APPLICATIONS OF THE RHODES EXPANSION TO THE CONSTRUCTION OF FREE SEMIGROUPS

by

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Drs, Gadjah Mada University, Indonesia, 1986

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIRMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

Mathematics and Statistics

SIMON FRASER UNIVERSITY

December, 1991

ISBN 0-315-78260-9

APPROVAL

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Degree: Master of Science (Mathematics)

Title of Thesis: Aplications of The Rhodes Expansion To The Construction of Free Semigroups

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December 19, 1991

Date Approved

Title of Thesis/Project/Extended Essay

APPLICATIONS OF THE RHODES EXPANSION

TO THE CONSTRUCTION OF FREE SEMIGROUPS

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Abstract

The Rhodes expansion $\hat{\mathbf{S}}^{\mathcal{L}}$ of a semigroup **S** is the set of all $<_{\mathcal{L}}$ -chains over **S** and $\hat{\mathbf{S}}^{\mathcal{L}}_{X}$ is the subsemigroup of $\hat{\mathbf{S}}^{\mathcal{L}}$ generated by sequences of the form (x), $x \in X$. When **S** is the free semilattice on X, then $\hat{\mathbf{S}}^{\mathcal{L}}_{X}$ is the free right regular band. If **S** is a zero semigroup then $\hat{\mathbf{S}}^{\mathcal{L}}_{X}$, $X = \mathbf{S} - \{0\}$, is an inflation of a right zero semigroup. In fact, the variety $\mathcal{IRZ}$ of inflations of right zero semigroups is the smallest variety properly containing $\mathcal{RZ} \vee \mathcal{Z}$ where $\mathcal{RZ}$ is the variety of right zero semigroups and $\mathcal{Z}$ is the variety of zero semigroups. Moreover if **S** is free in $\mathcal{Z}$ on X then $\hat{\mathbf{S}}^{\mathcal{L}}_{X}$ is free in $\mathcal{RZ} \vee \mathcal{Z}$ on X, and if we apply the right Rhodes expansion on $\hat{\mathbf{S}}^{\mathcal{L}}_{X}$, we have $(\widehat{\hat{\mathbf{S}}^{\mathcal{L}}_{X}})^{\mathcal{R}}$ is free in $\mathcal{RZ} \vee \mathcal{LZ} \vee \mathcal{Z}$, where $\mathcal{LZ}$ is the variety of left zero semigroups.

Acknowledgements

I would like to thank my Supervisor Dr. N.R. Reilly for his generous suggestion, guidance and encouragement. Without his help, this thesis could not have been written. I would also like to thank Indonesian Government, Higher Education Development Project, for financial support. Last, but not least, I would like to thank my wife Cicilia for her help.

Contents

Preface

The purpose of this thesis is to survey applications of the Rhodes Expansion to the construction of free semigroups.

In Chapter 1, we give the necessary definition and background. This includes a discussion of semigroups and varieties. We state, without proof, some theorems and lemmas. For more detail we refer the reader to [2], [3], [7], [8], [9] and [10].

In Chapter 2, we discuss the definiton of the Rhodes Expansion and its basic properties. Most of the results can be found in [1], [4] and [11]. However the proof presented here for Theorem 2.16 is new.

In Chapter 3, we provide the definition of Inflation of a Right Zero Semigroup. Some of the material in this chapter can be found in [6]. However the material concerning free object is new. The results here were obtained in collaboration with my supervisor.

Chapter 1

Introduction

We begin by presenting in this section some basic information concerning semigroups. Further information and proofs can be found in any of the basic standard texs, for example Clifford and Preston [3], Howie [7], Lallement [8], Petrich [9],[10], Burris and Sankappanavar [2].

Let $\mathbf{S}$ be a nonempty set with a binary operation $\cdot$. We say that $\mathbf{S}$ is a semigroup if $\cdot$ is associative, that is for all $x, y, z \in \mathbf{S}$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. When convenient, we will write $x \cdot y$ simply as xy.

A semigroup $\mathbf{S}$ is said to be a commutative semigroup if for all $x, y \in \mathbf{S}$, $xy = yx$.

If a semigroup $\mathbf{S}$ has an element 1 such that for all $x \in \mathbf{S}$, $x1 = 1x = x$, then $\mathbf{S}$ is called a semigroup with identity and 1 is an identity element of $\mathbf{S}$.

Let $\mathbf{S}$ be a semigroup. Then $\mathbf{S}^1$ is the semigroup obtained from $\mathbf{S}$ by adjoining an identity if necessary, that is

$$\mathbf{S}^1 = \begin{cases} \mathbf{S} & \text{if } \mathbf{S} \text{ has an identity element} \\ \mathbf{S} \cup \{1\} & \text{otherwise} \end{cases}$$

where the multiplication of S is extended to $\mathbf{S}^1$ by defining $1s = s1 = s$, for all $s \in \mathbf{S}$.

If a semigroup S contains an element 0 such that $x0 = 0x = 0$, for all $x \in \mathbf{S}$, then we say that S is a semigroup with zero and 0 is the zero element of S.

A semigroup S is a right zero semigroup if for all $x, y \in \mathbf{S}$ $xy = y$, and a left zero semigroup if for all $x, y \in \mathbf{S}$ $xy = x$.

A semigroup S is called a zero semigroup if for all $x, y \in \mathbf{S}$ $xy = c$ for some fixed element $c \in \mathbf{S}$.

A semigroup S is called band if every element of S is idempotent.

A nonempty subset T of a semigroup S is called a subsemigroup of S if for all $x, y \in T$ $xy \in T$.

Let A be a nonempty subset of a semigroup S. The intersection of all subsemigroups of S containing A is the subsemigroup $\langle A \rangle$ generated by A. It is easily shown that $\langle A \rangle = \{x \in \mathbf{S} : \exists a_1, \cdots, a_n \in A \text{ with } x = a_1 \cdots a_n\}$.

Let A be a nonempty subset of a semigroup S. Then A is called a left (right) ideal of S if $\mathbf{S}A \subseteq A$ ($A\mathbf{S} \subseteq A$), and a two-sided ideal, or simply an ideal, if it is both a left and a right ideal of S. An ideal A of a semigroup S is called a minimal ideal if for every ideal N of S with N $\subseteq$ A then N = A.

If ϕ is a mapping from a semigroup S into a semigroup $\mathbf{T}$, then ϕ is a homomorphism if for all $x, y \in \mathbf{S}$, $\phi(xy) = \phi x \ \phi y$. If ϕ is a homomorphism and one to one then it is called a monomorphism, and if it is both one to one and onto we call it an isomorphism. An isomorphism from S onto S is called an automorphism. A homomorphism is called an epimorphism if it is onto.

Let **S** and **T** be semigroups. The direct product of **S** and **T** is the cartesian product $\mathbf{S} \times \mathbf{T}$ together with the multiplication $(s,t)(s',t') = (ss',tt')$.

A binary relation ρ on a set X is a set of ordered pairs (x,y) where $x,y \in$ X. We write $x\rho y$ if $(x,y) \in \rho$. The equality relation on X, denoted by i_X, is defined by :

$$(x,y) \in i_X \text{ if and only if } x = y.$$

Let $B(X)$ be the set of all binary relations on X. If $\rho, \sigma \in B(X)$ then

$$\rho \circ \sigma \quad = \quad \{(x,y) \in X \times X : (\exists z \in X)(x,z) \in \rho \text{ and } (z,y) \in \sigma\}$$

Notation : $\rho^2 = \rho \circ \rho, \quad \rho^3 = \rho \circ \rho \circ \rho.$

If $\rho \in B(X)$, then the domain and the range of ρ are defined by :

$$\begin{aligned} \operatorname{dom}(\rho) &= \{x \in X : (\exists y \in X)(x,y) \in \rho\}, \\ \operatorname{ran}(\rho) &= \{y \in X : (\exists x \in X)(x,y) \in \rho\}. \end{aligned}$$

Let ρ be an any element of $B(X)$. The inverse of ρ, denoted by ρ^{-1}, is defined by:

$$\rho^{-1} = \{(y,x) \in X \times X : (x,y) \in \rho\}.$$

An equivalence relation R on a semigroup **S** is called left compatible if for all $s,t,a \in \mathbf{S}$, $(s,t) \in R$ implies that $(as,at) \in R$, and right compatible if for all $s,t,a \in \mathbf{S}$, $(s,t) \in R$ implies that $(sa,ta) \in R$. It is called compatible if it is both left and right compatible, that is for all $s, t, s', t' \in \mathbf{S}$, $(s,t) \in R$ and $(s',t') \in R$ implies that $(ss', tt') \in R$. A left (right) compatible equivalence relation is called a left (right) congruence. A compatible equivalence relation is called a congruence.

Theorem 1.1 *Let ρ be a congruence on a semigroup $\mathbf{S}$. Define a binary operation on the quotient set $\mathbf{S}/\rho$ as follows:*

$$(a\rho) \cdot (b\rho) = (ab)\rho.$$

Then $(\mathbf{S}/\rho, \cdot)$ is a semigroup.

Theorem 1.2 *Let I be an ideal of a semigroup $\mathbf{S}$. Then*

$$\rho_I = (I \times I) \cup i_{\mathbf{S}}$$

is a congruence on $\mathbf{S}$ and $\mathbf{S}/\rho_I = \{I\} \cup \{\{x\} : x \in \mathbf{S} - I\}$.

Let X be a nonempty set and X^+ be the set of all nonempty finite words $a_1a_2a_3 \cdots a_n$ over the alphabet X. Define a binary operation on X^+ by:

$$(a_1a_2 \cdots a_n)(b_1b_2 \cdots b_m) = (a_1a_2 \cdots a_nb_1b_2 \cdots b_m).$$

With respect to this operation, X^+ is a semigroup.

Theorem 1.3 *Let X be a nonempty set and $\mathbf{S}$ be a semigroup. If $\phi : X \longrightarrow \mathbf{S}$ is an arbitrary mapping then there exists a unique homomorphism $\psi : X^+ \longrightarrow \mathbf{S}$ such that $\psi|_X = \phi$.*

If $\mathbf{S}$ is a semigroup and $a \in \mathbf{S}$ then the principal left ideal generated by a is the smallest left ideal containing a, which is $\mathbf{S}a \cup \{a\} = \mathbf{S}^1 a$. Similarly the principal right ideal generated by a is the smallest right ideal containing a, that is $a\mathbf{S} \cup \{a\} = a\mathbf{S}^1$. The principal ideal generated by a is defined to be $\mathbf{S}^1 a \mathbf{S}^1$.

Let $\mathbf{S}$ be a semigroup. The equivalence relations $\mathcal{L}, \mathcal{R}, \mathcal{J}$ and $\mathcal{H}$ defined on $\mathbf{S}$ by

$$a \,\mathcal{L}\, b \text{ if and only if } \mathbf{S}^1 a = \mathbf{S}^1 b,$$
$$a \,\mathcal{R}\, b \text{ if and only if } a\mathbf{S}^1 = b\mathbf{S}^1,$$
$$a \,\mathcal{J}\, b \text{ if and only if } \mathbf{S}^1 a \mathbf{S}^1 = \mathbf{S}^1 b \mathbf{S}^1$$
$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}$$

are called Green's equivalence relations on $\mathbf{S}$.
The $\mathcal{L}$-class ($\mathcal{R}$-class, $\mathcal{H}$-class, $\mathcal{J}$-class) containing the element a will be written $L_a(R_a, H_a, J_a)$.

Lemma 1.4 *Let* $\mathbf{S}$ *be a semigroup and* $a \in \mathbf{S}$. *Then* H_a *is a subgroup if and only if* H_a *contains an idempotent.*

Lemma 1.5 *Let* $\mathbf{S}$ *be a semigroup and let* $a, b \in \mathbf{S}$. *Then*

$$a \,\mathcal{L}\, b \Leftrightarrow (\exists x, y \in \mathbf{S}^1)\ xa = b,\ yb = a.$$
$$a \,\mathcal{R}\, b \Leftrightarrow (\exists u, v \in \mathbf{S}^1)\ au = b,\ vb = a.$$
$$a \,\mathcal{J}\, b \Leftrightarrow (\exists x, y, u, v \in \mathbf{S}^1)\ xay = b,\ ubv = a.$$

Let L_a and R_a be the $\mathcal{L}$ and $\mathcal{R}$-classes containing a. Then

$$L_a \leq L_b \text{ if } \mathbf{S}^1 a \subseteq \mathbf{S}^1 b,$$
$$R_a \leq R_b \text{ if } a\mathbf{S}^1 \subseteq b\mathbf{S}^1.$$

Clearly $\leq$ is a partial order on the set of $\mathcal{L}$ and $\mathcal{R}$-classes of $\mathbf{S}$.

Let $s, t \in \mathbf{S}$. Then we write

$$s \leq_{\mathcal{L}} t \text{ if and only if } s \in \mathbf{S}^1 t.$$

If $s \leq_{\mathcal{L}} t$ and $t \leq_{\mathcal{L}} s$, then $s \equiv_{\mathcal{L}} t$. We write $s <_{\mathcal{L}} t$ if $s \leq_{\mathcal{L}} t$ and $s \not\equiv_{\mathcal{L}} t$.

We write $\leq_{\mathcal{L}}$ and $<_{\mathcal{L}}$ simply as $\leq$ and $<$, respectively.

The following is a simple but important observation concerning the relations $\mathcal{L}$ and $\mathcal{R}$.

Lemma 1.6 *For any semigroup* $\mathbf{S}$, $\mathcal{L}$ *is a right congruence and* $\mathcal{R}$ *is a left congruence.*

A semigroup $\mathbf{S}$ is a <u>right group</u> if and only if it is isomorphic to the direct product of a right zero semigroup and a group.

Lemma 1.7 *Let* $\mathbf{S}$ *be a semigroup. Then the following statements are equivalent.*

(i) $\mathbf{S}$ *is a right group.*

(ii) $\mathbf{S}$ *is a union of groups and the set of all idempotents* $E(\mathbf{S})$ *of* $\mathbf{S}$ *is a right zero semigroup.*

(iii) H_a *is a subgroup of* $\mathbf{S}$ *for all* $a \in \mathbf{S}$, $E(\mathbf{S})$ *is a right zero semigroup and* $\mathbf{S} \simeq E(\mathbf{S}) \times H_a$ *for all* $a \in \mathbf{S}$.

Let $\mathcal{V}$ be a class of semigroups. Let $\mathbf{S}$ be a semigroup in $\mathcal{V}$, X be a nonempty set, and $\phi : X \longrightarrow \mathbf{S}$ be a mapping. The pair $(\mathbf{S}, \phi)$ is a <u>free object in</u> $\mathcal{V}$ (<u>on</u> X), or a <u>relatively free object</u> (in $\mathcal{V}$) if for every $\mathbf{T} \in \mathcal{V}$

and any mapping $\psi : X \longrightarrow \mathbf{T}$ there exists a unique homomorphism $\bar{\psi} : \mathbf{S} \longrightarrow \mathbf{T}$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, "\psi"'] & \mathbf{S} \arrow[d, "\bar{\psi}"] \\ & \mathbf{T} \end{tikzcd}$$

For example, by Theorem 1.3, (X^+, ι) where ι is the embedding of X into X^+ is a free object on the set X in the class of all semigroups.
When we talk about a free object, the mapping ι is often omitted.

Let X^+ be the free semigroup on X. A pair of elements x, y of X^+ is called a (semigroup) identity to be written $x = y$. A semigroup **S** satisfies the identity $x = y$ if for any homomorphism $\psi : X^+ \longrightarrow \mathbf{S}$, we have $\psi x = \psi y$.

Let F be a nonempty family of identities. The class $\mathcal{V}$ of all semigroups satisfying the family of identities F is called the variety of semigroups determined by F, denoted by $\mathcal{V} = [F]$. If F $= \{x = y\}$, we will write $\mathcal{V} = [x = y]$.

For any variety $\mathcal{V}$ of semigroups and any nonempty set X, there exists a free object in $\mathcal{V}$ on X.

The following theorem is a special case of Birkhoff's Theorem (see [2]).

Theorem 1.8 *A class $\mathcal{V}$ of semigroups is a variety if and only if it is closed with respect to homomorphisms, subsemigroups and direct products.*

The varieties of semigroups constitute a lattice $(L, \wedge, \vee)$ with respect to the following operations:

$$\mathcal{U} \wedge \mathcal{V} = \mathcal{U} \cap \mathcal{V}$$
$$\mathcal{U} \vee \mathcal{V} = \cap \{\mathcal{W} : \mathcal{W} \in L,\ \mathcal{U}, \mathcal{V} \subseteq \mathcal{W}\}.$$

Alternatively,

$$\mathcal{U} \vee \mathcal{V} = \{S : S \text{ is a homomorphic image of a subsemigroup of } U \times V \text{ for some } U \in \mathcal{U},\ V \in \mathcal{V}\}.$$

Lemma 1.9 *Let $\mathcal{U}, \mathcal{V}$ be varieties of semigroups. Let FU(X) be a free object in $\mathcal{U}$. If FU(X) $\in \mathcal{V}$ for all $|X| < \infty$ then $\mathcal{U} \subseteq \mathcal{V}$.*

Proof: Let $u = v$ be any identity that holds in $\mathcal{V}$. Let $u = u(x_1, \cdots, x_n)$, $v = v(y_1, \cdots, y_m)$. Let $X = \{x_1, \cdots, x_n, y_1, \cdots, y_m\} = \{z_1, \cdots, z_k\}$. Write $u = u(z_1, \cdots, z_k), v = v(z_1, \cdots, z_k)$. Then $FU(X) \in \mathcal{V}$. Thus the identity $u(z_1, \cdots, z_k) = v(z_1, \cdots, z_k)$ holds in $FU(X)$.

Now let $\mathbf{S} \in \mathcal{U}$, and $a_1, \cdots, a_k \in \mathbf{S}$. Define $\psi : FU(X) \longrightarrow \mathbf{S}$ by:

$$\psi(z_i) = a_i,\ 1 \leq i \leq k.$$

Since $FU(X)$ is relatively free in $\mathcal{U}, \psi$ defines a unique homomorphism $FU(X) \longrightarrow \mathbf{S}$. Therefore

$$\begin{aligned} & u(z_1, \cdots, z_k) = v(z_1, \cdots, z_k) \\ & \Longrightarrow \psi(\ u(z_1, \cdots, z_k)\ ) = \psi(\ v(z_1, \cdots, z_k)\ ) \\ & \Longrightarrow u(\psi z_1, \cdots, \psi z_k) = v(\psi z_1, \cdots, \psi z_k) \\ & \Longrightarrow u(a_1, \cdots, a_k) = v(a_1, \cdots, a_k). \end{aligned}$$

Therefore $u = v$ is an identity for $\mathbf{S}$, that is $u = v$ is an identity for $\mathcal{U}$. Thus $\mathcal{U} \subseteq \mathcal{V}$. æ

Chapter 2

Rhodes Expansion

2.1 The Construction

Let $\mathbf{S}$ be a semigroup and define $\bar{\mathbf{S}}^{\mathcal{L}}$ to be the set of all $\leq_{\mathcal{L}}$ -chains over S, that is

$$\bar{\mathbf{S}}^{\mathcal{L}} = \{\ (s_n, \cdots, s_1)\ |s_i \in \mathbf{S},\ s_n \leq_{\mathcal{L}}\ \cdots\ \leq_{\mathcal{L}} s_1,\ n \geq 1\}.$$

Define a multiplication in $\bar{\mathbf{S}}^{\mathcal{L}}$ by:

$$(s_n, \cdots, s_1)(t_m, \cdots, t_1) = (s_n t_m, \cdots, s_1 t_m, t_m, \cdots, t_1).$$

It is easily verified that this is an associative operation so that $(\bar{\mathbf{S}}^{\mathcal{L}}, \cdot\ )$ is a semigroup. Since each occurrence of $\leq_{\mathcal{L}}$ is either $<_{\mathcal{L}}$ or $\equiv_{\mathcal{L}}$, we can now define a reduction of elements in $\bar{\mathbf{S}}^{\mathcal{L}}$ as follows:

If $s = (s_n, \cdots, s_1) \in \bar{\mathbf{S}}^{\mathcal{L}}$ and $s_{i+1} \equiv s_i$ for some $1 \leq i \leq n-1$, then an <u>elementary reduction</u> of s is defined to be:

$$(s_n, \cdots, s_{i+1}, s_i, \cdots, s_1) \longrightarrow (s_n, \cdots, s_{i+1}, s_{i-1}, \cdots, s_1).$$

That is, cancelling the element which is $\equiv_{\mathcal{L}}$ -equivalent to its successor on the left. If $s, t \in \bar{\mathbf{S}}^{\mathcal{L}}$ and s is obtained from t by applying a finite number of elementary reductions, then we say that s comes from t by reduction. If we cannot perform an elementary reduction on t, then we say that t is irreducible. Thus $s = (s_n, \cdots, s_1)$ is irreducible if and only if $s_n <_{\mathcal{L}} s_{n-1} <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1$. Clearly the process of reduction leads us to a unique irreducible element which we denote by $Red(s)$.

For convenience we will drop the subscript $\mathcal{L}$ and write $\leq, <$, and $\equiv$ instead of $\leq_{\mathcal{L}}$, $<_{\mathcal{L}}$,and $\equiv_{\mathcal{L}}$. Also we will include the relations $\leq, <$, or $\equiv$ when we are defining certain element of $\bar{\mathbf{S}}^{\mathcal{L}}$ for wich we already know the order of its components. For instance, if $s = (s_k, s_l, s_m, s_n, s_o) \in \bar{\mathbf{S}}^{\mathcal{L}}$ with $s_k \leq s_l \equiv s_m < s_n \leq s_o$ then we will write $s = (s_k \leq s_l \equiv s_m < s_n \leq s_o)$, and $Red(s) = Red(s_k \leq s_l < s_n \leq s_o)$.

Lemma 2.1 *Let s and t be any elements in $\bar{\mathbf{S}}^{\mathcal{L}}$. Then*

$$Red(s \cdot t) = Red(\ Red(s) \cdot Red(t)\ ),$$

where $\cdot$ denotes the multiplication in $\bar{\mathbf{S}}^{\mathcal{L}}$.

Proof: Let

$s = (x_{m1} \equiv x_{m2} \equiv \cdots \equiv x_{mr_m} < x_{(m-1)1} \equiv x_{(m-1)2} \equiv \cdots < x_{11} \equiv x_{12} \equiv \cdots \equiv x_{1r_1})$,

$t = (y_{n1} \equiv y_{n2} \equiv \cdots \equiv x_{ns_n} < y_{(n-1)1} \equiv y_{(n-1)2} \equiv \cdots < y_{11} \equiv y_{12} \equiv \cdots \equiv y_{1s_1})$ be elements of $\bar{\mathbf{S}}^{\mathcal{L}}$. By Lemma 1.5, $x_{ij}y_{n1}\ \mathcal{L}\ x_{ik}y_{n1}$ for all $1 \leq i \leq m$, $1 \leq j,\ k \leq r_i$. Hence

$$Red(s) = (x_{m1} < x_{(m-1)1} < \cdots < x_{11})$$
$$Red(t) = (y_{n1} < y_{(n-1)1} < \cdots < y_{11})$$

$Red(Red(s)Red(t)) = Red(x_{m1}y_{n1} \leq x_{(m-1)1}y_{n1} \leq \cdots \leq x_{11}y_{n1} \leq y_{n1} < y_{(n-1)1} < \cdots < y_{11})$.

$$\begin{aligned} Red(st) &= Red(x_{m1}y_{n1} \equiv x_{m2}y_{n1} \equiv \cdots \equiv x_{mr_m}y_{n1} \leq x_{(m-1)1}y_{n1} \equiv \\ &\quad x_{(m-1)2}y_{n1} \equiv \cdots \leq x_{11}y_{n1} \equiv x_{12}y_{n1} \equiv \cdots \equiv x_{1r_1}y_{n1} \leq y_{n1} \equiv \\ &\quad y_{n2} \equiv \cdots \equiv x_{ns_n} < y_{(n-1)1} \equiv y_{(n-1)2} \equiv \cdots < y_{11} \equiv y_{12} \equiv \\ &\quad \cdots \equiv y_{1s_1}) \\ &= Red(x_{m1}y_{n1} \leq x_{(m-1)1}y_{n1} \leq \cdots \leq x_{11}y_{n1} \leq y_{n1} \leq \cdots \leq y_{11}) \\ &= Red(Red(s)Red(t)). \end{aligned}$$

Definition 2.2 The <u>Rhodes Expansion $\hat{\mathbf{S}}^{\mathcal{L}}$</u> of a semigroup **S** is the set of all $<_{\mathcal{L}}$ -chains over **S**, that is the set of all irreducible elements of $\bar{\mathbf{S}}^{\mathcal{L}}$ together with the multiplication

$$s * t = Red(s \cdot t) \ \text{ for every } s, t \in \hat{\mathbf{S}}^{\mathcal{L}}$$

Where $\cdot$ denotes the multiplication in $\bar{\mathbf{S}}^{\mathcal{L}}$. This construction was introduced by J. Rhodes (see [4], chapter XII). The remainder of this and the next section is devoted to the basic properties of the Rhodes expansion, with the exception of Theorem 2.9, most result can be found in [1] and [4].

Lemma 2.3 *Let* **S** *be a semigroup. Then*

1. *$(\hat{\mathbf{S}}^{\mathcal{L}}, *)$ is a semigroup.*

2. *The mapping $s \longmapsto Red(s)$ is an epimorphism of $\bar{\mathbf{S}}^{\mathcal{L}}$ onto $\hat{\mathbf{S}}^{\mathcal{L}}$.*

3. *The mapping $\eta_s : (x_m, \cdots, x_1) \longrightarrow x_m$ is an epimorphism of $\hat{\mathbf{S}}^{\mathcal{L}}$ onto* **S**.

Proof: 1. Let s, t, u be any elements in $\hat{\mathbf{S}}^{\mathcal{L}}$. Then

$$\begin{aligned}
s * (t * u) &= s * (Red(t \cdot u)) \\
&= Red(s \cdot Red(t \cdot u)) \\
&= Red(Red(s) \cdot RedRed(t \cdot u)) \quad \text{(by Lemma 2.1)} \\
&= Red(Red(s) \cdot Red(t \cdot u)) \\
&= Red(s \cdot (t \cdot u)) \quad \text{(by Lemma 2.1)} \\
&= Red((s \cdot t) \cdot u) \quad (\text{associativity of } \bar{\mathbf{S}}^{\mathcal{L}}) \\
&= Red(Red(s \cdot t) \cdot Red(u)) \quad \text{(Lemma 2.1)} \\
&= Red(RedRed(s \cdot t) \cdot Red(u)) \\
&= Red(Red(s \cdot t) \cdot u) \quad \text{(Lemma 2.1)} \\
&= (Red(s \cdot t)) * u \\
&= (s * t) * u.
\end{aligned}$$

Thus $(\hat{\mathbf{S}}^{\mathcal{L}}, *)$ is a semigroup.

2. Let s and t be two elements in $\bar{\mathbf{S}}^{\mathcal{L}}$. Then $s \longmapsto Red(s)$, $t \longmapsto Red(t)$, and $st \longmapsto Red(st)$. By Lemma 2.1 and the Definition 2.2 :

$$\begin{aligned}
Red(s \cdot t) &= Red(\underbrace{Red(s)}_{\in \hat{\mathbf{S}}^{\mathcal{L}}} \cdot \underbrace{Red(t)}_{\in \hat{\mathbf{S}}^{\mathcal{L}}}) \\
&= Red(s) * Red(t)
\end{aligned}$$

Thus $s \longmapsto Red(s)$ is a homomorphism from $\bar{\mathbf{S}}^{\mathcal{L}}$ into $\hat{\mathbf{S}}^{\mathcal{L}}$. This is also an epimorphism since $Red(s) = s$ for every s in $\hat{\mathbf{S}}^{\mathcal{L}}$.

3. Let $s = (s_n, \cdots, s_1)$ and $t = (t_m, \cdots, t_1)$ be two elements in $\hat{\mathbf{S}}^{\mathcal{L}}$. Then $\eta_s(s) = s_n$, $\eta_s(t) = t_m$, and $s * t = Red(s_n t_m, \cdots, s_1 t_m, t_m, \cdots, t_1)$. Since the

reduction never changes the leftmost term of elements in $\bar{\mathbf{S}}^{\mathcal{L}}$, then $\eta_s(s * t) = \eta_s Red(s_n t_m, \cdots, s_1 t_m, t_m, \cdots, t_1) = s_n t_m = \eta_s(s) * \eta_s(t)$. Thus η_s is a homomorphism. Moreover η_s is surjective since for every $x_n \in \mathbf{S}$ we can find a chain $(x_n) \in \hat{\mathbf{S}}^{\mathcal{L}}$ such that $\eta_s(x_n) = x_n$.

In the light of Lemma 2.3 we will denote the multiplication in both $\bar{\mathbf{S}}^{\mathcal{L}}$ and $\hat{\mathbf{S}}^{\mathcal{L}}$ by juxtaposition.

Theorem 2.4 *$\hat{\mathbf{S}}^{\mathcal{L}}$ is generated by $\leq$ -chains of length 1, that is elements of the form $(s) \in \hat{\mathbf{S}}^{\mathcal{L}}$, with $s \in \mathbf{S}$.*

Proof : Let $(x_n < \cdots < x_1)$ be any element in $\hat{\mathbf{S}}^{\mathcal{L}}$.
For every $i = 1, 2, \cdots, (n-1)$, $x_{i+1} < x_i$, and therefore we can find $y_{i+1} \in \mathbf{S}$ such that $x_{i+1} = y_{i+1} x_i$. By induction on i we may conclude that $x_{i+1} = y_{i+1} y_i \cdots y_2 x_1$, for $1 \leq i \leq n-1$.
Thus

$$\begin{aligned}
(y_n) \cdot (y_{n-1}) \cdots (y_2) \cdot (x_1) &= Red(y_n \cdots y_2 x_1 \leq \cdots \leq y_2 x_1 \leq x_1) \\
&= Red(x_n \leq \cdots \leq x_2 \leq x_1) \\
&= Red(x_n < \cdots < x_2 < x_1) \\
&= (x_n < \cdots < x_2 < x_1)
\end{aligned}$$

Definition 2.5 For any subset A of a semigroup $\mathbf{S}$ with $\mathbf{S} = \langle A \rangle$,

$$\hat{\mathbf{S}}^{\mathcal{L}}_A = \langle (a) : a \in A \rangle.$$

Theorem 2.6 *Let A be a subset of a semigroup $\mathbf{S}$ with $\mathbf{S} = \langle A \rangle$. Let $s = (s_n < \cdots < s_1)$, and $t = (t_k < \cdots < t_1)$ be elements in $\hat{\mathbf{S}}^{\mathcal{L}}_A$.*

1. $s \leq t$ *in* $\hat{\mathbf{S}}^{\mathcal{L}}_A$ $(\hat{\mathbf{S}}^{\mathcal{L}})$ *iff* $n \geq k$, $s_k \equiv t_k$ *in* $\mathbf{S}$, $s_{k-1} = t_{k-1}, \cdots, s_1 = t_1$.

2. $s \equiv t$ *in* $\hat{\mathbf{S}}^{\mathcal{L}}_A$ $(\hat{\mathbf{S}}^{\mathcal{L}})$ *iff* $n = k$, $s_k \equiv t_k$ *in* $\mathbf{S}$, $s_{k-1} = t_{k-1}, \cdots, s_1 = t_1$.

3. $s < t$ *in* $\hat{\mathbf{S}}^{\mathcal{L}}_A$ $(\hat{\mathbf{S}}^{\mathcal{L}})$ *iff* $n > k$, $s_k \equiv t_k$ *in* $\mathbf{S}$, $s_{k-1} = t_{k-1}, \cdots, s_1 = t_1$.

Proof : 1. ($\Rightarrow$) Since $s \leq t$, then there exists $u = (u_m, \cdots, u_1) \in (\hat{\mathbf{S}}^{\mathcal{L}}_A)^1$ such that $s = ut$. Thus we have

$$(s_n < \cdots < s_1) = Red(u_m t_k \leq \cdots \leq u_1 t_k \leq t_k < \cdots < t_1)$$

By reading the sequences from right to left we obtain: $s_1 = t_1, \cdots, s_{k-1} = t_{k-1}$, $n \geq k$ and $s_k = u_i t_k \equiv t_k$ for some i, $1 \leq i \leq m$, that is $s_k \equiv t_k$.
($\Leftarrow$) Let $s = (s_n < \cdots < s_1) = (x_m) \cdots (x_1)$, q = maximum integer with $(x_q) \cdots (x_1) = (s_k < \cdots < s_1)$. Then

$$Red[(x_m) \cdots (x_{q+1})]\ (s_k < \cdots < s_1) = (s_n < \cdots < s_1)$$

Let $t = (t_k < \cdots < t_1) = (y_p) \cdots (y_1)$, $u \in \mathbf{S}$ be such that $s_k = u t_k$ and $u = u_r \cdots u_1$ for some $u_i \in A$. Then

$$\begin{aligned} (u_r) \cdots (u_1)\ (y_p) \cdots (y_1) &= (s_k \equiv \cdots \equiv t_k < t_{k-1} < \cdots < t_1) \\ Red((u_r) \cdots (u_1)\ (y_p) \cdots (y_1)) &= (s_k, t_{k-1}, \cdots, t_1) \\ &= (s_k, s_{k-1}, \cdots, s_1) \end{aligned}$$

Therefore $Red[(x_m) \cdots (x_{q+1})(u_r) \cdots (u_1)](y_p) \cdots (y_1) = (x_m) \cdots (x_{q+1})(s_k < \cdots < s_1)$
$= (s_n < \cdots < s_1)$

2. This follows from (1), since $s \equiv t$ iff $s \leq t$ and $t \leq s$.

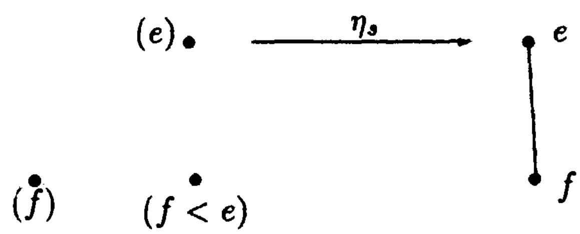
3. This follows from (1) and (2), since $s < t$ iff $s \leq t$ and $t \not\equiv s$.

The proof for $\hat{\mathbf{S}}^{\mathcal{L}}$ follows, since we can put $A = \mathbf{S}$.

Example

1. Let $\mathbf{S}$ be any left zero semigroup, that is for every $a, b \in \mathbf{S}$, $ab = a$. Since $a \leq b$ and $b \leq a$ then $\hat{\mathbf{S}}^{\mathcal{L}} = \{(a) : a \in \mathbf{S}\}$.

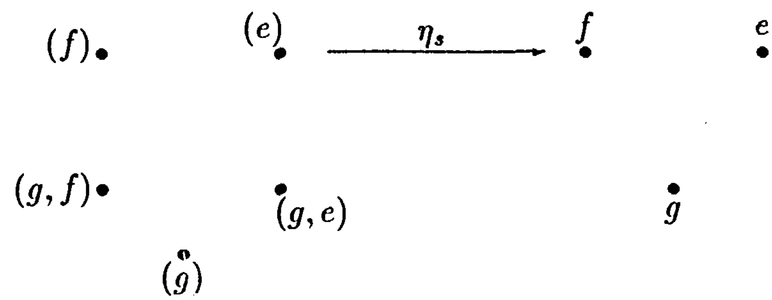
2. $\mathbf{S} = \{\ e^2 = e, f^2 = f, ef = fe = f\ \}$. Then $\hat{\mathbf{S}}^{\mathcal{L}} = \{(e), (f), (f < e)\}$. Moreover the homomorphism η_s from $\hat{\mathbf{S}}^{\mathcal{L}}$ onto $\mathbf{S}$ can be ilustrated as follows:



The set $\{(f), (f, e)\}$ is a right zero ideal in $\hat{\mathbf{S}}^{\mathcal{L}}$.

3. Let $\mathbf{S}$ be a semigroup generated by $\{e, f\}$ which satisfies $e^2 = e, f^2 = f, ef = fe = eg = ge = fg = gf = g$, that is $\mathbf{S} = \{e, f, ef = fe = g\}$. Then $\hat{\mathbf{S}}^{\mathcal{L}} = \{(e), (f), (g), (g, e), (g, f)\}$ with multiplication table:

	(e)	(f)	(g)	(g,e)	(g,f)
(e)	(e)	(g,f)	(g)	(g,e)	(g,f)
(f)	(g,e)	(f)	(g)	(g,e)	(g,f)
(g)	(g,e)	(g,f)	(g)	(g,e)	(g,f)
(g,e)	(g,e)	(g,f)	(g)	(g,e)	(g,f)
(g,f)	(g,e)	(g,f)	(g)	(g,e)	(g,f)



The set $\{(g),(g,f),(g,e)\}$ is a right zero ideal.

If A $= \{e,f\}$. Then

$$\begin{aligned}\hat{\mathbf{S}}^{\mathcal{L}}_A &= \langle (a) : a \in A\rangle \\ &= \langle (e),(f)\rangle \\ &= \{(e),(f),(g,e),(g,f)\}\end{aligned}$$

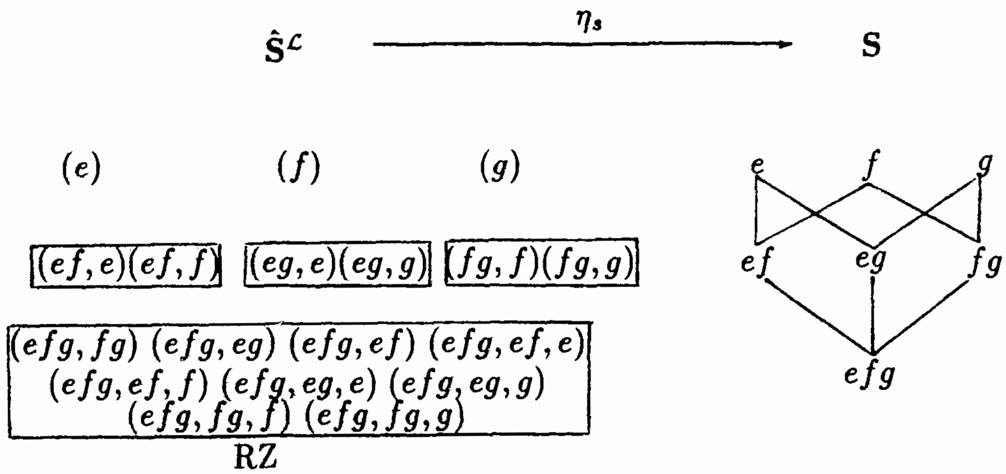
This example illustrates the fact that, in general, $\hat{\mathbf{S}}^{\mathcal{L}}_A$ is a proper sub-semigroup of $\hat{\mathbf{S}}^{\mathcal{L}}$.

4. Let S be a semigroup generated by $\{\ e,\ f,\ g\}$ which satisfies $e^2 = e,\ f^2 = f,\ g^2 = g,\ ef = fe,\ fg = gf,\ eg = ge,\ efg = gfe = fge = feg$

Then $\hat{\mathbf{S}}^{\mathcal{L}}$

$= \{(e),(f),(g),(ef,e),(ef,f),(fg,g),(fg,f),(eg,e),(eg,g),(efg,fg),$
$(efg,eg),(efg,ef),(efg,ef,e),(efg,ef,f),(efg,eg,e),(efg,eg,g),$
$(efg,fg,f),(efg,fg,g)\}$

And the homomorphism η_s from $\hat{\mathbf{S}}^{\mathcal{L}}$ to S can be described as follows:



If A= $\{e, f, g\} \subseteq \mathbf{S}$, then $\hat{\mathbf{S}}^{\mathcal{L}}_A = \langle\ (e), (f), (g)\ \rangle$

$= \{\ (e), (f), (g), (ef, f), (ef, e), (eg, e), (eg, g), (fg, f), (fg, g), (efg, fg, f),$
$(efg, fg, g), (efg, ef, f), (efg, ef, e), (efg, eg, e), (efg, eg, g)\}$.

2.2 Basic properties of the Rhodes Expansion

Throughout this section $\mathbf{S}$ is a fixed semigroup generated by a subset A and let $\eta_s : \hat{\mathbf{S}}^{\mathcal{L}}_A \longrightarrow \mathbf{S}$ be as defined in Lemma 2.3.

Lemma 2.7 *If* $\mathbf{G}$ *is a group, then* $\hat{\mathbf{G}}^{\mathcal{L}} \simeq \mathbf{G}$.

Proof: Note that $\hat{\mathbf{G}}^{\mathcal{L}} = \{\ (g_n, \cdots, g_1)\ :\ g_i \in \mathbf{G},\ n \geq 1\ \}$ and $g_i \equiv g_j$ for every g_i, $g_j \in \mathbf{G}$. Therefore

$$\begin{aligned} \hat{\mathbf{G}}^{\mathcal{L}} &= \{\ (g_i) : g_i \in \mathbf{G}\ \} \\ &\simeq \mathbf{G} \end{aligned}$$

Lemma 2.8 *For every idempotent e in* **S**, *$\eta_s{}^{-1}(e)$ is always a right zero semigroup.*

Proof: Suppose $s, t \in \eta_s{}^{-1}(e)$, that is $\eta_s(s) = \eta_s(t) = e$, and s, t have the form :

$s = (e, s_n, \cdots, s_1)$

$t = (e, t_k, \cdots, t_1)$

Then $st = (e, s_n, \cdots, s_1)\ (e, t_k, \cdots, t_1)$

$= Red(e^2, \cdots, s_1 e, e, t_k, \cdots, t_1)$

$= (e, t_k, \cdots, t_1)$

Moreover $\eta_s^{-1}(e)$ is a set of $\mathcal{R}$- equivalent idempotents of $\hat{\mathbf{S}}_A^{\mathcal{L}}$.

Theorem 2.9 *If G is a subgroup of* **S** *then $T = \eta_s^{-1}(G)$ is a right group and for any $a \in T$, $\eta_s|_{H_a} : H_a \longrightarrow G$ is an isomorphism.*

Proof: Let $E(T)$ be the set of all idempotents of T. We have that $e = (e_n, \cdots, e_1)$ is an idempotent in T if and only if

$$e^2 = (e_n^2, e_{n-1}e_n, \cdots, e_1 e_n, e_n, \cdots, e_1) = (e_n, \cdots, e_1),$$

that is $e_n^2 = e_n$. This can happen only if $e_n = 1$ (since G is a group and $e_n \in G$). Hence

$$E(T) = \{(1) \cup (1, e_{n-1}, \cdots, e_1) : 1 \text{ is the identity of G}\}.$$

Therefore $E(T) = \eta_s^{-1}(1)$ is right zero.

Let H_a, $a \in T$, be the $\mathcal{H}$-class containing a(in T). We want to show that there exists $e \in E(T)$ such that $e \in H_a$. Suppose $a = (a_m, \cdots, a_1)$ and let $e = (1, a_{m-1}, \cdots, a_1)$. Then $a_m \in G$ so that

$$
\begin{aligned}
a &= ea \\
e &= a(a_m^{-1}, a_{m-1}, \cdots, a_1) \\
a &= ae \\
e &= (a_m^{-1}, a_{m-1}, \cdots, a_1)(a_m, a_{m-1}, \cdots, a_1)
\end{aligned}
$$

That is $a\mathcal{R}e$ and $a\mathcal{L}e$. Thus $e^2 = e \in H_a$. According to Lemma 1.4, H_a is a group. Moreover $T = \cup_{a \in T} H_a$. Thus by Lemma 1.7 T is a right group.

Clearly $\eta_s|_{H_a}$ is a homomorphism of H_a into G.

Since $\eta_s^{-1}(1)$ is right zero then $\eta_s^{-1}(1) \cap H_a$ is right zero. Therefore $ker(\eta_s|_{H_a})$ is right zero. Thus

$$ker(\eta_s|_{H_a}) = \{1_{H_a}\}$$

Therefore $\eta_s|_{H_a}$ is ane-to-one.

It remains to show that $\eta_s|_{H_a}$ is onto.

Let e be the identity of H_a. (Note $H_a = H_e$). Let $g \in G$ and g' be such that $\eta_s(g') = g$.

<u>Claim</u> $g'e \in H_a = H_e$.

Let $e \;= (1, e_{n-1}, \cdots, e_1)$

$\quad g' = (g, g_{m-1}, \cdots, g_1)$

Then

$$
\begin{aligned}
g'e &= (g, g_{m-1}, \cdots, g_1, 1, e_{n-1}, \cdots, e_1) \\
&= (g, e_{n-1}, \cdots, e_1) \\
&\mathcal{L}(1, e_{n-1}, \cdots, e_1) = e \quad \text{(by Theorem 2.6)}
\end{aligned}
$$

On the other hand

$$(1, e_{n-1}, \cdots, e_1)(g, e_{n-1}, \cdots, e_1) = (g, e_{n-1}, \cdots, e_1)$$
$$(g, e_{n-1}, \cdots, e_1)(g^{-1}, e_{n-1}, \cdots, e_1) = (1, e_{n-1}, \cdots, e_1)$$

Thus $ge'\mathcal{L}e$ and $ge'\mathcal{R}e$, that is $ge' \in H_e$.

Also $\eta_s(g'e) = \eta_s g' \eta_s e = g1 = g$. Therefore $\eta_s|_{H_a}$ is surjective. Thus $\eta_s|_{H_a}$ is an isomorphism.

Corollary 2.10 *If G is a subroup of* S *with identity* 1 *then*

$$\eta_s^{-1}(G) \simeq \eta_s^{-1}(1) \times G.$$

Proof: This is an immediate consequence of Lemma 1.7 and Theorem 2.9.

Definition 2.11 A semigroup **S** has unambiguous $\mathcal{L}$-order if for every $s, t, u \in$ **S** with $s \leq t$ and $s \leq u$ we have $t \leq u$ or $u \leq t$.

Lemma 2.12 $\hat{\mathbf{S}}_A^{\mathcal{L}}$ *and* $\hat{\mathbf{S}}^{\mathcal{L}}$ *have unambiguous $\mathcal{L}$-order.*

Proof: Given $s = (s_n < \cdots < s_1)$, $t = (t_k < \cdots < t_1)$, $u = (u_l < \cdots < u_1) \in \hat{\mathbf{S}}_A^{\mathcal{L}}$ $(\hat{\mathbf{S}}^{\mathcal{L}})$ with $s \leq t$ and $s \leq u$ then by the Theorem 2.6 we have $n \geq k$, $s_k \equiv t_k$, $s_{k-1} = t_{k-1}, \cdots, s_1 = t_1$ and $n \geq l$, $s_l \equiv u_l$, $s_{l-1} = u_{l-1}, \cdots, s_1 = u_1$. Therefore if $k \leq l$, then $u \leq t$.

Repeating the Rhodes expansion produces nothing new :

Lemma 2.13 *Let $(\widehat{\hat{\mathbf{S}}_A^{\mathcal{L}}})_A^{\mathcal{L}}$ be the Rhodes Expansion of $\hat{\mathbf{S}}_A^{\mathcal{L}}$ defined by*

$$(\widehat{\hat{\mathbf{S}}_A^{\mathcal{L}}})_A^{\mathcal{L}} = \langle ((a)) : a \in A \rangle$$

that is the subsemigroup of $(\widehat{\hat{\mathbf{S}}^{\mathcal{L}}})^{\mathcal{L}}$ *generated by elements* $((a)) \in (\widehat{\hat{\mathbf{S}}^{\mathcal{L}}})^{\mathcal{L}}$, *where* $a \in A$, *then*

$$(\widehat{\hat{\mathbf{S}}^{\mathcal{L}}_A})^{\mathcal{L}}_A \simeq \hat{\mathbf{S}}^{\mathcal{L}}_A.$$

Proof: For any $a_i \in A$, $1 \leq i \leq n$, we have

$$\begin{aligned}
((a_n))\cdots((a_2))((a_1)) &= ((a_n))\cdots((a_3))\ \{((a_2))((a_1))\} \\
&= ((a_n))\cdots((a_3))\ Red((a_2)(a_1),(a_1)) \\
&= ((a_n))\cdots((a_3))\ Red[Red(a_2a_1,a_1),(a_1)] \\
&= \cdots \\
&= Red[Red(a_n\cdots a_1, a_{n-1}\cdots a_1,\cdots,a_1), \\
&\quad Red(a_{n-1}\cdots a_1, a_{n-2}\cdots a_1),\cdots, \\
&\quad Red(a_2a_1,a_1),(a_1)]
\end{aligned}$$

Thus we may assume that any element of $(\widehat{\hat{\mathbf{S}}^{\mathcal{L}}_A})^{\mathcal{L}}_A$ is of the form

$$Red[Red(s_n,\cdots,s_1) \leq Red(s_{n-1},\cdots,s_1) \leq \cdots \leq Red(s_2,s_1) \leq \ (s_1)]$$

Let η be the canonical morphism from $(\widehat{\hat{\mathbf{S}}^{\mathcal{L}}_A})^{\mathcal{L}}_A$ into $\hat{\mathbf{S}}^{\mathcal{L}}_A$ defined by:

$$s^* \overset{\eta}{\longmapsto} s, \text{ where}$$

$s^* = Red[Red(s_n,\cdots,s_1) \leq Red(s_{n-1},\cdots,s_1) \leq \cdots \leq Red(s_2,s_1) \leq \ (s_1)]$ and $s = Red(s_n,\cdots,s_1)$.

From Theorem 2.6 we have $s_k < s_{k-1}$ iff $Red(s_k,s_{k-1},\cdots,s_1) < Red(s_{k-1},\cdots,s_1)$. Therefore the two chains s^* and s have strict $<_{\mathcal{L}}$ in the same positions. Hence if $s = (a_m < \cdots < a_1)$ then

$s^* = ((a_m < \cdots < a_1), (a_{m-1} < \cdots < a_1), \cdots, (a_1))$. Thus s^* is uniquely determined by s. Therefore η is one-to-one.

In contrast to Lemma 2.13, if we apply the dual construction of the Rhodes expansion to the Rhodes expansion to obtain $(\widehat{\hat{\mathbf{S}}_A^{\mathcal{L}}})_A^{\mathcal{R}}$ then we may obtain something new. Indeed, N.R. Reilly showed in [11] that if **S** is the free semilattice on a countably infinite set A of generators then the semigroups

$$\hat{\mathbf{S}}_A^{\mathcal{L}}, \quad (\widehat{\hat{\mathbf{S}}_A^{\mathcal{L}}})_A^{\mathcal{R}}, \quad (\widehat{(\widehat{\hat{\mathbf{S}}_A^{\mathcal{L}}})_A^{\mathcal{R}}})_A^{\mathcal{L}},$$

are free in the (different) varieties that they generate.

2.3 Construction of Free Right Regular Bands

Definition 2.14 Let **S** be a semigroup. **S** is said to be a <u>semilattice</u> if **S** is commutative and every element of **S** is idempotent. Clearly the class $\mathcal{S}$ of semilattices is a variety defined by the identities $x^2 = x,\ xy = yx$.

Definition 2.15 Let **S** be a semigroup. **S** is said to be a <u>right regular band</u> (respectively, <u>regular band</u>) if for every $x, y, z \in \mathbf{S}$, $xyx = yx$ (respectively, $xyzx = xyxzx$). Left regular bands are defined dually.

In order to "locate" these varieties in the lattice of varieties of all bands let

$\mathcal{T}$ = variety of all trivial semigroups = $[x = y]$

$\mathcal{LZ}$ = variety of all left zero semigroups = $[xy = x]$

$\mathcal{RZ}$ = variety of all right zero semigroups = $[xy = y]$

$\mathcal{S}$ = variety of all semilattices = $[x^2 = x,\ xy = yx]$

$\mathcal{RB} = \mathcal{LZ} \vee \mathcal{RZ} = [x^2 = x,\ xyz = xz]$

$\mathcal{LNB} = \mathcal{S} \vee \mathcal{LZ} = [x^2 = x,\ xyz = xzy]$

$\mathcal{RNB} = \mathcal{S} \vee \mathcal{RZ} = [x^2 = x,\ zyx = yzx]$

$\mathcal{NB} = \mathcal{LNB} \vee \mathcal{RNB} = [x^2 = x,\ xaby = xbay]$

$\mathcal{LRB} = [x^2 = x,\ xyx = xy]$

$\mathcal{RRB} = [x^2 = x,\ xyx = yx]$

$\mathcal{REB} = [x^2 = x,\ xyzx = xyxzx]$

The lattice of subvarieties of regular bands is shown in the following diagram.

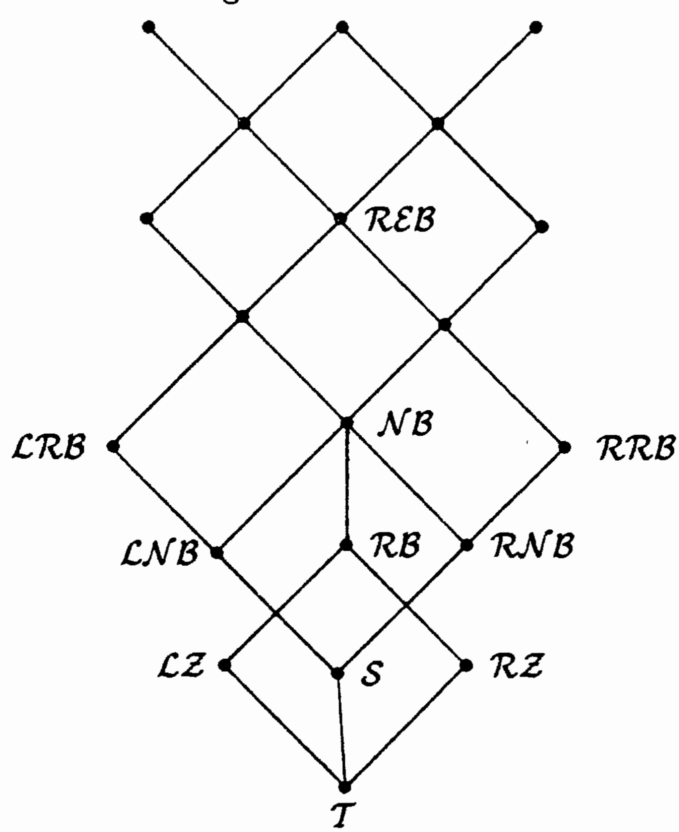


Diagram 2.1 : The lattice of proper varieties of bands

The following result is a particular case of a result in (Reilly [11]). However the proof is entirely different.

Theorem 2.16 *Let* $\mathbf{S}$ *be the free semilattice on* X*, and define a mapping* $\sigma : X \longrightarrow \hat{\mathbf{S}}_X^{\mathcal{L}}$ *by* $\sigma x = (x)$*, for all* $x \in X$*, then* $(\hat{\mathbf{S}}_X^{\mathcal{L}}, \sigma)$ *is the free right regular band on* X.

Proof: First we want to show that $\hat{\mathbf{S}}_X^{\mathcal{L}}$ is a band, that is for every $t \in \hat{\mathbf{S}}_X^{\mathcal{L}}$, $t^2 = t$.
Let $t = (t_n < \cdots < t_1) \in \hat{\mathbf{S}}_X^{\mathcal{L}}$, $t_i \in \mathbf{S}$. Then

$$\begin{aligned} t^2 &= Red({t_n}^2 \leq \cdots \leq t_1 t_n < t_n < \cdots < t_1) \\ &= Red(t_n \leq \cdots \leq t_1 t_n < t_n < \cdots < t_1) \ \ (\text{since } {t_n}^2 = t_n) \\ &= (t_n < \cdots < t_n) \\ &= t \end{aligned}$$

Thus for all $t \in \hat{\mathbf{S}}_X^{\mathcal{L}}$, $t^2 = t$ and $\hat{\mathbf{S}}_X^{\mathcal{L}}$ is a band.
Suppose $t, u \in \hat{\mathbf{S}}_X^{\mathcal{L}}$ with $t = (t_n, \cdots, t_1)$ and $u = (u_m, \cdots, u_1)$. Then

$$tu = Red(t_n u_m \leq \cdots \leq t_1 u_m \leq u_m < \cdots < u_1)$$

and

$$\begin{aligned} utu &= Red(u_m t_n u_m \leq \cdots \leq u_1 t_n u_m < t_n u_m \leq \cdots \leq t_1 u_m < u_m < \cdots < u_1) \\ &= Red(t_n u_m \leq \cdots \leq u_1 t_n u_m < t_n u_m \leq \cdots \leq t_1 u_m < u_m < \cdots < u_1) \\ &\quad (\text{since } \mathbf{S} \text{ is a semilattice}) \\ &= Red(t_n u_m \leq \cdots \leq t_1 u_m < u_m < \cdots < u_1) \end{aligned}$$

Thus for every $t, u \in \hat{\mathbf{S}}_X^{\mathcal{L}}$, $u\ t\ u = t\ u$ and $\hat{\mathbf{S}}_X^{\mathcal{L}}$ is a right regular band.

Now we want to show that $\hat{\mathbf{S}}_X^{\mathcal{L}}$ is the free regular band on X.
Let $t = (t_n, \cdots, t_1)$ be any element in $\hat{\mathbf{S}}_X^{\mathcal{L}}$. By the definition of $\hat{\mathbf{S}}_X^{\mathcal{L}}$, there

exist $x_i \in X,\ 1 \leq i \leq m$, such that $t = (x_m)\cdots(x_1)$.

From the fact that $\hat{\mathbf{S}}_X^{\mathcal{L}}$ is a right regular band, we can delete the left most element of any two identical x_i's until the reduced sequence obtained from this process contains distinct x_i's.

Thus we may assume that $x_1, \cdots, x_m$ are distinct. It then follows that $m = n$, since $(x_m)\cdots(x_1)$ is a sequence of length m and $(x_m)\cdots(x_1) = t = (t_n, \cdots, t_1)$.

Let $s_i \in X, i = 1, 2, \cdots, n$, be such that they are all distinct and $t = (s_n)\cdots(s_1)$.

We want to show that $x_i = s_i$ forall $i,\ 1 \leq i \leq n$. Since $t = (x_n)\cdots(x_1) = (s_n)\cdots(s_1)$, then

$$\begin{aligned} Red(x_n \cdots x_1, \cdots, x_2x_1, x_1) &= Red(s_n \cdots s_1, \cdots, s_1), \text{ that is} \\ (x_n \cdots x_1, \cdots, x_1) &= (s_n \cdots s_1, \cdots, s_1). \end{aligned}$$

By reading these two sequence from right to left, we have $x_1 = s_1,\ x_2x_1 = s_2s_1$. Since **S** is the free semilattice on X and $x_2x_1 = s_2s_1 \in \mathbf{S}$ with $x_1 = s_1$, then we conclude that $x_2 = s_2$. Continuing this process we have $x_i = s_i$ for all i. Thus every $t = (t_n, \cdots, t_1) \in \hat{\mathbf{S}}_X^{\mathcal{L}}$ can be written uniquely as $t = (x_n)\cdots(x_1)$ with $x_i \in X$ and $x_i \neq x_j$ for all $i \neq j$.

Let β be any mapping from X to any right regular band **B**.

Define $\gamma : \hat{\mathbf{S}}_X^{\mathcal{L}} \longrightarrow \mathbf{B}$ by: $\gamma t = (\beta t_n)\cdots(\beta t_1)$ for all $t = (t_n)\cdots(t_1) \in \hat{\mathbf{S}}_X^{\mathcal{L}}$, where $t = (t_n)\cdots(t_1)$ is the unique representation of t as a product of distinct elements of the form $(t_i),\ t_i \in X$.

$$
\begin{tikzcd}
X \arrow[r, "\sigma"] \arrow[dr, "\beta"'] & \hat{\mathbf{S}}_X^{\mathcal{C}} \arrow[d, "\gamma"] \\
& \mathbf{B}
\end{tikzcd}
$$

Clearly γ is well defined. We want to show that γ is a homomorphism. Let $a, b \in \hat{\mathbf{S}}_X^{\mathcal{C}}$ be such that $a = (t_n)\cdots(t_1)$, $b = (u_m)\cdots(u_1)$.

Case 1. $t_i \neq u_j$, $1 \leq i \leq n$, $1 \leq j \leq m$. Then

$$
\begin{aligned}
ab &= Red(t_n)\cdots(t_1)(u_m)\cdots(u_1) \\
&= (t_n)\cdots(t_1)(u_m)\cdots(u_1)
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma(ab) &= \underbrace{\beta(t_n)\cdots(\beta t_1)}_{\gamma a}\underbrace{(\beta u_m)\cdots(\beta u_1)}_{\gamma b} \\
&= \gamma a \gamma b
\end{aligned}
$$

Case 2. $C = \{t_1, \cdots t_n\} \cap \{u_1, \cdots, u_m\} \neq \emptyset$.

We proceed by induction on $|C|$, that is the number of variables that appear in both a and b. Suppose that $\gamma(ab) = (\gamma a)(\gamma b)$, for all a, b with $|C| \leq k$. Now consider a, b with $|C| = k + 1$. Let $t_i = u_j \in C$.

Then

$$
\begin{aligned}
\gamma a \gamma b &= (\beta t_n)\cdots(\beta t_{i+1})(\beta t_i)\cdots(\beta t_1)(\beta u_m)\cdots(\beta u_{j+1})(\beta t_i)(\beta u_{j-1})\cdots(\beta u_1) \\
&= (\beta t_n)\cdots(\beta t_{i+1})(\beta t_{i-1})\cdots(\beta t_1)(\beta u_m)\cdots(\beta u_{j+1})(\beta t_i)(\beta u_{j-1})\cdots(\beta u_1)
\end{aligned}
$$

(since $\mathbf{B}$ is a right regular band)

$$
\begin{aligned}
&= \gamma a^{*}\gamma b. \\
\text{where } a^{*} &= (t_n)\cdots(t_{i+1})(t_{i-1})\cdots(t_1) \text{ and} \\
ab &= Red\{(t_n)\cdots(t_1)(u_m)\cdots(u_{j+1})(t_i)(u_{j-1})\cdots(u_1)\} \\
&= Red\{(t_n)\cdots(t_{i+1})(t_{i-1})\cdots(t_1)(u_m)\cdots(u_{j+1})(t_i)(u_{j-1})\cdots(u_1)\} \\
&\quad \text{(by right regularity of } \hat{\mathbf{S}}_X^{\mathcal{L}})
\end{aligned}
$$

Thus we have $ab = a^{*}b$. By induction hypothesis $\gamma(a^{*}b) = \gamma a^{*}\gamma b$. Therefore $\gamma(ab) = \gamma a \gamma b$.

Since X generates $\hat{\mathbf{S}}_X^{\mathcal{L}}$, any homomorphism $\alpha : \hat{\mathbf{S}}_X^{\mathcal{L}} \longrightarrow \mathbf{B}$ which makes the above diagram commute must be such that $\alpha|_X = \gamma|_X$. Thus $\alpha = \gamma$. Therefore γ is unique and it is obvious that $\gamma \circ \sigma = \beta$. Thus $(\hat{\mathrm{S}}_X^{\mathcal{L}}, \sigma)$ is the free right regular band on X.

Chapter 3

Inflations of Right Zero Semigroups

3.1 The Variety $\mathcal{IRZ}$

Clearly the class $\mathcal{Z}$ of all zero semigroups is the variety of semigroups defined by the identity $xy = uv$. To give some idea as to where $\mathcal{Z}$ appears in the lattice L of varieties of semigroups, Diagram 3.1 presents a sublattice in L.

Definition 3.1 A semigroup S is an inflation of a right zero semigroup M if

(i) M is the minimum ideal S.

(ii) M is a right zero semigroup.

(iii) S/ρ_M is a zero semigroup.

It is straightforward to show that any ideal in a semigroup S that is a right zero semigroup is necessarily contained in every other ideal. Thus the term "minimum" could be deleted from part (i). However we leave it for the sake of emphasis.

The lattice of varieties of inflations of bands has been studied by Gerhard [6].

Recall that $\rho_M = M \times M \cup I_S$ or , equivalently, that ρ_M is the congruence on S defined by :

$$a\rho_M = \begin{cases} \{a\} & \text{if } a \notin M \\ M & \text{otherwise} \end{cases} \quad \text{for all } a \in \mathrm{S}.$$

Proposition 3.2 *Let* $\mathrm{S} \in \mathcal{Z}$ *and* $X = \mathrm{S} - \{0\}$. *Then* $\hat{\mathrm{S}}_X^{\mathcal{C}}$ *is an inflation of a right zero semigroup.*

Proof: For every $a, b \in X$, $ab = 0$ so that

$$\begin{aligned} (a)(b) &= (0,b), \quad (a)(0,b) = (0,b), \\ (a)(0,b) &= (0,b), \quad (0,b)(a) = (0,a). \end{aligned}$$

Since $\hat{\mathrm{S}}_X^{\mathcal{C}} = \langle (a) : a \in X \rangle$ we have

$$\hat{\mathrm{S}}_X^{\mathcal{C}} = \{(a), (0,a) : a \in X\}.$$

Let $M = \{(0,a) : a \in X\}$. For all $a, b \in X$, $(0,a)(0,b) = (0,b) \in M$,

$$(0,a)(b) = (0,b), \quad (b)(0,a) = (0,a).$$

Thus M is an ideal of $\hat{\mathrm{S}}_X^{\mathcal{C}}$. Since for every $(0,a),(0,b) \in M$, $(0,a)(0,b) = (0,b)$, M is a right zero semigroup, moreover M is the minimum ideal of $\hat{\mathrm{S}}_X^{\mathcal{C}}$.

Let $a, b \in X$. Then

$$\begin{aligned} a\rho_M \, b\rho_M &= ab\rho_M \\ &= (0, b)\rho_M \\ &= M \\ &\equiv 0 \end{aligned}$$

Thus $\hat{\mathbf{S}}_X^{\mathcal{L}}/\rho_M$ is a zero semigroup. Therefore $\hat{\mathbf{S}}_X^{\mathcal{L}}$ is an inflation of a right zero semigroup.

It is interesting to note the following simple properties of $\hat{\mathbf{S}}_X^{\mathcal{L}}$. If we apply the relation $\leq_{\mathcal{R}}$ on $\hat{\mathbf{S}}_X^{\mathcal{L}}$, then $(a) \leq_{\mathcal{R}} (b)$ if and only if $a = b$. In fact $(0, a) \leq_{\mathcal{R}} (b)$ and $(0, a) \leq_{\mathcal{R}} (0, b)$. Also for any $a, b \in \hat{\mathbf{S}}_X^{\mathcal{L}}$

$$\begin{aligned} a\mathcal{J}b &\Leftrightarrow a = xby \text{ and } b = tau \text{ for suitable } x, y, t, u \in (\hat{\mathbf{S}}_X^{\mathcal{L}})^1 \\ &\Leftrightarrow a = by \text{ and } b = au \text{ for suitable } y, u \in (\hat{\mathbf{S}}_X^{\mathcal{L}})^1 \\ &\Leftrightarrow a\mathcal{R}b \end{aligned}$$

Thus $\mathcal{R} = \mathcal{J}$.

Proposition 3.3 *Let* $\mathbf{A}_i$, $i \in I$, *be an inflation of a right zero semigroup* M_i. *Let*

$$\begin{aligned} A &= \prod_{i \in I} A_i \\ &= \{f : \ f : I \to \cup_{i \in I} A_i \text{ such that } f(i) \in A_i\} \end{aligned}$$

If multiplication in A *is defined by* $(fg)(i) = f(i)\, g(i)$, $i \in I$, *then* A *is an inflation of a right zero semigroup.*

<u>Proof</u>: Let $M = \{f \in A : f(i) \in M_i \text{ for all } i\}$. Let f be any element of M and g be any element of A. Then

$$(fg)(i) = \underbrace{f(i)}_{\in M_i}\underbrace{g(i)}_{\in A_i} \in M_i \text{ (since } M_i \text{is an ideal of } A_i)$$

Therefore $fg \in M$ and M is an ideal of A.

To see that M is a right zero semigroup, let f, g be two elements in M. Then

$$\begin{aligned}(fg)(i) &= \underbrace{f(i)}_{\in M_i}\underbrace{g(i)}_{\in M_i} \\ &= g(i) \text{ (since } M_i \text{ is right zero)}\end{aligned}$$

Thus $fg = g$ for every $f, g \in M$.

Let N be an ideal of A and $N \subseteq M$. Let f be any element in M. Then for every $g \in N$, $gf = f$, since M is a right zero semigroup while $f = gf \in N$, since N is an ideal. Therefore $M \subseteq N$ and $M = N$. Then M is the minimum ideal of A.

Let $f, g \in A$. Then $f(i), g(i) \in A_i$ for every $i \in I$. From the fact that A_i/ρ_{M_i} is a zero semigroup, we have

$$\begin{aligned}(fg)(i)\rho_{M_i} &= f(i)\rho_{M_i}\, g(i)\rho_{M_i} \\ &= M_i, \ (\forall i \in I)\end{aligned}$$

Thus $(fg)(i) \in M_i$ for every $i \in I$. Therefore $fg \in M$ (by the definition of M). Hence

$$\begin{aligned}f\rho_M\, g\rho_M &= fg\rho_M \\ &= M \\ &\equiv 0\end{aligned}$$

Thus A/ρ_M is a zero semigroup.

Proposition 3.4 *Let* S *be an inflation of a right zero semigroup. Let A be a homomorphic image of* S*. Then A is also an inflation of a right zero semigroup.*

Proof: Let f be any homomorphism from S onto A. Let M be the minimal ideal of S such that M is a right zero and S/ρ_M is a zero semigroup. Let $N = f(M)$.

Claim : N is an ideal of A.

Let a be any element in N and x be any element in A.

Since f maps S onto A and M onto N, there are $s_1, s_2 \in S$ with $s_1 \in M$ such that $f(s_1) = a$ and $f(s_2) = x$.

Therefore

$$\begin{aligned} ax &= f(s_1)\, f(s_2) \\ &= f(s_1 s_2) \text{ (since } f \text{ is a homomorphism)} \\ &\in N \quad \text{(since } s_1 \in M, s_2 \in S \text{ and } s_1 s_2 \in M) \end{aligned}$$

Thus $ax \in f(M) = N$ and, by symmetry, N is an ideal of A.

Let a, b be any elements in N. Since $N = f(M)$, there are x, y in M such that $f(x) = a$ and $f(y) = b$ and

$$\begin{aligned} ab &= f(x)\, f(y) \\ &= f(xy) \quad \text{(since } f \text{ is a homomorphism)} \\ &= f(y) \quad \text{(since } M \text{ is right zero)} \\ &= b \end{aligned}$$

Thus N is right zero.

Let P be an ideal of A such that $P \subseteq N$. Let a be any element in N. Then for every $b \in P$, $ba = a$, since N is a right zero semigroup while $a = ba \in P$, since P is an ideal. Therefore $P = N$ and N is the minimum ideal of A.

Let p, q be any elements in A and s, t be elements in **S** such that $f(s) = p$ and $f(t) = q$. From the fact that $\mathbf{S}/\rho_M$ is a zero semigroup, we have $st \in M$, therefore $f(st) \in N$. On the other hand $f(st) = f(s)\ f(t) = pq$. Thus for every elements $p, q \in A$, $pq \in N$ and, by symmetry, A/ρ_N is zero semigroup.

Proposition 3.5 *Let* **S** *be an inflation of a right zero semigroup and N be a subsemigroup of* **S**. *Then N is also an inflation of a right zero semigroup.*

Proof: Let M be the minimum ideal of **S**. Since $\mathbf{S}/\rho_M$ is a zero semigroup then for all $a, b \in \mathbf{S}$, we have $ab \in M$.

Therefore given $c \in N$ then $c^2 \in M$, that is $N \cap M \neq \emptyset$.

Claim : $N \cap M$ is an ideal of N.

Let a be any element in $N \cap M$ and x be any element in N. Then

$$\begin{aligned} ax &\in N \text{ (since } N \text{ is a subsemigroup) and} \\ ax &\in M \text{ (since } \mathbf{S}/\rho_M \text{ is a zero semigroup)} \end{aligned}$$

Thus $ax \in N \cap M$ and, by symmetry, $N \cap M$ is an ideal of N.

Claim : $N \cap M$ is the minimum ideal of N.

Let P be an ideal of N such that $P \subseteq N \cap M$. Let a be any element in P and b be any element in $N \cap M$. Then

$$ab \in P \text{ (since } P \text{ is an ideal) and}$$

$$ab = b \text{ (since } a, b \in M \text{ and } M \text{ is a right zero)}$$

Therefore $N \cap M \subseteq P$. Thus $N \cap M$ is the minimum ideal of N. It is obvious that $N \cap M$ is a right zero, since $N \cap M \subseteq M$. We want to show that $N/\rho_{N\cap M}$ is a zero semigroup, that is for all $a, b \in N$, $ab \in N \cap M$. Let a, b be elements in N. Then

$$\begin{aligned} ab &\in N \quad \text{(since } N \text{ is a subsemigroup of } \mathbf{S}\text{) and} \\ ab &\in M \quad \text{(since } a, b \in N \subseteq \mathbf{S} \text{ and } \mathbf{S}/\rho_M \text{ is a zero semigroup)} \end{aligned}$$

Therefore $ab \in N \cap M$. Thus $N/\rho_{N\cap M}$ is a zero semigroup.
We can summarize these observations in a theorem as follows:

Theorem 3.6 *The class $\mathcal{IRZ}$ of all inflations of right zero semigroups is a variety.*

Now we are looking for a basis of identities for $\mathcal{IRZ}$. This gives an alternative proof of the fact that $\mathcal{IRZ}$ is a variety.

Theorem 3.7 $\mathcal{IRZ} = [x(yz) = yz]$.

Proof: Let $\mathbf{S} \in \mathcal{IRZ}$. Then for all $a, b \in \mathrm{S}$, $ab \in M$ where M is an ideal and a right zero semigroup. Hence for all $a, b, c, d \in \mathbf{S}$ we have $(ab)(cd) = cd$, and

$$\begin{aligned} a(bc) &= a(bc)(bc) \\ &= (abc)(bc) \\ &= bc \end{aligned}$$

Therefore $\mathbf{S}$ satisfies the identity $x(yz) = yz$. Thus $\mathcal{IRZ} \subseteq [x(yz) = yz]$.

Let $\mathcal{V} = [x(yz) = yz]$. Let $T \in \mathcal{V}$ and $N = \{ab : a, b \in T\}$. Clearly N is an ideal of T.

<u>Claim</u> : N is a right zero semigroup.

For every $n, m \in N$ with $n = ab, m = cd$ where $a, b, c, d \in T$ we have

$$\begin{aligned} nm &= (ab)(cd) \\ &= cd \\ &= m \end{aligned}$$

as required.

Suppose that L is an ideal of T and $L \subseteq N$. Then

$$\begin{aligned} LN &\subseteq L \quad \text{(since } L \text{ is an ideal) and} \\ LN &= N \quad \text{(since } N \text{ is right zero)} \end{aligned}$$

Thus $N = L$ and N is the minimum ideal of T.

Since for every $a, b \in T$,

$$\begin{aligned} a\rho_N b\rho_N &= ab\rho_N \\ &= N \\ &\equiv 0, \end{aligned}$$

then T is an inflation of a right zero semigroup that is, $T \in \mathcal{IRZ}$. Thus $\mathcal{IRZ} = [x(yz) = yz]$.

We now provide another basis for $\mathcal{IRZ}$.

Theorem 3.8 $\mathcal{IRZ} = [(xy)(uv) = uv]$.

Proof: Let $\mathcal{U} = [(xy)(uv) = uv]$. It is clear that $\mathcal{U} \subseteq \mathcal{IRZ}$, since

$$\begin{aligned} x(yz) &= x(yz)(yz) \\ &= (x(yz))(yz) \\ &= yz \end{aligned}$$

On the other hand, let $\mathbf{S} \in \mathcal{IRZ}$ and M be the minimum (right zero) ideal of $\mathbf{S}$. Then for every $a, b, c, d \in \mathbf{S}$, we have $ab, cd \in M$. Since M is a right zero, then $(ab)(cd) = cd$. Thus $\mathbf{S} \in \mathcal{U} = [(xy)(uv) = (uv)]$. Therefore $\mathcal{IRZ} = [(xy)(uv) = (uv)]$.

3.2 Free Objects

Let X be a nonempty set and $\mathbf{S}$ be a semigroup defined by $\mathbf{S} = X \cup \{0\}$, with multiplication $ab = 0$, for all $a, b \in \mathbf{S}$. Let $i : X \longrightarrow \mathbf{S}$ with $i : x \longmapsto x$.

Theorem 3.9 *$(\mathbf{S}, i)$ is the free semigroup on X in $\mathcal{Z}$.*

Proof: Clearly $\mathbf{S} \in \mathcal{Z}$. Let $\mathbf{T}$ be any zero semigroup and $\psi : X \longrightarrow \mathbf{T}$ be any mapping. We want to show that there is a unique homomorphism $\beta : \mathbf{S} \longrightarrow \mathbf{T}$ such that the diagram below is commutative.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "\psi"'] & \mathbf{S} \arrow[d, "\beta"] \\ & \mathbf{T} \end{tikzcd}$$

Define $\beta : \mathbf{S} \longrightarrow \mathbf{T}$ by

$$\begin{aligned} \beta(x) &= \psi(x), \ \forall x \in X \text{ and} \\ \beta(0) &= 0 \end{aligned}$$

Since

$$\begin{aligned} \beta(xy) &= \beta(0) \\ &= 0 \\ &= \psi(x)\psi(y) \end{aligned}$$

then β is a homomorphism. Moreover β is unique, since every homomorphism $\alpha : \mathbf{S} \longrightarrow \mathbf{T}$ that makes the diagram commutative, $\alpha|_X = \beta|_X$, that is $\alpha = \beta$. Therefore $(\mathbf{S}, i)$ is the free semigroup on X in $\mathcal{Z}$.

From the Definition 2.5 we have $\hat{\mathbf{S}}_X^{\mathcal{L}} = \{(a), (0, a) : a \in X\}$ and if we set $M = \{(0, a)\}_{a \in X}$ then $\hat{\mathbf{S}}_X^{\mathcal{L}}$ is an inflation of the right zero semigroup M. Thus $\hat{\mathbf{S}}_X^{\mathcal{L}} \in \mathcal{IRZ}$. In fact we can do better. From Evans [5], we have $\mathcal{RZ} \vee \mathcal{Z} = [xy = zy]$ and the following diagram is a sublattice of the lattice generated by $\mathcal{RZ}, \mathcal{LZ}, \mathcal{S}$ and $\mathcal{Z}$.

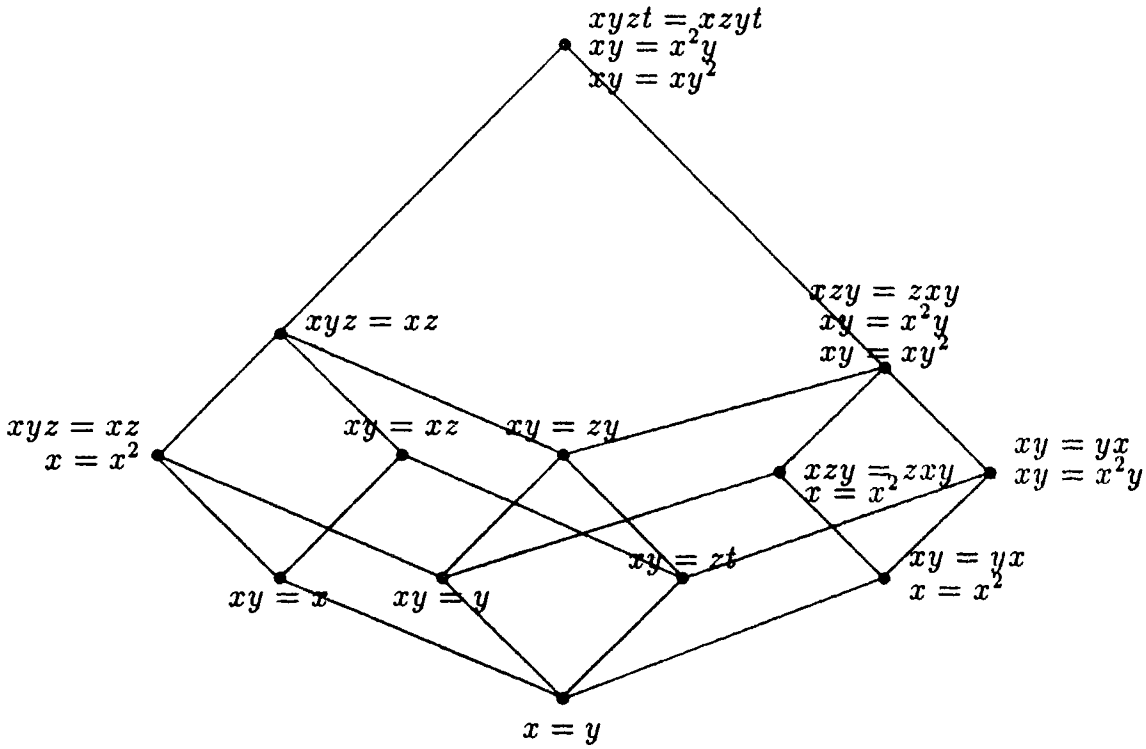


Diagram 3.1

The next results were obtained jointly with my supervisor.

Theorem 3.10 *Let X be a nonempty set and $(\mathbf{S}, i)$ be the free object in $\mathcal{Z}$ on X. Then $\hat{\mathbf{S}}_X^{\mathcal{L}}$ is the free object in $\mathcal{RZ} \vee \mathcal{Z}$ on X.*

Proof: From the Definition 2.5 we have $\hat{\mathbf{S}}_X^{\mathcal{L}} = \{(a), (0, a) : a \in X\}$. If we set $M = \{(0, a)\}$, then M is the minimum right zero ideal of $\hat{\mathbf{S}}_X^{\mathcal{L}}$. Therefore for every $y \in \hat{\mathbf{S}}_X^{\mathcal{L}}$, $y \in M$ or $y \notin M$.

If $y \in M$ then we have $xy = y = zy$.On the other hand if $y \notin M$ we have

$$(x)(y) = (0, y)$$

$$(0, a)(y) = (0, y)$$

Thus $\hat{\mathbf{S}}_X^{\mathcal{L}}$ satisfies $xy = zy$.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "\psi"'] & \hat{\mathbf{S}}_X^{\mathcal{C}} \arrow[d, "\beta"] \\ & \mathbf{T} \in [xy = zy] \end{tikzcd}$$

Let $\mathbf{T}$ be any element in $\mathcal{RZ} \vee \mathcal{Z}$. Let ψ be any mapping from X into $\mathbf{T}$. Let $w \in \mathbf{T}$. Define $\beta : \hat{\mathbf{S}}_X^{\mathcal{C}} \longrightarrow T$ by

$$\begin{aligned} \beta(x) &= \psi x \quad (x \in X) \\ \beta(0, a) &= w(\psi a) \end{aligned}$$

Clearly β is well defined. Let a and b be such that $a \in \hat{\mathbf{S}}_X^{\mathcal{C}}$, $b \in X$, then we have

$$\beta(a(b)) = \beta(0, b) = w\psi b$$
$$\beta(a)\beta(b) = **\, w\psi b = w\psi b.$$

If $a \in \hat{\mathbf{S}}_X^{\mathcal{C}}$, $b = (0, c)$, then

$$\beta((a)(b)) = \beta(0, c) = w\psi c$$
$$\beta(a)\beta(b) = **\, w\psi c.$$

Thus β is a homomorphism. Moreover β is unique, since every homomorphism $\alpha : \hat{\mathbf{S}}_X^{\mathcal{C}} \longrightarrow \mathbf{T}$ that makes the diagram commutative, $\alpha|_X = \beta|_X$, that is $\alpha = \beta$.

Thus $\hat{\mathbf{S}}_X^{\mathcal{C}}$ is the free object in $\mathcal{RZ} \vee \mathcal{Z}$ on X.

By applying the right Rhodes expansion on $\hat{\mathbf{S}}_X^{\mathcal{L}}$ we have the following theorem:

Theorem 3.11 $(\widehat{\hat{\mathbf{S}}_X^{\mathcal{L}}})^{\mathcal{R}}$ *is the free object in* $\mathcal{RZ} \vee \mathcal{LZ} \vee \mathcal{Z} = [xyz = xz]$.

Proof: The right Rhodes expansion of $\hat{\mathbf{S}}_X^{\mathcal{L}}$ has the form:

$$\begin{aligned}(\widehat{\hat{\mathbf{S}}_X^{\mathcal{L}}})^{\mathcal{R}} &= \langle ((a)) : a \in X \rangle \\ &= \{((a)), ((a) >_{\mathcal{R}} (0,b)) : a, b \in X\}.\end{aligned}$$

claim: $(\widehat{\hat{\mathbf{S}}_X^{\mathcal{L}}})^{\mathcal{R}} \in [xyz = xz]$.
Let $((a)), ((b)), ((x) > (0,y))$ and $((t) > (0,u))$ be any elements in $(\widehat{\hat{\mathbf{S}}_X^{\mathcal{L}}})^{\mathcal{R}}$. Then

$$\begin{aligned}((a))((b)) &= ((a) \geq (a)(b)) \\ &= ((a) > (0,b)), \\ ((a))((x) > (0,y)) &= ((a) \geq (a)(x) \geq (a)(0,y)) \\ &= ((a) \geq (0,x) \geq (0,y) \\ &= ((a) > (0,y)) \text{ since } (0,x) \equiv_{\mathcal{R}} (0,y), \\ ((x) > (0,y))((a)) &= ((x) > (0,y) \geq (0,y)(a)) \\ &= ((x) > (0,y) \geq (0,a)) \\ &= ((x) > (0,a)) \text{ and} \\ ((x) > (0,y))((t) > (0,u)) &= ((x) > (0,y) \geq (0,y)(t) \geq (0,y)(0,u)) \\ &= ((x) > (0,y) \geq (0,t) \geq (0,u)) \\ &= ((x) > (0,u)).\end{aligned}$$

Thus, by considering the above pattern, $(\widehat{\hat{\mathbf{S}}^{\mathcal{L}}_X})^{\mathcal{R}}$ satisfies the identity $xyz = xz$. Therefore $(\widehat{\hat{\mathbf{S}}^{\mathcal{L}}_X})^{\mathcal{R}} \in \mathcal{RZ} \vee \mathcal{LZ} \vee \mathcal{Z} = [xyz = xz]$.

Let $\mathbf{T}$ be any element in $[xyz = xz]$ and ψ be any mapping from X into $\mathbf{T}$.

$$
\begin{tikzcd}
X \arrow[r, "\iota"] \arrow[dr, "\psi"'] & (\widehat{\hat{\mathbf{S}}^{\mathcal{L}}_X})^{\mathcal{R}} \arrow[d, "\gamma"] \\
 & \mathbf{T} \in [xyz = xz]
\end{tikzcd}
$$

Define $\gamma : (\widehat{\hat{\mathbf{S}}^{\mathcal{L}}_X})^{\mathcal{R}} \longrightarrow \mathbf{T}$ by:

$$
\begin{aligned}
\gamma((a)) &= \psi a, \\
\gamma((a) > (0, b)) &= \psi a \psi b
\end{aligned}
$$

Clearly γ is well defined.

<u>Claim</u>: γ is a homomorphism.

For every $((a)), ((b)), ((x) > (0, y)), ((t) > (0, u))$ in $(\widehat{\hat{\mathbf{S}}^{\mathcal{L}}_X})^{\mathcal{R}}$ we have

$$
\begin{aligned}
\gamma(((a))((b))) &= \gamma((a) > (0, b)) \\
&= \psi a \psi b \\
&= \gamma((a))\gamma((b)) \\
\gamma(((a))((x) > (0, y))) &= \gamma((a) > (0, y)) \\
&= \psi a \psi y \\
\gamma((a))\gamma((x) > (0, y)) &= \psi a \psi x \psi y \\
&= \psi a \psi y \ \text{ since } \psi a, \psi x, \psi y \in \mathbf{T} \\
\gamma(((x) > (0, y))((a))) &= \gamma((x) > (0, a))
\end{aligned}
$$

$$
\begin{aligned}
&= \psi x \psi a \\
\gamma((x) > (0,y))\gamma((a)) &= \psi x \psi y \psi a \\
&= \psi x \psi a \\
\gamma(((x) > (0,y))((t) > (0,u))) &= \gamma((x) > (0,u)) \\
&= \psi x \psi u \\
\gamma((x) > (0,y))\gamma((t) > (0,u)) &= \psi x \psi y \psi t \psi u \\
&= \psi x \psi u.
\end{aligned}
$$

Thus γ is a homomorphism. Since every homorphism $\sigma : \widehat{(\hat{\mathbf{S}}_X^{\mathcal{L}})}^{\mathcal{R}} \longrightarrow \mathbf{T}$ that makes the above diagram commutes $\sigma|_X = \gamma|_X$, then γ is unique. Therefore $\widehat{(\hat{\mathbf{S}}_X^{\mathcal{L}})}^{\mathcal{R}}$ is the free object in $\mathcal{RZ} \vee \mathcal{LZ} \vee \mathcal{Z} = [xyz = xz]$.

Let $\mathbf{G} = X \cup M$, where $M = X \times X$, with multiplication defined by:

$$
ab = [a,b] \quad [a,b]c = [a,c]
$$
$$
[a,b][c,d] = [a,d] \quad a[b,c] = [a,c]
$$

It is easy to check that $\mathbf{G}$ is a semigroup.

Lemma 3.12 $(\mathbf{G}, \iota)$, *where* $\iota : x \longmapsto x$, *is a free object in* $\mathcal{RZ} \vee \mathcal{LZ} \vee \mathcal{Z}$ *on* X.

Proof: We need only to show that $\widehat{(\hat{\mathbf{S}}_X^{\mathcal{L}})}^{\mathcal{R}} \simeq \mathbf{G}$. Let ϕ be a mapping from $\mathbf{G}$ into $\widehat{(\hat{\mathbf{S}}_X^{\mathcal{L}})}^{\mathcal{R}}$ defined by :

$$
\begin{aligned}
\phi a &= ((a)), \\
\phi[a,b] &= ((a) > (0,b)).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\phi(ab) &= \phi[a, b] \\
&= ((a) > (0, b)) \\
&= \phi a \phi b. \\
\phi(a[b, c]) &= \phi[a, c] \\
&= ((a) > (0, c)) \\
&= \phi a \phi[b, c]. \\
\phi([b, c]a) &= \phi[b, a] \\
&= ((b) > (0, a)) \\
&= \phi[b, c]\phi a. \\
\phi([a, b][c, d]) &= \phi[a, d] \\
&= ((a) > (0, d)) \\
&= \phi[a, b]\phi[c, d].
\end{aligned}
$$

Thus ϕ is a homomorphism. Clearly ϕ is a bijection. Therefore ϕ is an isomorphism. Thus $(\widehat{\hat{\mathbf{S}}_X^{\mathcal{L}}})^{\mathcal{R}} \simeq \mathbf{G}$.

By applying the left Rhodes expansion on G we have the following lemma:

Lemma 3.13 $\hat{\mathbf{G}}_X^{\mathcal{L}} \simeq \mathbf{G}$.

Proof: By the Definition 2.5 we have

$$
\begin{aligned}
\hat{\mathbf{G}}_X^{\mathcal{L}} &= \langle (x) : x \in X \rangle \\
&= \{(x), ([x, y] < y) : x, y \in X\}.
\end{aligned}
$$

Define a mapping $\eta : \hat{\mathbf{G}}_X^{\mathcal{L}} \longrightarrow \mathbf{G}$ by

$$\begin{aligned} \eta(a) &= a, \\ \eta([a,b] < b) &= [a,b]. \end{aligned}$$

By Lemma 2.3, η is an epimorphism. It remains to show that η is one-to-one.

Clearly $\eta|_X$ is one-to-one and $\eta(a) \neq \eta([x,y] < y)$ for every $a, x, y \in X$. Suppose $\eta([a,b] < b) = \eta([c,d] < d)$ then $[a,b] = [c,d]$ therefore $a = c,\ b = d$. Thus $([a,b] < b) = ([c,d] < d)$. Thus η is one-to-one.

Furthermore, as a result of Theorem 2.13, if we apply the right Rhodes expansion on $\mathbf{G}$, we have $\hat{\mathbf{G}}_X^{\mathcal{R}} \simeq \mathbf{G}$.

3.3 Free Objects in $\mathcal{IRZ}$

Since inflations of right zero semigroups have arisen naturally in the study of Rhodes expansions it is interesting to determine the free objects in $\mathcal{IRZ}$.

Let $X \neq \emptyset$ and $M = X \times X$ with right zero multiplication, that is $(x,y)(s,t) = (s,t)$. Let $\mathbf{F} = X \cup M$ with multiplication :

$$xy = \begin{cases} (x,y) & \text{if } x, y \in X \\ y & \text{if } y \in M \\ (v,y) & \text{if } x = (u,v), y \in X \end{cases}$$

Lemma 3.14 : *$\mathbf{F}$ is an inflation of a right zero semigroup.*

<u>Proof</u>: First we consider associativity.

Case 1: $x, y, z \in X$. Then

$$\begin{aligned} x(yz) &= x(y,z) = (y,z) \\ (xy)z &= (x,y)z = (y,z). \end{aligned}$$

Case 2: $x = (t,u) \in M$, $y, z \in X$. Then

$$\begin{aligned} x(yz) &= (t,u)(y,z) = (y,z) \\ (xy)z &= (u,y)z = (y,z). \end{aligned}$$

Case 3: $x \in \mathbf{F}$, $y = (s,t) \in M$, $z \in X$. Then

$$\begin{aligned} x(yz) &= x(t,z) = (t,z) \\ (xy)z &= (s,t)z = (t,z). \end{aligned}$$

Case 4: $x, y \in \mathbf{F}$, $z \in M$. Then

$$\begin{aligned} x(yz) &= xz = z \\ (xy)z &= z. \end{aligned}$$

Thus $\mathbf{F}$ is a semigroup and clearly M is an ideal of $\mathbf{F}$ (by the definition of the multiplication in $\mathbf{F}$). It is also clear that M is a right zero semigroup. Next we show that M is the minimum ideal of $\mathbf{F}$.

Suppose that N is an ideal of $\mathbf{F}$ with $N \subseteq M$. Let (x,y) be any element in M. Then $N(x,y) = (x,y)$, by the definition of multiplication. But $N(x,y) \in N$, since N is an ideal. Therefore $(x,y) \in N$ for every $(x,y) \in M$. Thus M is the minimum ideal of $\mathbf{F}$.

Let $x\rho_M, y\rho_M \in \mathbf{F}/\rho_M$. Then

$$x\rho_M y\rho_M = xy\rho_M$$

$$\begin{aligned} &= M \text{ (since } xy \in M) \\ &\equiv 0 \end{aligned}$$

Therefore $\mathbf{F}/\rho_M$ is a zero semigroup and $\mathbf{F}$ is an inflation of a right zero semigroup.

Theorem 3.15 *If $\imath$ is a mapping from X to $\mathbf{F}$ defined by $\imath : x \longmapsto x$, then $(\mathbf{F}, \imath)$ is free in $\mathcal{IRZ}$ on X.*

<u>Proof</u>: It is easily seen that X generates $\mathbf{F}$. Let $\mathbf{T}$ be any inflation of a right zero semigroup and ψ be any mapping from X to $\mathbf{T}$.

$$\begin{tikzcd} X \arrow[r, "\imath"] \arrow[dr, "\psi"'] & \mathbf{F} \arrow[d, "\beta"] \\ & \mathbf{T} \end{tikzcd}$$

Define a mapping $\beta : \mathbf{F} \longrightarrow \mathbf{T}$ by

$$\begin{aligned} \beta(x) &= \psi(x) \;\; \forall x \in X \text{ and} \\ \beta(x, y) &= \beta(x)\beta(y). \end{aligned}$$

In order to establish that β is a homomorphism, we consider several cases.

<u>Case 1</u>: $x, y \in X$. Then

$$\begin{aligned} xy &= (x, y) \text{ and} \\ \beta(xy) &= \beta((x, y)) \\ &= \beta(x)\beta(y). \end{aligned}$$

Case 2: $x \in X,\ y = (u, v) \in M$. Then

$$\begin{aligned}\beta(xy) &= \beta(y) \\ &= \beta(u, v) \\ &= \beta(u)\beta(v).\end{aligned}$$

We know that $\beta(u)\beta(v) = \beta(x)(\beta(u)\beta(v))$, since $\beta(u)\beta(v)$ lies in the minimum ideal M' of $\mathbf{T}$, and M' is a right zero. Therefore

$$\begin{aligned}\beta(xy) &= \beta(x)(\beta(u)\beta(v)) \\ &= \beta(x)\beta(y).\end{aligned}$$

case 3: $x = (u, v) \in M,\ y \in X$. Then

$$\begin{aligned}\beta(xy) &= \beta((v, y)) \\ &= \beta(v)\beta(y) \\ &= \beta(u)(\beta(v)\beta(y)) \\ &= (\beta(u)\beta(v))\beta(y) \\ &= \beta(x)\beta(y).\end{aligned}$$

case 4: $x = (p, q),\ y = (s, t) \in M$. Then

$$\begin{aligned}\beta(xy) &= \beta(y) \\ &= \beta(s)\beta(t)\ \in \text{the minimum ideal } M' \text{ of } \mathbf{T} \\ &= (\beta(p)\beta(q))(\beta(s)\beta(t)) \\ &= \beta(x)\beta(y)\end{aligned}$$

Thus β is a homomorphism.

Since X generates $\mathbf{F}$, β is unique. Therefore $(\mathbf{F}, \imath)$ is the free inflation of a right zero semigroup on X.

The semigroup $\mathbf{F}$ as constructed above illustrates an interesting fact.

Lemma 3.16 $\mathcal{RZ} \vee \mathcal{Z} \subset_{\neq} \mathcal{IRZ}$.

Proof: Clearly $\mathcal{RZ} \cup \mathcal{Z} \subseteq \mathcal{IRZ}$. Therefore $\mathcal{RZ} \vee \mathcal{Z} \subseteq \mathcal{IRZ}$.
However $\mathbf{F} \in \mathcal{IRZ}$ and for any distinct elements $a, b, c \in X$,

$$ab = (a, b) \neq (c, b) = cb.$$

Thus $\mathbf{F} \notin \mathcal{RZ} \vee \mathcal{Z}$.

Theorem 3.17 *If $\mathcal{V}$ is a variety such that $\mathcal{RZ} \vee \mathcal{Z} \subset_{\neq} \mathcal{V} \subseteq \mathcal{IRZ}$, then $\mathcal{V} = \mathcal{IRZ}$.*

Proof: Let $\mathbf{S} \in \mathcal{V} - (\mathcal{RZ} \vee \mathcal{Z})$. Then $\mathbf{S} \in \mathcal{IRZ}$ but $\mathbf{S}$ does not satisfy the identity $xy = zy$. Consequently there exist $a, b, c \in \mathbf{S}$ with $ac \neq bc$. This implies that $c^2 \neq c$. Otherwise, since $\mathbf{S}$ is an inflation of a right zero semigroup we would have

$$ac = ac^2 = c^2 = bc^2 = bc$$

Which contradicts the choice of a, b and c. We must also have either $ac \neq c^2$ or $bc \neq c^2$. Without loss of generality we may assume that $ac \neq c^2$. Then also $a^2 \neq c^2$ since otherwise $ac = a^2c = c^3 = c^2$. Similarly $ac \neq ca$, since otherwise $c^2 = acc = cac = ac$.

Now, if $ac = a^2$ then we would have $ac = a(ac) = a^2c = acc = ac^2 = c^2$ which again contradicts the choice of a, b and c.

Thus we may assume that $ac \neq a^2$. Summarizing we have elements $a, b, c \in \mathbf{S}$ such that

$$ca \neq ac \neq bc,\ ac \neq c^2 \neq a^2,\ ac \neq a^2,\ c^2 \neq c. \qquad (1)$$

For any nonempty set X, let $FIRZ(X)$ denote the free semigroup in $\mathcal{IRZ}$ on X. By Lemma 1.9, it suffices to show that $FIRZ(X) \in \mathcal{V}$ for all finite nonempty sets X. So let $X = \{x_1, \cdots, x_n\}$ and consider $FIRZ(X)$. In $\mathbf{S}^{\binom{n^2}{2}} = \underbrace{\mathbf{S} \times \mathbf{S} \times \cdots \times \mathbf{S}}_{\binom{n^2}{2}}$ we will define $u_1, u_2, \cdots, u_n$ so that $x_i \longmapsto u_i$ defines an isomorphism of $FIRZ(x_1, x_2, \cdots, x_n)$ onto $U = \langle u_1, u_2, \cdots, u_n \rangle$. Each u_h can be thought of as an $\binom{n^2}{2}$ long vector with components indexed by $((i,j),(k,l))$ where $i < k$ or $i = k$ and $j < l$, $1 \leq i,j,k,l \leq n$. We wish to show that if $(i,j) \neq (k,l)$ then $u_iu_j \neq u_ku_l$ and also that $U \cap U^2 = \emptyset$. Let

$$P = \{((i,j),(k,l)) : \text{ where } i < k \text{ or } i = k \text{ and } j < l\}.$$

Let

$$\alpha : P \longmapsto \{1, 2, \cdots, \tbinom{n^2}{2}\} \text{ be a bijection.}$$

Let $1 \leq \delta \leq \binom{n^2}{2}$ and i,j,k,l be such that $\alpha((i,j),(k,l)) = \delta$. Let $(u_h)_\delta$ denote the δ^{th} component of u_h. We define $(u_i)_\delta, (u_j)_\delta, (u_k)_\delta$ and $(u_l)_\delta$, in various cases as follows : in all cases we define $(u_m)_\delta = c$ for $m \neq i,j,k,l$, this guarantees that $U \cap U^2 = \emptyset$.

Case I. $j = l$. Then $i \neq k$ so that we have the following subcases.

Case I(i). $j = l = i \neq k$. Define

$$(u_i)_\delta = (u_j)_\delta = (u_l)_\delta = c,\ (u_k)_\delta = a.$$

Case I(ii). $j = l = k \neq i$. Define

$$(u_i)_\delta = a,\ (u_j)_\delta = (u_k)_\delta = (u_l)_\delta = c.$$

Case I(iii). $i \neq j = l \neq k, \ i \neq k$. Define

$$(u_i)_\delta = a, \ \ (u_k)_\delta = b, \ \ (u_j)_\delta = (u_l)_\delta = c.$$

Case II. $i = k$. Then $j \neq l$ so that we have the following subcases.

Case II(i). $i = k = j \neq l$. Define

$$(u_i)_\delta = (u_j)_\delta = (u_k)_\delta = a, \ (u_l)_\delta = c.$$

Case II(ii). $i = k = l \neq j$. Define

$$(u_i)_\delta = (u_k)_\delta = (u_l)_\delta = a, \ (u_j)_\delta = c.$$

Case II(iii). $l \neq i = k \neq j, \ j \neq l$. Define

$$(u_i)_\delta = (u_j)_\delta = (u_k)_\delta = a, \ (u_l)_\delta = c.$$

Case III. $i \neq k, \ j \neq l$

Case III(i). $k \neq i = j \neq l, \ k = l$. Define

$$(u_i)_\delta = (u_j)_\delta = a, \ (u_k)_\delta = (u_l)_\delta = c.$$

Case III(ii). $k \neq i = j \neq l, \ k \neq l$.

$$(u_i)_\delta = (u_j)_\delta = (u_l)_\delta = c, \ (u_k)_\delta = a.$$

Case III(iii). $i \neq k, \ j \neq l = k, \ i \neq j, \ i \neq l$.

$$(u_j)_\delta = (u_k)_\delta = (u_l)_\delta = c, \ (u_i)_\delta = a.$$

Case III(iv). $i \neq k, \ j \neq l, \ i \neq j, \ k \neq l, \ i = l, \ j = k$.

$$(u_i)_\delta = (u_l)_\delta = a, \ (u_j)_\delta = (u_k)_\delta = c.$$

Case III(v). $i \neq k,\ j \neq l, i \neq j,\ k \neq l,\ i = l,\ j \neq k.$

$$(u_i)_\delta = (u_l)_\delta = a,\ (u_j)_\delta = (u_k)_\delta = c.$$

Case III(vi). $i \neq k,\ j \neq l,\ i \neq j,\ k \neq l,\ i \neq l,\ j = k.$

$$(u_i)_\delta = (u_l)_\delta = a,\ (u_j)_\delta = (u_k)_\delta = c.$$

In all cases it follows immediately from (1) that $(u_iu_j)_\delta \neq (u_ku_l)_\delta$. Thus, if $((i,j),(k,l)) \in P$ and $\alpha((i,j),(k,l)) = \delta$ then $(u_iu_j)_\delta \neq (u_ku_l)_\delta$. Therefore $u_iu_j \neq u_ku_l$.

Since each component of u_i, $i = 1, \cdots, n$, is an element of **S** and $\mathbf{S} \in \mathcal{IRZ}$ then $u_iu_ju_k = u_ju_k$. Therefore $U = \{u_i\} \cup \{u_iu_j\}$ and

$$\begin{aligned} |U| &= |\{u_1, u_2, \cdots, u_n\}| + |\{u_iu_j : i,j = 1,2,\cdots n\}| \\ &= n + n^2 \end{aligned}$$

Now define $\psi : FIRZ(x_1, \cdots, x_n) \longmapsto U$ to be the unique homomorphism such that $\psi(x_i) = u_i$. Since U is generated by $\{u_i : 1 \leq i \leq n\}$, ψ is an epimorphism. Since $Im\psi = U$ and $|Im\psi| = n + n^2 = |FIRZ(x_1, \cdots, x_n)|$, ψ must be one - to - one. Therefore ψ is an isomorphism.

Therefore $FIRZ(X) \in HSP\{\mathbf{S}\}$, that is $FIRZ(X) \in [\mathbf{S}]$.

Therefore $\mathcal{V} = \mathcal{IRZ}$.

Since $\mathcal{RZ} \vee \mathcal{Z}$ is a proper subvariety of $\mathcal{IRZ}$, it is interesting to consider the effect of the Rhodes expansion on free objects in $\mathcal{IRZ}$.

Let **F** be as constructed above.

Lemma 3.18 $\hat{\mathbf{F}}_X^{\mathcal{L}} \simeq \mathbf{F}$

Proof: From the Definition 2.5 we have

$$\begin{aligned} \hat{\mathbf{F}}_X^{\mathcal{L}} &= \langle\ (x)\ :x \in X\rangle \\ &= \{\ (x), ([x,y],y) : x,y \in X\ \} \end{aligned}$$

If we set $M' = \{\ ([x,y],y) : x,y \in X\ \}$ then

i. M' is the minimum ideal of $\hat{\mathbf{F}}_X^{\mathcal{L}}$.

ii. M' is a right zero semigroup.

iii. $\hat{\mathbf{F}}_X^{\mathcal{L}}/\rho_{M'}$ is a zero semigroup.

Thus $\hat{\mathbf{F}}_X^{\mathcal{L}}$ is an inflation of a right zero semigroup.

From Lemma 2.3 we have η_F is an epimorphism, hence we only need to show that η_F is one-to-one.

Clearly $\eta_F|_{\{(x)\}}$ is one-to-one and $\eta_F(x) \neq \eta_F([u,v],v)$ for every $x,u,v \in X$. Suppose $\eta_F([x,y],y) = \eta_F([u,v],v)$ then $[x,y] = [u,v]$. Therefore $x = u, y = v$. Thus $([x,y],y) = ([u,v],v)$. Therefore η_F is one-to-one.

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