Algebraic Hyperbolicity for Surfaces in Smooth Complete Toric Threefolds with Picard Rank 2 and 3

by

Sharon Robins

BS-MS, IISER Kolkata, 2018

Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of
Master of Science

in the
Department of Mathematics
Faculty of Science

SIMON FRASER UNIVERSITY
Summer 2020

Declaration of Committee

Name:	**Sharon Robins**
Degree:	**Master of Science (Mathematics)**
Thesis title:	**Algebraic Hyperbolicity for Surfaces in Smooth Complete Toric Threefolds with Picard Rank 2 and 3**
Committee:	**Chair:** Michael Monagan Professor, Mathematics
	Nathan Ilten Supervisor Associate Professor, Mathematics
	Nils Bruin Committee Member Professor, Mathematics
	Kalle Karu Examiner Professor, Mathematics University of British Columbia

Abstract

Algebraic hyperbolicity serves as a bridge between differential geometry and algebraic geometry. Generally, it is difficult to show that a given projective variety is algebraically hyperbolic. However, it was established recently that a very general surface of degree at least five in projective space is algebraically hyperbolic. In this thesis, we are interested in generalizing the study of surfaces in projective space to surfaces in toric threefolds with Picard rank 2 or 3. Towards this goal, we explored the combinatorial description of toric threefolds with Picard rank 2 and 3 by following the works of Kleinschmidt and Batyrev. Then we used the method of finding algebraically hyperbolic surfaces in toric threefolds by Haase and Ilten. As a result, we were able to determine several algebraically hyperbolic surfaces in each of these varieties.

Keywords: Algebraic geometry, Toric Variety, Hyperbolicity, Geometric genus

Acknowledgements

I would like to express my deep and sincere gratitude to my supervisor, Nathan Ilten, for providing guidance and feedback throughout this thesis. I am also thankful to my family and friends for being my side in this two year journey.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

Hyperbolicity has long been a topic of interest in the study of differential geometry. A smooth complex projective variety X is said to be Brody hyperbolic if there is no nonconstant holomorphic map $f : \mathbb{C} \to X$ [2]. The hyperbolicity of curves can be completely determined by their geometric genus (Definition 5.1). A curve is hyperbolic if and only if the geometric genus is at least 2. However, the determination of hyperbolicity becomes complicated as we go to higher dimensions. Demailly [6] introduced the notion of algebraic hyperbolicity (Definition 5.4) to improve the study of hyperbolicity. Compared to hyperbolicity, it is easier to determine algebraic hyperbolicity of varieties. Moreover, in light of Demailly's conjecture (Conjecture 5.8), which states the equivalence of hyperbolicity and algebraic hyperbolicity, we can focus on algebraic hyperbolicity of higher dimensional varieties. In this thesis, we study the algebraic hyperbolicity of surfaces.

There is good progress in understanding the hyperbolicity of surfaces in $\mathbb{P}^3$. It is known that a surface of degree at most four contains a curve of genus 0 or 1, and hence cannot be algebraically hyperbolic. Since some surfaces contain lines, we cannot expect all smooth surfaces of degree at least five to be algebraically hyperbolic. But it is still relevant to check almost every surface of a given degree in light of the definition of a very general surface (Definition 3.5).

Theorem 1.1 (Xu [23], Coskun and Riedl [3])**.** *A very general surface of degree at least five in $\mathbb{P}^3$ is algebraically hyperbolic.*

Haase and Ilten [13] initiated the study of algebraically hyperbolic surfaces in toric threefolds. *Toric varieties* (Definition 2.1) are a rich class of varieties that are easily accessible. In $\mathbb{P}^3$, it is natural to study the surfaces of same degree. For a general variety X, we will study a group $\mathrm{Pic}(X)$, called the Picard group (Definition 3.2) of X, and associate to every surface a class in $\mathrm{Pic}(X)$ generalizing the degree of a surface in $\mathbb{P}^3$. Additionally, We require this class to be basepoint free (Definition 3.4). The classes of basepoint free divisors generate a cone called the nef cone (Definition 3.26). The following theorem ensures the existence of hyperbolic surfaces of higher degrees.

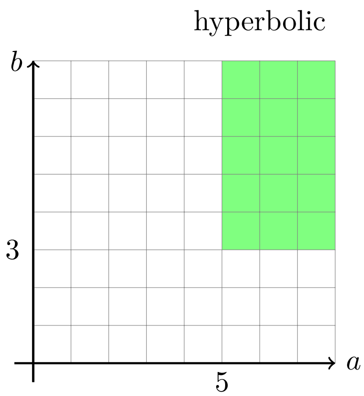


Figure 1.1: Algebraic hyperbolicity for a very general surface of the type $aH_1 + bH_2$ in $\mathbb{P}^2 \times \mathbb{P}^1$.

Theorem 1.2 (Haase and Ilten [13], Theorem 1.2)**.** *Let X be a smooth projective toric threefold with the nef cone $\operatorname{Nef}(X)$. There exists an ample divisor class H_0 such that for all divisors D whose class lies in $H_0 + \operatorname{Nef}(X)$, a very general surface $S \in |D|$ is algebraically hyperbolic.*

Example 1.3. Consider the variety $\mathbb{P}^2 \times \mathbb{P}^1$. Let H_1 be the pullback of the hyperplane class from $\mathbb{P}^2$ and let H_2 be the pullback of the hyperplane class from $\mathbb{P}^1$. The nef cone is generated by H_1 and H_2. A very general hypersurface in the class aH_1+bH_2 is algebraically hyperbolic if $a \geq 5$ and $b \geq 3$. See Figure 1.1. We refer the reader to [13, Example 6.1] for more details.

In this thesis, we will make Theorem 1.2 explicit in the case that X is a smooth toric threefold of Picard rank 2 or 3. Kleinschmidt [17] classified all smooth complete toric varieties with Picard rank 2. They are produced from the projectivization of decomposable toric vector bundles over $\mathbb{P}^1$ or $\mathbb{P}^2$ (Section 4.2). We will see a combinatorial interpretation of these varieties in Chapter 4.

We will get an explicit bound of algebraic hyperbolicity by imposing an additional condition on divisors, a configuration of divisors with connected sections (Theorem 5.19). We will obtain the following results.

Theorem 1.4. *Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(l)$ be a locally free sheaf on $\mathbb{P}^2$, let $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a projective toric bundle, F be the pullback of the hyperplane class, and ξ be the class of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. The nef cone of $\mathbb{P}(\mathcal{E})$ is generated by F and ξ. Let S be a very general surface in the class $\mathcal{O}(a\xi + bF)$.*

1. *Suppose $l = 0$. Then S is algebraically hyperbolic if $a \geq 3$ and $b \geq 5$.*
2. *Suppose $l = 1$. Then S is algebraically hyperbolic if $a \geq 3$ and $b \geq 4$ or $a = 2$ and $b \geq 7$ or $b = 0$ and $a \geq 6$.*

3. *Suppose $l = 2$ or 3. Then S is algebraically hyperbolic if $a \geq 3$ and $b \geq 4$ or $a = 2$ and $b \geq 7$ or $b = 0$ and $a \geq 4$.*

4. *Suppose $l \geq 4$. Then S is algebraically hyperbolic if $a \geq 3$ and $b \geq 4$ or $a = 2$ and $b \geq 7$ or $b = 0$ and $a \geq 3$.*

Theorem 1.5. *Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(l_1) \oplus \mathcal{O}_{\mathbb{P}^1}(l_2)$ be a locally free sheaf on $\mathbb{P}^1$, let $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ be a projective toric bundle, F be the pullback of the hyperplane class and ξ be the class of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. The nef cone of $\mathbb{P}(\mathcal{E})$ is generated by F and ξ. Let S be a very general surface in the class $\mathcal{O}(a\xi + bF)$.*

1. *Suppose $l_1 = l_2 = 0$. Then S is algebraically hyperbolic if $a \geq 5$ and $b \geq 3$.*

2. *Suppose $l_1 = 0, l_2 \geq 1$. Then S is algebraically hyperbolic if $a \geq 5$ and $b \geq 2$ or $a = 2$ and $b \geq 7$ or $b = 0$ and $a \geq 6$.*

3. *Suppose $l_1 \geq 1$. Then S is algebraically hyperbolic if $a \geq 5$ and $b \geq 0$.*

We will also obtain surfaces which are not hyperbolic. See Theorem 6.3 and Theorem 6.10. The bounds we obtain for algebraic hyperbolicity in these theorems are close to sharp, leaving only a few cases unresolved. Additionally, we study all smooth complete toric threefolds with Picard rank 3 classified by Batyrev. See Theorems 7.3, 7.11, 7.18, 7.25, 7.32 and 7.39.

In Chapter 2, we briefly review the basics of toric varieties needed later on. In Chapter 3, we review divisors and intersection theory. In Chapter 4, we recall the classification of toric varieties by Kleinschmidt and Batyrev. Chapter 5 introduces the notion of algebraic hyperbolicity and discusses conditions that decide algebraic hyperbolic surfaces. In Chapter 6 and 7, we will focus on finding the algebraically hyperbolic surfaces of those toric varieties described in Chapter 4.

Chapter 2

Toric Varieties

In this chapter, we will review the basics of toric varieties. We assume the reader is familiar with the basics of algebraic geometry. See [15] or Chapter 1 of [14]. Unless otherwise specified, by the term variety, we mean an irreducible quasi-projective variety. Because of the availability of textbooks [5] and [9] on toric varieties, we will only discuss the definitions important for this thesis. For convenience, we always follow the notations of [5].

A variety of the form $(\mathbb{C}^*)^n$ is known as a torus. It is a group under component-wise multiplication.

Definition 2.1. A toric variety X is an algebraic variety that contains a torus as an open dense subset with the action of the torus on itself extending to an action on X.

Example 2.2. The affine and projective spaces are toric varieties.

1. Consider $\mathbb{A}^n$ with the open dense subset $\{(t_1, \cdots, t_n) : t_i \neq 0 \text{ for all i}\} \cong (\mathbb{C}^*)^n$ where the torus action is given by

$$(\mathbb{C}^*)^n \times \mathbb{A}^n \to \mathbb{A}^n$$
$$(t_1, \cdots, t_n).(x_1, \cdots, x_n) \mapsto (t_1 x_1, \cdots, t_n x_n).$$

2. Consider $\mathbb{P}^n$ with the open dense subset $\{[1 : t_1 : \cdots : t_n] : t_i \neq 0\} \cong (\mathbb{C}^*)^n$ where the torus action is given by

$$(\mathbb{C}^*)^n \times \mathbb{P}^n \to \mathbb{P}^n$$
$$(t_1, \cdots, t_n).[x_0 : x_1 : \cdots : x_n] \mapsto [x_0 : t_1 x_1 : \cdots : t_n x_n].$$

Toric varieties are a rich class of varieties that are easily accessible. For the last few decades, there has been an increasing interest in studying various geometrical aspects of toric varieties. The geometry of a toric variety can be expressed by combinatorial data, which is generally easily determined. It is easy to compute the Picard group, nef cone, cohomology and many other geometric aspects of toric varieties using the combinatorial data.

2.1 Affine toric varieties

We begin with a brief review of affine varieties to fix notation and terminology. See [5, Chapter 1] for more details. Affine varieties are considered to be building blocks of all varieties. For us, an affine variety is an irreducible algebraic set. Let V be an affine variety with coordinate ring $\mathbb{C}[V]$. Hilbert's Nullstellensatz says that maximal ideals of $\mathbb{C}[V]$ are in one-to-one correspondence with points in V [14, Corollary 1.4]. Let R be a ring and $\mathrm{Specm}(R)$ be the set of all maximal ideals of R. We can write

$$V = \mathrm{Specm}(\mathbb{C}[V]).$$

This gives a correspondence between nilpotent-free finitely generated algebra over $\mathbb{C}$ and complex affine varieties [5, Lemma 1.0.1]. In this section, we will see the construction of a nilpotent-free finitely generated algebra from a combinatorial object called a cone.

Before defining a cone, we need to understand the character lattice of a torus. A character of a torus T is a morphism $\chi : T \to \mathbb{C}^*$ that is a group homomorphism. The characters of $(\mathbb{C}^*)^n$ form a group M isomorphic to $\mathbb{Z}^n$ [5, Section 1.1].

Let $M \cong \mathbb{Z}^n$ be a lattice of rank n, and $N = \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice. The pairing between $m \in M$ and $u \in N$ is denoted by $\langle m, u\rangle \in \mathbb{Z}$.

Definition 2.3. A rational polyhedral cone σ in the vector space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ is a subset of the form

$$\sigma = \mathrm{Cone}(u_1, \dots, u_s) = \{r_1u_1 + \ldots + r_su_s : \ r_i \geq 0\}$$

where $u_i \in N$.

We say σ is strongly convex if $\sigma \cap (-\sigma) = \{0\}$. Throughout the thesis, we use the term cone to denote a strongly convex rational polyhedral cone. The dimension of σ is defined as the dimension of the subspace spanned by σ in $N_{\mathbb{R}}$. A cone generated by one element is called a *ray* and is usually denoted by ρ. The semigroup $\rho \cap N$ is generated by a unique element denoted by u_ρ. We call u_ρ the *ray generator* of ρ.

For every cone $\sigma \subset N_{\mathbb{R}}$, we can associate a dual cone in $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. The dual cone is defined by

$$\sigma^\vee = \{m \in M_{\mathbb{R}} : \langle m, u\rangle \geq 0 \text{ for all } u \in \sigma\}.$$

A subset τ of σ is called a *face* if

$$\tau = \{u \in \sigma : \langle m, u\rangle = 0\}$$

for some $m \in \sigma^\vee$.

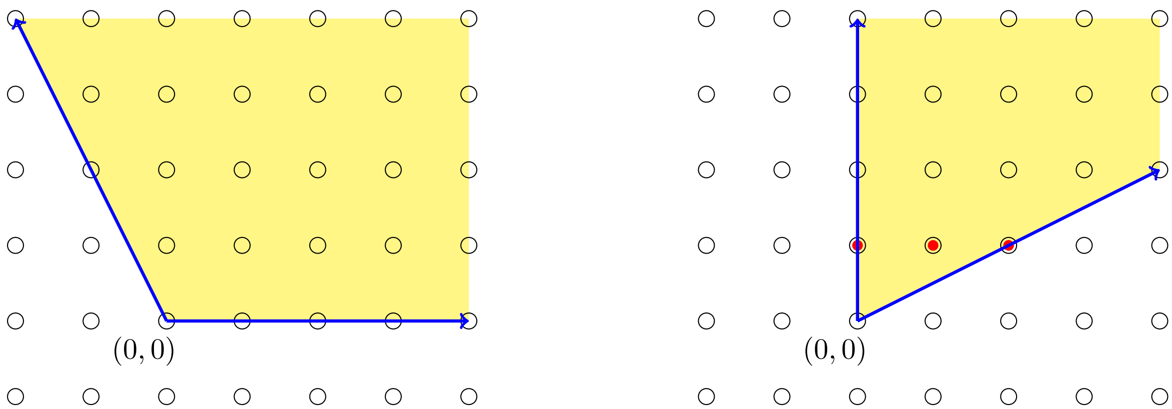
Theorem 2.4 (Gordan's lemma)**.** *Let σ be a rational polyhedral cone, then the semigroup $S_\sigma = \sigma^\vee \cap M$ is finitely generated.*

Proof. This is Proposition 1.2.17 from [5]. $\square$

Any additive semigroup S_σ determines a commutative $\mathbb{C}$-algebra $\mathbb{C}[S_\sigma]$. We construct it as a $\mathbb{C}$-vector space with basis the formal symbols $\{\chi^m : \ m \in S_\sigma\}$, and multiplication defined by

$$\chi^m . \chi^{m'} = \chi^{m+m'}.$$

Definition 2.5. We define the affine toric variety associated to σ by $U_\sigma := \mathrm{Specm}(\mathbb{C}[S_\sigma])$.



(a) σ (b) $\sigma^\vee$

Figure 2.1: An example of a cone and its dual cone

Example 2.6. Here is a simple example when $n = 2$. Fix a $\mathbb{Z}$-basis $\{e_1, e_2\}$ of N and let $\{e_1^*, e_2^*\}$ be the dual basis of M. Set $x = \chi^{e_1^*}$, $y = \chi^{e_2^*}$. Let $\sigma =$Cone$(e_1, -e_1 + 2e_2)$. Then we can show that the semigroup $\sigma^\vee \cap M$ is generated by $e_2^*, e_1^* + e_2^*, 2e_1^* + e_2^*$. See Figure 2.1. We have

$$U_\sigma = \mathrm{Specm}(\mathbb{C}[y, xy, x^2y]) \cong \mathrm{Specm}\left(\frac{\mathbb{C}[u, v, w]}{\langle v^2 - uw\rangle}\right) \cong \{(u, v, w) \in \mathbb{C}^3 : v^2 = uw\}.$$

2.2 General toric varieties

We discussed the building blocks in the previous section. This section will demonstrate how to glue them together appropriately to form new varieties with specific features. We begin by considering three important properties of a variety. We refer the reader to [5, Chapter 3].

Definition 2.7. Let X be a variety.

(a) We say X is normal if the local rings $\mathcal{O}_{X,p}$ are normal for all $p \in X$.

(b) We say X is separated if the image of the diagonal map $\Delta : X \to X \times X$ is Zariski closed in $X \times X$.

(c) We say X is complete if for every variety Z, the projection $\pi : X \times Z \to Z$ is a closed mapping in the Zariski topology.

Remark 2.8. Any projective variety is complete, and a variety is complete if and only if it is compact in the classical topology.

Next, we will define a fan and illustrate how to construct toric variety from this data.

Definition 2.9. A fan Σ in $N_{\mathbb{R}}$ is a collection of cones such that

(i) Every face of a cone $\sigma \in \Sigma$ is in Σ.

(ii) If $\sigma_1, \sigma_2 \in \Sigma$ then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

Furthermore, if Σ is a fan, then $\Sigma(r)$ is the set of r-dimensional cones of Σ. If τ is a face of σ, we have an open embedding $U_\tau \hookrightarrow U_\sigma$. Since $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2, $U_{\sigma_1 \cap \sigma_2}$ can be identified as an open subvariety of U_{σ_1} and U_{σ_2}. For a fan Σ in N, we can glue $\{U_\sigma : \sigma \in \Sigma\}$ together to obtain a normal separated toric variety

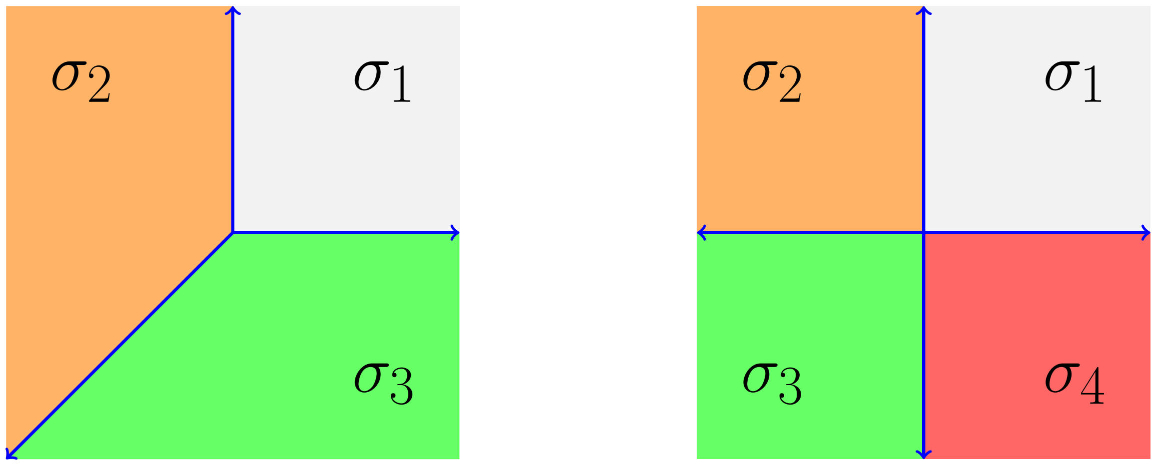
$$X_\Sigma := \bigcup_{\sigma \in \Sigma} U_\sigma.$$

Indeed, any normal separated toric variety X can be realized as a toric variety coming from a fan Σ. See [5, Section 3.1] for more details.

Remark 2.10. Since the affine toric variety corresponding to the trivial cone $\{0\}$ is the torus $T_N = N \otimes \mathbb{C}^* = \mathrm{Specm}(\mathbb{C}[M])$, we see that T_N is an affine open subset of X_Σ. The action of the torus on itself extends to an algebraic action of T_N on X_Σ.

Example 2.11. Let $\{e_1, e_2\}$ be a $\mathbb{Z}$-basis of N and let $\{e_1^*, e_2^*\}$ be the dual basis of M. Set $x = \chi^{e_1^*}$, $y = \chi^{e_2^*}$.

1. Let Σ be the fan consisting of $\sigma_1 = \mathrm{Cone}(e_1, e_2)$, $\sigma_2 = \mathrm{Cone}(e_2, -e_1 - e_2)$, $\sigma_3 = \mathrm{Cone}(-e_1 - e_2, e_1)$ and their faces. The corresponding toric variety is covered by open affine subsets $U_{\sigma_1} = \mathrm{Specm}(\mathbb{C}[x, y])$, $U_{\sigma_2} = \mathrm{Specm}(\mathbb{C}[x^{-1}, x^{-1}y])$ and $U_{\sigma_3} = \mathrm{Specm}(\mathbb{C}[xy^{-1}, y^{-1}])$. Let $[x_0 : x_1 : x_2]$ be the homogeneous coordinates on $\mathbb{P}^2$ and let $U_i \subset \mathbb{P}^2$ be the standard affine open subsets. Then $U_i \cong U_{\sigma_i}$ under the identification $x = x_1/x_0$ and $y = x_2/x_0$. Hence we have $X_\Sigma \cong \mathbb{P}^2$.



(a) Fan of $\mathbb{P}^2$ (b) Fan of $\mathbb{P}^1 \times \mathbb{P}^1$

Figure 2.2: Two fans in $\mathbb{R}^2$

2. Let Σ be the fan consisting of $\sigma_1 = \text{Cone}(e_1, e_2)$, $\sigma_2 = \text{Cone}(e_2, -e_1)$, $\sigma_3 = \text{Cone}(-e_1, -e_2)$, $\sigma_4 = \text{Cone}(-e_2, e_1)$ and their faces. The corresponding toric variety is covered by open affine subsets $U_{\sigma_1} = \text{Specm}(\mathbb{C}[x, y])$, $U_{\sigma_2} = \text{Specm}(\mathbb{C}[x^{-1}, y])$, $U_{\sigma_3} = \text{Specm}(\mathbb{C}[x^{-1}, y^{-1}])$ and $U_{\sigma_4} = \text{Specm}(\mathbb{C}[x, y^{-1}])$. The gluing of U_{σ_1} and U_{σ_4} (respectively U_{σ_2} and U_{σ_3}) along second coordinate gives $\mathbb{C} \times \mathbb{P}^1$. These two varieties glue by the map $x \to x^{-1}$ which gives $\mathbb{P}^1 \times \mathbb{P}^1$.

See Figure 2.2.

The properties of a fan Σ gives a lot of information about the geometry of the toric variety X_Σ.

Theorem 2.12. *Let X_Σ be the toric variety associated to a fan Σ in $N_\mathbb{R}$. Then:*

1. *X_Σ is complete if and only if $\bigcup\limits_{\sigma \in \Sigma} \sigma = N_\mathbb{R}$.*
2. *X_Σ is smooth if and only if the ray generators of every cone $\sigma \in \Sigma$ are part of a $\mathbb{Z}$-basis of N.*

Proof. This is Theorem 3.1.19 from [5]. $\square$

Example 2.13. By Theorem 2.12, it follows that the varieties $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ in Example 2.11 are smooth and complete.

Chapter 3

Divisors and Intersection theory

This chapter is concerned with divisors and intersection theory on toric varieties. First, we introduce divisors in a general setting and discuss the special case of smooth complete toric varieties. In Section 3.3, we will see the connection between divisors and polytopes. In Section 3.4, we show how to compute the intersection products on toric varieties.

3.1 Cartier divisors

Our main interest in this section is to introduce an abelian group $\mathrm{Pic}(X)$, called the Picard group of X, and associate to every hypersurface $S \subset X$, a class in $\mathrm{Pic}(X)$ generalizing the degree of a surface in $\mathbb{P}^3$. We refer the reader to [11, Chapter 11] and [5, Chapter 4] for more details.

Definition 3.1. A Cartier divisor on the normal variety X is an equivalence class D of collections $\{(U_i, f_i)\}_{i=0}^n$. Here, U_i form an open covering of X, and f_i are rational functions on U_i such that $f_i f_j^{-1}$ is a unit on $U_i \cap U_j$. Two collections $\{(U_i, f_i)\}_{i=0}^n$, $\{(V_j, g_j)\}_{j=0}^m$ are equivalent if $f_i g_j^{-1}$ is a unit on $U_i \cap V_j$ for all i, j.

The set of Cartier divisors is denoted by $\mathrm{CDiv}(X)$. We can see that it is an abelian group. In fact, given divisors $D = \{(U_i, f_i)\}$, $E = \{(V_i, g_i)\}$ we have $D + E = \{(U_i \cap V_j, f_i g_j)\}$. A Cartier divisor is called principal if it is equal to a divisor given by (X, f). Divisors D and E are linearly equivalent, written $D \sim E$, if their difference is a principal divisor.

Definition 3.2. The group of Cartier divisors modulo principal divisors is called the *Picard group* of X and is denoted by $\mathrm{Pic}(X)$. If D is a Cartier divisor, we write $[D] \in \mathrm{Pic}(X)$ for its equivalence class.

There is a more geometrically intuitive notion of divisors, called Weil divisors. A prime divisor is a codimension one subvariety and $\mathrm{Div}(X)$ is the free abelian group generated by the prime divisors on X. A Weil divisor is an element of $\mathrm{Div}(X)$.

For a normal variety X and a prime divisor V there exists a function called its valuation $v_V : \mathbb{C}(X)^* \to \mathbb{Z}$. Then $v_V(f)$ gives the order of vanishing of f along V. Moreover, $v_V(f)$ is zero for all but a finite number of prime divisors $V \subset X$. One may find more information on valuations from [5, Chapter 4].

Let us denote by $\mathrm{Div}_0(X)$ the subgroup of $\mathrm{Div}(X)$ consisting of principal divisors on X, i.e., divisors of the form

$$\mathrm{div}(f) := \sum_V v_V(f)V$$

for a nonzero rational function f on X. The group of Weil divisors modulo principal divisors is called the *Class group* of X and is denoted by $\mathrm{Cl}(X)$.

Remark 3.3. A Cartier divisor $D = \{(U_i, f_i)\}_{i=0}^n$ determines a Weil divisor as follows: If a prime divisor V has non empty intersection with U_i, set $a_V = v_V(f_i)$. It is possible that V has non empty intersection with U_j for some $i \neq j$. Since $f_i f_j^{-1}$ is a unit on $U_i \cap U_j$, we have that $v_V(f_i) = v_V(f_j)$. Since there are finitely many U_i, they globally glue together to give a Weil divisor

$$D = \sum_{U_i \cap V \neq \emptyset} a_V V.$$

We have $\mathrm{CDiv}(X) \subset \mathrm{Div}(X)$, and for a smooth variety, both notions of divisors agree, and $\mathrm{Pic}(X)$=$\mathrm{Cl}(X)$.

We will be dealing with smooth varieties, hence we use the term divisor. A divisor D is called effective if it is a nonnegative linear combination of prime divisors. We denote this by $D \geq 0$. The support of the divisor $D = \sum a_i D_i$ is the union of prime divisors appearing in D:

$$\mathrm{Supp}(D) = \bigcup_{a_i \neq 0} D_i.$$

Definition 3.4. A Cartier divisor D on a variety X is basepoint free if for every $p \in X$, there is a Cartier divisor D' in the divisor class of D such that $p \notin \mathrm{Supp}(D')$.

The complete linear system of D is defined to be

$$|D| = \{E \in \mathrm{CDiv}(X) |\ E \sim D,\ E \geq 0\}.$$

Thus, the complete linear system of D consists of all effective Cartier divisors on X linearly equivalent to D. Note that the complete linear systems in $\mathbb{P}^3$ are exactly surfaces of fixed degree [14, Proposition 6.4]. To D, we associate a vector space $L(D)$ over $\mathbb{C}$ as follows

$$L(D) = \{f \in \mathbb{C}(X)^* |\ \mathrm{div}(f)+D \geq 0\} \cup \{0\}.$$

It is immediate from the definition that it is a vector space. Further, we associate a projective variety, denote by $\mathbb{P}(L(D))$, to $L(D)$ via the following definition

$$\mathbb{P}\left(L(D)\right) := \left(L(D) \backslash \{0\}\right) / \sim,$$

where $\sim$ is the equivalence relation defined by $f \sim g$ if and only if $f = \lambda g$ where $\lambda \in \mathbb{C}^*$. There is a natural bijection between the elements of $|D|$ and the points in the projective space $\mathbb{P}(L(D))$. See [5, Exercise 6.0.11]. We will use the following definition for all our main results [7, p. 10].

Definition 3.5. A very general point of a variety X having a property P means that there is a countable union V of proper subvarieties of X such that every point p of $X \backslash V$ has the property P. Any member of $|D|$ corresponding to a very general point of $\mathbb{P}(L(D))$ is said to be a very general member of $|D|$.

3.2 Torus-invariant divisors

Let X_Σ be the toric variety of a fan Σ in $N_\mathbb{R}$ with $\dim(N_\mathbb{R}) = n$. Then X_Σ is a normal variety of dimension n. We will be primarily concerned with subvarieties on a toric variety X_Σ that are mapped to themselves by the torus action. We refer the reader to [5, Chapter 3] for more details.

By [5, Theorem 3.2.6], a k-dimensional cone σ of Σ corresponds to a $(n-k)$-dimensional subvariety $V(\sigma)$ in X_Σ. For $\rho_i \in \Sigma(1)$, we will denote it by D_i rather than $V(\rho_i)$ to point out that it is a divisor. The complement

$$X_\Sigma \backslash T = \bigcup_{\rho_i \in \Sigma(1)} D_i$$

is called the toric boundary of X_Σ. For $m \in M$, the character χ^m is a rational function on X_Σ, and its divisor is given by [5, Proposition 4.1.2]

$$\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho.$$

Here is how torus-invariant divisors relate to the Picard group:

Lemma 3.6. *Let X_Σ be a smooth complete toric variety, then we have the short exact sequence*

$$0 \to M \to \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho \to \mathrm{Pic}(X_\Sigma) \to 0,$$

where the first map is $m \mapsto \mathrm{div}(\chi^m)$, and the second map sends a torus invariant divisor to its divisor class in $\mathrm{Pic}(X_\Sigma)$.

Proof. Since X_Σ is smooth, $\mathrm{Pic}(X_\Sigma)$ =$\mathrm{Cl}(X_\Sigma)$. The statement then follows from (5, Theorem 4.2.1). $\square$

Example 3.7. Let $r \in \mathbb{N}$ and consider the fan Σ_r in $\mathbb{R}^2$ consisting of four maximal cones σ_i shown in Figure 3.1. We call X_{Σ_r} the Hirzebruch surface $\mathcal{H}_r$. Let D_1, D_2, D_3, D_4 denotes

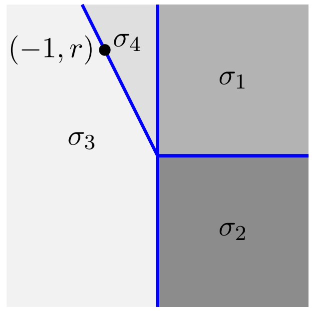


Figure 3.1: A fan Σ_r

the divisors corresponding to the ray generators $u_1 = e_1, u_2 = e_2, u_3 = -e_1 + re_2, u_4 = -e_2$. By Lemma 3.6, we have the following relations

$$\begin{aligned}\operatorname{div}(\chi^{e_1}) &= D_1 - D_3 \sim 0 \\ \operatorname{div}(\chi^{e_2}) &= D_2 + rD_3 - D_4 \sim 0.\end{aligned}$$

Thus, $\operatorname{Pic}(\mathcal{H}_r) \cong \mathbb{Z}[D_3] + \mathbb{Z}[D_4]$.

Definition 3.8. Let X_Σ be a toric variety. The rank of $\operatorname{Pic}(X_\Sigma)$ is called the Picard rank of X_Σ.

Remark 3.9. For a general variety, the Picard rank is the rank of the Neron-Severi group [19, Definition 1.1.15]. For a toric variety, the Neron-Severi group coincide with the Picard group [5, Proposition 6.3.15].

Next, we introduce a tool for explicit computations.

Definition 3.10. Let Σ be a complete fan. A support function is a function $\varphi : N_\mathbb{R} \to \mathbb{R}$ that is linear on each cone of Σ.

Let $D = \sum a_\rho D_\rho$ be a divisor on X_Σ. We define a function φ_D such that

$$\varphi_D(u_\rho) = -a_\rho \text{ for all } \rho \in \Sigma(1). \tag{3.1}$$

It can be extended uniquely to a support function on $N_\mathbb{R}$. We will return to support functions in Theorem 4.5, where we will use them to give elegant criteria for a divisor to be basepoint free.

Every normal variety X has a specific Weil divisor called the canonical divisor class, and K_X denotes any representative [5, Definition 8.0.20]. Later on, we will see that the canonical divisor of a smooth variety plays a particularly important role. For the toric variety X_Σ, there is a natural choice of representative [5, Theorem 8.2.3]. It is provided by

$$K_{X_\Sigma} = \sum_{\rho \in \Sigma(1)} -D_\rho. \tag{3.2}$$

3.3 Polytopes

In this section, we investigate the relation between divisors and polytopes. See [5] or [9, Chapter 3] for more details. Polytopes play a significant role in studying toric varieties.

Let M and N be dual lattices with associated vector spaces $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$.

Definition 3.11. A convex polytope $P \subset M_{\mathbb{R}}$ is the convex hull of a finite set of points in $M_{\mathbb{R}}$. It is called a lattice polytope if it is the convex hull of a finite set $S \subset M$.

Throughout the thesis, we use the term polytope to denote a convex polytope. The dimension of a polytope P is the dimension of the subspace spanned by the differences $\{m_1 - m_2 : m_1, m_2 \in P\}$. An affine half-space of $M_{\mathbb{R}}$ is a subset of the form

$$H^+_{u,b} = \{m \in M_{\mathbb{R}} : \langle m, u\rangle \geq b\} \quad \text{for} \quad u \in N_{\mathbb{R}} \backslash \{0\} \quad \text{and} \quad b \in \mathbb{R}.$$

A polytope can be written as a finite intersection of affine half-spaces. A subset $Q \subset P$ is a face of P, written $Q < P$, if there are $u \in N_{\mathbb{R}} \backslash \{0\}$ and $b \in \mathbb{R}$ with

$$P \subset H^+_{u,b} \quad \text{and} \quad Q = \{m \in P : \langle m, u\rangle = b\}.$$

We also consider P as an improper face. The faces of dimensions 0, 1, $\dim(P) - 1$ are called vertices, edges, and facets, respectively. Thus, in particular, the vertices are the minimal non-empty faces, and the facets are the maximal proper faces.

Polytopes arise naturally when dealing with toric varieties.

Definition 3.12. Let $D = \sum a_\rho D_\rho$ be a Cartier divisor on X_Σ. We define

$$P(D) := \{v \in M_{\mathbb{R}} : \langle v, u_\rho\rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\}.$$

Remark 3.13. If Σ is a complete fan, then $P(D)$ is a polytope [5, Proposition 4.3.8].

Example 3.14. Consider the variety $\mathbb{P}^2$ discussed in Example 2.11. Let $D = 3D_0$ be a divisor on $\mathbb{P}^2$, where D_0 is the divisor corresponding to the ray generated by $-e_1 - e_2$. Then a point $(x, y) \in P(D)$ if and only if

$$\begin{aligned} x &\geq 0, \\ y &\geq 0, \\ -x - y &\geq -3. \end{aligned}$$

See Figure 3.2.

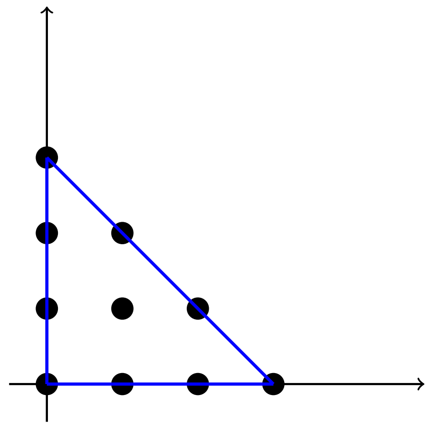


Figure 3.2: The polytope of Example 3.14

Remark 3.15. After choosing a basis $e_1^*, \cdots, e_n^*$ for M, we can consider $P(D) \subset \mathbb{R}^n$. We usually present it in the form

$$P(D) = P(A,t) = \{x \in \mathbb{R}^n \ : \ Ax \geq t\} \text{ for some } A \in \mathbb{R}^{m \times n},\ t \in \mathbb{R}^m.$$

The main feature of $P(D)$ is that the lattice points of $P(D)$ determine a basis for the vector space $L(D)$.

Theorem 3.16. *If D is a torus-invariant divisor on X_Σ, then*

$$L(D) = \bigoplus_{m \in P(D) \cap M} \mathbb{C} \cdot \chi^m.$$

Proof. This is Proposition 4.3 from [5]. □

Given two polytopes $P, Q \in M_{\mathbb{R}}$, we define their Minkowski sum

$$P + Q := \{x + y : x \in P,\ y \in Q\}.$$

The following proposition shows how a divisor's geometric properties reflect the combinatorial properties of the polytope $P(D)$.

Proposition 3.17. *Let D and E be divisors on a complete toric variety X_Σ. Then:*

(i) $P(kD) = kP(D)$ *for* $k \geq 0$.

(ii) $P(D + \operatorname{div}(\chi^m)) = P(D) - m$.

(iii) $P(D) + P(E) \subset P(D + E)$.

(iv) If D and E are basepoint free, then $P(D) + P(E) = P(D + E)$.

Proof. See [5, Exercise 4.3.2, Theorem 6.1.7]. □

3.4 Intersection theory

Counting the number of intersection points with a given curve and surface is vital in our main tools for finding algebraic hyperbolicity. We restrict our attention to how to compute the intersection product for a toric threefold. For the general theory, see [10] or [7]. For toric varieties, see [5, Section 6.3, 12.5] and [9, Chapter 5].

We will start this section by recalling Bezout's theorem [7, Theorem 1.1]: given two projective plane curves C and D of degree d and e which intersect transversely (Definition 3.20), then the number of intersection points is de. The beauty of this theorem is not about the number de, but simply the fact that this cardinality of intersection does not depend on the particular choices of C and D subject to the hypothesis that they meet transversely. This theory can be generalized. We begin by the following definition.

Definition 3.18. On a variety X, the Chow group $A_k(X)$ is defined to be the free abelian group on the k-dimensional subvarieties of X, modulo the subgroup generated by the cycles of the form $\operatorname{div}(f)$, where f is a nonzero rational function on a $(k+1)$-dimensional subvariety of X. A k-cycle on X is a finite formal sum $\sum n_i V_i$, where V_i are k-dimensional subvarieties of X, and the n_i are integers. If A is a cycle, we write $[A] \in A_k(X)$ for its equivalence class.

Remark 3.19. We have $A_{n-1}(X) = \mathrm{Cl}(X)$.

Definition 3.20. We say that subvarieties A, B of a variety X intersect transversely at a point p if A, B and X are all smooth at p and the tangent spaces to A and B at p together span the tangent space to X. We say that subvarieties $A, B \subset X$ intersect generically transversely if there is a dense set of points in the intersection at which they are transverse. We extend the terminology to cycles by saying that two cycles $A = \sum n_i A_i$ and $B = \sum m_j B_j$ are generically transverse if each A_i is generically transverse to each B_j.

Theorem 3.21 (Moving Lemma)**.** *Given classes $\alpha \in A_k(X), \beta \in A_l(X)$ in the Chow group of a smooth projective variety X, we can find representative cycles α', β' intersecting generically transversely. Moreover, the class of the intersection of these cycles is independent of the choice of α', β'.*

Proof. This is Theorem 1.6 from [7]. □

Using Theorem 3.21, it is easy to define the intersection product on the Chow groups of a smooth variety: $\alpha \cdot \beta$ is defined to be the class $\alpha' \cap \beta'$.

Theorem 3.22. *Let X be a smooth projective variety and set $A^k(X) = A_{n-k}(X)$. Then there exists a unique intersection product on $A(X) = \oplus A^k(X)$ satisfying the following condition: if two subvarieties A, B of X are generically transverse, then*

$$[A] \cdot [B] = [A \cap B].$$

This structure makes $A(X)$ into an associative, commutative ring, graded by codimension, called the Chow ring of X.

Proof. This is Theorem 1.5 from [7]. □

The elements of the Chow group $A_0(X)$ can be seen as the formal sum of points in X. If X is complete, then there is a degree map [7, Proposition 1.21]

$$\begin{aligned} \deg : A_0(X) &\to \mathbb{Z} \\ \sum n_P P &\mapsto \sum n_P. \end{aligned}$$

Definition 3.23. Let $A \in A^k(X), B \in A^{n-k}(X)$. Then $\deg([A] \cdot [B])$ is defined as the intersection number. If C is a curve and D is a divisor, by abuse of notation we will write $C \cdot D$ in place of $\deg([C] \cdot [D])$.

Generally, it is not easy to describe Chow groups. But for a toric variety, we have the following result.

Lemma 3.24. *The Chow groups $A_k(X)$ of a toric variety $X = X_\Sigma$ are generated by the classes of $V(\sigma)$ corresponding to the cones σ of dimension $n - k$ of Σ.*

Proof. See [9, p.96]. □

Hence, it is enough to find the intersection product of torus invariant subvarieties for a toric variety.

Theorem 3.25. *Let Σ be a smooth complete fan. Let $\rho \in \Sigma(1)$ and σ be a cone of Σ not containing ρ. Then,*

$$[D_\rho] \cdot [V(\tau)] = \begin{cases} [V(\gamma)] & \text{if} \quad \gamma = \rho + \tau \in \Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. See [5, Lemma 12.5.2] or [9, p.98]. □

We are interested in calculating the intersection products in smooth projective toric threefolds. Using the above theorem, we can find all intersection products in a toric threefold.

3.5 Positivity properties of divisors

In this section, we will discuss several kinds of positivity of a Cartier divisor. Roughly speaking, the positivity properties of a divisor D relate to the vector space $L(D)$ having a large dimension. See [19] and [5] for more details.

Definition 3.26. Let X_Σ be a projective toric variety. The nef cone

$$\mathrm{Nef}(X_\Sigma) \subset \mathrm{Pic}(X_\Sigma) \otimes \mathbb{R},$$

is the convex cone in $\mathrm{Pic}(X_\Sigma) \otimes \mathbb{R}$ spanned by the classes of all basepoint free divisors. The effective cone

$$\mathrm{Eff}(X_\Sigma) \subset \mathrm{Pic}(X_\Sigma) \otimes \mathbb{R},$$

is the convex cone spanned by the classes of all effective divisors.

By [5, Lemma 15.1.8], we have

$$\mathrm{Eff}(X_\Sigma) = \mathrm{Cone}\left([D_\rho] : \rho \in \Sigma(1)\right).$$

In the next chapter, we will discuss a criterion to determine the basepoint free divisors (Theorem 4.5). In light of Kleiman's numerical criterion of ampleness [5, Theorem 6.3.22] and [19, Theorem 2.2.26], we may consider the following definitions.

Definition 3.27. Let D be a Cartier divisor on a projective toric variety X_Σ. Then

1. D is said to be ample if its class is in the interior of the nef cone.

2. D is said to be big if its class is in the interior of the effective cone.

Moreover, a basepoint free divisor D is big if and only if $\dim(P(D)) = \dim(X_\Sigma)$ [5, Lemma 9.3.9].

Chapter 4

Classification of toric varieties

Previously, we have seen that a toric variety can be described by a fan. We want to look at this from a slightly different point of view. In this chapter, we will give an alternate description of toric varieties by *primitive collections*. Primitive collections were first introduced by Batyrev[1]. This chapter aims to introduce the classification of smooth complete toric threefolds with Picard rank 2 and 3 up to primitive collections. We will revisit all these varieties in Chapters 6 and 7 after discussing enough material to understand algebraic hyperbolicity in the next chapter.

4.1 Primitive collections

In this section, we will see the definition of primitive collection and a tool that concretely describes the nef cone. See [5, Chapter 6] or [1] for more details.

Definition 4.1. Let Σ be a fan. A subset $\{\rho_1, \rho_2, \cdots, \rho_k\} \subset \Sigma(1)$ is called a primitive collection if $\{\rho_1, \rho_2, \cdots, \rho_k\}$ is not contained in a single cone of Σ, but every proper subset is.

Example 4.2. Let us describe the fan Σ of $\mathbb{P}^2 \times \mathbb{P}^1$. Here, Σ is generated by the ray generators $u_1 = e_1, u_2 = e_2, u_3 = -e_1 - e_2, u_4 = e_3, u_5 = -e_3$. Each u_i gives a ray ρ_i. See Figure 4.1. The fan Σ contains six maximal cones given by $\mathrm{Cone}(e_1, e_2, e_3)$, $\mathrm{Cone}(e_1, e_2, -e_3)$, $\mathrm{Cone}(e_1, -e_1-e_2, e_3)$, $\mathrm{Cone}(e_1, -e_1-e_2, -e_3)$, $\mathrm{Cone}(e_2, -e_1-e_2, e_3)$, $\mathrm{Cone}(e_2, -e_1-e_2, -e_3)$, nine two-dimensional cones and five one-dimensional cones given by rays. Hence, all primitive collections of Σ are given by $\{\rho_1, \rho_2, \rho_3\}$ and $\{\rho_4, \rho_5\}$. In fact, if we know the rays and all possible primitive collections of a fan, then we can describe the fan completely.

Definition 4.3. A fan is called a splitting fan if there is no intersection between any two primitive collections.

Remark 4.4. The fan of $\mathbb{P}^2 \times \mathbb{P}^1$ is splitting.

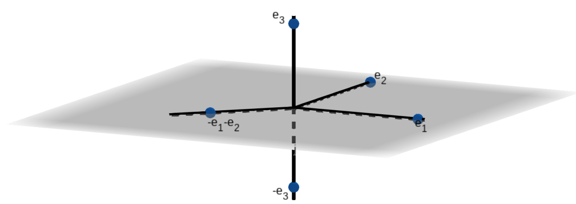


Figure 4.1: Fan of $\mathbb{P}^2 \times \mathbb{P}^1$

Recall the definition of basepoint free divisor (Definition 3.4) from Chapter 3. The knowledge of primitive collections will allow us to compute the nef cone of a given variety. For this, we use the support function (Definition 3.10) associated with the divisor.

Theorem 4.5. *Let X_Σ be a smooth projective toric variety. Then*

1. *A Cartier divisor D is basepoint free if and only if the support function φ_D satisfies*

$$\varphi_D(u_{\rho_1} + \cdots + u_{\rho_k}) \geq \varphi_D(u_{\rho_1}) + \cdots + \varphi_D(u_{\rho_k})$$

for all primitive collections $\{\rho_1, \cdots, \rho_k\}$.

2. *A Cartier divisor is ample if and only if the support function φ_D satisfies*

$$\varphi_D(u_{\rho_1} + \cdots + u_{\rho_k}) > \varphi_D(u_{\rho_1}) + \cdots + \varphi_D(u_{\rho_k})$$

for all primitive collections $\{\rho_1, \cdots, \rho_k\}$.

Proof. This is Theorem 6.4.9 from [5]. $\square$

Example 4.6. Consider the variety $\mathcal{H}_r$ constructed from the fan in Example 3.7 again. The only primitive collections are $\{\rho_1, \rho_3\}$ and $\{\rho_2, \rho_4\}$. We can see that any divisor is linearly equivalent to a divisor $D = aD_3 + bD_4$. Applying the previous theorem, we can see that D is nef if and only if $a, b \geq 0$ and ample if and only if $a, b > 0$.

4.2 Toric projective bundles

In this section, we briefly cover the description of the fan of a projective toric bundle. Roughly speaking, a projective bundle over a projective variety X can be thought of as fibres isomorphic to projective space glued together into another projective variety. For a discussion of sheaf theory and projective bundles, we refer the reader to [14, Chapter 2.1, 2.5, 2.7]. Details of the fan of a toric projective bundle can be found in [5, Chapter 7]. We begin this section with the definition of a sheaf.

Definition 4.7. A sheaf $\mathcal{F}$ on a topological space X consists of the following data,

1. for every open subset $U \subset X$ a set $\mathcal{F}(U)$,
2. for each pair of open sets $V \subset U$ a map $\mathrm{res}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ called the restriction map,

such that the following conditions hold:

(a) for open subsets $W \subset V \subset U \subset X$, we have $\mathrm{res}_{V,W} \circ \mathrm{res}_{U,V} = \mathrm{res}_{U,W}$ and $\mathrm{res}_{U,U} = \mathrm{Id}$,

(b) if $\{U_i\}$ is an open covering of an open subset U and if $\mathrm{res}_{U,U_i}(s) = \mathrm{res}_{U,U_i}(t)$ for all i then $s = t$,

(c) if $\{U_i\}$ is an open covering of an open subset U and if $s_i \in \mathcal{F}(U_i)$ satisfy $\mathrm{res}_{U_i,U_i \cap U_j}(s_i) = \mathrm{res}_{U_j,U_i \cap U_j}(s_j)$ for all i, j, then there exists $s \in \mathcal{F}(U)$ such that $\mathrm{res}_{U,U_i}(s) = s_i$ for all i.

Remark 4.8. We define a sheaf of abelian groups, or a sheaf of rings, by replacing the word 'set' in the definition by 'abelian group', or 'ring' respectively.

Example 4.9. Let X be a variety. For each open set $U \subset X$, let $\mathcal{O}_X(U)$ be the ring of regular functions from U to $\mathbb{C}$, and for each $V \subset U$, let $\mathrm{res}_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ be the restriction map in the usual sense. Then $\mathcal{O}_X$ is a sheaf of rings on X, called the structure sheaf [14, Example 1.0.1].

Definition 4.10. A sheaf of $\mathcal{O}_X$-modules is a sheaf $\mathcal{F}$ on X, such that for each open set $U \subset X$, the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$ module, and for each inclusion of open sets $V \subset U$, the restriction homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

Example 4.11. Each divisor D on X determines a sheaf of $\mathcal{O}_X$-modules $\mathcal{O}_X(D)$ as follows

$$\mathcal{O}_X(D)(U) = \{f \in \mathbb{C}(X)^* | (\,\mathrm{div}(f) + D)|_U \geq 0\} \cup \{0\}.$$

A locally free sheaf of rank r on a variety X is a sheaf of $\mathcal{O}_X$-modules locally isomorphic to a direct sum of r-copies of $\mathcal{O}_X$. Let $\mathcal{E} = \mathcal{O}_{X_\Sigma}(E_0) \oplus ... \oplus \mathcal{O}(E_r)$ be a locally free sheaf of rank $r+1$ defined by torus invariant divisors $E_i = \sum_{\rho \in \Sigma(1)} a_{i\rho} D_\rho$ on X_Σ. Then we can construct a projective bundle $\pi : \mathbb{P}(\mathcal{E}) \to X_\Sigma$. For the construction, see [5, Chapter 7] or [14, Chapter 2]. For our purposes, we do not need the construction explicitly. The following theorem will explain the fan $\Sigma_\mathcal{E}$ of this toric variety.

Theorem 4.12. *Let $e_1, \cdots, e_r$ be a basis for $\mathbb{R}^r$. Set $e_0 = -e_1 - \cdots - e_r$ and $F_j = \mathrm{Cone}(e_0, \cdots, \overset{\wedge}{e_j}, \cdots, e_r)$. Given $\sigma_i \in \Sigma$, consider the cones $\sigma_{ij} \in N_{\mathbb{R}} \times \mathbb{R}^r$ provided by*

$$\sigma_{ij} = \mathrm{Cone}\Big(u_\rho + (a_{1\rho} - a_{0\rho})e_1 + \cdots + (a_{r\rho} - a_{0\rho})e_r | \quad \rho \in \sigma_i(1)\Big) + F_j,$$

where the last addition is the Minkowski sum. Then the cones σ_{ij} and their faces form the fan $\Sigma_{\mathcal{E}}$.

Proof. This is Proposition 7.3.3 from [5]. □

Example 4.13 (Hirzebruch surfaces)**.** Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r)$. Then the fan $\Sigma_{\mathcal{E}}$ has ray generators given by $u_1 = e_1, u_2 = e_2, u_3 = -e_1 + re_2, u_4 = -e_2$. The only primitive collections are $\{\rho_1, \rho_3\}$ and $\{\rho_2, \rho_4\}$.

4.3 Kleinschmidt's classification

Kleinschmidt [17] classified all smooth complete toric varieties with Picard rank 2. It turns out that all such varieties are projectivization of decomposable bundles over a projective space of a smaller dimension. Later, Kleinschmidt and Sturmfels [18] proved that every smooth complete toric variety of rank at most 3 is necessarily projective. In this section, we will discuss the case of toric threefolds.

Theorem 4.14. *Let X_Σ be a smooth complete toric threefold with Picard rank 2. Then either $X_\Sigma \cong \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(l_1) \oplus \mathcal{O}_{\mathbb{P}^1}(l_2)\right)$ with $l_2 \geq l_1 \geq 0$ or $X_\Sigma \cong \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(l)\right)$ with $l \geq 0$.*

Proof. This is Theorem 7.3.7 from [5]. □

The original statement of Kleinschmidt is applicable to varieties of arbitrary dimension with Picard rank 2. We will explain the fan description in the next theorem.

Theorem 4.15.

1. *The fan of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(l_1) \oplus \mathcal{O}_{\mathbb{P}^1}(l_2))$ has ray generators $u_1 = e_1, u_2 = e_2, u_3 = -e_1 - e_2, u_4 = e_3$ and $u_5 = l_1e_1 + l_2e_2 - e_3$. The only primitive collections are $\{\rho_1, \rho_2, \rho_3\}$ and $\{\rho_4, \rho_5\}$.*

2. *The fan of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(l))$ has ray generators $u_1 = e_1, u_2 = -e_1, u_3 = e_2, u_4 = e_3$ and $u_5 = le_1 - e_2 - e_3$. The only primitive collections are $\{\rho_1, \rho_2\}$ and $\{\rho_3, \rho_4, \rho_5\}$.*

Proof. This is Example 7.3.5 from [5]. □

Remark 4.16. We can also conclude from [5, Example 7.3.5] that smooth complete two dimensional toric varieties with Picard rank 2 are Hirzebruch surfaces $\mathcal{H}_r = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r))$.

4.4 Batyrev's classification

Batyrev completely classified smooth complete toric varieties with Picard rank 3 in terms of primitive collections. He showed that the number of primitive collections of its generators is 3 or 5 [1, Theorem 5.7].

Theorem 4.17. *Let X_Σ be a smooth complete toric threefold of Picard rank 3 with a splitting fan. Then $X_\Sigma \cong \mathbb{P}(\mathcal{O}_{\mathcal{H}_r} \oplus \mathcal{O}_{\mathcal{H}_r}(aD_3 + bD_4))$ with $a \geq 0$.*

Proof. In this case, the associated toric variety is isomorphic to the projectivization of a decomposable bundle of rank 2 over a smooth complete toric surface with Picard rank 2 [1]. By Remark 4.16, it is clear that smooth complete surfaces are given by Hirzebruch surfaces $\mathcal{H}_r$. Hence, we can take $\mathcal{E} = \mathcal{O}_{\mathcal{H}_r}(a_1D_3 + b_1D_4) \oplus \mathcal{O}_{\mathcal{H}_r}(a_2D_3 + b_2D_4)$. Without loss of generality, take $a_2 \geq a_1$ and $\mathcal{L} = \mathcal{O}_{\mathcal{H}_r}(-a_1D_3 - b_1D_4))$. Then $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E} \otimes \mathcal{L})$ by Lemma 7.9 [14] and the result follows. □

Theorem 4.18. *The fan Σ of $\mathbb{P}(\mathcal{O}_{\mathcal{H}_r} \oplus \mathcal{O}_{\mathcal{H}_r}(aD_3 + bD_4))$ has ray generators $u_1 = e_1, u_2 = -e_1 + re_2 + ae_3, u_3 = e_2, u_4 = -e_2 + be_3, u_5 = e_3, u_6 = -e_3$. The only primitive collections are $\{\rho_1, \rho_2\}, \{\rho_3, \rho_4\}$ and $\{\rho_5, \rho_6\}$.*

Proof. Maximal cones in $\mathcal{H}_r$ are given by $\sigma_1 = \mathrm{Cone}(e_1, e_2)$, $\sigma_2 = \mathrm{Cone}(e_2, -e_1 + re_2)$, $\sigma_3 = \mathrm{Cone}(-e_1 + re_2, -e_2)$ and $\sigma_4 = \mathrm{Cone}(-e_2, e_1)$. It is easy to see that D_3 and D_4 generate the Picard group. Hence, we can take $E_0 = 0$ and $E_1 = aD_3 + bD_4$ with $a \geq 0$. Take $F_0 = e_3$ and $F_1 = -e_3$. Then the result follows from Theorem 4.12. □

Theorem 4.19. *Let X_Σ be a smooth projective toric threefold with Picard rank 3 which does not have a splitting fan. Then its ray generators can be partitioned into 5 non-empty sets $X_0, X_1, \cdots, X_4$ in such a way that the primitive collections are $X_i \cup X_{i+1}$, where $i \in \mathbb{Z}/5\mathbb{Z}$.*

Proof. See [1, Theorem 6.6]. □

Following [1], we can list all the possibilities in Theorem 4.19.

(i) $X_0 = \{e_1, e_2\}, X_1 = \{-e_1 - e_2 + (b+1)e_3\}, X_3 = \{-e_3\}, X_4 = \{e_3\}, X_5 = \{-e_1 - e_2 + be_3\}$ where $b \geq 0$.

(ii) $X_0 = \{e_1\}, X_1 = \{-e_1 - e_2 + (b+1)e_3, e_2\}, X_3 = \{-e_3\}, X_4 = \{e_3\}, X_5 = \{-e_1 + be_3\}$ where $b \geq 0$.

(iii) $X_0 = \{e_1\}, X_1 = \{-e_1 + ce_2 + (b+1)e_3\}, X_3 = \{-e_2 - e_3, e_2\}, X_4 = \{e_3\}, X_5 = \{-e_1 + ce_2 + (b+1)e_3\}$ where $b, c \geq 0$.

(iv) $X_0 = \{e_1\}, X_1 = \{-e_1 + (c+1)e_2 + (b+1)e_3\}, X_3 = \{-e_2 - e_3\}, X_4 = \{e_2, e_3\}, X_5 = \{-e_1 + ce_2 + be_3\}$ where $b, c \geq 0$.

(v) $X_0 = \{e_1\}, X_1 = \{-e_1 + (b+1)e_3\}, X_3 = \{-e_3\}, X_4 = \{e_3\}, X_5 = \{-e_1 - e_2 + be_3, e_2\}$ where $b \geq 0$.

Chapter 5

Algebraic hyperbolicity

In this chapter, we discuss the geometric genus of a curve and how it is related to the degree of the curve using the notion of algebraic hyperbolicity. In Section 5.2, we define connected sections, and in Section 5.3, we discuss the main tool of finding algebraically hyperbolic surfaces in toric threefolds by Haase and Ilten [13]. Finally, in Section 5.4, we see when a curve is considered to be a complete intersection of hypersurfaces.

5.1 Geometric genus and hyperbolicity

This section aims to define algebraic hyperbolicity and highlight the connection to Brody hyperbolicity. Any discussion about algebraic hyperbolicity should start by defining the geometric genus of a curve. By a curve C, we mean an irreducible projective variety of dimension 1. We denote its canonical divisor by K_C. See [14, Section 2.8] for details.

Definition 5.1. Let C be a smooth curve with a canonical divisor K_C. Then the geometric genus of X is defined by

$$g = \dim_{\mathbb{C}} \left(L(K_C) \right).$$

Example 5.2. Every elliptic curve has an invariant differential ω with no zeros or poles. Hence 0 is a canonical divisor and $\dim_{\mathbb{C}} \left(L(0) \right) = \dim_{\mathbb{C}} \mathbb{C} = 1$.

Remark 5.3. Every irreducible projective curve admits a birational map to a unique non-singular model [8, Theorem 7.5.3]. Hence, we extend the definition of the geometric genus to singular curves by defining it as the geometric genus of the nonsingular model.

For a smooth plane curve of degree d we know the geometric genus g is $\dfrac{(d-1)(d-2)}{2}$. If it has singularities, it is modified by an error factor [8, Proposition 8.3.5]. Naturally, the following question arises:

Question: Given a curve C in a surface S of degree d, how small is the geometric genus of C?

This motivates the definition of algebraic hyperbolicity.

Definition 5.4. A smooth complex projective variety X is algebraically hyperbolic if there exists an ample divisor H on X and some $\epsilon > 0$ such that any curve $C \subset X$ of geometric genus $g(C)$ satisfies

$$2g(C) - 2 \geq \epsilon(C \cdot H).$$

Remark 5.5. The intersection number $C \cdot H$ is known as the degree of the curve C with respect to the ample divisor H.

Example 5.6. Let X be a curve of genus $g \geq 2$. Then it is algebraically hyperbolic.

Definition 5.7. A smooth complex projective variety X is said to be Brody hyperbolic if there is no nonconstant holomorphic map from $\mathbb{C}$ to X.

Demailly [6] proved that Brody hyperbolicity implies algebraic hyperbolicity and conjectured the converse:

Conjecture 5.8. *A smooth complex projective variety X is Brody hyperbolic if it is algebraically hyperbolic.*

5.2 Connected sections

This section aims to introduce the definition of connected sections and provide combinatorial criteria to determine a configuration of divisors with connected sections from the fan's description. Later, we will see the main tool Theorem 5.19, which requires a configuration of divisors with connected sections. We closely follow the notation in [13].

Definition 5.9. Let D, E be effective, non-trivial torus invariant divisors on a toric variety X with $D - E \geq 0$. The *section graph* for D, E is the graph G whose vertex set is $V(G) = P(E) \cap M$ and where $a, b \in V(G)$ are connected by an edge if and only if there exists $a', b' \in P(D - E) \cap M$ such that $a + a' = b + b'$ in $P(D) \cap M$.

Definition 5.10. We say the configuration (D, E) has the integer decomposition property (IDP) if

$$(P(E) \cap M) + (P(D - E) \cap M) = P(D) \cap M.$$

Definition 5.11. A configuration of divisors (D, E) has connected sections if

(i) The section graph G is connected;

(ii) The configuration (D, E) has IDP.

Theorem 5.12. *If Σ is a splitting fan and E, E' are nef divisors on X_Σ, then the configuration $(E + E', E)$ has IDP.*

Proof. This is Corollary 4.2 from [16]. $\square$

Remark 5.13. If Σ is not a splitting fan, then the above statement is not true in general. See [20]. Recall from Theorem 4.19 that we have non-splitting fans for all smooth complete toric threefold of Picard rank 3 with 5 primitive collections. We will treat each of those cases individually and extend the Theorem 5.12 to all smooth toric threefold of Picard rank 3. See Lemmas 7.12, 7.19, 7.26, 7.33 and 7.40.

Example 5.14. Consider the variety $\mathcal{H}_r$ for $r = 1$ constructed from the fan in Example 3.7. Let $D = D_3 + 2D_4$, $E = D_3 + D_4$ and $E' = D_4$. From the Example 4.6, we can see that E and E' are nef. Hence, by Theorem 5.12, the configuration (D, E) has IDP. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad t = \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \quad s = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

Then, $P(E) = P(A, t)$ and $P(E') = P(A, s)$. From Figure 5.1, we can see that the section graph is connected. Hence (D, E) has connected sections.

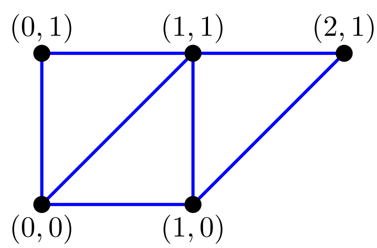


Figure 5.1: The section graph for Example 5.14

If we know a configuration of divisors has IDP; it is enough to check that the associated graph is connected to verify connected sections. We will discuss a combinatorial criterion that ensures the associated graph is connected.

We will first translate the information of the fan into a toric ideal. Let X_Σ be a smooth complete toric threefold. Recall the short exact sequence (Lemma 3.6)

$$0 \longrightarrow M \xrightarrow{i} \mathbb{Z}^{\Sigma[1]} \xrightarrow{\pi} \mathrm{Pic}(X) \to 0.$$

After choosing a basis $\{e_1^*, e_2^*, e_3^*\}$ for M and a basis for $\mathrm{Pic}(X)$, we get the short exact sequence

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{i} \mathbb{Z}^r \xrightarrow{\pi} \mathbb{Z}^k \to 0.$$

Note that $k = r - 3$. We can represent the map π by a matrix $A = (a_{ij})$. Consider the $\mathbb{C}$-algebra homomorphism given by

$$\begin{aligned} \alpha : \mathbb{C}[x_1, \cdots, x_r] &\to \mathbb{C}[y_1, y_1^{-1}, \cdots, y_k, y_k^{-1}] \\ x_i &\mapsto y_1^{a_{1i}} \cdots y_k^{a_{ki}}. \end{aligned}$$

We define the toric ideal $I_A = \ker(\alpha)$.

Lemma 5.15. *The toric ideal I_A associated with the matrix A is the ideal in $\mathbb{C}[x_1, \cdots, x_r]$ generated by binomials $x^{v^+} - x^{v^-}$ for $v^+, v^- \in \mathbb{Z}^r_{\geq 0}$ with $Av^+ = Av^-$.*

Proof. This is Proposition 1.1.9 from [5]. $\square$

We identify a vector $v \in \ker(A) \cap \mathbb{Z}^r$ with the binomial $x^{v^+} - x^{v^-}$ where $v_i^+ = \max(v_i, 0)$ and $v_i^- = -\min(v_i, 0)$.

Definition 5.16. We say that a subset $\mathcal{G} \subset \ker(A) \cap \mathbb{Z}^r$ is a Markov basis if the corresponding binomials generate I_A.

Lemma 5.17. *The Krull dimension of $\mathbb{C}[x_1, \cdots, x_r]/I_A$ is equal to* $\operatorname{rank}(A)$.

Proof. This is Lemma 4.2 from [22]. $\square$

Proposition 5.18. *Let $(E + E', E)$ be an IDP pair of divisors on X. Set*

$$\mathcal{G} := \left(i\left(P(E')\right) \cap \mathbb{Z}^r\right) - \left(i\left(P(E')\right) \cap \mathbb{Z}^r\right).$$

If $\mathcal{G}$ is a Markov basis for I_A, then the configuration $(E + E', E)$ has connected sections.

Proof. This is Proposition 4.5 from [13]. $\square$

5.3 Main tools

We are in a position to describe the main tools for finding algebraically hyperbolic surfaces. Let X be the toric variety of a fan Σ. By the toric boundary of X, we mean the complement in X of the open torus orbit T.

Theorem 5.19. *Let (D, E) be non-trivial basepoint free torus invariant divisors on a smooth complete toric threefold X. Assume that this configuration has connected sections and that D is big. Let $S \in |D|$ be a very general surface and $C \subset S$ any curve that is not contained in the toric boundary of X. Then the geometric genus g of C satisfies*

$$2g - 2 \geq C \cdot (E + K_X).$$

Proof. This is Theorem 3.6 from [13]. $\square$

Even though the above theorem gives a bound for most of the curves, we need to determine the genus of finitely many curves that lie in the boundary. Polytopes associated to the divisors again will play a crucial role here.

Lemma 5.20 (Curves in the boundary)**.** *Let $S \in |D|$ be a general surface, where D is a big and basepoint free divisor. If $C \subset S$ is an curve contained in the toric boundary of X, then $C = S \cap D_\rho$ for some $\rho \in \Sigma(1)$ corresponding to a face $F < P(D)$ on which the ray ρ takes its minimum. Then, the geometric genus of C equals the number of interior lattice points of F viewed in a 2-d ambient space.*

Proof. This is Lemma 4.1 from [13]. □

Remark 5.21. If we do not assume that D is big, then the geometric genus of C is at most the number of interior lattice points of F. Since D is non-trivial but not big, $\dim(P(D))$ is either 1 or 2. In this case, there is always a ray ρ such that it takes its minimum on a 1-d face. Hence, there is always a genus 0 curve in the boundary.

Corollary 5.22. *Let X be a smooth projective toric threefold. Then a very general surface $S \in |D|$ is algebraically hyperbolic if it has the following two properties:*

1. *All curves in the toric boundary have genus at least two.*

2. *There exists an ample divisor H and an $\epsilon_0 > 0$ such that any curve not in the toric boundary satisfies*
$$2g(C) - 2 \geq \epsilon_0(C \cdot H).$$

Proof. Let S be any very general surface in $|D|$. Let $C_1, \ldots, C_k$ be all finitely many curves in the boundary. Take ϵ to be minimum among ϵ_0 and $\dfrac{1}{C_i \cdot H}$ for $i = 1, \cdots, k$. Then, we have
$$2g(C) - 2 \geq \epsilon(C \cdot H),$$
for every curve in S.

□

5.4 Noether-Lefschetz Theorem

If we know a bit more about the curves in a threefold, we get better bounds for algebraic hyperbolicity. We will discuss a criterion to obtain better lower bounds on the intersection numbers of our curves C with divisors on X. We say a variety of codimension r is a complete intersection if it is cut out by the intersection of r hypersurfaces.

Theorem 5.23 (Classical Noether-Lefschetz theorem)**.** *If $S_d \subset \mathbb{P}^3$ is a very general surface of degree $d \geq 4$, then the restriction map $\mathrm{Pic}(\mathbb{P}^3) \to \mathrm{Pic}(S_d)$ is an isomorphism. Furthermore, every curve $C \subset S_d$ is a complete intersection.*

Proof. This is Theorem 1 from [12]. $\square$

Historically, different approaches were involved in extending the Noether-Lefschetz theory to other varieties. Under favourable conditions, the Noether-Lefschetz theorem can be extended to hypersurfaces in three-dimensional varieties [21].

Theorem 5.24 (Generalized Noether-Lefschetz theorem)**.** *Let X be a smooth threefold, and D be a divisor such that $D + K_X$ is basepoint free. Then for a very general surface $S \in |D|$, the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(S)$ is an isomorphism.*

Proof. This is a particular case of Theorem 1 from [21]. $\square$

Chapter 6

Algebraically hyperbolic surfaces with Picard rank 2

In this chapter, we discuss the algebraic hyperbolicity of a very general surface in smooth complete toric threefolds with Picard rank 2. Kleinschmidt's classification of smooth complete toric threefolds with Picard rank 2 was discussed in Section 4.3.

6.1 Case 1: Projective bundles over $\mathbb{P}^2$

Consider the variety $X_\Sigma \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(l))$. We recall the description of the fan described in Theorem 4.15 again. Here, Σ is generated by ray generators $e_1, -e_1, e_2, e_3, le_1 - e_2 - e_3$ with $l \geq 0$ and corresponding rays denoted by $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5$ and divisors by D_1, D_2, D_3, D_4, D_5. It has two primitive collections $\{e_1, -e_1\}$ and $\{e_2, e_3, le_1 - e_2 - e_3\}$.

Lemma 6.1. *The Picard group* $\mathrm{Pic}(X)$ *is generated by the classes of* D_2 *and* D_3*.*

Proof. By Lemma 3.6, the Picard group is generated by the classes of D_i subject to the following relations:

$$\begin{aligned} D_1 - D_2 + lD_3 &\sim 0 \\ D_3 - D_5 &\sim 0 \\ D_4 - D_5 &\sim 0. \end{aligned} \tag{6.1}$$

Thus, $\mathrm{Pic}(X) = \mathbb{Z}[D_2] \bigoplus \mathbb{Z}[D_3]$. □

Lemma 6.2. *The Nef cone is generated by the classes of* D_2 *and* D_3*, whereas the effective cone is generated by the classes of* D_1 *and* D_3*.*

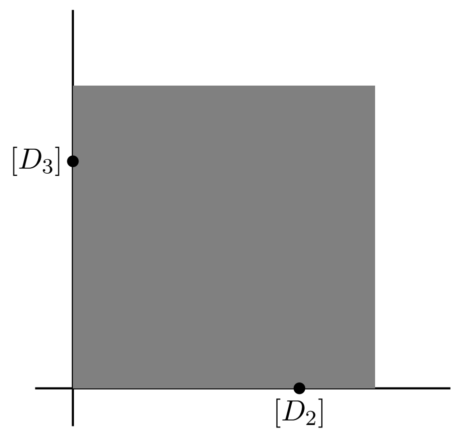
Proof. Let $D = aD_2 + bD_3$. If D is nef then we have (Theorem 4.5),

$$\varphi_D(0) \geq \varphi_D(e_1) + \varphi_D(-e_1)$$

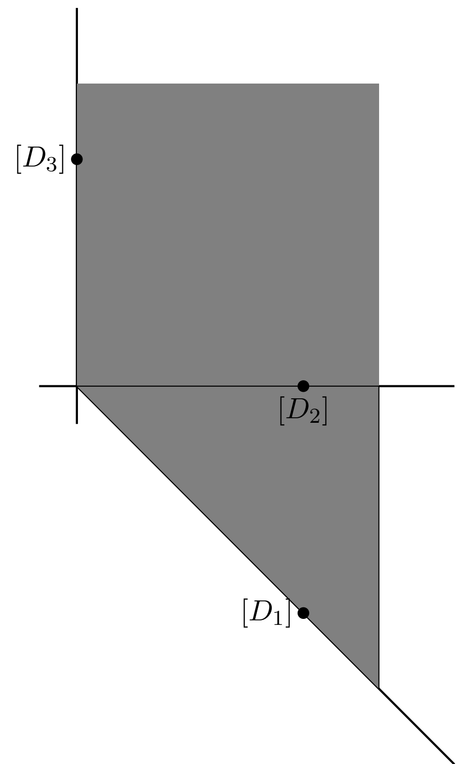
and it follows that $a \geq 0$. Also,

$$\varphi_D(le_1) \geq \varphi_D(e_2) + \varphi_D(e_3) + \varphi_D(le_1 - e_2 - e_3)$$

and it follows that $b \geq 0$. Thus, the nef cone is generated by the classes of D_2 and D_3. Using (6.1), one can easily see that the effective cone is generated by the classes of D_1 and D_3. See Figure 6.1. $\square$



(a) Nef cone



(b) Effective cone

Figure 6.1: Nef and effective cone of $X_\Sigma \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(l))$

Theorem 6.3. *(First main result) Let* $X_\Sigma \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(l))$ *and* S *be a very general surface in the class* $aD_2 + bD_3$*.*

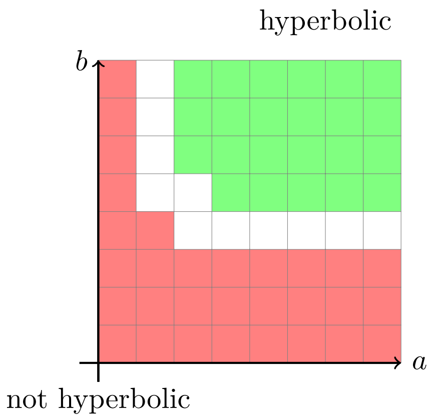
1. *Suppose* $l = 0$*. Then* S *is algebraically hyperbolic if* $a \geq 3$ *and* $b \geq 5$ *or* $a = 2$ *and* $b \geq 5$ *or* $a \geq 3$ *and* $b = 4$*. Moreover,* S *is not algebraically hyperbolic if* $a \leq 1$ *or* $b \leq 3$ *or* $a = 2$ *and* $b = 4$*.*

2. *Suppose* $l = 1$*. Then* S *is algebraically hyperbolic if* $a \geq 3$ *and* $b \geq 4$ *or* $a = 2$ *and* $b \geq 7$ *or* $b = 0$ *and* $a \geq 6$*. Moreover,* S *is not algebraically hyperbolic if* $a \leq 1$ *or* $1 \leq b \leq 3$ *or* $a = 2, 3$ *and* $b = 0$*.*

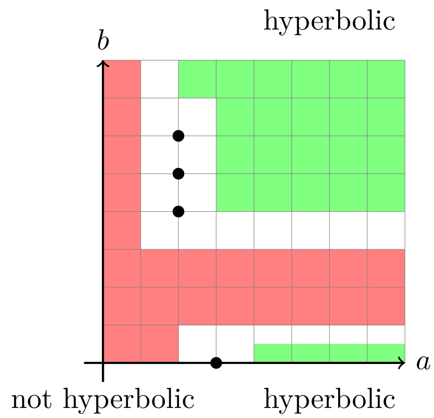
3. *Suppose $l = 2$. Then S is algebraically hyperbolic if $a \geq 3$ and $b \geq 4$ or $a = 2$ and $b \geq 7$ or $b = 0$ and $a \geq 4$. Moreover, S is not algebraically hyperbolic if $a \leq 1$ or $1 \leq b \leq 3$ or $a = 2$ and $b = 0$.*

4. *Suppose $l = 3$. Then S is algebraically hyperbolic if $a \geq 3$ and $b \geq 4$ or $a = 2$ and $b \geq 7$ or $b = 0$ and $a \geq 4$. Moreover, S is not algebraically hyperbolic if $a \leq 1$ or $1 \leq b \leq 3$.*

5. *Suppose $l \geq 4$, then S is algebraically hyperbolic if $a \geq 3$ and $b \geq 4$ or $a = 2$ and $b \geq 7$ or $b = 0$ and $a \geq 3$. Moreover, S is not algebraically hyperbolic if $a \leq 1$ or $1 \leq b \leq 3$.*



(a) $l = 0$



(b) $l = 2$

Figure 6.2: Algebraic hyperbolicity for very general surface of the type $aD_2 + bD_3$ in $X_\Sigma \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(l))$, for $l = 0$ and 2.

Remark 6.4. Note that $[D_3]$ is the same as the pullback of the hyperplane class from $\mathbb{P}^2$, and $[D_2]$ is the same as ξ the class of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. See [16].

Before proving the above theorem, we need to find a collection of divisors with connected sections discussed in Section 5.2. The Canonical divisor of X_Σ is given by (3.2)

$$\begin{aligned} K_X &= -D_1 - D_2 - D_3 - D_4 - D_5 \\ &\sim -2D_2 + (l-3)D_3. \end{aligned}$$

After choosing a basis $\{e_1^*, e_2^*, e_3^*\}$ for M and a basis for $\mathrm{Pic}(X)$ as in Lemma 6.1, we get the short exact sequence

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{i} \mathbb{Z}^5 \xrightarrow{\pi} \mathbb{Z}^2 \to 0.$$

We can represent the map π by the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -l & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Lemma 6.5. *A Markov Basis for A is given by the rows of the matrix*

$$\begin{pmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & l \end{pmatrix}.$$

Proof. Let $I = \langle x_3 - x_4, x_3 - x_5, x_1x_5^l - x_2\rangle$ and I_A be the toric ideal associated with the matrix A. It is enough to show that $I = I_A$. Clearly, $I \subset I_A$. Note that

$$\begin{aligned} \frac{\mathbb{C}[x_1, x_2, x_3, x_4, x_5]}{\langle x_3 - x_4, x_3 - x_5, x_1x_5^l - x_2\rangle} &\cong \frac{\mathbb{C}[x_1, x_2, x_3, x_5]}{\langle x_3 - x_5, x_1x_5^l - x_2\rangle} \\ &\cong \frac{\mathbb{C}[x_1, x_2, x_5]}{\langle x_1x_5^l - x_2\rangle} \\ &\cong \mathbb{C}[x_1, x_5]. \end{aligned}$$

Hence I is a prime ideal. Moreover, $\dim(\mathbb{C}[x_1, ..., x_5]/I) = 2 = \mathrm{rank}(A)$. Hence, $I = I_A$ by Lemma 5.17. $\square$

Lemma 6.6. *Let $l \geq 1$, $D = aD_2 + bD_3$, $E' = D_2$ and $E = (a-1)D_2 + bD_3$. Then (D, E) has connected sections.*

Proof. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ l & -1 & -1 \end{pmatrix}, \quad t_1 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 0 \\ -a \\ -b \\ 0 \\ 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 0 \\ -a+1 \\ -b \\ 0 \\ 0 \end{pmatrix}.$$

Then $P(E') = P(A, t_1)$, $P(D) = P(A, t_2)$ and $P(E) = P(A, t_3)$. Clearly the polytope $P(E')$ contains the points $(0, 0, 0), (1, 0, 0), (1, l-1, 1), (1, l, 0), (1, l-1, 0) \in \mathbb{Z}^3$. Let $G = i(P(E') - i(P(E'))$. Then G contains the vectors

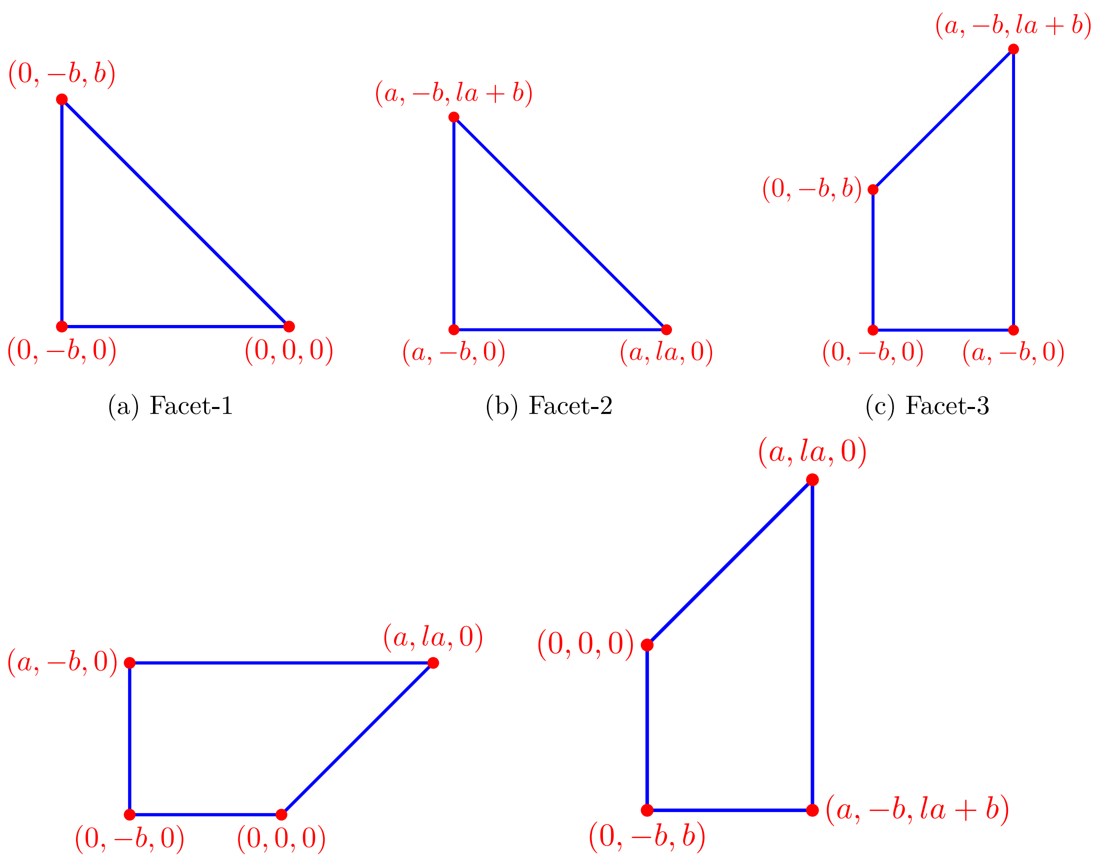
$$\begin{aligned} i(1, 0, 0) - i(0, 0, 0)) &= (1, -1, 0, 0, l) - (0, 0, 0, 0, 0) = (1, -1, 0, 0, l) \\ i(1, l, 0) - i(1, l-1, 1)) &= (1, -1, l, 0, 0) - (1, -1, l-1, 1, 0) = (0, 0, 1, -1, 0) \\ i(1, l-1, 1) - i(1, l-1, 0) &= (1, -1, l-1, 1, 0) - (1, -1, l-1, 0, 1) = (0, 0, 0, 1, -1). \end{aligned}$$

Thus by Lemma 6.5, G is a Markov basis for A. Note that $P(D) \cap \mathbb{Z}^3 = P(E) \cap \mathbb{Z}^3 + P(E') \cap \mathbb{Z}^3$ by Theorem 5.12. Hence by Proposition 5.18, configuration (D, E) has connected sections. $\square$

Lemma 6.7 (Curves in the boundary)**.** *Let $X_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(l))$ and S be a very general surface in the class $aD_2 + bD_3$ with $a \geq 1$.*

1. *Let $l = 1$. Then S contains a curve of genus 0 or 1 if $a = 1$ or $1 \leq b \leq 3$ or $a = 2, 3$ and $b = 0$.*
2. *Let $l = 2$. Then S contains a curve of genus 0 or 1 if $a = 1$ or $1 \leq b \leq 3$ or $a = 2$ and $b = 0$.*
3. *Let $l \geq 3$. Then S contains a curve of genus 0 or 1 if $a = 1$ or $1 \leq b \leq 3$.*
4. *For all other cases, every curve in the toric boundary has the geometric genus at least 2.*

Proof. Since $a \geq 1$, D is big. We will analyze each of the facets of $P(D)$. See Figure 6.3.



(a) Facet-1 (b) Facet-2 (c) Facet-3

(d) Facet-4 (e) Facet-5

Figure 6.3: Facets of Poltope $P(aD_2 + bD_3)$ when $a, b \geq 1$.

1. **Facet 1:** Interior is given by the equations $x = 0$, $y > -b$, $z > 0$ and $lx - y - z > 0$. It is easy to find the integer values for x, y, z satisfies the above equations. Indeed there are $(b-1)(b-2)/2$ solutions if $b \geq 1$. If $b = 0$, the restriction is the trivial divisor, and there is no curve in the intersection.

2. **Facet 2:** Interior is given by the equations $x = a$, $y > -b$, $z > 0$ and $la - y - z > 0$. Thus, we have $(la + b - 2)(la + b - 1)/2$ interior lattice points.

3. **Facet 3:** Interior is given by the equations $y = -b$, $0 < x < a$, $z > 0$ and $lx + b - z > 0$. Thus, we have $(a-1)(b-1) + la(a-1)/2$ interior lattice points.

4. **Facet 4:** Interior is given by the equations $z = 0$, $0 < x < a$, $y > -b$ and $lx > y$. Thus, we have $(a-1)(b-1) + la(a-1)/2$ interior lattice points.

5. **Facet 5:** Interior is given by the equations $lx - y - z = 0$, $0 < x < a$, $y > -b$ and $z > 0$. Thus, we have $(a-1)(b-1) + la(a-1)/2$ interior lattice points.

Thus by Lemma 5.20, we have the results.

□

Proof of Theorem 6.3. The case $l = 0$ is discussed in [13, Example 6.1] and [4, Theorem 1.1]. Hence, we can assume $l \geq 1$. Let $D = aD_2 + bD_3$ with $a \geq 1$ and $b \geq 0$ and $E = (a-1)D_2 + bD_3$. Then D is big, and the configuration (D, E) has connected sections by Theorem 6.6. Applying Theorem 5.19, for any curve C not contained in the toric boundary on a very general surface S in $|D|$ we have,

$$2g - 2 \geq C \cdot ((a-3)D_2 + (b + l - 3)D_3). \tag{6.2}$$

By Theorem 5.24, if $a \geq 2$ and $b \geq 3 - l$, then the natural restriction map $\mathrm{Pic}(X_\Sigma) \to \mathrm{Pic}(S)$ is an isomorphism. Thus, any curve C is rationally equivalent to the complete intersection of S with a divisor in the class $cD_1 + dD_3$. By Theorem 3.25, and using the notations from Table 6.1, we can write

$$C \sim bcV(\tau_1) + adV(\tau_4) + bdV(\tau_7).$$

If C is not contained in the boundary, an intersection number calculation yields

$$2g - 2 \geq cb(b + l - 3) + d(la(a-3) + a(b + l - 3) + (a-3)b).$$

The degree of such a curve C with respect to ample class $H = D_2 + D_3$ is given by

$$\deg(C) = cb + d(a + b + al).$$

Let $\epsilon_0 = \dfrac{1}{a+b+al}$. Then we have

$$2g - 2 \geq \epsilon_0 \cdot \deg(C).$$

By combining Lemma 6.7 and Corollary 5.22, we have the results.

□

τ	$D_1.V(\tau)$	$D_2.V(\tau)$	$D_3.V(\tau)$	$D_4.V(\tau)$	$D_5.V(\tau)$
$\tau_1 =$Cone(ρ_1, ρ_3)	$-l$	0	1	1	1
$\tau_2 =$ Cone(ρ_1, ρ_4)	$-l$	0	1	1	1
$\tau_3 =$ Cone(ρ_1, ρ_5)	$-l$	0	1	1	1
$\tau_4 =$ Cone(ρ_2, ρ_3)	0	l	1	1	1
$\tau_5 =$ Cone(ρ_2, ρ_4)	0	l	1	1	1
$\tau_6 =$ Cone(ρ_2, ρ_5)	0	l	1	1	1
$\tau_7 =$ Cone(ρ_3, ρ_4)	1	1	0	0	0
$\tau_8 =$ Cone(ρ_3, ρ_5)	1	1	0	0	0
$\tau_9 =$ Cone(ρ_4, ρ_5)	1	1	0	0	0

Table 6.1: Intersection numbers in $X_\Sigma \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(l))$

6.2 Case 2: Projective bundles over $\mathbb{P}^1$

We proceed in a similar fashion to Section 6.1. Consider the variety $X_\Sigma \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(l_1) \oplus \mathcal{O}_{\mathbb{P}^1}(l_2))$. We recall the description of the fan described in Theorem 4.15 again. Here, Σ is generated by ray generators $e_1, e_2, -e_1 - e_2, e_3, l_1 e_1 + l_2 e_2 - e_3$ with $0 \leq l_1 \leq l_2$ and corresponding rays denoted by $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5$ and divisors by D_1, D_2, D_3, D_4, D_5. It has two primitive collections $\{e_1, e_2, -e_1 - e_2\}$ and $\{e_3, le_1 + a_2 e_2 - e_3\}$.

Lemma 6.8. *The Picard group* Pic(X) *is generated by the classes of D_3 and D_4.*

Proof. By Lemma 3.6, the Picard group is generated by the classes of D_i subject to the following relations:

$$\begin{aligned} D_1 &\sim D_3 - l_1 D_5 \\ D_2 &\sim D_3 - l_2 D_5 \\ D_4 &\sim D_5. \end{aligned} \tag{6.3}$$

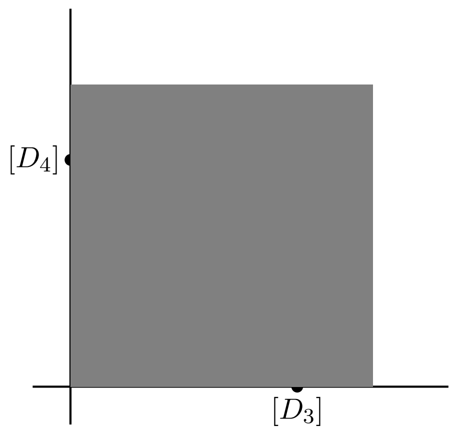
Thus, $\text{Pic}(X) = \mathbb{Z}[D_3] \bigoplus \mathbb{Z}[D_4]$. □

Lemma 6.9. *The Nef cone is generated by the classes D_3 and D_4, whereas the effective cone is generated by the classes of D_2 and D_4.*

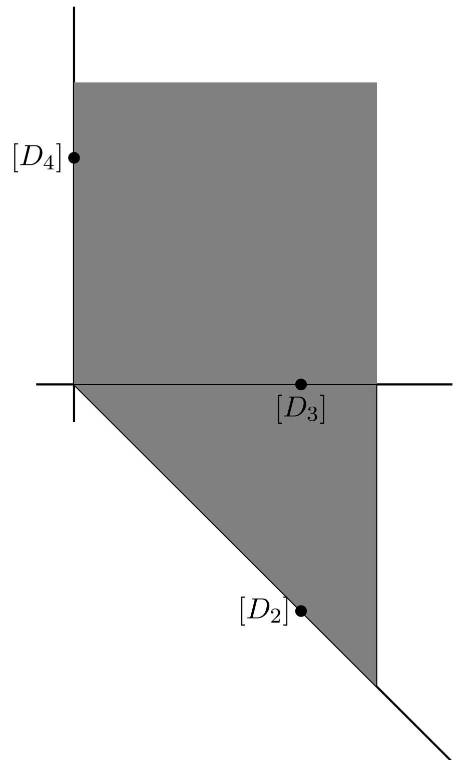
Proof. Let $D = aD_3 + bD_4$. If D is nef, then we have (Theorem 4.5),

$$\varphi_D(0) \geq \varphi_D(e_1) + \varphi_D(e_2) + \varphi_D(-e_1 - e_2)$$
$$\varphi_D(l_1e_1 + l_2e_2) \geq \varphi_D(l_1e_1 + l_2e_2 - e_3) + \varphi_D(e_3).$$

Hence, it follows that $a, b \geq 0$. Thus, the nef cone is generated by the classes of D_3 and D_4. Using (6.3), one can easily see that the effective cone is generated by the classes of D_2 and D_4. See Figure 6.4. $\square$



(a) Nef cone



(b) Effective cone

Figure 6.4: Nef and effective cone of $X_\Sigma \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(l_1) \oplus \mathcal{O}_{\mathbb{P}^1}(l_2))$

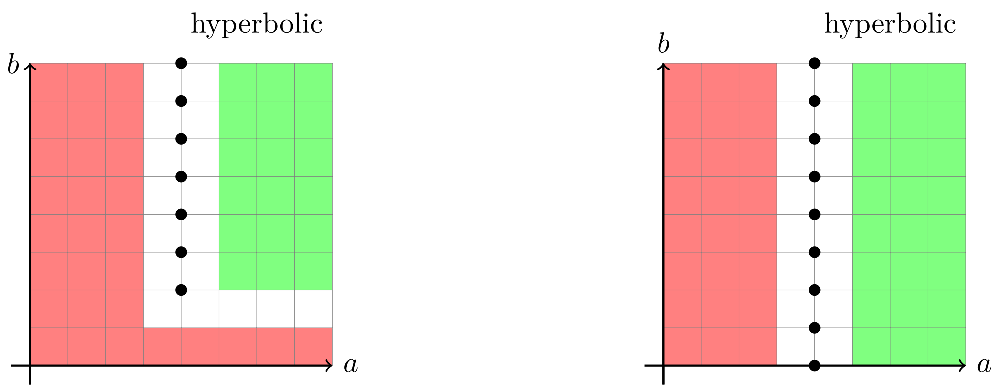
Theorem 6.10. *(Second main result) Let $X_\Sigma \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(l_1) \oplus \mathcal{O}_{\mathbb{P}^1}(l_2))$ and S be a very general surface in the class $aD_3 + bD_4$.*

1. *Suppose $l_1 = l_2 = 0$. Then S is algebraically hyperbolic if $a \geq 5$ and $b \geq 3$ or $a = 4$ and $b \geq 3$ or $a \geq 5$ and $b = 2$. Moreover, S is not algebraically hyperbolic if $a \leq 3$ or $b \leq 1$ or $a = 4$ and $b = 2$.*

2. *Suppose $l_1 = 0, l_2 \geq 1$. Then S is algebraically hyperbolic if $a \geq 5$ and $b \geq 2$. Moreover, S is not algebraically hyperbolic if $a \leq 3$ or $b \leq 1$.*

3. *Suppose $l_1 \geq 1$. Then S is algebraically hyperbolic if $a \geq 5$. Moreover, S is not algebraically hyperbolic if $a \leq 3$.*

Remark 6.11. Note that $[D_4]$ is the same as the pullback of the hyperplane class from $\mathbb{P}^1$, and $[D_3]$ is the same as ξ the class of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. See [16].



(a) $l_1 = 0, l_2 \geq 1$

(b) $l_1 \geq 1$

Figure 6.5: Algebraic hyperbolicity for a very general surface of the type $aD_3 + bD_4$ in $X_\Sigma \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(l_1) \oplus \mathcal{O}_{\mathbb{P}^1}(l_2))$

The canonical divisor is given by (3.2)

$$\begin{aligned} K_X &= -D_1 - D_2 - D_3 - D_4 - D_5 \\ &\sim -3D_3 + (l + a_2 - 2)D_4. \end{aligned}$$

We have the short exact sequence

$$0 \longrightarrow M \xrightarrow{i} \mathbb{Z}^{\Sigma[1]} \xrightarrow{\pi} \mathrm{Pic}(X) \to 0.$$

After choosing a basis $\{e_1^*, e_2^*, e_3^*\}$ for M and a basis for $\mathrm{Pic}(X)$ as in Lemma 1, we get the short exact sequence

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{i} \mathbb{Z}^5 \xrightarrow{\pi} \mathbb{Z}^2 \to 0.$$

We can represent the map π by the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ -l_1 & -l_2 & 0 & 1 & 1 \end{pmatrix}.$$

Lemma 6.12. *A Markov Basis for A is given by the rows of the matrix*

$$\begin{pmatrix} 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & l_1 - l_2 \\ 1 & 0 & -1 & 0 & l_1 \end{pmatrix}.$$

Proof. Let $I = \langle x_4 - x_5, x_1 - x_2 x_5^{(l_2-l_1)}, x_1 x_5^{l_1} - x_3 \rangle$, and I_A is the toric ideal associated with the matrix A. It is enough to show that $I = I_A$. Clearly, $I \subset I_A$. Note that

$$\begin{aligned} \frac{\mathbb{C}[x_1, x_2, x_3, x_4, x_5]}{\langle x_4 - x_5, x_1 - x_2 x_5^{(l_2-l_1)}, x_1 x_5^{l_1} - x_3 \rangle} &\cong \frac{\mathbb{C}[x_1, x_2, x_3, x_5]}{\langle x_1 - x_2 x_5^{(l_2-l)}, x_1 x_5^{l_1} - x_3 \rangle} \\ &\cong \frac{\mathbb{C}[x_2, x_3, x_5]}{\langle x_2 x_5^{l_2} - x_3 \rangle} \\ &\cong \mathbb{C}[x_2, x_5]. \end{aligned}$$

Hence I is a prime ideal. Moreover $\dim(\mathbb{C}[x_1, ..., x_5]/I) = 2 = \text{rank}(A)$. Hence $I = I_A$ by Lemma 5.17.

□

Theorem 6.13. *Let $l_2 \geq 1$, $D = aD_3 + bD_4$, $E' = D_3$ and $E = (a-1)D_3 + bD_4$. Then (D, E) has connected sections.*

Proof. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \\ l_1 & l_2 & -1 \end{pmatrix}, \quad t_1 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 0 \\ 0 \\ -a \\ -b \\ 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 0 \\ 0 \\ -a+1 \\ -b \\ 0 \end{pmatrix}.$$

Then $P(E') = P(A, t_1)$, $P(D) = P(A, t_2)$ and $P(E) = P(A, t_3)$. Clearly the polytope $P(E')$ contains the points $(0,0,0), (1,0,0), (0,1,0), (0,1,1) \in \mathbb{Z}^3$. Let $G = i(P(E') - i(P(E'))$. Then G contains the vectors

$$\begin{aligned} i((0,1,1) - i(0,1,0)) &= (0,0,0,1,-1) \\ i((1,0,0) - i(0,1,0)) &= (1,-1,0,0,l_1 - l_2) \\ i((1,0,0) - i(0,0,0)) &= (1,0,-1,0,l_1). \end{aligned}$$

Thus by Lemma 6.12, G is a Markov basis for A. Note that $P(D) \cap \mathbb{Z}^3 = P(E) \cap \mathbb{Z}^3 + P(E') \cap \mathbb{Z}^3$ by Theorem 5.12. Hence by Proposition 5.18, configuration (D, E) has connected sections. □

Lemma 6.14 (Curves in the boundary). *Let $X_\Sigma \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(l_1) \oplus \mathcal{O}_{\mathbb{P}^1}(l_2))$ with $l_2 \geq 1$ and S be a very general surface in the class $aD_3 + bD_4$ with $a \geq 1$.*

1. *Let $l_1 = 0$. Then S contains a curve of genus 0 or 1 if $1 \leq a \leq 3$ or $b = 1$.*

2. *Let $l_1 \geq 1$. Then S contains a curve of genus 0 or 1 if $1 \leq a \leq 3$.*

3. *For all other cases, every curve in the toric boundary has the geometric genus at least 2.*

Proof. Since $a \geq 1$, D is big. We will analyze each of the facets of $P(D)$.

1. **Facet 1:** Interior is given by the equations $z = -b$, $x + y < a$, $x > 0$ and $y > 0$. If $a \geq 1$, we have $(a-1)(a-2)/2$ interior lattice points. If $a = 0$, the restriction is the trivial divisor, and there is no curve in the intersection.

2. **Facet 2:** Interior is given by the equations $l_1x + l_2y - z = 0$, $x + y < a$, $x > 0$ and $y > 0$. If $a \geq 1$, we have $(a-1)(a-2)/2$ interior lattice points. If $a = 0$, the restriction is the trivial divisor, and there is no curve in the intersection.

3. **Facet 3:** Interior is given by the equations $y = 0$, $z > -b$, $x + y < a$, $x > 0$ and $l_1x + l_2y - z > 0$. If $b \geq 1$ or $b = 0$ and $l_1 \geq 1$, then we have $(a-1)(b-1) + l_1a(a-1)/2$ interior lattice points. If $b = 0$ and $l_1 = 0$, there are no interior lattice points.

4. **Facet 4:** Interior is given by the equations $x = 0$, $z > -b$, $x + y < a$, $y > 0$ and $l_1x + l_2y - z > 0$. If $a, b \geq 1$ or $b = 0$ and $l_2 \geq 1$, we have $(a-1)(b-1) + l_2a(a-1)/2$ interior lattice points. If $a = 0$ or $b = 0$ and $l_2 = 0$, then the restriction is the trivial divisor, and there is no curve in the intersection.

5. **Facet 5:** Interior is given by the equations $x + y = a$, $z > -b$, $x > 0$, $y > 0$ and $l_1x + l_2y - z > 0$. If $a, b \geq 1$ or $b = 0$ and $l_2 \geq 1$, we have more interior lattice points than facet 4. If $a = 0$ or $b = 0$ and $l_2 = 0$, then the restriction is the trivial divisor, and there is no curve in the intersection.

Thus by Lemma 5.20, we have the results. □

Proof of Theorem 6.10. The case $l_1 = l_2 = 0$ is discussed in [13, Example 6.1]. Hence, we can assume $l_2 \geq 1$ Let $D = aD_3 + bD_4$ with $a \geq 1$ and $b \geq 0$ and $E = (a-1)D_3 + bD_4$. Then D is big, and the configuration (D, E) has connected sections by Theorem 6.13. Applying Theorem 5.19, for any curve C not contained in the toric boundary on a very general surface S in $|D|$ we have,

$$2g - 2 \geq C \cdot ((a-4)D_3 + (b + l_1 + l_2 - 2)D_4. \tag{6.4}$$

τ	$D_1.V(\tau)$	$D_2.V(\tau)$	$D_3.V(\tau)$	$D_4.V(\tau)$	$D_5.V(\tau)$
$\tau_1 = \text{Cone}(\rho_1, \rho_2)$	$-l_1$	$-l_2$	0	1	1
$\tau_2 = \text{Cone}(\rho_1, \rho_3)$	$l_2 - l_1$	0	l_2	1	1
$\tau_3 = \text{Cone}(\rho_1, \rho_4)$	1	1	1	0	0
$\tau_4 = \text{Cone}(\rho_1, \rho_5)$	1	1	1	0	0
$\tau_5 = \text{Cone}(\rho_2, \rho_3)$	0	$l_1 - l_2$	l_1	1	1
$\tau_6 = \text{Cone}(\rho_2, \rho_4)$	1	1	1	0	0
$\tau_7 = \text{Cone}(\rho_2, \rho_5)$	1	1	1	0	0
$\tau_8 = \text{Cone}(\rho_3, \rho_4)$	1	1	1	0	0
$\tau_9 = \text{Cone}(\rho_3, \rho_5)$	1	1	1	0	0

Table 6.2: Intersection numbers in $X_\Sigma \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(l_1) \oplus \mathcal{O}_{\mathbb{P}^1}(l_2))$.

By Theorem 5.24 , if $a \geq 3$ and $b \geq 2 - l_1 - l_2$, then the natural restriction map $\text{Pic}(X_\Sigma) \to \text{Pic}(S)$ is an isomorphism. Thus, any curve C is rationally equivalent to the complete intersection of S with a divisor in the class $cD_2 + dD_4$. By Theorem 3.25, and using the notations from Table 6.2, we can write

$$C \sim acV(\tau_5) + bcV(\tau_6) + adV(\tau_8).$$

If C is not contained in the boundary, an intersection number calculation yields

$$2g - 2 \geq c(l_1 a(a-4) + a(b + l_1 + l_2 - 2) + b(a-4)) + da(a-4).$$

The degree of such a curve C with respect to ample class $H = D_3 + D_4$ is given by

$$\deg(C) = c(l_1 a + b + a) + da.$$

Let $\epsilon_0 = \dfrac{1}{l_1 a + b + a}$. Then we have

$$2g - 2 \geq \epsilon_0 \cdot \deg(C).$$

By combining Lemma 6.14 and Corollary 5.22, we have the results. $\square$

Chapter 7

Algebraically hyperbolic surfaces with Picard rank 3

In this chapter, we discuss the algebraic hyperbolicity of a very general surface in all smooth complete toric threefold with Picard rank 3. Batyrev's classification of smooth complete toric threefolds with Picard rank 3 was discussed in Section 4.4.

7.1 Fan with three primitive collections

Again, we proceed in a similar fashion to Section 6.1. In this case, the associated smooth toric variety is isomorphic to projectivization of a decomposable bundle over a smooth toric variety of smaller dimensions with Picard rank 2. Hence, we consider the projectivization of vector bundle $\mathcal{O}_{\mathcal{H}_r} \oplus \mathcal{O}_{\mathcal{H}_r}(a,b)$ with $a \geq 0$ over Hirzebruch surface $\mathcal{H}_r$. We recall the description of the fan described in Theorem 4.18. Here, Σ is generated by ray generators e_1,$-e_1+re_2+ae_3$,e_2,-e_2+be_3,e_3,$-e_3$ and corresponding rays denoted by $\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6$ and divisors by D_1,D_2,D_3,D_4,D_5,D_6. It has three primitive collections $\{e_1,-e_1+re_2+ae_3\},\{e_2,-e_2+be_3\}$ and $\{e_3,-e_3\}$.

Lemma 7.1. *The Picard group* $\mathrm{Pic}(X)$ *is generated by the classes of* D_1, D_4 *and* D_6.

Proof. By Lemma 3.6, the Picard group is generated by the classes of D_i subject to the following relations:

$$\begin{aligned} D_1 - D_2 &\sim 0 \\ rD_2 + D_3 - D_4 &\sim 0 \\ aD_2 + bD_4 + D_5 - D_6 &\sim 0. \end{aligned} \tag{7.1}$$

Thus, $\mathrm{Pic}(X) = \mathbb{Z}[D_1] \bigoplus \mathbb{Z}[D_4] \bigoplus \mathbb{Z}[D_6]$. □

Lemma 7.2. *1. If $b \geq 0$, then the Nef cone is generated by D_1, D_4 and D_6, whereas the effective cone is generated by D_1, D_3 and D_5.*

2. *If $b < 0$, then the Nef cone is generated by D_1, D_4 and $D_6 - bD_4$, whereas the effective cone is generated by D_1, D_3, D_5 and D_6.*

Proof. Let $D = dD_1 + eD_4 + fD_6$. If D is nef then we have (Theorem 4.5),

$$\begin{aligned} \varphi_D(re_2 + ae_3) &\geq \varphi_D(e_1) + \varphi_D(-e_1 + re_2 + ae_3) \\ \varphi_D(be_3) &\geq \varphi_D(e_2) + \varphi_D(-e_2 + be_3) \\ \varphi_D(0) &\geq \varphi_D(e_3) + \varphi_D(-e_3). \end{aligned}$$

Subcase 1: If $b \geq 0$, it follows that $d \geq 0$, $f \geq 0$ and $e \geq 0$. Thus, the nef cone is generated by the classes of D_1, D_4 and D_6. Using (7.1), it is easy to see that effective cone is generated by the classes of D_1, D_3 and D_5.

Subcase 2: If $b < 0$, it follows that $d \geq 0$, $f \geq 0$ and $e \geq bf$. Thus, the nef cone is generated by the classes of D_1, D_4 and $D_6 - bD_4$. Using (7.1), it is easy to see that effective cone is generated by the classes of D_1, D_3, D_5 and D_6.

□

Theorem 7.3. *Let X_Σ be a smooth toric threefold associated with the fan in Theorem 4.18. Let $b \geq 0$ and S be a very general surface in the class $dD_1 + eD_4 + fD_6$.*

1. *If $d \geq 4 - a - r, e \geq 4 - b$ and $f \geq 3$, then S is algebraically hyperbolic.*

2. *If $d \geq 4 - a - r, e \geq 3 - b$ and $f \geq 4$, then S is algebraically hyperbolic.*

3. *If $d \geq 3 - a - r, e \geq 4 - b$ and $f \geq 4$, then S is algebraically hyperbolic.*

4. *If $e = 1$ or $f = 1$, then S is not algebraically hyperbolic.*

Let $b < 0$, and S be a very general surface S in the class $dD_1 + eD_4 + f(D_6 - bD_4)$.

1. *If $d \geq 4 - a - r$, $e \geq 2$ and $f \geq 4$, then S is algebraically hyperbolic.*

2. *If $e = 1$ or $f = 1$, then S is not algebraically hyperbolic.*

Before proving the theorem, we need to find a collection of divisors with connected sections discussed in chapter 5. The Canonical divisor of X_Σ is given by (3.2)

$$\begin{aligned} K_{X_\Sigma} &= -D_1 - D_2 - D_3 - D_4 - D_5 - D_6 \\ &\sim (-2 + a + r)D_1 + (-2 + b)D_4 + -2D_6. \end{aligned}$$

After choosing a basis $\{e_1^*, e_2^*, e_3^*\}$ for M and a basis for $\mathrm{Pic}(X)$ as in Lemma 6.1, we get the short exact sequence

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{i} \mathbb{Z}^6 \xrightarrow{\pi} \mathbb{Z}^3 \to 0.$$

We can represent the map π by the matrix

$$A = \begin{pmatrix} 1 & 1 & -r & 0 & -a & 0 \\ 0 & 0 & 1 & 1 & -b & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Lemma 7.4. *A Markov Basis for A is given by the rows of the matrix*

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ r & 0 & 1 & -1 & 0 & 0 \\ a+br & 0 & b & 0 & 1 & -1 \end{pmatrix}.$$

Proof. We need to consider three cases. We will show for one case, and other cases can prove similarly.

Case 1: Let $b \geq 0$ and $I = \langle x_1 - x_2, x_1^r x_3 - x_4, x_1^{a+br} x_3^b x_5 - x_6 \rangle$ and I_A be the toric ideal associated with the matrix A. It is enough to show that $I = I_A$. Clearly, $I \subset I_A$. Note that,

$$\begin{aligned} \frac{\mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]}{\langle x_1 - x_2, x_1^r x_3 - x_4, x_1^{a+br} x_3^b x_5 - x_6 \rangle} &\cong \frac{\mathbb{C}[x_1, x_3, x_4, x_5, x_6]}{\langle x_1^r x_3 - x_4, x_1^{a+br} x_3^b x_5 - x_6 \rangle} \\ &\cong \frac{\mathbb{C}[x_1, x_3, x_5, x_6]}{\langle x_1^{a+br} x_3^b x_5 - x_6 \rangle} \\ &\cong \mathbb{C}[x_1, x_3, x_5]. \end{aligned}$$

Hence, I is a prime ideal. Moreover $\dim(\mathbb{C}[x_1, ..., x_6]/I) = 3 = \mathrm{rank}(A)$. Hence, $I = I_A$ by Lemma 5.17. We can prove similarly in cases 2 and 3 that the same vectors to be the Markov basis.

Case 2: If $b < 0$ and $a + br \geq 0$, consider $I = \langle x_1 - x_2, x_1^r x_3 - x_4, x_1^{a+br} x_5 - x_3^{-b} x_6 \rangle$.

Case 3: If $b < 0$ and $a + br < 0$, consider $I = \langle x_1 - x_2, x_1^r x_3 - x_4, x_5 - x_3^{-b} x_6 x_1^{-a-br} \rangle$.

□

Lemma 7.5. *Let $b \geq 0$, $D = dD_1 + eD_4 + fD_6$, $E' = D_1 + D_4 + D_6$ and $E = (d-1)D_1 + (e-1)D_4 + (f-1)D_6$ with $d, e, f \geq 2$. Then (D, E) has connected sections.*

Proof. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & r & a \\ 0 & 1 & 0 \\ 0 & -1 & b \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad t_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} -d \\ 0 \\ 0 \\ -e \\ 0 \\ -f \end{pmatrix}, \quad t_3 = \begin{pmatrix} -d+1 \\ 0 \\ 0 \\ -e+1 \\ 0 \\ -f+1 \end{pmatrix}.$$

Then $P(E') = P(A, t_1)$, $P(D) = P(A, t_2)$ and $P(E) = P(A, t_3)$. Clearly $P(E')$ contains the points $(0,0,0), (-1,0,0), (r,1,0), (a+br, b, 1) \in \mathbb{Z}^3$. Let $G = i(P(E') - i(P(E'))$. Then G contains the vectors

$$\begin{aligned} i(0,0,0) - i(-1,0,0) &= (1,-1,0,0,0,0) \\ i(r,1,0) - i(0,0,0) &= (r,0,1,-1,0,0) \\ i(a+br,b,1) - i(0,0,0) &= (a+br,0,b,0,1,-1). \end{aligned}$$

Thus, by Lemma 7.4, G is a Markov basis for A. Note that $P(D) \cap \mathbb{Z}^3 = P(E) \cap \mathbb{Z}^3 + P(E') \cap \mathbb{Z}^3$ by Theorem 5.12. Hence by Proposition 5.18, configuration (D, E) has connected sections. $\square$

Lemma 7.6. *Let $b < 0$, $D = dD_1 + (e - bf)D_4 + fD_6$, $E' = D_1 - bD_4 + D_6$ and $E = (d-1)D_1 + (e - bf + b)D_4 + (f-1)D_6$. Then (D, E) has connected sections.*

Proof. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & r & a \\ 0 & 1 & 0 \\ 0 & -1 & b \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad t_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ b \\ 0 \\ -1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} -d \\ 0 \\ 0 \\ -e+bf \\ 0 \\ -f \end{pmatrix}, \quad t_3 = \begin{pmatrix} -d+1 \\ 0 \\ 0 \\ -e+fb-b \\ 0 \\ -f+1 \end{pmatrix}.$$

Then $P(E') = P(A, t_1)$, $P(D) = P(A, t_2)$ and $P(E) = P(A, t_3)$. Clearly $P(E')$ contains the points $(0,0,0), (-1,0,0), (r,1,0), (-br,-b,1), (a,0,1) \in \mathbb{Z}^3$. Let $G = i(P(E') - i(P(E'))$. Then G contains the vectors

$$\begin{aligned} i(0,0,0) - i(-1,0,0) &= (1,-1,0,0,0,0) \\ i(r,1,0) - i(0,0,0) &= (r,0,1,-1,0,0) \\ i(-br,-b,0) - i(a,0,1) &= (a+br,0,b,0,1,-1). \end{aligned}$$

Thus, by Lemma 7.4, G is a Markov basis for A. Note that $P(D) \cap \mathbb{Z}^3 = P(E) \cap \mathbb{Z}^3 + P(E') \cap \mathbb{Z}^3$ by Theorem 5.12. Hence by Proposition 5.18, configuration (D, E) has connected sections. $\square$

Lemma 7.7 (Curves in the boundary)**.** *Let $X_\Sigma \cong \mathbb{P}(\mathcal{O}_{H_r} \oplus \mathcal{O}_{H_r}(aD_3 + bD_4))$ and S be a very general surface in the class $dD_1 + eD_4 + fD_6$ with $d, e, f \geq 1$. If $f = 1$ or $e = 1$, then S contains a curve of genus 0 or 1. If $d, e, f \geq 3$, then every curve in the toric boundary has at least genus 2.*

Proof. Since $d, e, f \geq 1$, D is big. We will analyze each of the facets of $P(D)$.

1. **Facet 1:** Interior is given by the equations $x = -d$, $0 < z < f$ and $0 < y < bz + e$. Thus, we have $b(f-1)f/2 + (e-1)(f-1)$ interior lattice points.

2. **Facet 2:** Interior is given by the equations $x = ry + az$, $0 < z < f$ and $0 < y < bz + e$. Thus, we have $(f-1)(e-1) + bf(f-1)/2$ interior lattice points.

3. **Facet 3:** Interior is given by the equations $y = 0$, $0 < z < f$,and $0 < x < az$. Thus, we have $(f-1)(d-1) + a(f-1)f/2$ interior lattice points.

4. **Facet 4:** Interior is given by the equations $y = bz + e$, $0 < z < f$,and $0 < x < re + z(rb + a)$. Thus, we have $(f-1)(d-1+re) + (rb+a)(f-1)f/2$ interior lattice points.

5. **Facet 5:** Interior is given by the equations $z = 0$, $0 < y < e$ and $-d < x < ry$. Thus, we have $(e-1)(d-1) + r(e-1)e/2$ interior lattice points.

6. **Facet 6:** Interior is given by the equations $z = f$, $0 < y < bf+e$ and $-d < x < ry+af$. Thus, we have $(bf+e-1)(d+af-1) + r(bf+e-1)(bf+e)/2$ interior lattice points.

Thus by Lemma 5.20, we have the results. $\square$

Lemma 7.8 (Curves in the boundary)**.** *Let $X_\Sigma \cong \mathbb{P}(\mathcal{O}_{H_r} \oplus \mathcal{O}_{H_r}(aD_3 + bD_4))$ and S be a very general surface in the class $dD_1 + eD_4 + f(D_6 - bD_4)$.*

Proof. We will analyze each of the facets of $P(D)$.

1. **Facet 1:** Interior is given by the equations $x = -d$, $0 < z < f$ and $0 < y < e - bf + bz$. Thus, we have $(e-1)(f-1) + (-b)(f-1)f/2$ interior lattice points.

2. **Facet 2:** Interior is given by the equations $x = ry + az$, $0 < z < f$ and $0 < y < bz + e - bf$. Thus, we have $(e-1)(f-1) + (-b)(f-1)f/2$ interior lattice points.

3. **Facet 3:** Interior is given by the equations $y = 0$, $0 < z < f$,and $-d < x < az$. Thus, we have $(f-1)(d-1) + a(f-1)f/2$ interior lattice points.

4. **Facet 4:** Interior is given by the equations $y = bz + e - bf$, $0 < z < f$,and $-d < x < ry + az$. Thus, we have at least$(f-1)(d-1+re) + (re+a)(f-1)f/2$ interior lattice points.

5. **Facet 5:** Interior is given by the equations $z = 0$, $0 < y < e - bf$ and $-d < x < ry$. Thus, we have at least $(e - bf - 1)(d-1) + r(e - bf - 1)(e - bf)/2$ interior lattice points.

6. **Facet 6:** Interior is given by the equations $z = f$, $0 < y < e-$ and $-d < x < ry + af$. Thus, we have at least $(e-1)(d + af - 1) + r(e-1)(e)/2$ interior lattice points.

Thus, by Lemma 5.20, we have the results. $\square$

Proof of Theorem 7.3. We will consider two cases.

Subcase 1: Let $b \geq 0$, then choose $D = dD_1 + eD_4 + fD_6$ and $E = (d-1)D_1 + (e-1)D_3 + (f-1)D_6$ with $d, e, f \geq 1$. Then D is big, and the configuration (D, E) has connected sections by Theorem 7.5. Applying Theorem 5.19, for any curve C not contained in the toric boundary on a very general surface S in $|D|$ we have,

$$2g - 2 \geq C \cdot ((d + a + r - 3)D_1 + (e + b - 3)D_4 + (f - 3)D_6). \tag{7.2}$$

By Theorem 5.24, if $d \geq 2 - a - r$, $e \geq 2 - b$ and $f \geq 2$, then the natural restriction map $\mathrm{Pic}(X_\Sigma) \to \mathrm{Pic}(S)$ is an isomorphism. Thus, any curve C is rationally equivalent to the complete intersection of S with a divisor in the class $jD_1 + kD_3 + lD_5$. By Theorem 3.25, and using the notations from Table 7.1, we can write

$$C \sim dkV(\tau_1) + dlV(\tau_4) + ejV(\tau_6) + elV(\tau_{12}) + jfV(\tau_8) + kfV(\tau_{10}) + fl(aV(\tau_8) + bV(\tau_{12})).$$

If C is not contained in the boundary, an intersection number calculation yields

$$\begin{aligned} 2g - 2 \geq\ & j(bf(f-3) + e(f-3) + f(e+b-3)) + \\ & k(d(f-3) + f(d+a+r-3) + +af(f-3)) + \\ & l((d+af)(e+b-3) + (d+af)b(f-3)) + (e+bf)(d+a+r-3) + (e+bf)r(e+b-3) + (e+bf)(a+br)(f-3)). \end{aligned}$$

The degree of such a curve C with respect to ample class $H = D_1 + D_4 + D_6$ is given by

$$\deg(C) = j(bf + f + e) + k(f + d + af) + l((d + af)(1 + b) + (e + bf)(1 + r + a + br)).$$

If $e \geq 4 - b$, $d \geq 4 - a - r$ and $f \geq 3$ then choose

$$\epsilon_0 = \frac{1}{f + (d + af)(1 + b) + (e + bf)(1 + r + a + br)}.$$

τ	$D_1.V(\tau)$	$D_2.V(\tau)$	$D_3.V(\tau)$	$D_4.V(\tau)$	$D_5.V(\tau)$	$D_6.V(\tau)$
$\tau_1 =$Cone(ρ_1, ρ_3)	0	0	0	0	1	1
$\tau_2 =$Cone(ρ_1, ρ_4)	0	0	0	0	1	1
$\tau_3 =$Cone(ρ_1, ρ_5)	0	0	1	1	$-b$	0
$\tau_4 =$Cone(ρ_1, ρ_6)	0	0	1	1	0	b
$\tau_5 =$Cone(ρ_2, ρ_3)	0	0	0	0	1	1
$\tau_6 =$Cone(ρ_2, ρ_4)	0	0	0	0	1	1
$\tau_7 =$Cone(ρ_2, ρ_5)	0	0	1	1	$-b$	0
$\tau_8 =$Cone(ρ_2, ρ_6)	0	0	1	1	0	b
$\tau_9 =$Cone(ρ_3, ρ_5)	1	1	$-r$	0	$-a$	0
$\tau_{10} =$Cone(ρ_3, ρ_6)	1	1	$-r$	0	0	a
$\tau_{11} =$Cone(ρ_4, ρ_5)	1	1	0	r	$-a-br$	0
$\tau_{12} =$Cone(ρ_4, ρ_6)	1	1	0	r	0	$a+br$

Table 7.1: Intersection numbers in X_Σ associated with the fan in Theorem 4.18

Then we have

$$2g-2 \geq jf + kf + l(d + af + (e+bf)(1+r)) \geq \epsilon_0.\ \deg C.$$

Similarly, we can take $e \geq 3-b$, $d \geq 4-a-r$ and $f \geq 4$ or $e \geq 4-b$, $d \geq 3-a-r$ and $f \geq 4$. By combining Lemma 7.7 and Corollary 5.22, we have the results.

Subcase 2: Let $b < 0$, then choose $D = dD_1 + (e-b)D_4 + fD_6$ and $E = (d-1)D_1 + (e-bf+b)D_4 + (f-1)D_6$ with $d, e, f \geq 1$. Then D is big, and the configuration (D, E) has connected sections by Theorem 7.6. For any curve C that not contained in the toric boundary on a very general surface S in $|D|$ we have,

$$2g-2 \geq C.((d+a+r-3)D_1 + (e+2b-fb-2)D_4 + (f-3)D_6).$$

If $d \geq 4-a-r$, $e \geq 2$ and $f \geq 4$, then choose $\epsilon_0 = 1$.Then we have

$$2g-2 \geq \epsilon_0(C \cdot H).$$

By combining Lemma 7.8 and Corollary 5.22, we have the results.

□

7.2 Fan with five primitive collections: Case 1

Again, we proceed in a similar fashion to Section 6.1. Recall the description of the fan described in 4.19. Here, Σ generated by the ray generators $e_1, e_2, -e_1-e_2+(b+1)e_3, -e_3, e_3, -e_1-e_2+be_3$ with $b \geq 0$ and corresponding divisors associated to rays given by $D_1, D_2, D_3, D_4, D_5, D_6$ respectively. The five primitive collections are given in 4.19.

Lemma 7.9. *The Picard group* $\mathrm{Pic}(X)$ *is generated by the classes of* D_1, D_4 *and* D_6*.*

Proof. By Lemma 3.6, the Picard group is generated by the classes of D_i subject to the following relations:

$$\begin{aligned} D_1 - D_3 - D_6 &\sim 0 \\ D_2 - D_3 - D_6 &\sim 0 \\ (b+1)D_3 - D_4 + D_5 + bD_6 &\sim 0. \end{aligned} \tag{7.3}$$

Thus, $\mathrm{Pic}\, X_\Sigma = \mathbb{Z}[D_1] \bigoplus \mathbb{Z}[D_4] \bigoplus \mathbb{Z}[D_6]$. □

Lemma 7.10. *The Nef cone is generated by the classes of* D_1, D_4 *and* $D_4 + D_6$*, whereas the effective cone is generated by the classes of* D_3, D_5 *and* D_6*.*

Proof. Let $D = dD_1 + eD_4 + fD_6$. If D is nef, then we have (Theorem 4.5),

$$\begin{aligned} \varphi_D((b+1)e_3) &\geq \varphi_D(e_1) + \varphi_D(e_2) + \varphi_D(-e_1 - e_2 + (b+1)e_3) \\ \varphi_D(-e_1 - e_2 + be_3) &\geq \varphi_D(-e_1 - e_2 + (b+1)e_3) + \varphi_D(-e_3) \\ \varphi_D(0) &\geq \varphi_D(e_3) + \varphi_D(-e_3) \\ \varphi_D(-e_1 - e_2 + (b+1)e_3) &\geq \varphi_D(-e_1 - e_2 + be_3) + \varphi_D(e_3) \\ \varphi_D(be_3) &\geq \varphi_D(-e_1 - e_2 + be_3) + \varphi_D(e_2). \end{aligned}$$

It follows that $d \geq 0, e \geq f, e \geq 0, f \geq 0$ and $f + d \geq 0$. Thus, the nef cone is generated by the classes of D_1, D_4 and $D_4 + D_6$. Using (7.3), it is easy to see that effective cone is generated by the classes of D_3, D_5 and D_6. □

Theorem 7.11. *Let* X_Σ *be the toric variety described by the fan as in Theorem 4.19, Case 1. Let* S *be a very general surface in the class* $dD_1+(e+f)D_4+fD_6$*. Then* S *is algebraically hyperbolic if* $d \geq 4, e \geq 2$ *and* $f \geq 3$*. Moreover,* S *is not algebraically hyperbolic if* $1 \leq d \leq 3$*.*

To prove the theorem, we need a collection of divisors with connected sections.

Lemma 7.12. *Let* $D = dD_1 + (e + f)D_4 + fD_6$*,* $E = d'D_1 + (e' + f')D_4 + f'D_6$ *with* $d \geq d' \geq 0$*,* $e \geq e' \geq 0$ *and* $f \geq f' \geq 0$*. Then*

$$P(D) \cap \mathbb{Z}^3 = P(E) \cap \mathbb{Z}^3 + P(D - E) \cap \mathbb{Z}^3.$$

Proof. Let

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & \text{b+1} \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & -1 & \text{b} \end{pmatrix}, \quad t_1 = \begin{pmatrix} -d \\ 0 \\ 0 \\ -e-f \\ 0 \\ -f \end{pmatrix}, \quad s_1 = \begin{pmatrix} -d' \\ 0 \\ 0 \\ -e'-f' \\ 0 \\ -f' \end{pmatrix}.$$

Then, we have $P(D) = P(A_1, t_1)$, $P(E) = P(A_1, s_1)$ and $P(D-E) = P(A_1, t_1 - s_1)$. Let $(x_0, y_0, z_0) \in P(D) \cap \mathbb{Z}^3$. We proceed by two cases.

Case 1: If $0 \le z_0 \le f$, then choose $0 \le z' \le f'$ and $0 \le z'' \le f - f'$ such that $z = z' + z''$. Let

$$A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} -d \\ 0 \\ -(b+1)z_0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} -d' \\ 0 \\ -(b+1)z' \end{pmatrix}.$$

Then, $P_1 = P(D) \cap \{z = z_0\} = P(A_2, t_2)$, $P_2 = P(E) \cap \{z = z'\} = P(A_2, s_2)$ and $P_3 = P(D-E) \cap \{z = z''\} = P(A_2, t_2 - s_2)$. It can easily check that these polytopes correspond to nef divisors on $\mathbb{P}^2$ and $P_1 = P_2 + P_3$. Hence, by Theorem 5.12, it has IDP. Then we can find $(x', y') \in P_2 \cap \mathbb{Z}^2$ and $(x'' + y'') \in P_3 \cap \mathbb{Z}^2$ such that $(x_0, y_0) = (x', y') + (x'', y'')$.

Case 2: If $f \le z_0 \le f + e$, then choose $f' \le z' \le f' + e'$ and $f - f' \le z'' \le f - f' + e - e'$ such that $z = z' + z''$. Let

$$A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad t_3 = \begin{pmatrix} -d \\ 0 \\ -bz_0 - f \end{pmatrix}, \quad s_3 = \begin{pmatrix} -d' \\ 0 \\ -bz' - f \end{pmatrix}.$$

Then, $P_1 = P(D) \cap \{z = z_0\} = P(A_3, t_3)$, $P_2 = P(E') \cap \{z = z'\} = P(A_3, s_3)$ and $P_3 = P(E) \cap \{z = z''\} = P(A_3, t_3 - s_3)$. It can easily check that these polytopes correspond to nef divisors on $\mathbb{P}^2$ and $P_1 = P_2 + P_3$. Hence, by Theorem 5.12, it has IDP. Then we can find $(x', y') \in P_2 \cap \mathbb{Z}^2$ and $(x'' + y'') \in P_3 \cap \mathbb{Z}^2$ such that $(x_0, y_0) = (x', y') + (x'', y'')$.

$\square$

The canonical divisor is given by (Theorem 3.6)

$$\begin{aligned} K_X &= -D_1 - D_2 - D_3 - D_4 - D_5 - D_6 \\ &\sim (b-2)D_1 - 2D_4 - D_6. \end{aligned}$$

After choosing a basis $\{e_1^*, e_2^*, e_3^*\}$ for M and a basis for $\mathrm{Pic}(X)$ as in Lemma 7.9, we get the short exact sequence

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{i} \mathbb{Z}^6 \xrightarrow{\pi} \mathbb{Z}^3 \to 0.$$

We can represent the map π by the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & -b-1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}.$$

Theorem 7.13. *A Markov Basis for A is given by the rows of the matrix*

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & b+1 & -1 & 1 & b \end{pmatrix}.$$

Proof. Let $I = \langle x_1 - x_3x_6, x_2 - x_3x_6, x_3^{b+1}x_5x_6^b - x_4 \rangle$, and I_A is the toric ideal associated with the matrix A. It is enough to show that $I = I_A$. Clearly, $I \subset I_A$. Note that

$$\begin{aligned} \frac{\mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]}{\langle x_1 - x_3x_6, x_2 - x_3x_6, x_3^{b+1}x_5x_6^b - x_4 \rangle} &\cong \frac{\mathbb{C}[x_2, x_3, x_4, x_5, x_6]}{\langle x_2 - x_3x_6, x_3^{b+1}x_5x_6^b - x_4 \rangle} \\ &\cong \frac{\mathbb{C}[x_3, x_4, x_5, x_6]}{\langle x_3^{b+1}x_5x_6^b - x_4 \rangle} \\ &\cong \mathbb{C}[x_3, x_5, x_6]. \end{aligned}$$

Hence I is a prime ideal. Moreover, $\dim(\mathbb{C}[x_1, ..., x_6]/I) = 3 = \mathrm{rank}(A)$. Hence, $I = I_A$ by Lemma 5.17. □

Lemma 7.14. *Let $D = dD_1 + (e+f)D_3 + fD_6$, $E' = D_3 + D_6$ and $E = dD_1 + (e+f-1)D_3 + (f-1)D_6$ with $f \geq 1$. Then (D, E) has connected sections.*

Proof. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & \text{b+1} \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & -1 & \text{b} \end{pmatrix}, \quad t_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} -d \\ 0 \\ 0 \\ -e-f \\ 0 \\ -f \end{pmatrix}, \quad t_3 = \begin{pmatrix} -d+1 \\ 0 \\ 0 \\ -e-f+1 \\ 0 \\ -f \end{pmatrix}.$$

Then $P(E') = P(A, t_1)$, $P(D) = P(A, t_2)$ and $P(E) = P(A, t_3)$. Clearly the polytope $P(E')$ contains the points $(0,0,0), (0,0,1), (0,1,1), (1,0,1) \in \mathbb{Z}^3$. Let $G = i(P(E') - i(P(E'))$. Then G contains the vectors

$$\begin{aligned} i((1,0,1)) - i((0,0,1)) &= (1,-1,0,0,0,-1) \\ i((0,1,1)) - i((0,0,1)) &= (0,1,-1,0,0,-1) \\ i((0,0,1)) - i((0,0,0)) &= (0,0,b+1,-1,1,b). \end{aligned}$$

Thus by Lemma 7.13, G is a Markov basis for A. Note that $P(D) \cap \mathbb{Z}^3 = P(E) \cap \mathbb{Z}^3 + P(E') \cap \mathbb{Z}^3$ by Theorem 7.12. Hence by Proposition 5.18, configuration (D, E) has connected sections. $\square$

Lemma 7.15. *If $1 \leq f \leq 3$, then S contains curves of genus 0 or 1 and hence cannot be algebraically hyperbolic. If $d \geq 4, e \geq 2$ and $f \geq 2$, then every curve in the boundary has the genus at least two.*

Proof. We will analyze the facets.

1. **Facet 1:** Interior is given by the equations $x = -d$, $y \geq 0$, $-x - y + (b+1)z > 0$, $-z > -e - f$, $z > 0$ and $-x - y + bz > -f$. Thus, we have, at least $d(e + f - 1)$ solutions if $e + f \geq 2$.

2. **Facet 2:** Interior is given by the equations $y = 0$, $x > -d$, $-x - y + (b+1)z > 0$, $-z > -e - f$, $z \geq 0$ and $-x - y + bz > -f$. Thus, we have at least $d(e + f - 1)$ solutions if $e + f \geq 2$.

3. **Facet 3:** Interior is given by the equations $-x - y + (b+1)z = 0$, $x > -d$, $y > 0$, $z > 0$ and $-x - y + bz > -f$. Thus, we have at least $d(f - 1)$ solutions if $f \geq 1$.

4. **Facet 4:** Interior is given by the equations $z = e + f$, $x > -d$, $y > 0$ and $-x - y + bz > -f$. Thus we have at least $\dfrac{(d+f-2)(d+f-1)}{2}$ solutions if $d + f \geq 1$.

5. **Facet 5:** Interior is given by the equations $z = 0$, $x > -d$, $y > 0$ and $-x - y + (b+1)z > 0$. Thus, we have $\dfrac{(d-2)(d-1)}{2}$ solutions if $d \geq 1$.

6. **Facet 6:** Interior is given by the equations $-x-y+bz=-f$, $x>-d$, $y>0$, $-x-y+(b+1)z>0$ and $-z>-e-f$. Thus we have at least, $(e-1)(d+f-1)$ solutions if $e\geq 1$.

Thus by Lemma 5.20, we have the results. $\square$

Proof of Theorem 7.11. Let $D=dD_1+(e+f)D_4+fD_6$ and $E=(d-1)D_1+(e+f-1)D_4+(f-1)D_6$ with $f\geq 1, e\geq 1$ and $d\geq 0$. Then D is big, and the configuration (D,E) has connected sections by Theorem 7.14. Applying Theorem 5.19, for any curve C not contained in the toric boundary on a very general surface S in $|D|$ we have,

$$2g-2\geq C\cdot((d+b-2)D_1+(e+f-3)D_4+(f-2)D_6). \tag{7.4}$$

Let $H=D_1+2D_3+D_6$. If $d\geq 4, e\geq 2, f\geq 3$, then we have

$$2g-2\geq C\cdot H. \tag{7.5}$$

Applying (7.5) and Lemma 7.15 on Corollary 5.22, we have the results.

$\square$

7.3 Fan with five primitive collections: Case 2

Again, we proceed in a similar fashion to Section 6.1. Recall the description of the fan described in 4.19. Here, Σ generated by the ray generators $e_1, -e_1-e_2+(b+1)e_3, e_2, -e_3, e_3, -e_1+be_3$ with $b\geq 0$ and corresponding divisors associated to rays given by $D_1, D_2, D_3, D_4, D_5, D_6$ respectively. The five primitive collections are given in 4.19.

Lemma 7.16. *The Picard group* $\mathrm{Pic}(X)$ *is generated by the classes of* D_1, D_4 *and* D_6.

Proof. By Lemma 3.6, the Picard group is generated by the classes of D_i subject to the following relations:

$$\begin{aligned} D_1-D_2-D_6 &\sim 0\\ D_2-D_3 &\sim 0\\ (b+1)D_2-D_4+D_5+bD_6 &\sim 0. \end{aligned} \tag{7.6}$$

Thus, $\mathrm{Pic}\,X_\Sigma=\mathbb{Z}[D_1]\bigoplus\mathbb{Z}[D_4]\bigoplus\mathbb{Z}[D_6]$. $\square$

Lemma 7.17. *The Nef cone is generated by the classes of* D_1, D_4 *and* D_4+D_6, *whereas the effective cone is generated by the classes of* D_2, D_5 *and* D_6.

Proof. Let $D = dD_1 + eD_4 + fD_6$. If D is nef then it satisfies following relations (Theorem 4.5),

$$\begin{aligned}
\varphi_D((b+1)e_3) &\geq \varphi_D(e_1) + \varphi_D(e_2) + \varphi_D(-e_1 - e_2 + (b+1)e_3) \\
\varphi_D(-e_1 + be_3) &\geq \varphi_D(-e_1 - e_2 + (b+1)e_3) + \varphi_D(e_2) + \varphi_D(-e_3) \\
\varphi_D(0) &\geq \varphi_D(e_3) + \varphi_D(-e_3) \\
\varphi_D(-e_1 + (b+1)e_3) &\geq \varphi_D(-e_1 + be_3) + \varphi_D(e_3) \\
\varphi_D(be_3) &\geq \varphi_D(-e_1 + be_3) + \varphi_D(e_1).
\end{aligned}$$

It follows that $d \geq 0, e \geq f, e \geq 0, f \geq 0$ and $f + d \geq 0$. Thus, the nef cone is generated by the classes of D_1, D_4 and $D_4 + D_6$. Using (7.6), it is easy to see that effective cone is generated by the classes of D_2, D_5 and D_6. $\square$

Theorem 7.18. *Let X_Σ be the toric variety described by the fan as in Theorem 4.19, Case 2. Let S be a very general surface in the class $dD_1+(e+f)D_4+fD_6$. Then S is algebraically hyperbolic if $d \geq 4, e \geq 4$ and $f \geq 1$. Moreover, S is not algebraically hyperbolic if $1 \leq d \leq 3$ or $1 \leq e \leq 3$.*

To prove the theorem, we need a collection of divisors with connected sections.

Lemma 7.19. *Let $D = dD_1 + (e+f)D_4 + fD_6$, $E = d'D_1 + (e'+f')D_4 + f'D_6$ with $d \geq d' \geq 0$, $e \geq e' \geq 0$ and $f \geq f' \geq 0$. Then*

$$P(D) \cap \mathbb{Z}^3 = P(E) \cap \mathbb{Z}^3 + P(D - E) \cap \mathbb{Z}^3.$$

Proof. Let

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & \mathrm{b+1} \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 0 & \mathrm{b} \end{pmatrix}, \quad t_1 = \begin{pmatrix} -d \\ 0 \\ 0 \\ -e-f \\ 0 \\ -f \end{pmatrix}, \quad s_1 = \begin{pmatrix} -d' \\ 0 \\ 0 \\ -e'-f' \\ 0 \\ -f' \end{pmatrix}.$$

Then, we have $P(D) = P(A_1, t_1)$, $P(E) = P(A_1, s_1)$ and $P(D-E) = P(A_1, t_1 - s_1)$. Let $(x_0, y_0, z_0) \in P(D) \cap \mathbb{Z}^3$. We proceed by two cases.

Case 1: If $0 \leq z_0 \leq f$, then choose $0 \leq z' \leq f'$ and $0 \leq z'' \leq f - f'$ such that $z = z' + z''$. Let

$$A_2 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} -d \\ -(b+1)z_0 \\ 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} -d' \\ -(b+1)z' \\ 0 \end{pmatrix}.$$

Then, $P_1 = P(D) \cap \{z = z_0\} = P(A_2, t_2)$, $P_2 = P(E) \cap \{z = z'\} = P(A_2, s_2)$ and $P_3 = P(D-E) \cap \{z = z''\} = P(A_2, t_2 - s_2)$. It can easily check that these polytopes correspond to nef divisors on $\mathbb{P}^2$ and $P_1 = P_2 + P_3$. Hence, by Theorem 5.12, it has IDP. Then we can find $(x', y') \in P_2 \cap \mathbb{Z}^2$ and $(x'' + y'') \in P_3 \cap \mathbb{Z}^2$ such that $(x_0, y_0) = (x', y') + (x'', y'')$.

Case 2: If $f \leq z_0 \leq f + e$, then choose $f' \leq z' \leq f' + e'$ and $f - f' \leq z'' \leq f - f' + e - e'$ such that $z = z' + z''$. Let

$$A_3 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} -d \\ -(b+1)z_0 \\ 0 \\ -bz_0 - f \end{pmatrix}, \quad s_3 = \begin{pmatrix} -d' \\ -(b+1)z' \\ 0 \\ -bz' - f' \end{pmatrix}.$$

Then, $P_1 = P(D) \cap \{z = z_0\} = P(A_3, t_3)$, $P_2 = P(E') \cap \{z = z'\} = P(A_3, s_3)$ and $P_3 = P(E) \cap \{z = z''\} = P(A_3, t_3 - s_3)$. It can easily check that these polytopes correspond to nef divisors on Hirzebruch surface $\mathcal{H}_1$ and $P_1 = P_2 + P_3$. Hence, by Theorem 5.12, it has IDP. Then we can find $(x', y') \in P_2 \cap \mathbb{Z}^2$ and $(x'' + y'') \in P_3 \cap \mathbb{Z}^2$ such that $(x_0, y_0) = (x', y') + (x'', y'')$.

□

The canonical divisor is given by (Theorem 3.6)

$$\begin{aligned} K_X &= -D_1 - D_2 - D_3 - D_4 - D_5 - D_6 \\ &\sim (b-2)D_1 - 2D_4. \end{aligned}$$

After choosing a basis $\{e_1^*, e_2^*, e_3^*\}$ for M and a basis for $\mathrm{Pic}(X)$ as in Lemma 7.16, we get the short exact sequence

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{i} \mathbb{Z}^6 \xrightarrow{\pi} \mathbb{Z}^3 \to 0.$$

We can represent the map π by the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & -b-1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 & 1 & 1 \end{pmatrix}$$

Theorem 7.20. *A Markov Basis for A is given by the rows of the matrix*

$$\begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ b & 1 & 0 & -1 & 1 & 0 \end{pmatrix}$$

Proof. Let $I = \langle x_2 - x_3, x_1 - x_2x_6, x_1^b x_2 x_5 - x_4 \rangle$ and I_A is the toric ideal associated with the matrix A. It is enough to show that $I = I_A$. Clearly, $I \subset I_A$. Note that

$$\begin{aligned} \frac{\mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]}{\langle x_2 - x_3, x_1 - x_2x_6, x_1^b x_2 x_5 - x_4 \rangle} &\cong \frac{\mathbb{C}[x_1, x_2, x_4, x_5, x_6]}{\langle x_1 - x_2x_6, x_1^b x_2 x_5 - x_4 \rangle} \\ &\cong \frac{\mathbb{C}[x_2, x_4, x_5, x_6]}{\langle x_1^b x_2 x_5 - x_4 \rangle} \\ &\cong \mathbb{C}[x_2, x_5, x_6]. \end{aligned}$$

Hence I is a prime ideal. Moreover, $\dim(\mathbb{C}[x_1, ..., x_5]/I) = 3 = \text{rank}(A)$. Hence, $I = I_A$ by Lemma 5.17. ☐

Lemma 7.21. *Let $D = dD_1 + (e+f)D_3 + fD_6$, $E' = D_3 + D_6$ and $E = dD_1 + (e+f-1)D_3 + (f-1)D_6$ with $f \geq 1$. Then (D, E) has connected sections.*

Proof. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & \text{b+1} \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 0 & \text{b} \end{pmatrix}, \quad t_1 = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} -d \\ 0 \\ -e-f \\ 0 \\ 0 \\ -f \end{pmatrix}, \quad t_3 = \begin{pmatrix} -d+1 \\ 0 \\ -e-f+1 \\ 0 \\ 0 \\ -f \end{pmatrix}.$$

Then $P(E') = P(A, t_1)$, $P(D) = P(A, t_2)$ and $P(E) = P(A, t_3)$. Clearly the polytope $P(E')$ contains the points $(0,0,0), (-1,0,0), (b,0,1), (b,1,1) \in \mathbb{Z}^3$. Let $G = i(P(E') - i(P(E'))$. Then G contains the vectors

$$\begin{aligned} i((0,0,0)) - i((-1,0,0)) &= (1,-1,0,0,0,-1) \\ i((b,0,1)) - i((0,0,0)) &= (b,1,0,-1,1,0) \\ i((b,0,1)) - i((b,1,1)) &= (0,1,-1,0,0,0). \end{aligned}$$

Thus by Lemma 7.20, G is a Markov basis for A. Note that $P(D) \cap \mathbb{Z}^3 = P(E) \cap \mathbb{Z}^3 + P(E') \cap \mathbb{Z}^3$ by Theorem 7.19. Hence by Proposition 5.18, configuration (D, E) has connected sections. ☐

Lemma 7.22. *If $1 \leq d \leq 3$ or $1 \leq e3$, then S contains curves of genus 0 or 1 and cannot be algebraically hyperbolic. If $e, d \geq 4$, then every curve in the boundary has the genus at least two.*

Proof. We will analyze each of the facets.

1. **Facet 1:** Interior is given by the equations $x = -d$, $y > 0$, $z > 0$, $-x-y+(b+1)z > 0$ and $z < e+f$. Thus, we have $\frac{(e+f-1)((b+1)(e+f)-2)}{2}$ solutions if $e+f \geq 1$. If $e+f = 0$, there is no curve in the intersection between S and D_1.

2. **Facet 2:** Interior is given by the equations $-x-y+(b+1)z = 0$, $x > 0$, $y > 0$, $z > 0$, $z < e+f$ and $-x+bz > -f$. Thus, we have at least $e(d+f-1)+\frac{(d+f-2)(d+f-1)}{2}$ solutions if $d+f \geq 1$.

3. **Facet 3:** Interior is given by the equations $y = 0$, $x > -d$, $-x-y+(b+1)z > 0$, $z > 0$, $z < e+f$ and $-x+bz > -f$. Thus, we have at least $(e-1)(d-1)$ solutions if $e \geq 1$ and $d \geq 1$.

4. **Facet 4:** Interior is given by the equations $z = e+f$, $x > -d$, $-x-y+(b+1)z > 0$, $y > 0$ and $-x+bz > -f$. Thus we have at least $e(d+f-1)+\frac{(f+d-2)(f+d-1)}{2}$ solutions.

5. **Facet 5:** Interior is given by the equations $z = 0$, $x > -d$, $-x-y(b+1)z > 0$ and $y > 0$. Thus we have, $\frac{(d-2)(d-1)}{2}$ solutions if $d \geq 1$. If $d = 0$, there is no curve in the intersection between S and D_5.

6. **Facet 6:** Interior is given by the equations $-x+bz = -f$, $-x-y+(b+1)z > 0$, $y > 0$ and $-z > -e-f$. Thus we have, $\frac{(e-2)(e-1)}{2}$ solutions if $e \geq 1$. If $e = 0$, there is no curve in the intersection between S and D_6.

Thus by Lemma 5.20, we have the results. $\square$

Proof of Theorem 7.18. Let $D = dD_1 + (e+f)D_4 + fD_6$ and $E = (d-1)D_1 + (e+f-1)D_4 + fD_6$ with $d \geq 1, e \geq 1$ and $f \geq 0$. Then D is big, and the configuration (D, E) has connected sections by Theorem 7.21. Applying Theorem 5.19, for any curve C not contained in the toric boundary on a very general surface S in $|D|$ we have,

$$2g-2 \geq C \cdot ((d+b-3)D_1 + (e+f-3)D_4 + fD_6). \tag{7.7}$$

Let $H = D_1 + 2D_4 + D_6$. If $d \geq 4, e \geq 4, f \geq 1$, then we have

$$2g-2 \geq C \cdot H. \tag{7.8}$$

Applying (7.8) and Lemma 7.22 on Corollary 5.22, we have the results. $\square$

7.4 Fan with five primitive collections: Case 3

Again, we proceed in a similar fashion to Section 6.1. Recall the description of the fan described in 4.19. Here, Σ generated by the ray generators $e_1, -e_1 + ce_2 + (b+1)e_3, -e_2 - e_3, e_2, e_3, -e_1 + ce_2 + be_3$ with $c, b \geq 0$ and corresponding divisors associated to rays given by $D_1, D_2, D_3, D_4, D_5, D_6$ respectively. The five primitive collections are given in 4.19.

Lemma 7.23. *The Picard group* $\mathrm{Pic}(X)$ *is generated by the classes of* D_1, D_3 *and* D_6*.*

Proof. By Lemma 3.6, the Picard group is generated by the classes of D_i subject to the following relations:

$$\begin{aligned} D_1 - D_2 - D_6 &\sim 0 \\ cD_2 - D_3 + D_4 + cD_6 &\sim 0 \\ (b+1)D_2 - D_3 + D_5 + bD_6 &\sim 0. \end{aligned} \tag{7.9}$$

Thus, $\mathrm{Pic}\, X_\Sigma = \mathbb{Z}[D_1] \bigoplus \mathbb{Z}[D_3] \bigoplus \mathbb{Z}[D_6]$. □

Lemma 7.24. *The Nef cone is generated by the classes of* D_1, D_3 *and* $D_3 + D_6$*, whereas the effective cone is generated by the classes of* D_2, D_4, D_5 *and* D_6*.*

Proof. Let $D = dD_1 + eD_3 + fD_6$. If D is nef, then we have (Theorem 4.5),

$$\begin{aligned} \varphi_D(ce_2 + (b+1)e_3) &\geq \varphi_D(e_1) + \varphi_D(-e_1 + ce_2 + (b+1)e_3) \\ \varphi_D(-e_1 + ce_2 + be_3) &\geq \varphi_D(-e_1 + ce_2 + (b+1)e_3) + \varphi_D(-e_2 - e_3) + \varphi_D(e_2) \\ \varphi_D(0) &\geq \varphi_D(-e_2 - e_3) + \varphi_D(e_2) + \varphi_D(e_3) \\ \varphi_D(-e_1 + ce_2 + (b+1)e_3) &\geq \varphi_D(e_3) + \varphi_D(-e_1 + ce_2 + be_3) \\ \varphi_D(ce_2 + be_3) &\geq \varphi_D(-e_1 + ce_2 + be_3) + \varphi_D(e_1). \end{aligned}$$

It follows that $d \geq 0, f \geq 0, e \geq 0$ and $e \geq f$. Thus, the nef cone is generated by the classes of D_1, D_3 and $D_3 + D_6$. Using (7.9), one can see that the effective cone is generated by the classes D_2, D_4, D_5 and D_6. □

Theorem 7.25. *Let* X_Σ *be the toric variety described by the fan as in Theorem 4.19, Case 3. Let* S *be a very general surface in the class* $dD_1 + (e+f)D_3 + fD_6$*. Then* S *is algebraically hyperbolic if* $f \geq 2, e \geq 4$ *and* $d \geq 2 - b - c$*. Moreover,* S *is not algebraically hyperbolic if* $1 \leq e \leq 3$ *or* $f = 1$*.*

Lemma 7.26. *Let* $D = dD_1 + (e+f)D_3 + fD_6$*,* $E = d'D_1 + (e' + f')D_3 + f'D_6$ *with* $d \geq d' \geq 0$*,* $e \geq e' \geq 0$ *and* $f \geq f' \geq 0$*. Then*

$$P(D) \cap \mathbb{Z}^3 = P(E) \cap \mathbb{Z}^3 + P(D-E) \cap \mathbb{Z}^3.$$

Proof. Let

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & \mathrm{c} & \mathrm{b+1} \\ 0 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & \mathrm{c} & \mathrm{b} \end{pmatrix}, \quad t_1 = \begin{pmatrix} -d \\ 0 \\ -e-f \\ 0 \\ 0 \\ -f \end{pmatrix}, \quad s_1 = \begin{pmatrix} -d' \\ 0 \\ -e'-f' \\ 0 \\ 0 \\ -f' \end{pmatrix}.$$

Then, we have $P(D) = P(A_1, t_1)$, $P(E) = P(A_1, s_1)$ and $P(D-E) = P(A_1, t_1 - s_1)$. Let $(x_0, y_0, z_0) \in P(D) \cap \mathbb{Z}^3$. We proceed by two cases.

Case 1: If $0 \le z_0 \le f$**.** Then choose integers $0 \le z' \le f'$ and $0 \le z'' \le f - f'$ such that $z = z' + z''$. Pick any integers $0 \le y' \le e' + f' - z'$ and $0 \le y'' \le e - e' + f - f' - z''$ such that $y_0 = y' + y''$. Let

$$A_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} -d \\ -cy_0 - (b+1)z_0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} -d' \\ -cy' - (b+1)z' \end{pmatrix}.$$

Then $P_1 = P(D) \cap \{z = z_0, y = y_0\} = P(A_2, t_2)$, $P_2 = P(E) \cap \{z = z', y = y'\} = P(A_2, s_2)$ and $P_3 = P(D-E) \cap \{z = z'', y = y''\} = P(A_2, t_2 - s_2)$. It can easily check that these polytopes correspond to nef divisors on $\mathbb{P}^1$ and $P_1 = P_2 + P_3$. Hence, by Theorem 5.12, it has IDP. Then we can find $x' \in P_2 \cap \mathbb{Z}^2$ and $x'' \in P_3 \cap \mathbb{Z}^2$ such that $x = x' + x''$.

Case 2: If $f \le z_0 \le f+e$, then choose $f' \le z' \le f'+e'$ and $f-f' \le z'' \le f-f'+e-e'$ such that $z = z'+z''$. Pick any integers $0 \le y' \le e'+f'-z'$ and $0 \le y'' \le e-e'+f-f'-z''$ such that $y_0 = y' + y''$. Let

$$A_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t_3 = \begin{pmatrix} -d \\ -cy_0 - bz_0 - f \end{pmatrix}, \quad s_3 = \begin{pmatrix} -d' \\ -cy' - bz' - f' \end{pmatrix}.$$

Then $P_1 = P(D) \cap \{z = z_0, y = y_0\} = P(A_3, t_3)$, $P_2 = P(E') \cap \{z = z', y = y'\} = P(A_3, s_3)$ and $P_3 = P(E) \cap \{z = z'', y = y''\} = P(A_3, t_3 - s_3)$. It can easily check that these polytopes correspond to nef divisors on $\mathbb{P}^1$ and $P_1 = P_2 + P_3$. Hence, by Theorem 5.12, it has IDP. Then we can find $x' \in P_2 \cap \mathbb{Z}$ and $x'' \in P_3 \cap \mathbb{Z}$ such that $x_0 = x' + x''$.

$\square$

The canonical divisor is given by (3.6)

$$\begin{aligned} K_X &= -D_1 - D_2 - D_3 - D_4 - D_5 - D_6 \\ &\sim (b+c-1)D_1 - 3D_3 - D_6. \end{aligned}$$

After choosing a basis $\{e_1, e_2, e_3\}$ for M and a basis for $\mathrm{Pic}(X)$ as in Lemma 7.23, we get the short exact sequence

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{i} \mathbb{Z}^6 \xrightarrow{\pi} \mathbb{Z}^3 \to 0.$$

We can represent the map π by the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & -c & -b-1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Theorem 7.27. *A Markov Basis for A is given by the rows of the matrix*

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & b+1 & -1 & 0 & 1 & b \\ 0 & b+1-c & 0 & -1 & 1 & b-c \end{pmatrix}.$$

Proof. Let $I_1 = \langle x_1 - x_2x_6, x_1^{(b+1)}x_5x_6^b - x_3, x_5x_2^{(b+1-c)}x_6^{(b-c)} - x_4\rangle$, $I_2 = \langle x_1 - x_2x_6, x_1^{(b+1)}x_5x_6^b - x_3, x_5 - x_4x_2^{(c-b-1)}x_6^{(c-b)}\rangle$ and I_A is the toric ideal associated with the matrix A. It is enough to show that $I_1 = I_A$ if $b \geq c$ and $I_2 = I_A$ if $b < c$. We will show for case $b \geq c$, and the second case can prove similarly. Clearly, $I_1 \subset I_A$. Note that

$$\begin{aligned} \frac{\mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]}{\langle x_1 - x_2x_6, x_1^{(b+1)}x_5x_6^b - x_3, x_5x_2^{(b+1-c)}x_6^{(b-c)} - x_4\rangle} &\cong \frac{\mathbb{C}[x_2, x_3, x_4, x_5, x_6]}{\langle x_1^{(b+1)}x_5x_6^b - x_3, x_5x_2^{(b+1-c)}x_6^{(b-c)} - x_4\rangle} \\ &\cong \frac{\mathbb{C}[x_2, x_4, x_5, x_6]}{\langle x_5x_2^{(b+1-c)}x_6^{(b-c)} - x_4\rangle} \\ &\cong \mathbb{C}[x_2, x_5, x_6]. \end{aligned}$$

Hence I_1 is a prime ideal. Moreover $\dim(\mathbb{C}[x_1, ..., x_5, x_6]/I_1) = 3 = \mathrm{rank}(A)$. Hence, $I_1 = I_A$ by Lemma 5.17. □

Lemma 7.28. *Let $D = dD_1 + (e+f)D_3 + fD_6$, $E' = D_1 + D_3$ and $E = (d-1)D_1 + (e+f-1)D_3$ with $d, e \geq 1$. Then (D, E) has connected sections.*

Proof. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & \mathrm{c} & \mathrm{b+1} \\ 0 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & \mathrm{c} & \mathrm{b} \end{pmatrix}, \quad t_1 = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} -d \\ 0 \\ -e-f \\ 0 \\ 0 \\ -f \end{pmatrix}, \quad t_3 = \begin{pmatrix} -d+1 \\ 0 \\ -e-f+1 \\ 0 \\ 0 \\ -f \end{pmatrix}.$$

Then $P(E') = P(A, t_1)$, $P(D) = P(A, t_2)$ and $P(E) = P(A, t_3)$. Clearly the polytope $P(E')$ contains the points $(0,0,0),(0,0,1),(0,1,0),(-1,0,0) \in \mathbb{Z}^3$. Let $G = i(P(E') - i(P(E'))$. Then G contains the vectors

$$\begin{aligned} i(0,0,0) - i(-1,0,0)) &= (1,-1,0,0,0,1) \\ i(0,0,0) - i(0,1,0)) &= (0,-c,1,-1,0,-c) \\ i(0,0,1) - i(0,1,0) &= (0,b+1-c,0,-1,1,b-c). \end{aligned}$$

Thus by Lemma 6.5, G is a Markov basis for A. Note that $P(D) \cap \mathbb{Z}^3 = P(E) \cap \mathbb{Z}^3 + P(E') \cap \mathbb{Z}^3$ by Theorem 5.12. Hence by Proposition 5.18, configuration (D, E) has connected sections. $\square$

Lemma 7.29. *If $1 \le b \le 3$ or $a \le 1$, then S contains curves of genus 0 or 1 and cannot be algebraically hyperbolic. If $e \ge 4$ and $f \ge 2$, then every curve in the boundary has the genus at least two.*

Proof. We will analyze each of the facets of $P(D)$.

1. **Facet 1:** Interior is given by the equations $x = 0$, $y > 0$, $z > 0$ and $y + z < e + f$. It is easy to find the integer values for x, y, z satisfies the above equations. Indeed there are $\dfrac{(e+f-1)(e+f-2)}{2}$ solutions if $e + f \ge 1$. If $e + f = 0$, there is no curve in the intersection.

2. **Facet 2:** Interior is given by the equations $-x+cy+(b+1)z = -d$, $-y-z > -e-f$, $y > 0$, $z > 0$ and $-x + cy + bz > -d - f$. Thus, we have $e(f-1) + \dfrac{(f-1)(f-2)}{2}$ solutions if $f \ge 1$. If $f = 0$, there is no curve in the intersection.

3. **Facet 3:** Interior is given by the equations $y = 0$, $x > 0$, $-x - y + (b+1)z > -e$, $z > 0$, $z < d + f$ and $-x + bz > -e - f$. Thus, we have at least $(e+f-1)(d+f-1)$ solutions if $f \ge 1$.

4. **Facet 4:** Interior is given by the equations $-z = -d - f$, $x > 0$, $-x + (b+1)z > -e$, $y > 0$ and $-x + (b+1)z > -e - f$. Thus we have $(d+f)(b(d+f)+e-1) + \dfrac{b(d+f)+(e-2))(b(d+f)+e-1)}{2}$ solutions.

5. **Facet 5:** Interior is given by the equations $z = 0$, $x > 0$, $-x - y(b+1)z > -e$ and $y > 0$. Thus we have, $\dfrac{(e-2)(e-1)}{2}$ solutions if $e \geq 1$.
6. **Facet 6:** Interior is given by the equations $-x + bz = -e - f$, $-x - y + (b+1)z > -e$, $y > 0$ and $-z > -d - f$. Thus we have, $\dfrac{(e-2)(e-1)}{2}$ solutions if $e \geq 1$.

Thus by Lemma 5.20, we have the results. □

Proof of Theorem 7.25. Let $D = dD_1 + (e+f)D_3 + fD_6$ and $E = (d-1)D_1 + (e+f-1)D_3 + fD_6$ with $d \geq 1, e \geq 1$ and $f \geq 0$. Then D is big, and the configuration (D, E) has connected sections by Theorem 7.28. Applying Theorem 5.19, for any curve C not contained in the toric boundary on a very general surface S in $|D|$ we have,

$$2g - 2 \geq C \cdot ((d+b+c-2)D_1 + (e+f-4)D_3 + (f-1)D_6). \tag{7.10}$$

Let $H = D_1 + 2D_3 + D_6$. If $d \geq 4, e \geq 4, f \geq 1$, then we have

$$2g - 2 \geq C \cdot H. \tag{7.11}$$

Applying (7.11) and Lemma 7.29 on Corollary 5.22, we have the results. □

7.5 Fan with five primitive collections: Case 4

Again, we proceed in a similar fashion to Section 6.1. Recall the description of the fan described in 4.19. Here, Σ generated by the ray generators $e_1, -e_1+(c+1)e_2+(b+1)e_3, -e_2-e_3, e_2, e_3, -e_1 + ce_2 + be_3$ with $c, b \geq 0$ and corresponding divisors associated to rays given by $D_1, D_2, D_3, D_4, D_5, D_6$ respectively. The five primitive collections are given in 4.19.

Lemma 7.30. *The Picard group* $\mathrm{Pic}(X)$ *is generated by the classes of* D_1, D_3 *and* D_6*.*

Proof. By Lemma 3.6, the Picard group is generated by the classes of D_i subject to the following relations:

$$\begin{aligned} D_1 - D_2 - D_6 &\sim 0 \\ (c+1)D_2 - D_3 + D_4 + cD_6 &\sim 0 \\ (b+1)D_2 - D_3 + D_5 + bD_6 &\sim 0. \end{aligned} \tag{7.12}$$

Thus, $\mathrm{Pic}\, X_\Sigma = \mathbb{Z}[D_2] \bigoplus \mathbb{Z}[D_3] \bigoplus \mathbb{Z}[D_6]$. □

Lemma 7.31. *The Nef cone is generated by the classes of* D_1, D_3 *and* $D_3 + D_6$*, whereas the effective cone is generated by th classes of* D_2, D_4, D_5 *and* D_6*.*

Proof. Let $D = dD_1 + eD_3 + fD_6$. If D is nef, then we have (Theorem 4.5),

$$\begin{aligned}
\varphi_D(ce_2 + (b+1)e_3) &\geq \varphi_D(e_1) + \varphi_D(-e_1 + ce_2 + (b+1)e_3) \\
\varphi_D(-e_1 + ce_2 + be_3) &\geq \varphi_D(-e_1 + (c+1)e_2 + (b+1)e_3) + \varphi_D(-e_2 - e_3) \\
\varphi_D(0) &\geq \varphi_D(-e_2 - e_3) + \varphi_D(e_2) + \varphi_D(e_3) \\
\varphi_D(-e_1 + (c+1)e_2 + (b+1)e_3) &\geq \varphi_D(e_2) + \varphi_D(e_3) + \varphi_D(-e_1 + ce_2 + be_3) \\
\varphi_D(ce_2 + be_3) &\geq \varphi_D(-e_1 + ce_2 + be_3) + \varphi_D(e_1).
\end{aligned}$$

It follows that $d \geq 0, e \geq f, e \geq 0, e + f \geq 0$ and $f + d \geq 0$. Thus, the nef cone is generated by the classes of D_1, D_3 and $D_3 + D_6$. Using (7.12), one can see that the effective cone is generated by the classes D_2, D_4, D_5 and D_6. $\square$

Theorem 7.32. *Let X_Σ be the toric variety described by the fan as in Theorem 4.19, Case 4. Let S be a very general surface in the class $dD_1 + (e+f)D_3 + fD_6$. Then S is algebraically hyperbolic if $f \geq 4, e \geq 2$ and $d \geq 1 - b - c$. Moreover, S is not algebraically hyperbolic if $1 \leq f \leq 3$ or $e = 1$.*

Lemma 7.33. *Let $D = dD_1 + (e+f)D_3 + fD_6$, $E = d'D_1 + (e' + f')D_3 + f'D_6$ with $d \geq d' \geq 0$, $e \geq e' \geq 0$ and $f \geq f' \geq 0$. Then*

$$P(D) \cap \mathbb{Z}^3 = P(E) \cap \mathbb{Z}^3 + P(D - E) \cap \mathbb{Z}^3.$$

Proof. Let

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & \text{b+1} \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & -1 & \text{b} \\ 0 & 1 & 0 \end{pmatrix}, \quad t_1 = \begin{pmatrix} -d \\ 0 \\ -e-f \\ 0 \\ 0 \\ -f \end{pmatrix}, \quad s_1 = \begin{pmatrix} -d' \\ 0 \\ -e'-f' \\ 0 \\ 0 \\ -f' \end{pmatrix}.$$

Then, we have $P(D) = P(A_1, t_1)$, $P(E) = P(A_1, s_1)$ and $P(D - E) = P(A_1, t_1 - s_1)$. Let $(x_0, y_0, z_0) \in P(D) \cap \mathbb{Z}^3$. We proceed by two cases.

Case 1: If $0 \leq z_0 \leq f$, then choose integers $0 \leq z' \leq f'$ and $0 \leq z'' \leq f - f'$ such that $z = z' + z''$. Pick any integers $0 \leq y' \leq e' + f' - z'$ and $0 \leq y'' \leq e - e' + f - f' - z''$ such that $y_0 = y' + y''$. Let

$$A_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} -d \\ -cy_0 - (b+1)z_0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} -d' \\ -cy' - (b+1)z' \end{pmatrix}.$$

Then, $P_1 = P(D) \cap \{z = z_0, y = y_0\} = P(A_2, t_2)$, $P_2 = P(E) \cap \{z = z', y = y'\} = P(A_2, s_2)$ and $P_3 = P(D-E) \cap \{z = z'', y = y''\} = P(A_2, t_2 - s_2)$. It can easily check that these polytopes correspond to nef divisors on $\mathbb{P}^1$ and $P_1 = P_2 + P_3$. Hence, by Theorem 5.12, it has IDP. Then we can find $x' \in P_2 \cap \mathbb{Z}^2$ and $x'' \in P_3 \cap \mathbb{Z}^2$ such that $x = x' + x''$.

Case 2: If $f \leq z_0 \leq f+e$, then choose $f' \leq z' \leq f'+e'$ and $f-f' \leq z'' \leq f-f'+e-e'$ such that $z = z'+z''$. Pick any integers $0 \leq y' \leq e'+f'-z'$ and $0 \leq y'' \leq e-e'+f-f'-z''$ such that $y_0 = y'+y''$. Let

$$A_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t_3 = \begin{pmatrix} -d \\ -cy_0 - bz_0 - f \end{pmatrix}, \quad s_3 = \begin{pmatrix} -d' \\ -cy' - bz' - f' \end{pmatrix}.$$

Then, $P_1 = P(D) \cap \{z = z_0, y = y_0\} = P(A_3, t_3)$, $P_2 = P(E') \cap \{z = z', y = y'\} = P(A_3, s_3)$ and $P_3 = P(E) \cap \{z = z'', y = y''\} = P(A_3, t_3 - s_3)$. It can easily check that these polytopes correspond to nef divisors on $\mathbb{P}^1$ and $P_1 = P_2 + P_3$. Hence, by Theorem 5.12, it has IDP. Then we can find $x' \in P_2 \cap \mathbb{Z}$ and $x'' \in P_3 \cap \mathbb{Z}$ such that $x_0 = x' + x''$.

□

The canonical divisor is given by (3.6)

$$\begin{aligned} K_X &= -D_1 - D_2 - D_3 - D_4 - D_5 - D_6 \\ &\sim (b+c)D_1 - 3D_3 - 2D_6. \end{aligned}$$

After choosing a basis $\{e_1^*, e_2^*, e_3^*\}$ for M and a basis for $\mathrm{Pic}(X)$ as in Lemma 7.30, we get the short exact sequence

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{i} \mathbb{Z}^6 \xrightarrow{\pi} \mathbb{Z}^3 \to 0.$$

We can represent the map π by the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & -c-1 & -b-1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Theorem 7.34. *A Markov Basis for A is given by the rows of the matrix*

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & b+1 & -1 & 0 & 1 & b \\ 0 & b-c & 0 & -1 & 1 & b-c \end{pmatrix}.$$

Proof. Let $I_1 = \langle x_1 - x_2x_6, x_2^{b+1}x_5x_6^b - x_3, x_4 - x_5x_2^{b-c}x_6^{b-c}\rangle$, $I_2 = \langle x_1 - x_2x_6, x_2^{b+1}x_5x_6^b - x_3, x_5 - x_4x_2^{c-b}x_6^{c-b}\rangle$ and I_A is the toric ideal associated with the matrix A. It is enough to show that $I_1 = I_A$ if $b \geq c$ and $I_2 = I_A$ if $b < c$. We will show for case $b \geq c$ and the second case can prove similarly. Clearly, $I_1 \subset I_A$. Note that

$$\begin{aligned}\frac{\mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]}{\langle x_1 - x_2x_6, x_2^{b+1}x_5x_6^b - x_3, x_4 - x_5x_2^{b-c}x_6^{b-c}\rangle} &\cong \frac{\mathbb{C}[x_2, x_3, x_4, x_5, x_6]}{\langle x_2^{b+1}x_5x_6^b - x_3, x_4 - x_5x_2^{b-c}x_6^{b-c}\rangle} \\ &\cong \frac{\mathbb{C}[x_2, x_4, x_5, x_6]}{\langle x_4 - x_5x_2^{b-c}x_6^{b-c}\rangle} \\ &\cong \mathbb{C}[x_2, x_5, x_6].\end{aligned}$$

Hence, I_1 is a prime ideal. Moreover, $\dim(\mathbb{C}[x_1, ..., x_5]/I_1) = 3 = \text{rank}(A)$. Hence, $I_1 = I_A$ by Lemma 5.17. □

Lemma 7.35. *Let $D = dD_1 + (e + f)D_3 + fD_6$, $E' = D_1 + D_3$ and $E = (d-1)D_1 + (e + f - 1)D_3$ with $d, e \geq 1$. Then (D, E) has connected sections.*

Proof. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & \text{c+1} & \text{b+1} \\ 0 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & \text{c} & \text{b} \end{pmatrix}, \quad t_1 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} -d \\ 0 \\ -e-f \\ 0 \\ 0 \\ -f \end{pmatrix}, \quad t_3 = \begin{pmatrix} -d \\ 0 \\ -e-f+1 \\ 0 \\ 0 \\ -f+1 \end{pmatrix}.$$

Then $P(E') = P(A, t_1)$, $P(D) = P(A, t_2)$ and $P(E) = P(A, t_3)$. Clearly the polytope $P(E')$ contains the points $(0, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0) \in \mathbb{Z}^3$. Let $G = i(P(E') - i(P(E'))$. Then G contains the vectors

$$\begin{aligned} i(1, 1, 0) - i(0, 1, 0)) &= (1, -1, 0, 0, 0, -1) \\ i(0, 0, 1) - i(0, 0, 0)) &= (0, b+1, -1, 0, 1, b) \\ i(0, 0, 1) - i(0, 1, 0) &= (0, b-c, 0, -1, 1, b-c). \end{aligned}$$

Thus by Lemma 7.34, G is a Markov basis for A. Note that $P(D) \cap \mathbb{Z}^3 = P(E) \cap \mathbb{Z}^3 + P(E') \cap \mathbb{Z}^3$ by Theorem 7.33. Hence by Proposition 5.18, configuration (D, E) has connected sections. □

Lemma 7.36. *If $1 \leq f \leq 3$ or $e \leq 1$, then S contains curves of genus 0 or 1 and cannot be algebraically hyperbolic. If $f \geq 4$ and $e \geq 2$, then every curve in the boundary has the genus at least two.*

Proof. We will analyze each of the facets of $P(D)$.

1. **Facet 1:** Interior is given by the equations $x = -d$, $-y - z > -e - f$, $y > 0$ and $z > 0$. Thus, we have $\dfrac{(e+f-1)(e+f-2)}{2}$ solutions if $e + f \geq 1$. If $e + f = 0$, there is no curve in the intersection.

2. **Facet 2:** Interior is given by the equations $-x + (c+1)y + (b+1)z = 0$,$y > 0$, $z > 0$ and $-x + cy + bz > -f$. Thus, we have $\dfrac{(f-1)(f-2)}{2}$ solutions if $f \geq 1$. If $f = 0$, there is no curve in the intersection.

3. **Facet 3:** Interior is given by the equations $y + z = e + f$, $x > -d$, $y > 0$, $z > 0$ and $-x + cy + bz > -f$. Thus, we have at least $(e+f-2)(d+f-1)$ solutions if $e+f \geq 2$ and $d + f \geq 1$.

4. **Facet 4:** Interior is given by the equations $y = 0$, $x > -d$, $-x+(c+1)y+(b+1)z > 0$, $-y-z > -e-f$, $z > 0$ and $-x+cy+bz > -f$. Thus, we have at least $\dfrac{(f-1)(f-2)}{2}$ solutions if $f \geq 1$.

5. **Facet 5:** Interior is given by the equations $z = 0$, $x > -d$, $-x+(c+1)y+(b+1)z > 0$, $-y-z > -e-f$, $y > 0$ and $-x+cy+bz > -f$. Thus, we have at least $\dfrac{(f-2)(f-1)}{2}$ solutions if $f \geq 1$.

6. **Facet 6:** Interior is given by the equations $-x+cy+bz = -f$, $-x+(c+1)y+(b+1)z > 0$, $-y-z > -e-f$, $y > 0$ and $z > 0$. Thus, we have $f(e-1) + \dfrac{(e-2)(e-1)}{2}$ solutions if $e \geq 1$. If $e = 0$ and $f \geq 1$, there are no interior lattice points.

Thus by Lemma 5.20, we have the results. □

Proof of Theorem 7.32. Let $D = dD_1 + (e+f)D_3 + fD_6$ and $E = dD_1 + (e+f-1)D_3 + (f-1)D_6$ with $d \geq 0, e \geq 1$ and $f \geq 1$. Then D is big, and the configuration (D, E) has connected sections by Theorem 7.35. Applying Theorem 5.19, for any curve C not contained in the toric boundary on a very general surface S in $|D|$ we have,

$$2g - 2 \geq C \cdot ((d+b+c)D_1 + (e+f-4)D_3 + (f-3)D_6). \tag{7.13}$$

Let $H = D_1 + 2D_3 + D_6$. If $d \geq 1 - b - c, e \geq 2, f \geq 4$, then we have

$$2g - 2 \geq C \cdot H. \tag{7.14}$$

Applying (7.14) and Lemma 7.36 on Corollary 5.22, we have the results. □

7.6 Fan with five primitive collections: Case 5

Again, we proceed in a similar fashion to Section 6.1. Recall the description of the fan described in 4.19. Here, Σ generated by the ray generators $e_1, -e_1+(b+1)e_3, -e_3, e_3, -e_1-e_2+be_3, e_2$ with $b \geq 0$ and corresponding divisors associated to rays given by $D_1, D_2, D_3, D_4, D_5, D_6$ respectively. The five primitive collections are given in 4.19.

Lemma 7.37. *The Picard group* $\operatorname{Pic}(X)$ *is generated by the classes of* D_1, D_3 *and* D_6*.*

Proof. By Lemma 3.6, the Picard group is generated by the classes of D_i subject to the following relations:

$$\begin{aligned} D_1 - D_2 - D_5 &\sim 0 \\ -D_5 + D_6 &\sim 0 \\ (b+1)D_2 - D_3 + D_4 + bD_5 &\sim 0. \end{aligned} \tag{7.15}$$

Thus, $\operatorname{Pic} X_\Sigma = \mathbb{Z}[D_1] \bigoplus \mathbb{Z}[D_3] \bigoplus \mathbb{Z}[D_6]$. □

Lemma 7.38. *The Nef cone is generated by the classes of* D_1, D_3 *and* $D_3 + D_6$*, whereas the effective cone is generated by the classes of* D_2, D_4 *and* D_5*.*

Proof. Let $D = dD_1 + eD_3 + fD_6$. If D is nef then it satisfies following relations (Theorem 4.5),

$$\begin{aligned} \varphi_D((b+1)e_3) &\geq \varphi_D(e_1) + \varphi_D(-e_1 + (b+1)e_3) \\ \varphi_D(-e_1 + be_3) &\geq \varphi_D(-e_1 + (b+1)e_3) + \varphi_D(-e_3) \\ \varphi_D(0) &\geq \varphi_D(e_3) + \varphi_D(-e_3) \\ \varphi_D(-e_1 + (b+1)e_3) &\geq \varphi_D(-e_1 - e_2 + be_3) + \varphi_D(e_2) + \varphi_D(e_3) \\ \varphi_D(be_3) &\geq \varphi_D(-e_1 - e_2 + be_3) + \varphi_D(e_2) + \varphi_D(e_1). \end{aligned}$$

It follows that $d \geq 0, e \geq 0, f \geq 0$ and $e \geq f$. Thus, the nef cone is generated by the classes of D_1, D_3 and $D_3 + D_6$. Using (7.15), one can see that the effective cone is generated by the classes D_2, D_4 and D_5. □

Theorem 7.39. *Let* X_Σ *be the toric variety described by the fan as in Theorem 4.19, Case 5. Let* S *be a very general surface in the class* $dD_1 + (e+f)D_3 + fD_6$*. Then* S *is algebraically hyperbolic if* $d \geq 2, e \geq 1$ *and* $f \geq 4$*. Moreover,* S *is not algebraically hyperbolic if* $d = 1$ *or* $1 \leq f \leq 3$*.*

Lemma 7.40. *Let* $D = dD_1 + (e+f)D_3 + fD_6$*,* $E = d'D_1 + (e'+f')D_3 + f'D_6$ *with* $d \geq d' \geq 0$*,* $e \geq e' \geq 0$ *and* $f \geq f' \geq 0$*. Then*

$$P(D) \cap \mathbb{Z}^3 = P(E) \cap \mathbb{Z}^3 + P(D-E) \cap \mathbb{Z}^3.$$

Proof. Let

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & \text{b+1} \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & -1 & \text{b} \\ 0 & 1 & 0 \end{pmatrix}, \quad t_1 = \begin{pmatrix} -d \\ 0 \\ -e-f \\ 0 \\ 0 \\ -f \end{pmatrix}, \quad s_1 = \begin{pmatrix} -d' \\ 0 \\ -e'-f' \\ 0 \\ 0 \\ -f' \end{pmatrix}.$$

Then, we have $P(D) = P(A_1, t_1)$, $P(E) = P(A_1, s_1)$ and $P(D-E) = P(A_1, t_1 - s_1)$. Let $(x_0, y_0, z_0) \in P(D) \cap \mathbb{Z}^3$. We proceed by two cases.

Case 1: If $0 \le z_0 \le f$, then choose $0 \le z' \le f'$ and $0 \le z'' \le f - f'$ such that $z = z' + z''$. Let

$$A_3 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad t_3 = \begin{pmatrix} -d \\ -(b+1)z_0 \\ -bz_0 \\ -f \end{pmatrix}, \quad s_3 = \begin{pmatrix} -d' \\ -(b+1)z' \\ -bz' \\ -f' \end{pmatrix}.$$

Then, $P_1 = P(D) \cap \{z = z_0\} = P(A_2, t_2)$, $P_2 = P(E) \cap \{z = z'\} = P(A_2, s_2)$ and $P_3 = P(D-E) \cap \{z = z''\} = P(A_2, t_2 - s_2)$. It can easily check that these polytopes correspond to nef divisors on Hirzebruch surface $\mathcal{H}_1$ and $P_1 = P_2 + P_3$. Hence, by Theorem 5.12, it has IDP. Then we can find $(x', y') \in P_2 \cap \mathbb{Z}^2$ and $(x'' + y'') \in P_3 \cap \mathbb{Z}^2$ such that $(x_0, y_0) = (x', y') + (x'', y'')$.

Case 2: If $f \le z_0 \le f + e$, then choose $f' \le z' \le f' + e'$ and $f - f' \le z'' \le f - f' + e - e'$ such that $z = z' + z''$. Let

$$A_2 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} -d \\ -bz_0 \\ -f \end{pmatrix}, \quad s_2 = \begin{pmatrix} -d' \\ -bz' \\ -f' \end{pmatrix}.$$

Then, $P_1 = P(D) \cap \{z = z_0\} = P(A_3, t_3)$, $P_2 = P(E') \cap \{z = z'\} = P(A_3, s_3)$ and $P_3 = P(E) \cap \{z = z''\} = P(A_3, t_3 - s_3)$. It can easily check that these polytopes correspond to nef divisors on $\mathbb{P}^2$ and $P_1 = P_2 + P_3$. Hence, by Theorem 5.12, it has IDP. Then we can find $(x', y') \in P_2 \cap \mathbb{Z}^2$ and $(x'' + y'') \in P_3 \cap \mathbb{Z}^2$ such that $(x_0, y_0) = (x', y') + (x'', y'')$.

$\square$

The canonical divisor of X_Σ is given by (3.6)

$$\begin{aligned} K_X &= -D_1 - D_2 - D_3 - D_4 - D_5 - D_6 \\ &\sim (b-1)D_1 - 2D_3 - 2D_6. \end{aligned}$$

After choosing the basis $\{e_1^*, e_2^*, e_3^*\}$ for M and a basis for $\mathrm{Pic}(X)$ as in Lemma 7.37, we get the short exact sequence

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{i} \mathbb{Z}^6 \xrightarrow{\pi} \mathbb{Z}^3 \to 0.$$

We can represent the map π by the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & -b-1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Theorem 7.41. *A Markov Basis for A is given by*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ b & 1 & -1 & 1 & 0 & 0 \end{pmatrix}.$$

Proof. Let $I = \langle x_5 - x_6, x_1 - x_2x_5, x_1^b x_2 x_4 - x_3 \rangle$ and I_A is the toric ideal associated with matrix A. It is enough to show that $I{=}I_A$. Clearly, $I \subset I_A$. Note that

$$\begin{aligned} \frac{\mathbb{C}[x_1,x_2,x_3,x_4,x_5,x_6]}{\langle x_5 - x_6, x_1 - x_2x_5, x_1^b x_2 x_4 - x_3 \rangle} &\cong \frac{\mathbb{C}[x_1,x_2,x_3,x_4,x_5]}{\langle x_1 - x_2x_5, x_1^b x_2 x_4 - x_3 \rangle} \\ &\cong \frac{\mathbb{C}[x_2,x_3,x_4,x_5,x_6]}{\langle x_1^b x_2 x_4 - x_3 \rangle} \\ &\cong \mathbb{C}[x_2,x_4,x_5]. \end{aligned}$$

Hence, I is a prime ideal. Moreover $\dim(\mathbb{C}[x_1,...,x_5]/I) = 3 = \mathrm{rank}(A)$. Hence, $I = I_A$ by Lemma 5.17. ∎

Lemma 7.42. *Let $D = dD_1 + (e+f)D_3 + fD_6$, $E' = D_3 + D_6$ and $E = dD_1 + (e+f-1)D_3 + (f-1)D_6$ with $f \geq 1$. Then (D,E) has connected sections.*

Proof. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & \text{b+1} \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & -1 & \text{b} \\ 0 & 1 & 0 \end{pmatrix}, \quad t_1 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} -d \\ 0 \\ -e-f \\ 0 \\ 0 \\ -f \end{pmatrix}, \quad t_3 = \begin{pmatrix} -d \\ 0 \\ -e-f+1 \\ 0 \\ 0 \\ -f+1 \end{pmatrix}.$$

Then $P(E') = P(A, t_1)$, $P(D) = P(A, t_2)$ and $P(E) = P(A, t_3)$. Clearly the polytope $P(E')$ contains the points $(0,0,0), (0,-1,0), (b,0,1), (b+1,0,1) \in \mathbb{Z}^3$ and their images are $(0,0,0,0,0,0), (0,0,0,0,1,-1), (b,1,-1,1,0,0), (b+1,0,-1,1,-1,0) \in \mathbb{Z}^6$ respectively. Let $G = i(P(E') - i(P(E'))$. Then G contains the vectors

$$\begin{aligned} (0,0,0,0,0) - (0,0,0,0,1,-1) &= (0,0,0,0,-1,1) \\ (b,1,-1,1,0,0) - (0,0,0,0,0) &= (b,1,-1,1,0,0) \\ (b+1,1,-1,1,0,0) - (b,1,-1,1,0,0) &= (1,-1,0,0,-1,0). \end{aligned}$$

Thus by Lemma 7.41, G is a Markov basis for A. Note that $P(D) \cap \mathbb{Z}^3 = P(E) \cap \mathbb{Z}^3 + P(E') \cap \mathbb{Z}^3$ by Theorem 7.40. Hence by Proposition 5.18, configuration (D, E) has connected sections. □

Lemma 7.43. *If $1 \le f \le 3$ or $d = 1$, then S contains curves of genus 0 or 1 and hence cannot be algebraically hyperbolic. If $f \ge 4$ and $d \ge 2$, then every curve in the boundary has the genus at least two.*

Proof. We will analyze the facets.

1. **Facet 1:** Interior is given by the equations $x = -d$, $-z > -e-f$, $z > 0$, $-x-y+bz > 0$ and $y > -f$. There are $(f+e-1)(f+d-1) + \dfrac{b(e+f-1)(e+f)}{2}$ solutions if $f \ge 1$.
2. **Facet 2:** Interior is given by the equations $-x + (b+1)z = 0$, $z > 0$, $-x-y+bz > 0$, and $y > -f$. Thus, we have $\dfrac{(f-2)(f-1)}{2}$ solutions if $f \ge 1$. If $f = 0$, there is no curve in the intersection.
3. **Facet 3:** Interior is given by the equations $z = e+f$, $x > -d$, $-x-y+bz > 0$, and $y > -f$. Thus, we have at least $\dfrac{(f-2)(f-1)}{2}$ solutions if $f \ge 1$. If $f = 0$, there are $\dfrac{(be+d-2)(be+d-1)}{2}$ solutions.
4. **Facet 4:** Interior is given by the equations $z = 0$, $x > -d$, $-x + (b+1)z > 0$, $-x-y+bz > 0$ and $y > -f$. Thus we have $(d-1)f + \dfrac{(d-2)(d-1)}{2}$ solutions if

$d \geq 1$. If $d = 0, f \geq 1$, there are no interior lattice points. If $d, f = 0, e \geq 1$, there is no curve in the intersection.

5. **Facet 5:** Interior is given by the equations $-x - y + bz = 0$, $x > -d$, $-z > -e - f$, $z > 0$, $-x + (b+1)z > 0$ and $y > -f$. Thus we have at least, $\dfrac{(f-2)(f-1)}{2}$ solutions if $f \geq 1$.

6. **Facet 6:** Interior is given by the equations $y = -f$, $x > -d$, $-x - y + bz > 0$, $-x + (b+1)z > 0$, $-z > -e - f$ and $z > 0$. Thus we have at least, $\dfrac{(f-2)(f-1)}{2}$ solutions if $f \geq 1$.

Thus by Lemma 5.20, we have the results. □

Proof of Theorem 7.39. Let $D = dD_1 + (e+f)D_3 + fD_6$ and $E = (d-1)D_1 + (e+f-1)D_3 + fD_6$ with $d \geq 1, e \geq 1$ and $f \geq 0$. Then D is big, and the configuration (D, E) has connected sections by Theorem 7.42. Applying Theorem 5.19, for any curve C not contained in the toric boundary on a very general surface S in $|D|$ we have,

$$2g - 2 \geq C \cdot ((d+b-1)D_1 + (e+f-3)D_3 + (f-3)D_6). \tag{7.16}$$

Let $H = D_1 + 2D_3 + D_6$. If $d \geq 2, e \geq 1, f \geq 4$, then we have

$$2g - 2 \geq C \cdot H. \tag{7.17}$$

Applying (7.17) and Lemma 7.43 on Corollary 5.22 we have the results. □

Bibliography

[1] Victor V. Batyrev. On the classification of smooth projective toric varieties. *Tohoku Math. J. (2)*, 43(4):569–585, 1991.

[2] Damian Brotbek. On the hyperbolicity of general hypersurfaces. *Publ. Math. Inst. Hautes Études Sci.*, 126:1–34, 2017.

[3] Izzet Coskun and Eric Riedl. Algebraic hyperbolicity of the very general quintic surface in $\mathbb{P}^3$. *Adv. Math.*, 350:1314–1323, 2019.

[4] Izzet Coskun and Eric Riedl. Algebraic hyperbolicity of very general surfaces. *arXiv preprint arXiv:1912.07689*, 2019.

[5] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[6] Jean-Pierre Demailly. Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 285–360. Amer. Math. Soc., Providence, RI, 1997.

[7] David Eisenbud and Joe Harris. *3264 and all that—a second course in algebraic geometry*. Cambridge University Press, Cambridge, 2016.

[8] William Fulton. *Algebraic curves*. Advanced Book Classics. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989. An introduction to algebraic geometry, Notes written with the collaboration of Richard Weiss, Reprint of 1969 original.

[9] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[10] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[11] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Vieweg + Teubner, Wiesbaden, 2010. Schemes with examples and exercises.

[12] Phillip Griffiths and Joe Harris. On the noether-lefschetz theorem and some remarks on codimension-two cycles. *Mathematische Annalen*, 271(1):31–51, 1985.

[13] Christian Haase and Nathan Ilten. Algebraic hyperbolicity for surfaces in toric threefolds, 2019.

[14] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[15] Brendan Hassett. *Introduction to algebraic geometry*. Cambridge University Press, Cambridge, 2007.

[16] Atsushi Ikeda. Subvarieties of generic hypersurfaces in a nonsingular projective toric variety. *Math. Z.*, 263(4):923–937, 2009.

[17] Peter Kleinschmidt. A classification of toric varieties with few generators. *Aequationes mathematicae*, 35(2-3):254–266, 1988.

[18] Peter Kleinschmidt and Bernd Sturmfels. Smooth toric varieties with small Picard number are projective. *Topology*, 30(2):289–299, 1991.

[19] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[20] Tadao Oda. Problems on minkowski sums of convex lattice polytopes. *arXiv preprint arXiv:0812.1418*, 2008.

[21] G. V. Ravindra and V. Srinivas. The Noether-Lefschetz theorem for the divisor class group. *J. Algebra*, 322(9):3373–3391, 2009.

[22] Bernd Sturmfels. *Gröbner bases and convex polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1996.

[23] Geng Xu. Subvarieties of general hypersurfaces in projective space. *J. Differential Geom.*, 39(1):139–172, 1994.