# DELIGNE-LUSZTIG CHARACTER THEORY FOR GENERAL LINEAR GROUPS OF RANK 2 

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## Abstract

In this thesis after providing the necessary background in algebraic geometry and algebraic groups we expose the character theory of Deligne and Lusztig for finite groups of Lie type. Then we work out the theory in the case of $G L\left(2, \mathbb{F}_{q}\right)$, where $q$ is an odd prime power, and compare the results with the known character tables.

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## Notation

$\mathbb{Z} \quad$ the integers
$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ the rational, real and complex numbers
$\mathbb{Q}_{\ell} \quad$ the $\ell$-adic numbers
$\mathbb{F}_{q} \quad$ the finite field with $q$ elements
$|A| \quad$ the cardinality of a set $A$
$M^{\top} \quad$ the transpose of matrix $M$
${ }^{g} A \quad g A g^{-1}$
$A^{g} \quad g^{-1} A g$
$\operatorname{Irr}(A) \quad$ the set of irreducible characters of the finite group $A$
$C_{A}(B)$ the centralizer of $B$ in $A$
$N_{A}(B)$ the normalizer of $B$ in $A$
$(A, A)$ the commutator subgroup of $A$
$Z(A) \quad$ the center of the group $A$
$|A: B| \quad$ the index of subgroup $B$ in $A$
$K \quad$ algebraically closed field
$K[X] \quad$ the coordinate ring of the variety $X$ over $K$
$\mathfrak{S}_{n} \quad$ symmetric group on $n$ elements

## Preface

This thesis is concerned with the representation theory of a special class of finite groups called finite groups of Lie type (Examples include $G L\left(n, \mathbb{F}_{q}\right)$ and $S p\left(2 n, \mathbb{F}_{q}\right)$ ). These groups are of theoretical importance since they form the only infinite family of finite simple groups whose representation theory is rather nontrivial. The case of $P S L\left(2, \mathbb{F}_{p}\right)$ was already done by Frobenius himself back in 1896 [8]. The next general result was due to Green [10]. Green succeeded to combine several available techniques to construct the irreducible characters of the general linear groups: $G L\left(n, \mathbb{F}_{q}\right)$. In 1968 Srinivasan [17] did the same for the group $S p\left(4, \mathbb{F}_{q}\right)$. Based on these data MacDonald conjectured that there should be a correspondence between irreducible representations of a finite group of Lie type and characters of maximal tori in general position. Further evidence for this conjecture was provided when the character table of $G_{2}\left(\mathbb{F}_{q}\right)$ was computed by Chang and Ree in [5]. Finally the conjecture was proven by Deligne and Lusztig in [6]. Their construction relies on deep cohomological methods in algebraic geometry. The objective of this thesis is to provide a quick overview of the concepts and the construction introduced by Deligne and Lusztig while keeping the technicalities at a minimum. There are already a number of very good expositions of the subject ([4], [7]), and the author of these lines does not think that he can do much better in that regard. Therefore I have tried to express the theory as clearly as possible while avoiding the technical proofs that might not help the understanding of the subject.

The representations we consider are always over an algebraically closed field $K$ of characteristic zero. The usual choice is $K=\mathbb{C}$, however we take $K=\overline{\mathbb{Q}}_{\ell}$, this gives the same result since $\overline{\mathbb{Q}}_{\ell}$ is isomorphic to $\mathbb{C}$.

The reader should be familiar with representation theory of finite groups, (see [15] and [9]) and basics concepts of algebraic geometry.

## Part I

## Tools

## Chapter 1

## Algebraic Geometry

### 1.1 Affine Varieties over Finite Fields

Fix a prime $p$ and let $q=p^{m}$ where $m \geq 1$ is an integer.
Definition 1.1. An affine variety $X$ over $\overline{\mathbb{F}}_{p}$ is said to have an $\mathbb{F}_{q}$-structure (or to be defined over $\mathbb{F}_{q}$ ) if there exist an $\mathbb{F}_{q}$-subalgebra $A_{0}$ of $A=\overline{\mathbb{F}}_{p}[X]$ of finite type over $\mathbb{F}_{q}$ (i.e. it is finitely generated as a $\mathbb{F}_{q}$-module), such that $A \cong A_{0} \otimes_{\mathbb{F}_{q}} \overline{\mathbb{F}}_{p}$. If $X$ and $Y$ are affine varieties defined over $\mathbb{F}_{q}$ by $\mathbb{F}_{q}$-subalgebras $A_{0}, B_{0}$ a morphism $\phi: X \rightarrow Y$ is a $\mathbb{F}_{q}$-morphism if there is a homomorphism of $\mathbb{F}_{q}$-algebras $\phi_{0}^{*}: B_{0} \rightarrow A_{0}$ such that the homomorphism $\phi^{*}: B \rightarrow A$ defining $\phi$ is $\phi_{0}^{*} \otimes 1$.

Definition 1.2. Suppose $X$ is an affine variety with an $\mathbb{F}_{q}$-structure. This means that we can write $\overline{\mathbb{F}}_{p}[X]=A \cong A_{0} \otimes_{\mathbb{F}_{q}} \overline{\mathbb{F}}_{p}$. The homomorphism $a \otimes \lambda \mapsto a^{q} \otimes \lambda$ of $A$ into $A$ defines a $\mathbb{F}_{q}$-morphism $F: X \rightarrow X$. This is called the Frobenius morphism corresponding to an $\mathbb{F}_{q}$-structure $A_{0}$. It is evident that every $\mathbb{F}_{q}$-structure defines a Frobenius morphism. We also define $X^{F}$ to be the set of fixed points of $F$.

We list some useful properties of Frobenius morphisms:
Proposition 1.1 ([7] 3.6). Let $X$ be an affine variety defined over $\overline{\mathbb{F}}_{p}$ with an $\mathbb{F}_{q}$-structure and the associated Frobenius morphism $F$.
(i) Let $\phi$ be an automorphism of $X$ such that $(\phi F)^{n}=F^{n}$ for some positive integer $n$, then $\phi F$ is the Frobenius morphism corresponding to some $\mathbb{F}_{q}$-structure of $X$.
(ii) If $F^{\prime}$ is another Frobenius morphism corresponding to an $\mathbb{F}_{q}$-structure over $X$, there exist a positive integer $n$ such that $F^{n}=F^{\prime n}$.
(iii) $F^{n}$ is the Frobenius morphism corresponding to some $\mathbb{F}_{q^{n}}$-structure over $X$.
(iv) Any closed subvariety of a variety defined over $\mathbb{F}_{q}$ is defined over a finite extension of $\mathbb{F}_{q}$. Any morphism from a variety defined over $\mathbb{F}_{q}$ to another one is defined over a finite extension of $\mathbb{F}_{q}$.
(v) The F-orbits in the set of points of $X$ and the set $X^{F}$ of fixed points of $F$ are finite.

## $1.2 \quad$-adic Cohomology

Let $X$ be an affine algebraic variety over the field $\overline{\mathbb{F}}_{p}$ and $\ell$ be a prime number different from $p$. One can associate ([1], [2], [3]) to $X$ finite dimensional $\overline{\mathbb{Q}}_{\ell}$-vector spaces $H_{c}^{i}\left(X, \overline{\mathbb{Q}}_{\ell}\right)$. They are is called the $i$ th $\ell$-adic cohomology group of $X$ with compact support. A good introduction to the subject is provided both in [4] and [7]. $H_{c}^{i}\left(X, \overline{\mathbb{Q}}_{\ell}\right)$ satisfy several properties which are essential for the construction of Deligne-Lusztig:

Proposition $1.2([7] 10.1) . H_{c}^{i}\left(X, \overline{\mathbb{Q}}_{\ell}\right)=0$ if $i \notin\{0, \cdots, 2 \operatorname{dim} X\}$.
Proposition 1.3 ([7] 10.2). Any finite morphism $f: X \rightarrow X$ induces a linear endomorphism of $H_{c}^{i}\left(X, \bar{Q}_{\ell}\right)$ for any $i$, and this correspondence is functorial; a Frobenius endomorphism induces an automorphism of this space.

In particular we have: $(f g)^{*}=g^{*} f^{*}$ for all $f, g \in \boldsymbol{A u t}(X)$. Thus the map $f \mapsto\left(f^{*}\right)^{-1}$ is a representation of the group $\operatorname{Aut}(X)$ in the module $H_{c}^{i}\left(X, \overline{\mathbb{Q}}_{\ell}\right)$.

Definition 1.3. Define the virtual vector space: $H_{c}^{*}\left(X, \overline{\mathbb{Q}}_{\ell}\right)=\sum_{i}(-1)^{i} H_{c}^{i}\left(X, \overline{\mathbb{Q}}_{\ell}\right)$, and let $g \in \operatorname{Aut}(X)$ be of finite order. We define the Lefschetz number of $g$ on $X$ as:

$$
\mathcal{L}(g, X)=\operatorname{trace}\left(\left(g^{*}\right)^{-1}, H_{c}^{*}\left(X, \bar{Q}_{\ell}\right)\right)
$$

Theorem 1.1 (Grothendieck Trace Formula [2], [7] 10.4). Let $F$ be the Frobenius morphism associated to some $\mathbb{F}_{q}$-structure on $X$. Then:

$$
\left|X^{F}\right|=\operatorname{trace}\left(F^{*}, H_{c}^{*}\left(X, \overline{\mathbb{Q}}_{\ell}\right)\right)
$$

Corollary 1.1 ([7] 10.5). Let $X$ be an affine variety with an $\mathbb{F}_{q}$-structure and associated Frobenius morphism $F$. Suppose $g \in \mathbf{A u t}(X)$ is an $F$-stable automorphism of finite order. Then we have: $\mathcal{L}(g, X)=f(z)_{z=\infty}$, where $f(z)$ is the formal series: $-\sum_{n=1}^{\infty} \mid X^{F^{n} g^{-1} \mid z^{n}}$.

Proof. Since $g$ is of finite order by Proposition 1.1 for every $n, F^{n} g^{-1}$ is also a Frobenius morphism $X$. We apply Grothendieck's Trace Formula to $F^{n} g^{-1}$ :

$$
\begin{aligned}
\left|X^{F^{n} g^{-1}}\right| & =\operatorname{trace}\left(\left(F^{n} g^{-1}\right)^{*}, H_{c}^{*}\left(X, \overline{\mathbb{Q}}_{\ell}\right)\right) \\
& =\operatorname{trace}\left(\left(g^{-1}\right)^{*}\left(F^{n}\right)^{*}, H_{c}^{*}\left(X, \overline{\mathbb{Q}}_{\ell}\right)\right) \\
& =\operatorname{trace}\left(\left(g^{*}\right)^{-1}\left(F^{*}\right)^{n}, H_{c}^{*}\left(X, \overline{\mathbb{Q}}_{\ell}\right)\right) .
\end{aligned}
$$

Since $g$ is an $F$-stable automorphism, $F$ and $g$ commute as morphism of varieties. Hence the maps $F^{*}$ and $\left(g^{*}\right)^{-1}$ also commute and so there is a basis of $\oplus_{j} H_{c}^{j}\left(X, \overline{\mathbb{Q}}_{\ell}\right)$ in which both have triangular form. Let $\lambda_{1}, \cdots, \lambda_{k}$ be the eigenvalues of $F^{*}$ and $\mu_{1}, \cdots, \mu_{k}$ be those of $\left(g^{*}\right)^{-1}$ and let $\varepsilon_{j}= \pm 1$ be the sign of $H_{c}^{j}\left(X, \overline{\mathbb{Q}}_{\ell}\right)$ in $H_{c}^{*}\left(X, \overline{\mathbb{Q}}_{\ell}\right)$ in which $\lambda_{j}$ and $\mu_{j}$ are eigenvalues. Now we have:

$$
\begin{aligned}
f(z) & =-\sum_{n=1}^{\infty}\left|X^{F^{n} g^{-1}}\right| z^{n} \\
& =-\sum_{n=1}^{\infty}\left(\sum_{j=1}^{k} \varepsilon_{j} \mu_{j} \lambda_{j}^{n}\right) z^{n} \\
& =\sum_{j=1}^{k} \varepsilon_{j} \mu_{j} \frac{-\lambda_{j} z}{1-\lambda_{j} z}
\end{aligned}
$$

This shows that $f(z)_{z=\infty}$ is equal to $\sum_{j=1}^{k} \varepsilon_{j} \mu_{j}$ which by definition is $\mathcal{L}(g, X)$.
With Corollary 1.1 in hand we can prove many properties of Lefschetz numbers. We prove two which we need:

Corollary 1.2. The Lefschetz number $\mathcal{L}(g, X)$ is a rational integer independent of $\ell$.
Proof. The independence of $\ell$ is a clear consequence of Corollary 1.1. Moreover the proof shows that $f(z) \in \overline{\mathbb{Q}}_{\ell}(z)$. But since it is a formal series with integer coefficients, it has to be in $\mathbb{Q}(z)$. So we have $\mathcal{L}(g, X) \in \mathbb{Q}$. But a Lefschetz number is an algebraic integer since it is the character value of the finite group generated by $g$. Thus we have $\mathcal{L}(g, X) \in \mathbb{Z}$.

Proposition 1.4. Assume that $X$ is a finite set and $g: X \rightarrow X$ is an automorphism. Then $\mathcal{L}(g, X)=\left|X^{g}\right|$.

Proof. We can assume that the Frobenius $F$ acts on $X$ as the identity, otherwise we can replace it by some power. This implies:

$$
\mathcal{L}(g, X)=\left(-\sum_{n=1}^{\infty}\left|X^{g^{-1}}\right| z^{n}\right)_{z=\infty}=\left|X^{g^{-1}}\right|\left(\frac{-z}{1-z}\right)_{z=\infty}=\left|X^{g}\right| .
$$

## Chapter 2

## Algebraic Groups

### 2.1 Basics

An algebraic group over an algebraically closed field $K$ is an affine variety $G$ with a group structure, such that the maps $\mu: G \times G \rightarrow G$ and $i: G \rightarrow G$ given by multiplication and inversion are both morphisms of varieties.

Example 2.1. The affine line $\mathbb{A}^{1}(K)$ with addition is an algebraic group. It is usually called the additive group and is denoted by $\mathbb{G}_{a}$. The open subset of nonzero points of $\mathbb{A}^{1}(K)$ is a group with multiplication. We can realize this as an algebraic variety:

$$
\{X Y-1=0\} \subset \mathbb{A}^{2}(K)
$$

This group is called the multiplicative group and is denoted by $\mathbb{G}_{m}$.
Example 2.2. We can identify the set $M(n, K)$ of all $n$ by $n$ matrices with entries in $K$ with $\mathbb{A}^{n^{2}}(K)$. The general linear group, $G L(n, K)$, is the open set defined by the equation $\operatorname{det} A \neq 0$, this set is also an algebraic variety:

$$
\{Z(\operatorname{det} A)-1=0\} \subset \mathbb{A}^{n^{2}+1}(K) .
$$

(Actually with a similar construction it one can show that the complement of an affine hypersurface is an affine variety itself). Matrix multiplication and inversion are morphisms of varieties.

### 2.1.1 Identity Component

Being an algebraic group is a strong assumption, it forces the algebraic groups to behave nicely as an affine variety:

Theorem 2.1 ([11] §7.3). Let $G^{\circ}$ be the connected component of the identity in the algebraic group $G$. Then:
(i) $G^{\circ}$ is a normal subgroup of finite index in $G$ whose cosets are the connected as well as the irreducible, components of $G$.
(ii) Every closed subgroup of finite index contains $G^{\circ}$.

### 2.1.2 Homomorphisms

A homomorphism of algebraic groups (or homomorphism for short) is a group homomorphism which is also a morphism of varieties.

Proposition 2.1 ([11] §7.4). Let $\phi: G \rightarrow G^{\prime}$ be a homomorphism of algebraic groups. Then
(i) $\operatorname{Ker}(\phi)$ is a closed subgroup of $G$.
(ii) $\operatorname{Im}(\phi)$ is a closed subgroup of $G^{\prime}$.
(iii) $\phi\left(G^{\circ}\right)=\phi(G)^{\circ}$.
(iv) $\operatorname{dim}(G)=\operatorname{dim}(\operatorname{Ker}(\phi))+\operatorname{dim}(\operatorname{Im}(\phi))$.

### 2.1.3 Closed Subgroups

Not every subgroup of an algebraic group is an algebraic group itself. However if a subgroup is closed under the Zarisky topology then it has the structure of an algebraic group. Therefore it is useful to know which constructions in abstract group theory give us closed subgroups. The following is true:

Theorem $2.2([11] \S 8.2) . C_{G}(H), N_{G}(H)$ are closed subgroups of $G$ if $H$ is a closed subgroup of $G$.

### 2.1.4 Linearization

Closed subgroups of $G L(n, K)$ are called linear algebraic groups. Here is some examples of linear algebraic groups:

- The group $S L(n, K)$ consists of matrices with determinant 1 in $G L(n, K)$.
- The group $D(n, K)$ of diagonal matrices in $G L(n, K)$.
- The group $T(n, K)$ of upper triangular matrices in $G L(n, K)$.
- The group $U(n, K)$ of unipotent triangular matrices, the subgroup of $T(n, K)$ whose elements have diagonal entries equal to 1 .

Every linear algebraic group is an algebraic group. The converse is also true:
Theorem 2.3 ([11] §8.6). Let $G$ be an algebraic group. Then $G$ is isomorphic to a closed subgroup of some $G L(n, K)$.

### 2.1.5 Characters and Representations

Definition 2.1. Let $V$ be a finite dimensional vector space over $K$. A rational representation of $G$ in $V$ is a homomorphism of algebraic groups $r: G \rightarrow G L(V)$.

Definition 2.2. A character of an algebraic group is a homomorphism from $G$ to $\mathbb{G}_{m}$. One example is: det : $G L(n, K) \rightarrow \mathbb{G}_{m}$. The set of characters is denoted by $X^{*}(G)$ and it has a natural structure of an abelian group also note that $X^{*}(G)$ can be viewed as a subgroup of $K[G]$. A homomorphism $\gamma: \mathbb{G}_{m} \rightarrow G$ is called a cocharacter of $G$. We denote by $X_{*}(G)$ the set of cocharacters, if $G$ is commutative $X_{*}(G)$ also has a structure of an abelian group.

Characters arise in connection with rational representations. If $r: G \rightarrow G L(V) \mathrm{a}$ rational representation of $G$ then for each $\chi \in X^{*}(G)$, define:

$$
V_{\chi}=\{v \in V: \forall g \in G r(g) v=\chi(g) v\}
$$

obviously this is a $G$-stable subspace of $V$ (possibly 0 ). The characters corresponding to nonzero subspaces of $V$ are called the weights of $G$ in $V$; a nonzero vector in $V_{\chi}$ is called a weight vector. Conversely if $v$ is any nonzero vector in $V$ which spans a $G$-stable subspace of $V$, then it is clear that $g \bullet v=\chi(g) v$ defines a character of $G$.

Lemma 2.1 ([11] §11.4). Let $r: G \rightarrow G L(V)$ be a rational representation. Then the subspaces $V_{\chi}, \chi \in X^{*}(G)$, are linearly independent; in particular, only finitely many of them are nonzero.

Proof. Otherwise choose $n$ minimal nonzero vectors $v_{i} \in V_{\chi_{i}}$ for distinct $\chi_{i}, 1 \leq i \leq n$ such that $\sum_{i} v_{i}=0$. Note that since we assumed that $v_{i}$ are nonzero $n \geq 2$. Since the $\chi_{i}$ are distinct there is some $x \in G$ such that $\chi_{1}(x) \neq \chi_{2}(x)$. On the other hand we have:

$$
0=r(x)\left(\sum_{i} v_{i}\right)=\sum_{i} \chi_{i}(x) v_{i}
$$

Therefore we have:

$$
\sum_{i} \chi_{1}(x)^{-1} \chi_{i}(x) v_{i}=0
$$

The coefficient of $v_{2}$ is different from 1 so we can subtract the above equation from the original one to obtain a nontrivial dependence involving at most $n-1$ characters. This contradicts the choice of $n$.

### 2.1.6 Jordan Decomposition

Let $V$ be a finite dimensional vector space over $K$. An endomorphism $E$ of $V$ is semisimple if there is a basis of $V$ consisting of eigenvectors of $E$. We say $E$ is nilpotent if $E^{n}=0$ for some $n$ and unipotent if $E-1$ is nilpotent. Clearly if $E$ is nilpotent, then its minimum polynomial divides $T^{m}$ for some $m$, and so the eigenvalues are all zero. From linear algebra, we know that the converse is also true, and so $E$ is unipotent if and only if its eigenvalues in $K$ all equal 1.

Theorem 2.4 ([14] §10). Let $G$ be an algebraic group over a perfect field $k$. For any $g \in G(k)$ there exist unique elements $s, u \in G(k)$ such that:
(i) $g=s u=u s$
(ii) for all representations $\phi: G \rightarrow G L(V), \phi(s)$ is semisimple and $\phi(u)$ is unipotent.
$s$ and $u$ are called the semisimple and unipotent parts of $g$, and $g=s u$ is the Jordan decomposition of $g$. Also $G_{u}\left(G_{s}\right)$ denotes the set of all unipotent (semisimple) elements in $G$.

### 2.2 Diagonalizable Groups and Tori

Definition 2.3. An algebraic group isomorphic to some $D(n, K)$ is called a torus. An algebraic group is diagonalizable if it is isomorphic to a closed subgroup of some $D(n, K)$.

### 2.2.1 Characters of Tori

We will show that tori have many characters. The coordinate functions:

$$
\chi_{i}: \operatorname{diag}\left(a_{1}, \cdots, a_{n}\right) \mapsto a_{i}
$$

are all characters of $D(n, K)$. We want to show that $X^{*}(D(n, K))$ is a free abelian group of rank $n$.

Lemma 2.2 ([11] §16.1). Let $G$ be a group, $X$ the set of all homomorphisms from $G$ to $K^{\times}$. Then $X$ is a linearly independent subset of all $K$-valued functions on $G$.

Proof. Suppose not and let $\chi_{1}, \cdots, \chi_{n} \in X$ be linearly dependent with $n$ as small as possible:

$$
\sum_{i=1}^{n-1} a_{i} \chi_{i}+\chi_{n}=0
$$

This means that $a_{i}$ are nonzero and $n \geq 2$. Since $\chi_{1} \neq \chi_{n}$ we can find $y \in G$ such that $\chi_{1}(y) \neq \chi_{n}(y)$. For arbitrary $x \in G$ we have:

$$
\sum_{i=1}^{n-1} a_{i} \chi_{i}(x y)+\chi_{n}(x y)=0
$$

which gives us:

$$
\begin{equation*}
\sum_{i=1}^{n-1} a_{i} \chi_{i}(x) \chi_{i}(y)+\chi_{n}(x) \chi_{n}(y)=0 . \tag{2.1}
\end{equation*}
$$

On the other hand we have:

$$
\chi_{n}(y)\left(\sum_{i=1}^{n-1} a_{i} \chi_{i}(x)+\chi_{n}(x)\right)=0
$$

which implies:

$$
\begin{equation*}
\sum_{i=1}^{n-1} a_{i} \chi_{i}(x) \chi_{n}(y)+\chi_{n}(x) \chi_{n}(y)=0 \tag{2.2}
\end{equation*}
$$

Now if we subtract the equations 2.1 and 2.2 we are left with:

$$
\sum_{i=1}^{n-1} a_{i}\left(\chi_{i}(y)-\chi_{n}(y)\right) \chi_{i}=0
$$

where not all coefficients are zero. This contradicts the minimality of $n$.
$D(n, K)$ can be considered as the open set in $\mathbb{A}^{n}(K)$ consisting of the points without zero in their coordinates. This implies:

$$
K[D(n, K)]=K\left[\chi_{1}, \chi_{1}^{-1}, \cdots, \chi_{n}, \chi_{n}^{-1}\right] .
$$

The monomials $\chi_{1}^{a_{1}} \cdots \chi_{n}^{a_{n}}$ with $a_{i} \in \mathbb{Z}$ are all characters of $D(n, K)$ and they form a $K$-basis of $K[D(n, K)]$. Let $\Xi$ be a character of $D(n, K)$ which is not a monomial character. $\Xi$ is a regular function therefore it belongs to $K[D(n, K)]$ but the monomials form a K-basis for $K[D(n, K)]$, we have that $\Xi$ is a K-linear combination of monomials. However, then the set consisting of the monomial characters union $\{\Xi\}$ is not K-linearly independent, contradicting Lemma 2.2. So $X^{*}(D(n, K))$ is an abelian group generated by $\chi_{i}$. This abelian group is in fact free to see this suppose there is a character $\chi_{1}^{a_{1}} \cdots \chi_{n}^{a_{n}}$ has finite order, i.e. there exist an integer $M \geq 1$ such that:

$$
\left(\chi_{1}^{a_{1}} \cdots \chi_{n}^{a_{n}}\right)^{M}=1
$$

Let $\operatorname{diag}(1, \cdots, 1, \lambda) \in D(n, K)$, where $\lambda$ is an arbitrary element in $K$. Now we have:

$$
1=1(\operatorname{diag}(1, \cdots, 1, \lambda))=\left(\chi_{1}^{a_{1}} \cdots \chi_{n}^{a_{n}}\right)^{M}(\operatorname{diag}(1, \cdots, 1, \lambda))=\lambda^{a_{n} M}
$$

Since $\lambda$ was arbitrary we conclude that $a_{n}=0$. Similarly we can show that all $a_{i}$ are zero hence $X^{*}(D(n, K)) \cong \mathbb{Z}^{n}$.

### 2.2.2 Duality of $X^{*}$ and $X_{*}$

A cocharacter $\gamma: \mathbb{G}_{m} \rightarrow D(n, K)$ composed with a character $\chi: D(n, K) \rightarrow \mathbb{G}_{m}$ gives a homomorphism of algebraic groups: $(\chi \circ \gamma): \mathbb{G}_{m} \rightarrow \mathbb{G}_{m}$. This homomorphism is an element of $X^{*}\left(\mathbb{G}_{m}\right) \cong \mathbb{Z}$, so by 2.2.1 there is an integer denoted by $\langle\chi, \gamma\rangle$ such that $(\chi \circ \gamma)(a)=a^{(\chi, \gamma)}$. With this pairing we have the following isomorphisms:

$$
\begin{aligned}
& X^{*}(D(n, K)) \cong \operatorname{Hom}\left(X_{*}(D(n, K)), \mathbb{Z}\right) \chi \mapsto\langle\chi, \cdot\rangle \\
& X_{*}(D(n, K)) \cong \boldsymbol{\operatorname { H o m }}\left(X^{*}(D(n, K)), \mathbb{Z}\right) \gamma \mapsto\langle\cdot, \gamma\rangle
\end{aligned}
$$

We will prove the second isomorphism the proof of the first is similar. By the given map we have an injection of $X_{*}(D(n, K)) \hookrightarrow \operatorname{Hom}\left(X^{*}(D(n, K), \mathbb{Z})\right.$. Now we will prove that is surjective. Let $f \in \operatorname{Hom}\left(X^{*}(D(n, K), \mathbb{Z})\right.$ be an arbitrary element. Set $a_{i}:=f\left(\chi_{i}\right)$. Now consider the cocharacter $\gamma_{f}$ which is defined as follows:

$$
\gamma_{f}(\lambda)=\operatorname{diag}\left(\lambda^{a_{1}}, \cdots, \lambda^{a_{n}}\right)
$$

Now we see that:

$$
\lambda^{\left(\chi_{i}, \gamma_{f}\right\rangle}=\left(\chi_{i} \circ \gamma_{f}\right)(\lambda)=\chi_{i}\left(\boldsymbol{\operatorname { d i a g }}\left(\lambda^{a_{1}}, \cdots, \lambda^{a_{n}}\right)\right)=\lambda^{a_{i}}
$$

So $\left\langle\chi_{i}, \gamma_{f}\right\rangle=a_{i}$. Therefore $f$ is represented by $\left\langle\cdot, \gamma_{f}\right\rangle$. Because of the two isomorphisms we say the abelian groups $X^{*}(D(n, K))$ and $X_{*}(D(n, K))$ are dual abelian groups.

### 2.2.3 Classification

Theorem 2.5 ([11] §16.2). Let $D$ be a diagonalizable group. Then $D=D^{\circ} \times H$, where $D^{\circ}$ is a torus and $H$ is a finite group of order prime to $p$, where $p$ is the characteristic exponent of $K$. In particular, a connected diagonalizable group is a torus.

Theorem 2.6 ([11] §16.3). Let $D$ be a diagonalizable subgroup of an algebraic group $G$. Then: $N_{G}^{\circ}(D)=C_{G}^{\circ}(D)$.

### 2.3 Solvable Groups

Suppose $A, B$ are subgroups of an abstract group $G$. The subgroup $(A, B)$ generated by all elements of the form $a b a^{-1} b^{-1}$ for $a \in A, b \in B$ is called the commutator of $A$ and $B$. Generally the commutator subgroup of two closed subgroups of an algebraic group is not closed itself. However the following is true:

Proposition 2.2 ([11] §17.2). Let $A, B$ be closed subgroups of an algebraic group $G$.
(i) If $A$ is connected, then $(A, B)$ is closed and connected.
(ii) If $A, B$ are normal in $G$, then $(A, B)$ is closed and normal in $G$.

An immediate consequence is that the derived group of a connected algebraic group $G$ is closed, normal and connected. We say an algebraic group is solvable if its derived series terminates in the trivial group. This series is defined as follows:

$$
\mathcal{D}^{0} G=G, \mathcal{D}^{i+1} G=\left(\mathcal{D}^{i} G, \mathcal{D}^{i} G\right)
$$

Lemma 2.3. An algebraic group $G$ is solvable if and only if there is a sequence of algebraic groups:

$$
G \supset G_{1} \supset \cdots \supset G_{n}=\{e\},
$$

with $G_{i+1}$ normal in $G_{i}$ for each $i$ and $G_{i} / G_{i+1}$ is commutative.
Proof. If $G$ is solvable, then the derived series satisfies the above conditions. For the converse note that $G_{1} \supset \mathcal{D} G$ therefore $G_{2} \supset \mathcal{D}^{2} G, \ldots$

Example 2.3. $T(n, K)$ is solvable. By the above lemma we have to build a normal series with abelian quotients. Set $G_{1}=U(n, K)$. Now let $G_{r}$ be the subgroup of $G_{1}$ such that $a_{i j}=0$ for $0<j-i \leq r$. Now the map:

$$
a_{i j} \mapsto\left(a_{1, r+2}, \cdots, a_{i, r+i+1}, \cdots\right)
$$

from $G_{r}$ to onto $\mathbb{G}_{a} \times \mathbb{G}_{a} \times \cdots$ is a homomorphism with $G_{r+1}$ as kernel. The following example is illuminating:

$$
\left\{\left[\begin{array}{ccc}
* & * & * \\
0 & * & * \\
0 & 0 & *
\end{array}\right]\right\} \supset\left\{\left[\begin{array}{ccc}
1 & * & * \\
0 & 1 & * \\
0 & 0 & 1
\end{array}\right]\right\} \supset\left\{\left[\begin{array}{ccc}
1 & 0 & * \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]\right\} \supset\left\{\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]\right\}
$$

The following theorem is a kind of converse. Every connected solvable algebraic group is a closed subgroup of some $T(n, K)$ for some $n$ :

Theorem 2.7 ([11] §17.6 Lie-Kolchin). Let $G$ be a connected solvable subgroup of $G L(V)$ where $V$ is a nonzero finite dimensional vector space over $K$. Then $G$ has a common eigenvector in $V$.

In a connected solvable algebraic group we have the following decomposition:
Theorem 2.8 ([11] §19.3). Let $G$ be a connected solvable algebraic group. then:
(i) $G_{u}$ is a closed connected normal subgroup of $G$ including $(G, G)$, and $G_{u}$ has a chain of closed connected subgroups, each normal in $G$ and each of codimension one in the next.
(ii) The maximal tori of $G$ are conjugate in $G$, and if $T$ is one of these then we have the semidirect product: $G=T G_{u}$.

### 2.4 Borel Subgroups

A Borel subgroup of $G$ is a closed connected solvable subgroup of $G$, which is maximal for these properties. Every algebraic group has a Borel subgroup since the set of closed connected solvable subgroups of $G$ has a maximal element.

Example 2.4. $T(n, K)$ is a Borel subgroup of $G L(n, K)$. It is closed and connected and we have already shown that it is solvable. Now suppose $A$ is a closed connected solvable subgroup of $G L(n, K)$, we wish to show that it is isomorphic to some subgroup of $T(n, K)$, this would show the maximality of $T(n, K)$. First note that $G L(n, K)=G L\left(K^{n}\right)$. Now by Theorem 2.7, $A$ has a common eigenvector $v$ in $K^{n}$. Now consider $W=V /\langle v\rangle . A$ acts on $W$ and $\operatorname{dim} W=n-1$ by induction we can assume that there is a basis for $W$ in which the elements of $A$ become upper triangular. This basis combined with $v$ gives us a basis of $V$ with respect to which the elements of $A$ are upper triangular. Since change of a basis is conjugation by a fixed element this imples that there exist $g \in G L(n, K)$ such that ${ }^{9} A \subset T(n, K)$. So $A$ is isomorphic to some subgroup of $T(n, K)$ this shows that $T(n, K)$ is a Borel subgroup.

Theorem 2.9 ([11] §21.3). Let $B$ be any Borel subgroup of $G$. Then $G / B$ is a projective variety, and all other Borel subgroups are conjugate to $B$.

A maximal torus of $G$ is a torus of $G$ that is not strictly contained in another torus. The rank of a group is the dimension of its maximal torus. By virtue of the following result this is well defined:

Corollary 2.1 ([11] §21.3). The maximal tori of $G$ are those of the Borel subgroups of $G$, and are all conjugate.

Remark 2.1. Note that Theorem 2.8 gives a semidirect decomposition of Borel subgroups. Given a maximal torus $T$ in a Borel subgroup $B$ we can write a semidirect product $B=T B_{u}$ where $B_{u}$ is the unipotent part of $B$, we also denote it by $U_{B}$ or just $U$.

Theorem 2.10 ([11] §22.3). Let $S$ be a torus in $G$ then $C_{G}(S)$ is connected.
Let $S$ be any torus in $G$ the group $W(G, S)=N_{G}(S) / C_{G}(S)$ is called the Weyl group of $G$ relative to $S . W(G, S)$ is a finite group, to see this note that by Theorem 2.6 we have: $N_{G}^{\circ}(S)=C_{G}^{\circ}(S)$ on the other hand Theorem 2.10 shows that the centralizer of any torus is
connected therefore the $W(G, S)$ is the quotient $N_{G}(S) / N_{G}^{\circ}(S)$ and so it has to be finite by Theorem 2.1. Since all maximal tori are conjugate, their Weyl groups are isomorphic, this group is simply called the Weyl group of $G$ and is usually denoted by $W(G)$ or $W$ when there is no ambiguity about the underlying group.

Theorem 2.11 ([11] §23.1). Let $B$ be a Borel subgroup of $G$. Then $N_{G}(B)=B$.
Let $\mathcal{B}$ be the set of all Borel subgroups and fix $B_{0} \in \mathcal{B}$. the map $x B_{0} \mapsto{ }^{x} B_{0}$ is bijection from $G / B_{0}$ onto $\mathcal{B}$. Using by Theorem 2.11 we have:

$$
x B_{0}=y B_{0} \Leftrightarrow x^{-1} y \in B_{0} \Leftrightarrow x^{-1} y \in N_{G}\left(B_{0}\right) \Leftrightarrow{ }^{x^{-1} y} B_{0}=B_{0} \Leftrightarrow{ }^{x} B_{0}={ }^{y} B_{0}
$$

which shows that the map is both well defined and 1-1. Being surjective is obvious in view of Theorem 2.9. Therefore the set of all Borel subgroups has the structure of a projective variety.

Example 2.5. The variety of all Borel subgroups of $G L(2, K)$ is isomorphic to $\mathbb{P}^{1}(K)$. To see this note that $G$ has a natural transitive action on the set of lines in $K^{2} . B_{0}=T(2, K)$ is the stabilizer of the line [ ${ }_{0}^{*}$ ]. Now every other Borel subgroup can be written as ${ }^{x} B_{0}$ for some $x \in G L(2, K)$ and it fixes the line $x\left[{ }_{0}^{*}\right]$. So we can establish a bijection between $G L(2, K) / T(2, K)$ and $\mathbb{P}^{1}(K)$.

### 2.5 Reductive Groups

The identity component of the intersection of all Borel subgroups is called the radical of $G$ and is denoted by $R(G)$. The unipotent radical of $G\left(R_{u}(G)\right)$ is the unipotent subgroup of the radical. We call a non-trivial connected group semisimple if its radical is trivial and reductive if its unipotent radical is trivial.

Being reductive is a strong assumption. We list some of its consequences:
Theorem 2.12 ([11] §26.2). Let $G$ be reductive and $S$ any subtorus of $G$. Then $C_{G}(S)$ is reductive. If $S$ is maximal then $C_{G}(S)=S$.

Theorem 2.13 ([11] §26.2). Let $G$ be reductive. For any Borel subgroup $B$ containing a fixed torus $T$ there exist a Borel subgroup $B^{-}$such that $B \cap B^{-}=T$.

Theorem 2.14 (Bruhat Decomposition [11] §28.3). A reductive group $G$ can be written as the disjoint union of double cosets: $G=\coprod_{w \in W} B \dot{w} B$, where $B$ is a fixed Borel subgroup and $W$ is the Weyl group of $G$. Moreover $B \dot{w} B=B \dot{v} B$ if and only if $v=w$ in $W$.

Moreover we can classify all connected reductive goups over an arbitrary algebraically closed field $K$. To do this we need to define root datum:

Definition 2.4 (Root Datum). A root datum is a quadruple $\Psi=\left(X, R, X^{\vee}, R^{\vee}\right)$, where:
(a) $X$ and $X^{\vee}$ are free abelian groups of finite rank, in duality by a pairing $X \times X^{\vee} \rightarrow \mathbb{Z}$ denoted by $\langle$,$\rangle .$
(b) $R$ and $R^{\vee}$ are finite subsets of $X$ and $X^{\vee}$, and we are given a bijection $\alpha \mapsto \alpha^{\vee}$ of $R$ onto $R^{\vee}$ such that: $\left\langle\alpha, \alpha^{\vee}\right\rangle=2$ for $\alpha \in R$.
(c) For $\alpha \in R$ we define endomorphisms $s_{\alpha}$ and $s_{\alpha}^{\vee}$ of $X$ and $X^{\vee}$ by:

$$
\begin{array}{ll}
\forall x \in X: & s_{\alpha}(x)=x-\left\langle x, \alpha^{\vee}\right\rangle \alpha \\
\forall y \in X^{\vee}: & s_{\alpha}^{\vee}(y)=y-\langle\alpha, y\rangle \alpha^{\vee}
\end{array}
$$

Now if $\alpha \in R$ then $s_{\alpha}(R)=R$ and $s_{\alpha}^{\vee}\left(R^{\vee}\right)=R^{\vee}$.
From condition (b) we have $s_{\alpha}^{2}=1$ and $s_{\alpha}(\alpha)=-\alpha$. The Weyl group $W=W(\Psi)$ of $\Psi$ is the group of automorphisms of $X$ generated by the set $\left\{s_{\alpha}: \alpha \in R\right\}$. By the symmetry between $X$ and $X^{\vee}$ we see that $\Psi^{\vee}=\left(X^{\vee}, R^{\vee}, X, R\right)$ is a root data too. It is called the dual root datum. $R$ is the set of roots of $\Psi$ and $R^{\vee}$ is the set of coroots of $\Psi$. There is a natural notion of isomorphism of root data, given two isomorphic root data $\left(X, R, X^{\vee}, R^{\vee}\right),\left(X_{1}, R, X_{1}^{\vee}, R_{1}^{\vee}\right)$ means that we have an isomorphism $X_{1} \cong X$ that maps $R_{1}$ into $R$ and its dual maps $R^{\vee}$ onto $R_{1}^{\vee}$.

Example 2.6. Take $X=X^{\vee}=\mathbb{Z}^{n}$ with the standard pairing and $R=R^{\vee}=\left\{e_{i}-e_{j}, i \neq j\right\}$ where $e_{i}$ denotes the usual orthonormal unit vectors. Note that:

$$
\begin{aligned}
\left\langle e_{i}-e_{j}, e_{i}-e_{j}\right\rangle & =\left\langle e_{i}, e_{i}-e_{j}\right\rangle+\left\langle e_{j}, e_{i}-e_{j}\right\rangle \\
& =\left\langle e_{i}, e_{i}\right\rangle-\left\langle e_{i}, e_{j}\right\rangle-\left\langle e_{j}, e_{i}\right\rangle+\left\langle e_{j}, e_{j}\right\rangle \\
& =2
\end{aligned}
$$

By $s_{i j}$ denote the endomorphism associated to $e_{i}-e_{j}$ we have:

$$
\begin{aligned}
s_{i j}\left(e_{k}\right) & =e_{k}-\left\langle e_{k}, e_{i}-e_{j}\right\rangle\left(e_{i}-e_{j}\right) \\
& =e_{k}-\left(\left\langle e_{k}, e_{i}\right\rangle-\left\langle e_{k}, e_{j}\right\rangle\right)\left(e_{i}-e_{j}\right) \\
& =e_{k}-\left(\delta_{k i}-\delta_{k j}\right)\left(e_{i}-e_{j}\right) \\
& = \begin{cases}e_{k} & k \neq i, j \\
e_{j} & k=i \\
e_{i} & k=j\end{cases}
\end{aligned}
$$

We see that $s_{i j}$ is just the reflection which fixes everything orthogonal to $e_{i}-e_{j}$. Also $s_{i j}(R)=R$ and $s_{i j}=s_{j i}$. The map $s_{i j} \mapsto(i j)$ gives us an isomorphism from $W$ to $\mathfrak{S}_{n}$.

Given an arbitrary connected reductive group $G$ and a maximal torus $T$ of $G$ we can construct a root datum: $\left(X^{*}(T), \Psi, X_{*}(T), \Psi^{\vee}\right)$ in a way that the Weyl group of the root datum and the Weyl group $W(T)$ are isomorphic. Since all maximal tori in $G$ are conjugate this root datum is independent of $T$. In fact the associated root data characterizes $G$ up to isomorphism and every possible root datum is the root data of some reductive group (See [16] 9.6.2 and 10.1.1). Here we give the example of the case when $G=G L(n, K)$

Example 2.7 (Root Datum of $G L(n, K)$ ). Let $G=G L(n, K)$ and $T=D(n, K)$. By our previous discussion we know that $\operatorname{Hom}\left(T, \mathbb{G}_{m}\right)$ and $\operatorname{Hom}\left(\mathbb{G}_{m}, T\right)$ are dual free abelian groups. Now $R \subset \boldsymbol{\operatorname { H o m }}\left(T, \mathbb{G}_{m}\right)$ is a finite set, they are given by:

$$
\alpha_{i j}: \operatorname{diag}\left(\lambda_{1}, \lambda_{2}, \cdots, \lambda_{n}\right) \mapsto \lambda_{i} \lambda_{j}^{-1}
$$

The coroots $R^{\vee} \subset \operatorname{Hom}\left(\mathbb{G}_{m}, T\right)$ are given by:

$$
\alpha_{i j}^{\vee}: \lambda \mapsto \operatorname{diag}\left(1, \cdots, 1, \lambda, 1, \cdots, 1, \lambda^{-1}, 1, \cdots, 1\right)
$$

where $\lambda$ is in the $i$ th column and $\lambda^{-1}$ is in the $j$ th column. Now we compute:

$$
\begin{aligned}
\lambda^{\left\langle\alpha_{i j}, \alpha_{i j}^{\vee}\right\rangle} & =\left(\alpha_{i j} \circ \alpha_{i j}^{\vee}\right)(\lambda) \\
& =\alpha_{i j}\left(\operatorname{diag}\left(1, \cdots, 1, \lambda, 1, \cdots, 1, \lambda^{-1}, 1, \cdots, 1\right)\right) \\
& =\lambda\left(\lambda^{-1}\right)^{-1}=\lambda^{2}
\end{aligned}
$$

Finally we have to compute the action of $s_{\alpha_{i j}}, s_{\alpha_{i j}}^{\vee}$. We have seen that the coordinate characters form a basis for $\operatorname{Hom}\left(T, \mathbb{G}_{m}\right)$. But first we compute the inner products:

$$
\begin{aligned}
\lambda^{\left\langle\chi_{k}, \alpha_{i j}^{\vee}\right\rangle} & =\left(\chi_{k} \circ \alpha_{i j}^{\vee}\right)(\lambda) \\
& =\chi_{k}\left(\operatorname{diag}\left(1, \cdots, 1, \lambda, 1, \cdots, 1, \lambda^{-1}, 1, \cdots, 1\right)\right) \\
& = \begin{cases}1 & k \neq i, j \\
\lambda & k=i \\
\lambda^{-1} & k=j\end{cases} \\
\Rightarrow\left\langle\chi_{k}, \alpha_{i j}^{\vee}\right\rangle & = \begin{cases}0 & k \neq i, j \\
1 & k=i \\
-1 & k=j\end{cases}
\end{aligned}
$$

Now we have:

$$
\begin{aligned}
s_{\alpha_{i j}}\left(\chi_{k}\right) & =\chi_{k}\left(\alpha_{i j}\right)^{-\left\langle\chi_{k}, \alpha_{i j}^{\vee}\right\rangle} \\
& =\chi_{k}\left(\chi_{i} \chi_{j}^{-1}\right)^{-\left\langle\chi_{k}, \alpha_{i j}^{\vee}\right\rangle} \\
& =\chi_{k} \chi_{i}^{-\left(\chi_{k}, \alpha_{i j}^{\vee}\right\rangle} \chi_{j}^{\left(\chi_{k}, \alpha_{i j}^{\vee}\right\rangle} \\
& = \begin{cases}\chi_{k} \chi_{i}^{0} \chi_{j}^{0}=\chi_{k} & k \neq i, j \\
\chi_{k} \chi_{k}^{-1} \chi_{j}^{1}=\chi_{j} & k=i \\
\chi_{k} \chi_{i}^{1} \chi_{k}^{-1}=\chi_{i} & k=j\end{cases}
\end{aligned}
$$

Therefore we see that that $s_{\alpha_{i j}}(R)=R$. The proof of $s_{\alpha_{i j}}^{\vee}\left(R^{\vee}\right)=R^{\vee}$ is similar. The calculations for the root datum of $G L(n, K)$ were the same as the calculations in Example 2.6. In fact the map $\chi_{i}, \chi_{i}^{\vee} \mapsto e_{i}$ is an isomorphism of root data. In particular this shows that the Weyl group of $G L(n, K)$ is $\mathfrak{S}_{n}$.

### 2.6 Finite groups of Lie type

Definition 2.5. An algebraic group over $\overline{\mathbb{F}}_{p}$ is said to be defined over $\mathbb{F}_{q}$ if it has an $\mathbb{F}_{q}$-structure such that the associated Frobenius morphism is a group homomorphism. The finite groups arising as fixed points of the Frobenius morphism are called finite groups of Lie type.

Example 2.8. Consider the group $G L(n, K)$ with the following $\mathbb{F}_{q}$-structure

$$
\mathbb{F}_{q}\left[T_{i j}, \operatorname{det}\left(T_{i j}\right)^{-1}\right] \otimes_{\mathbb{F}_{q}} \widetilde{\mathbb{F}}_{q} \cong \overline{\mathbb{F}}_{p}\left[T_{i j}, \operatorname{det}\left(T_{i j}\right)^{-1}\right]
$$

The corresponding Frobenius morphism is defined by $\left(\lambda_{i j}\right) \mapsto\left(\lambda_{i j}^{q}\right)$. Its fixed points over $\mathbb{F}_{q}$ form the group $G L\left(n, \mathbb{F}_{q}\right)$. Any embedding of an algebraic group $G$ into $G L(n, K)$ as above defines a standard Frobenius endomorphism on $G$ by restriction of the endomorphism of $G L(n, K)$ defined by $\lambda_{i j} \mapsto \lambda_{i j}^{q}$. But there are other examples of rational structures on algebraic groups, for instance the unitary group is $G L(n, K)^{F^{\prime}}$ where $F^{\prime}$ is the endomorphism defined by:

$$
F^{\prime}(x)=F\left(\left(x^{\top}\right)^{-1}\right),
$$

with $F$ being the standard endomorphism on $G L(n, K)$.
The following theorem is very important in the theory of finite groups of Lie type:
Theorem 2.15 (Lang-Steinberg; [18] §10). Suppose $G$ is a connected group and $F$ is a surjective endomorphism of $G$ with finitely many fixed points. Then the Lang map $\mathbb{L}: G \rightarrow G$ defined by $\mathbb{L}(g)=g^{-1} F(g)$ is surjective.

Example 2.9. Lang-Steinberg Theorem holds for the standard Frobenius and $G=T(2, K)$. For an arbitrary element: $h=\left[\begin{array}{ll}a & b \\ 0 & d\end{array}\right] \in T(2, K)$ we have to find $g=\left[\begin{array}{ll}x & y \\ 0 & w\end{array}\right] \in T(2, K)$ such that:

$$
\begin{aligned}
{\left[\begin{array}{cc}
x & y \\
0 & w
\end{array}\right]^{-1}\left[\begin{array}{cc}
x^{q} & y^{q} \\
0 & w^{q}
\end{array}\right] } & =\left[\begin{array}{ll}
a & b \\
0 & d
\end{array}\right] \\
{\left[\begin{array}{cc}
x^{q} & y^{q} \\
0 & w^{q}
\end{array}\right] } & =\left[\begin{array}{cc}
x & y \\
0 & w
\end{array}\right]\left[\begin{array}{cc}
a & b \\
0 & d
\end{array}\right] \\
{\left[\begin{array}{cc}
x^{q} & y^{q} \\
0 & w^{q}
\end{array}\right] } & =\left[\begin{array}{cc}
a x & b x+d y \\
0 & d w
\end{array}\right]
\end{aligned}
$$

This means that finding $g$ is equivalent to finding $x, y, w$ such that:

$$
a x=x^{q}, d w=w^{q}, y^{q}=b x+d y, x, w \neq 0 .
$$

Pick nonzero solution of $T^{q}-a T=0, T^{q}-d T=0$ for $x, w$ respectively. Then pick any value for $y$ such that it satisfies $T^{q}=b x+d T$ for the chosen $x$. It is obvious that these three values for $x, w, y$ satisfy all the conditions.

Example 2.10. Now we prove the Lang-Steinberg Theorem in the case where $F$ is the standard Frobenius $x \mapsto x^{q}$ and $G=G L(2, K)$. Let $h=\left[\begin{array}{cc}a & b \\ c & d\end{array}\right] \in G L(2, K)$ be arbitrary we have to find $g=\left[\begin{array}{cc}x & y \\ z & w\end{array}\right] \in G L(2, K)$ such that: $g^{-1} F(g)=h$. We have already proved the Lang-Steinberg Theorem for $T(2, K)$ so we can assume that $c \neq 0$. Now we have:

$$
\begin{aligned}
{\left[\begin{array}{cc}
x & y \\
z & w
\end{array}\right]^{-1}\left[\begin{array}{cc}
x^{q} & y^{q} \\
z^{q} & w^{q}
\end{array}\right] } & =\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \\
{\left[\begin{array}{cc}
x^{q} & y^{q} \\
z^{q} & w^{q}
\end{array}\right] } & =\left[\begin{array}{cc}
x & y \\
z & w
\end{array}\right]\left[\begin{array}{cc}
a & b \\
c & d
\end{array}\right] \\
{\left[\begin{array}{cc}
x^{q} & y^{q} \\
z^{q} & w^{q}
\end{array}\right] } & =\left[\begin{array}{ll}
a x+c y & b x+d y \\
a z+c w & b z+d w
\end{array}\right]
\end{aligned}
$$

This is equivalent to finding $x, y, z, w$ such that:

$$
\left\{\begin{array}{l}
x^{q}=a x+c y \\
y^{q}=b x+d y \\
z^{q}=a z+c w \\
w^{q}=b z+d w \\
x w-y z \neq 0
\end{array}\right.
$$

Since $c \neq 0$ we can write $y=c^{-1}\left(x^{q}-a x\right), w=c^{-1}\left(z^{q}-a z\right)$ and therefore we have to find $x, z$ such that:

$$
\left\{\begin{array}{l}
x^{q^{2}}-\left(a^{q}+d c^{q-1}\right) x^{q}+\left(a d c^{q-1}-c^{q} b\right) x=0 \\
z^{q^{2}}-\left(a^{q}+d c^{q-1}\right) z^{q}+\left(a d c^{q-1}-c^{q} b\right) z=0 \\
x z\left(z^{q-1}-x^{q-1}\right) \neq 0
\end{array}\right.
$$

Now suppose that there is no such $x, z$ that satisfy all the above equations. This means that for all nonzero solutions $\lambda_{1}, \lambda_{2}$ of

$$
f(T)=T^{q^{2}}-\left(a^{q}+d c^{q-1}\right) T^{q}+\left(a d c^{q-1}-c^{q} b\right) T=0
$$

satisfy $\lambda_{1}^{q-1}=\lambda_{2}^{q-1}$. In other words all nonzero solutions of $f(T)=0$ satisfy the equation $T^{q-1}=\mu$ where $\mu$ is a constant depending only on $a, b, c, d$. This means that $f(T)=0$ has at most $q-1$ distinct nonzero solutions. However:

$$
f^{\prime}(T)=a d c^{q-1}-c^{q} b=c^{q-1}(a d-b c)
$$

is nonzero and constant therefore $f(T)$ can not have multiple roots so $f(T)$ has $q^{2}-1$ roots but $q^{2}-1>q-1$; this contradiction shows that we can find $x, z$ with the desired properties.

Some consequences of Lang-Steinberg Theorem:
Corollary 2.2. Let $V$ be an algebraic variety defined over $\mathbb{F}_{q}$, and let $G$ be a connected algebraic group defined over $\mathbb{F}_{q}$ acting on $V$ by a morphism which is defined over $\mathbb{F}_{q}$. Then any $F$-stable orbit contains an $F$-stable point.

Proof. Let $v$ be a point in an $F$-stable orbit, so we have: $F(v)=g \bullet v$ for some element $g \in G$. By Lang's theorem the element $g^{-1}$ can be written as $h^{-1} F(h)$ with $h \in G$. So we have:

$$
F(v)=g \bullet v=\left(h^{-1} F(h)\right)^{-1} \bullet v=\left(F(h)^{-1} h\right) \bullet v
$$

therefore we have: $F(h) \bullet F(v)=h \bullet v$. This implies: $F(h \bullet v)=h \bullet v$. Hence $h \bullet v$ is the $F$-fixed point in the orbit of $v$.

Corollary 2.3. Suppose $H$ is a closed connected subgroup of the algebraic group $G$ and that both are defined over $\mathbb{F}_{q}$. Then $(G / H)^{F} \cong G^{F} / H^{F}$.

Proof. By Corollary 2.2 any rational left $H$ coset contains a rational point. So the natural $\operatorname{map}(G / H)^{F} \rightarrow G^{F} / H^{F}$ is surjective. It is injective since if $x, y \in G^{F}$ are in the same coset then $x^{-1} y \in H^{F}$.

A maximal torus of $G^{F}$ is defined to be a subgroup of the form $T^{F}$ where $T$ is an $F$ stable maximal torus of $G$. Although every maximal torus of $G$ lies in a Borel subgroup of $G$ it need not be true that every $F$-stable maximal torus of $G$ lies in an $F$-stable Borel subgroup of $G$. An $F$-stable maximal torus of $G$ is called maximally split if it lies in an $F$-stable Borel subgroup of $G^{F}$.

Along the same lines we can prove:
Corollary 2.4 ([7] 3.15). Let $G$ be a connected algebraic group defined over $\mathbb{F}_{q}$.
(i) $G$ has an $F$-stable Borel subgroup and any two $F$-stable Borel subgroups are conjugate by an element of $G^{F}$.
(ii) Any F-stable Borel subgroup contains an F-stable maximal torus.
(iii) F-stable maximal tori contained in $F$-stable Borel subgroups are conjugate.
(iv) Any F-stable conjugacy class of $G$ contains an $F$-stable element.
(v) Any F-stable semisimple element lies in an F-stable maximal torus.

Definition 2.6. Let $A$ be a group on which the Frobenius $F$ acts. $F$-conjugation in $A$ is the action of $A$ on itself given by: $x \bullet a=x a F(x)^{-1}$. If $F$ acts trivially, $F$-conjugacy reduces to ordinary conjugacy. Define:

$$
C_{F, A}(a)=\left\{x \in A: x^{-1} a F(x)=a\right\} .
$$

This is a subgroup of $A$ called the $F$-centralizer of $a$.
Proposition 2.3 ([7] 3.21). Let $V$ be an algebraic variety defined over $\mathbb{F}_{q}$, and let $G$ be a connected algebraic group defined over $\mathbb{F}_{q}$ acting on $V$ by a morphism which is defined over $\mathbb{F}_{q}$. Let $\mathcal{O}$ be an $F$-stable orbit in $V$ and let $x$ be an element of $\mathcal{O}^{F}$, such a point exist by Corollary 2.2. Now we have:
(i) Let $g \in G$ then $g x \in \mathcal{O}^{F}$ if and only if $\mathbb{L}(g) \in \operatorname{Stab}_{G}(x)$.
(ii) There is a well defined map which sends the $G^{F}$-orbit of $g x \in \mathcal{O}^{F}$ to the $F$-conjugacy class of the image of $\mathbb{L}(g)$ in $\mathbf{S t a b}_{G}(x) / \mathbf{S t a b}_{G}(x)^{\circ}$ and it is a bijection.

Corollary 2.5. Let $T$ be a given $F$-stable maximal torus of a connected reductive group $G$, defined over $\mathbb{F}_{q}$. The $G^{F}$-conjugacy classes of maximal tori of $G$ are parametrized by the $F$-conjugacy classes of $N_{G}(T) / N_{G}(T)^{\circ}=W(T)$.

Theorem 2.16 ([4] 3.3.6). Let $T$ be an $F$-stable maximal torus of $G$ obtained from the maximally split torus $T_{0}$ by twisting with $w$. Then $N_{G}(T)^{F} / T^{F} \cong C_{F, W}(w)$.
Proof. Let $T={ }^{T} T_{0}$. Then we have:

$$
F\left(g_{0}\right)={ }^{F(g} t_{t_{0}}={ }^{g \dot{w}} F\left(t_{0}\right)={ }^{g}\left(\left(F\left(t_{0}\right)\right)^{w^{-1}}\right)
$$

Thus $F$ acts on $T$ as $w^{-1} \circ F$ acts on $T_{0}$. Now since $T$ is connected by Corollary 2.3: $N_{G}(T)^{F} / T^{F} \cong\left(N_{G}(T) / T\right)^{F}$. Conjugation by $g$ transforms $T$ to $T_{0}, N_{G}\left(T_{0}\right)$ to $N_{G}(T)$ and $N_{G}(T)^{F} / T^{F}$ to $N_{G}\left(T_{0}\right)^{F} / T_{0}^{F}$. It also transforms $\left(N_{G}(T) / T\right)^{F}$ to $\left(N_{G}\left(T_{0}\right) / T_{0}\right)^{w^{-1} \circ F}$. Thus $\left(N_{G}(T) / T\right)^{F}$ is isomorphic to subgroup of $W$ fixed by $w^{-1} \circ F$. Now suppose $x \in W$ is fixed by $w^{-1} \circ F$. This means that:

$$
\left(w^{-1} \circ F\right)(x)=x \Leftrightarrow w F(x) w^{-1}=x \Leftrightarrow x^{-1} w F(x)=w \Leftrightarrow x \in C_{F, W}(w) .
$$

## Part II

## DL Characters

## Chapter 3

## Construction

Let $A$ and $B$ be two finite groups. A bimodule $M$ with a left $\overline{\mathbb{Q}}_{\ell}[A]$-action and a right $\overline{\mathbb{Q}}_{\ell}[B]$-action is called a $A$-module- $B$. Such a module gives a functor from the category of left $\overline{\mathbb{Q}}_{\ell}[B]$-modules to the category of left $\overline{\mathbb{Q}}_{\ell}[A]$-modules. For an arbitrary left $\overline{\mathbb{Q}}_{\ell}[B]$-module $E$ we define:

$$
R(M \mid B, A)(E)=M \otimes_{\overline{\mathbb{Q}}_{\ell}[B]} E .
$$

This is a left $\overline{\mathbb{Q}}_{\ell}[A]$-module where $A$ acts on $M \otimes_{\overline{\mathbb{Q}}_{\ell}[B]} E$ through its action on $M . R(M \mid B, A)$ is called generalized induction functor and the following example explains this naming:

Example 3.1. If $B$ is a subgroup of $A$ and $M=\overline{\mathbb{Q}}_{\ell}[A]$ on which $A$ acts by left translation and $B$ by right translation then:

$$
R(M \mid B, A)(E)=\overline{\mathbb{Q}}_{\ell}[A] \otimes_{\overline{\mathbb{Q}}_{\ell}[B]} E=\operatorname{Ind}_{B}^{A}(E) .
$$

The modules we will consider will always be (virtual) vector spaces of finite dimension. This hypothesis is necessary in particular for the next proposition which generalizes the character formula for induced characters.

Proposition 3.1 ([7] 4.5). Let $M$ be a $A$-module- $B$ and $E$ be an $B$-module then for $a \in A$ we have:

$$
\operatorname{trace}(a, R(M \mid B, A)(E))=\frac{1}{|B|} \sum_{b \in B} \operatorname{trace}\left(\left(a, b^{-1}\right), M\right) \operatorname{trace}(b, E) .
$$

Proof.

$$
e=\frac{1}{|B|} \sum_{b \in B} b^{-1} \otimes b
$$

is an idempotent of the algebra $\overline{\mathbb{Q}}_{\ell}[B \times B] \cong \overline{\mathbb{Q}}_{\ell}[B] \otimes \overline{\mathbb{Q}}_{\ell}[B]$. Its image in the representation of $B \times B$ on the tensor product $M \otimes_{\bar{Q}_{\ell}} E$ is a projector whose kernel is the subspace $N$ generated by the elements of the form $m b \otimes x-m \otimes b x$. To see this suppose $\sum_{i} m_{i} \otimes x_{i}$ lies in the kernel this means $\sum_{i} \sum_{b \in B} m_{i} b^{-1} \otimes b x_{i}=0$. Now we can write:

$$
\sum_{i} m_{i} \otimes x_{i}=\frac{1}{|B|} \sum_{b \in B} \sum_{i}\left(m_{i} \otimes x_{i}-m_{i} b^{-1} \otimes b x_{i}\right)
$$

$M \otimes_{\overline{\mathbb{Q}}_{\ell}[B]} E$ is the quotient of $M \otimes_{\overline{\mathbb{Q}}_{\ell}} E$ by the subspace $N$. To see this it is enough to recall the construction of the tensor product both these tensor product are constructed as the quotients of $\mathbb{Z}$-module of formal linear combinations of $M \times E$ with coefficients in $\mathbb{Z}$. The kernel of this quotient is the sub-Z $\mathbb{Z}$-module generated by elements:

$$
\left\{\begin{array}{l}
\left(m_{1}+m_{2}, x\right)-\left(m_{1}, x\right)-\left(m_{2}, x\right) \\
\left(m, x_{1}+x_{2}\right)-\left(m, x_{1}\right)-\left(m, x_{2}\right) \\
(m \lambda, x)-(m, \lambda x)
\end{array}\right.
$$

where $m, m_{1}, m_{2} \in M, x, x_{1}, x_{2} \in E$ and $\lambda$ is in the ring we are tensoring over. In this setting it is obvious that the kernel is larger in the case $M \otimes_{\overline{\mathbb{Q}}_{l}[B]} E$ and contains the kernel in the case of $M \otimes_{\bar{Q}_{e}} E$. Now we get:

$$
\begin{aligned}
\operatorname{trace}(a, R(M \mid B, A)(E)) & =\operatorname{trace}\left(a \otimes 1, M \otimes_{\overline{\mathbb{Q}}_{\ell}[B]} E\right) \\
& =\operatorname{trace}\left(a \otimes 1,\left(M \otimes_{\overline{\mathbb{Q}}_{\ell}} E\right) / N\right) \\
& =\operatorname{trace}\left(a \otimes 1, e\left(M \otimes_{\overline{\mathbb{Q}}_{\ell}} E\right)\right) \\
& =\operatorname{trace}\left((a \otimes 1) e, M \otimes_{\overline{\mathbb{Q}}_{\ell}} E\right) \\
& =\operatorname{trace}\left(\frac{1}{|B|} \sum_{b \in B}\left(a, b^{-1}\right) \otimes b, M \otimes_{\overline{\mathbb{Q}}_{\ell}} E\right) \\
& =\frac{1}{|B|} \sum_{b \in B} \operatorname{trace}\left(\left(a, b^{-1}\right), M\right) \operatorname{trace}(b, E) .
\end{aligned}
$$

Let $G$ be a reductive group defined over $\overline{\mathbb{F}}_{q}$ with the associated Frobenius map $F$. Given an $F$-stable maximal torus $T$ in $G$ the construction of Deligne-Lusztig is a generalized induction character from $T^{F}$ to $G^{F}$. To define a generalized induction functor we need a
bimodule, in this case this is provided by $\ell$-adic cohomology. Choose a Borel subgroup $B$ containing $T$. We have a semidirect product $B=T U_{B}$ where $U_{B}$ is the unipotent radical of $B$. The affine variety:

$$
S_{B}=\mathbb{L}^{-1}\left(U_{B}\right) \subset G
$$

is stable under left translation by elements in $G^{F}$, let $g \in G^{F}$ and $x \in S_{B}$ :

$$
\mathbb{L}(g x)=(g x)^{-1} F(g x)=x^{-1} g^{-1} F(g) F(x)=x^{-1} F(x) \in U_{B} \Rightarrow x \in S_{B} .
$$

It is also stable under right translation by elements of $T^{F}$ :

$$
\mathbb{L}(x t)=(t x)^{-1} F(x t)=t^{-1} x^{-1} F(x) t \in t^{-1} U_{B} t=U_{B} \Rightarrow x \in S_{B} .
$$

Therefore we can define a left action of $G^{F}$ and a right action of $T^{F}$ on $S_{B}$. By Proposition 1.3 the vector spaces $H_{c}^{i}\left(S_{B}, \overline{\mathbb{Q}}_{\ell}\right)$ become left $\overline{\mathbb{Q}}_{\ell}\left[G^{F}\right]$-module and right $\overline{\mathbb{Q}}_{\ell}\left[T^{F}\right]$-module. Therefore the virtual vector space $H_{c}^{*}\left(S_{B}, \overline{\mathbb{Q}}_{\ell}\right)$ is a left $\overline{\mathbb{Q}}_{\ell}\left[G^{F}\right]$-module and a right $\overline{\mathbb{Q}}_{\ell}\left[T^{F}\right]$ module. This is the bimodule we are looking for, denote it by $M_{B}$. Let $\theta \in \operatorname{Hom}\left(T^{F}, \overline{\mathbb{Q}}_{\ell}\right)$, we define:

$$
R_{T \subset B}^{G}(\theta)^{1}=R\left(M_{B} \mid T^{F}, G^{F}\right)(\theta)=M_{B} \otimes_{\overline{\mathbb{Q}}_{\ell}\left[T^{F}\right]} \theta
$$

By Proposition 3.1 its character at $g \in G^{F}$ is given by:

$$
\begin{aligned}
R_{T C B}^{G}(\theta)(g) & =\frac{1}{\left|T^{F}\right|} \sum_{t \in T^{F}} \operatorname{trace}\left(\left(g, t^{-1}\right), H_{c}^{*}\left(S_{B}, \overline{\mathbb{Q}}_{\ell}\right)\right) \operatorname{trace}(t, \theta) \\
& =\frac{1}{\left|T^{F}\right|} \sum_{t \in T^{F}} \mathcal{L}\left(\left(g, t^{-1}\right), S_{B}\right) \theta(t)
\end{aligned}
$$

The characters $R_{T \subset B}^{G}(\theta)$, where $\theta \in \operatorname{Irr}\left(T^{F}\right)$ are called DL characters ${ }^{2}$; they were first defined by Deligne and Lusztig in [6].

[^0]
## Chapter 4

## Properties

In this chapter we will investigate the properties of the DL characters $R_{T \subset B}^{G}(\theta)$ defined in the previous chapter.

### 4.1 Inner Products

Let $T, S$ be $F$-stable maximal tori in $G$. Set:

$$
N(T, S)=\left\{g \in G: T^{g}=S\right\}
$$

This set is clearly a union of right cosets of $T$. We now define:

$$
W(T, S)=\{T g: g \in N(T, S)\}
$$

Since $T$ and $S$ are $F$-stable $N(T, S)$ will also be $F$-stable and so there will be an induced action of $F$ on $W(T, S)$.

Lemma 4.1. There is a bijection between $W(T, S)^{F}$ and the set of right cosets of $T^{F}$ in $N(T, S)^{F^{\prime}}$.

Proof. $W(T, S)^{F}$ consists of the set of $F$-stable cosets of $T$ in $N(T, S)$. Now every $F$-stable coset $T g$ contains an $F$-stable element. To see this note that $F(T g)=T g$ and so $F(g) g^{-1} \in$ $T$. By Lang's theorem there exist a $t \in T$ such that $F(g) g^{-1}=t^{-1} F(t)=F(t) t^{-1}$. Thus $t^{-1} g$ is an $F$-stable element in $T g$. The set of $F$-stable elements in $T g$ is a coset of $T^{F}$ in $N(T, S)^{F}$. Conversely every right coset of $T^{F}$ in $N(T, S)^{F}$ lies in a unique $F$-stable right coset of $T$ in $N(T, S)$ which itself is an element of $W(T, S)$.
$W(T, S)^{F}$ is an empty set unless $T$ and $S$ are conjugate by an element in $G^{F}$. Choose representatives $\dot{w} \in N(T, S)^{F}$ for the elements $w \in W(T, S)^{F}$. Given $\theta \in \operatorname{lrr}\left(T^{F}\right)$ we define a character ${ }^{w} \theta$ of $S^{F}$ by:

$$
{ }^{w^{\prime}} \theta(s)=\theta\left(\dot{w} s \dot{w}^{-1}\right) .
$$

Thanks to the above lemma this definition does not depend on the chosen representatives. To see this choose two representatives of $w: m=t_{1} g, n=t_{2} g$ we have:

$$
\begin{aligned}
& { }^{w_{\theta}}(s)=\theta\left(m s m^{-1}\right)=\theta\left(t_{1} g s g^{-1} t_{1}^{-1}\right) \\
& { }^{w_{\theta}}(s)=\theta\left(n s n^{-1}\right)=\theta\left(t_{2} g s g^{-1} t_{2}^{-1}\right)
\end{aligned}
$$

Now by the above lemma $s^{\prime}=g s g^{-1}$ lies in $T^{F}$. And since the group is abelian both are equal to $\theta\left(s^{\prime}\right)$.

Theorem 4.1 (Orthogonality; [4] 7.3.4). With the above notation we have:

$$
\left\langle R_{T \subset B}^{G}(\theta), R_{S \subset B^{\prime}}^{G}(\eta)\right\rangle=\left|\left\{w \in W(T, S)^{F}:{ }^{w} \theta=\eta\right\}\right| .
$$

Corollary 4.1. $R_{T \subset B}^{G}(\theta)$ is independent of $B$.
Proof. If $f, f^{\prime}$ are two characters of $G^{F}$ then the equalities $\langle f, f\rangle=\left\langle f, f^{\prime}\right\rangle=\left\langle f^{\prime}, f^{\prime}\right\rangle$ imply $\left\langle f-f^{\prime}, f-f^{\prime}\right\rangle=0$ hence $f=f^{\prime}$. We apply this to $f=R_{T \subset B}^{G}(\theta), f^{\prime}=R_{T \subset B^{\prime}}^{G}(\theta)$ where $B$ and $B^{\prime}$ are two Borel subgroups containing $T$. The orthogonality theorem shows that: $\langle f, f\rangle=\left\langle f, f^{\prime}\right\rangle=\left\langle f^{\prime}, f^{\prime}\right\rangle$ and the corollary follows.

Remark 4.1. From now on we will use the symbol $R_{T}^{G}(\theta)$ instead of $R_{T \subset B}^{G}(\theta)$.
Definition 4.1. Suppose $\theta \in \operatorname{Irr}\left(T^{F}\right)$ is such that ${ }^{w} \theta=\theta$ implies $w=1$ for $w \in W(T)^{F}$. Then $\theta$ is said to be in general position.
Corollary 4.2. If $\theta$ is in general position then $\pm R_{T}^{G}(\theta)$ is an irreducible character of $G^{F}$.
Proof. We have: $\left\langle R_{T}^{G}(\theta), R_{T}^{G}(\theta)\right\rangle=\left|\left\{w \in W(T)^{F}:{ }^{w} \theta=\theta\right\}\right|$. If $\theta$ is in general position this means: $\left\langle R_{T}^{G}(\theta), R_{T}^{G}(\theta)\right\rangle=1$. Since $R_{T}^{G}(\theta)$ is a virtual character this means that either $R_{T}^{G}(\theta)$ or $-R_{T}^{G}(\theta)$ is an irreducible character.

Corollary 4.3. If the $F$-stable maximal tori $T, S$ of $G$ are not $G^{F}$-conjugate then:

$$
\left\langle R_{T}^{G}(\theta), R_{S}^{G}(\eta)\right\rangle=0
$$

Proof. If the $F$-stable maximal tori $T, S$ of $G$ are not $G^{F}$-conjugate then $W(T, S)^{F}$ is empty and the result follows from the Orthogonality Theorem.

### 4.2 Computing DL Characters

### 4.2.1 Character Formula

Definition 4.2. Denote $R_{T}^{G}(1)$ by $Q_{T}^{G}$ and call it the Green function attached to $T$.
Remark 4.2. In [10], Green defined these functions for $G L\left(n, \mathbb{F}_{q}\right)$ using combinatorial methods, hence the naming. His methods led to determination of the character table of $G L\left(n, \mathbb{F}_{q}\right)$ but could not be generalized to the case of an arbitrary finite group of Lie type. For a given $n$ he explicitly defined a set of polynomials $Q_{\pi_{2}}^{\pi_{1}}$ indexed by two partitions of $n: \pi_{1}, \pi_{2}$. To see how this matches with our Green functions one should note that in $G L\left(n, \mathbb{F}_{q}\right)$ the unipotent classes as well as the conjugacy classes of $F$-stable maximal tori can be indexed by partitions of $n$, now $Q_{\pi_{2}}^{\pi_{1}}(q)$ gives the value of the Green function attached to the corresponding maximal $F$-stable torus on the corresponding unipotent class of $G L\left(n, \mathbb{F}_{q}\right)$.

The next theorem expresses the values of all DL characters in terms of Green functions:
Theorem 4.2 ([4] 7.2.8). Let $g=s u$ be the Jordan decomposition of $g \in G^{F}$. Then

$$
R_{T}^{G}(\theta)(g)=\frac{1}{\left|C_{G}^{\circ}(s)^{F}\right|} \sum_{\substack{x \in G^{F} \\ x^{-1} s x \in T^{F}}} \theta\left(x s x^{-1}\right) Q_{x T x^{-1}}^{C_{G}^{\circ}(s)}(u)
$$

Remark 4.3. First we have to check that the right side expression makes sense. For a fixed $x \in G^{F}$ let $x^{-1} s x=t_{0} \in T$ then we have:

$$
{ }_{x t x^{-1}}^{s}=x t x^{-1} s x t^{-1} x^{-1}=x t t_{0} t^{-1} x^{-1}=x t_{0} x^{-1}=x x^{-1} s x x^{-1}=s
$$

So $x T x^{-1} \subset C_{G}(s)$. Since it is connected we can conclude that $x T x^{-1} \subset C_{G}^{\circ}(s)$. Next we have to show that $x T x^{-1}$ is an $F$-stable maximal torus of $C_{G}^{\mathrm{o}}(s)$. It is a maximal torus in $C_{G}^{\circ}(s)$ because it is a maximal torus in $G$. To see that it is $F$-stable we just note that both $x$ and $T$ are $F$-stable. Also each unipotent element of $C_{G}(s)$ lies in $C_{G}^{\circ}(s)([12, \S 1.12])$ therefore $u \in C_{G}^{\circ}(s)$. Hence the right side is well defined.
Remark 4.4. Let $g \in G^{F}$ and let $g=s u$ be its Jordan decomposition. If the $G^{F}$-conjugacy class of $s$ does not intersect with $T^{F}$ then $R_{T}^{G}(g)=0$.
Remark 4.5. Let $g=z u$ be the Jordan decomposition of an element in $G^{F}$ such that $z$ lies in the center. In this case we have:

$$
R_{T}^{G}(\theta)(g)=\frac{1}{\left|G^{F}\right|} \sum_{x \in G^{F}} \theta(z) Q_{T}^{G}(u)=\theta(z) Q_{T}^{G}(u)
$$

### 4.2.2 Computing Green Functions

The character formula reduces the problem of computing the values of all DL characters to that of computing the Green functions. We can try to compute the Green functions using the formula given in the previous chapter however there is an easier interpretation of Green functions. To explain this we need some preparation:

Proposition 4.1 ([4] 7.7.1). Let $B_{0}$ be a fixed Borel subgroup of $G$. $G$ acts on $G / B_{0} \times G / B_{0}$ by left multiplication. The orbits are in bijective correspondence with elements of $W\left(T_{0}\right)=$ $N\left(T_{0}\right) / T_{0}$, where $T_{0}$ is a maximal torus in $B_{0}$.

Proof. Let $\left(x B_{0}, y B_{0}\right) \in G / B_{0} \times G / B_{0}$. This element lies in the same orbit as ( $B_{0}, x^{-1} y B_{0}$ ). Now by Bruhat decomposition $x^{-1} y \in B_{0} \dot{w} B_{0}$ for a unique $w \in W\left(T_{0}\right)$. Let $x^{-1} y=b \dot{w} b^{\prime}$ for $b, b^{\prime} \in B_{0}$. Then $\left(B_{0}, x^{-1} y B_{0}\right)=\left(B_{0}, b \dot{w} B_{0}\right)$, and this lies in the same orbit as $\left(B_{0}, \dot{w} B_{0}\right)$. Thus every orbit contains an element of the form ( $B_{0}, \dot{w} B_{0}$ ). Conversely suppose ( $B_{0}, \dot{w} B_{0}$ ) lies in the same orbit as $\left(B_{0}, \dot{w}^{\prime} B_{0}\right)$. Then $\left(B_{0}, \dot{w} B_{0}\right)=\left(x B_{0}, x \dot{w}^{\prime} B_{0}\right)$ for some $x \in G$. Thus $B_{0}=x B_{0}$ and $\dot{w}^{\prime} B_{0}=x \dot{w} B_{0}$. Hence $x \in B_{0}$ and $\dot{w}^{\prime} \in B_{0} \dot{w} B_{0}$. It follows that $w=w^{\prime}$.

Proposition 4.2 ([4] 7.7.2). $G$ acts on $\mathcal{B} \times \mathcal{B}$ by conjugation, where $\mathcal{B}$ is the variety of Borel subgroups in $G$. The orbits are in bijective correspondence with elements of $W\left(T_{0}\right)=$ $N\left(T_{0}\right) / T_{0}$, where $T_{0}$ is a maximal torus in $B_{0}$. Each orbit contains a unique element of the form ( $B_{0},{ }^{\dot{w}} B_{0}$ ), where $\dot{w}$ is a representative for $w \in W(T)$.

Definition 4.3. By $\mathcal{O}_{w}$ we denote the $G$-orbit in $\mathcal{B} \times \mathcal{B}$ corresponding to $w \in W(T)$. It is the set of all pairs $\left(B_{1}, B_{2}\right) \in \mathcal{B} \times \mathcal{B}$ such that $B_{1}, B_{2}$ are in the same orbit as $\left(B_{0},{ }^{w} B_{0}\right)$. We say $B_{1}, B_{2} \in \mathcal{B}$ are in relative position $w$ if $\left(B_{1}, B_{2}\right) \in \mathcal{O}_{w}$. Now $X(w)$ is defined to be the set of all $B \in \mathcal{B}$ such that $(B, F(B)) \in \mathcal{O}_{w} . X(w)$ is a locally closed subset (i.e. it is the intersection of an open set with a closed set) of $\mathcal{B}$ and so inherits a structure of an algebraic variety. It is called the DL variety attached to $w^{1}$.
$G^{F}$ acts on DL varieties, to see this let $B \in X(w)$ and $g \in G^{F}$. Then $(B, F(B)) \in \mathcal{O}_{w}$ and $\left({ }^{g} B, F\left({ }^{g} B\right)\right)=\left({ }^{g} B,{ }^{g} F(B)\right)$ lies in the same $G$-orbit as $(B, F(B))$. Thus $\left({ }^{g} B, F\left({ }^{g} B\right)\right) \in \mathcal{O}_{w}$, and so ${ }^{g} B \in X(w)$. With the respect to this action we have the following:

Theorem 4.3 ([4] 7.7.11). Let $T$ be an $F$-stable maximal torus of $G$ obtained from a maximally split torus by twisting by $w \in W$. Then $R_{T}^{G}(1)(g)=\mathcal{L}(g, X(w))$, for all $g \in G^{F}$.

[^1]Example 4.1. $X(e)$ is the set of all Borel subgroups $B$ of $G$, such that $(B, F(B))$ is in the same orbit as $(B, B)$. In other words $X(e)$ is the set of $F$-stable Borel subgroups of $G$. This is the set of fixed points of Frobenius on $\mathcal{B}$ and so by Proposition 1.1 it is finite. By Proposition 1.4 and Corollary 2.3 we have:

$$
R_{T}^{G}(1)=\mathcal{L}(g, X(e))=\left|X(e)^{g}\right|=\left|\left((G / B)^{F}\right)^{g}\right|=\left|\left(G^{F} / B^{F}\right)^{g}\right|
$$

Since $g$ acts by left translation we conclude that: $R_{T}^{G}(1)=\operatorname{Ind}_{B^{F}}^{G^{F}}(1)$.
The next theorem shows that more is true:
Theorem 4.4 ([4] 7.2.4). Let $T$ be a maximally split $F$-stable torus and $B$ an $F$-stable Borel subgroup of $G$ containing $T$. Let $\theta \in \operatorname{Irr}\left(T^{F}\right)$ and $\theta_{B^{F}}$ be the one dimensional representation of $B^{F}$ which extends $\theta$ and has $U_{B}^{F}$ in the kernel. Then we have:

$$
R_{T}^{G}(\theta)=\operatorname{lnd}_{B^{F}}^{G^{F}}\left(\theta_{B^{F}}\right)
$$

Example 4.2 (DL varieties for $G L(2)$ ). From Example 2.7 we know that the Weyl group of $G L(2)$ has two elements: $W=\{e, s\}$. We have already determined $X(e)$ in the general case, it is the set of fixed points of the Frobenius. In Example 2.5 we have seen that $G / B$ is isomorphic to the projective line $\mathbb{P}^{1}\left(\overline{\mathbb{F}}_{p}\right)$ and by definition $X(e)$ is the $F$-stable points of $\mathbb{P}^{1}\left(\overline{\mathbb{F}}_{p}\right)$. In other words $X(e)=\mathbb{P}^{1}\left(\mathbb{F}_{q}\right)$. So it remains to determine the DL variety $X(s)$ we claim:

$$
X(s)=\mathbb{P}^{1}\left(\overline{\mathbb{F}}_{p}\right) \backslash X(e) .
$$

To see this suppose $B$ is not $F$-stable then $(B, F(B))$ does not belong to $\mathcal{O}_{e}$. But by Proposition 4.2 there are only two orbits in $\mathbb{P}^{1}\left(\overline{\mathbb{F}}_{p}\right) \times \mathbb{P}^{1}\left(\overline{\mathbb{F}}_{p}\right)$ hence $(B, F(B)) \in \mathcal{O}_{s}$.

### 4.3 Geometric Conjugacy

The next natural question is how the reducible DL characters decompose into irreducible characters of $G^{F}$ and to which extent the elements of $\operatorname{Irr}\left(G^{F}\right)$ are covered by DL characters. Here is the answer to the second question:

Corollary 4.4 ([4] 7.5.8). For any irreducible character $\pi$ of $G^{F}$, there exist an $F$-stable maximal torus $T \subset G$ and a character $\theta \in \operatorname{lrr}\left(T^{F}\right)$ such that $\left\langle\pi, R_{T}^{G}(\theta)\right\rangle \neq 0$.

The answer to the first question turns out to be hard. Corollary 4.3 shows $R_{T}^{G}(\theta)$ and $R_{S}^{G}(\eta)$ are orthogonal if $T$ and $S$ are not $G^{F}$-conjugate, but since they are virtual characters this does not imply that they do not have any irreducible components in common. We will give a condition for two DL characters to have a common irreducible component. We define an equivalence relation between the pairs ( $T, \theta$ ) where $T$ is an $F$-stable maximal torus and $\theta \in \operatorname{Irr}\left(T^{F}\right)$. This relation depends not only on the finite group $G^{F}$ but on the following chain of finite subgroups of $G$ :

$$
G^{F} \subset G^{F^{2}} \subset G^{F^{3}} \subset \cdots
$$

Also we define the norm $\operatorname{map} \mathcal{N}_{F^{n} / F}$ as follows:

$$
\left\{\begin{array}{l}
\mathcal{N}_{F^{n} / F}: T^{F^{n}} \rightarrow T^{F} \\
t \mapsto t F(t) F^{2}(t) \cdots F^{n-1}(t)
\end{array}\right.
$$

Definition 4.4. Let $T, S$ be $F$-stable maximal tori of $G$ and $\theta \in \operatorname{Irr}\left(T^{F}\right), \eta \in \operatorname{Irr}\left(S^{F}\right)$. The pairs $(T, \theta)$ and $(S, \eta)$ are called geometrically conjugate if for some $n>0$ there is an element $g \in G^{F^{n}}$ which conjugates $T$ to $S$ and $\theta \circ \mathcal{N}_{F^{n} / F} \in \operatorname{Irr}\left(T^{F^{n}}\right)$ to $\eta \circ \mathcal{N}_{F^{n} / F} \in \operatorname{Irr}\left(S^{F^{n}}\right)$.

With this definition we can state:
Theorem 4.5 ([4] 7.3.8). Let $T, S$ be $F$-stable maximal tori of $G$ and $\theta \in \operatorname{Irr}\left(T^{F}\right), \eta \in$ $\operatorname{lrr}\left(S^{F}\right)$. suppose $(T, \theta)$ and $(S, \eta)$ are not geometrically conjugate. Then $R_{T}^{G}(\theta)$ and $R_{S}^{G}(\eta)$ have no irreducible component in common.

Example 4.3 (Unipotent Characters). Any two pairs $(T, 1),(S, 1)$ where $T, S$ are $F$-stable maximal tori are geometrically conjugate. To see this note that all tori are conjugate in $G$ so there is $g \in G$ such that $T={ }^{g} S$. It is obvious that there is an $n>0$ such that $g \in G^{F^{n}}$ and $g$ will transform the unit character to unit character. This together with Theorem 4.5 implies that all Green functions have the same irreducible components. Theses characters are called the unipotent characters of $G^{F}$.

## Part III

## An Example

## Chapter 5

## Computations for $G L(2)$

In this chapter:
$\propto q=p^{m}$, where $p$ is an odd prime and $m \geq 1$ is an integer.
$\infty F$ the standard Frobenius associated to $q$.
$\omega G=G L\left(2, \overline{\mathbb{F}}_{p}\right)$.

- $B=T\left(2, \overline{\mathbb{F}}_{p}\right)$.
- $T=D\left(2, \widetilde{\mathbb{F}}_{p}\right)$.
- $W=N_{G}(T) / T$.


### 5.1 F-stable Maximal Tori

Fix the maximal $F$-stable torus $T$. By Corollary $2.5 F$-conjugacy classes in $W=\{e, s\}$ determine the the $G^{F}$-conjugacy classes of $F$-stable maximal tori. Now by definition the $F$-conjugacy class of $w$ is all elements of the form $x w F(x)^{-1}$. If $w=e$ then its $F$-conjugacy class consists only of $e$. Now since $F$-conjugacy classes are orbits of an action we conclude that the $\{s\}$ is the other $F$-conjugacy class. So we get two classes of $F$-stable maximal tori corresponding to elements of the Weyl group. The class corresponding to $e$ (which from now we will denote it by $T_{e}$ ) is the the maximally split torus. $T_{e}^{F}$ is the set of fixed points
of $F$ on $T$. Pick an arbitrary element in $t \in T$. Now $F(t)=t$ implies:

$$
\left[\begin{array}{cc}
\lambda & 0 \\
0 & \mu
\end{array}\right]=t=F(t)=\left[\begin{array}{cc}
\lambda^{q} & 0 \\
0 & \mu^{q}
\end{array}\right]
$$

Therefore $\lambda=\lambda^{q}, \mu=\mu^{q}$ so $\lambda, \mu \in \mathbb{F}_{q}^{\times}$. This implies $T_{e}^{F} \cong \mathbb{F}_{q}^{\times} \times \mathbb{F}_{q}^{\times}$. By the proof of Theorem 2.16 the torus orresponding to $s$ (denoted by $T_{s}$ ) is isomorphic to the group of fixed points of $s^{-1} \circ F$ on $T$. For an arbitrary element $t \in T$ this means that $F\left(t^{s}\right)=t$. In other words:

$$
\left[\begin{array}{cc}
\lambda & 0 \\
0 & \mu
\end{array}\right]=t=F\left(t^{s}\right)=F\left(\left[\begin{array}{cc}
\mu & 0 \\
0 & \lambda
\end{array}\right]\right)=\left[\begin{array}{cc}
\mu^{q} & 0 \\
0 & \lambda^{q}
\end{array}\right] .
$$

Therefore $\lambda=\mu^{q}$ and $\mu=\lambda^{q}$. This implies: $\lambda^{q^{2}}=\lambda$. Therefore:

$$
T^{s^{-1} \circ F}=\left\{\left[\begin{array}{cc}
\lambda & 0 \\
0 & \lambda^{q}
\end{array}\right]: \lambda \in \mathbb{F}_{q^{2}}^{\times}\right\}
$$

$T^{s^{-1} \circ F}$ is isomorphic to $\mathbb{F}_{q^{2}}^{\times}$via the map: $\left[\begin{array}{cc}\lambda & 0 \\ 0 & \lambda^{q}\end{array}\right] \mapsto \lambda$. We have to find the $F$-stable form of this group. To do this we note that: $\mathbb{F}_{q^{2}}=\mathbb{F}_{q}(\sqrt{\tau})$ where $\tau$ is a generator for the cyclic group $\mathbb{F}_{q}^{\times}$. Therefore we can write $\lambda$ as $a+b \sqrt{\tau}$ where $a, b \in \mathbb{F}_{q}$. Now:

$$
\lambda^{q}=(a+b \sqrt{\tau})^{q}=a^{q}+b^{q}(\sqrt{\tau})^{q}=a-b \sqrt{\tau} .
$$

Now conjugation by $\left[\begin{array}{cc}\sqrt{\tau} & -\sqrt{\tau} \\ 1 & 1\end{array}\right]$ maps $T^{s^{-1} \circ F}$ to the the $F$-stable form:

$$
T_{s}^{F}=\left\{\left[\begin{array}{cc}
a & b \tau \\
b & a
\end{array}\right]: a, b \in \mathbb{F}_{q}, a^{2}-\tau b^{2} \neq 0\right\}
$$

### 5.2 Conjugacy Classes

We wish to determine all the conjugacy classes in $G^{F}$. First we determine the semisimple classes in $G^{F}$. From Corollary 2.4 we know that every semisimple element lies in an $F$-stable maximal torus.

- An element of center is of the form $Z_{a}=\left[\begin{array}{cc}a & 0 \\ 0 & a\end{array}\right]$ such that $a \in \mathbb{F}_{q}^{\times}$. The central elements lie in both the split and the nonsplit torus. Each element in the center determines a conjugacy class. This way we will get $q-1$ conjugacy classes each of size 1 .
- Consider elements of the split torus that are not central. These elements are of the form $C_{a, b}^{1}=\left[\begin{array}{ll}a & 0 \\ 0 & b\end{array}\right]$ where $a, b \in \mathbb{F}_{q}^{\times}$are distinct. $C_{a, b}^{1}$ is conjugate to $C_{b, a}^{1}$ via the matrix $\left[\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right]$ which is representative of the nontrivial element of Weyl group of $T_{e}$. The number of these classes are $\frac{1}{2}(q-1)(q-2)$. The centralizer is the torus $T_{e}^{F}$ itself. So each class is of size $q(q+1)$.
- Finally consider the noncentral elements of the nonsplit torus. These elements are of form $C_{a, b}^{2}=\left[\begin{array}{cc}a & \tau b \\ b & a\end{array}\right]$ where $a \in \mathbb{F}_{q}$ and $b \in \mathbb{F}_{q}^{\times}$. Again $C_{a, b}^{2}$ is conjugate to $C_{a,-b}^{2}$ via the action of the Weyl group. So we get $\frac{1}{2} q(q+1)$ classes. The centralizer is $T_{s}^{F}$. So the size of each class is $q(q-1)$.

Now we know all the semisimple classes. Consider the element: $\left[\begin{array}{cc}1 & 1 \\ 0 & 1\end{array}\right]$. By multiplying the elements of the center by this element we obtain $q-1$ classes: $U_{a}=\left[\begin{array}{lll}a & 1 \\ 0 & a\end{array}\right]$ for $a \in \mathbb{F}_{q}^{\times}$. The centralizer of this elements is the matriccs of the form $\left[\begin{array}{cc}a & * \\ 0 & a\end{array}\right]$ for $a \in \mathbb{F}_{q}^{\times}$. So the size of each class is $q^{2}-1$. Now a simple counting argument shows that we have exhausted all the classes in $G^{F}$.

### 5.3 DL Characters from $T_{e}$

An element of $\operatorname{Irr}\left(T_{e}^{F}\right)$ is given by a pair $(\alpha, \beta)$ of characters of $\mathbb{F}_{q}^{\times}$:

$$
(\alpha, \beta)\left(\left[\begin{array}{ll}
a & 0 \\
0 & b
\end{array}\right]\right)=\alpha(a) \beta(b)
$$

We want to see which ones are irreducible, we have to compute the action of the Weyl group. Pick representatives for the elements in the Weyl group. $\dot{e}=\left[\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right]$ and $\dot{s}=\left[\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right]$. Since the identity element acts trivially we just compute the action of $s$ :

$$
\begin{aligned}
s_{((\alpha, \beta))}\left(\left[\begin{array}{ll}
a & 0 \\
0 & b
\end{array}\right]\right) & =(\alpha, \beta)\left(\left[\begin{array}{ll}
a & 0 \\
0 & b
\end{array}\right]^{s}\right)=(\alpha, \beta)\left(\left[\begin{array}{ll}
b & 0 \\
0 & a
\end{array}\right]\right) \\
& =\alpha(b) \beta(a)=(\beta, \alpha)\left(\left[\begin{array}{ll}
a & 0 \\
0 & b
\end{array}\right]\right)
\end{aligned}
$$

In other words $s$ acts by exchanging $\alpha$ and $\beta$. By the Orthogonality Theorem the DL characters $R_{T_{e}}^{G}(\alpha, \beta)$ are all distinct when $\{\alpha, \beta\}$ runs over non-ordered pairs of characters of $\mathbb{F}_{q}^{\times}$so we get $\frac{1}{2} q(q-1)$ characters of the form $R_{T_{e}}^{G}(\alpha, \beta)$. These virtual characters have
norm 1 when $\alpha \neq \beta$ otherwise they have norm 2. Since $T_{e}$ is contained in the $F$-stable Borel subgroup $B$ we can use Theorem 4.4. This gives us:

$$
R_{T_{e}}^{G}(\alpha, \beta)=\operatorname{Ind}_{B^{F}}^{G^{F}}\left((\alpha, \beta)_{B^{F}}\right)
$$

where $(\alpha, \beta)_{B^{F}}$ is extended to a character of $B^{F}$ as follows:

$$
(\alpha, \beta)_{B^{F}}\left(\left[\begin{array}{ll}
a & * \\
0 & b
\end{array}\right]\right)=\alpha(a) \beta(b)
$$

These are exactly the characters $W_{\alpha, \beta}$ in $\S 5.2$ [9]. Also this shows that when $\alpha \neq \beta$ then $R_{T_{e}}^{G}(\alpha, \beta)$ are actual characters with norm 1, in other words they are irreducible characters of $G^{F}$.

### 5.4 DL Characters from $T_{s}$

Since $T_{s}$ is not contained in any $F$-stable Borel subgroup the DL characters constructed from $T_{s}$ are not in the form of induction. Therefore we have to rely on the theory we have developed for DL characters. An element $\varphi \in \operatorname{Irr}\left(T_{s}^{F}\right)$ is a character of $\mathbb{F}_{q^{2}}^{\times} . \dot{e}=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$, $\dot{s}=\left[\begin{array}{cc}-1 & 0 \\ 0 & 1\end{array}\right]$ are representatives for the Weyl group. We compute the action of $s$ :

$$
\begin{aligned}
{ }^{s} \varphi\left(\left[\begin{array}{cc}
a & \tau b \\
b & a
\end{array}\right]\right) & =\varphi\left(\left[\begin{array}{cc}
a & \tau b \\
b & a
\end{array}\right]^{s}\right)=\varphi\left(\left[\begin{array}{cc}
a & -\tau b \\
-b & a
\end{array}\right]\right) \\
& =\varphi(a-b \sqrt{\tau})=\varphi\left((a+b \sqrt{\tau})^{q}\right)=\varphi^{q}(a+b \sqrt{\tau}) \\
& =\varphi^{q}\left(\left[\begin{array}{cc}
a & \tau b \\
b & a
\end{array}\right]\right)
\end{aligned}
$$

So $s$ acts by sending $\varphi$ to $\varphi^{q}$. By the Orthogonality Theorem the DL characters $R_{T_{s}}^{G}(\varphi)$ are all distinct when $\varphi$ runs over representatives of $\operatorname{lrr}\left(\mathbb{F}_{q^{2}}^{\times}\right)$modulo $\varphi \equiv \varphi^{q}$ (because we have: $\left.R_{T_{s}}^{G}(\varphi)=-R_{T_{s}}^{G}\left(\varphi^{q}\right)\right)$. So we get $\frac{1}{2}\left(q^{2}-1\right)$ characters. These characters are irreducible unless $\varphi=\varphi^{q}$ and when they are not irreducible they have two irreducible components.

### 5.4.1 Computing $Q_{T_{s}}^{G}$

Now we compute the Green function associated to $T_{s}$ by Theorem 4.3 and Corollary 1.1 we have:

$$
Q_{T_{s}}^{G}(g)=R_{T_{s}}^{G}(1)(g)=\mathcal{L}(g, X(s))=\left(-\sum_{n=1}^{\infty} \mid X(s)^{F^{n} g^{-1} \mid z^{n}}\right)_{z=\infty}
$$

So we have to count the number of fixed points of $F^{n} g^{-1}$ in $X(s)$ for various $g$. In Example 4.2 we have seen that $X(s)$ is the compliment of $X(e)=\mathbb{P}^{1}\left(\mathbb{F}_{q}\right)$ in $\mathbb{P}^{1}\left(\overline{\mathbb{F}}_{p}\right)$ so this is equivalent to counting the fixed points of $F^{n} g^{-1}$ in $\mathbb{P}^{1}\left(\overline{\mathbb{F}}_{p}\right)$ and then deducting the number of fixed points which are $F$-stable. For the projective line we have:

$$
\mathbb{P}^{1}\left(\overline{\mathbb{F}}_{p}\right)=\left\{\left[\begin{array}{l}
\mathbf{p} \\
1
\end{array}\right]: \mathbf{p} \in \mathbb{A}^{1}\left(\overline{\mathbb{F}}_{p}\right)\right\} \cup\left\{\left[\begin{array}{l}
1 \\
0
\end{array}\right]\right\} .
$$

The points of the form $\left[\begin{array}{l}p \\ 1\end{array}\right]$ are usually called the affine or the finite part. While the point $\left[\begin{array}{l}1 \\ 0\end{array}\right]$ is also called infinity $(\infty)$.

- Let $g=\left[\begin{array}{ll}a & 0 \\ 0 & a\end{array}\right]$. $g$ fixes $\left[\begin{array}{l}1 \\ 0\end{array}\right]$ so we look at the affine part:

$$
\begin{aligned}
& F^{n}\left[\begin{array}{ll}
a & 0 \\
0 & a
\end{array}\right]^{-1}\left[\begin{array}{l}
1 \\
0
\end{array}\right]=F^{n}\left[\begin{array}{c}
a^{-1} \\
0
\end{array}\right]=F^{n}\left[\begin{array}{l}
1 \\
0
\end{array}\right]=\left[\begin{array}{l}
1 \\
0
\end{array}\right] \\
& F^{n}\left[\begin{array}{ll}
a & 0 \\
0 & a
\end{array}\right]^{-1}\left[\begin{array}{l}
\mathbf{p} \\
1
\end{array}\right]=F^{n}\left[\begin{array}{c}
a^{-1} \mathbf{p} \\
a^{-1}
\end{array}\right]=F^{n}\left[\begin{array}{l}
\mathbf{p} \\
1
\end{array}\right]=\left[\begin{array}{c}
\mathbf{p}^{q^{n}} \\
1
\end{array}\right]
\end{aligned}
$$

We have $q^{n}+1$ fixed points however all $F$-stable points were among them so: $\left|X(s)^{F^{n} g^{-1}}\right|=$ $q^{n}+1-q-1=q^{n}-q$. Therefore:

$$
\begin{aligned}
\mathcal{L}\left(\left[\begin{array}{ll}
a & 0 \\
0 & a
\end{array}\right], X(s)\right) & =\left(-\sum_{n=1}^{\infty}\left(q^{n}-q\right) z^{n}\right)_{z=\infty} \\
& =\left(-\sum_{n=1}^{\infty} q^{n} z^{n}+q \sum_{n=1}^{\infty} z^{n}\right)_{z=\infty} \\
& =\left(\frac{-q z}{1-q z}+\frac{q z}{1-z}\right)_{z=\infty} \\
& =1-q
\end{aligned}
$$

- Let $g=\left[\begin{array}{ll}a & 0 \\ 0 & b\end{array}\right]$ :

$$
\begin{aligned}
& F^{n}\left[\begin{array}{ll}
a & 0 \\
0 & b
\end{array}\right]^{-1}\left[\begin{array}{l}
1 \\
0
\end{array}\right]=F^{n}\left[\begin{array}{c}
a^{-1} \\
0
\end{array}\right]=F^{n}\left[\begin{array}{l}
1 \\
0
\end{array}\right]=\left[\begin{array}{l}
1 \\
0
\end{array}\right] \\
& F^{n}\left[\begin{array}{ll}
a & 0 \\
0 & b
\end{array}\right]^{-1}\left[\begin{array}{l}
\mathbf{p} \\
1
\end{array}\right]=F^{n}\left[\begin{array}{c}
a^{-1} \mathbf{p} \\
b^{-1}
\end{array}\right]=F^{n}\left[\begin{array}{c}
b a^{-1} \mathbf{p} \\
1
\end{array}\right]=\left[\begin{array}{c}
b a^{-1} \mathbf{p}^{q^{n}} \\
1
\end{array}\right]
\end{aligned}
$$

We should have $b a^{-1} \mathbf{p}^{q^{n}}=\mathbf{p}$ since $a \neq b$ the nonzero solutions are not $F$-stable. We have $q^{n}+1$ fixed points in $\mathbb{P}^{1}\left(\overline{\mathbb{F}}_{p}\right)$ only two of them $\left(\left[\begin{array}{l}1 \\ 0\end{array}\right]\right.$, [ $\left.\left[\begin{array}{l}0 \\ 1\end{array}\right]\right)$ are $F$-stable. Therefore $\left|X(s)^{F^{n} g^{-1}}\right|=q^{n}-1:$

$$
\begin{aligned}
\mathcal{L}\left(\left[\begin{array}{ll}
a & 0 \\
0 & b
\end{array}\right], X(s)\right) & =\left(-\sum_{n=1}^{\infty}\left(q^{n}-1\right) z^{n}\right)_{z=\infty} \\
& =\left(-\sum_{n=1}^{\infty} q^{n} z^{n}+\sum_{n=1}^{\infty} z^{n}\right)_{z=\infty} \\
& =\left(\frac{-q z}{1-q z}+\frac{z}{1-z}\right)_{z=\infty} \\
& =0
\end{aligned}
$$

- Let: $g=\left[\begin{array}{cc}a & \tau b \\ b & a\end{array}\right], b \neq 0$

$$
\begin{aligned}
F^{n}\left[\begin{array}{cc}
a & \tau b \\
b & a
\end{array}\right]^{-1}\left[\begin{array}{l}
1 \\
0
\end{array}\right] & =F^{n}\left(a^{2}-\tau b^{2}\right)^{-1}\left[\begin{array}{cc}
a & -\tau b \\
-b & a
\end{array}\right]\left[\begin{array}{l}
1 \\
0
\end{array}\right] \\
& =F^{n}\left[\begin{array}{c}
-b^{-1} a \\
1
\end{array}\right]=\left[\begin{array}{c}
-b^{-1} a \\
1
\end{array}\right] \\
F^{n}\left[\begin{array}{cc}
a & \tau b \\
b & a
\end{array}\right]^{-1}\left[\begin{array}{l}
\mathbf{p} \\
1
\end{array}\right] & =F^{n}\left(a^{2}-\tau b^{2}\right)^{-1}\left[\begin{array}{cc}
a & -\tau b \\
-b & a
\end{array}\right]\left[\begin{array}{l}
\mathbf{p} \\
1
\end{array}\right] \\
& =F^{n}\left[\begin{array}{c}
a \mathbf{p}-\tau b \\
-b \mathbf{p}+a
\end{array}\right] \\
& =F^{n}\left[\begin{array}{c}
(a \mathbf{p}-\tau b)(-b \mathbf{p}+a)^{-1} \\
1
\end{array}\right] \\
& =\left[\begin{array}{c}
(a \mathbf{p}-\tau b)^{q^{n}}(-b \mathbf{p}+a)^{-q^{n}} \\
1
\end{array}\right]
\end{aligned}
$$

We can assume $a \neq b \mathbf{p}$, otherwise we have $\infty$ and the point $\mathbf{p}$ is not fixed. This gives us the following equation:

$$
a \mathbf{p}^{q^{n}}-\tau b=\left(-b \mathbf{p}^{q^{n}}+a\right) \mathbf{p}
$$

which is equivalent to:

$$
b \mathbf{p}^{q^{n}+1}+a \underline{\mathbf{p}}^{q^{n}}-a \mathbf{p}-\tau b=0
$$

This has $q^{n}+1$ solutions and all solutions lie in $X(s)$. To see this suppose $\mathbf{p}_{0}$ is an $F$-stable solution, this assumption would turn the above equation into:

$$
b \mathbf{p}_{0}^{2}+a \mathbf{p}_{0}-a \mathbf{p}_{0}-\tau b=b \mathbf{p}_{0}^{2}-\tau b=0
$$

Since $b \neq 0$ this means $\mathbf{p}_{0}^{2}=\tau$ but this is impossible since $\tau$ is a generator for $\mathbb{F}_{q}^{\times}$. This contradiction shows that all solutions lie in $X(s)$. Hence $\left|X(s)^{F^{n} g^{-1}}\right|=q^{n}+1$. This gives us:

$$
\begin{aligned}
\mathcal{L}\left(\left[\begin{array}{cc}
a & \tau b \\
b & a
\end{array}\right], X(s)\right) & =\left(-\sum_{n=1}^{\infty}\left(q^{n}+1\right) z^{n}\right)_{z=\infty} \\
& =\left(-\sum_{n=1}^{\infty} q^{n} z^{n}-\sum_{n=1}^{\infty} z^{n}\right)_{z=\infty} \\
& =\left(\frac{-q z}{1-q z}+\frac{-z}{1-z}\right)_{z=\infty} \\
& =2
\end{aligned}
$$

- Finally suppose $g=\left[\begin{array}{cc}a & 1 \\ 0 & a\end{array}\right]$ :

$$
\begin{aligned}
& F^{n}\left[\begin{array}{ll}
a & 1 \\
0 & a
\end{array}\right]^{-1}\left[\begin{array}{l}
1 \\
0
\end{array}\right]=F^{n}\left[\begin{array}{l}
a \\
0
\end{array}\right]=\left[\begin{array}{l}
1 \\
0
\end{array}\right] \\
& F^{n}\left[\begin{array}{ll}
a & 1 \\
0 & a
\end{array}\right]^{-1}\left[\begin{array}{l}
\mathbf{p} \\
1
\end{array}\right]=F^{n}\left[\begin{array}{c}
a^{-1} \mathbf{p}-1 \\
a^{-1}
\end{array}\right]=F^{n}\left[\begin{array}{c}
\mathbf{p}-a \\
1
\end{array}\right]=\left[\begin{array}{c}
\mathbf{p}^{q^{n}}-a \\
1
\end{array}\right]
\end{aligned}
$$

This gives us the equation: $\mathbf{p}^{q^{n}}-\mathbf{p}-a=0$ this has $q^{n}$ solutions and all of the solutions lie in $X(s)$ since $F(\mathbf{p})=\mathbf{p}$ would imply $a=0$. In total there are $q^{n}+1$ fixed points in $\mathbb{P}^{1}\left(\overline{\mathbb{F}}_{p}\right)$ and $q^{n}$ lie in $X(s)$.

$$
\mathcal{L}\left(\left[\begin{array}{ll}
a & 1 \\
0 & a
\end{array}\right], X(s)\right)=\left(-\sum_{n=1}^{\infty} q^{n} z^{n}\right)_{z=\infty}=\left(\frac{-q z}{1-q z}\right)_{z=\infty}=1
$$

### 5.4.2 Values for DL Characters

We compute the values of these DL characters. For the classes of the form $Z_{a}$ we use Remark 4.4:

$$
R_{T_{s}}^{G}(\varphi)\left(\left[\begin{array}{ll}
a & 0 \\
0 & a
\end{array}\right]\right)=\varphi\left(\left[\begin{array}{ll}
a & 0 \\
0 & a
\end{array}\right]\right) Q_{T_{s}}^{G}(1)=(1-q) \varphi(a)
$$

The virtual characters $R_{T_{s}}^{G}(\varphi)$ are all zero on the classes $C_{a, b}^{1}$ by Re mark 4.5. For classes $C_{a, b}^{2}$ we use Theorem 4.2:

$$
\begin{align*}
& R_{T_{s}}^{G}(\varphi)\left(\left[\begin{array}{cc}
a & \tau b \\
b & a
\end{array}\right]\right)=\frac{1}{\left|C_{G}^{\circ}\left(\left[\begin{array}{cc}
a & \tau b \\
b & a
\end{array}\right]\right)^{F}\right|} \sum_{x \in G^{F}} \varphi\left(\left[\begin{array}{cc}
a & \tau b \\
b & a
\end{array}\right]^{x}\right) Q_{x T_{e} x^{-1}}^{C_{G}^{\circ}\left(\left[\begin{array}{cc}
a & \tau b
\end{array}\right]\right)}(1)  \tag{1}\\
& x^{-1}\left[\begin{array}{cc}
a & \tau b \\
b & a
\end{array}\right] x \in T_{s}^{F} \\
& =\frac{1}{\left|T_{s}^{F}\right|} \sum_{\substack{x \in G^{F} \\
x^{-1} \\
a \uparrow b \\
b \\
b \\
a}} \sum_{x \in T_{s}^{F}} \varphi\left(\left[\begin{array}{cc}
a & \tau b \\
b & a
\end{array}\right]^{x}\right) Q_{x T_{s} x^{-1}}^{T_{s}}(1) \\
& =\frac{1}{\left|T_{s}^{F}\right|} \sum_{x \in N_{G}\left(T_{s}\right)^{F}} \varphi\left(\left[\begin{array}{cc}
a & \tau b \\
b & a
\end{array}\right]^{x}\right) Q_{T_{s}}^{T_{s}}(1) \\
& =\frac{1}{\left|T_{s}^{F}\right|} \sum_{x \in s T_{s}^{F} \cup T_{s}^{F}} \varphi\left(\left[\begin{array}{cc}
a & \tau b \\
b & a
\end{array}\right]^{x}\right) \\
& =\frac{1}{\left|T_{s}^{F}\right|}\left(\sum_{x \in s T_{s}^{F}} \varphi\left(\left[\begin{array}{cc}
a & \tau b \\
b & a
\end{array}\right]^{x}\right)+\sum_{x \in T_{s}^{F}} \varphi\left(\left[\begin{array}{cc}
a & \tau b \\
b & a
\end{array}\right]^{x}\right)\right) \\
& =\frac{1}{\left|T_{s}^{F}\right|}\left(\sum_{x \in s T_{s}^{F}} \varphi\left(\left[\begin{array}{cc}
a & -\tau b \\
-b & a
\end{array}\right]\right)+\sum_{x \in T_{s}^{F}} \varphi\left(\left[\begin{array}{cc}
a & \tau b \\
b & a
\end{array}\right]\right)\right) \\
& =\frac{1}{\left|T_{s}^{F}\right|}\left(\left|T_{s}^{F}\right| \varphi(a-b \sqrt{\tau})+\left|T_{s}^{F}\right| \varphi(a-b \sqrt{\tau})\right) \\
& =\varphi(a-b \sqrt{\tau})+\varphi(a+b \sqrt{\tau})
\end{align*}
$$

Table 5.1: DL Characters of $G L\left(2, \mathbb{F}_{q}\right)$

| Classes | $Z_{a}$ | $C_{a, b}^{1}$ | $C_{a, b}^{2}$ | $U_{a}$ |
| :---: | :---: | :---: | :---: | :---: |
| No. | $q-1$ | $\frac{1}{2}(q-1)(q-2)$ | $\frac{1}{2} q(q-1)$ | $q-1$ |
| Size | 1 | $q(q+1)$ | $q(q-1)$ | $q^{2}-1$ |
| $R_{T_{e}}^{G}(\alpha, \beta)$ | $(q+1) \alpha(a) \beta(a)$ | $\alpha(a) \beta(b)+\alpha(b) \beta(a)$ | 0 | $\alpha(a) \beta(a)$ |
| $R_{T_{s}}^{G}(\varphi)$ | $(1-q) \varphi(a)$ | 0 | $\left(\varphi+\varphi^{q}\right)(a+b \sqrt{\tau})$ | $\varphi(a)$ |

Remark 5.1. Obtaining the irreducible characters arising from the nonsplit torus is one of the major advantage of this construction. Previously obtaining these characters were much harder as there is no natural counterpart for induction from a Borel subgroup. The arguments for getting these characters were usually ad hoc and involved guess work as it is seen in $\S 5.2$ [9].

### 5.5 Geometric Conjugacy and Irreducibles

We got $\frac{1}{2}(q-1)(q-2)$ irreducible characters from the maximal torus $T_{e}^{F}$ and $\frac{1}{2} q(q-1)$ irreducible characters from the maximal torus $T_{s}^{F}$. So in total we have $(q-1)^{2}$ irreducible characters which is comparable to the total number of irreducibles, $\left(q^{2}-1\right)$. If we look at the character table of $G^{F}$ in $\S 5.2$ [9] we see that the only characters that we have missed are $V_{\alpha}, U_{\alpha}$ which arise as components of $W_{\alpha, \alpha}$ which is the same as $R_{T_{e}}^{G}(\alpha, \alpha)$. So $V_{\alpha}, U_{\alpha}$ together with the irreducible DL characters are all the irreducible characters of $G^{F}$. It is interesting to see how the set of irreducible characters of $G^{F}$ is partitioned by geometric conjugacy classes:

- We already saw in Example 4.3 that $\left(T_{e}, 1\right)$ and $\left(T_{s}, 1\right)$ are geometrically conjugate therefore $R_{T_{e}}^{G}(1)$ and $R_{T_{s}}^{G}(1)$ have the same irreducible components. Now it is easy to check that the trivial representation occurs as component of $R_{T_{e}}^{G}(1)=\operatorname{Ind}_{B^{F}}^{G^{F}}(1)$ therefore we have:

$$
R_{T_{e}}^{G}(1)=1+\chi
$$

where $\chi$ is the character of an irreducible representation of dimension $q$. Now $R_{T_{s}}^{G}(1)$ is of norm 2, orthogonal to $\operatorname{Ind}_{B^{F}}^{G^{F}}(1)$ and with the same irreducible components hence
we can conclude that $R_{T_{s}}^{G}(1)= \pm(\chi-1)$. Checking the values shows that:

$$
R_{T_{s}}^{G}(1)=1-\chi
$$

- By Theorem 4.5 we see that the conjugacy class of $\left(T_{e},(\alpha, \beta)\right)$ and $\left(T_{s},(\omega)\right)$ consists of a single pair when they have norm 1 .
- Now consider the representation $R_{T_{e}}^{G}(\alpha, \alpha)$. One notices that:

$$
\begin{aligned}
R_{T_{e}}^{G}(\alpha, \alpha) & =(\alpha \circ \operatorname{det}) \cdot R_{T_{e}}^{G}(1) \\
& =(\alpha \circ \operatorname{det}) \cdot(1+\chi)=(\alpha \circ \operatorname{det})+(\alpha \circ \operatorname{det}) \cdot \chi
\end{aligned}
$$

the only remaining DL characters are $R_{T_{s}}^{G}(\omega)$ where $\omega^{q}=\omega$. If $\omega \in \operatorname{lrr}\left(\mathbb{F}_{q^{2}}^{\times}\right)$is of order $q-1$, there exist $\alpha \in \operatorname{Irr}\left(\mathbb{F}_{q}^{\times}\right)$such that: $\omega=\alpha \circ \mathcal{N}_{F^{2} / F}$. This establishes that the remaining pairs $\left(\left(T_{e},(\alpha, \alpha)\right)\right.$ and $\left.\left(T_{s}, \omega\right),\left(\omega=\omega^{q}\right)\right)$ are geometrically conjugate and so they have the same irreducible components as $R_{T_{e}}^{G}(\alpha, \alpha)$. Similar to the first case we have:

$$
R_{T_{s}}^{G}(\omega)=(\alpha \circ \operatorname{det})-(\alpha \circ \operatorname{det}) \cdot \chi, \quad \omega=\alpha \circ \mathcal{N}_{F^{2} / F}
$$

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[^0]:    ${ }^{1}$ This notation does not reflect the choice of Frobenius morphism $F$, however for the sake of simplicity we will omit it.
    ${ }^{2}$ DL stands for Deligne-Lusztig.

[^1]:    ${ }^{1}$ DL stands for Deligne-Lusztig.

