A NEW EXACT SOLUTION OF THE INTERIOR EINSTEIN FIELD EQUATIONS WITH STRESS-ENERGY TENSOR OF SEGRE CLASS [111,1]

by

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ABSTRACT

A method is presented for creating exact solutions of the interior Einstein field equations when the Segre class of the stress-energy tensor is prescribed. The "g" and "T" methods of Synge are combined in a way which takes advantage of the algebraic structure of the stress-energy tensor. The field equations in orthogonal coordinates are written for the case when the stress-energy tensor is of the algebraically general Segre class [111,1] and its eigenvectors are aligned with the coordinate vectors. A new exact solution of these equations is found which in special instances satisfies the strong energy conditions.

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I. Introduction

Motivation

From the beginning of the theory of General Relativity there has been a sustained search for new exact solutions of the Einstein field equations

$$(1.1) \qquad G_{ij} = -8\pi T_{ij}.$$

The solutions of (1.1) which are of the most interest are the exact solutions i.e. solutions which satisfy (1.1) together with appropriate side conditions. The side conditions will generally reflect properties of physical situations which we wish to model (boundary conditions, energy conditions). The system (1.1) is a quasilinear, coupled system of ten second order partial differential equations for ten unknown functions g_{ij} of four variables. Adding to the complexity of (1.1) is the fact that the global topology of space-time is unknown hence the boundary conditions are arbitrary. The question of what boundary conditions to use is still not completely resolved. If we wish the exact solutions to have reasonable physical interpretations then we must arrange that T_{ij}, the stress-energy tensor, satisfies certain conditions. If T_{ij} describes macroscopic matter with physically plausible properties such as everywhere nonnegative energy density, nonspacelike momentum transfers, and pressures rather than tensions, then the difficulties posed by the system (1.1) become enormous. The nonlinearity of the partial differential equations (1.1) is the most vexing feature from a mathematical viewpoint.

As discussed in Synge [1;p184] there is a variety of approaches that may be adopted when looking for solutions of (1.1). Systems of equations (1.1) may be classified into two types depending upon the nature of the stress-energy tensor. Wherever the stress-energy tensor is zero the system (1.1) describes a <u>vacuum</u> space-time. For this case the equations (1.1) reduce to

$$(1.2) \qquad R_{ij} = 0$$

where R_{ij} is the Ricci tensor. The vacuum field equations have been extensively studied and in many special cases the solutions of (1.2) are completely known [2].

When the stress-energy tensor is nonzero the system (1.1) describes a region of space-time which is a body in the sense used by Das [3]. In this case we will say that the equations (1.1) describe an <u>interior</u> space-time. The interior field equations have been studied mainly for the simplest models of macroscopic matter i.e. dust, perfect fluids, wave fields, and other idealized equations of state. As a consequence of these simplifications there are no realistic exact solutions of the interior field equations for interesting simple astronomical situations such as rotating stars, nonspherical stars and multibody systems. The interior field equations will be the exclusive focus of our attention in the pages to follow.

In this thesis our principal objective is to write and, if possible, solve the system of equations

$$(1.3) \qquad G_{ij} = -8\pi T_{ij}, \qquad T_{ij} \neq 0,$$

where the stress-energy tensor is "algebraically general". We will impose certain conditions which guarantee that the solutions we will find are physically reasonable. In pursuit of this objective we will find it necessary to develop in detail a method of solving (1.3). The method which we will use is derived from the works of Sygne [1;p184] and Petrov [4;p323]. The selection of this method will be seen to be consistent with our goals of generality and reality implied previously. The process of finding a methodology is a long and difficult one. The recent book by Buchdahl [5] presents many of the perplexities which were encountered in formulating our method.

Methodology

The method which we shall use is a compromise between two methods outlined by Sygne coupled with an algebraic classification of the stress-energy tensor. In the first of these two methods, the "T-method", one proceeds by choosing a coordinate system and then specifying the stress-energy tensor, T_{ij}, as a set of ten functions. In this approach the system (1.3) is viewed as a difficult system of partial differential equations. The T-method cannot completely satisfy our demand of physical reality in the fullest sense, as a solution found in this manner is quite unlikely to bear any resemblance to the matter distribution to be modelled when the problem was first formulated.

The difficulty in specifying the "source" stems directly from the fact that General Relativity is a field theory in which the role of field variables is played by the metric tensor. The intrinsic linkage between the geometry of space-time and the matter-energy distribution expressed in (1.3) rules out any a priori notion of "distance" in space-time. This absence of "distance" prevents one from ascribing a physical interpretation of the stress-energy tensor before the metric tensor is found from (1.3). This problem has usually been avoided by assuming that the metric is "almost flat" i.e. pressures and densities are nonrelativistic. The reason for this hypothesis is that with it we can specify the geometrical distribution of matter before we have solved (1.3). In the presence of extreme pressures and densities this assumption cannot be used.

The second method, the "g-method", simply requires one to prescribe ten sufficiently smooth functions g_{ij} as the metric tensor provided that the correct signature is maintained everywhere. In this approach the equations (1.3) are regarded as definitions of the components of the stress-energy tensor. The g-method is unsatisfactory since the calculated stress-energy tensor is unlikely to have physically reasonable properties.

The heart of the difficulties in these two approaches lies in the uncertain role of the coordinates with which we specify either the geometry of space-time or the matter-energy distribution. Mathematically, the coordinates serve as a set of markers of events in space-time. Central to any process of physically interpreting the equations (1.1) is the concept of a local frame of reference. The natural language to express the physical content of the field equations is that of the tetrad calculus. A choice of local reference frame is a physical one and is usually adapted to the problem at hand. In principle, the tetrad components of the tensors which appear in (1.1) are susceptible to physical measurement. For the interior field equations we will see that the best local reference frame is one which contains the eigenvectors of the stress-energy tensor. We will call any such local reference frame the natural frame of T_{ij}.

These considerations lead us to wonder if a hybrid method, which combines the advantages of the T-method and the g-method and is expressed in the tetrad calculus, might allow some progress towards our goals. We will call this hybrid method the "mixed method".

The mixed method allows us to specify the algebraic/relativistic structure of the physical stress-energy tensor T_{AB} with respect to the flat metric tensor η_{AB}. Naturally we will find that when the tetrad is the "generalized eigentetrad" of T_{AB}

then the corresponding tetrad field equations will assume their simplest form. Further simplifying hypotheses may be made before we partition the tetrad field equations for a nonempty space-time:

$$(1.4) \qquad G_{AB} = -8\pi T_{AB}, \qquad T_{AB} \neq 0.$$

With respect to its generalized eigentetrad, the physical components of the stress-energy tensor are either constant, having values zero or one, or they are nontrivial scalar functions of position. It will be seen in the third chapter that there are at most four nonconstant physical components (related to the four eigenvalues) of T_{AB}. We can partition (1.4) into two subsystems using as a criterion the nature of the components of T_{AB}. The subsystem corresponding to trivial (zero or one) components will yield a system of partial differential equations which we will attempt to solve. The subsystem corresponding to nontrivial components of T_{AB} will be viewed as a set of <u>definitions</u> of these components. For the defined components of the stress-energy tensor to be physically reasonable we must impose a set of differential inequalities. These differential inequalities will force the solutions found by the mixed method to satisfy the strong energy conditions [6;p88]. Although any of the energy conditions could be imposed on the solutions in this manner, there is no guarantee that the matter-energy distribution corresponding to T_{AB} is realized in nature.

In this thesis we have not imposed an equation of state in finding the new interior solution. An equation of state may be adjoined to the "reduced" system of partial differential equations which we attempt to solve in the mixed method. If the equation of state is a nonlinear relation between the energy density and the principal stresses then the resulting system of partial differential equations becomes extremely complicated. The equations of state usually imposed in General Relativity are simple linear equations as a result of this problem.

Description of Results

The second chapter contains most of the mathematical background necessary to develop and employ the mixed method outlined above. Some parts of the theory of differential manifolds are developed in more detail than we employ; this was done with a view to further work based on the results of this thesis. The tensor calculus is described in this abstract setting. In order that one may freely use any notation system for the tensor calculus or the tetrad calculus, an appendix to this chapter is given which provides a scheme for converting tensor equations from one notation system to another.

The third chapter contains the algebraic tools necessary to classify the stress-energy tensor. Several definitions from linear algebra are generalized to our needs. Several conditions are found on the secular equation of the stress-energy tensor which force the existence of four real eigenvalues at a point in space-time. A short summary of the Segre characteristic and the Plebanski classification of the stress-energy tensor completes description of the mixed method. The "Principal Axes" theorem for classifying a second rank symmetric tensor with respect to an indefinite metric concludes this chapter.

The fourth chapter is a collection of three known exact solutions which have been treated by a specialization of the mixed method. The specialization of the mixed method which we use is to require that the generalized eigentetrad is aligned with the coordinate tetrad. These examples illustrate the application of the mixed method on simple solutions. Other solutions have also been used to test the tetrad formulation of the equations (1.1) but these have not been included since they do not satisfy the alignment hypothesis.

In the fifth chapter the new exact solution which we find is presented. A new family of interior solutions for stress-energy tensors of Segre class [111,1] is found and several parametric subfamilies are shown to satisfy the strong energy conditions. In view of the complexity of the solution, adequate description of a possible physical source appears difficult. A second family of new solutions is also found but these are even more complex and have not been analyzed in detail. Both families of solutions are found via the mixed method under the simplifying hypothesis that the generalized eigentetrad is aligned with the coordinate tetrad. Some of the more lengthy calculations which would only obscure the application of the mixed method are omitted.

Possibilities for Further Research

There are many directions in which further work based on this thesis could proceed. The mixed method could be extended as a "generalization" of the Geroch-Held-Penrose formalism [2]. The G.H.P. formalism is a powerful tool which has been used primarily to investigate the vacuum field equations. The use of the generalized eigentetrad for the Segre class [11,2] and its algebraic degeneracies produces the same tetrad field equations in the mixed method as the G.H.P. formalism. However, the G.H.P. formalism uses a tetrad consisting of two null and two spacelike vectors for the Segre class [111,1] and its algebraic degeneracies. The mixed method uses a tetrad of eigenvectors for the Segre class [111,1] none of which can be null. It seems to us that the mixed method can lead to a useful extension of this formalism for the investigation of the interior field equations.

To develop a scheme for searching for interior and vacuum solutions simultaneously it will be necessary to carefully study the interface (boundary) between the regions of space-time where the stress-energy tensor is zero or not. This is the reason for the detailed discussion of piecewise differentiable maps in the second chapter. The specification of just where this interface occurs in physical terms will entail a careful examination of the "hypersurfaces of discontinuity" of physical quantities.

Perhaps one way to find this interface is by examining the hypersurfaces across which the Segre class of the stress-energy tensor changes. A study in this direction of the hypersurfaces where the Petrov type of the Weyl tensor changes has been carried out by Case [7].

Another way in which we may possibly extend the results of the fifth chapter is by extensively studying the partial differential equation

$$(1.5) \qquad U_{xy} + F(x,y)U = 0$$

where $F(x,y)$ is an arbitrary function of class C^3. This equation is found to have solutions for a very special choice of $F(x,y)$. The general solution is unkown even for simple choices of $F(x,y)$.

All of the work in the last two chapters is a consequence of the simplifying hypothesis that the generalized eigentetrad of T_{AB} is aligned with the coordinate tetrad. In these cases we always used orthogonal coordinate systems. Other relative orientations introduced by using nonorthogonal coordinates of these two tetrads will produce systems of partial differential equations of great complexity, however solutions of these systems may more accurately model realistic physical matter distributions.

The process of completing the generalized eigentetrad when the stress-energy tensor has less than four real eigenvectors could be varied so that special vectors such as Killing vectors, recurrent vectors, or other special vectors are incorporated. Even in the cases where the stress-energy tensor has a proper eigentetrad there may be some indeterminacy due to repeated eigenvalues.

Finally, the generalized eigentetrads of the stress-energy tensor should be of use in computing global topological invariants of space-time. Invariants such as the Chern class have been the subject of intensive research in theoretical physics recently. The canonical form of the stress-energy tensor with respect to its generalized eigentetrad could simplify the computation of these invariants. These simplifications should lead to a deeper understanding of just what physical significance these invariants are and what role they could play in the search for a unification between Quantum Physics and and General Relativity.

II. Differential Manifolds, Tensor Calculus, and the Tetrad Calculus

Differential Manifolds

In order to formulate the mathematical structures which are used to model space-time in General Relativity, a condensed exposition of terminology, notation, and conventions is required. The main analytical tools we shall use are the tensor calculus on manifolds and the tetrad calculus. A small background of abstract manifold theory and the theory of fibre bundles is required to place these tools in a unified context.

A n-dimensional topological manifold is defined as a separable Hausdorff topological space M such that every point in M has an open neighbourhood which is homeomorphic to an open subset of R^n (we assume that R has the usual topology induced by the Euclidean metric). A n-dimensional coordinate chart on M, abbreviated to "n-coordinate chart", is a pair (U,u) where U is an open subset of M and u is a homeomorphism of U onto an open subset of R^n. For any n-coordinate chart (U,u) on M and for each i, $1 \leq i \leq n$, define the i-th coordinate function of u, x^i_u, so that $x^i_u : U \longrightarrow R$ is given by

$$(2.1) \qquad x^i_u = Pr_i \circ u$$

Here the i-th canonical projection of R onto R is denoted by Pr_i.

In general, n-dimensional topological manifolds cannot be globally coordinatized by the use of a single n-coordinate chart. Simple examples of this point are the spheres S^n, $n \geq 1$, and the tori T^n, $n \geq 2$. If (U,u) and (V,v) are two n-coordinate charts on M then it is a reasonable requirement that the coordinates of a point P in $U \cap V$ be unchanged in the transition from one n-coordinate chart to the other. Two n-coordinate charts (U,u) and (V,v) are said to be $\underline{C^r\text{-compatible}}$, $r \geq 0$, if either

(a) $U \cap V = \emptyset$, or

(b) $U \cap V \neq \emptyset$ and both of the coordinate transition maps

(2.2) $\quad v \circ u^{-1} : u(U \cap V) \text{--->} v(U \cap V)$

$\quad u \circ v^{-1} : v(U \cap V) \text{--->} u(U \cap V)$

are of class C^r when viewed as maps from R^n to R^n. The notion of C^r-compatibility is illustrated in Figure 1.

Let A be an arbitrary set of indices. A collection

$\mathcal{A} = \{(U_a, u_a) : a \text{ in } A\}$

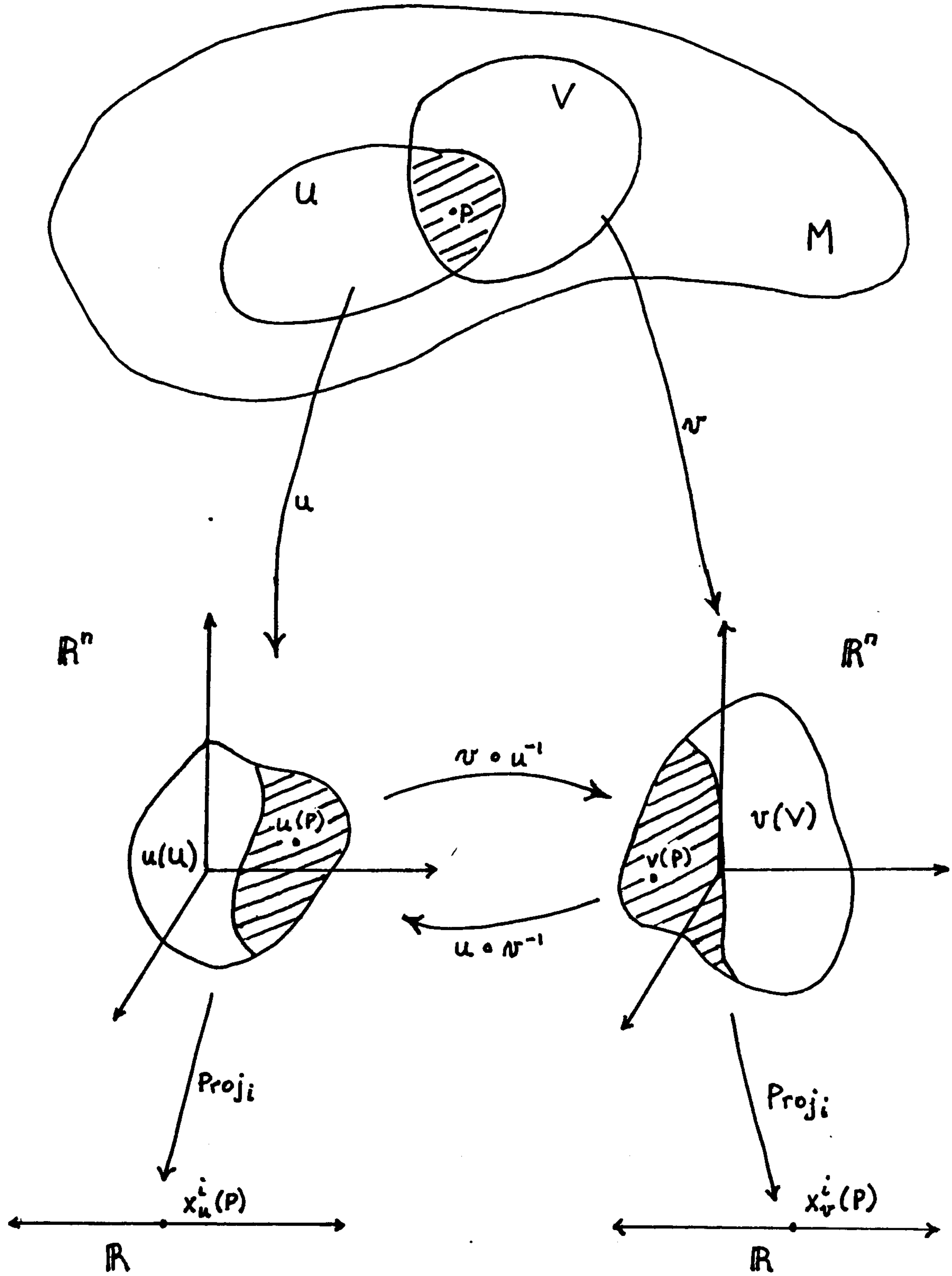
of n-coordinate charts on M is called a $\underline{C^r\text{-subatlas on } M}$ if

(a) $\{U_a : a \text{ in } A\}$ is an open covering of M,

(b) for each (a,b) in A×A the n-coordinate charts (U_a, u_a) and (V_b, v_b) are C^r-compatible.

Two C^r-subatlases $\mathcal{A}$ and $\mathcal{B}$ are $\underline{C^r\text{-equivalent}}$ if $\mathcal{A} \cup \mathcal{B}$ is a C^r-subatlas. It is easy to show that C^r-equivalence is an

Figure 1: An Overview of C^r-compatibility



equivalence relation on the class of C^r-subatlases of M. Each C^r-equivalence class of C^r-subatlases of M is partially ordered by inclusion. A maximal element in a C^r-equivalence class of C^r-subatlases on M is called a *C^r-atlas on M*. By a lemma of Spivak [8] it can be shown that each equivalence class $\langle \mathfrak{A} \rangle$ has a unique maximal element hence there is an injective correspondence between C^r-equivalence classes $\langle \mathfrak{A} \rangle$ and C^r-atlases. Each C^r-equivalence class, $\langle \mathfrak{A} \rangle$, is called a *differentiable structure of class C^r on M*. We agree to use the unique C^r-atlas to represent any particular C^r-differentiable structure on M.

Each C^r-differentiable structure, $r \geq 1$, is known to contain a differentiable structure of class C^∞[9]. Results of Kervaire [10] and Smale [11] show that a differentiable structure of class C^0 does not always contain a C^1-differentiable structure if the dimension of the manifold M is greater than or equal to four. To avoid these problems we replace C^r, $r \geq 1$ by C^∞ as is done in most texts on differential manifolds and differential geometry.

A n-dimensional *differentiable manifold M of class C^r*, $r \geq 1$, is a pair $(M, \langle \mathfrak{A} \rangle)$ where M is a separable, Hausdorff n-dimensional topological manifold and $\langle \mathfrak{A} \rangle$ is a differentiable structure of class C^r on M. When $r = 0$ $(M, \langle \mathfrak{A} \rangle)$ is simply called a *topological manifold*.

We define a C^r-subatlas $\mathfrak{A}$ to be *oriented* if $r \geq 1$ and for each pair (a,b) in A×A, the Jacobian of the transition map $u \circ u^{-1}$ is positive wherever it is defined. A differentiable structure

$\langle\mathfrak{A}\rangle$ is <u>oriented</u> if each subatlas in $\langle\mathfrak{A}\rangle$ is oriented and if the union of any two subatlases is also oriented.

Let M be an m-dimensional differentiable manifold of class C^r, $r\geq 1$, with C^r-atlas

$\mathfrak{A} = \{(U_a, u_a): a \text{ in } A\}$, A an arbitrary index set.

Let N be an n-dimensional differentiable manifold of class C^s, $s\geq 1$, with C^s-atlas

$\mathfrak{B} = \{(V_b, v_b): b \text{ in } B\}$, B an arbitrary index set.

A continuous map $F:M \longrightarrow N$ is of class C^k where $k\leq \min(m,n)$ if for each pair (a,b) in $A\times B$, the coordinate representation of F,

$\tilde{F}_{ab}: u\ (U_a \cap F^{-1}[V_b]) \longrightarrow v_a(V_b \cap F(U_a))$

defined by

$$\text{(2.3)} \qquad \tilde{F}_{ab} = v_b \circ F \circ u_a^{-1}$$

is in $C^k(R^m;R^n)$, the space of functions of class C^k from R^m into R^n. Figure 2 illustrates this definition. We say that F is in $C^k(M;N)$ to indicate that F is of class C^k from M into N.

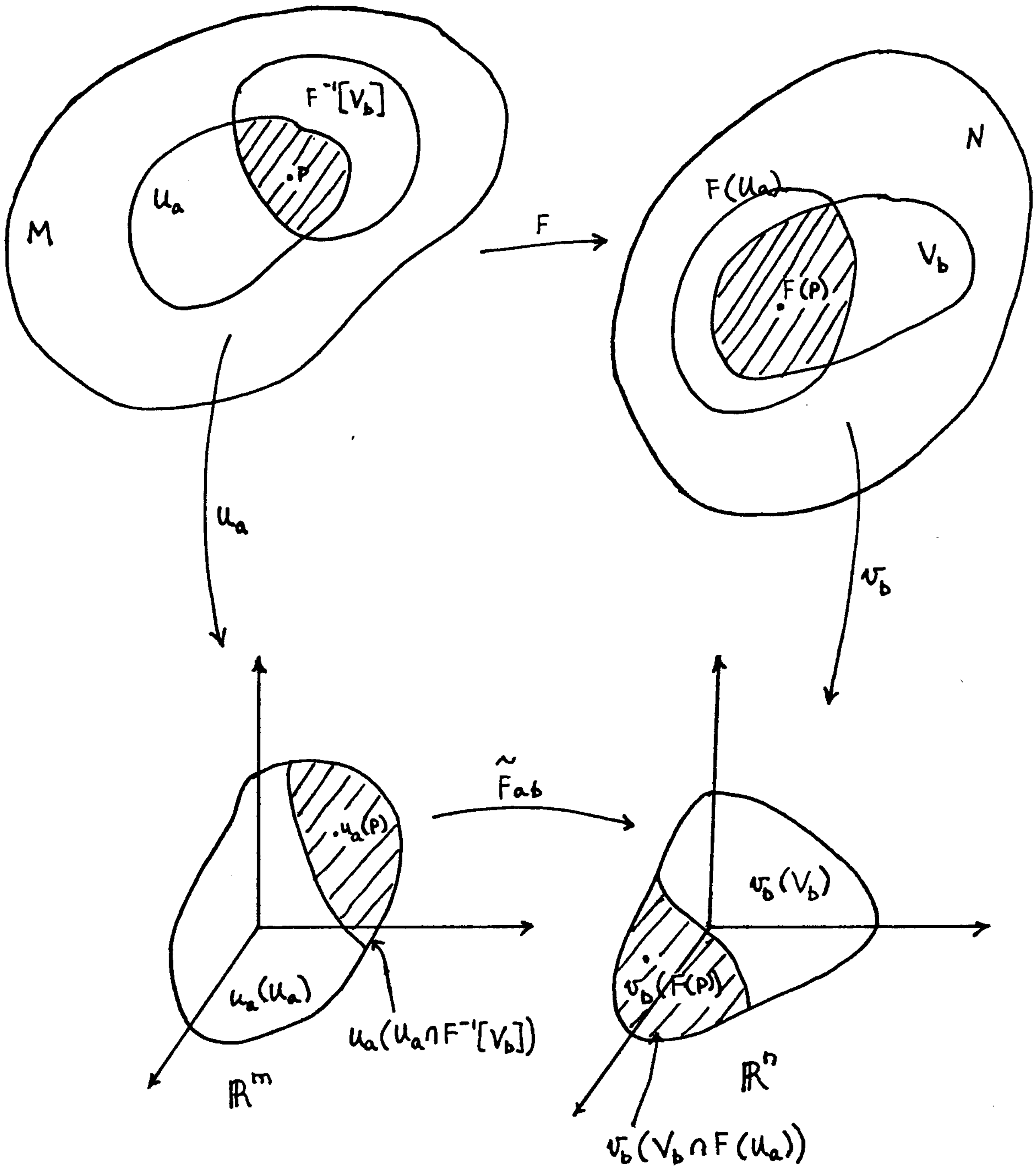
A map F in $C^k(M;N)$ is called a <u>C^k-diffeomorphism</u> if there is a map F^{-1} in $C^k(N;M)$ such that

(a) $F\circ F^{-1}$ = identity on N, and

(b) $F^{-1}\circ F$ = identity on M.

Since the physical entities of interest in General Relativity may exhibit various types of discontinuity we will define more general spaces of maps which will formalize the types of discontinuity which are of interest. The following definitions

Figure 2: The Coordinate Representation of a Function



are drawn from the work cf Abraham [12].

Let D be open in R^m, E be a subset of R^n, and let $F:D \dashrightarrow E$. If

$$\{g_i : R^m \longrightarrow R, \ i \text{ in } I\}$$

is a finite family of functions with each g_i in $C^1(R^m;R)$ then for each i in I we define a parametrized family of hypersurfaces $S_i(c)$, $-\varepsilon<c<\varepsilon$, where $\varepsilon>0$, and

$$S_i(c) = g_i^{-1}(c).$$

Define F to be <u>piecewise of class C^q on D</u> if and only if

(a) F is defined and of class C^q in D-S, where S is a finite union of closed hypersurfaces $S_i(0)$ in D;and

(b) the parametrized family of functions $F \restriction S_i(c)$ converges uniformly to a bounded limit function on each $S_i(0)$ as c tends to zero through both negative and positive values. The hypersurfaces $S_i(0)$ are the <u>hypersurfaces of discontinuity</u> of F. For a fixed

$$S = \bigcup_{i \in I} S_i(0)$$

the space of piecewise C^q maps from D into E is denoted $C^q_S(D;E)$. If F is in $C^p(D;E)$ and is also in $C^q_S(D;E)$ with p<q we write that

$$F \in C^{p,q}_S(D;E).$$

Given a differentiable manifold $(M,\langle\alpha\rangle)$ of class C^r we say that a function $F:M \longrightarrow R$ is in $C^{p,q}_S(M;R)$ if

(a) p<q<r,

(b) S is a finite union of closed hypersurfaces S in M where each $S_i = g_i^{-1}(0)$ for some g_i in $C^1(M;R)$,

(c) F is in $C^p(M;R)$, and

(d) for each m-coordinate chart (U_a, u_a) in $\mathcal{A}$ the function

$$F \circ u_a^{-1} : u_a(U_a) \text{---}> R$$

is in $C^q_{u_a(S)}(R^n; R)$. The extension of this definition to maps $F: M \text{---}> N$ where $(M, \langle\mathcal{A}\rangle)$ is a C^r-manifold and $(N, \langle\mathcal{B}\rangle)$ is a C^s-manifold is straightforward. We say F is in $C^{p,q}_S(M;N)$ if

(a) $p<q<\min(r,s)$,

(b) S is a union of closed hypersurfaces S_i in M where each $S_i = g_i^{-1}(0)$ for some g_i in $C^1(M;R)$

(c) F is in $C^p(M;N)$, and

(d) for each pair (a,b) in A×B the coordinate representation for F, $\tilde{F}_{ab}$ defined in (2.3), is in $C^{p,q}_{u_a(S)}(R^m;R^n)$.

Figure 3 illustrates this complicated definition.

There are several equivalent means of defining tangent vectors and the tangent space on a differential manifold [13], [14], [15]. Define a <u>tangent vector</u> of M at P to be a linear map $V_P : C^1(M;R) \text{---}> R$ such that

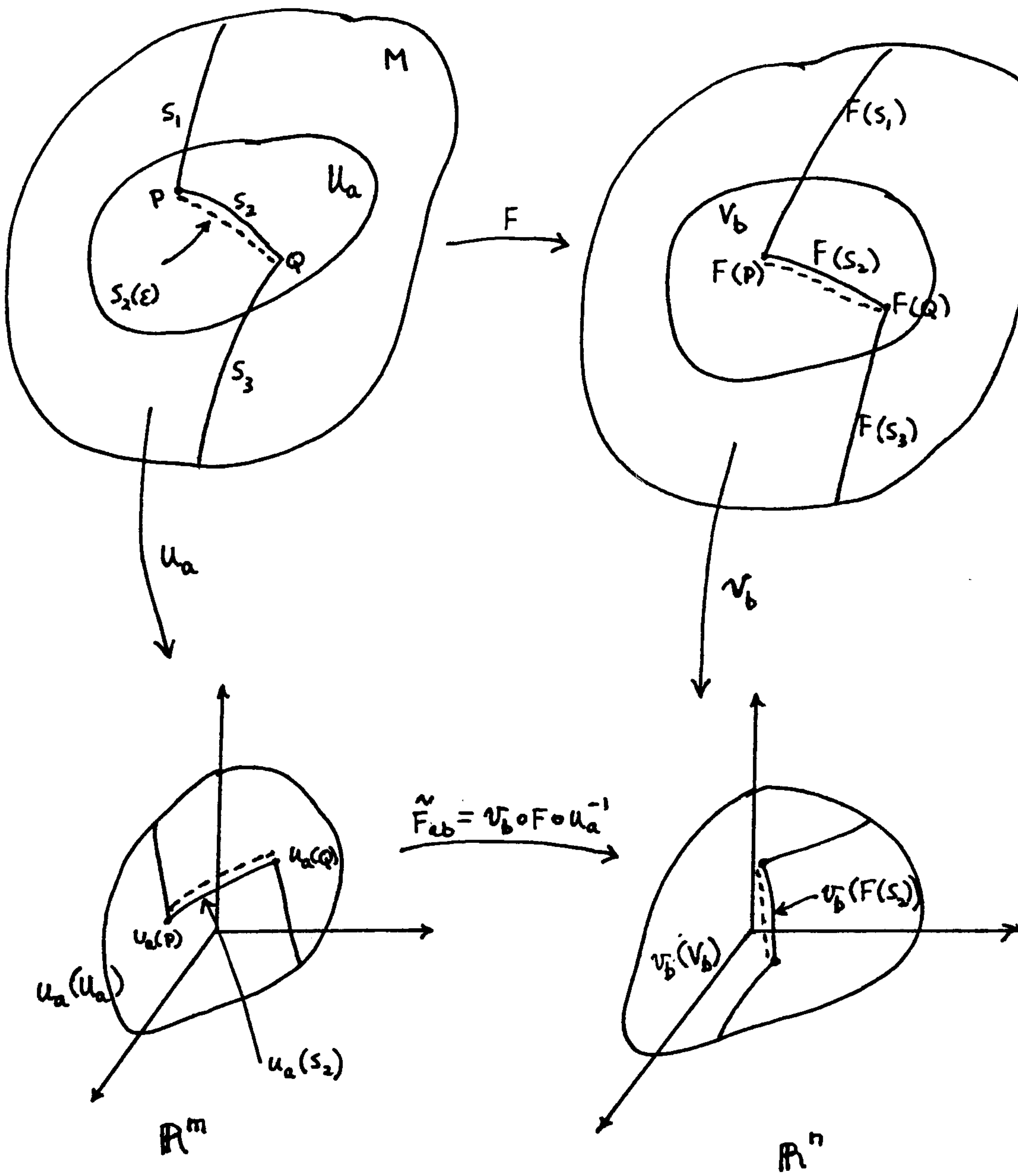
(2.4) $$V_P(fg) = V_P(f)\,g(P) + f(P)\,V_P(g)$$

for all f,g in $C^1(M;R)$.

$T_P M = \{V : V \text{ is a tangent vector}\}$ is a vector space of dimension equal to that of M. The vector space $T_P M$ is called the <u>tangent space to M at P</u> .

Let E, B, F be differential manifolds of class $C^k, k\geq 1$, respectively of dimensions m+n, n, m. Let $\mathcal{H}$ be a surjection in $C^k(E;B)$. If $\{U_a : a \text{ in } A\}$ is an open covering of B and

Figure 3: Piecewise Differentiable Maps on Mani-
folds



$$\phi_a : U_a \times F \dashrightarrow \pi^{-1}[U_a]$$

is a family of C^k-diffeomorphisms such that

$$\pi \circ \phi_a(P,V) = P$$

for all P in U_a, and all V in F, then the system $\{(U_a, \phi_a)\}$ is a *local decomposition of* π and we say that π has the *local product property with respect to F*. Since ϕ_a is a C^k-diffeomorphism there is a C^k-map

$$\lambda_a : \pi^{-1}[U_a] \dashrightarrow F$$

such that

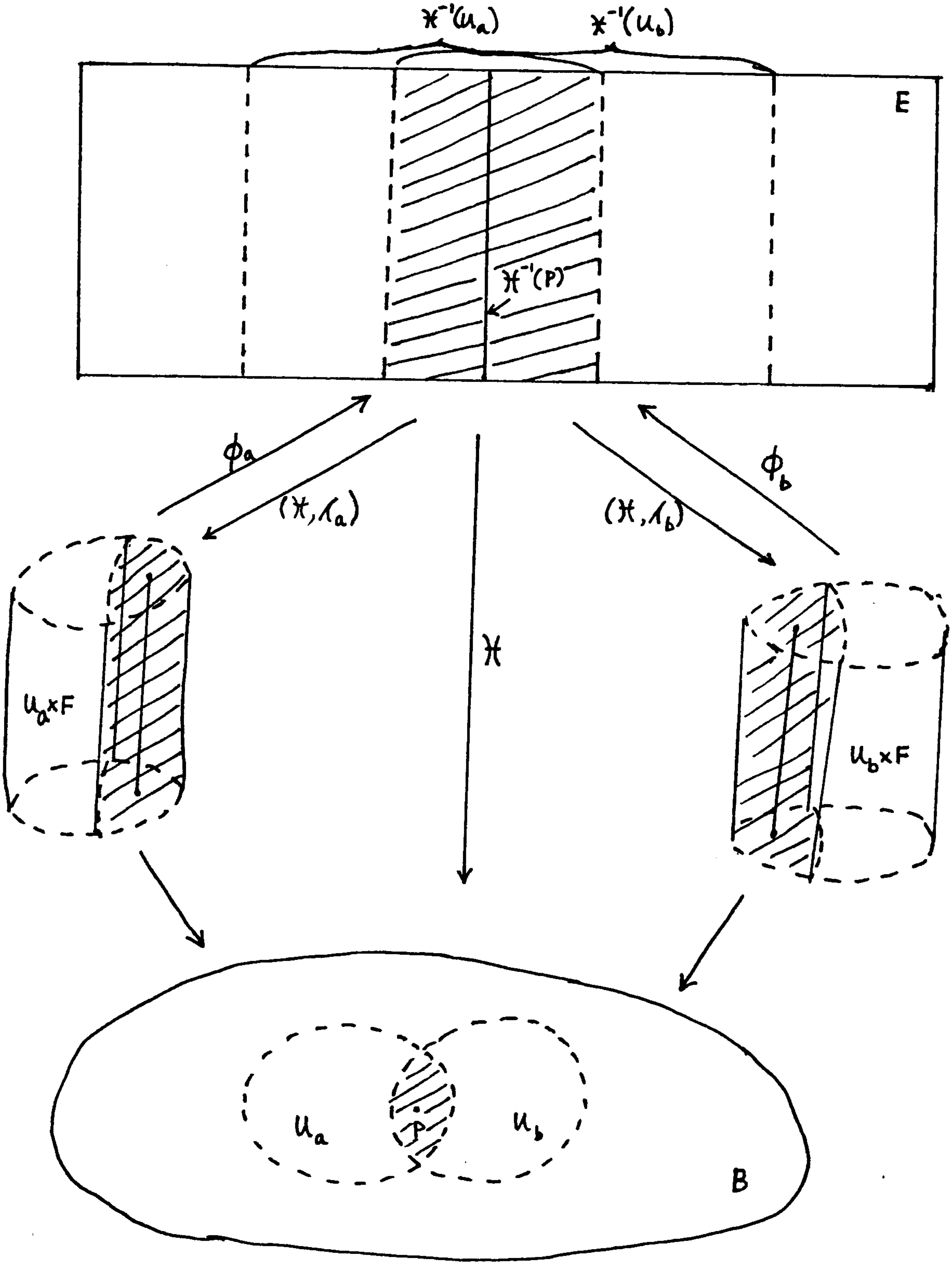
$$(\pi, \lambda_a) : \pi^{-1}[U_a] \dashrightarrow U_a \times F$$

is a diffeomorphism. The C^k-map (π, λ_a) is called a *bundle chart on E over* U_a. The pair (π, λ_a) is also called a *local trivialization of* π *over* U_a. Note that it is important *not* to confuse a local decomposition of π with the notationally similiar concept of a subatlas on B.

A *differentiable fibre bundle of class* C^k is a quadruple $[E, \pi, B, F]$ in which π has the local product property with respect to F for some open covering $\{U_a : a \text{ in } A\}$ of B. We say that E is the *total space*, π is the *bundle projection*, B is the *base space*, and F is the *typical fibre*. For each P in B, the set $E_P = \pi^{-1}[P]$ is called the *typical fibre* over P. E_P can be shown to be a closed submanifold embedded in E and that E is diffeomorphic to F [16]. Figure 4 illustrates some of the features of the complicated structure of the fibre bundle definition.

If $U_a \cap U_b \neq \emptyset$, $a \neq b$, the two local trivializations of π, (π, λ_a) and (π, λ_b), may not agree on $\pi^{-1}[U_a \cap U_b]$. For a fixed P in $U_a \cap U_b$

Figure 4: Fibre Bundle Structure



the maps $\lambda_{a,p} = \lambda_a | E_p$ and $\lambda_{b,p} = \lambda_b | E_p$ are C^k-diffeomorphisms from E_p to F. The maps $\lambda_{a,p} \circ \lambda_{b,p}^{-1}$ and $\lambda_{b,p} \circ \lambda_{a,p}^{-1}$ are C^k-diffeomorphisms of F which allow transition from one local trivialization to the other.

The group of diffeomorphisms of F is very large, so usually a subgroup is selected as the allowable transition diffeomorphisms which relate different local trivializations of χ. The subgroup G of diffeomorphisms on F which is allowed is called the _structural group_ of the fibre bundle. This process of reducing the group of admissible transition diffeomorphisms is of fundamental importance in some of the latest work in differential geometry and theoretical physics.

If S is a finite union of closed hypersurfaces in B then a _cross-section of class C^k on E_ is a map f in $C^k(B;E)$ such that $\chi \circ f$ = identity on B, and $k \geq \max(p,q)$. Many useful structures such as vector fields, tensor fields, frames of reference, and other structures can be elegantly written as sections in an appropriate bundle.

Topology of Differential Manifolds

The point set topology on a differential manifold is built into its definition. Additional topological properties which may be desired must be explicitly hypothesized. The hypotheses that manifolds are Hausdorff and separable are made to avoid pathologies. It can be shown that a separable Hausdorff topological space is locally compact. Separability and local compactness imply that all differential manifolds are paracompact. Since every differential manifold is paracompact they all admit partitions of unity. Partitions of unity are essential in formulating a theory of integration on manifolds. Thus the hypotheses on the topology of a differential manifold allow many of the ideas and concepts of calculus on Euclidean spaces to be "lifted" up to differential manifolds.

The global topology of a differential manifold must be explicitly postulated. The usual way to determine the global topology of a differential manifold is by giving a suitable subatlas. Often the global topology is not specified explicitly but merely restricted by the existence of special structures on the manifold i.e. line-element fields, spinor fields.

<u>Space-time M_4</u>

In General Relativity we assume that M_4 is a noncompact, connected, oriented manifold of class C^4 with a second countable, Hausdorff topology. We also assume that the dimension of M_4 is four. Note that a poor selection of coordinates a subatlas of class C^r, $r<4$, may be imposed on M_4. With such a subatlas functions on M_4 may exhibit discontinuities which are solely the result of coordinate discontinuities. This fact is of particular importance in General Relativity since we work almost exclusively with coordinate representations of functions on M_4.

We also assume that M_4 is endowed with a pseudo-Riemannian metric i.e. there is a map g of class $C^{1,3}$ on M_4 into the space of bilinear forms on $T_P M_4$ such that for each P in M_4, g(P) is non-degenerate and of signature -2. By a well-known result of Geroch [17] it is known that the existence of g together with the Hausdorff condition on the topology of M_4 imply that M_4 is paracompact.

The principal mathematical structure of interest in General Relativity is not the underlying manifold structure of M_4 but rather the differential geometric structure which is induced on M by the selection of the pseudo-Riemannian metric g_{ij}. Each pair $(T_P M_4, g(P))$ realizes the correspondence principle that General Relativity "goes over" to Special Relativity in the limit of infinitesimal regions near P.

Let P be a point in M_4. We define $T_P^* M_4$ to be the space of linear functionals on $T_P M_4$. Both $T_P M_4$ and $T_P^* M_4$ are isomorphic to R^4. These isomorphisms are determined by a selection of bases in both $T_P M_4$ and $T_P^* M_4$. The bases may be either "holonomic" if they are derived from some system of local coordinates near P or they may be "nonholonomic" if they are not so derived. For each pair of nonnegative integers (r,s) we define

$$T_P^{r,s} M_4 = \{F: (T_P^* M_4)^r \times (T_P M_4)^s \dashrightarrow R, \text{ F is multilinear}\}.$$

$T_P^{r,s} M_4$ can be shown to be a real vector space of dimension 4^{r+s}.

The <u>tensor bundle of type (r,s) over M</u> is $[T^{r,s}M_4, \pi, M_4, T^{r,s}R^4]$ where

$$T^{r,s}M_4 = \bigcup_{m \in M_4} T_P^{r,s} M_4,$$

is the disjoint union over M_4 of the tensor spaces $T^{r,s}M_4$. The bundle projection π is the natural projection and $T^{r,s}R^4$ is the space of tensors of type (r,s) on R^4. A <u>tensor field t of type (r,s)</u> on M_4 of class $C^{p,q}_S$, $0\leq p<q$, is a section in $T^{r,s}M_4$ of class $C^{p,q}_S$ i.e. t is in $C^{p,q}_S(M_4;T^{r,s}M_4)$. Each local trivialization of χ determines a local representation for a section t. Transitions between local trivializations of χ induce transitions between local representations of sections.

Notation Conventions for Tensor Calculus

There are a large number of books on coordinate tensor calculus and its use in the formulation of differential geometry. Unfortunately, many of these books differ in the placement of indices and even in the definitions of many of the tensors which we will use later. To avoid any ambiguities in the placement of indices on these tensors we have summarized the notation which we will use in future.

The following notation conventions will be used throughout this thesis:

(a) small latin letters assume the range {1,2,3,4} when used as indices;

(b) capital latin letters assume the range {1,2,3,4} when used as tetrad labels.

The Einstein summation convention holds for all types of indices when they appear in "up-down" pairs in a tensor expression. If the summation is to be suspended for an "up-down" pair of indices it will be explicitly stated. The symbols "()" and "[ ]" placed around a set of n indices of the same level will denote symmetrization and skew-symmetrization respectively. Both processes incorporate the factor 1/(n!). Any indices, within the range of "()" or "[ ]" which are to be excluded from the indicated process, will be enclosed in bars "| |". Partial differentiation is denoted by a comma which precedes the index of the coordinate function with respect to which we are differentiating. Covariant differentiation will be denoted by a semicolon ";" preceding the index of differentiation.

To enable easy transition between different notation systems encountered in various books, we have defined six parameters $e_i = +1$ or -1, $1 \le i \le 6$ which characterize all of the more usual notation conventions. The parameters e_i are defined to agree with four parameters defined by Ernst [18] for a similiar purpose. The e_i, $1 \le i \le 6$ are defined by the following relations:

(a) $e_1 \, \mathrm{signature}(g) = 2$,

(b) $e_2 V_i R^{i}{}_{\cdot jkl} = V_{j;lk} - V_{j;kl}$,

(c) $e_3 R_{ij} = R^{m}{}_{\cdot imj}$,

(d) $e_4 G_{ij} = -e_1 e_2 e_3 8\pi T_{ij}$,

(e) $e_5 = +1$ if space-time indices range over $\{0,1,2,3\}$,

-1 otherwise ,

(f) $e_6 \eta_{ijkl} = \sqrt{|\det g_{ij}|} \, \mathrm{sign}(ijkl)$.

In Appendix 1 relations are given which relate some of the e_i, $1 \le i \le 6$, to the Wheeler-Misner Thorne classification of notation conventions used in General Relativity[19]. Methods are given there which allow tensor equations to be converted from one convention to another.

In this thesis the choice $e_i=-1$, $1\leq i\leq 5$, $e_6=+1$ will be used exclusively. This choice closely parallels the notation of Eisenhart [20] except we use different punctuation to denote derivatives of various types.

Coordinate Tensor Analysis on M

For computational purposes the realization of tensor fields as sections in various tensor bundles is impractical. Classically, at P in M_4, a tensor field T of type (r,s) is represented by a set of 4^{r+s} quantities $T^{i_1\dots i_r}{}_{j_1\dots j_s}$ which are its components with respect to a choice of local coordinates on M_4. The selection of local coordinates near P induces a natural choice of bases on T_pM_4 (the holonomic basis associated with the local coordinates). Since M_4 is endowed with a pseudo-Riemannian metric, any selection of a basis on T_pM_4 induces a dual basis on $T_p^*M_4$. This pair of bases associated with the local coordinate system near P produces the representation $T^{i_1\dots i_r}{}_{j_1\dots j_s}$ of the tensor T.

Let (U,u) and (V,v) be two coordinate charts such that P is in $U\cap V$. Each set of coordinate functions $\{x_u^i : 1\leq i\leq 4\}$ and $\{x_v^i : 1\leq i\leq 4\}$ associated with these charts induce a holonomic basis in T_pM_4. The coordinate transition maps $u\circ v^{-1}$ and $v\circ u^{-1}$ induce transformations of coordinate bases in T_pM_4. The maps $(u\circ v^{-1})_*$ and $(v\circ u^{-1})_*$ are the maps which transform the bases under the coordinate transformations above. The two maps just defined with the subscript "stars" are called the Jacobian transformations of the coordinate transitions. Similiarly the coordinate charts (U,u) and (V,v) induce (via the pseudo-Riemannian metric) dual bases on $T_p^* M_4$ which we write $\{dx_u^i : 1 \le i \le 4\}$ and $\{dx_v^i : 1 \le i \le 4\}$ respectively. The maps $(u \circ v^{-1})^*$ and $(v \circ u^{-1})^*$ transform the dual bases on $T_p^* M_4$. The "starred" maps, where the superscript star denotes pull-back, induce transition maps between diferent representations of T (each representation is associated with a local trivialization of $\mathfrak{X}$). Figure 5 should clarify some of the relations above.

With these relationships in mind the transformation law for a tensor T of type (r,s) at P is written as

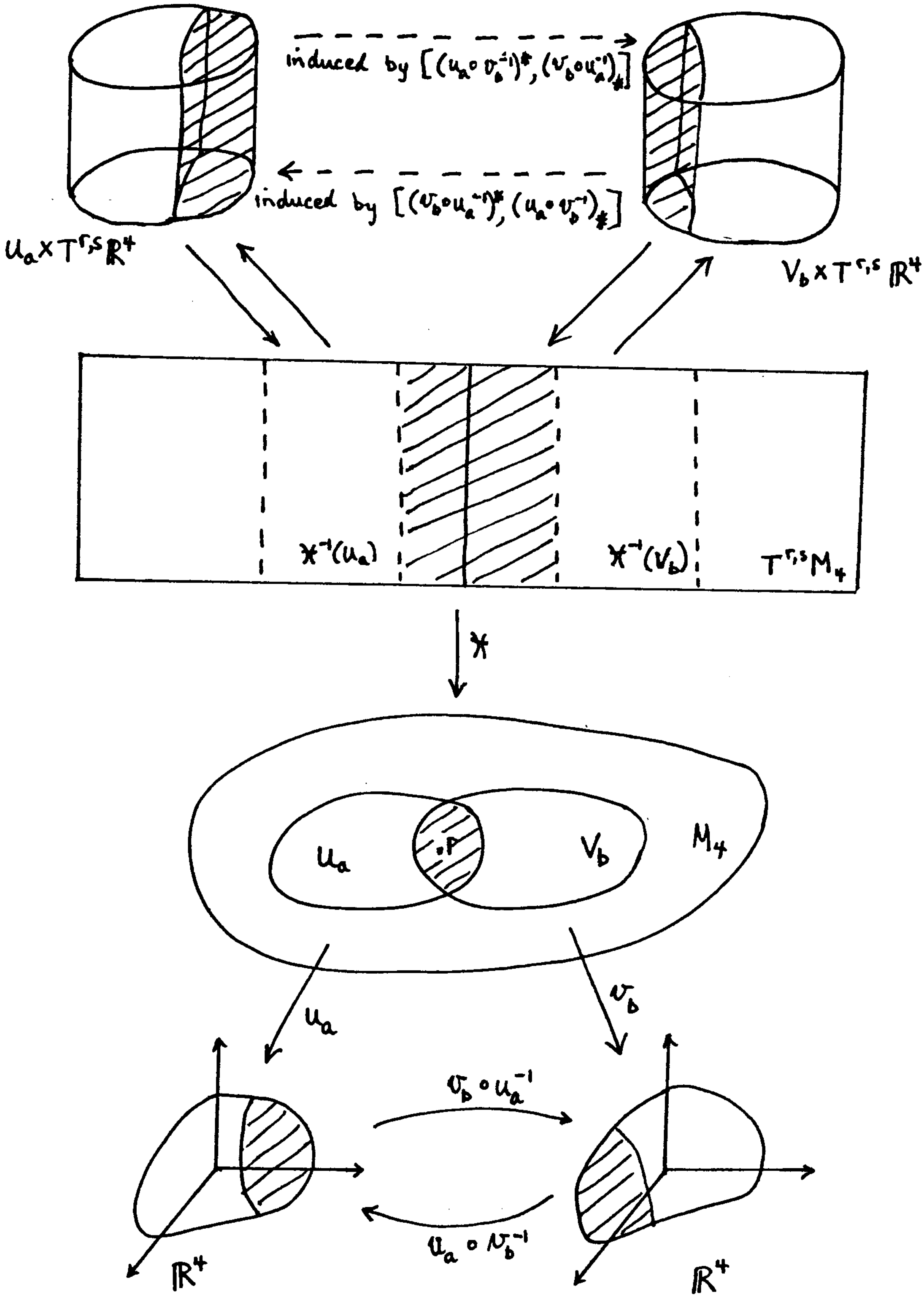
$$(2.4) \qquad T^{i_1 \cdots\cdots i_r}{}_{j_1 \cdots j_s}(v(P)) = \frac{\partial x_v^{i_1}}{\partial x_u^{k_1}} \cdots \frac{\partial x_v^{i_r}}{\partial x_u^{k_r}} \frac{\partial x_u^{l_1}}{\partial x_v^{j_1}} \cdots T^{k_1 \cdots\cdots k_r}{}_{l_1 \cdots l_s}(u(P))$$

where we have implicitly assumed the Einstein summation convention for repeated indices. The transformation law (2.4) represents the change of representation of T induced by the coordinate transition map $v \circ u^{-1}$. The importance of the metric tensor in General Relativity and the need to do exeedingly complicated calculations based on the Einstein field equations force us to employ the coordinate tensor calculus almost exclusively in later chapters.

The Christoffel symbols of the second kind associated with a symmetric metric tensor g_{ij} are

$$(2.5) \qquad \left\{ {i \atop jk} \right\} \equiv g^{il}\,[\,g_{jl,k} + g_{lk,j} - g_{jk,l}\,]/2.$$

Figure 5: The Tensor Bundle Structure on Space-Time



where g^{il} is found from the equation

$$g^{il} g_{lj} = \delta^{i}_{\cdot j},$$

where $\delta^{i}_{\cdot j}$ denotes the Kronecker tensor. The partial covariant derivative of a tensor $T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}$ with respect to x^k is given by

$$(2.6) \qquad T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s;k} = T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s,k} + \sum_{\alpha=1}^{r} \left\{ {i_\alpha \atop a k} \right\} T^{i_1 \cdots i_{\alpha-1} a i_{\alpha+1} \cdots i_r}{}_{j_1 \cdots j_s} - \sum_{\beta=1}^{s} \left\{ {b \atop j_\beta k} \right\} T^{i_1 \cdots i_r}{}_{j_1 \cdots j_{\beta-1} b j_{\beta+1} \cdots j_s}$$

The Riemann curvature tensor $R^{i}_{\cdot jkl}$ is defined in terms of the Christoffel symbols of the second kind as

$$(2.7) \qquad R^{i}_{\cdot jkl} = \left\{ {i \atop j l} \right\}_{,k} - \left\{ {i \atop j k} \right\}_{,l} + \left\{ {i \atop n k} \right\} \left\{ {n \atop j l} \right\} - \left\{ {i \atop n l} \right\} \left\{ {n \atop j k} \right\}.$$

Contracting on the first and fourth indices we produce the Ricci tensor

$$(2.8) \qquad R_{ij} = R^{m}_{\cdot ijm}\,.$$

The Einstein tensor is defined by

$$(2.9) \qquad G_{ij} = R_{ij} - (R/2)\, g_{ij}$$

where R, the curvature scalar is given by

$$(2.10) \qquad R = R^{i}_{\cdot i}.$$

The Einstein field equations (1.1) summarize the relationship between the differential geometry of M_4 and the matter-energy distribution on M_4.

Tetrad Calculus on M_4

Of fundamental importance in studying General Relativity is the careful distinction between the role of coordinates and the role of physical frames of reference [21],[22]. The axiom of general covariance in General Relativity essentially asserts that coordinates have no real physical role. On the other hand, frames of reference have real physical interpretations and may induce extraneous effects in laboratory experiments. This contrasts favourably to the use of coordinates whose physical meaning is often obscure or nonexistent [23]. More precisely, a frame of reference is a linearly independent set of four vector fields $\{e_A^i : 1 \le A \le 4\}$. The index i refers to the components of the e_A relative to some coordinate basis which we have prescribed. The set $\{e_A^i : 1 \le A \le 4\}$ is also called a linearly independent tetrad (four-leg, vierbein). If the members of the tetrad are pseudo-orthonormal i.e.

$$g_{ij}\, e_A{}^i\, e_B{}^j = \eta_{AB} \tag{2.11}$$

where

$$(2.12)\quad \eta_{AB} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

then the tetrad may be interpreted physically as follows:

(a) $\{e_1, e_2, e_3\}$ is an "orthonormal" triad of "rigid" rods,

(b) e_4 may be represented as a "standard" clock.

The equations (2.11) play a central role in the development of the tetrad calculus.

From the canonical forms which will be given for the stress-energy tensor in the next chapter it will become apparent that for the most general uses of the mixed method we will need to consider tetrads of greater generality than pseudo-orthonormal ones. We are forced to consider tetrads which include two null vectors. For this purpose we extend the notion of pseudo-orthonormality to that of quasi-orthonormality. A tetrad $\{e_A^i : 1 \leq A \leq 4\}$ is said to be <u>quasi-orthonormal</u> if it is either pseudo-orthonormal or it satisfies (2.11) with the flat metric tensor

$$(2.13)\quad \eta_{AB} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

where we assume the null vectors in tetrad are labelled e_3, e_4.

Let P be a point in M_4 and (U,u) be a local coordinate system at P. If $T^{i_1 \ldots i_r}{}_{j_1 \ldots j_s}(u(P))$ is a tensor of type (r,s) then the corresponding Lorentz tensor is found by projecting the coordinate tensor onto a suitable pseudo-orthonormal (quasi-orthonormal) tetrad defined on (U,u). Thus

$$(2.14) \qquad T^{A_1 \ldots A_r}{}_{B_1 \ldots B_s} = T^{i_1 \ldots i_r}{}_{j_1 \ldots j_s}\, e^{A_1}{}_{i_1} \ldots e^{A_r}{}_{i_r} e_{B_1}{}^{j_1} \ldots e_{B_s}{}^{j_s}$$

where the $e^A{}_i$ are found from

$$(2.15) \qquad e^A{}_i\, e_B{}^i = \delta^A{}_B.$$

Clearly this projection may be undone by projecting onto the dual tetrad $\{e^A{}_i : 1 \leq A \leq 4\}$ defined in (2.15). Any algebraic identity among coordinate tensors has a corresponding analogue among the Lorentz tensors [21]. The tetrad labels behave like tensor indices under change of tetrad but act like labels under change of coordinates. The dual statement is also true.

Lorentz indices (tetrad labels) are raised by the use of η_{AB} defined by (2.12) or (2.13) depending on the tetrad type. We apply the summation convention to up-down pairs of Lorentz indices and generally carry most of the notation conventions used for coordinate tensors. When a tetrad label, having a fixed value in its range, is raised or lowered, the sign of the factor in which the label appears changes if the label value is 1, 2, 3 and remains the same if the label value is 4.

The quantities in the tetrad calculus which are the analogues of the Christoffel symbols of the second are called the Ricci rotation coeffients. For an arbitrary tetrad they are defined by

$$(2.16)\qquad \gamma_{ABC} = e_{Ai;j}\, e_B{}^i\, e_C{}^j .$$

The Ricci rotation coefficients for a pseudo-orthonormal tetrad also satisfy the identity [20]

$$(2.17)\qquad \gamma_{ABC} = -\gamma_{BAC} .$$

An extensive discussion of the Ricci rotation coefficients may be found in Schouten [24]. Direct calculation of the Ricci rotation coefficients is very tedious due to the presence of a covariant differentiation in the definition (2.16). However, employing the skew-symmetry (2.17) this task is eased by the introduction of the quantities [25]

$$(2.18)\qquad \lambda_{ABC} = (e_{Ai,j} - e_{Aj,i})\, e_B{}^i\, e_C{}^j .$$

This equation allows the calculation of the γ_{ABC} using only partial differentiations with respect to the coordinates. The Ricci rotation coefficients are given in terms of the λ_{ABC} by

(2.19) $\quad \gamma_{ABC} = (\lambda_{ABC} + \lambda_{BCA} - \lambda_{CAB})/2$

The Lorentz covariant derivative is defined on Lorentz tensors by [21]

(2.20) $$T^{A_1\ldots A_r}{}_{B_1\ldots B_s\|K} = T^{A_1\ldots A_r}{}_{B_1\ldots B_s|K} + \sum_{n=1}^{r} \gamma_E{}^{A_n}{}_{\cdot C}\, T^{A_1\ldots A_{n-1} E A_{n+1}\ldots A_r}{}_{B_1\ldots B_s} - \sum_{m=1}^{s} \gamma_{B_m}{}^{E}{}_{\cdot C}\, T^{A_1\ldots A_r}{}_{B_1\ldots B_{m-1} E B_{m+1}\ldots B_s}.$$

In (2.20) we use a double stroke to denote the Lorentz covariant differentiation and a single stroke to denote the directional derivative. The directional derivative is defined by

(2.21) $$T^{A_1\ldots A_r}{}_{B_1\ldots B_s|C} = T^{A_1\ldots A_r}{}_{B_1\ldots B_s,i}\, e_C{}^{i}.$$

Referring to Eisennart we find the tetrad form of the Riemann tensor is

(2.22) $$R_{ABCD} = 2\gamma_{AB[C|D]} + 2\gamma_{ABN}\gamma^{N}{}_{\cdot[CD]} + \gamma_{ANC}\,\gamma^{N}{}_{\cdot BD} - \gamma_{AND}\,\gamma^{N}{}_{\cdot BC}.$$

Raising the Lorentz index A and contracting on the first and fourth indices we get the tetrad form of the Ricci tensor

(2.23) $$R_{BC} = \gamma^{M}{}_{\cdot BC|M} - \gamma^{M}{}_{\cdot BM|C} + \gamma^{M}{}_{BN}\gamma^{N}{}_{\cdot CM} - \gamma^{M}{}_{\cdot BC}\gamma^{N}{}_{\cdot BN}.$$

Advantages of the Tetrad Calculus

There are several important features that the tetrad calculus has the coordinate tensor calculus does not. The principal advantage of the tetrad calculus is that the components of any tensor representing a physical quantity may be measured (in principle) when projected onto a tetrad. Furthermore, when the tetrad is pseudo-orthonormal with the timelike vector pointing to the future, the tetrad components of physical quantities transform as tensors over the proper isochronous Lorentz group L^{+}_{+}.

The tetrad calculus is gaining in popularity and usefulness in theoretical physics. The tetrad calculus allows the possibility of describing gravitational radiation in terms of "optical" scalars. Other applications of the tetrad calculus include the classification of interior solutions by the algebraic structure of the Ricci tensor or, as in this thesis, of the stress-energy tensor. Recent work in attempting to express General Relativity as a gauge theory has made essential use of tetrad methods. The tetrad calculus and the coordinate calculus have been combined to form a "bicovariant" calculus which has been used in General Relativity by Treder [26].

III. Algebraic Classification of the Stress-Energy Tensor

Algebraic Classification of Second Rank Symmetric Tensors

As part of the "mixed" method we will incorporate a scheme of algebraically classifying the stress-energy tensor with respect to the metric tensor. A short search of the literature shows that there at least ten different methods available for algebraically classifying a symmetric tensor of rank two with respect to the metric tensor. Clearly the choice of a classification scheme poses a problem particularly if we wish it to be compatible with the aims and techniques of the mixed method. Any such scheme should be compatible with the tetrad calculus since most of our calculations will be carried out in that language. The methods which appear to be the most suited to our purposes are the schemes of Petrov [4], based on the Segre characteristic, and that of Plebanski [27] which is a refinement of Petrov's.

Algebraic Preliminaries

The problem of algebraically classifying a second rank, symmetric real tensor on a Riemannian space is considerably simpler than the same problem on a pseudo-Riemannian space. Complications which may arise in the pseudo-Riemannian case include the possibility of non-real eigenvalues and non-simple elementary divisors. A more subtle difficulty, as pointed out by Synge [28], is that <u>before</u> the metric tensor is known we have no right to <u>assume</u> that T_{ij}, $T^{i}_{\cdot j}$, and T^{ij} are related to each other as mathematical objects. It is the <u>physical</u> content of the field equations (1.1) which allows us to assume that these different tensors are really just different representations of the same physical object. Corresponding to this statement we have (T_{ij}, g_{ij}), $(T^{i}_{\cdot j}, \delta^{i}_{\cdot j})$, and (T^{ij}, g^{ij}) are equivalent as pairs of quadratic forms since g_{ij} is nonsingular. The tensor T_{ij} is associated with an endomorphism of the tangent space at any point in space-time. It is this endomorphism that we will classify. The algebraic invariants of the endomorphism are also the algebraic invariants of T_{ij} with respect to g_{ij}.

Let $T: T_P M_4 \dashrightarrow T_P M_4$ be the endomorphism associated with T_{ij} at P. We say that a nonzero vector V in $T_P M_4$ is an <u>eigenvector</u> of T with eigenvalue λ if

(3.1) $\quad T(V) = \lambda V.$

The polynomial $C_T(x)$ defined by the equation

(3.2) $\quad C_T(x) = \det [T - xI],$

where I is the identity endomorphism, is called the <u>characteristic polynomial of T</u>. The roots of the equation

$$C_T(x) = 0 \tag{3.3}$$

are called the <u>eigenvalues</u> of T.

When T represents the stress-energy tensor the eigenvalues λ are the principal stresses if the corresponding eigenvectors are spacelike, or the energy flux density if the corresponding eigenvector is timelike. Certain classes of stress-energy tensors do not have timelike eigenvectors. For these stress-energy tensors the physical significance of the eigenvalues and of the eigenvectors is unclear. Plebanski [27] has shown that if a stress-energy tensor satisfies certain energy conditions for macroscopic matter then it must have real eigenvalues. If we are to find "realistic" solutions of the field equations (1.3) then we should ensure that the stress-energy tensor has four real eigenvalues. A direct approach to this problem will be applied. From now on we will always assume that we are working on a four-dimensional vector space.

Since $T_P M_4$ is of dimension four over R we see that the characteristic polynomial is of degree four. Consider the general quartic equation over R i.e.

$$x^4 + Ax^3 + Bx^2 + Cx + D = 0 \tag{3.4}$$

where A, B, C, D are real numbers. By the transformation

$$y = x + A/4 \tag{3.5}$$

we find that (3.4) reduces to

$$y^4 + Py^2 + Qy + R = 0 \tag{3.6}$$

where

$$P = B - 3A^2/8,$$

$$Q = A^3/8 - AB/2 + C,$$

$$R = D + BA^2/16 - AC/4 - 3A^4/256 .$$

Let y_1, y_2, y_3, y_4 denote the roots of (3.6). Following the argument of Kochendorffer [29] we set

$$\begin{aligned} z_1 &= (y_1 + y_2)(y_3 + y_4), \\ z_2 &= (y_1 + y_3)(y_2 + y_4), \\ z_3 &= (y_1 + y_4)(y_2 + y_3). \end{aligned} \tag{3.8}$$

From (3.6) we see that $y_1 + y_2 + y_3 + y_4 = 0$ so together with

$$\begin{aligned} &y_1 y_2 y_3 y_4 = R, \\ &y_1 y_2 + y_1 y_3 + y_1 y_4 + y_2 y_3 + y_2 y_4 + y_3 y_4 = P, \\ &y_1 y_2 y_3 + y_1 y_2 y_4 + y_1 y_3 y_4 + y_2 y_3 y_4 = -Q \end{aligned} \tag{3.9}$$

we get

$$(w-z_1)(w-z_2)(w-z_3) = w^3 - 2Pw^2 + (P^2 - 4R)w + Q^2 \tag{3.10}$$

which follows from the definitions of the z_j, $1 \le j \le 3$, and the Basis Theorem for Symmetric Polynomials (Marcus [30]). The polynomial

$$w^3 - 2Pw^2 + (P^2 - 4R)w + Q^2$$

is the cubic resolvent of (3.6). The roots z_j, $1 \le j \le 3$, of

$$w^3 - 2Pw^2 + (P^2 - 4R)w + Q^2 = 0 \tag{3.11}$$

can be used to find the roots y_i, $1 \le i \le 4$, by the formulae

$$\begin{aligned}
y_1 &= [(-z_1)^{\frac{1}{2}} + (-z_2)^{\frac{1}{2}} + (-z_3)^{\frac{1}{2}}]/2, \\
y_2 &= [(-z_1)^{\frac{1}{2}} - (-z_2)^{\frac{1}{2}} - (-z_3)^{\frac{1}{2}}]/2, \\
y_3 &= [-(-z_1)^{\frac{1}{2}} + (-z_2)^{\frac{1}{2}} - (-z_3)^{\frac{1}{2}}]/2, \\
y_4 &= [-(-z_1)^{\frac{1}{2}} - (-z_2)^{\frac{1}{2}} + (-z_3)^{\frac{1}{2}}]/2,
\end{aligned}$$

where the sign ambiguity in $(-z_j)^{\frac{1}{2}}$, $1 \le j \le 3$, is resolved by the relation [31]

$$(-z_1)^{\frac{1}{2}}(-z_2)^{\frac{1}{2}}(-z_3)^{\frac{1}{2}} = -Q. \tag{3.12}$$

These transformations have reduced the problem of finding the roots of (3.4) to finding the roots of (3.11). Set

$$v = w - 2P/3 \tag{3.13}$$

in (3.11) to produce

$$(3.14) \qquad v^3 + av + b = 0$$

where

$$(3.15) \qquad a = -P^2/3 - 4R$$

$$(3.16) \qquad b = 2P^3/27 - 8PR/3 + Q^2$$

The solutions of (3.14) are found by the use of the Cardano-Tartaglia formulae [32].

We are now in a position to impose conditions on the coefficients of (3.4) which will guarantee that the roots of (3.4) are real. Remembering that A is real we see from (3.5) that the y_i, $1 \le i \le 4$, are real if the x_i are. By a simple argument each y_i is real if and only if the z_j, $1 \le j \le 3$, are real and nonpositive. From (3.7) we see that for the z_j to be real we need the roots v_j, $1 \le j \le 3$, to be real. This restriction on the roots of (3.14) means that the discriminant of (3.14), defined by

$$\Delta = 4a^3 + 27b^2,$$

must satisfy the inequality

$$\Delta \le 0.$$

The roots of (3.14) when $\Delta < 0$ are given by the formulae

$$v_1 = 2\sqrt{(-a/3)}\ \cos\theta$$

$$(3.17) \qquad v_2 = 2\sqrt{(-a/3)}\ \cos(\theta + \tfrac{2\pi}{3})$$

$$v_3 = 2\sqrt{(-a/3)}\ \cos(\theta + \tfrac{4\pi}{3})$$

where θ is chosen to that $0 \le 3\theta \le \pi$ and

$$(3.18) \qquad \theta = \begin{cases} (1/3)\tan^{-1}(-\sqrt{-\Delta/27b^2}) \cap [0, \pi/6) & \text{if } b < 0, \\ \frac{\pi}{6} & \text{if } b = 0, \\ (1/3)\tan^{-1}(-\sqrt{-\Delta/27b^2}) \cap (\pi/6, \pi/3] & \text{if } b > 0. \end{cases}$$

Using these roots in terms of the coefficients in (3.4) we get the roots z_j as

$$z_1 = (2/3)\sqrt{B^2+12D-3AC}\ \cos\theta + 2B/3 - A^2/4,$$

$$z_2 = (2/3)\sqrt{B^2+12D-3AC}\ \cos(\theta + \tfrac{2\pi}{3}) + 2B/3 - A^2/4,$$

$$z_3 = (2/3)\sqrt{B^2+12D-3AC}\ \cos(\theta + \tfrac{4\pi}{3}) + 2B/3 - A^2/4.$$

We will use the condition that $z_j \le 0$ to find conditions on the coefficients of (3.4) to force (3.4) to have real distinct roots. First we need a lemma.

<u>Lemma 1</u>: If $\Delta < 0$ then

(a) $a < 0$, and

(b) $B^2 + 12D - 3AC > 0$.

The proof of this lemma follows from the definition of Δ, (3.16), (3.17), and the definitions of the coefficients P, Q, and R. Finally we are in a position to state the following theorem.

<u>Theorem 1</u> : For the quartic equation (3.4) to have four distinct real roots it is sufficient that the conditions

(a) $\Delta < 0$,

(b) $3A^2 - 8B > 0$

(c) $3A^4 - 16A^2B - 256D + 64AC \geq 0$.

The proof of this theorem is a straightforward but lengthy argument based on the formulae for the roots z , given above, and the lemma.

The generalization of the theorem above to the case when $\Delta \leq 0$, so the roots are not necessarily distinct, is not easily accessible by direct methods. However Chaundy [33] has derived the following theorem by means of algebraic geometry.

Theorem 2 : Necessary and sufficient conditions for (3.4) to have four real roots are

(a) $\Delta \leq 0$,

(b) $3A^2 - 8B > 0$,

(c) $3A^4 - 16A^2B + 16B^2 + 64AC - 64D \geq 0$.

Characterization of the Endomorphism T

It is well-known that the eigenvalues and the eigenvectors of a linear transformation T do not provide sufficient information to characterize its similarity class. However a knowledge of these quantities together with other algebraic invariants will allow a complete characterization of the similarity class of T. Choose a basis for T_pM_4, then the endomorphism T will have a matrix relative to the basis. The determination of similarity invariants of a matrix is well-known. We summarize this theory by drawing extensively on the text of Mal'cev[34].

A matrix over the real ring, $R[\lambda]$, of polynomials in one indeterminate, λ, is called a <u>λ-matrix</u>. We define two λ-matrices to be <u>λ-equivalent</u> if one may be transformed into the other by a sequence of operations of the types:

(a) multiplication of a row (column) by a nonzero real number,

(b) addition of one row (column), multiplied by an element in $R[\lambda]$, to another row (column).

It is clear that λ-equivalence is an equivalence relation on the set of λ-matrices over R. The λ-equivalence classes have canonical representatives of the form

$$(3.19) \qquad \begin{bmatrix} f_1(\lambda) & 0 & 0 & 0 \\ 0 & f_2(\lambda) & 0 & 0 \\ 0 & 0 & f_3(\lambda) & 0 \\ 0 & 0 & 0 & f_4(\lambda) \end{bmatrix}$$

where $f_k(\lambda)$ is a divisor of $f_{k+1}(\lambda)$ and all nonzero $f_k(\lambda)$ are monic. Every λ-matrix can be reduced by the operations (a) and (b) to a canonical form of the type above.

For any fixed λ-matrix A we have the following definition:

$$(3.20) \quad D_k(\lambda) = \begin{cases} 1 & \text{if } k = 0, \\ g(\lambda) & \text{if not all } k \times k \text{ minors are 0 and } k \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

for $k = 0, 1, \ldots,$ order(A). The polynomial $g(\lambda)$ is the monic divisor of greatest degree of the $k\times k$ minors of A. The polynomials $D_k(\lambda)$ and $f_k(\lambda)$ are related by

Theorem 3(Mal'cev): If A is a λ-matrix of order 4 and there is a number r, $1\le r\le 4$, so that $D_k(\lambda) = 0$ for $k = 1,\ldots,r$ and $D_{r+1}(\lambda) = 0$ then the polynomials $f_k(\lambda)$ above are given by

$$(3.21)\quad f_k(\lambda) = \begin{cases} D_k(\lambda)/D_{k-1}(\lambda) & k = 1,2,\ldots,r \\ \\ 0 & r < k \le 4. \end{cases}$$

The polynomials $f_k(\lambda)$ are called the *invariant factors of A*. The number r is the rank of A. The irreducible factors over R of the nonconstant invariant factors of A are called the *real elementary divisors of A*.

Theorem 4(Mal'cev): The set of real elementary divisors along with the rank and order of a λ-matrix A completely determine the invariant factors of A, hence constitute a complete set of invariants for λ-equivalence.

Finally, to link the notion of λ-equivalence with similiarity we have the

Theorem 5(Mal'cev): Two matrices A and B over R are similiar if and only if their characteristic λ-matrices $A - \lambda I$ and $B - \lambda I$

are λ-equivalent.

To distinguish the possibility of having elementary divisors which are irreducible over R we will say that T is <u>admissible</u> if the characteristic equation of T satisfies the hypotheses of Theorem 2. From the previous two theorems we see that to classify the stress-energy tensor we need to find its elementary divisors.

For an admissible T we can write the system of elementary divisors in the pattern

$$
\begin{array}{c}
(\lambda - \lambda_1)^{p_{11}}, \ (\lambda - \lambda_1)^{p_{12}}, \ldots\ldots\ldots, (\lambda - \lambda_1)^{p_{1l_1}} \\
(\lambda - \lambda_2)^{p_{21}}, \ (\lambda - \lambda_2)^{p_{22}}, \ldots\ldots\ldots, (\lambda - \lambda_2)^{p_{2l_2}} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(\lambda - \lambda_m)^{p_{m1}}, \ (\lambda - \lambda_m)^{p_{m2}}, \ldots\ldots\ldots, (\lambda - \lambda_m)^{p_{ml_m}}
\end{array}
\tag{3.22}
$$

where $\sum_{k=1}^{m} l_k = \text{rank } T \le 4$, and the λ_k, $k = 1,\ldots,m$ are the distinct real eigenvalues of T. If the algebraic multiplicity of the eigenvalue λ_k is $n_k \le 4$ then the numbers p_{kj} , $j=1,\ldots,l_k$ satisfy the inequalities

$$1 \le p_{k(j+1)} \le p_{kj}$$

and

$$\sum_{j=1}^{l_k} p_{kj} = n_k.$$

For an admissible T we define the <u>real Segre characteristic of T</u> as the symbol

(3.23) $$[(p_{11},\ldots,p_{1\ell_1}),(p_{21},\ldots,p_{2\ell_2}),\ldots,(p_{m1},\ldots p_{m\ell_m})].$$

Since the real Segre characteristic is difficult to compute we define the <u>real Weyr characteristic</u> and give an algorithm relating the two characteristics.

For an admissible T we recursively define the numbers q_{kj} for each distinct real eigenvalue λ_k by

(3.24) $$q_{kj} = \begin{cases} 0 & j = 0, \\ \dim_{\mathbb{R}} N(T - \lambda_k I)^j - q_{k(j-1)} & j \geq 1. \end{cases}$$

where $N(T - \lambda_k I)^j$ is the null space of $(T - \lambda_k I)^j$. The real Weyr characteristic of T is the symbol with nonzero q_{kj}

(3.25) $$\{(q_{11},q_{12},\ldots),(q_{21},\ldots),\ldots,(q_{m1},\ldots\ldots)\}.$$

We can relate the Weyr characteristic to the Segre characteristic by the following algorithm:

(a) for each distinct eigenvalue λ_k of T we create an array of dots (column by column) so that there are q_{kj} dots in the j-th column counting from the top;

(b) the Segre characteristic numbers p_{kj}, corresponding to the eigenvalues λ_k, are the numbers of dots in the rows.

Classification of the Stress-Energy Tensor

The first of the algebraic classifications of the stress--energy tensor that we will discuss is the Segre classification described in Petrov[4], Plebanski[27]. Bromwich[35] has shown that there are at most nineteen Segre characteristics which may be assigned to a pair of tensors $(T_{\ \ },g_{\ \ })$. Among these nineteen cases there is one which is "degenerate" in the sense that both tensors in the pair are singular. Excluding the degenerate case we can extend the Segre symbol to include the symbols "$Z\bar{Z}$" which indicate the existence of an irreducible quadratic elementary divisor.

The condition that T is admissible excludes the characteristics with "$Z\bar{Z}$". Admissibility of T reduces the number of possible Segre characteristics for the stress-energy to sixteen. Modulo algebraic degeneracies (repeated eigenvalues) the possible Segre characteristics are of the types:

(a) [111,1],

(b) [11,2],

(c) [1,3],

(d) [2,2],

(e) [4],

where we use the convention that the last digit represents the elementary divisor corresponding to a timelike or null eigenvector.

Cormack and Hall[36] have shown that the type (d) and its degeneracy together with (e) are not compatible with the symmetry of the stress-energy tensor and the inequality $\det(g\ ) < 0$. Collinson and Shaw[37] have shown that the type (c) and its degeneracy cannot satisfy reasonable energy conditions on the stress-energy tensor i.e. reference frames may be found in which the mass density is negative.

The types (a) and (b) admit eleven distinct symbols ccunting their algebraic degeneracies. Canonical forms for the coordinate tensor formulation of T are given in [4], [27], and [38].

For the type (a) we have

$$\begin{bmatrix} -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

For the type (b) we have

$$\begin{bmatrix} -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & \lambda & \gamma \end{bmatrix}$$

where $\gamma = 1$ or -1. For the type (b) Plebanski[27] has shown that a necessary condition for the stress-energy tensor to satisfy the energy conditions is for $\gamma = 1$.

The Plebanski classification is essentially a refinement of the Segre classification. To each second rank trace-free symmetric tensor there is assigned a symbol of the form

$[n_1 U_1 - n_2 U_2 - \ldots\ldots]_{(q_1 - q_2 - \ldots\ldots)}$.

The symbols U have values given by

$$U_i = \begin{cases} Z\bar{Z} & \text{if the i-th eigenvalue is complex,} \\ T & \text{if the i-th eigenvalue is real} \\ & \text{and the eigenvector is timelike,} \\ S & \text{if the i-th eigenvalue is real} \\ & \text{and the eigenvector is spacelike,} \\ N & \text{if the i-th eigenvalue is real} \\ & \text{and the eigenvector is null.} \end{cases}$$

The number n_i is the algebraic multiplicity of the i-th eigenvalue and the number q_i is the degree of the elementary divisor of maximal degree corresponding to the i-th eigenvalue. We shall adopt the convention that non-spacelike entries in the symbol will be recorded last. The Segre characteristic is easily found from the "subscript" part of the Plebanski symbol.

Since stress-energy tensors with nonzero trace are included in the scope of this thesis we extend the Plebanski classification to such tensors. This extension does not allow the transition from trace-free tensors to non-zero trace tensors to be carried through while preserving the Plebanski symbol. The algebraic multiplicities of the eigenvalues and the relativistic type of the eigenvectors are preserved but the Segre class may not be preserved.

Completion of the Eigentetrad and the Principal Axes Theorem

In general T will not have four eigenvectors at a point P in M_4. This will happen when T has a non-simple elementary divisor. Even in the cases when T does have four simple elementary divisors the corresponding eigenvectors may not be determined due to algebraic degeneracy. In order to apply the mixed method it is crucial that the "natural" frame of T_{ij} be used. First we consider the problem of completing the set of eigenvectors of T_{ij} to a full tetrad when there is a non-simple elementary divisor.

A theorem of Woods(cited in Wong[38]) has the following

Corollary: In a 4-dimensional pseudo-Riemannian space there exist sets of r mutually orthogonal linearly independent null vectors if $r \leq 2$. When $r > 2$ there no such sets. The number of real null vectors in each set cannot exceed the integer $\min\{p, 4-p\}$ where $2p-4$ is the signature of the metric tensor.

Together with the result of Petrov[4] that a non-simple elementary divisor corresponds to a null eigenvector this corollary shows that for a stress-energy tensor of Segre type [11,2] we cannot even hope to have a pseudo-orthonormal eigentetrad. We are thus led to consider the problem of completing the set of eigenvectors of T when there are fewer than four eigenvectors and one of them is null.

A theorem of Lense (cited in Wong[38]) provides the solution to this problem.

Theorem (Lense): A given set of r+s linearly independent mutually orthogonal vectors in an n-dimensional pseudo-Riemannian space, r of which are null and s of which are non-null, can always be normalized and completed to a quasi-orthonormal basis with q null vectors if $r \geq 0$, $s \geq 0$, $2r + s \leq n$, and $(1/2)(n - s) \geq q \geq r$.

This theorem guarantees that the set of eigenvectors of T can always can be completed to a quasi-orthonormal tetrad in the case of Segre class [11,2]. To find canonical forms for T of Segre class [11,2] with respect to its generalized eigentetrad we appeal to the theory of elementary divisors. From the text of Petrov [4] we may derive the following theorem.

Principal Axes Theorem on M_4: Let T_{ij} be a symmetric tensor of rank two on a pseudo-Riemannian space (M_4, g_{ij}). Let the Segre characteristic of T_{ij} with respect to g_{ij} be [111,1]. Then the canonical form of the pair of tensors (T_{ij}, g_{ij}) is given by

$$T_{ij} = -\lambda_1 e_{1i} e_{1j} - \lambda_2 e_{2i} e_{2j} - \lambda_3 e_{3i} e_{3j} + \lambda_4 e_{4i} e_{4j},$$
$$g_{ij} = -e_{1i} e_{1j} - e_{2i} e_{2j} - e_{3i} e_{3j} + e_{4i} e_{4j}.$$

If the Segre characteristic of T_{ij} with respect to g_{ij} is [11,2] then the pair of tensors $(T_{i}^{\;j}, g_{i}^{\;j})$ may be represented

$$T_{i}^{\;j} = -\lambda_1 e_{1i} e_{1}^{\;j} - \lambda_2 e_{2i} e_{2}^{\;j} + \lambda_3 (e_{3i} e_{4}^{\;j} + e_{4i} e_{3}^{\;j}) + \gamma e_{4i} e_{4}^{\;j},$$

$$g_{i}^{\;j} = -e_{1i} e_{1}^{\;j} - e_{2i} e_{2}^{\;j} + e_{3i} e_{4}^{\;j} + e_{3i} e_{4}^{\;j}.$$

In the above canonical forms we have used λ_i, $1 \leq i \leq 4$, to represent the eigenvalues of $T_{i}^{\;j}$ with respect to $g_{i}^{\;j}$. The parameter γ may take the value 1 or -1.

IV. Application of the Mixed Method to Algebraically Degenerate Stress-Energy Tensors

Special Hypotheses used with the Mixed Method

In this chapter the mixed method is used to verify some well-known solutions of the Einstein field equations. The three examples considered are algebraically degenerate in the sense that the stress-energy tensors will have repeated eigenvalues. The stress-energy tensors of the examples are in the Segre classes [(111),1] and [(111,1)]. We also have imposed the hypothesis that all tetrads are "aligned" in the sense that the vectors of a tetrad are parallel to the vectors of a holonomic tetrad. This hypothesis restricts the examples to ones known in orthogonal coordinates.

The metric tensor is known before the application of the mixed method so we will be able to find the aligned tetrad without solving any partial differential equations. In each example the Ricci rotation coefficients are computed using the indirect method indicated in Chapter 2. Then lengthy, but straightforward, calculations will verify the tetrad formulation of the field equations.

Stress-Energy Tensor of Segre Class [(111),1]

Suppose that the stress-energy tensor is of Segre class [(111),1]. This means that an eigentetrad of the stress-energy tensor is pseudo-orthonormal. There are two distinct eigenvalues p and P of multiplicities three and one respectively. The eigenvalue P corresponds to a timelike eigenvector e_4. The eigenvalue p has an infinity of pseudo-orthonormal triads of spacelike eigenvectors associated with it. By choosing an orthogonal system of coordinates and applying the alignment hypothesis we are able to select a unique triad of eigenvectors corresponding to p. Let this triad of spacelike eigenvectors be written $\{e_1,e_2,e_3\}$. In the orthogonal coordinates we write the field equations as

$$G_{ij} = -8\pi[(P + p)e_{4i}e_{4j} - pg_{ij})]. \tag{4.1}$$

where the e_{4i} are the covariant components of the eigenvector e_4. From (4.1) and the definition of the Einstein tensor we calculate that the curvature scalar is

$$R = 8\pi(P - 3p). \tag{4.2}$$

Using (4.1), (4.2), and the definition of the Einstein tensor we can write (4.1) as

$$R_{ij} = 4\pi(P-p)g_{ij} - 8\pi(P+p)e_{4i}e_{4j}. \tag{4.3}$$

Projecting (4.3) onto the eigentetrad $\{e^{i}_{A}: 1 \le A \le 4\}$ yields

$$(4.4) \qquad R_{AB} = 4\pi(P-p)\eta_{AB} - 8\pi(P+p)\eta_{4A}\eta_{4B}.$$

The equations (4.4) are the tetrad equations which will be verified.

The solution which we will use to verify (4.4) is the interior Schwarzschild solution for a homogeneous sphere of perfect fluid [40]. In the coordinates (r,θ,ϕ,t) we find the metric tensor g_{ij} is given by

$$(4.5) \quad ds^2 = -dr^2/(1-qr^2) - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + (L-\sqrt{1-qr^2})^2\, dt^2$$

where

$q = (8/3)\pi P$,

P = "constant density" of the perfect fluid,

$L = (3/2)\sqrt{1-qa^2}$,

a = radius of the sphere of perfect fluid.

From the metric (4.5) we can easily find the eigentetrad to be

$e_1 = \sqrt{1-qr^2}\, \frac{\partial}{\partial r}$,

$e_2 = (1/r)\, \frac{\partial}{\partial \theta}$,

$e_3 = (1/r\sin\theta)\, \frac{\partial}{\partial \phi}$,

$e_4 = [\, 2/(2L-\sqrt{1-qr^2})\,]\, \frac{\partial}{\partial t}$.

The 1-forms dual to this tetrad are found by using (4.5) to lower the coordinate superscript index i. The 1-forms are found to be

$\tilde{e}_1 = (-1/\sqrt{1-qr^2})\ dr,$

$\tilde{e}_2 = -r\ d\theta,$

$\tilde{e}_3 = -r\ \sin\theta\ d\phi,$

$\tilde{e}_4 = (1/2)(2L-\sqrt{1-qr^2})\ dt.$

The independent nonzero Ricci rotation coefficients of the eigentetrad above are

$\gamma_{122} = (1/r)\sqrt{1-qr^2},$

$\gamma_{133} = (1/r)\sqrt{1-qr^2},$

$\gamma_{144} = -qr/(2L-\sqrt{1-qr^2}),$

$\gamma_{233} = (1/r)\ \cot\theta.$

Computing the Lorentz-covariant Ricci tensor from the equation (2.23) we find that the nonzero components are

$R_{11} = 4\pi\,(P-p)\,\eta_{11},$

$R_{22} = 4\pi\,(P-p)\,\eta_{22},$

$R_{33} = 4\pi\,(P-p)\,\eta_{33},$

$R_{44} = 4\pi\,(P-p)\eta_{44} - 8\pi\,(P+p)\eta_{44}\eta_{44}.$

The details of these calculations are omitted. It is a straightforward task to check that results above are consistent with the equations (4.4).

Stress-Energy Tensor of Segre Class [(111,1)]

The cosmological version of the Einstein field equations are written as

(4.8) $$G_{ij} + \Lambda g_{ij} = -8\pi T_{ij}$$

where Λ is the cosmological constant. A simple transposition shows that we may regard (4.8) as interior equations with

$$(4.9) \qquad \tilde{T}_{ij} = T_{ij} + (\Lambda g_{ij})/8\pi.$$

If we set $T_{ij}=0$ and solve the field equations which result, then for the Schwarzschild prcblem the cosmological solution [40] in coordinates (r,θ,ϕ,t), we get

$$(4.10) \qquad ds^2 = -dr^2/(1-ar^2) - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2 + (1-ar^2)dt^2$$

where $a = \Lambda/3$. From (4.9) and (4.10) the eigentetrad in terms of the holonomic tetrad is

$$(4.11) \qquad \begin{aligned} e_1 &= \sqrt{(1-ar^2)}\,\frac{\partial}{\partial r}, \\ e_2 &= (1/r)\,\frac{\partial}{\partial \theta}, \\ e_3 &= 1/(r\sin\theta)\,\frac{\partial}{\partial \phi}, \\ e_4 &= 1/\sqrt{(1-ar^2)}\,\frac{\partial}{\partial t}. \end{aligned}$$

From (4.11) and (4.10) the 1-forms dual to this tetrad are

$$(4.12) \qquad \begin{aligned} \tilde{e}_1 &= -1/\sqrt{1-ar^2}\; dr, \\ \tilde{e}_2 &= -r\; d\theta, \\ \tilde{e}_3 &= -r\sin\theta\; d\phi, \\ \tilde{e}_4 &= \sqrt{1-ar^2}\; dt. \end{aligned}$$

Using $T_{ij}^{\cdot} = 0$, and (4.8) we find the tetrad form of the field equations is

$$R_{AB} = \Lambda \eta_{AB}. \tag{4.13}$$

Applying the definitions of Chapter 2 we find the nonzero Ricci rotation coefficients are

$$\begin{aligned} \gamma_{122} &= (1/r)\sqrt{1-ar^2}, \\ \gamma_{133} &= (1/r)\sqrt{1-ar^2}, \\ \gamma_{144} &= ar/\sqrt{1-ar^2}, \\ \gamma_{233} &= (\cot\theta)/r. \end{aligned} \tag{4.14}$$

By a lengthy but straightforward application of (2.23) the nonzero Lorentz components of the Ricci tensor are found to be

$$R_{AB} = \begin{cases} 0 & A \neq B, \\ -3a & A=1,2,3 \\ 3a & A=4. \end{cases} \tag{4.15}$$

Recalling the definition of η_{AB} it is clear that (4.15) agrees with the field equations (4.13). The unique eigenvalue of the modified stress-energy tensor T is $\Lambda/8\pi$.

As a further example of a stress-energy tensor in Segre class [(111,1)] we will use the Einstein-de Sitter solution of the cosmological field equations with $\Lambda > 0$, and the mass density zero. The Einstein-de Sitter metric is given by

(4.16) $$ds^2 = -\exp(2at)dx^2 - \exp(2at)dy^2 - \exp(2at)dz^2 + dt^2,$$

where $a^2 = \Lambda/3$. The eigenvalue λ is zero since T is identically zero as a consequence of the energy conditions [6]. Since the cosmological constant Λ is assumed to be nonzero we get tetrad field equations which are of the same form as (4.13). From (4.16) we find the eigenvectors of the "stress-energy" tensor $T_{ij} = \frac{-\Lambda g_{ij}}{8\pi}$ to be

(4.17)
$$\begin{aligned} e_1 &= \exp(-at)\frac{\partial}{\partial x}, \\ e_2 &= \exp(-at)\frac{\partial}{\partial y}, \\ e_3 &= \exp(-at)\frac{\partial}{\partial z}, \\ e_4 &= \frac{\partial}{\partial t}. \end{aligned}$$

We find the corresponding 1-forms to be

(4.18)
$$\begin{aligned} \tilde{e}_1 &= -\exp(at)\ dx, \\ \tilde{e}_2 &= -\exp(at)\ dy, \\ \tilde{e}_3 &= -\exp(at)\ dz, \\ \tilde{e}_4 &= dt. \end{aligned}$$

The Ricci rotation coefficients are found to be

$$\gamma_{141} = a,$$

(4.18) $\gamma^{242} = a,$

$\gamma^{343} = a.$

Applying the definition (2.23) the tetrad components of the Ricci tensor are

$$R^{11} = -3a^2,$$

(4.19) $$R^{22} = -3a^2,$$

$$R^{33} = -3a^2,$$

$$R^{44} = 3a^2.$$

All the other Ricci tensor components are zero. From the definition of η_{AB} we see that the field equations are satisfied.

V. A New Interior Solution for a Stress-Energy Tensor of Segre Class [111,1]

The Field Equations

Consider a stress-energy tensor of Segre class [111,1] which has the canonical form

$$(5.1) \qquad T_{ij} = -\lambda_1 e_{1i} e_{1j} - \lambda_2 e_{2i} e_{2j} - \lambda_3 e_{3i} e_{3j} + \lambda_4 e_{4i} e_{4j}$$

where $\{e_A{}^i : 1 \le A \le 4\}$ is the eigentetrad of the stress-energy tensor and λ_1, λ_2, λ_3, and λ_4 are its eigenvalues. We assume that the eigenvectors are labelled so that λ_4 is the eigenvalue of the timelike eigenvector e_4. The field equations (1.1) can be contracted to show that the curvature scalar R is

$$(5.2) \qquad R = 8\pi(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) .$$

Using (5.2) and projecting the field equations (1.1) onto the pseudo-orthonormal tetrad of eigenvectors we find

$$(5.3) \qquad R_{AB} = 8\pi[\lambda_1 \eta_{1A}\eta_{1B} + \lambda_2 \eta_{2A}\eta_{2B} + \lambda_3 \eta_{3A}\eta_{3B} - \lambda_4 \eta_{4A}\eta_{4B}] + 4\pi\eta_{AB}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) .$$

Now two very restrictive hypotheses are imposed to simplify the problem. First assume that a solution of (5.3) exists which has the form

$$(5.4)\qquad ds^2 = -\exp(-2a)\,dx^2 - \exp(-2b)\,dy^2 - \exp(-2c)\,dz^2 + \exp(-2d)\,dt^2$$

where a, b, c, and d are C^3-functions of x, y, z, t. Secondly we assume that the eigentetrad is aligned with the holonomic basis which is pseudo-orthogonal by the first hypothesis. The eigen-tetrad can be written

$$(5.5)\qquad \begin{aligned} e_1 &= -\exp(-a)\,\frac{\partial}{\partial x}\,, \\ e_2 &= -\exp(-b)\,\frac{\partial}{\partial y}\,, \\ e_3 &= -\exp(-c)\,\frac{\partial}{\partial z}\,, \\ e_4 &= -\exp(-d)\,\frac{\partial}{\partial t}\,. \end{aligned}$$

Writing the 1-forms dual to the eigentetrad by (2.15) we get

$$(5.6)\qquad \begin{aligned} \tilde{e}_1 &= \exp(a)\,dx, \\ \tilde{e}_2 &= \exp(b)\,dy, \\ \tilde{e}_3 &= \exp(c)\,dz, \\ \tilde{e}_4 &= \exp(d)\,dt. \end{aligned}$$

The Ricci rotation coefficients are computed from (5.5) and (5.6) (2.23) to produce (we raise the tetrad indices by the rules in Chapter 2)

$$\begin{aligned}
&\gamma^{211} = -a_2\exp(-b) && \gamma^{322} = -b_3\exp(-c)\\
&\gamma^{122} = -b_1\exp(-a) && \gamma^{233} = -c_2\exp(-b)\\
(5.7)\qquad &\gamma^{311} = -a_3\exp(-c) && \gamma^{422} = -b_4\exp(-d)\\
&\gamma^{133} = -c_1\exp(-a) && \gamma^{244} = d_2\exp(-b)\\
&\gamma^{411} = -a_4\exp(-d) && \gamma^{433} = -c_4\exp(-d)\\
&\gamma^{144} = d_1\exp(-a) && \gamma^{344} = d_3\exp(-c)
\end{aligned}$$

In (5.7) the numerical subscripts on the functions a, b, c, d are to denote partial differentiaticn with respect to the indicated variable. We label the coordinates by the scheme $(x^1, x^2, x^3, x^4) \equiv (x, y, z, t)$.
A straightforward application of (2.23) together with the rules for raising tetrad indices gives

$$\begin{aligned}
(5.8)\qquad R^{11} = &- \gamma^{211|2} - \gamma^{311|3} + \gamma^{144|1} + \gamma^{411|4} - \gamma^{122|1} - \gamma^{133|1}\\
&+ [\gamma^{122}]^2 + [\gamma^{133}]^2 + [\gamma^{144}]^2 + [\gamma^{211}]^2 + [\gamma^{311}]^2\\
&- [\gamma^{411}]^2 + \gamma^{211}\gamma^{233} - \gamma^{211}\gamma^{244} + \gamma^{311}\gamma^{322}\\
&- \gamma^{311}\gamma^{344} - \gamma^{411}\gamma^{422} - \gamma^{411}\gamma^{433}
\end{aligned}$$

$$\begin{aligned}
(5.9)\qquad R^{12} = &-\gamma^{133|2} + \gamma^{144|2} + \gamma^{144}\gamma^{244} + \gamma^{133}\gamma^{233} - \gamma^{122}\gamma^{233}\\
&+ \gamma^{122}\gamma^{244}
\end{aligned}$$

$$\begin{aligned}
(5.10)\qquad R^{13} = &-\gamma^{122|3} + \gamma^{144|3} + \gamma^{122}\gamma^{233} + \gamma^{144}\gamma^{344} - \gamma^{133}\gamma^{322}\\
&+ \gamma^{133}\gamma^{344}
\end{aligned}$$

(5.11) $$R^{14} = -\gamma_{122|4} - \gamma_{133|4} + \gamma_{122}\gamma_{422} + \gamma_{133}\gamma_{433} - \gamma_{144}\gamma_{422} - \gamma_{144}\gamma_{433}$$

(5.12) $$R^{22} = -\gamma_{122|1} - \gamma_{322|3} + \gamma_{422|4} - \gamma_{211|2} - \gamma_{233|2} + \gamma_{244|2} + [\gamma_{211}]^2 + [\gamma_{233}]^2 + [\gamma_{244}]^2 + [\gamma_{122}]^2 + [\gamma_{322}]^2 - [\gamma_{422}]^2 + \gamma_{122}\gamma_{133} - \gamma_{122}\gamma_{144} + \gamma_{322}\gamma_{311} - \gamma_{322}\gamma_{344} - \gamma_{422}\gamma_{411} - \gamma_{422}\gamma_{433}$$

(5.13) $$R^{23} = -\gamma_{211|3} + \gamma_{244|3} + \gamma_{211}\gamma_{311} + \gamma_{244}\gamma_{344} - \gamma_{233}\gamma_{311} + \gamma_{233}\gamma_{344}$$

(5.14) $$R^{24} = -\gamma_{211|4} - \gamma_{233|4} + \gamma_{211}\gamma_{411} + \gamma_{233}\gamma_{344} + \gamma_{411}\gamma_{244} - \gamma_{244}\gamma_{433}$$

(5.15) $$R^{33} = -\gamma_{133|1} - \gamma_{233|2} + \gamma_{433|4} - \gamma_{311|3} - \gamma_{322|3} + \gamma_{344|3} + [\gamma_{322}]^2 + [\gamma_{344}]^2 + [\gamma_{311}]^2 + [\gamma_{133}]^2 + [\gamma_{233}]^2 - [\gamma_{433}]^2 + \gamma_{122}\gamma_{133} - \gamma_{133}\gamma_{144} + \gamma_{233}\gamma_{211} - \gamma_{233}\gamma_{244} - \gamma_{433}\gamma_{422} - \gamma_{433}\gamma_{411}$$

(5.16) $$R^{34} = -\gamma_{311|4} - \gamma_{322|4} + \gamma_{311}\gamma_{411} + \gamma_{322}\gamma_{422} + \gamma_{344}\gamma_{411} + \gamma_{344}\gamma_{422}$$

(5.17) $$R^{44} = -\gamma_{144|1} - \gamma_{244|2} - \gamma_{344|3} - \gamma_{411|4} - \gamma_{422|2} - \gamma_{433|4} + [\gamma_{411}]^2 + [\gamma_{422}]^2 + [\gamma_{433}]^2 - [\gamma_{144}]^2 - [\gamma_{244}]^2 - [\gamma_{344}]^2 + \gamma_{144}\gamma_{122} + \gamma_{144}\gamma_{133} + \gamma_{244}\gamma_{211} + \gamma_{244}\gamma_{233} + \gamma_{344}\gamma_{311} + \gamma_{344}\gamma_{322}$$

From (5.3) we find (after solving a simple system of linear equations for the λ_k in terms of the R^{AB}'s) that the eigenvalues are given by

$$
\begin{aligned}
\lambda_1 &= (1/16\pi)[R^{11} - R^{22} - R^{33} + R^{44}] \\
\lambda_2 &= (1/16\pi)[- R^{11} + R^{22} - R^{33} + R^{44}] \\
\lambda_3 &= (1/16\pi)[- R^{11} - R^{22} + R^{33} + R^{44}] \\
\lambda_4 &= (1/16\pi)[- R^{11} - R^{22} - R^{33} - R^{44}]
\end{aligned} \tag{5.18}
$$

The equations (5.8) to (5.18) allow us to find the eigenvalues in terms of the γ^{ABC}. We find them to be

$$
\begin{aligned}
8\pi\lambda_1 = {} & \gamma^{322}{}_{|3} - \gamma^{422}{}_{|4} + \gamma^{233}{}_{|2} - \gamma^{244}{}_{|2} - \gamma^{433}{}_{|4} - \gamma^{344}{}_{|3} \\
& + [\gamma^{422}]^2 + [\gamma^{433}]^2 - [\gamma^{233}]^2 - [\gamma^{244}]^2 \\
& - [\gamma^{322}]^2 - \gamma^{122}\gamma^{133} + \gamma^{122}\gamma^{144} + \gamma^{322}\gamma^{344} \\
& + \gamma^{422}\gamma^{433} + \gamma^{133}\gamma^{144} + \gamma^{233}\gamma^{244}
\end{aligned} \tag{5.19}
$$

$$
\begin{aligned}
8\pi\lambda_2 = {} & \gamma^{311}{}_{|3} - \gamma^{411}{}_{|4} + \gamma^{133}{}_{|1} - \gamma^{144}{}_{|1} - \gamma^{433}{}_{|4} - \gamma^{344}{}_{|3} \\
& + [\gamma^{411}]^2 + [\gamma^{433}]^2 - [\gamma^{133}]^2 - [\gamma^{144}]^2 - [\gamma^{311}]^2 \\
& - [\gamma^{344}]^2 - \gamma^{211}\gamma^{233} + \gamma^{211}\gamma^{244} + \gamma^{311}\gamma^{344} \\
& + \gamma^{411}\gamma^{433} + \gamma^{133}\gamma^{144} + \gamma^{233}\gamma^{244}
\end{aligned} \tag{5.20}
$$

$$
\begin{aligned}
8\pi\lambda_3 = {} & \gamma^{211}{}_{|2} - \gamma^{411}{}_{|4} + \gamma^{122}{}_{|1} - \gamma^{144}{}_{|1} - \gamma^{422}{}_{|4} - \gamma^{244}{}_{|2} \\
& + [\gamma^{411}]^2 + [\gamma^{422}]^2 - [\gamma^{122}]^2 - [\gamma^{144}]^2 - [\gamma^{211}]^2 \\
& - [\gamma^{244}]^2 - \gamma^{311}\gamma^{322} + \gamma^{211}\gamma^{244} + \gamma^{311}\gamma^{344} \\
& + \gamma^{411}\gamma^{422} + \gamma^{122}\gamma^{144} + \gamma^{322}\gamma^{344}
\end{aligned} \tag{5.21}
$$

$$(5.22) \qquad 8\pi \lambda_4 = \gamma^{211|2} + \gamma^{311|3} + \gamma^{122|1} + \gamma^{133|1} + \gamma^{322|3} + \gamma^{233|2}$$
$$- [\gamma^{122}]^2 - [\gamma^{133}]^2 - [\gamma^{211}]^2 - [\gamma^{233}]^2$$
$$- [\gamma^{311}]^2 - [\gamma^{322}]^2 + \gamma^{411}\gamma^{422} + \gamma^{411}\gamma^{433}$$
$$+ \gamma^{422}\gamma^{433} - \gamma^{211}\gamma^{233} - \gamma^{122}\gamma^{133} - \gamma^{311}\gamma^{322}.$$

From the "off-diagonal" elements of R^{AB} we find the system of equations which we will try to solve.

$$(5.23) \qquad \gamma^{144|2} - \gamma^{133|2} + \gamma^{122}\gamma^{244} - \gamma^{122}\gamma^{233} + \gamma^{133}\gamma^{233}$$
$$+ \gamma^{144}\gamma^{244} = 0$$

$$(5.24) \qquad \gamma^{144|3} - \gamma^{122|3} + \gamma^{133}\gamma^{344} - \gamma^{133}\gamma^{322} + \gamma^{122}\gamma^{322}$$
$$+ \gamma^{144}\gamma^{344} = 0$$

$$(5.25) \qquad \gamma^{244|3} - \gamma^{211|3} + \gamma^{233}\gamma^{344} - \gamma^{233}\gamma^{311} + \gamma^{211}\gamma^{311}$$
$$+ \gamma^{244}\gamma^{344} = 0$$

$$(5.26) \qquad -\gamma^{122|4} - \gamma^{133|4} + \gamma^{144}\gamma^{422} + \gamma^{144}\gamma^{433} + \gamma^{122}\gamma^{422}$$
$$+ \gamma^{133}\gamma^{433} = 0$$

$$(5.27) \qquad -\gamma^{211|4} - \gamma^{233|4} + \gamma^{244}\gamma^{411} + \gamma^{244}\gamma^{433} + \gamma^{211}\gamma^{411}$$
$$+ \gamma^{233}\gamma^{433} = 0$$

$$(5.28) \qquad -\gamma^{311|4} - \gamma^{322|4} + \gamma^{344}\gamma^{411} + \gamma^{344}\gamma^{422} + \gamma^{311}\gamma^{411}$$
$$+ \gamma^{322}\gamma^{422} = 0$$

Now using the expressions for the γ^{ABC} which we computed in (5.7) and the rules for moving tetrad indices mentioned in Chapter 2 we can find the equations (5.23) to (5.28) in coordinate form.

$$
\begin{aligned}
&c_{12} + d_{12} + c_1 c_2 - c_1 a_2 + d_1 d_2 - d_1 a_2 - b_1 c_2 - b_1 d_2 = 0, \\
&b_{13} + d_{13} + b_1 b_3 - b_1 a_3 + d_1 d_3 - d_1 a_3 - c_1 b_3 - c_1 d_3 = 0, \\
&a_{23} + d_{23} + a_2 a_3 - a_2 b_3 + d_2 d_3 - d_2 b_3 - c_2 a_3 - c_2 d_3 = 0, \\
&b_{14} + c_{14} + b_1 b_4 - b_1 a_4 + c_1 c_4 - c_1 a_4 - d_1 b_4 - d_1 c_4 = 0, \\
&a_{24} + c_{24} + a_2 a_4 - a_2 b_4 + c_2 c_4 - c_2 b_4 - d_2 a_4 - d_2 c_4 = 0, \\
&a_{34} + b_{34} + a_3 a_4 - a_3 c_4 + b_3 b_4 - b_3 c_4 - d_3 a_4 - d_3 b_4 = 0.
\end{aligned} \tag{5.29}
$$

Similiarly we can convert equations (5.19) to (5.22) in the same way.

$$
\begin{aligned}
8\pi\chi_1 = &-\exp(-2a)[b_1 c_1 + b_1 d_1 + c_1 d_1] \\
&-\exp(-2b)[c_{22} + d_{22} - b_2 c_2 - b_2 d_2 + c_2^2 + d_2^2 + c_2 d_2] \\
&-\exp(-2c)[b_{33} + d_{33} - b_3 c_3 - d_3 c_3 + b_3^2 + d_3^2 + b_3 d_3] \\
&+\exp(-2d)[b_{44} + c_{44} - b_4 d_4 - c_4 d_4 + b_4^2 + c_4^2 + b_4 c_4]
\end{aligned} \tag{5.30}
$$

$$
\begin{aligned}
8\pi\chi_2 = &-\exp(-2a)[c_{11} + d_{11} - a_1 c_1 - a_1 d_1 + c_1^2 + d_1^2 + c_1 d_1] \\
&-\exp(-2b)[a_2 c_2 + a_2 d_2 + c_2 d_2] \\
&-\exp(-2c)[a_{33} + d_{33} - a_3 c_3 - c_3 d_3 + a_3^2 + d_3^2 + a_3 d_3] \\
&+\exp(-2d)[a_{44} + c_{44} - a_4 d_4 - c_4 d_4 + a_4^2 + c_4^2 + a_4 c_4]
\end{aligned} \tag{5.31}
$$

$$
\begin{aligned}
8\pi\chi_3 = &-\exp(-2a)[b_{11} + d_{11} - a_1 b_1 - a_1 d_1 + b_1^2 + d_1^2 + b_1 d_1] \\
&-\exp(-2b)[a_{22} + b_{22} - a_2 b_2 - b_2 d_2 + a_2^2 + d_2^2 + a_2 d_2]
\end{aligned} \tag{5.32}
$$

$$-\exp(-2c)[a_3b_3 + a_3d_3 + b_3d_3]$$
$$+\exp(-2d)[a_{44} + b_{44} - a_4d_4 - b_4d_4 + a_4^2 + c_4^2 + a_4b_4]$$

(5.33) $$8\pi\chi_4 = -\exp(-2a)[b_{11} + c_{11} - a_1b_1 - a_1c_1 + b_1^2 + c_1^2 + b_1c_1]$$
$$-\exp(-2b)[a_{22} + c_{22} - a_2b_2 - b_2c_2 + a_2^2 + c_2^2 + a_2c_2]$$
$$-\exp(-2c)[a_{33} + b_{33} - a_3c_3 - b_3c_3 + a_3^2 + b_3^2 + a_3b_3]$$
$$+\exp(-2d)[a_4b_4 + a_4c_4 + b_4c_4]$$

Using the equations (5.29) together with (5.30> to (5.33) we were able to verify the conservation equations

$$T^{AB}{}_{|B} = T^{AB}{}_{|B} - \gamma^{A}{}_{DB}\, T^{DB} - \gamma^{B}{}_{DB}\, T^{AD}$$
$$= 0.$$

These calculations are extremely long and have been omitted. Each conservation equation invclves approximately 250 terms. An attempt to calculate the Riemann tensor by the use of FORMAC73 programs was made. In one instance for an unaligned tetrad a single component of the Riemann tensor was on the order of 2000 terms.

Solution of the Off-Diagonal Equations

In this section we will try to find a solution of the off--diagonal equations (5.29). They are are a systen of six quasi-linear partial differential equations for four unknown functions a, b, c, d each of four variables. We start by making a very strong hypothesis i.e. that all of the unknown functions do not depend on the variable x^4. This assumption implies that any solutions which we find will be static since $\frac{\partial}{\partial x^4}$ will be a Killing vector which is "hypersurface orthogonal." Transform the equations (5.29) by a change of dependent variables

$$
\begin{aligned}
a &= \tilde{a} - d, \\
b &= \tilde{b} - d, \\
c &= \tilde{c} - d, \\
d &= \tilde{d}.
\end{aligned}
\tag{5.34}
$$

The transformation (5.34) reduces the system (5.29) to

$$
\begin{aligned}
&\tilde{c}_{12} + \tilde{c}_1 \tilde{c}_2 - \tilde{a}_1 \tilde{c}_2 - \tilde{b}_1 \tilde{c}_2 + 2\tilde{d}_1 \tilde{d}_2 = 0, \\
&\tilde{b}_{13} + \tilde{b}_1 \tilde{b}_3 - \tilde{c}_1 \tilde{b}_3 - \tilde{a}_1 \tilde{b}_3 + 2\tilde{d}_1 \tilde{d}_3 = 0, \\
&\tilde{a}_{23} + \tilde{a}_2 \tilde{a}_3 - \tilde{b}_2 \tilde{a}_3 - \tilde{c}_2 \tilde{a}_3 + 2\tilde{d}_2 \tilde{d}_3 = 0.
\end{aligned}
\tag{5.35}
$$

Transforming the dependent variables once more by the transformation

$$
\begin{aligned}
\tilde{a} &= \ln|F|, \\
\tilde{b} &= \ln|G|, \\
\tilde{c} &= \ln|H|,
\end{aligned}
\tag{5.36}
$$

we get the system

$$
(5.37)\qquad
\begin{aligned}
&F_{23} - (G_3/G)F_2 - (H_2/H)F_3 + 2\tilde{d}_2\tilde{d}_3F = 0,\\
&G_{13} - (H_1/H)G_3 - (F_3/F)G_1 + 2\tilde{d}_1\tilde{d}_3G = 0,\\
&H_{12} - (F_2/F)H_1 - (G_1/G)H_2 + 2\tilde{d}_1\tilde{d}_2H = 0.
\end{aligned}
$$

The system of equations (5.37) is very difficult to solve for functions F, G, H of three variables even when the function d is freely prescribed as a nontrivial function of three variables. We will assume as a working hypothesis that F, G, and H have the following dependence on the three variables x^1, x^2, x^3.

$$
(5.38)\qquad
\begin{aligned}
&F = F(x^2, x^3),\\
&G = G(x^3),\\
&H = H(x^2).
\end{aligned}
$$

With this hypothesis the equations (5.37) reduce to

$$(5.39)\qquad F_{23} - (G_3/G)F_2 - (H_2/H)F_3 + 2\tilde{d}_2\tilde{d}_3F = 0,$$

$$(5.40)\qquad 2\tilde{d}_1\tilde{d}_3G = 0,$$

$$(5.41)\qquad 2\tilde{d}_1\tilde{d}_2H = 0.$$

As a consequence of this working hypothesis we must make some similiar hypothesis about d i.e. we assume that $\tilde{d} = \tilde{d}(x^2, x^3)$. The equation (5.39) has the form

$$(5.42)\qquad U_{23} + A(x^3)U_2 + B(x^2)U_3 + C(x^2, x^3)U = 0$$

where

$$A(x^3) = -(G_3/G),$$

$$B(x^2) = -(H_2/H),$$

$$C(x^2,x^3) = 2\tilde{d}_2\tilde{d}_3.$$

The differential form $W = A\,dx^3 + B\,dx^2$ is exact hence there is a function $P(x^2,x^3)$ such that $dP = W$. The function P is found after a simple integration to be

$$P(x^2,x^3) = -\ln|GH|.$$

Notice that there is no arbitrary function of the variable x^1 added to P. This is consistent with the working hypothesis made earlier. Now make a last transformation of the dependent variable given by [42]

$$F(x^2,x^3) = V(x^2,x^3)\exp(-P(x^2,x^3)). \tag{5.43}$$

From (5.43) the equation (5.39) reduces to

$$V_{23} + [2\tilde{d}_2\tilde{d}_3 - (G_3/G)(H_2/H)]V = 0. \tag{5.44}$$

If $\tilde{d}(x^2,x^3) = (1/p)\ln|H| + (1/q)\ln|G|$ with $pq = 2$ then equation (5.44) becomes

(5.45) $\quad V_{23} = 0.$

The solution of (5.45) is

(5.46) $\quad V(x^2,x^3) = f(x^2) + g(x^3),$

where f and g are arbitrary functions of class C^3. Reversing this long trail of transformations we find a solution for (5.29) is

(5.46) $\quad a(x^2,x^3) = \ln|[f(x^2)+g(x^3)][H(x^2)]^{\frac{p-1}{p}}[G(x^3)]^{\frac{q-1}{q}}|,$

(5.47) $\quad b(x^2,x^3) = \ln|[H(x^2)]^{-\frac{1}{p}}[G(x^3)]^{\frac{q-1}{q}}|,$

(5.48) $\quad c(x^2,x^3) = \ln|[H(x^2)]^{\frac{p-1}{p}}[G(x^3)]^{-\frac{1}{q}}|,$

(5.49) $\quad d(x^2,x^3) = (1/p)\ln|H| + (1/q)\ln|G|,$

where the parameters p and q must satisfy $pq = 2$. Another solution cf the equation (5.42) leads to a more complex form for the functions a, b, c, d. This solution is in Appendix 2.

The Energy Conditions

From the text of Hawking and Ellis [6] we find that the weak energy conditions (in our signature) are written as

(5.50) $\lambda_4 \geq 0,$

(5.51) $\lambda_4 - \lambda_1 \geq 0,$

(5.52) $\lambda_4 - \lambda_2 \geq 0,$

(5.53) $\lambda_4 - \lambda_3 \geq 0.$

These inequalities express the physical statement that the energy density is everywhere non-negative. Translating these inequalities in terms of the expressions (5.30) to (5.33) for the eigenvalues λ_i we find using the solution found in the previous section

(5.54)
$$\begin{aligned} 8\pi\lambda_4 = &-\exp(-2a)[b_{11} + c_{11} - a_1 b_1 - a_1 c_1 + b_1^2 + b_1 c_1 + c_1^2] \\ &-\exp(-2b)[a_{22} + c_{22} - a_2 b_2 + a_2 c_2 - b_2 c_2 + c_2^2 + a_2^2] \\ &-\exp(-2c)[a_{33} - a_3 c_3 + a_3 b_3 + b_3^2 - b_3 c_3 + b_{33} + a_3^2] \\ &\geq 0, \end{aligned}$$

(5.55)
$$\begin{aligned} 8[\lambda_4 - \lambda_1] = &\exp(-2a)[-b_{11} - c_{11} + a_1 b_1 + a_1 c_1 - b_1^2 - c_1^2 + b_1 d_1 + c_1 d_1] \\ &+ \exp(-2b)[-a_{22} + a_2 b_2 - a_2^2 - a_2 c_2 + d_{22} + d_2^2 - b_2 d_2 + c_2 d_2] \\ &+ \exp(-2c)[b_3 d_3 + d_3^2 + d_{33} - c_3 d_3 - a_{33} + a_3 c_3 - a_3^2 - a_3 b_3] \\ &\geq 0, \end{aligned}$$

(5.56)
$$8[\lambda_4 - \lambda_2] = \exp(-2a)[d_{11} - a_1 d_1 + d_1^2 + c_1 d_1 - b_{11} + a_1 b_1 - b_1^2 - b_1 c_1]$$

$$+ \exp(-2b)[a_2 d_2 + c_2 d_2 - a_{22} - c_{22} + a_2 b_2 - a_2^2 + b_2 c_2 - c_2^2]$$
$$+ \exp(-2c)[a_3 d_3 + d_{33} - c_3 d_3 + d_3^2 - a_3 b_3 - b_{33} + b_3 c_3 - b_3^2]$$
$$\geq 0,$$

(5.57) $$8\pi[\lambda_4 - \lambda_3] = \exp(-2a)[-c_{11} + a_1 c_1 - b_1 c_1 - c_1^2 + d_{11} - a_1 d_1 + b_1 d_1 + d_1^2]$$
$$+ \exp(-2b)[-c_{22} - a_2 c_2 + b_2 c_2 - c_2^2 + a_2 d_2 + d_2^2 - b_2 d_2 + d_{22}]$$
$$+ \exp(-2c)[-a_{33} + a_3 c_3 - a_3^2 - b_{33} + b_3 c_3 - b_3^2 + b_3 d_3 + a_3 d_3]$$
$$\geq 0.$$

From Hawking and Ellis [6] we also find energy conditions called the strong energy conditions. The strong energy conditions are a statement of the physically reasonable requirement that the mass-energy density measured by a local observer is positive and that all momentum vectors are non-spacelike. These energy conditions for stress-energy tensors of Segre class [111,1] are written as (5.51) to (5.53) together with the condition

(5.58) $$S = 8\pi[\lambda_4 - \lambda_1 - \lambda_2 - \lambda_3] \geq 0.$$

Again using the solution from the previous section we find that (5.58) becomes

(5.59) $$S = 2\exp(-2a)[b_1 d_1 + c_1 d_1 + d_{11} + d_1^2 - a_1 d_1]$$
$$+ 2\exp(-2b)[d_{22} - b_2 d_2 + d_2^2 + c_2 d_2 + a_2 d_2]$$
$$+ 2\exp(-2c)[b_3 d_3 + d_{33} + d_3^2 - c_3 d_3 + a_3 d_3]$$

$\geq 0.$

We want to force the solution we have found to satisfy the strong energy conditions elaborated above. There is no known technique of solving differential inequalities like (5.55) to (5.59) other than trial and error. We will suppose that the arbitrary functions in (5.45) to (5.49) are of the simple forms below.

$$
\begin{aligned}
& f(x^2) = \text{constant } K, \\
(5.60) \qquad & g(x^3) = \text{constant } L, \\
& H(x^2) = A(x^2)^n, \\
& G(x^3) = B(x^3)^m,
\end{aligned}
$$

where A and B are nonzero constants. Since the strong energy conditions imply the weak energy conditions we will use (5.54) to find appropriate ranges of the exponents m, n in (5.60) above. Inequality (5.54) shows that we require

$$
\begin{aligned}
(5.61) \qquad & [(2p-2)/p]HH'' + [(p-1)/p]^2(H')^2 + (2-p)GG'' \\
& + [(2-p)/2]^2(G')^2 \leq 0,
\end{aligned}
$$

where $pq = 2$. If we assume that $p = 3$ so that $q = 2/3$ then a straighforward analysis of the inequality (5.61) will show us that allowable ranges of the exponents m, n are

$m \in R - [0,4/3],$

$n \in (0,3/4].$

For the choice $n = 1/3$, $m = 2$ it is a simple task to verify that (5.54) is satisfied wherever all the functions are defined. This choice of parameters also satisfies all the inequalities (5.55) to (5.58). Thus we see that this 3-parameter family of solutions satisfies the weak energy conditions. The strong energy conditions also hold for this choice of parameters.

Finally we write the metric tensor for the special values that we assigned to the parameters above.

$$(5.62)\quad ds^2 = -(A^{-4/3} B^{7/9} y^{-4/9} z^{14/9})\,dx^2/(K+L)^2 - (A^{2/3} B y^{2/9} z^2)\,dy^2$$
$$- (A^{-4/3} B^3 y^{-4/9} z^6)\,dz^2 + (A^{-2/3} B^{-3} y^{-2/9} z^{-6})\,dt^2.$$

Appendix 1: Transformations between Systems of Notation used in General Relativity

Generalizing work done by Ernst [18] and Misner et al. [19] we have found that almost all systems of notation in current use in General Relativity can be classified by six parameters $\{e_i : 1 \le i \le 6\}$. The parameters take the values +1 or −1 depending on the conventions used. The five parameters

$\{e_1, e_2, e_3, e_5, e_6\}$

are independent and deal only with the definitions of mathematical symbolism. The parameter e_4 is used as an auxiliary parameter in order to unify this classification scheme with that of Ernst. The parameters $\{e_1, e_2, e_3, e_4\}$ are identical with those of Ernst. The three parameters defined in Misner et al. will be denoted W_1, W_2, W_3.

The sets of parameters $\{e_1, e_2, e_3, e_4\}$ and $\{W_1, W_2, W_3\}$ are sufficient to transform any tensor expression which does not involve the Levi-Civita tensor density (transformations of tensor expressions which involve the Levi-Civita tensor density use the parameters e_5, e_6). The definitions of the e_i, $1 \le i \le 6$, are as follows:

(a) e_1 signature(g) = +2,

(b) $e_2 V_i R^{i}{}_{jkl} = V_{j;lk} - V_{j;kl}$,

(c) $e_3 R_{ij} = R^{m}{}_{imj}$,

(d) $e_4 G_{ij} = -e_1 e_2 e_3 8\pi T_{ij}$,

(e) $e_5 = +1$ if space-time indices run over $\{0,1,2,3\}$,

−1 otherwise

(f) $e_6 \eta_{ijkl} = \sqrt{|g|}\ \mathrm{sign}(i,j,k,l)$ where (i,j,k,l) denotes a permutation in S, and $|g|$ denotes the absolute value of det(g).

The parameters W_1, W_2, W_3 are defined by

(a) $W_1\ \mathrm{signature}(g) = +2$,

(b) $W_2 V_i\ R^{i}_{\ jkl} = V_{j;kl} - V_{j;lk}$,

(c) $W_3 G_{ij} = 8\pi T_{ij}$,

(d) $W_2 W_3 R_{ij} = R^{m}_{\ imj}$.

From these definiticns it is straightforward to find the relations

$e_1 = W_1$

$e_2 = -W_2$

$e_3 = W_2 W_3$

$e_4 = W_1$.

To transform tensor equations in one notation system (primed system) to another notation system (unprimed) it is necessary to calculate six parameters a_i, $1 \le i \le 6$, which are defined in terms of the e-parameters by the relation

$a_i = e_i e_i'$, $1 \le i \le 6$.

Once these conversion parameters are known then tensor equations in the primed notation may be converted to tensor equations in the unprimed notation by use of the following relations:

$g'_{ij} = a_1 g_{ij}$,

$g'^{ij} = a_1 g^{ij}$,

$g' = g$ where g denotes the determinant of g ,

$\eta'_{ij} = a_1 \eta_{ij}$,

$\eta'^{ij} = a_1 \eta^{ij}$,

$[ij,k]' = a_1 [ij,k]$,

$\left\{{k \atop ij}\right\}' = \left\{{k \atop ij}\right\}$,

$B'^{k}_{\cdot ij} = a_1 B^{k}_{\cdot ij}$ where $B^{k}_{\cdot ij} = \left\{{k \atop ij}\right\} - \left\{{k \atop ji}\right\}$,

$R'_{ijkl} = a_1 a_2 R_{ijkl}$, (insert one factor of a for each index raised)

$R'^{\cdot\cdot}_{ij} = a_2 a_3 R^{\cdot\cdot}_{ij}$, (insert one factor of a for each index raised)

$R' = a_1 a_2 a_3 R$

$G'^{\cdot\cdot}_{ij} = a_2 a_3 G^{\cdot\cdot}_{ij}$,

$S'^{\cdot\cdot}_{ij} = a_2 a_3 S^{\cdot\cdot}_{ij}$, where $S^{\cdot\cdot}_{ij} = R^{\cdot\cdot}_{ij} - (R/4) g^{\cdot\cdot}_{ij}$.

The conformal curvature tensor $C^{i}_{\cdot jkl}$ and the projective curvature tensor $W^{i}_{\cdot jkl}$ transform the same way as the Riemann tensor under changes in ϵ_1, ϵ_2 but misbehave under changes in ϵ_3.

The differential parameters $\Delta_1\phi$ and $\Delta_2\phi$ transform by the relations:

$\Delta_1'\phi = a_1 \Delta_1\phi$,

$\Delta_2'\phi = a_1 \Delta_2\phi$.

If $h_A^{\dot{i}}$ is an orthonormal tetrad then the transformation relations are

$h'^{\dot{i}}_A = a_1 h^{\dot{i}}_A$,

$h'_{A\dot{i}} = h_{A\dot{i}}$,

$h'^{A}_{\dot{i}} = a_1 h^{A}_{\dot{i}}$, and

$h'^{A\dot{i}} = h^{A\dot{i}}$.

The Ricci rotation coefficients transform by

$\gamma'_{ABC} = \gamma_{ABC}$,

$$\gamma'^{A\cdot}_{\cdot BC} = a_1 \gamma^{A}_{\cdot BC}.$$

The directional derivatives transform as

$$\phi'_{|A} = a_1 \phi_{|A},$$

$$\phi'^{|A} = \phi^{|A}.$$

For equations which involve the dualizing operation we use the transformation format

$$*'(T'^{\cdots}{}_{\cdots}) = a_5 a_6 {}^{*}(T'^{\cdots}{}_{\cdots}).$$

All other transformations between tensor equations should be derivable from the above relations.

Appendix 2: Another Solution of the Off-Diagonal Equations

If we approach the equations (5.29) with the ansatz

$$a = a(x^2, x^3),$$

$$b = b(x^2, x^3),$$

$$c = c(x^2, x^3),$$

$$d = d(x^4),$$

we find that the system (5.29) collapses into

$$a_{23} + a_2 a_3 - a_2 b_3 - a_3 c_2 = 0$$

with the other equations becoming trivial identities. Preforming similiar transformations as in Chapter 5 we reduce the equation above to the form

$$U_{23} - (P_{23} + P_2 P_3) U = 0$$

where P is as in Chapter 5. Writing the expression in parentheses as

(1) $$P_{23} + P_2 P_3 = \lambda,$$

where $\lambda = \lambda(x^2, x^3)$, we see that if

(2) $$U_{23} - \lambda U = 0,$$

then by prescribing λ we may construct solutions of the off-diagonal equations. Set $\lambda = n(n+1)/(x^2+x^3)^2$. Assume a solution of (1) exists of the form $f = f(x^2+x^3)$. Then (1) reduces to

$$f'' + (f')^2 = n(n+1)/v^2$$

where we have set $v = x^2 + x^3$. Transforming the dependent variable by $w = f'$ we get

$$(w')^2 + w^2 = n(n+1)/v^2, \tag{3}$$

which we recognize as a Riccati differential equation. The solution U of (2) is related to the solution of (3) by

$$w(v) = U'/U.$$

This last equation is an Euler equation with characteristic roots $-n$, $n+1$. The point $v=0$ is a regular singular point therefore the general solution of the auxiliary equation for (3) is [44]

$$U(v) = K_1|v|^{-n} + K_2|v|^{n+1}.$$

Reversing the trail of transformations we find that

$$f(x^2,x^3) = \ln|K_1|x^2+x^3|^{-n} + K_2|x^2+x^3|^{n+1}| + K_3.$$

From [43] we find that (2) has as a general solution

$$U = \sum_{r=0}^{n} \{(-1)^r (n+r)!/(n-r)!r! (x^2+x^3)^{-r} [f^{(n-r)}(x^2) + g^{(n-r)}(x^3)]\}$$

where f and g are arbitrary functions of class C^n. From these results complicated expressions for a, b, c, d may be derived. This family of solutions of (5.29) has not been tested for any of the energy conditions due to their extreme complexity.

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