,g,\bar{g}) = (\{\bar{H}_i\}_{i\neq 1} \cup (\bar{H}_1)_{v_2} \cup \bar{\Lambda},\bar{h},h)$.

and $\phi(\{\bar{G}_i\}_{i\neq 1} \cup (\bar{G}_1)_{u_2} \cup \bar{\Omega},\bar{g},g) = (\bar{H}_1 \cup \Lambda,h,\bar{h})$ (25)

$\psi(G_{u_3}) = H_{v_3}$ implies either

(β_1): $\psi(g) = h$ and $\psi(\bar{g}) = \bar{h}$

or

(β_2): $\psi(g) = \bar{h}$ and $\psi(\bar{g}) = h$.

(β_1) implies $\psi(\bar{G}_1,\bar{g}) = (\bar{H}_1,\bar{h})$ and

$\psi(\Omega,g,\bar{g}) = (\Lambda,h,\bar{h})$. (26)

(β_2) implies $\psi(\bar{G}_1 \cup \Omega,g,\bar{g}) = (\{\bar{H}_i\}_{i\neq k} \cup (\bar{H}_k)_{v_3} \cup \bar{\Lambda},\bar{h},h)$

and $\psi(\{\bar{G}_i\}_{i\neq j} \cup (\bar{G}_j)_{u_3} \cup \bar{\Omega},g,\bar{g}) = (\bar{H}_1 \cup \Lambda,\bar{h},h)$. (27)

If (α_1) holds, then since $\gamma = 1$ and from (24),

$\phi((\bar{G}_1)_{u_1},\bar{g}) = ((\bar{H}_1)_{v_1},\bar{h})$.

Thus from (24) $((\bar{G}_1)_{u_1} \cup K,\bar{g}) \simeq ((\bar{H}_1)_{v_1} \cup L,\bar{h})$.

So from (22) and by an argument analogus to that used in Case 2(a), $(\bar{T},g) \simeq (\bar{T},h)$.

Combining this with results from (22) and (24) gives $G \simeq H$.

If (β_1) holds a similar argument to (α_1) holding shows $G \simeq H$.

Otherwise, both (α_2) and (β_2) hold. From (22) $\bar{\Gamma}$ and $\tilde{\Gamma}$ have the same number of radial limbs. Since $\bar{\gamma} = 2$ either

$$\bar{\Gamma} = \{\bar{G}_1, \bar{G}_2\} \text{ and } \tilde{\Gamma} = \{\bar{H}_1, \bar{H}_2\}$$

if the p-vertices are in different limbs or

$$\bar{\Gamma} = \{\bar{G}_1\} \text{ and } \tilde{\Gamma} = \{\bar{H}_1\}$$

if both p-vertices are in the same limb.

If $\bar{\Gamma} = \{\bar{G}_1, \bar{G}_2\}$ and $\tilde{\Gamma} = \{\bar{H}_1, \bar{H}_2\}$, then from (25)

$$\phi(G_1, \bar{g}) = (\bar{H}_2, h),\ \phi(\Omega, g, \bar{g}) = (\{(\bar{H}_1)_{v_2}\} \cup \Lambda, \bar{h}, h),$$

$$\phi(\bar{G}_2, g) = (\bar{H}_1, \bar{h}) \text{ and } \phi(\{(\bar{G}_1)_{u_2}\} \cup \Omega, g, \bar{g}) = (\Lambda, \bar{h}, h).$$

and from (27) $\psi(G_1, \bar{g}) = (\bar{H}_1, h)$ and $\psi(\bar{G}_1, g) = (\bar{H}_1, \bar{h})$.

$$\text{Hence} \quad \phi\psi^{-1}(\bar{H}_1, h) = \phi(G_1, \bar{g}) = (\bar{H}_2, h) \tag{28}$$

$$\text{and} \quad \phi^{-1}\psi(\bar{G}_1, g) = \phi^{-1}(\bar{H}_1, \bar{h}) = (\bar{G}_2, g). \tag{29}$$

$$\text{By (28)} \quad ((\bar{H}_1)_h, \bar{h}) \simeq ((\bar{G}_1)_{\bar{g}}, g) \simeq ((\bar{H}_2)_h, \bar{h}) \tag{30}$$

and from (22) one of the following must hold since $\bar{\gamma} = 2$:

$$\begin{aligned} &((\bar{G}_1)_g, \bar{g}) \simeq ((\bar{H}_1)_h, \bar{h}),\ ((\bar{G}_1)_g, \bar{g}) \simeq ((\bar{H}_2)_h, \bar{h}), \\ &((\bar{G}_2)_g, \bar{g}) \simeq ((\bar{H}_1)_h, \bar{h}) \text{ or } ((\bar{G}_2)_g, \bar{g}) \simeq ((\bar{H}_2)_h, \bar{h}). \end{aligned} \tag{31}$$

Now (30) and (31) imply either

$$((\bar{G}_1)_g, \bar{g}) \simeq ((\bar{G}_1)_{\bar{g}}, g) \text{ or } ((\bar{G}_2)_g, \bar{g}) \simeq ((\bar{G}_1)_{\bar{g}}, g).$$

$$\text{From (9)} \quad (\bar{G}_1, g) \simeq (\bar{G}_1, \bar{g}) \text{ or } (\bar{G}_2, g) \simeq (\bar{G}_1, \bar{g}). \tag{32}$$

(28), (29) and (32) show that

$$\begin{aligned} &(\bar{G}_1, \bar{g}) \simeq (\bar{H}_1, \bar{h}) \simeq (\bar{G}_1, g) \simeq (\bar{H}_1, h) \simeq \\ &\qquad (\bar{G}_2, g) \simeq (\bar{H}_2, h) \end{aligned} \tag{33}$$

which in turn implies that

$$((\bar{G}_1)_{u_2},g) \simeq ((\bar{H}_1)_{v_2},h) \tag{34}$$

since $\bar{G}_1$ contains one p-vertex.

From (22), (33) and (34) we have that

$$\begin{aligned}(\{\bar{H}_i\}_{i\neq 1} \cup (\bar{H}_1)_{v_2} \cup \bar{\Lambda},\bar{h},h) \simeq \\ (\{\bar{G}_i\}_{i\neq 1} \cup (\bar{G}_1)_{u_2} \cup \bar{\Omega},\bar{g},g)\end{aligned} \tag{35}$$

since $(\bar{\Gamma},g) \simeq (\bar{T},h)$ implies $(\bar{\Gamma},g,\bar{g}) \simeq (\bar{T},h,\bar{h})$

and $(\Gamma,\bar{g}) \simeq (T,\bar{h})$ implies $(\Gamma,g,\bar{g}) \simeq (T,h,\bar{h})$.

From (25) and (35) $(G_1 \cup \Omega,g,\bar{g}) \simeq (\bar{H}_1 \cup \Lambda,h,\bar{h})$. (36)

Thus from (36) and (33) $(\Omega,g,\bar{g}) \simeq (\Lambda,h,\bar{h})$. (37)

Using (33) which implies all the radial limbs are isomorphic and (22) and (37) we arrive at $G \simeq H$.

If $\bar{\Gamma} = \{\bar{G}_1\}$, $\bar{T} = \{\bar{H}_1\}$ then from (25) and (27)

$$\phi(\Omega,g,\bar{g}) = (\bar{\Lambda},\bar{h},h) \quad \text{and} \quad \phi(\bar{\Omega},g,\bar{g}) = (\Lambda,\bar{h},h) \tag{38}$$

since $u_2,u_3 \in \bar{G}_1$.

Also from (25) and (27)

$$\begin{aligned}\phi^{-1}((\bar{H}_1)_{v_2},\bar{h},h) &= (\bar{G}_1,g,\bar{g}) = \psi^{-1}((\bar{H}_1)_{v_3},\bar{h},h) \\ \phi((\bar{G}_1)_{u_2},g,\bar{g}) &= (\bar{H}_1,\bar{h},h) = \psi((\bar{G}_1)_{u_3},g,\bar{g})\end{aligned} \tag{39}$$

since $v_2,v_3 \in \bar{H}_1$ and $u_2,u_3 \in \bar{G}_1$.

By (22) $(\bar{\Omega},g,\bar{g}) \simeq (\bar{\Lambda},h,\bar{h})$ and hence using (38)

$$(\bar{\Omega},g,\bar{g}) \simeq (\bar{\Lambda},h,\bar{h}) \simeq (\Omega,\bar{g},g) \simeq (\Lambda,\bar{h},h). \tag{40}$$

Also from (22) either $((\bar{G}_1)_g,\bar{g}) \simeq ((\bar{H}_1)_h,\bar{h})$ or

$$((\bar{G}_1)_g,\bar{g}) \simeq ((\tilde{H}_1)_{v_1} \cup L,\bar{h}).$$

If $((\bar{G}_1)_g,\bar{g}) \simeq ((\bar{H}_1)_h,\bar{h})$ then $(\tilde{G}_1,g) \simeq (\tilde{H}_1,h)$ and under this isomorphism u_2 corresponds to v_2 or to v_3. Therefore either $((\tilde{G}_1)_{u_2},g) \simeq ((\tilde{H}_1)_{v_2},h)$ or $((\tilde{G}_1)_{u_2},g) \simeq ((\tilde{H}_1)_{v_3},h)$. In both cases (39) shows that $(\bar{G}_1,\bar{g}) \simeq (\bar{H}_1,\bar{h})$. Hence $G \simeq H$.

If $((\bar{G}_1)_g,\bar{g}) \simeq ((\bar{H}_1)_{v_1} \cup L,\bar{h})$ then

$|E(\bar{G}_1)| = |E(\bar{H}_1)| + |E(L)| - 1.$

But from (39) $|E(\bar{G}_1)| = |E(\bar{H}_1)| + 1.$

Therefore $|E(L)| = 2$ and so L has just two edges, one being the central edge. By (40) the same is true of K, $\bar{K}$, and $\bar{L}$. As in the proof of the central case 2(b) we may construct the graphs $\bar{G}_1$ and $\bar{H}_1$, and the graphs $\tilde{H}_1$ and $\tilde{G}_1$ and show that $G \simeq H$. In fact G and H are paths of length $2r - 1$ with an edge attached to each vertex except two vertices at one end and one vertex at the other end.

(c) $\bar{\gamma} = 1$: From (22) $\theta(\Omega,g,\bar{g}) = (\Lambda,h,\bar{h})$ and

$$\theta((\tilde{G}_1)_{u_1} \cup K \cup (\bar{G}_1)_g,\bar{g}) = ((\tilde{H}_1)_{v_1} \cup L \cup (\bar{H}_1)_h,\bar{h}). \quad (41)$$

$\phi(G_{u_2}) = H_{v_2}$ implies $\phi(\Omega,g,\bar{g}) = (\Lambda,h,\bar{h})$ and

$$\phi((\bar{G}_1)_{u_2} \cup \bar{K} \cup (\tilde{G}_1)_{\bar{g}},g) = ((\bar{H}_1)_{v_2} \cup \bar{L} \cup (\tilde{H}_1)_{\bar{h}},h). \quad (42)$$

If $(\tilde{G}_1,\bar{g}) \simeq (\tilde{H}_1.\bar{h})$, then by (8) and (22)

$$((\tilde{G}_1)_{u_1} \cup K,\bar{g}) \simeq ((\tilde{H}_1)_{v_1} \cup L,\bar{h})$$

and so by (41) $((\bar{G}_1)_g,\bar{g}) \simeq ((\bar{H}_1)_h,\bar{h})$

Thus by (9) $(\bar{G}_1,g) \simeq (\bar{H}_1,h)$. Hence $G \simeq H$.

Similarly, if $(\bar{G}_1,g) \simeq (\bar{H}_1,h)$, then (8), (22) and (42) imply that $(G_1,\bar{g}) \simeq (H_1,\bar{h})$ and $G \simeq H$.

Otherwise from (41) and (42)

$$((H_1)_{v_1} \cup L,\bar{h}) \simeq ((\bar{G}_1)_g,\bar{g}) \text{ and}$$
$$((\bar{G}_1)_{u_2} \cup \bar{K},g) \simeq ((H_1)_{\bar{h}},h), \tag{43}$$

and $((G_1)_{u_1} \cup K,\bar{g}) \simeq ((\bar{H}_1)_h,\bar{h})$ and

$$((\bar{H}_1)_{v_2} \cup \bar{L},h) \simeq ((G_1)_{\bar{g}},g). \tag{44}$$

From (43) $|E(H_1)| + |E(L)| = |E(\bar{G}_1)| + 1$

and $|E(\bar{G}_1)| + |E(\bar{K})| = |E(H_1)| + 1$.

Therefore, $|E(L)| + |E(\bar{K})| = 2$.

Since $|E(L)| \geq 1$ and $|E(\bar{K})| \geq 1$ then $|E(L)| = |E(\bar{K})| = 1$ and $L,\bar{K}$ each consist solely of the central edge, which is common to all limbs. Hence Λ and $\bar{\Omega}$ are empty. By (43), $\bar{G}_1$ and H_1 are paths of length r. Similarly, by (44), G_1 and $\bar{H}_1$ are paths of length r. Therefore G and H are both paths of length 2r - 1. Hence $G \simeq H$.

This completes the proof of the bicentral case of Theorem 1.10 and thus the proof of Theorem 1.10.

Theorem 1.10 has as a corollary Theorem 1.9 since p-vertices are mapped onto p-vertices in Theorem 1.9 also. Theorem 1.9 in turn yields the solution to the edge problem in the case when G and H are trees, since in the edge problem an edge joined to a pendant vertex must be mapped onto an edge with a pendant vertex. Thus pendant vertices are mapped onto pendant vertices satisfying the hypothesis of Theorem 1.9.

B. Manvel [11] has also given a proof of Theorem 1.9 that he was able to modify to prove the following which is the only result of this type for trees.

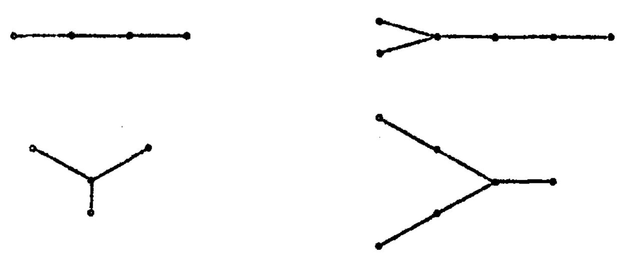
Let X(G) denote a subset of V(G). Define an equivalence relation 'R' on the set X(G) by:

$$u \text{ 'R' } v \quad \text{for } u,v \in X(G) \text{ if and only if } G_u \simeq G_v.$$

Let any element in a particular equivalence class represent that equivalence class and let M(X) denote the set of representatives of the equivalence classes of the set X(G).

<u>Theorem 1.17</u> - Except in the two cases illustrated in Figure 8, a tree T with at least three vertices is uniquely reconstructible by the subgraphs $\{T_v\}_{v \in M(S)}$ where

$S(T) = \{u \in V(T): d(u) = 1\}$.



(a) (b)

<u>Figure 8</u>

§1.3 Connected Graphs With Cycles

As stated earlier, at the time of writing of this paper there is no known solution of the vertex problem for connected graphs with cycles. However, in this section we present results for subclasses of connected graphs with cycles.

Definition 1.12 - In a regular graph G all vertices have the same degree. The complete graph K_p of order p has every pair of its vertices adjacent and so is regular of degree p - 1.

Theorem 1.18 - A regular graph G can be reconstructed from the collection $\{G_v\}_{v \varepsilon V(G)}$.

Proof - The first step is to find a method of determining from the collection $\{G_v\}_{v \varepsilon V(G)}$ whether or not G is a regular graph.

From the subgraphs G_v, we know the number of vertices, say n, in G. The number m of edges is also easily determined. With m_v denoting the number of edges in G_v, the degree of v in G is $d(v) = m - m_v$. Adding these n equations we obtain

$$\sum_{v \varepsilon V(G)} d(v) = nm - \sum_{v \varepsilon V(G)} m_v \qquad (1)$$

But
$$\sum_{v \varepsilon V(G)} d(v) = 2m \qquad (2)$$

since each edge is incident with two vertices.

Thus from (1) and (2)

$$m = \frac{\sum_{v \varepsilon V(G)} m_v}{n - 2}$$

Since $d(v) = m - m_v$, we obtain the degrees of the vertices of G. Thus we determine whether or not G is regular.

If G is regular of degree k, then we reconstruct G by taking any G_v, adding the vertex v and adding k edges, one edge from each vertex with degree k - 1 in G_v to v.

<u>Definition 1.13</u> - A connected graph G is <u>separable</u> if it contains a cut-vertex. A <u>block</u> is a maximal connected subgraph that is not separable. A <u>pendant block</u> of G is a block of G that contains exactly one cut-vertex of G.

We now present Bondy's [3] result that the vertex problem holds for separable graphs with no pendant vertices. Thus we assume G and H are separable graphs of order n with no pendant vertices for the following lemma and theorem.

<u>Lemma 1.19</u> - Suppose G has blocks $B_1, B_2, \cdots, B_s$ (where $s > 1$) and H has blocks $C_1, C_2, \cdots, C_t$. Then $s = t$ and the blocks can be relabeled so that $B_i \cong C_i$ $(1 \leq i \leq s)$.

<u>Proof</u> - Let B_i have order b_i and C_i have order c_i. Moreover, we say the order of B is less than the order of C if $|V(B)| < |V(C)|$ or if $|V(B)| = |V(C)|$ and $|E(B)| < |E(C)|$.

We may assume that $b_1 \geq b_2 \geq \cdots \geq b_s$ and $c_1 \geq c_2 \geq \cdots \geq c_t$. The proof will be by induction. Suppose we have already shown that $B_1 \cong C_1$, $B_2 \cong C_2, \cdots, B_k \cong C_k$. We may suppose that $b_{k+1} \geq c_{k+1}$. Take $Y \cong B_{k+1}$. If Y occurs γ times in $\bigcup_{i=1}^{k} B_i$, then Y also occurs γ times in $\bigcup_{i=1}^{k} C_i$ since Y cannot be isomorphic to a subgraph that contains vertices in more than one block. Now Y occurs at least $\gamma + 1$ times in G since $Y \cong B_{k+1}$. Hence by Kelly's Lemma (1.7), Y occurs at least once in $\bigcup_{i=k+1}^{t} C_i$. Therefore Y is isomorphic to a subgraph of C_j for some $j > k$, i.e., the order of Y $\leq$ order of C_j. But

order of Y = order of $B_{k+1} = b_{k+1} \geq c_{k+1} \geq c_j$ = order of C_j.

Hence $B_{k+1} \cong C_{k+1}$. Induction is started by the same argument with $k = 0$. Therefore $s = t$ and $B_i \cong C_i$ for all i.

Theorem 1.20 - If G and H are separable graphs without pendant vertices then the vertex problem holds.

Proof - Let $B_1, B_2, \cdots, B_t$ be the pendant blocks of G ($t \geq 2$) and let B_i have order b_i. Suppose $b_1 \leq b_i$ ($2 \leq i \leq t$) and let u be the cut-vertex joining B_1 to the rest of G. Write $G_1 = G_{(V(B_1)-\{u\})}$ and denote by G_1^s the graph obtained from G_1 by adding s isolated vertices and joining each to u by one edge. Then G_1^1 is a proper subgraph of G and, hence, by Kelly's Lemma (1.7), H has a subgraph $H_1^1 \simeq G_1^1$, i.e., $\psi(G_1^1) = H_1^1$ and $\psi(u) = v$ say. (Note: u and v are not necessarily corresponding vertices). Let p be the pendant vertex of H_1^1 and let H_1 be the subgraph of H obtained by deleting p from H_1^1. Then it is clear that $H_1 = \psi(G_1)$ and by Lemma 1.19 H_1 has one block fewer than H. If we call that block C_1 then $C_1 \simeq B_1$. Now H is obtained from H_1^1 by adding $b_1 - 2$ vertices and some edges. Since no pendant block of H has order less than B_1 it is easy to see that those edges can only be incident with v,p and the $b_1 - 2$ vertices added to H_1^1. Thus v is a cut-vertex of H and it follows that the subgraph of H on v,p and the $b_1 - 2$ vertices added to H_1^1 is isomorphic to C_1. It now remains to show that there is an isomorphism of B_1 and C_1 mapping u onto v. Denote by B_1^1 the graph obtained from B_1 by adding an isolated vertex and joining it to u by one edge. Define C_1^1 analogously. It will suffice to prove that $B_1^1 \simeq C_1^1$.

Obviously $G_u \simeq H_v$, so u and v have the same degree by Lemma 1.8. Let $d(u) = r + s$ where the degree of u in G_1 is r and in B_1 is s. Hence $r,s > 1$. By the definition of H_1 it is clear that the degree of v in H_1 is r; hence the degree of v in C_1 is s. If G_1^s has α subgraphs isomorphic to B_1^1 then G has $\alpha + r$ subgraphs isomorphic to B_1^1 since B_1 is a block of G. Thus H has $\alpha + r$ subgraphs isomorphic to B_1^1 by Kelly's Lemma (1.7) and H_1^s has s subgraphs isomorphic to B_1^1 since $H_1^s \simeq G_1^s$. It follows that C_1^r has r subgraphs isomorphic to B_1^1 and hence that $B_1^1 \simeq C_1^1$ since $B_1 \simeq C_1$.

In the light of Theorems 1.4, 1.6 and 1.20 the graphs for which the vertex problem hasn't been solved can be broken into two major classes: (1) connected graphs with cycles and pendant vertices and (2) connected graphs with cycles and without cut-vertices, i.e., blocks. At the time of writing this paper we know of no work having been done on the vertex problem for class (2) graphs.

<u>Definition 1.14</u> - A <u>cactus</u> is a connected graph G such that each block of G is either an edge or a cycle.

The following result proved by Geller and Manvel [4] and independently by Greenwell and Hemminger [5] presents us a solution of the vertex problem for a subclass of class (1) graphs.

<u>Theorem 1.21</u> - If G and H are cacti, then the vertex problem holds for G and H.

Greenwell and Hemminger [5], and Bondy [3] in their attempt to solve the vertex problem for class (1) graphs proposed the following conjecture.

Conjecture C - The Pendant Vertex Problem - If G and H are connected graphs with pendant vertices S(G) and S(H), respectively, $|S(G)| \geq 1$, and $\sigma: S(G) \to S(H)$ is a one-to-one onto function such that $G_v \simeq H_{\sigma(v)}$ for all $v \in S(G)$, then $G \simeq H$.

For this conjecture to be true we must obviously have $|S(G)| > 2$. P. Neumann pointed out a counterexample for the conjecture when $|S(G)| = 3$. We illustrate the counterexample in Figure 9. Neumann also claims that he has found a counterexample when $|S(G)| = 4$.

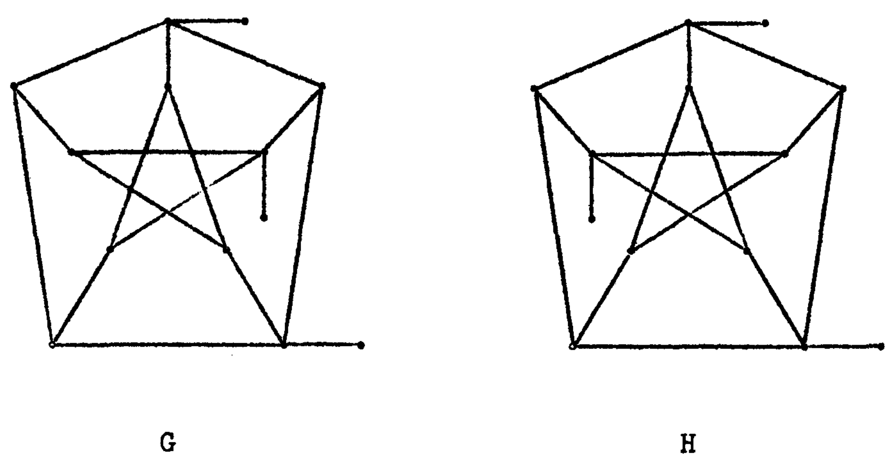


Figure 9

Greenwell and Hemminger [5] proved the following theorem.

Theorem 1.22 - If G and H are cacti with pendant vertices, then the pendant vertex problem holds for G and H.

Definition 1.15 - A vertex of G, having degree d, will be called a left end-vertex if no vertex of G has degree d - 1. We denote the set of degrees of the vertices of G by D(G). i.e., the degree set.

P. Zweig [20] gives some other classes of graphs for which the vertex problem holds by proving the following theorem which gives the conditions under which $G_{v_i} \simeq H_{\sigma(v_i)}$ for any value of i and coupled with D(G) = D(H) implies that $G \simeq H$.

Theorem 1.23 - Let G and H denote graphs that both have the degree set D. Assume that there is a vertex u_1 of G that is adjacent only to left end-vertices. If there is a vertex v_1 of H with the same degree, say d, as u_1, and if α is an isomorphism of G_{u_1} onto H_{v_1} then there is an isomorphism $\bar{\alpha}$ of G onto H defined by $\bar{\alpha}(u_1) = v_1$ and $\bar{\alpha}(u_i) = \alpha(u_i)$, $i \neq 1$.

Proof - Let u denote a vertex of G to which u_1 is adjacent. In G_{u_1}, u has degree k - 1 for some k, as does its image, say v, in H_{v_1}. But no vertex in H has degree k - 1, since $(k - 1) \notin D$, and G and H have the same degree sets. Hence v_1 is adjacent to v in H. This method gives d vertices of H to which v_1 is adjacent, one corresponding to each vertex to which u_1 is adjacent. Since by hypothesis $d = d(v_1)$, there are no other vertices in H to which v_1 is adjacent. Thus u_1 not adjacent to u_i implies that v_1 is not adjacent to the α-image of u_i, and the extension of α to $\bar{\alpha}$ is an isomorphism of G onto H.

The corresponding theorem for the reconstruction problem states that a graph is reconstructible from its degree set D(G) and a subgraph G_u, $u \in V(G)$, if the degree of u is d and G_u contains d vertices with degrees not in D(G). G is then reconstructed by adding a vertex to G_u and joining it to each vertex having a degree not in D(G).

<u>Corollary 1.24</u> - Any regular graph G of order n is reconstructible from any one of its $(n - 1)$ order subgraphs.

<u>Proof</u> - Each vertex is a left end-vertex since D(G) contains only one element.

<u>Definition 1.16</u> - A <u>trail</u> is a walk in which all edges are distinct. A graph G is called <u>Eulerian</u> if it has a spanning closed trail that contains all of its edges.

Euler proved that a graph G is Eulerian if and only if every vertex of G has even degree. Thus every vertex of an Eulerian graph G is a left end-vertex and we obtain the following corollary.

Corollary 1.25 - Any Eulerian graph G of order n is reconstructible from any of its (n - 1) order subgraphs.

Definition 1.17 - Psuedo regular graph G of degree n is one whose degree set is $\{1,n\}$.

Corollary 1.26 - A psuedo regular graph G of degree n is reconstructible.

Proof - If $n \neq 2$, every vertex of G is a left end-vertex. If $n = 2$ the graph is a path.

We conclude this chapter with some results from false proofs that the vertex problem holds.

D. Netherwood presented an erroneous proof of a stronger conjecture related to Conjecture A. The conjecture he claimed to have proved is the following.

Conjecture D - If G and H are graphs, $|V(G)| > 3$, and $\sigma: V(G) \to V(H)$ is a one-to-one onto function such that $G_v \simeq H_{\sigma(v)}$ for all $v \in M(V)$, then $G \simeq H$.

Manvel [11] proved this conjecture holds for trees, with the exception of two trees, by proving Theorem 1.17 for $v \in M(S)$.

<u>Definition 1.18</u> - Two vertices of a graph G are <u>similar</u> if there is an automorphism of the graph G mapping one onto the other.

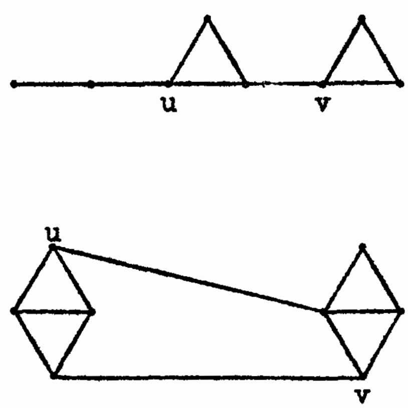
<u>Theorem 1.27</u> - If u and v are similar vertices of G, then $G_u \simeq G_v$.

<u>Proof</u> - Let α: $G \to G$ be an automorphism such that u is mapped onto v. Now if we restrict α to vertices of G_u, then α is an isomorphism of G_u onto G_v. Therefore $G_u \simeq G_v$.

An incorrect proof of the vertex problem used the converse of Theorem 1.27, namely, that if $G_u \simeq G_v$, then u and v are similar vertices. That the latter is not true is shown by the two counterexamples G and G' in Figure 10. The two vertices u, v in Figure 10 are vertices that are not similar but are such that $G_u \simeq G_v$ and $G'_u \simeq G'_v$.

G:

G':



<u>Figure 10</u>

The Graph G in Figure 10 with 8 vertices and 9 edges is the smallest possible counterexample in terms of the number of vertices and then of edges in a graph. An inspection of the diagrams of all graphs with 7 or less vertices does not yield a counterexample, and the same methods of exhaustion serve to prove that there does not exist any counterexample with 8 vertices and fewer than 9 edges.

This confirms the result that the vertex problem holds for all graphs with at most 7 vertices.

CHAPTER 2 - THE EDGE PROBLEM

The purpose of this chapter is to show that the edge problem is a special case of the vertex problem. The first results we present suggest that there might be some connection between the edge problem and the vertex problem.

Lemma 2.1 - If the conditions of the edge problem are satisfied, then every type of edge proper subgraph which occurs in G or H occurs the same number of times in both, and e and $\sigma(e)$ are edges in the same number of these subgraphs for all $e \in E(G)$.

To illustrate the formulation of these problems in reconstruction terms we give that version (with proof) of Lemma 2.1

Lemma 2.1' - Let the family of graphs $\{G^e\}_{e \in E(G)}$ be given. Then the number of edge proper subgraphs of G isomorphic to a given graph is determined by the $\{G^e\}_{e \in E(G)}$ as well as the number of these that contain a given edge of G.

Proof - Let T be a graph with j edges, $1 \leq j < |E(G)|$. Let α be the number of subgraphs of G that are isomorphic to T and α_i the number of subgraphs of G that are isomorphic to T and that contain the element e_i of E(G), where $E(G) = \{e_i : i = 1, \cdots, m\}$. Then $\alpha - \alpha_i$ is known since it is the number of subgraphs of G^{e_i} that are isomorphic to T.

Thus from the collection $\{G^e\}_{e \in E(G)}$ and T we know m, j and $\alpha - \alpha_i$ for each i = 1,···,m. Therefore, we know M where

$$M = \sum_{i=1}^{m} (\alpha - \alpha_i) = m\alpha - \sum_{i=1}^{m} \alpha_i .$$

But $\sum_{i=1}^{m} \alpha_i = j\alpha$ so $M = (m - j)\alpha$. But $m \neq j$ so $\alpha = M/(m - j)$ is known and hence $\alpha_i = \alpha - (\alpha - \alpha_i)$ is known for each i = 1,···,m.

Definition 2.1 - A graph S is a k-star if $|V(S)| = k + 1$ and for some $v \in V(S)$, $E(S) = \{<v,w>: w$ is an element of $V(S)$ and $w \neq v\}$. A graph is a forest if it has no circuits, so that each component is a tree.

Theorem 2.2 - The collection $\{G^e\}_{e \in E(G)}$ determines whether or not G is connected if $|E(G)| > 3$.

Proof - If G^e is a star plus one isolated vertex for each e in E(G), then G is a star and hence is connected.

If G^e is a forest with exactly two trees for all e in E(G) and for some e neither component is an isolated vertex, then G is a tree and not a star.

If G^e is connected for some e in E(G), then G is connected and not a tree.

Since the converse of these statements also holds, we have the theorem.

<u>Theorem 2.3</u> - The edge problem is true for disconnected graphs having at least two non trivial connected components.

<u>Proof</u> - The method of determining the connected components on two or more vertices is similar to the technique used in the proof of Lemma 1.19 except that Lemma 2.1 is used instead of Lemma 1.7. The theorem follows immediately.

<u>Definition 2.2</u> - If G is a graph, then the <u>line graph</u> of G, denoted by L(G), is the graph V(L(G)) = E(G) and with $\langle e_1, e_2 \rangle$ in E(L(G)) if and only if e_1 and e_2 are adjacent in G.

Hemminger in [5] and [12] proves that the edge problem is true for G if and only if the vertex problem is true for L(G). Thus proving that the edge problem is a special case of the vertex problem. Hence we present some results for line graphs and then present Hemminger's proof.

<u>Theorem 2.4</u> - (Whitney [19]or see page 248 of [17]]) - If G and H are connected graphs other than triangles, then $G \simeq H$ if and only if $L(G) \simeq L(H)$.

<u>Lemma 2.5</u> - Let G be a graph. Then $L(G^e) = (L(G))_e$ for all e in G.

<u>Proof</u> - We have $V(L(G^e)) = E(G) - \{e\} = V((L(G))_e)$ and $\langle e_1, e_2 \rangle$ is in $E(L(G^e))$ if and only if $e_1, e_2 \neq e$ and e_1 and e_2 are adjacent in G, that is if and only if $\langle e_1, e_2 \rangle$ is in

$E((L(G))_e)$. Therefore $E(L(G^e)) = E((L(G))_e)$. Thus $L(G^e) = (L(G))_e$.

<u>Theorem 2.6</u> - (Hemminger) - The edge problem is true for G if and only if the vertex problem is true for L(G).

<u>Proof</u> - Suppose the vertex problem is true for line graphs. Let G and H be graphs, $|E(G)| > 3$ and let $\sigma: E(G) \to E(H)$ be a one-to-one onto function such that $G^e \simeq H^{\sigma(e)}$ for all e in E(G). By Theorem 2.3 if G is disconnected then $G \simeq H$. So suppose G is connected. By Lemma 2.5 we have $(L(G))_e = L(G^e) \simeq L(H^{\sigma(e)}) = (L(H))_{\sigma(e)}$ for all e in E(G). But then $\sigma: V(L(G)) \to V(L(H))$ is a one-to-one onto function such that $(L(G))_e \simeq (L(H))_{\sigma(e)}$ for all e in V(L(G)) and $|V(L(G))| > 2$. So by our assumpton $L(G) \simeq L(H)$. Since G and H are connected L(G) and L(H) are connected. Hence by Theorem 2.4, since $|E(G)| > 3$, we have $G \simeq H$.

Conversely, suppose the edge problem is true for graphs. Let G and H be graphs with $|V(L(G))| > 2$, and let $\sigma: V(L(G)) \to V(L(H))$ be a one-to-one onto function such that $L(G)_e \simeq L(H)_{\sigma(e)}$ for all e in V(L(G)). If L(G) is disconnected, then by Theorem 1.4 $L(G) \simeq L(H)$. So suppose that L(G) and L(H) are connected.

If $|V(L(G))| > 5$, then $(L(G))_e \simeq (L(H))_{\sigma(e)}$ and Lemma 2.5 imply $L(G^e) \simeq L(H^{\sigma(e)})$ for all e in V(L(G)). From Whitney's Theorem (2.4) we have $G^e \simeq H^{\sigma(e)}$ for all e in

$V(L(G))$. Since the edge problem holds $G \simeq H$.

If $|V(L(G))| = 4$, it is easy to check by exhaustion that there are not two connected graphs G and H such that $|E(G)| = |E(H)| = 4$, $G \not\simeq H$ and $(L(G))_e \simeq (L(H))_{\sigma(e)}$ for all e in $V(L(G))$. Thus $(L(G))_e \simeq (L(H))_{\sigma(e)}$ implies $G \simeq H$ which in turn implies $L(G) \simeq L(H)$.

If $|V(L(G))| = 3$ and $G \not\simeq H$, then the only possible connected graphs G and H can be, such that $(L(G))_e \simeq (L(H))_{\sigma(e)}$ for all e in $V(L(G))$, are a triangle and a 3-star. But both of these graphs have a triangle as line graphs. Thus $L(G) \simeq L(H)$. If $G \simeq H$ then $L(G) \simeq L(H)$.

Thus $L(G) \simeq L(H)$ for $|V(L(G))| > 2$.

CHAPTER 3 - TOURNAMENTS

In [11], Harary and Palmer prove that the edge problem holds for all tournaments and the vertex problem holds for tournaments that are not strong. Before presenting these results we give the definitions of tournament and strong tournament.

Definition 3.1 - Let G be a directed graph. If $e = \langle u,v \rangle \in E(G)$ we say u defeats v or v is defeated by u. An element e of E(G) is called a directed edge of G.

Definition 3.2 - A tournament T is a directed graph such that each pair of distinct vertices, say u and v, is joined by one and only one of the directed edges $\langle u,v \rangle$ or $\langle v,u \rangle$.

Definition 3.3 - Two vertices u and v of a directed graph are said to be strongly connected if there is a path from u to v and a path from v to u. A tournament T is a strong tournament if each pair of distinct vertices of T is strongly connected.

Definition 3.4 - The score of a vertex v_i in T is the number of vertices v_i defeats in T or the outdegree of v_i. We denote the score of v_i by either s_i or $od(v_i)$. A vertex which defeats every vertex in T is called a transmitter. A vertex which is defeated by every vertex in T is called a receiver. The indegree of a vertex v is the number of vertices it is defeated by in T which we denote by id(v).

Theorem 3.1 - (Harary and Palmer [11]) - Let T be a tournament with $|V(T)| = n$, then T can be reconstructed from the collection of subgraphs $\{T^e\}_{e \varepsilon E(T)}$ for $n \geq 3$.

Proof - Clearly T has a receiver if and only if some T^e has a vertex v with $id(v) = n - 1$. If T does not have a receiver, then $s_1 \geq 1$ where we order the vertices v so the scores satisfy $s_1 \leq s_2 \leq \cdots \leq s_n$. Choose T^e such that there exists a vertex $u \varepsilon T^e$ with $od(u) = s_1 - 1$. Let v be the other vertex of T^e with total degree $n - 2$. Then T is obtained by adding the directed edge $\langle u,v \rangle$. If T does have a receiver, choose T^e with no vertices w such that $id(w) = n - 1$. Then T^e was obtained from T by deleting an edge which is incident with the receiver. Let v be a vertex of T^e with $id(v) = n - 2$ and $od(v) = 0$. Let u be the other vertex of T^e with total degree $n - 2$. Without loss of generality we can assume that v was the receiver of T and so T is obtained by adding the directed edge $\langle u,v \rangle$ to T^e.

Definition 3.5 - A tournament T is said to be transitive if $\langle u,v \rangle \varepsilon E(T)$ and $\langle v,w \rangle \varepsilon E(T)$ imply $\langle u,w \rangle \varepsilon E(T)$ where u, v and w are distinct.

The vertex problem does not hold for all tournaments.

If $|V(T)| = 3$ and T is either a cyclic triple or a transitive triple, then the collection $\{T_v\}_{v\epsilon V(T)}$ is the same in both cases. Hence T is not determined by the T_v.

If $|V(T)| = 4$, there are four tournaments, see Figure 11, and only two of them can be reconstructed. One is strong, see Figure 11(a), and has score sequence (1,1,2,2) and $\{T_v\}_{v\epsilon V(T)}$ has exactly two cyclic triples. One is transitive, see Figure 11(b), and has score sequence (0,1,2,3) and of course $\{T_v\}_{v\epsilon V(T)}$ has no cyclic triples. The other two have score sequence (0,2,2,2) and (1,1,1,3) and each collection $\{T_v\}_{v\epsilon V(T)}$ has exactly one cyclic triple. Hence in the latter two cases T is not determined by the T_v.

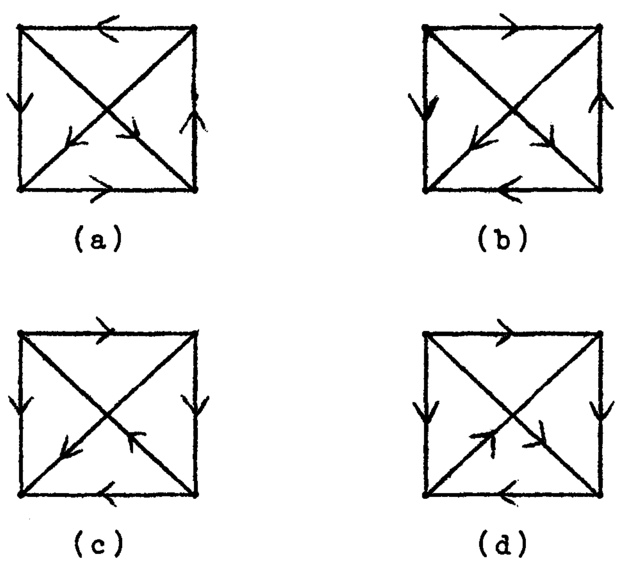


Figure 11

If $|V(T)| \geq 5$ the vertex problem holds for T if T is not a strong tournament. Before presenting Harary and Palmer's [11] proof of this we state some results that will be required in the proof.

The following theorem due to Moon and Moser is proved in [16, page 6].

Theorem 3.2 - Each vertex of a strong tournament T with $|V(T)| = n$ is contained in some cycle of length k, for $k = 3, 4, \cdots, n$.

The following results are proved in [11].

Theorem 3.3 - A tournament T with at least four vertices is strong if and only if it has neither a transmitter nor a receiver and for some vertex v, T_v is strong where $v \in V(T)$.

Theorem 3.4 - A tournament T with at least five vertices has a transmitter if and only if at least four of the T_v have a transmitter where $v \in V(T)$.

If G and H are two graphs with no vertices in common, the new graph G + H is obtained by joining each vertex of G with each vertex of H by an edge. When these additional edges are to be directed from G to H we indicate this by writing $G \mathbin{+\!\!\rightarrow} H$.

Theorem 3.5 - If T is a tournament with at least five vertices such that one of the T_v, say T_{v_1}, where $v \in V(T)$, does not have a transmitter and at least four of the T_v have a transmitter, then $T = v_1 + \rightarrow T_{v_1}$.

Proof - By Theorem 3.4, T has a transmitter, say v. Then T_u has a transmitter whenever $u \neq v$. Hence $v = v_1$ and $T = v_1 + \rightarrow T_{v_1}$.

Theorem 3.6 - If T is a tournament with at least five vertices and each T_v has a transmitter, then T can be reconstructed from the T_v where $v \in V(T)$ and $|V(T)| = n$.

Proof - We shall show that there is a largest integer m with $2 \leq m \leq n$ such that for a suitable labeling of the T_v the following conditions hold:

(1) each T_v has vertices of score $n - 2, n - 3, \cdots, n - m$.

(2) $T_{v_1}, T_{v_2}, \cdots, T_{v_m}$ do not have a vertex of score $n - (m + 1)$ but $T_{v_{m+1}}, T_{v_{m+2}}, \cdots, T_{v_n}$ do not have such a vertex.

(3) $T_{v_1}, T_{v_2}, \cdots, T_{v_m}$ are all isomorphic and $T = v_1 + \rightarrow T_{v_1}$.

By Theorem 3.4, T has a transmitter, say v_1. Since each T_v has a transmitter, T must have a vertex, say v_2, of score n - 2 (see Figure 12).

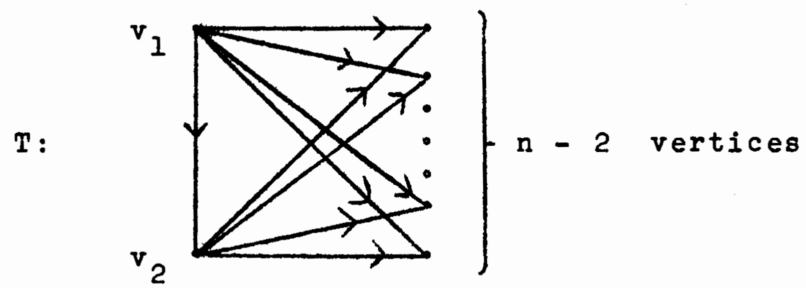


Figure 12

Now there are two possibilities:

Case 1 - None of the other vertices v_i with $i \geq 3$ has score n - 3. Then T_{v_1} and T_{v_2} do not have vertices of score n - 3 but for each $i \geq 3$, T_{v_i} does (namely v_2). Each T_v has a vertex of score n - 2. Clearly T_{v_1} and T_{v_2} are isomorphic, $T = v_1 + \rightarrow T_{v_1}$ and $m = 2$.

Case 2 - Some vertex, say v_3, has score n - 3. Then $m \geq 3$ and again there are two possibilities. If none of the v_i with $i \geq 4$ has score n - 4, then T_{v_1}, T_{v_2} and T_{v_3} do not have vertices of score n - 4 but for $i \geq 4$ each T_{v_i} does (namely v_3). Each T_v has vertices of score n - 2 and n - 3. Clearly T_{v_1}, T_{v_2} and T_{v_3} are isomorphic, $T = v_1 + \rightarrow T_{v_1}$ and $m = 3$. Otherwise some vertex, say v_4, has score n - 4 and

and $m \geq 4$. Continuing in this way we obtain (1), (2) and (3) and hence T can be reconstructed.

Each of the Theorems 3.4, 3.5 and 3.6 has a directional dual. The dual theorems are obtained by replacing the word transmitter by receiver.

<u>Definition 3.6</u> - Given a directed graph G define a new directed graph G^0, called the <u>condensation</u> of G, whose vertices are the strongly connected components of G, say $S_1, S_2, \cdots, S_r$ and for which $\langle S_i, S_j \rangle \in E(G^0)$ if there is an edge in G from a vertex of S_i to a vertex of S_j.

<u>Theorem 3.7</u> - (Harary and Palmer [11]) - Let T be a tournament that is not strong and $|V(T)| = n$, then T can be reconstructed from the collection of subgraphs $\{T_v\}_{v \in V(T)}$ if $n \geq 5$.

<u>Proof</u> - Using Theorem 3.4 and its dual, we can tell from the T_v whether or not T has a transmitter or a receiver. Then using Theorem 3.3 we can tell from the T_v whether or not T is strong. If T is not strong and has a transmitter or a receiver, T can be reconstructed by Theorems 3.5 and 3.6 or their directional duals.

Assume T is not strong and has neither a transmitter nor a receiver. Thus T must contain at least six vertices. Let the strong components of T be $S_1, S_2, \cdots, S_r$ with S_1 the transmitter and S_2 the receiver in T^0. The number of vertices in the component S_i is denoted by $|S_i|$. Since T does not have a transmitter, $|S_1| \geqq 3$ and since T does not have a receiver $|S_2| \geqq 3$. For each $i = 1,\cdots,n$ let $S_1^i,\cdots,S_r^i$ be the strong components of T_{v_i} with S_1^i and S_2^i the transmitter and receiver respectively of $T_{v_i}^0$.

Choose the notation so that $|S_1^1| \geqq |S_1^i|$ and $|S_2^2| \geqq |S_2^i|$ for all i. Then S_1 and S_1^1 are isomorphic and S_2 and S_2^2 are isomorphic. If $|S_1^1| + |S_2^2| = n$, then $T = S_1^1 + \to S_2^2$. Otherwise, the number of components of T is greater than two.

If $|S_1| \geqq 4$ then by Theorem 3.2 there is a cycle of length $|S_1| - 1$ in S_1. Therefore there is a vertex v in S_1 such that $(S_1)_v$ is a strong tournament. Then we can choose T_{v_3} with $|S_1^3| = |S_1| - 1$. Now delete all of the vertices of S_1^3 from T_{v_3} to obtain $(T_{v_3})_{S_1^3}$, a subtournament of T_{v_3}. It is clear that $T = S_1 + \to (T_{v_3})_{S_1^3}$. Similarly T can be reconstructed if $|S_2| \geqq 4$.

If $|S_1| = |S_2| = 3$ then both S_1 and S_3 are cyclic triples. Therefore if v is a vertex of S_1 then T_v has a transmitter. Choose T_{v_3} so that it has a transmitter. Let u be the transmitter of T_{v_3} and let v be the vertex of T_{v_3} which is adjacent to every vertex of T_{v_3} except u, (see Figure 13(a)). Then T is obtained by adding a vertex v_3 to T_{v_3} with v_3 adjacent from v and v_3 adjacent to all other vertices of T_{v_3} (see Figure 13(b)).

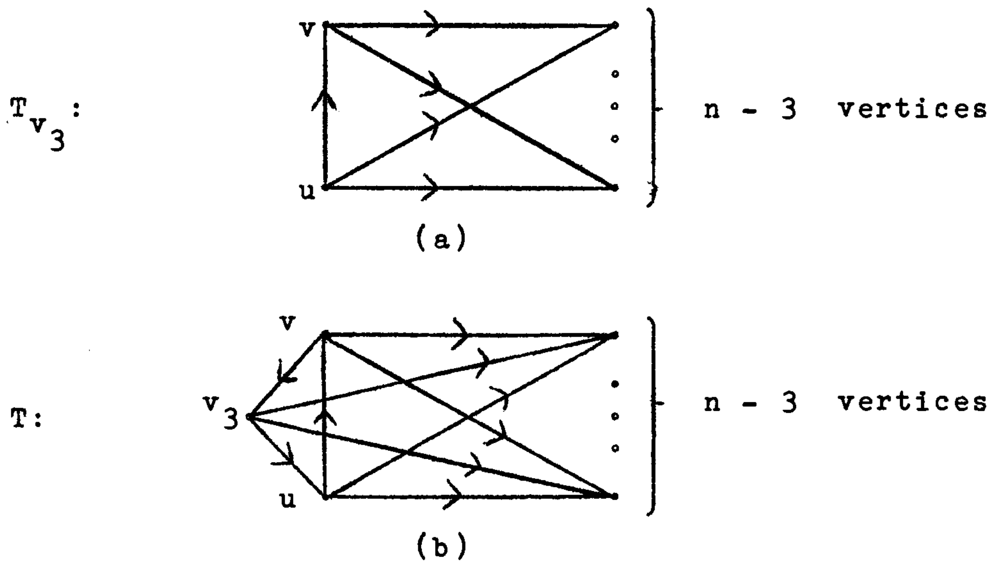


Figure 13

In [11] Harary and Palmer state that they believe the vertex problem will hold for strong tournaments. For this to be true the order of the tournaments will have to be greater than six as a counterexample for strong tournaments of order 5 has been found (Figure 14) and we have found by exhaustion two counterexamples for order 6 strong tournaments. We present the counterexamples in Figure 15. We believe that Beineke and Parker [1] have also found the same two counterexamples.

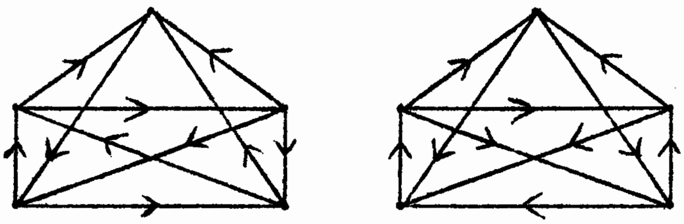
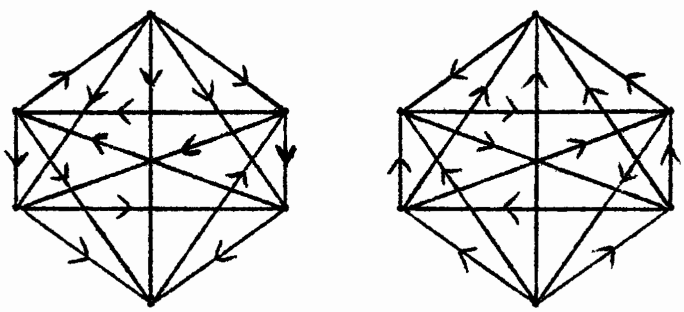
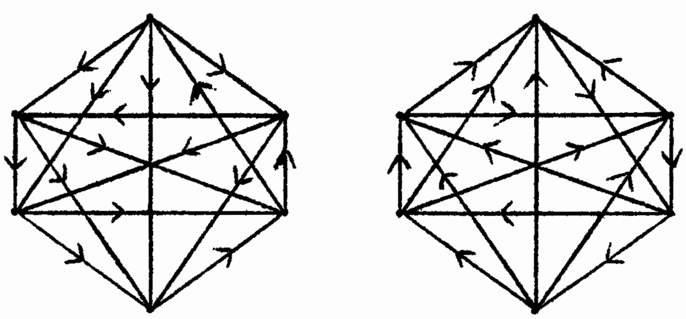


Figure 14



Counterexample (1)



Counterexample (2)

Figure 15

<u>Definition 3.7</u> - Given a directed graph G with vertices $u_1,\cdots,u_n$, the <u>converse</u> G^* of G is the directed graph with the same vertices satisfying $\langle u_i,u_j\rangle \in E(G^*)$ if and only if $\langle u_j,u_i\rangle \in E(G)$.

We observe in Figure 15 that the two graphs in counterexample (1) are the converses of each other and also the two graphs in counterexample (2) are the converses of each other. Thus this leads to the following conjecture.

<u>Conjecture E</u> - Let S and T be tournaments of order n, $n \neq 5$, such that $S_{u_i} \simeq T_{v_i}$, for $i = 1,\cdots,n$ and some labelings of the vertices of S and T, then either $T \simeq S$ or $T^* \simeq S$.

For order 4 tournaments this is obviously true as there is only one strong tournament of order 4. The conjecture is false for order 5 tournaments as the two tournaments in Figure 14 are not converses of each other. As stated above the conjecture holds for order 6 tournaments. To check the conjecture for order 7 tournaments by exhaustion becomes very long and tedious. It would be very interesting to check this case with a computer if possible.

J. Moon claims the following theorem, related to Conjecture E, has been proved. We believe a proof of the theorem is to appear in R. Goldberg's Doctor of Philosophy Dissertation.

Theorem 3.8 - If S and T are two order n tournaments such that there exists a mapping from the edges of S onto edges of T for which 3-cycles and 4-cycles are mapped to 3-cycles and 4-cycles, respectively, (and preimages of 3-cycles and 4-cycles are 3-cycles and 4-cycles, respectively) then either $S \simeq T$ or $S \simeq T^*$.

CHAPTER 4 - LABELED GRAPHS

Since the general vertex problem appears to be very difficult it seems desirable to modify it in some way. One possibility is to try to prove some weakened, but still general, form. A way to do this is to consider graphs with distinct labels on some of the vertices.

Definition 4.1 - A graph G with distinct labels on some of the vertices is called a partially labeled graph. If all the vertices of G have distinct labels we have a labeled graph.

If we look at partially labeled graphs, we can usually find pairs of graphs of order n which do not share k subgraphs, and other pairs which do not have k common subgraphs. We therefore define r(n,p) as the number of subgraphs G_v required to distinguish order n graphs with p vertices unlabeled.

Harary and Manvel [8] have established the exact values of r(n,p) for all $n \leq 7$ and also for $p \leq 4$. We now present some of their results.

Considering the simple class of graphs in Figure 16, Harary and Manvel found a remarkably good lower bound for r(n,p). We let [ ] denote the greatest integer function.

G and H in Figure 16 have [n/2] + 1 isomorphic maximal subgraphs as follows:

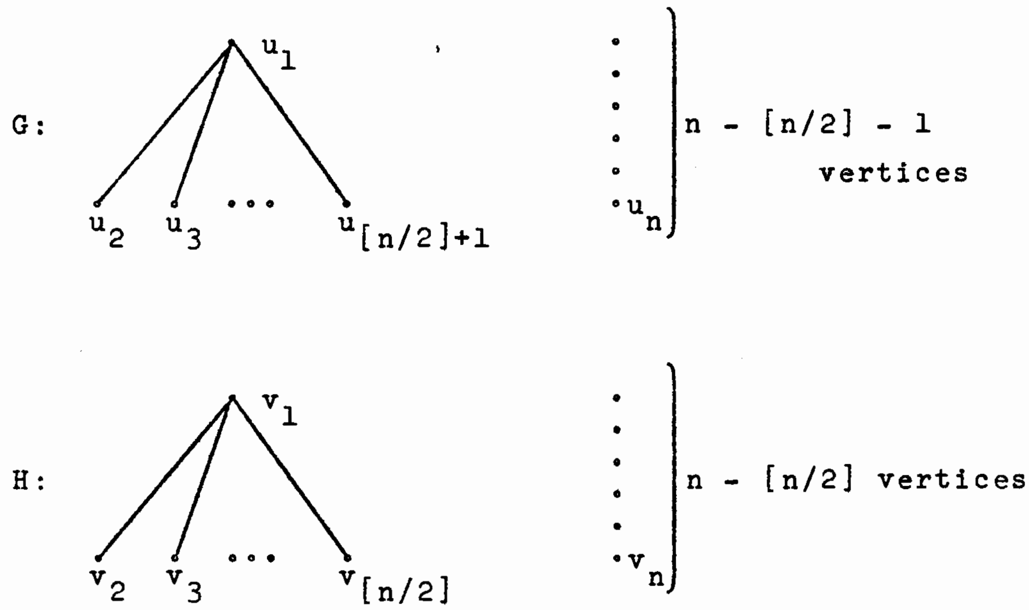


Figure 16

$$G_{u_1} \simeq H_{v_1}$$

and $G_{u_{i+1}} \simeq H_{v_{n-i+1}}$, $i = 1, \cdots, [n/2]$.

This gives us the bound for unlabeled graphs.

$$r(n,n) \geq [n/2] + 2 \qquad (1)$$

To obtain bounds for more labeled vertices, we need only note that when we add k labeled, isolated vertices to G and H and label the vertices u_1 and v_1 we obtain non isomorphic graphs of order n + k with n - 1 unlabeled vertices having [n/2] + 1 common subgraphs. Hence we see that

$$r(n + k, n - 1) \geq [n/2] + 2$$

or more conveniently

$$r(n,p) \geq [(p + 1)/2] + 2, \; n > p > 0. \qquad (2)$$

The bounds (1) and (2) are very often best possible.

The known cases in which they can be improved are discussed now.

The value $r(5,5) \geq 4$ may be improved to 5 by the pair of order 5 graphs in Figure 17. There are three other such pairs of order 5 graphs.

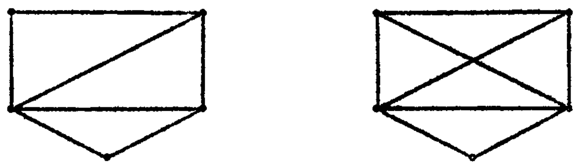


Figure 17

A computer search of order 6 graphs produced a unique pair of order 6 graphs with five common subgraphs shown in Figure 18. This example shows that $r(6,6) \geq 6$.

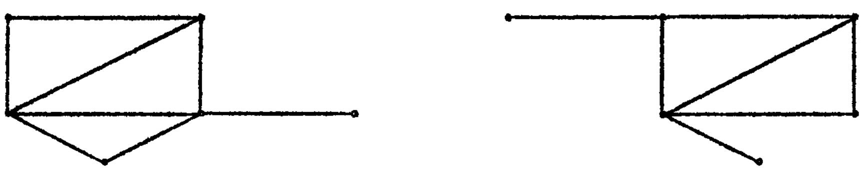


Figure 18

A similar computer search of order 7 graphs produced several pairs of graphs with five common subgraphs, including the one shown in Figure 19. Thus we see that $r(7,7)$ is at least 6.

Figure 19

If a labeled isolated vertex is added to each graph of Figure 18, we obtain an example which shows that $r(7,6) \geq 6$.

We now obtain upper bounds for $r(n.p)$ when $n \leq 7$ and $p \leq 4$. Kelly [13] and Harary [9] verified by exhaustion that

$r(n,n) \leq n, \; 3 \leq n \leq 7.$

Furthermore, since no vertex of the two graphs of Figure 18 can be labeled without reducing the number of isomorphic pairs of subgraphs below five, we see that $r(6,5) \leq 5$. For order 7 graphs the computer found no pair with six common subgraphs so $r(7,7) \leq 6$. Since $r(7,6)$ is at most $r(7,7)$ we see that $r(7,6) \leq 6$. Finally, by examining all labelings of two vertices of all the pairs of order 7 graphs with five common subgraphs, we notice that the number of common subgraphs is reduced to below five; thus,

$r(7,5) \leq 5.$

All other cases for $n \leq 7$ involve at most four unlabeled vertices and are settled by the following theorems. We first introduce some special notation.

Let U and L be the subgraphs of G induced by its unlabeled and labeled vertices respectively. We denote the set of labeled vertices adjacent to an unlabeled vertex u by N(u).

<u>Lemma 4.1</u> - If u, v and w are labeled vertices and u is adjacent to all (or none) of the vertices of U then G is reconstructible from G_u, G_v and G_w.

<u>Proof</u> - From G_v and G_w we can see which labeled vertices are adjacent to u. Furthermore, G_v shows us that u is adjacent to all (or none) unlabeled vertices. Thus G is reconstructed from G_u by inserting u, joining u to the appropiate labeled vertices and joining (or not joining) u to the vertices of U.

<u>Theorem 4.2</u> - (Harary and Manvel [8]) - A graph G with at most two unlabeled vertices is reconstructible from any three of its subgraphs G_v, $v \in V(G)$, that is, $r(n,0) \leq 3$, $r(n,1) \leq 3$ and $r(n,2) \leq 3$.

<u>Proof</u> - Suppose first that all vertices of G are labeled and we have deleted u, t and w. From G_u we can read off all edges not adjacent to u, from G_t those not adjacent to t and from G_w those not adjacent to w. Since no edge is adjacent to u, t and w we have all edges and so have G.

If just one vertex is unlabeled that vertex is thereby distinguished and we are thus dealing again with a graph which is labeled for all practical purposes.

Finally, if there are two unlabeled vertices r and s we must distinguish three cases. If r and s are both deleted we have L from G_r, N(r) from G_s and N(s) from G_r. Since G_u, u a labeled vertex in G, shows whether or not r and s are adjacent we have G. If r is deleted along with two labeled vertices, say u and t, then we find L and N(s) from G_r and determine N(r) and the presence or absence of the edge <r,s> from G_u and G_t. If the labeled vertices u, t and w are deleted then we have U and by the first part of this theorem L. If any one of the three labeled vertices u, t, or w is adjacent to neither or both of r and s we are done by Lemma 4.1. Hence assume that each of u, t and w is adjacent to exactly one unlabeled vertex. But then, in any case, r and s can be distinguished in G_u, G_t and G_w, so we are again done by the first part of this theorem.

<u>Lemma 4.3</u> - Any graph G with 3 or 4 unlabeled vertices is reconstructible from the subgraphs G_v where v is an unlabeled vertex.

<u>Proof</u> - From the subgraphs G_v we can easily find the subgraph L and also the labeled neighborhoods (sets of adjacent labeled vertices) of all the unlabeled vertices. In order to reconstruct G we must find the subgraph U with the neighborhoods assigned to the proper vertices. Thus the proof reduces to showing that a graph on 3 or 4 vertices, perhaps with a partial labeling, can be reconstructed from its subgraphs G_v so as to display the labeling. But this is a straightforward exercise since there are only 14 graphs with 3 or 4 vertices.

<u>Theorem 4.4</u> - (Harary and Manvel [8]) - A graph G with at most 4 unlabeled vertices is reconstructible from any 4 of its subgraphs G_v, $v \in V(G)$, that is, $r(n,3) \leq 4$ and $r(n,4) \leq 4$ for $n \geq 4$.

<u>Proof</u> - If two or fewer vertices are unlabeled any 3 G_v will do as shown in Theorem 4.2. Thus we proceed to deal with graphs with three or four unlabeled vertices. In almost every case, we can easily reconstruct the labeled subgraph L and the labeled neighborhoods of the unlabeled vertices. The difficult step is to reconstruct U and distinguish which vertex receives which neighborhood.

Consider a graph with three unlabeled vertices r, s and t and delete the vertices v_1, v_2, v_3 and v_4. There are four cases to consider which correspond to one, two, three or four labeled vertices being deleted.

<u>Case 1</u> - $v_1 = r$, $v_2 = s$, and $v_3 = t$. That is all unlabeled vertices are among the deleted vertices. Then Lemma 4.3 shows that G can be reconstructed.

<u>Case 2</u> - $v_1 = r$, $v_2 = s$, but t is not deleted. From G_r and G_s we can find L and the sets N(r), N(s) and N(t) of labeled vertices adjacent to r, s and t respectively. We can also decide whether both (or neither) of r and s are adjacent to t. If so we can reconstruct G since G_{v_3} gives us the number of edges in U and hence shows whether or not r

and s themselves are adjacent. If exactly one of r and s is adjacent to t (say r) and there are two edges in U, then G_r can be completed to G by inserting r adjacent to both unlabeled vertices and the appropriate labeled ones. But if there is only one edge in U, then G_s can be similarly completed to G by inserting s adjacent only to the labeled vertices in N(s).

<u>Case 3</u> - v_1 = r but s and t are not deleted. By Lemma 4.1 if v_2, v_3 or v_4 is adjacent to all or none of r, s and t we are done. So suppose each of v_2, v_3 and v_4 is adjacent to either one or two of r, s, and t. If v_2 is adjacent to s, but not to r or t then s is essentially labeled (recognized) in G_r, G_{v_3} and G_{v_4} so we can use Theorem 4.2. This is also true if v_2 is adjacent to r and t but not to s. Since similar arguments work for t, we may assume that v_2 is adjacent (non-adjacent) to s and t if and only if it is non-adjacent (adjacent) to r and similarly for v_3 and v_4. Since we know which of v_2, v_3 and v_4 are adjacent to s and t we can immediately reconstruct G from G_{v_2} since we obtain L from G_r and s and t are distinguished from r in G_{v_2} by their relationship to v_3.

<u>Case 4</u> - r, s and t are not deleted. Again by Lemma 4.1

we may assume that each v_1, v_2, v_3 and v_4 is adjacent to just one cr two of r, s and t. If v_1 is adjacent to r and not to s or t, then r is essentially labeled in G_{v_2}, G_{v_3} and G_{v_4} so we can apply Theorem 4.2. On the other hand if v_1 is adjacent to r and s and not to t, then t is essentially labeled in G_{v_2}, G_{v_3} and G_{v_4}, so Theorem 4.2 works in this case also.

This completes the proof for graphs with three unlabeled vertices. The proof of graphs with four unlabeled vertices is similar but more tedious and thus we omit it. The proof is given in [8].

We summarize the known values of r(n,p) in Table 1.

n \ p	0	1	2	3	4	5	6	7
3	3	3	3	3				
4	3	3	3	4	4			
5	3	3	3	4	4	5		
6	3	3	3	4	4	5	6	
7	3	3	3	4	4	5	6	6
↓	↓	↓	↓	↓	↓			

Note: An arrow ↓ denotes values are known.

Table 1 - Known values of r(n,p).

So far the vertex problem has been considered for graphs with small numbers of unlabeled vertices. A more interesting but also more difficult problem is the reconstruction of graphs with only a few labeled vertices. The reconstruction of arbitrary graphs with one or two labeled vertices would certainly be a major breakthrough.

Harary and Manvel concluded another conjecture suggested by Table 1, which displays the fact for $n = 7$ (but not for $n < 7$) there is no graph which actually requires all seven subgraphs G_v for reconstruction purposes.

<u>Sharp Reconstruction Conjecture</u> - For $|V(G)| \geq 7$ every graph G with n vertices requires fewer than n of its subgraphs G_v, $v \in V(G)$, i.e., $r(n,n) < n$ whenever $n \geq 7$. Furthermore, for all $p > 0$, there exists an integer $n = n(p)$ such that $r(n,n) < n - p$.

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