Learning and Teaching Visualization Tasks in Secondary Mathematics Classrooms

by

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Abstract

In the new British Columbia's Mathematics curriculum, students are expected to "visualize to explore mathematical concepts." Although research shows that spatial skills can improve with targeted intervention, oftentimes these skills are considered to be difficult to transmit and to assess. In this study, students in Mathematics 8 and Mathematics 9/10 classes engaged in visualization tasks inspired by Caleb Gattegno's practice. They imagined a scenario described by the teacher, drew a sketch of the scenario, and discussed their solution with peers. Through drawing, the students developed strategies including showing hidden edges of three-dimensional shapes, and stepwise rotation of shapes and learning how to draw loci. Through discourse, they stated conjectures and arguments, applying their own mathematical agency. Students developed awareness of definitions, conventions and properties of shapes and the ability to consider multiple constraints of task and to generalize. Additionally, they developed awareness of their own and their peers' thinking.

Keywords: Visualization; visualization task; spatial reasoning; drawing; awareness; agency

In loving memory of my grandmothers, Anne Shaffer and Ina Herman.

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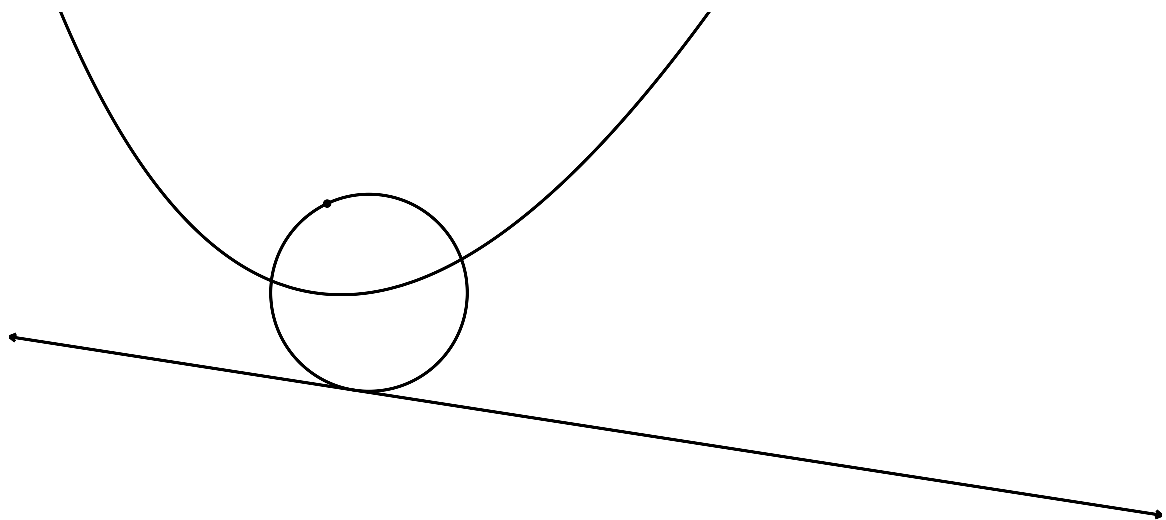
Glossary

Agency	The ability to define and redefine the world. Student awareness of their own knowledge authority with a sense of being authors of mathematical ideas.
Awareness	A shift in the structure of attention.
Awareness-in-action	The powers of construal or acting in the material world.
Awareness-in-discipline	The articulation and formalisation of awareness-in-action.
Imagery	What happens inside a person's mind when she is thinking, planning, considering and reflecting.
Imagine-draw-talk routine	Students imagine a described scenario with closed eyes, draw what they imagine and then discuss their drawing with a peer.
Spatial Reasoning	The ability to perform mental manipulations of visual stimuli, the ability to transform spatial forms into other visual arrangements, an awareness of the structural features of spatial forms and the analytical thinking required to find relationships and solve problems.
Visualization	The ability to form imagery related to a concept or task involving motion in two or three dimensions that can be subsequently drawn and described.
Visualization Task	Short tasks that invite students to imagine and then draw as well as describe a solution involving a two or three-dimensional image.



Chapter 1.

Introduction

In Spring 2018, as a student of the Masters of Secondary Mathematics program at Simon Fraser University, I partook in Nathalie Sinclair's Geometry course, Math 604. During the first class, she asked us to close our eyes and she invited us to visualize a scenario that she described in detail. Imagining Nathalie's intended description was difficult but captured my curiosity. I found myself independently and repeatedly considering and reconsidering the situation she had described even though it was challenging and I was unsure about the accuracy of my solution. Nathalie presented the visualization task in a way that allowed us to interact with mathematical ideas in a new way. Her task was a disturbance (Mason, 2002b) that ultimately led me to make a deliberate change to my teaching practice and to include visualization tasks.

I have taught mathematics to students in grades 6 through 12 for the past seven years in two different independent schools. I worked for one year at a private school in Montreal before moving to British Columbia to teach mathematics and science at a private school in the lower mainland where I currently teach and conducted my research.

In my experience teaching, I had noticed that my students often relied on rote procedures or applying formulas without connection to visual representations. For instance, I noticed that some students would add a series of seemingly randomly multiplied numbers when determining the surface area of a prism. They did not always account for the areas of all faces of the prism in question. I had also noticed that students were not always comfortable with visual representations of concepts such as integer tiles for integer addition and subtraction or algebra tiles for multiplying polynomials. They seemed to show a preference for memorizing algorithms rather than applying a meaning-making visual model. In all these cases, I intuitively felt that if students were to think in terms of visual models it would support their deeper understanding of mathematical concepts.

Many of the earlier visualization tasks in Nathalie's class were very difficult for me and I did not solve them independently. However, by the end of the course I felt more comfortable interpreting the verbal instructions and remembering these as I generated my own image that I was able to sketch. I was able to solve some of the later tasks partially or completely and the experience was personally gratifying.

I found engaging with visualization tasks to be a unique experience since my usual experience with visual imagery in mathematics involved observing an image rather than generating an image from a verbal description. The translation from language to a visual offered some mystery as to what image would ultimately be constructed. I therefore appreciated developing my own spatial reasoning through these tasks. Based on my own positive experience with visualization tasks, I wondered how or where I might implement these tasks with my own students.

However, I did have some reservations about implementation: With limited curriculum content devoted to geometry, I questioned whether I should set aside the time needed to allow my students to engage with visualization tasks. I also wondered how it might be possible to assess student visualization and whether I should be assessing their visualization capacity at all. Finally, I worried that my students might not find the tasks as engaging as I had or necessarily see their utility and value in the context of the mathematics curricula they were learning.

I decided nevertheless to implement visualization tasks in with my Mathematics 8 and Mathematics 9/10 classes and to examine the impact of these tasks on my students and their learning.

Chapter 2.

Literature Review

2.1. Visualization Task Origins: Gattegno's Practice

The visualization tasks I referred to in Chapter one were inspired by work done by mathematics researcher Caleb Gattegno (1987), renowned for his claim that "only awareness is educable" (p. 110). Gattegno (1965) describes in his work a practice he used to allow students to become aware of imagery:

> In a number of experiments I have asked my classes to consider with their eyes shut some situation in their minds which I generate by instructing them to produce some images and act mentally upon them. By doing so it became possible to perceive in the mental situation a number of distinctive features which when talked about sounded strangely like the geometrical statements we read in books as theorems or problems. (p. 22)

To Gattegno, this was a powerful teaching method because students "generate chapters rather than memorize them and […] use as stuff merely imagery, something everybody owns" (p. 23). Giving students the opportunity to make use of their own imagining powers allows these to flourish rather than diminish (lest their own imagining not be engaged).

Mason (2002a) refers to mental imagery as what happens inside a person's mind when she is thinking, planning, considering and reflecting and posits that this process can be visual. He suggests that the power of mental imagery or imagining is "the most fundamental of all the many powers which children possess" (p. 1). Imagery allows students to look "through the particular to the general" (p. 3). Mason (2002a) emphasises that "the core of mathematics and of mathematical thinking is the expression of generality" (p. 12). During a discussion about mental imagery, a student's attention can be shifted by a remark made by another student or by the teacher so that she becomes "more explicitly aware of some features and less aware of others" (Mason, 1998, p. 254). When certain details are stressed and others are ignored, abstraction and generalization can take place. Therefore, Gattegno's tasks not only support student access of their own mental imagery, but the development of their ability to generalize and to think abstractly.

2.2. Visualization and Mathematics

Visualization is an area of interest in a range of fields including mathematics education and has been researched for several decades (Clements, 2014). Presmeg (2006) describes visualization as “the processes of construction and transforming both visual mental imagery and the inscriptions of a spatial nature” (p. 206). For Arcavi, it is “a method of seeing the unseen” (2003, p. 216). Clements et al. (2004) hold that in visualization, the image is dynamic, “not a ‘picture in the head’ [but rather, something] more abstract [and] more malleable than a picture” (p. 284).

Tahta (1989) outlines three powers or capabilities students employ when working with space that can be applied to visualizing: *imagining*, seeing what is said, *construing*, seeing what is drawn or saying what is seen and *figuring*, drawing what is seen. Although it is impossible to access students’ imagining or seeing directly, they can share their construing as verbal descriptions and their figuring as drawing. I propose my own working definition of visualization as the ability to form mental imagery related to a concept or task involving motion in two or three dimensions that can be subsequently drawn and described.

As Wheatley (1990) remarks, “conceiving of a diagram and then drawing it requires imagery” (p. 11). Practice drawing their own imagery allows students to develop their spatial sense (Sinclair et al., 2019). For this reason, my definition includes drawing as an observable phenomenon. Mason (2002a) suggests that describing is a useful way for students to express what they discern and to express their ideas clearly and concisely. He labels this process “say what you see.” Because language can provide information and reasoning additional to a diagrammatic representation, I also include description as part of my definition of visualization.

In the new British Columbia competency-driven mathematics curriculum students are expected to “visualize to explore mathematical concepts” in every grade from kindergarten through grade 12 (British Columbia Ministry of Education, 2016). The Ontario Ministry of Education as well as the National Research Council support focus on spatial reasoning in schools in recent publications (National Research Council, 2006; Ontario Ministry of Education, 2014).

A variety of areas of the brain are involved in mathematics learning including specific regions involved in visual processing (Boaler et al., 2016). Neural networks in the parietal cortex play a role in both numerical and visual-spatial tasks (Hubbard et al., 2009). Boaler advocates that visual learning of mathematics is imperative for all students and not necessarily only those who tend to visualize or prefer to do so.

Various studies have linked spatial skills with representation of number and arithmetic skills (Booth & Siegler, 2008; Gunderson et al., 2012; Siegler & Booth, 2004). Gunderson, Ramirez, Beilock and Levine (2012) hypothesise that spatial representation of numbers (on the number line) supports student development of number sense based on their finding that 5-year-olds' spatial skills, mediated by their number-line knowledge at age 6 predicted their performance on a symbolic calculation task at age 8. Other studies have also drawn connections between spatial and patterning skills (Mulligan, 2015; Rittle-Johnson et al., 2018).

Recent research findings emphasize pedagogical advantages of visualization. Spatial visualization skills have been linked to mathematics performance (Mix & Cheng, 2012). Verdine, Irwin, Golinkoff and Hirsh-Pasek (2014) found a relationship between spatial and mathematics skills in preschool aged children. Casey, Lombardi, Pollock, Fineman and Pezaris (2017) found that first-grade spatial skills on spatial visualization and mental rotation tasks predicted complex math reasoning skills in fifth grade. Later, in high school, spatial skills predict successful education and vocational outcomes in STEM fields (Wai et al., 2009). Mix and Cheng (2012) have asserted that "the relation between spatial skills and math is so well established that it no longer makes sense to ask whether they are related. Rather, we need to know why the two are connected – the causal mechanisms and shared processes…" (p. 206).

Wheatley (1991) emphasizes the importance of students developing their own imagery through visualization. He highlights that, "students can memorize facts and become proficient in demonstrating procedures, but they may not be giving meaning to their mathematical activity" (p. 35) as they do when they form mental imagery. He cites an example where a student with low spatial ability was challenged to form a rectangle using all 36 Unifix cubes, only finding a 7 x 5 rectangle with one cube left over rather than working to find rectangles using all 36 cubes. In contrast, students with high spatial ability were often able to solve similar non-routine problems creatively. Supporting student facility with visualization and development of imagery can support their meaning-making in various mathematical situations. Wheatley has developed pedagogical practices to allow for student development of imagery through Quick Draw exercises which will be described in more details in 2.5.

Visualization supports meaning-making in problem solving according to Piggott and Pumfrey (2008) as well. They present three reasons why students need to visualize when engaging in mathematical problem solving, generally: to step into a problem, to model and to plan ahead. Students apply discrete skills during visualization including: internalisation of a situation including an image's salient features, identification of a useful representation of a problem, comparison of two representations of the same problem, connection of processes or underpinning structure to build visualization and description of one's own visualization as a means of communication with others.

2.3. Visualization Challenges

Much early research in visualization examined student preferences towards either visualizing or reasoning in alternative ways. Suwarsono (1982) developed the Mathematical Processing Instrument (MPI) to determine learner preference for verbal-analytic approaches versus visual responses to solving mathematics tasks. Comparison of verbalizers to visualizers led various researchers who implemented this tool throughout the 1980s to find that non-visualizers achieve better results in mathematics and even in spatial tasks compared to visualizers (Lean & Clements, 1981; Presmeg, 1986; Suwarsono, 1982).

Presmeg's work (1986) sought to determine advantages and disadvantages of visualization in supporting mathematical achievement. She noted that visualizers especially benefitted from dynamic imagery and visuals serving abstract functions as well as their possible mnemonic effect. However, inflexible thinking when an uncontrollable image arose, lack of coupling of visual with analytical thought and introduction of irrelevant detail in one-case concrete diagrams were disadvantageous for visualizers. More recent work by Aspinwall and Shaw (2002) further points to the generation of uncontrollable images, false images that persist for students, during visualization. Emphasis on the importance of analytic thought in concert with visualization has also appeared in the literature (Presmeg & Balderas-Cañas, 2001;

Zazkis et al., 1996).Presmeg found that the most effective teachers for visualizer students applied both visual and non-visual methods. Integration of different methods might help guide students toward more abstract thinking, circumventing the disadvantage of student fixation on concrete or object images.

More recently, researchers have distinguished between two groups of visualizers who perform differently on spatial imagery tasks and in mathematical problem solving: spatial visualizers and object visualizers (Haciomeroglu & Lavenia, 2017; Hegarty & Kozhevnikov, 1999; Kozhevnikov et al., 2005). Spatial visualizers “generate and process images analytically, part by part [whereas object visualizers] encode and process objects as a single perceptual unit” (Kozhevnikov et al., 2005, p. 723). While spatial visualizers perhaps apply Presmeg’s dynamic and abstract visuals, object visualizers might tend to generate uncontrollable images with irrelevant detail and lacking the relevant analytic thought needed for solving mathematical and spatial tasks.

There is evidence that many students and teachers tend to favour algorithmic thinking over visual thinking. Eisenberg and Dreyfus (1991) found that a majority of calculus students incorrectly responded to tasks easily addressed using visual solutions. They suggest one reason for avoidance of visual thinking by students is its inherent higher cognitive demands. Because information in diagrams is explicit – more cognitive processing is required to make sense of visual representation compared to analytic representation such as symbolic notation where information is implicit and does not necessarily need to be unpacked. Helping students overcome this inherent challenge of visualization may ultimately benefit students and allow them to access many advantages of visualization previously described.

2.4. Spatial Reasoning

According to the Cambridge dictionary, the adjective ‘spatial’ refers to position, area or the size of objects. Spatial imagery, “representation of spatial relationships between parts of an object and the location of objects in space and their movement” differs from visual imagery, “the shape, colour or brightness” of an object (Hegarty & Kozhevnikov, 1999, p. 685). This distinction is mirrored at the neuropsychological level: the spatial relations and object pathways are functionally and anatomically separate; the former runs from the occipital lobe up to the posterior parietal lobe and the latter runs from the occipital lobe down to the inferior temporal lobe (Kozhevnikov et al., 2005).

Spatial reasoning, according to the National Research Council (2006), involves the location and movement of objects and ourselves, either mentally or physically, in space. It is not a single ability or process but refers to a considerable number of concepts, tools and processes. Davis and his Spatial Reasoning Study Group (2015) characterize the complexity of spatial reasoning as: locating, orienting, decomposing/recomposing, shifting dimensions, balancing, diagramming, symmetrizing, navigating, transforming, comparing, scaling, sensing and visualizing. However, he emphasizes that spatial reasoning cannot be parsed into discrete sub-skills. Arcavi (2003) offers an integrated definition of spatial reasoning as “the ability to perform mental manipulations of visual stimuli, the ability to transform spatial forms into other visual arrangements, an awareness of the structural features of spatial forms and the analytical thinking required to find relationships and solve problems” (p. 217).

Spatial reasoning involves understanding relationships within and between spatial structures as well as representation and communication about them (Ontario Ministry of Education, 2014). In the context of school mathematics, spatial reasoning may be considered challenging to transmit or to assess because “it is not easily reducible to alphanumeric modes of communication” (Davis et al., 2015, p. 12). This may be the reason this competency is not explicitly included in many North American curricula (National Research Council, 2006); British Columbia is no exception (British Columbia Ministry of Education, 2016). Consequentially, the role of spatial reasoning receives limited attention in classroom practice (Mulligan et al., 2018).

Spatial skills are not as fixed as once previously believed (Newcombe, 2010). Uttal et al.’s (2013) meta-analysis of 206 spatial training studies conducted over a 25-year period to conclude that spatial skills are subject to improvement with practice and targeted interventions (Davis et al., 2015). Strategies to improve spatial reasoning for young children listed by the authors include construction play, puzzle play, drawing tasks and paper folding. Although it is clear that student spatial reasoning has the potential to grow, Newcombe (2010) suggests that “we don’t know which [strategy] is most effective” to tap into this potential (p.32).

Spatial reasoning is beginning to receive more attention in curricula and consequently in classrooms. Tasks and interventions developed by the Spatial Reasoning Study Group include involve symmetry, congruence and mental rotation for students in kindergarten through grade two (Bruce et al., 2015). Through touchscreen and drawing tasks, students were able to develop an understanding of transformational movement involved in symmetry as well as the notion of equidistance in symmetry. In an activity to determine twelve unique Pentominoes, students used visualization to assess their congruence through mental transformations such as rotation. They were also able to mentally imagine whether Pentominoes folded into "boxes without tops." Engaging students in mental rotation tasks such as rebuilding structures made of linking cubes upside-down allowed students to develop their ability to generate, retain and manipulate objects and symbols or "mathematical imagination."

As noted previously (2.2), Tahta (1989) presents *imagining*, seeing what is said, *construing*, seeing what is drawn or saying what is seen and *figuring*, drawing what is seen as powers used when working with space. Visualization tasks offer students the opportunity to apply each of these powers and therefore perhaps to develop their spatial reasoning.

2.5. Drawing and Diagrams

The use of diagrams in mathematics extends historically at least to the ancient Greeks whose two-dimensional diagrams were drawn on papyrus (Cellucci, 2019). Most notably, Euclid's *Elements* presents theorems about two and three-dimensional geometry with diagrams. Greek diagrams played a fundamental role in mathematics, sometimes serving as 'proofs without words' (Van Maanen, 2006). For some time in the eighteenth century, diagrams fell out of favour with mathematicians such as Leibniz, Kant and other mathematicians of the era whose philosophy dictated that proof was necessarily independent of diagram (Cellucci, 2019). Most recently in the twentieth century, a renewed acceptance and interest in diagrams returned to mathematics. Diagrams are now ubiquitous in mathematics texts at all levels. They are highlighted as a strategy in the mathematical problem-solving process as defined in the second step of Pólya's (1957) *How to Solve it*: "draw a diagram."

Drawing skills have been correlated with mathematic performance in recent years by cognitive psychologists (Carlson et al., 2013; Grissmer et al., 2010). The ability of students at age 4½ to draw a human figure has been correlated to the mathematical ability at age 12 (Malanchini et al., 2016). Spatial awareness is required in drawing a human figure since this task requires awareness of proportionality, use of space, symmetry and number of body parts. Drawing requires awareness of spatial relations within and between objects (Sinclair et al., 2019). Students whose drawings show visuo-spatial relations of problems solve problems more successfully than those whose drawings emphasize pictorial details (Hegarty & Kozhevnikov, 1999). Drawing therefore emphasizes abstraction analogously to Mason's imagery (2002a). Some pictorial details can be excluded so that salient spatial features can be emphasized.

In mathematics education research, much research on student drawing has focused on student generation of diagrams for the purpose of problem solving (Bremigan, 2005; Nunokawa, 2006). Diagrams allow students to better understand and find patterns in problems and instruction in diagramming allows students to improve in drawing diagrams and develop conceptual imagery needed for problem-solving (Yancey et al., 1989).

Besides use as an effective problem-solving strategy, drawing of diagrams also has a role to play in the development of spatial reasoning in the context of mathematics education. Diagrams are inherently 2D in nature and are often used to represent 3D objects. Translating between 2D and 3D is challenging for students and requires exposure to learned conventions (Francis & Whiteley, 2015). Furthermore, the same 2D diagram can represent multiple different 3D objects. This creates ambiguity in a classroom: "what the teacher "sees" (interprets) may not be what many of the students interpret, and what one student "sees" is different from what another student "sees" (Francis & Whiteley, 2015, p. 124).

Diagrams can also allow aspects of motion in a mathematical situation to be recorded and captured on a static diagram. Even in an era where ways to capture motion through media such as videotape are prevalent, drawing remains the most accessible way for students to record their ideas and actions. McGarvey et al. (2015) explored how students show motion through drawing to represent a student taking part in a relay race and also the dynamic motion of a triangle in *Sketchpad,* a mathematical visualization software. Movement in the relay race is represented in different ways including multiple images, arrows, and different distances between repeated images. Students who drew dynamic movement of triangles tended to either represent multiple different shapes or continuous movement from one shape to another, often with arrows. In both cases, students represented a range of triangles beyond the prototypical triangle showing development of their spatial reasoning. Drawing supported students in making meaning of the definition of a triangle

Wheatley and Reynolds (1999) developed “Quick Draw” or “Image Maker” exercises to specifically encourage student formation and transformation of mental images. In their practice, a teacher shows a geometric line drawing for three seconds on an overhead projector to students. The students then draw their constructed mental image from memory. They share what they saw and describe how they drew their sketch. Drawing is an integral part of Wheatley’s practice as it is the first time a student transfers and communicates their mental image. Through this type of activity, students develop geometric concepts and vocabulary, take part in imaging, make sense of their activity, and develop self-confidence as well as a more positive attitude toward mathematics. Student drawings as well as discussion provide teachers with information about ways of student knowing. Sinclair et al. (2019) have found “the very act of drawing […] changes the way students see (spatially reason about) geometric images” in such Quick Draw exercises (p. 250). Through tasks involving Quick Draw exercises, students in their study attended to properties of congruent shapes, symmetry and structures such as grids.

2.6. Awareness

For Gattegno (1987), awareness allows the employment of an individual’s own powers. In his visualization tasks, he described a strong sense of awareness developed by participants:

> […] there was an element of conviction accompanying the statement that could only come from the inner sense of truth of the people involved. In this way they actually believed they were making a statement that was universally valid, the more so because everyone got the same insight into the situation. (1965, p. 22)

Mason (1998) describes the development of a new awareness for an individual as a "shift in (her own) structure of attention" (p. 252) He distinguishes between *awareness-in-action* and *awareness-in-discipline. Awareness-in-action* is defined as "the powers of construal or acting in the material world." These are employed subconsciously to enable previously learned skills such as walking, talking, reading, counting, and many others. Imagining is considered an *awareness-in-action*. Drawing and describing what is imagined are actions students can take in the material world because they possess the power of imagining.

Awareness-in-discipline "enables articulation and formalisation of awareness-in-action" (p. 256). Mason draws a connection of *awareness-in-discipline* to Gattegno's awareness of awareness that allows disciplines such as mathematics or the sciences to arise. Becoming aware of the powers of mental imagery allows the discipline of geometry to arise, for instance. Mason indicates that it is *awareness-in-discipline* that enables a shift in the structure of attention of students through formalization of generalizations, relationships, assertions, and justifications. Reasoning might be considered an *awareness-in-discipline*. Describing why a drawing was drawn in a certain way might be considered *awareness-in-discipline* as well.

Mason cautions that routine exercises strengthen *awarenesses-in-action* to the detriment of *awarenesses-in-discipline* because student attention is focused on the doing rather than on the construing taking place. Instead, "tasks which provoke students into rehearsing or exercising skills, but which at the same time attract their attention away from the skill to be automated" (p. 259) can support development of this higher level of awareness. Students engaged in textbook practice questions might only develop an *awareness-in-action*. Those who have the opportunity to engage in visualization tasks, through which they have the opportunity to look through the particular to the general (Mason, 2002a), might be more likely to develop their *awareness-in-discipline.*

Similar to Mason, Wheeler (2001) describes student development of a new awareness as a moment of perceptual difference that occurs with "a new stage in the mental structure of a situation" (p. 51). Wheeler proposes potential mediators of awareness including imagery (as well as language, notation and graphical representation). He notes that few people have studied mediation of imagery between the mathematizer and the situation s/he is mathematising and how imagery can change awareness.

2.7. Agency

Agency, the ability to define and redefine the world (Freire, 1994), is a critical factor in student learning. Mathematical agency gives “students awareness of their knowledge authority [and] supports the development of a sense of being authors of mathematical ideas” (Martinez & Ramirez, 2018). Students who express mathematical agency can autonomously engage in mathematical tasks through their own thinking or through cooperation with peers. They rely on their own thinking to draw conclusions. The locus of authority shifts from the teacher to the students.

Imagery is something that everyone owns (Gattegno, 1965). Jencks and Peck (1972) hold that mental imagery “provides [students] with a basis for trusting their own reasoning, and it brings them a giant step closer to being independent” (p. 644) and able to express their own mathematical agency. Students who receive experiences that allow them to build necessary conceptual images learn in “deeper and more penetrating ways” compared with students who do not and whose learning is instead “rote and superficial” (p. 644). Because mental imagery requires independent thought, experiences that encourage visualization may naturally promote student agency.

Secondary students show a preference for mathematics learning environments where they can “create[e] initial thought and ideas or [take] established ideas and extend them” (Boaler, 2002, p. 45). Student discourse is a further means of promoting their own agency (Bell & Pape, 2012). Dialogic interactions between students allows for disagreement and negotiation of meaning to take place between students and for students to think actively while discussing new ideas with their peers.

Dewey (1944) promoted learning through “opportunity [for students] to employ [their] own powers in activities that have meaning” (p. 203). Mason further asserted the necessity for students to use their own powers, lest these diminish if not applied (Mason, 2002a). Employing Tahta’s (1989) powers associated with spatiality: *imagining* (seeing what is said), *construing* (seeing what is drawn or saying what is seen) and *figuring* (drawing what is seen) through visualization tasks may support student expression of their own mathematical agency. This may occur through application of their own mental imagery (imagining power) as well as in collaborative discourse with their peers.

2.8. Research Questions

I decided to implement short visualization tasks that invite students to imagine, draw and describe a solution involving a two- or three-dimensional image. I believed that my intentional implementation of this practice might help my students become more comfortable with visual imagery and applying this to mathematical situations.

My research focused on how students engaged with visualization tasks through drawing (Tahta's figuring) and describing (Tahta's construing). I sought to examine how implementation of visualization tasks might influence student awareness of geometric properties relevant to the tasks. Finally, I wanted to learn about students' perspectives about engaging with the tasks and with peer collaboration that took place.

My research questions include:

1. What types of drawing strategies do individual students develop when they engage in visualization tasks?
2. How does discourse support student awareness of geometric properties when they engage in visualization tasks?
3. What geometric properties do students develop awareness of through their (a) drawing activity and (b) discourse over time when they engage in visualization tasks?
4. What mathematical skills or competencies do students develop through their (a) drawing activity and (b) discourse over time when they engage in visualization tasks?
5. How do students describe their individual experience of engaging with visualization tasks and with peer collaboration?

Chapter 3.

Methods

3.1. Setting/Participants

I wanted to explore implementation of visualization tasks as a means of altering my own practice as well as engaging in educational research. Therefore, I selected mathematics classes in my own teaching assignment at a growing private school in the Lower Mainland. The school was started by several parents of students and began with a mission centred around the arts. Recently, the school has expanded this focus to integrate science and technology alongside this initial focus.

The school size during the year the study was conducted, 2018-2019, was approximately 200 students, with the majority in the elementary years. Secondary classes in my care were smaller in size and included Mathematics 8 with eight students and Mathematics 9/10 with nine students. I decided to complete the research study with both these groups since I taught these groups linear courses that would span September through June and had the opportunity to work with these students throughout the year.

The 2018-2019 school year was marked by a school move from a facility housing grades four through twelve into a brand new, purpose-built school, where these grades joined grades JK through three moving in from a separate facility in the spring. Preparation and packing for the move, followed by on-going construction in the new building shaped the year and frequently impacted teaching and work with students. Time was dedicated to packing and preparing to move from September through March when the move took place. Construction on the new building continued after we had moved in and proceeded right until the end of the school year.

3.2. Data Collection

Data collection for the study involved both visualization tasks conducted in the classroom as well as follow-up tasks conducted at the end of the study with individual students. Individuals were also interviewed about their experience in learning mathematics and with visualization tasks.

3.2.1. Classroom Tasks

The visualization tasks implemented invited my students to imagine and then draw as well as describe a solution involving a two or three-dimensional image. Students first imagined a scenario I described orally with closed eyes. I repeated the description at least twice. The students then drew a response to the task individually. They next compared their drawing to that of one or several peers and had an opportunity to draw a further refined solution after some discussion with the partner or partners. Finally, we debriefed the task together through a group discussion. Through this *imagine-draw-talk* routine, the students engaged with twelve classroom visualization tasks between November 2018 and June 2019.

The chosen routine was influenced by Gattegno's visualization tasks and Mason's tasks involving imagery. The aural, linguistic delivery of a description ofa scenario for students to construct was the same as the work done by both math educators. However, Gattegno's (1965) and Mason's (2002a) work relied on solely verbal communication from participants to relay their imagery back to the researchers.

For the in class visualization tasks I conducted, the *draw* component of the routine was the first time students transferred and communicated their mental imagery constructed based on an aural scenario. They generated a diagram as a means of communicating their visualization. This was supported verbally during their discussion with peers and in a class debrief only after they had in engaged in drawing their own initial diagram.

In visualization, the image is dynamic, "not a 'picture in the head' [but rather, something] more abstract [and] more malleable than a picture" (Clements et al., 2004, p. 284). Although a mentally constructed image is flexible, drawing commits a fixed representation to paper. Multiple drawings can allow the represented imagery to develop and change with time. Diagramming can allow students to extend and develop new ideas (de Freitas & Sinclair, 2012).

An example of the script for Task 9, the *hanging cube task* follows. The scripts for all tasks are included in Appendix A. Changes made that were improvised when reading this task to the Mathematics 9/10 class are shown in parentheses. These changes arose naturally as I made sense of the tasks for students in my teacher role.

Pre-task: You can close your eyes. Close your eyes. You will have time to silently visualize with your eyes closed. And I will repeat the instructions twice.

Task: Imagine a cube that's hanging from a string from one of its vertices (or uh, one of the corners of the cube). Halfway down between the top and the bottom vertices of the dangling cube, I slice the cube in half, horizontally. What is the shape that is cut?

Post-Task: Now, on the top of your handout, draw what you imagined, individually.

Now, discuss with a peer what you imagined.

As students discussed with peers, I circulated and sometimes asked questions to gain a better understanding of student thinking or to facilitate discussion between students. When it seemed like most students had had a chance to share their own thinking with a partner, I prompted a group discussion allowing different students to share their solutions with the class. Sometimes, I requested that students share a verbal description with their peers. Often, I allowed students to draw on the white board as a way of showing their thinking.

Tasks 1 through 7 were implement in class as pilot tasks prior to the study. I videotaped tasks 8 through 12 and these were part of the study. I recorded my own notes and observations before, during and after task implementation. I planned to attend to the correctness of student drawings as well as the types of awareness that emerged in their drawings and verbally in discourse for individuals and for the group during each classroom task. I also aimed to observe the affective disposition of students and their level of engagement during the lesson.

Following each task, I collected the written work of each student participant including their individual drawing, their adjusted drawing after discussion with a peer and their final drawing after the class debrief. I also photographed drawings made by students on the whiteboard during task debriefs. Sometimes, I did not have an opportunity to take notes during or after a given task. When I re-watched the video, I recorded further notes and observations that I had not noticed while conducting the lesson, paying attention to the same aspects listed above. I also took the time to examine individual student drawings more closely at this time.

3.2.2. Interview Tasks and Questions

I chose to interview individual students to focus on their individual experiences with visualization tasks in a format in which my attention felt less divided than during implementation of tasks in my classes. I was also interested in observing how drawing strategies and awareness of geometric properties developed for individuals over time. Interviews were conducted after task 12 was implemented (with Robin, Cameron, Devon, Jordan, Leslie, Jess and Alex). Each interview included two novel visualization tasks. Tasks 13 and 14 (Appendix B) were selected as analogous tasks to tasks 9 and 10 to observe student changes following engagement in visualization tasks in the group setting. Interview tasks were videotaped and student written work in response to each task was collected. Students selected for interviews had either participated significantly during specific task debriefs or had produced drawings with a range of accuracies, either highly accurate or very different from expected solutions. The students also represented a range of mathematical abilities in each grade-level group. These students were also selected based on their availability and willingness to take part in an interview during a lunch hour.

To learn more about individual student experiences with learning mathematics and with visualization tasks, I conducted interviews with selected students. I asked each participant five questions in the same order.

1. What do you enjoy about learning mathematics?
2. What parts, if any, do you not enjoy about learning mathematics?
3. What is your opinion about the visualization tasks so far?
4. Which task is one you found to be the most challenging so far? Why?
5. Did collaborating with peers results in you making changes to your drawings?

Interviews were conducted after task 12 was implemented (with Robin, Cameron, Devon, Jordan, Leslie, Jess and Alex). Each student interview began with these questions before the students were engaged with the visualization tasks. This was to create an informal atmosphere and give the students a chance to become comfortable with being interviewed before engaging in the tasks. The same questions were included in each interview. The interviews were videotaped to observe student drawing taking place over time. During the interviews, I responded to student answers, often asking follow-up questions relevant to their responses.

3.3. Data Analysis

Implementation of visualization tasks was a change to my teaching practice. I worked reflexively, journaling about tasks before, during and after implementation. In several earlier tasks, I explored asking students to describe their solutions to tasks without gestures during the debrief. Working with two groups of students afforded me the opportunity to learn from implementation with the first group that experienced a given task and to make adjustments with the second. For most tasks, I implemented first with the Mathematics 9/10 group and then with the Mathematics 8 group. Adjustments included altering the task script as well as choices made for directing the debrief conversation. Sometimes, adjustments were improvised as I attempted to clarify and emphasize task parameters for students.

I selected two tasks that were implemented later in the study to analyse: Task 9 (*hanging cube task*) and Task 10 (*parabola locus task*). These tasks were selected because they resulted in class debriefs where various students shared their thinking and a number of unique individual student drawings were generated. A two-dimensional locus task and a three-dimensional task were included in the analysis. These tasks took place later on in the study when the students had developed some familiarity with the *imaging-draw-talk* routine. I had also developed teaching experience with responding to students during debriefs at this point in the study.

Class debrief conversations of implementation of tasks 9 and 10 with both Mathematics 8 and Mathematics 9/10 classes were transcribed. I selected instances when student awareness about geometric properties changed during the discussion.

Student drawings, both on the whiteboard and on individual student papers, were examined. Analogous tasks drawings were compared to find how they may have changed over time within a lesson and between a lesson and the final interview tasks. Instances when student conjectures, justifications or drawings may have been altered following engagement in discussion with a peer and/or the class debrief were noted.

Follow-up interviews with Robin, Cameron, Skyler, Jordan, Leslie, Jess and Alex were transcribed. The responses of each interviewee were summarized to overview mathematics they do and do not enjoy, their opinion about visualization tasks and how these tasks allowed them to collaborate with peers. I focused on student experiences with the visualization tasks and peer collaboration and looked for both unique experiences as well as recurring trends related to each of these.

Student interview drawings for task 14 (*tetrahedron*) and task 13 (*ellipse locus*) were compared with their individual drawings for tasks 9 (*hanging cube*) and 10 (*parabola locus*), respectively. The *tetrahedron task* was compared with the *hanging cube task* because both involved finding the cross-sectional shape of a three-dimensional shape. The *ellipse locus task* was compared with the *parabola locus task* because both tasks involved motion and a locus. Changes in how students responded to the tasks and features they were able to include in their drawings both within an individual task and between a classroom task and the follow-up interview task were examined

Each drawing could be categorized as an expected or unexpected solution in response to the task script. Unexpected solutions, in particular, provided insight into student awareness about geometric properties and constraints presented in a given task. Student drawings were analysed for evidence of awareness of hidden edges in three-dimensional shapes, rotation strategies, understanding of locus, awareness of multiple task constraints and evidence of generalization.

Transcripts of student discourse during task debriefs and interviews were examined for further information about student awareness of geometric properties, in particular, student conjecture and argument during classroom debrief. Student responses to the follow-up interview question “Which task is one you found to be the most challenging so far? Why?” were examined. If these included either tasks 9 or 10, this information was cross-referenced with student drawings and discourse to find what types of tasks students perceived to be challenging and how this compared with their own work on such tasks.

Finally, student responses to interview questions were revisited with the aim of better understanding their own sense of awareness of the tasks, their own thinking and their peers' thinking. Evidence of student mathematical agency during the tasks or through their description of their experience with the tasks was also explored.

Chapter 4.

Results and Preliminary Analysis

Results data included student drawings produced during classroom tasks and during individual interviews as well as transcripts of classroom tasks and interviews showing discourse and student verbal descriptions of their work. Comparisons of students' drawings were structured in two ways: comparison between students for each task and comparison between tasks for individual students. In particular, the three-dimensional tasks were compared with one another as were the two-dimensional tasks. Transcripts were examined to gain further insight into student ideas and understanding apparent in the drawings that took place over time.

This study involved a small sample of students whose experiences with visualization tasks were individual and unique in many ways. Chapter 4 will present data organized by student to provide an overview of each student's individual experience and to present comparisons between tasks for individual students and to allow areas of development in their drawing and spatial reasoning as well as their individual experiences to be emphasized. Chapters 5 will present additional discussion based on comparisons made between student drawings for specific tasks. Findings taken from classroom discourse will also be highlighted. Chapter 6 will synthesize themes that arose based on student experiences.

4.1. Student Experiences with Visualization Tasks

Seven students participated in individual interviews that followed their classroom visualization task experiences. In this chapter, I present these seven students' drawings and experiences in the *hanging cube* and *parabola locus* classroom tasks as well as in the *tetrahedron* and *ellipse locus* interview tasks. Student perspectives about visualization tasks as well as working with their peers are also included. These were gained based on their responses to interview questions about student experience with visualization tasks (interview questions 3 and 4) and about collaboration with peers during the tasks (interview question 5). The students whose drawings and spatial reasoning developed especially evidently are presented earlier in the chapter.

Data for each student will be shared in the following order: first, the student will be briefly introduced. The tasks are ordered to allow for comparison between three- and two-dimensional tasks and their drawings: the *hanging cube* task, the *tetrahedron* task, the *parabola locus* task and the *ellipse locus* tasks. Finally, an overview of each student's experience with visualization tasks will be shared.

4.1.1. Alex (Grade 9/10 Class)

Alex is a student who joined the school this year for her grade 9 year. I have taught her only for one year, in Foundations of Mathematics and Pre-Calculus 10. Alex is an exceptionally hard-working student. She is willing to ask questions when she does not understand a concept and is able to make progress and has been very successful in the course. Alex sometimes lacks specific vocabulary and language to explain her ideas. She also takes time to process new vocabulary and terms. However, she is very willing to learn, and her comprehension of mathematical terms has grown a lot this year. Alex is very supportive of her peers and is always willing to collaborate and help her fellow students during group problem-solving.

Alex was willing to partake in several individual interviews related to visualization tasks as the research progressed. Therefore, she engaged with more tasks of this nature than any other student in the study. She was always quite involved in the tasks and eager to share her ideas in class with her peers and in group discussion.

Of all the students in the study, Alex demonstrated the most evident growth in her drawings in response to visualization tasks. Her determination of both the cross-section of a three-dimensional shape and a locus clearly improved over the course of the study.

Alex's Hanging Cube

Alex's drawing of the *hanging cube task* includes her own exploration including multiple drawings including an octahedron, a cube hanging shown from different angles, and a cube with a rectangular cross-section shown with hidden edges (Figure 4-1). The progression from octahedron (Figure 4-1a left) to cube (Figure 4-1a centre) shows that Alex became aware that a cube's vertices have three edges that meet rather than four through her drawing.

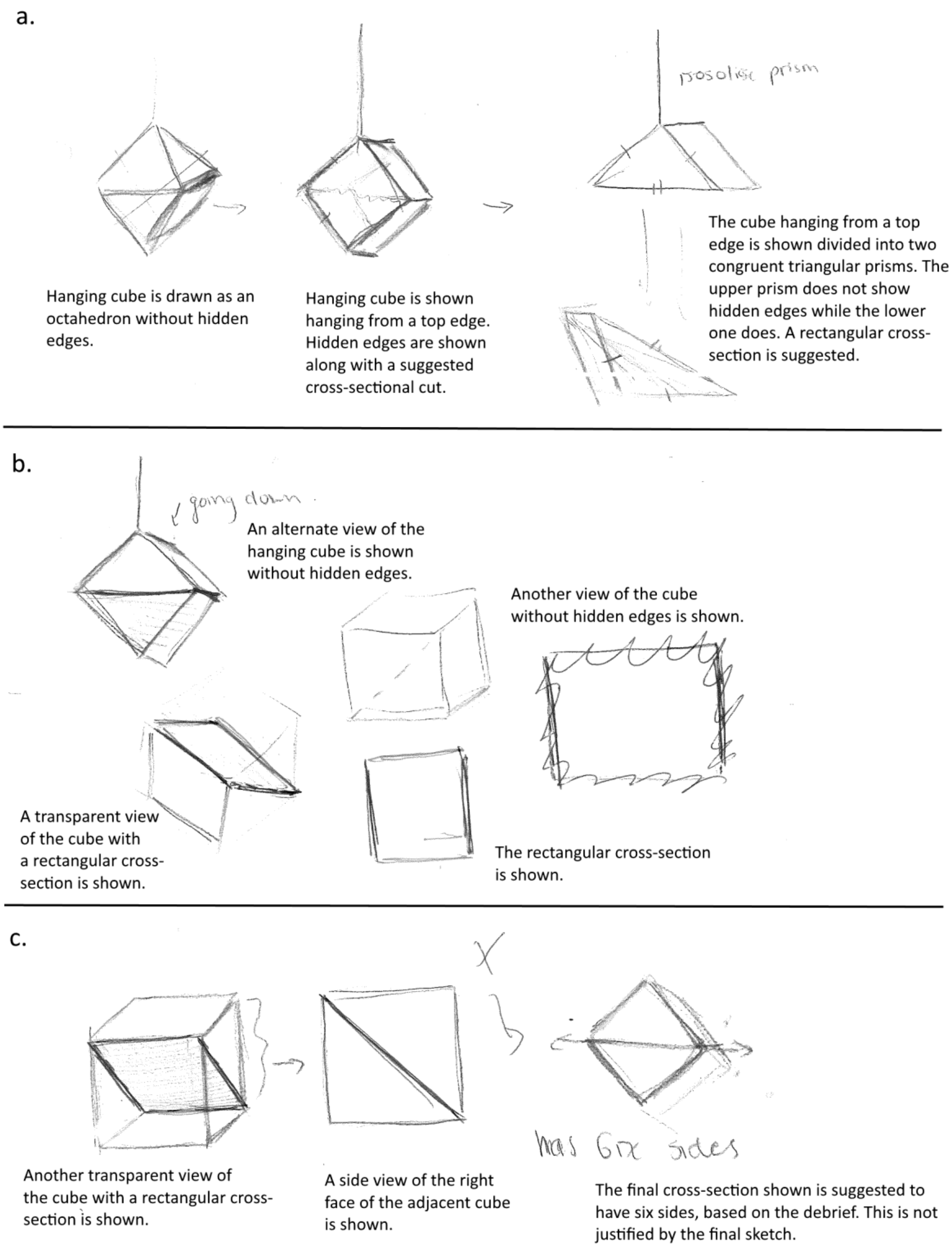


Figure 4-1 Alex's Hanging Cube

a. Initial octahedron and then cube; b. Adjusted cube with a rectangular cross-section; c. Final result suggests the cross-section has six sides but still shows a rectangle.

Although Alex did not develop a cross-section that was horizontal, as required by constraints of this task, she was able to determine a horizontal cross-section for the *tetrahedron task* in the follow-up interview, described in the next section.

Alex's Tetrahedron

Alex drew three different views of the tetrahedron as she rotated it from resting face onto an edge (Figure 4-2c). She indicated how the tetrahedron rotated by denoting the original right bottom edge (CD) with a thicker line and the original left bottom edge (BC) with an arrow on all three views. These markers helped her orient her own view of the structure and to describe its orientation when sharing her drawing.

Alex: Yeah, I tried to like, bold the sides [AB, BC and CD].

Ms. H.: So this [bold CD] side is moving to here [C'D']?

Alex: Yeah. Yeah.

Ms. H.: Ok. So you started with it in the position from the – exactly like before [ABCD] and then you're trying to…

Alex: Redraw it to [be balanced on] like the edge [B'C']. Yeah.

Ms. H.: Um. I see. So you kind of like, rotated it.

Alex: Yeah so like it turned like this way, [counterclockwise]. And after [A'B'C'D'] I tried to draw it from like the side perspective [A''C''D''] which was... Similar to..

Ms. H.: Oh. Is that [A''C''D''] like looking from this side?

Alex: Yeah. And, yes, um, I'm not really sure what I imagined [inaudible questions]. Hmmm... [Draws more]

Alex was able to describe how she rotated the tetrahedron to become aware of properties of the shape in different positions. She was able to generate a cross-section (WXYZ) that was parallel to the floor (Figure 4-2c). The strategy she implemented showing the rotation in steps may have helped her become aware of the position and orientation of the features of the tetrahedron in different positions. She was able to recognize that the horizontal cross-section passed through four faces of the shape.

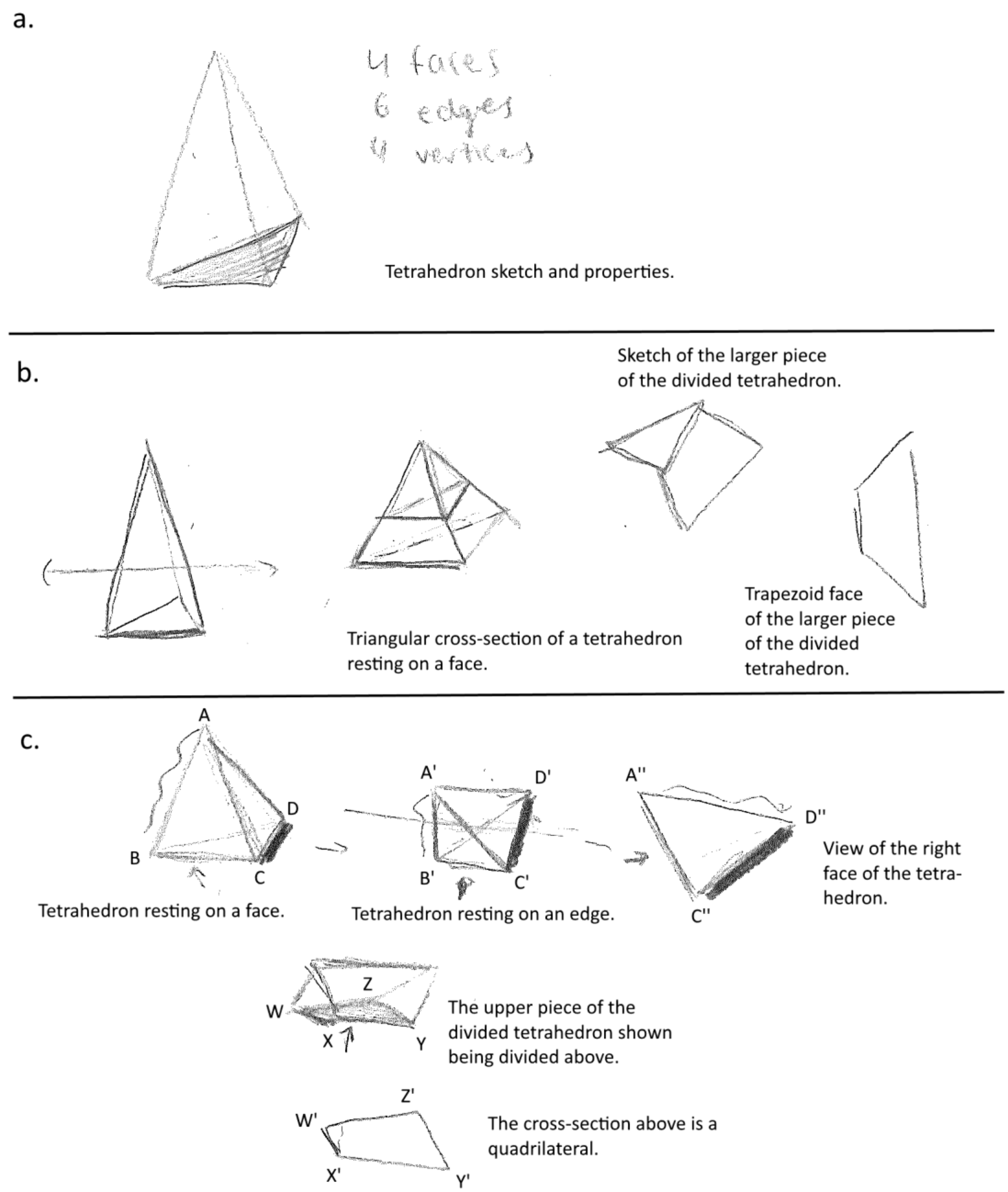


Figure 4-2 Alex's Tetrahedron
a. Tetrahedron sketch and properties; b. Cross-section of a tetrahedron resting on a face; c. Cross-section of a tetrahedron resting on an edge is a quadrilateral.

Alex's cross-section was accurately sketched showing all hidden edges on the structure, notably edge BD (Figure 4-2c). By showing all four sides of the cross-sectional cut, WXYZ, she was able to identify it to be a quadrilateral. She pointed to these four sides (WX, XY, YZ and ZW) using her pencil during the debrief dialog described below.

Alex based the angles of her predicted quadrilateral on the view of this polygon provided in the three-dimensional drawing.

> Ms. H.: Uh, which one, which face is the cross-section?
>
> Alex: Uh, this one [points to face WXYZ].
>
> Ms. H.: Um, how many sides does it have?
>
> Alex: Um, four?
>
> Ms. H.: How do you know?
>
> Alex: Um, because I believe it has at least one at the first [WX] face, and after it would be here [XY] and here [YZ] and after it will go back [to ZW] [traces each side on the drawing as this is described with her pencil].
>
> Ms. H.: Oh, ok. And I'm just trying to see the length between this drawing…Um so it has four sides. Is it, what kind of quadrilateral?
>
> Alex: A weird rectangle? I tried to draw it here [W′X′Y′Z′].

Alex's ability to describe the rotation of her tetrahedron as well as her indication of four faces through which the cross-section passed demonstrates her *awareness-in-discipline*, an awareness which enabled her "shift in the structure of attention" about properties of the shape after it was rotated (Mason, 1998, p. 252). The tetrahedron's cross-section was cut through its four faces to generate a quadrilateral cross-section.

Alex was able to generate a solution that adhered to all constraints of the task, in particular, a cross-section that was parallel to the floor (WXYZ). Her drawing strategies included showing multiple steps involved in the rotation of the tetrahedron and showing hidden edges of the tetrahedron. These were implemented in both the *hanging cube task* and the *tetrahedron task*. However, Alex only showed systematic stepwise rotation and a cross-section drawn to be accurately oriented parallel to the floor in the *tetrahedron task*. Alex was therefore able to accurately discover that the cross-section was a quadrilateral (which she named a 'weird rectangle.')

Alex's drawing strategies involved inclusion of hidden edges of the tetrahedron (Figure 4-2) and systematic and stepwise rotation of the figure (Figure 4-2b). She developed the ability to adhere to multiple constraints in this task by selecting a horizontal cross-section (Figure 4-2c). Therefore, she developed new awareness of geometric properties of a tetrahedron through this cross-section visualization task.

Alex's Parabola Locus

Alex's original solution to the *parabola locus task* demonstrated her understanding that point P or Q was included in the area of the moving circle (Figure 4-3a and b left). Only after discussion with a peer and the class debrief did she adjust her own understanding of a circle 'passing through a point R or point S' to indicate that its circumference passed through the point (Figure 4-3b right and c).

Alex's engagement in discourse with peers changed her understanding of the convention of 'passing through a point' and she was able to transfer this new understanding when working on the *ellipse locus task*.

Alex's Ellipse Locus

Alex applied her adjusted understanding of the convention of 'passing through a point' as developed during the *parabola locus task* to the *ellipse locus task* (Figure 4-4b). Every inner circle's circumference passed through point Q while also remaining tangent to the outer circle (Figure 4-4b). Alex independently sketched an accurate and precise elliptical locus.

Alex adjusted her own solution to this task based on her understanding of the definition of tangency which changed over the course of her engagement with the task. She explained changes made between Figure 4-4a to Figure 4-4b without prompting during our debrief.

Alex: Um, so at first, I was thinking about the circle like the point and after I guess didn't know what tangent meant; I thought tangent like meant intersecting…

Ms. H.: Yeah…

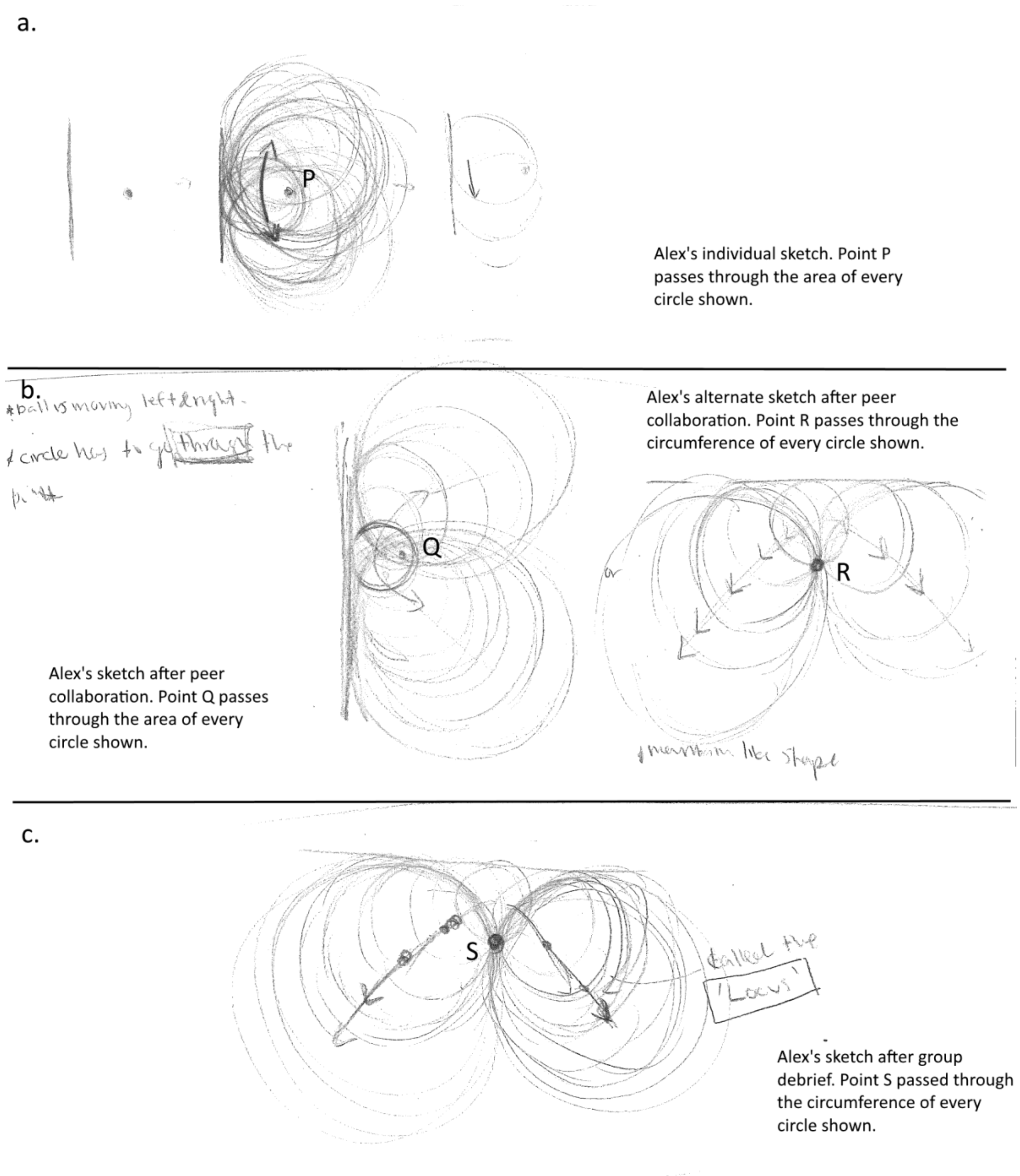


Figure 4-3 Alex's Parabola Locus
a. Initial sketch shows point P passing through all circle areas c; b. Sketch after peer collaboration shows point Q passing through all circle areas or point R passing through all circle circumferences; c. Sketch after class debrief shows point S passing through all circle circumferences.

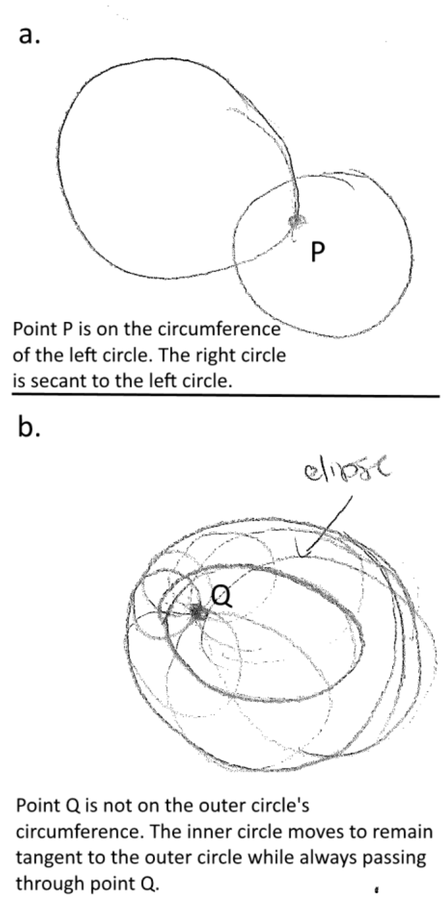


Figure 4-4 Alex's Ellipse Locus
a. First sketch shows point P on the left circle’s circumference and the right circle secant to the left circle; b. Second sketch shows point Q not on the outer circle’s circumference and the inner circle remaining tangent to the outer circle while passing through point Q.

Alex: … so that’s why I drew that [Figure 4-4a] and after you explained that [tangent] doesn’t [mean intersecting]. So it [the inner circle] might be the inside of it [the outer circle] so… this and after it [the inner circle] was like tangent to the [outer] circle [draws Figure 4-4b] yeah. And if it [the inner tangent circle] would rotate around I guess it could like get bigger like that and like after smaller but it always touches the point [Q] – [and] the circumference of the first [outer] circle.

Alex’s explanation of the changes in sketch from Figure 4-4a to Figure 4-4b based on her adjustment of the definition of tangency demonstrates further *awareness-in-discipline* (Mason, 1998) as she was able to recognize how her understanding of the definition of tangency shifted and impacted her solution to the task.

Alex's understanding of the definition of tangency and the 'passing through a point' convention and the represent development of her understanding of geometric definitions and conventions through her engagement in the visualization tasks.

Alex's Experience

Alex explained that visualization tasks were enjoyable because they allowed her to express her own sense of agency and allowed her to think in a different way.

> I think they're really fun because I've never used my brain like that before, I guess, like to imagine like certain shapes where to go or what pattern they form so **I think it's [an] interesting outcome I can make in my brain** [my emphasis]…. It's different because uhh, I guess we use a lot of geometry which I we don't use in other classes, uhh, I guess besides art but we don't really, like, visualize what pattern they form and like what to do with them I guess.

Secondary students show a preference for mathematics learning environments where they can "create[e] initial thoughts and ideas or [take] established ideas and extend them" (Boaler, 2002, p. 45). Alex referred to an outcome she could "make in [her] own brain" pointing to her own agency at play during visualization tasks which allowed her to create her own initial thoughts that could be extended through discussion with peers.

For Alex, discourse with peers helped her interpret verbal aspects of the task.

> They, they would show me how they interpreted the uh, like question first. 'cause most of the times I like would like not hear what you said or like misunderstand. And after they would say "oh like the solution actually can't" – I was like "oh yeah you're right" and I changed my answer because they said that or their illustration was different and they would like hold it up and say "oh because blah blah blah" and I would say "yeah."

Student discourse is a further means of promoting their own agency (Bell & Pape, 2012). Alex was able to develop her *parabola locus task* through discussion with peers.

During visualization, uncontrollable images that arise can lead toward inflexible thinking (Aspinwall & Shaw, 2002; Presmeg, 1986). Discussion and working with peers' ideas allowed Alex to change her solution, particularly in the *parabola locus task*, and therefore to remain flexible in her thinking.

4.1.2. Leslie (Grade 8 Class)

Leslie is a student I have taught for three years in grades 6, 7 and 8. She is a student who has been very apprehensive and cautious about approaching mathematical tasks. She was not confident in her own thinking and was quite uncomfortable and nervous when stuck on solving a given task. Leslie's problem-solving skills and confidence in her own thinking and reasoning have especially increased this year. Her achievement in Mathematics 8 has been exceptional. Her persistence in group problem-solving tasks has grown impressively.

Leslie often expresses that visual tasks are challenging. She has excellent reasoning and justification skills but appears to be less comfortable with geometric and visual situations. She has an excellent memory and very much appreciates algorithms. However, she always insists on thoroughly understanding *why* an algorithm works before she is willing to accept and apply it.

Leslie was actively involved in discourse with peers and developed her own new insights during classroom visualization tasks. She was able to develop her understanding of how a non-square cross-section of a cube could exist as well as develop an understanding of locus through engagement in visualization tasks.

Leslie's Hanging Cube

During the classroom activity of the *hanging cube task*, Leslie suggested that her chosen cross-section (ABCD) was square (Figure 4-5). Although the expected task solution would be a horizontal, hexagonal cross-section, the focus of our discussion is on Leslie's understanding of the non-horizontal rectangular cross-section that she had selected (ABCD).

Leslie sketched both halves of the divided cube as two congruent triangular prisms showing a two-dimensional rectangular cross-section (ABCD) drawn between them (Figure 4-5). Despite its rectangular shape, Leslie indicated in writing that this cross-section (ABCD) was square. She provided verbal justification for this conjecture during the class debrief following teacher prompting.

Leslie: It's [ABCD is] a square.

Ms. H.: Wait. Okay wait. So Leslie, how do you know?

Leslie: Because, because like all the sides [of the cube] are the same length…

Ms. H.: Okay so…

Leslie: So that would mean that the cross-section[‘s sides AB, BC, CD, and DA] would also be the same.

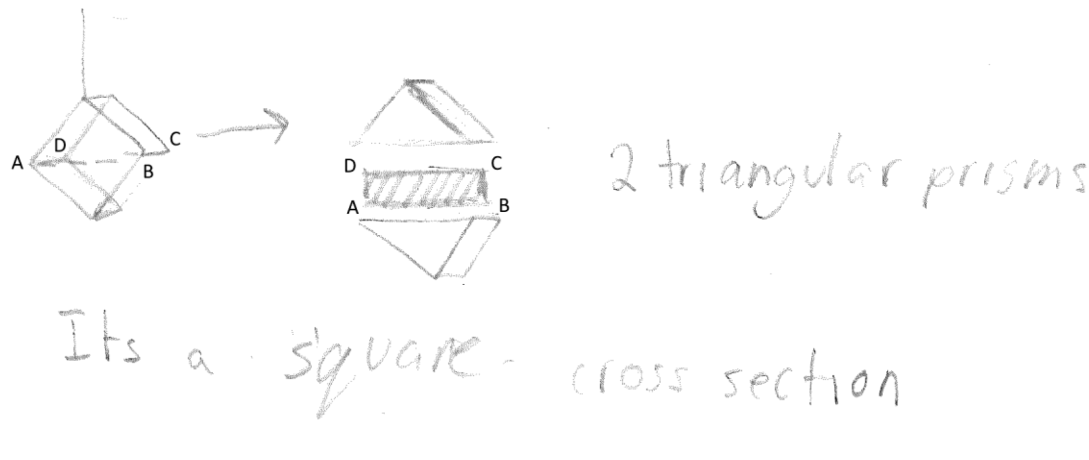


Figure 4-5 Leslie's Hanging Cube
Sketch shows a cube hanging from a top edge with transparent edges. It also shows how this is cut into two congruent triangular prisms. Although a rectangle is drawn between the two prisms suggesting this to be the cross-section, she has written that this cross-section is square.

When the physical cube was brought for the students to examine, Leslie had the opportunity to construe, or *say what was seen* (Tahta, 1989), and thereby adjust her conjecture. She wondered aloud whether the cross-section of the cube she had selected (ABCD) was rectangular rather than square. Then, she repeated the statement “it’s a rectangle” twice amongst her peers, suggesting that she was becoming more and more convinced that her new conjecture was correct.

Leslie: Oh, **is it [ABCD] a rectangle?** [My emphasis].

Jess: Square.

Devon: No, it [ABCD] makes a square.

Leslie: No, **it’s [ABCD] a rectangle.** [My emphasis].

Ms. H.: ‘cause I guess, so, this, this length…

Devon: Nah, it makes a square.

Leslie: **It's a rectangle!** [My emphasis].

Seeing a physical cube brought by the teacher allowed for Leslie to articulate her observation: Leslie had become aware of the dimensions of the chosen cross-section where one pair of sides (BC and DA) spanned the length of the cube's sides and another pair of sides (AB and CD) spanned the length of the diagonals of the cube's faces. She was able to suggest why the cross-section she had selected (ABCD) was rectangular rather than square during the debrief.

Ms. H.: How do you know that they're [AB and BC are] the same [length]? [This question was directed at Jordan who conjectured that the cross-section was square.]

Leslie: Yeah, but it's [AB is] in the middle [of the face] so it's not using any of the [sides of the] faces.

Jess: So it's [AB is] like the diagonal.

Leslie seemed to suggest that the cross-sectional rectangle (ABCD) included two sides (AB and CD) that are "in the middle" of the faces rather than on their sides. Jess clarified Leslie's statements by stating outright that one of these sides (AB) might be the diagonal. The students did not explicitly state that the diagonals were longer than the sides of the faces, but this understanding was implied. This analytic explanation allowed her to discover that a cube could possess a non-square cross-section that was even though all sides of the cube are the same length.

Through her observation of a physical model of a cube and through discourse, Leslie was able to state a geometric property of cubes: a cube can possess a cross-section that differs in shape from its faces. She had not previously been aware of this property.

Leslie's Tetrahedron

Leslie sketched a tetrahedron as a square-based pyramid on top of a triangular prism (Figure 4-6). Although she was briefly shown a square-based pyramid as an example, she was still unsure of how to define and imagine a triangular-based pyramid.

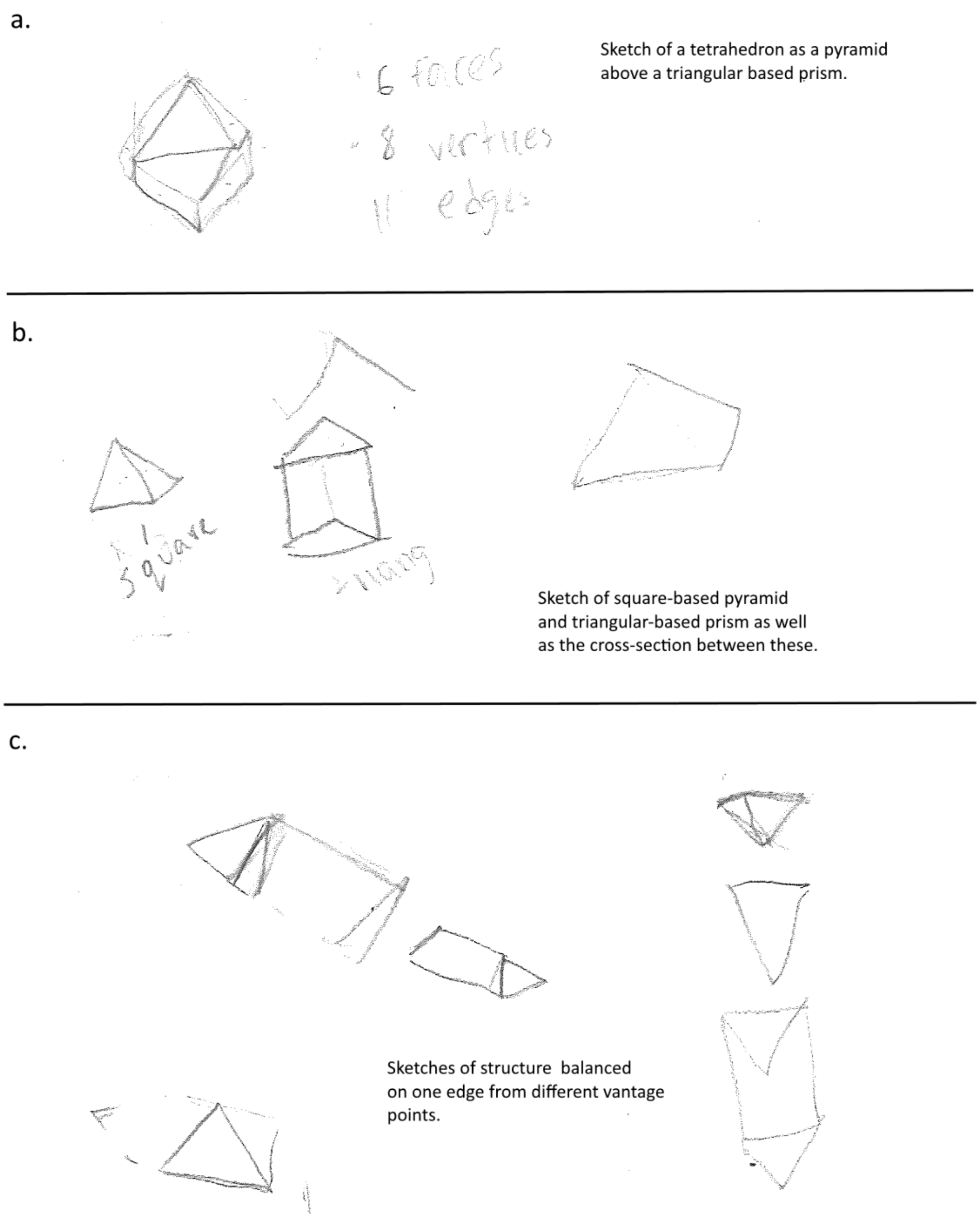


Figure 4-6 Leslie's Tetrahedron
a. Sketch of a tetrahedron as a square-based pyramid connected to a triangular-based prism; b. Cross-section of her structure; c. Sketches of the structure balanced on one edge.

I tried to draw it, 'cause like, 'cause it's a pyramid. But it doesn't have to be like [gestures horizontally indicating the base] square down, so I just like moved it 'cause on that it's three vertices so it's a square and so it's three vertices and it's a triangular prism at the bottom as the base, then that can connect possibly?

Leslie's conception of the tetrahedron significantly impacted her ideas generated in the further cross-section tasks. Therefore, it is difficult to draw conclusions about the development of her awareness of geometric properties of tetrahedra from this task.

Leslie's Parabola Locus

During the *parabola locus task*, Leslie's initial drawings included one possible circle passing through the point and tangent to the line (Figure 4-7, Figure 4-8).

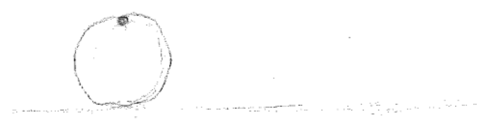


Figure 4-7 Leslie's Parabola Locus

Even when prompted, Leslie only found one circle tangent to a line and passing through a point at a time. Leslie did not see multiple ways a circle could meet the multiple constraints in this task: Remaining circular, passing through the point and staying tangent to the line.

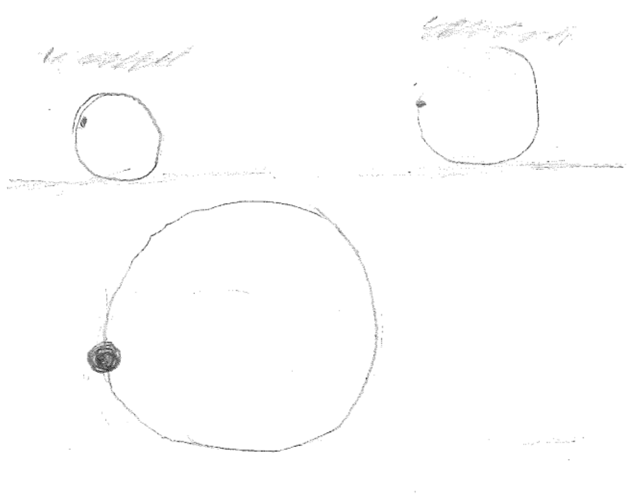


Figure 4-8 Leslie's Alternate Parabola Loci

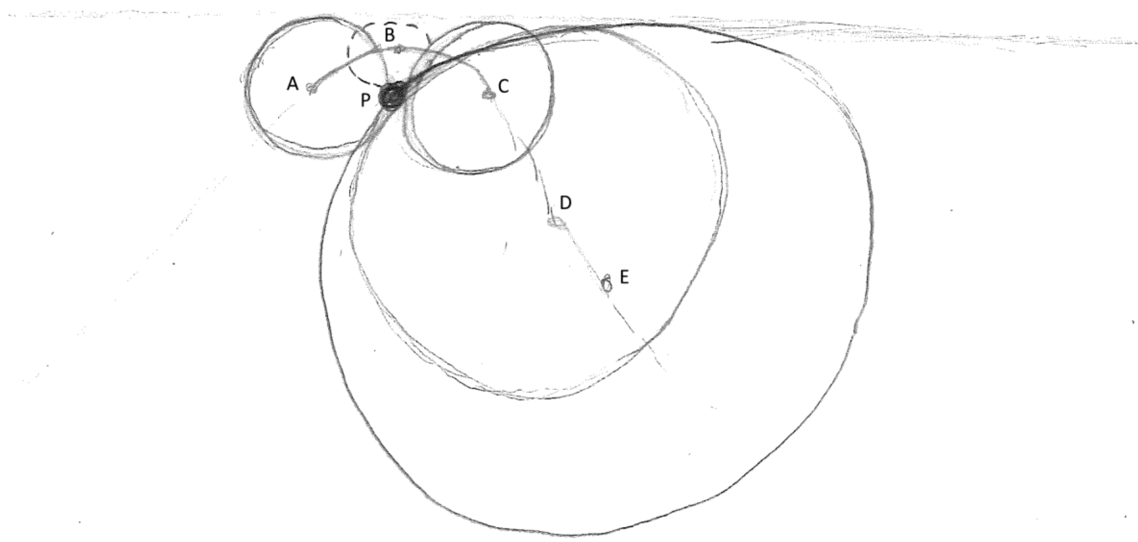


Figure 4-9 Jess's Collaborative Parabola Locus
Circles A, B and C were initially sketched without a locus. Jess later added the locus, added circles D and E, and extended the locus through these circles..

When she had the opportunity to examine Jess's proposed drawing (Figure 4-9), Leslie asked, "How does this one [circle A] move to this one [circle C]?" Jess, Devon and I all suggested to Leslie that circle A could change sizes to maintain tangency to the line while always passing through point P.

Ms. H.: Oh! Well you can imagine the circle can change sizes.

Devon: Yeah. So it goes little and big.

Jess: So it [circle A] goes [gestures] and then small [as circle B] to go pass through and then it's even bigger [as circle C] and then big [gestures circle getting bigger].

Leslie: Ohh, to pass through!

Jess: So technically, I have [inaudible] this. So it's like nnnnnn [sound effect and gestures circle shrinking.]

Devon: Oh! So it like gets smaller and…

Leslie: …it gets bigger, and yeah.

Through discourse with the group, Leslie's statement that "…it gets bigger" suggests her own developing awareness that the circle could change sizes in this task from circle B to circle C.

Although Leslie seemed to agree that circle A, B and C could pass through point P and remain tangent to the line simultaneously, she noted that “[the circle’s diameter] can’t be too big […] ‘cause then it won’t go through the line.”

Leslie suggested that an oval or ellipse that might result if the moving circle became larger while maintaining tangency to the line.

Ms. H.: So, I guess what would happen if the circle keeps moving this way [gestures on paper along the line away from the point].

Leslie: How do you move it that way though?

Jess: You just make a big… [gestures with one hand to indicate a big circle].

Ms. H.: I guess we could extend the line.

Leslie: Yeah, but then if we make [the circle] bigger, then it won’t be equal on both sides [of the line].

Ms. H.: Yeah, uh…

Jess: Yeah.

Ms. H.: Well, what would…

Jess: Both circles equal both sides.

Leslie: It’s gonna get bigger and bigger on this side [away from the line] but it can’t get bigger on that side [toward the line].

Jess: Then it’s an oval…

Ms. H.: Oh.

Leslie: So it’s not a circle anymore.

Leslie was not able to visualize how circles with larger diameters such as circles D and E could maintain tangency and pass through the point until Jess later added these to the sketch. Leslie’s initial sketches of a unique circle passing through the point and tangent to the line (Figure 4-7 and Figure 4-8) were still fixed in her understanding of the situation.

Leslie observed actively as Jess was able to extend her drawing by adding two additional larger circles (D and E) to the right of the first three she had drawn and extending the locus through them (Figure 4-9).

Ms. H.: Oh yeah. So if you kept moving this circle, like, to the side [gestures to the right] and it got bigger and bigger and bigger where would this path move to? Do you want to add? [moves paper toward Jess]

Leslie: Continue. Please.

Ms. H.: Ok. [Nods]

Jess: Yeah. [Jess extends the locus through the two additional circles after indicating their centres first]

Leslie watched Jess's work carefully as she was drawing, pointing to the drawing as Jess extended the locus (Figure 4-9).

She was able to retrospectively reflect on how collaboration with Jess allowed her to develop awareness of how the circle could change sizes to remain tangent to the line while passing through the point during her final interview.

> I think it was with the circle getting bigger and bigger, it was Jess, she did it and then she started to explain how it stays tangent to the line but it's getting progressively bigger so yeah.

Leslie actively discussed and observed how the moving circle changed sizes in the *parabola locus task*. She was able to apply this new awareness to the *ellipse locus task* which follows.

Leslie's Ellipse Locus

Leslie was able to sketch multiple positions where the inner circle (circles A, B and C) was tangent to the outer circle (Figure 4-10b). Circle B dilated as it moved to the position of circles C and D to maintain tangency to the outer circle while passing through point Q.

Leslie was able to think more flexibly about the *ellipse locus task* without encountering an uncontrollable image as she had in the *parabola locus task*. She independently drew the path of the locus of the centre of the circle demonstrating her new awareness of locus.

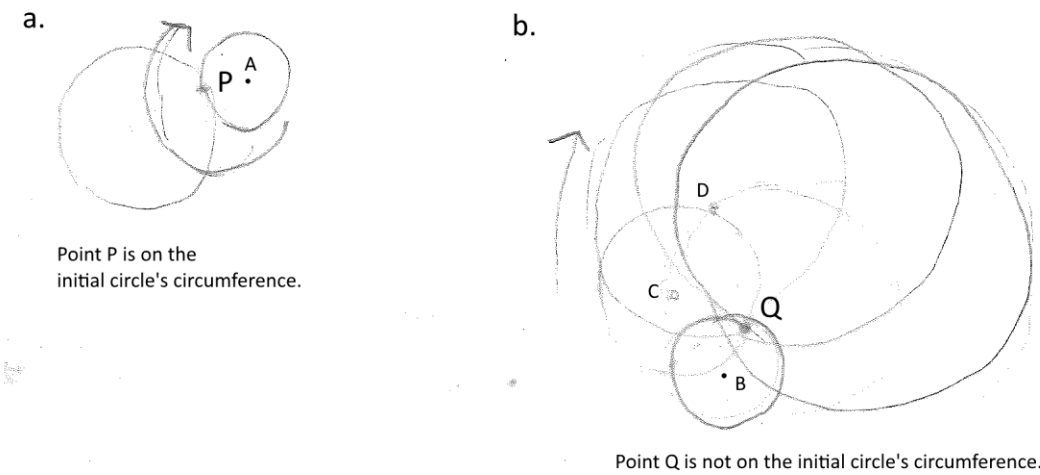


Figure 4-10 Leslie's Ellipse Locus
a. First sketch where Point P is on the circumference; b. Second sketch where Point Q is not on the circumference.

Leslie made two attempts to sketch a solution to the *ellipse locus task*. I asked her to articulate what she changed between the two sketches (Figure 4-10).

Ms. H.: Out of curiosity what did you change between the first (Figure 4-10a) and the second [drawing] (Figure 4-10b)?

Leslie: I put this one [circle A Figure 4-10a] on the outside of it [the leftmost circle], and so… but then I heard that it moves around the circumference of the thing and this [points to circle A] doesn't really move around it, it kind of just stays on the edge. And this one [Figure 4-10b] [inaudible]…

Ms. H.: Were there any other differences?

Leslie: This one [Figure 4-10a] I kind of just, this one it stayed the same and it's [circle A is] just moving around the point [P] while this one's [circle B is] moving around the circumference [of the outer circle] and it's getting bigger because it doesn't just stay the same size because then it won't be touching the side.

Ms. H.: I was wondering how did you decide where to put the point [P]?

Leslie: I kind of just thought because it can't be in the middle, so I just chose a different point [P] that's close to the edge of the circle so that it's tangent.

Ms. H.: And how did you choose on this one [point Q]?

Leslie: Um, also not in the middle. So I just put it in a place and then to make sure that the circle [B] stays tangent to the outer circle.

Ms. H.: Do you know what path the centre of the circle would trace?

Leslie: Of like here [draws] – it sort of [would] make a mini circle inside.

Leslie's ability to describe to her change of the position of point P on the circle's circumference to point Q inside the circle to meet the constraints of the *ellipse locus task* shows her own *awareness-in-discipline*. She realized that her first solution (Figure 4-10a) conflicted with the constraint that the inner circle would "roll around the centre of the outer circle" and was able to adjust her solution to accommodate this constraint. Her thinking to checking various constraints is a development in her analytic thought, a necessary competency that supports successful visualization (Presmeg & Balderas-Cañas, 2001; Zazkis et al., 1996).

Through this task, Leslie demonstrated understanding of locus and the ability to verify multiple constraints through her drawing and description.

Leslie's Experience

Leslie indicated that visualization tasks are very different from what is typically the focus of mathematics class.

> I think that well, some of them, what we do doesn't really correspond to what we're learning in class so it kind of confuses you on what we're learning in class, but sometimes when we do those it helps me kind of relax from doing other stuff. But yeah, and I think that they're, I sometimes I don't understand them because it doesn't really relate to what we're learning in class sometimes, so I don't know how to do it so I kind of guess.

Leslie suggested that it is possible that visualization tasks may be helpful but was not certain of exactly how. She suggested one possibility may have been practicing mental mathematics.

> It may help me when I'm doing a problem and I don't have related, a sheet of paper to write down on, it can help me with figuring stuff in my mind maybe.

Leslie agreed that peer collaboration changed her thinking about solving visualization tasks.

> Yeah, sometimes they heard a specific part that sort of sparked something and they did something a little differently than mine and then I started thinking about it more and then they started explaining it and then I sort of got it and I changed mine.

Discussion and working with peers' ideas allowed Leslie to extend her solution to the *parabola locus task*. She was initially unsure of how a circle tangent to the line and passing through the point could change positions and therefore change sizes while meeting these two constraints. However, access to Jess's ideas may have allowed her to ultimately solve the *ellipse locus task* independently.

4.1.3. Devon (Grade 8 Class)

Devon is a student I have taught for three years in grades 6, 7 and 8. She is a student who describes herself as having a lot of difficulty with mathematics, especially during the first year I taught her. She often takes a lot of time to process and think, especially when it involves any mental numerical mathematical calculations. In my observations, Devon is comfortable with tasks with a visual or geometric component and especially enjoys drawing diagrams either in her notebook or on the white board. Devon has made incredible gains in her mathematics learning in three years and has completed Mathematics 8 very successfully. She still does not necessarily describe herself as liking mathematics during informal conversations.

Similar to Leslie, Devon was able to develop her understanding of how three-dimensional shapes could have cross-sections that differed in shape from their faces through engagement in visualization tasks. She also developed an understanding of how a circle could maintain tangency to a line or curve while passing through a point in locus tasks.

Devon's Hanging Cube

Devon sketched the Hanging Cube such that it hung from an edge rather than a single vertex similar to Leslie (Figure 4-11). Initially, she indicated that her chosen cross-section was a square both on her drawing and verbally. She erased and changed this response to "rectangle" (shown below) following discussion with peers.

> Ms. H.: How do you know that [all sides of the cross-section are] the same?

Devon: Because all the faces [of the cube] have the same width.

Leslie: Yeah, but [part of the cross-section is] in the middle [of the face] so it's not using any of the [sides of the] faces.

Jess: So it's like the diagonal.

Ms. H.: So, this is a face, but…

Jordan: Oh my god! [holds face]

Devon: Oh, naw… [erases square label and changes it to rectangle on her drawing in Figure 4-11]

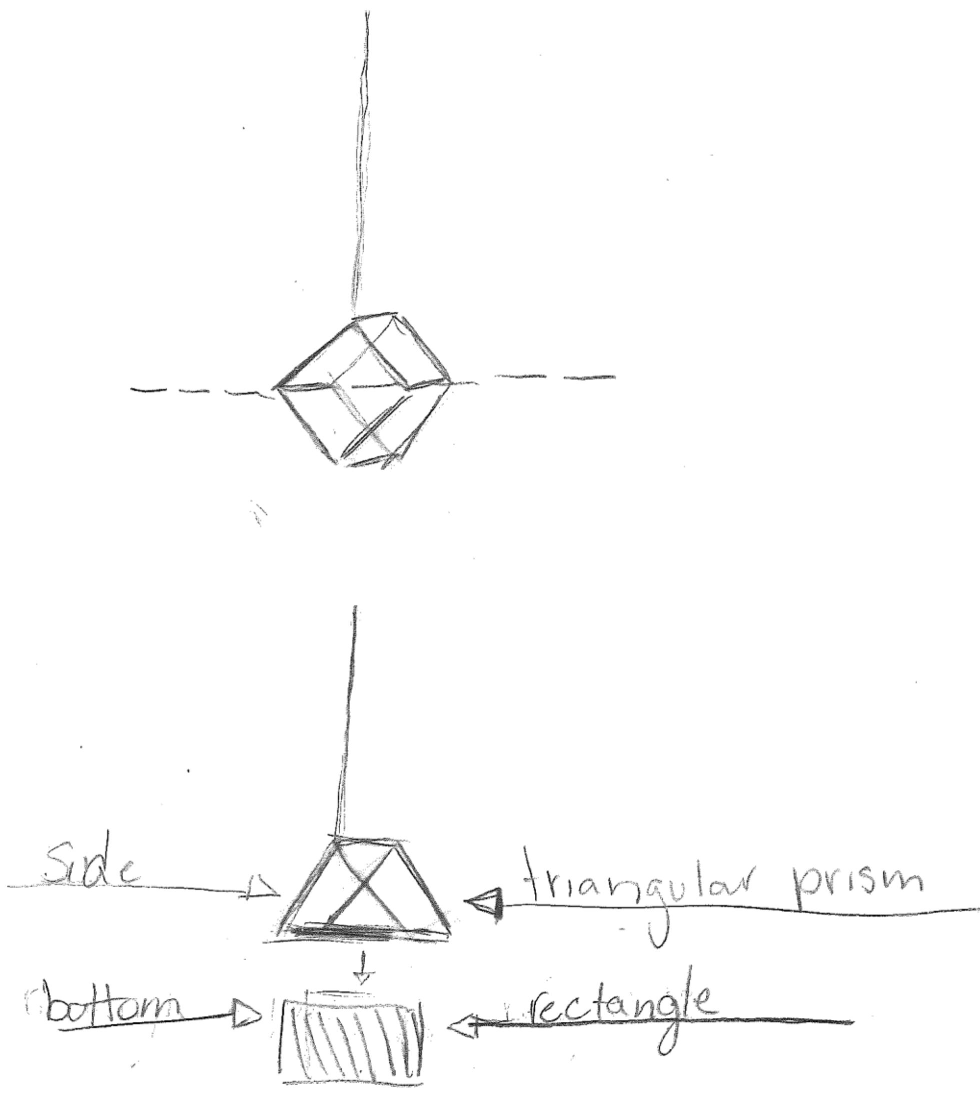


Figure 4-11 Devon's Hanging Cube
Cross-section of her cube was initially noted to be square but erased this and was later changed to a rectangle during the group debrief.

The group discussion allowed Devon to become aware that the cross-section she had drawn was rectangular and not square. Leslie and Jess drew Devon's attention to the diagonals of each face that were longer than the face's sides. The cross-section selected included one pair of edges of the cube and one pair of diagonals of the faces; therefore, it was rectangular. Devon applied a new label to her drawing to include her change in awareness of this geometric property of a cube.

Devon's Tetrahedron

Devon accurately drew a tetrahedron balancing both on its face (Figure 4-12a and b) and on its edge (Figure 4-12c) including all hidden edges. She was able to accurately sketch and determine a horizontal triangular cross-sectional cut on the tetrahedron balanced on its face (Figure 4-12b).

Devon highlighted the left edge of the tetrahedron (AB) with a bold line and indicated that she had rotated the tetrahedron such that AB became the bottom edge (A'B' in Figure 4-12c). Her sketch of the tetrahedron resting on this bottom edge showed all invisible edges from this new vantage point. Devon selected a cross-section that cut the tetrahedron in half through one of its highest (C') and lowest vertices (A'). Devon was able to state that the cross-section (A'C'D) was a triangle in the interview. Her drawing accurately showed that this triangle was acute and isosceles and distinct in shape from the tetrahedron's faces.

Devon's drawing strategies involved inclusion of hidden edges in the drawing of the shape and showing stepwise and systematic rotation of the tetrahedron. She developed awareness of a geometric property of tetrahedra: she was able to determine a cross-section that was distinct in shape from its faces.

In her final interview, Devon reflected upon the group's selection of rectangular cross-section in the *hanging cube task* that was not horizontal as required by the constraints of the task (Figure 4-11):

> I would say [the most challenging task was] one of the ones where there was a cube and one of the vertices was attached to a string and you have to split it in half. And at first we just thought you would end up with two triangles, but we didn't think about, you're splitting it down the middle and the cube's kind of being held at an angle 'cause of the string, so, like, we didn't think about that at first.

The next step for Devon to further extend her understanding of the task and geometric properties of the tetrahedron would be to potentially explore a horizontal cross-section of this shape, keeping this criterion given in the task in mind.

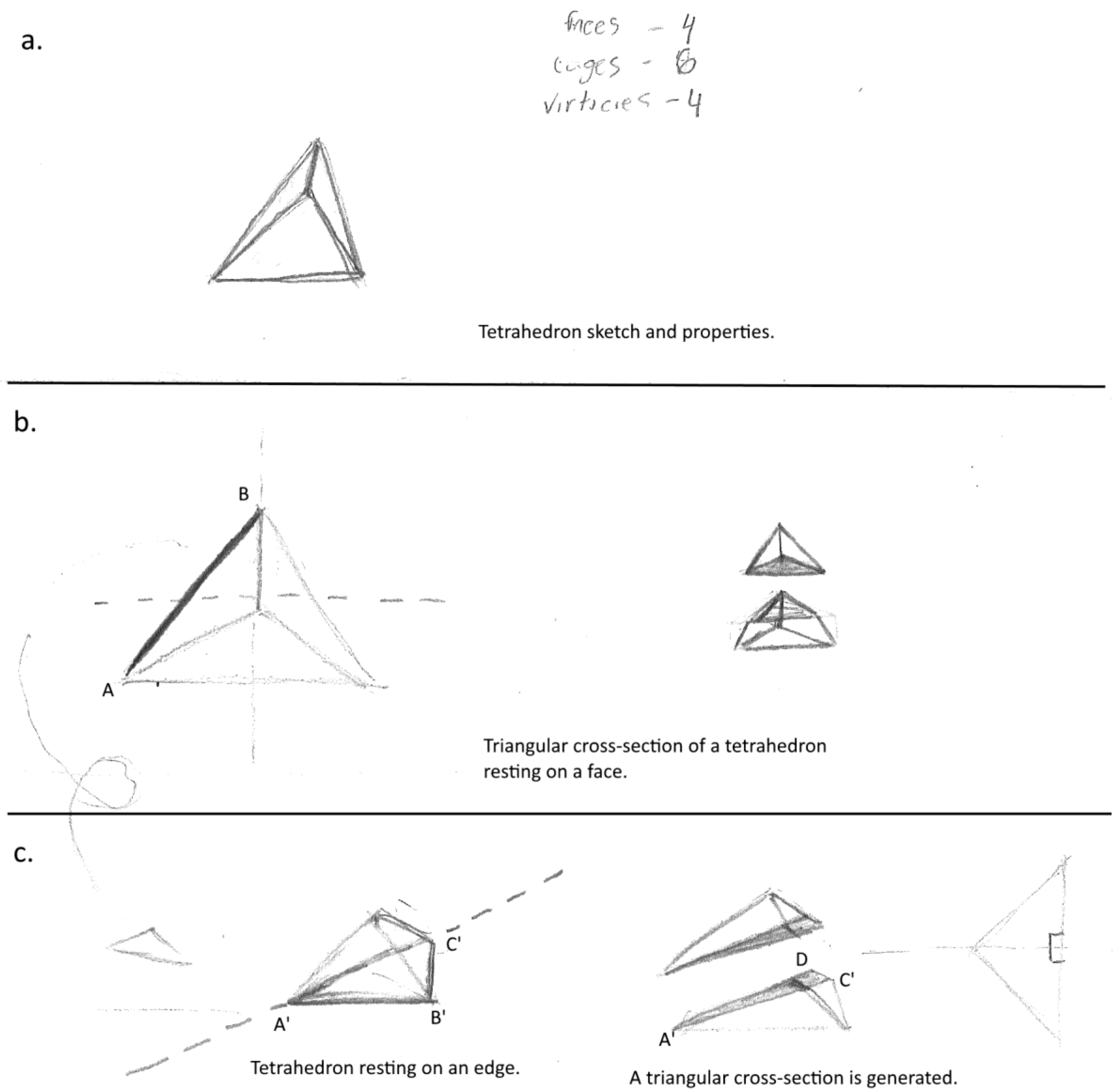


Figure 4-12 Devon's Tetrahedron
a. Tetrahedron sketch and properties; b. Cross-section of a tetrahedron resting on a face; c. Cross-section of a tetrahedron resting on an edge

Devon's Parabola Locus

Devon's personal understanding of definitions in the task impacted her sketch of the *parabola locus task* (Figure 4-13). Devon drew the line as a line segment (AB).

Therefore, her circle was able to move around the segment ultimately tracing both of its sides. Her locus (dashed line) followed the path the centre of the circle traced.

Devon illustrated her ‘point’ as two line segments meeting at the lower circle’s centre (CDE). She articulated uncertainty about how this “point” moved.

> Devon: And, um, I sort of imagined it said like draw the path in the middle of the circle. So the circle moves all the way around the line but it stays, it stays matching the line at all times and there’s my point (CDE). I wasn’t sure, does the point (CDE) move with the circle or not?

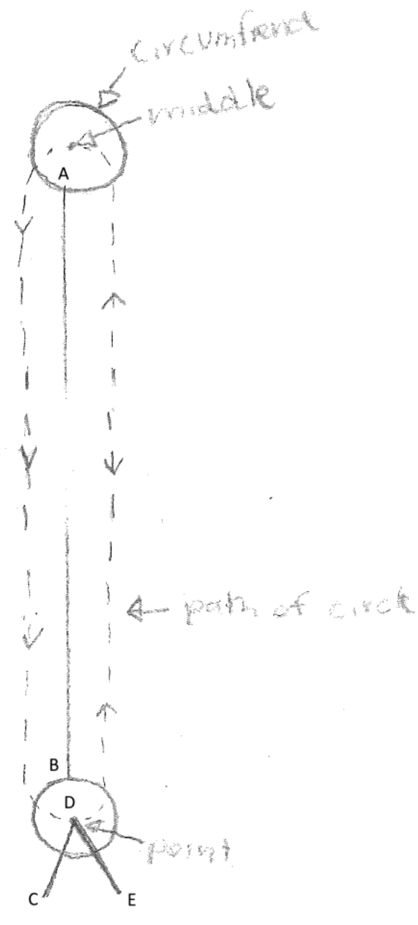


Figure 4-13 Devon's Parabola Locus

Devon later engaged in discussion with Leslie and Jess and helped these two students extend Jess’s proposed drawing (Figure 4-9). This allowed Devon to develop her understanding of generally accepted geometric definitions and conventions required to solve the task as expected.

Jess: [Points to her drawing (Figure 4-9)] Mine makes sense.

Ms. H.: Does hers make sense?

Devon: Um, so the circle goes like this – little [circle B], then big [circle C]!

Devon later interpreted Jess's drawing (Figure 4-9). She was able to ask questions about how the drawing could be extended.

Devon: The only thing is then when it's [the moving circle is] up here [away from the line] and it's big it's not touching the line.

Ms. H.: Uh, is it tangent to the line?

Devon: But what if it's like…this, here [gestures away from the line]?

Devon was able to ask questions about how circle A could move along the line while maintaining tangency but also pass through point P.

Ms. H.: Uh, oh I guess, well then, huh…

Devon: Yeah. Make it big until it touches the line.

Her suggestion to "make it big until it touches the line" suggested her own developing ideas and visualization based on Jess's sketch.

Devon provided feedback to Jess after she added the locus through circles A, B and C. After Jess observed out loud "that's weird," Devon replied, "It's gorgeous," offering positive feedback to her classmate.

Devon demonstrated the ability to interact with Jess's drawing and her peers' ideas in the *parabola locus task* primarily as an observer and commentator. Growth in her ability to find ways to visually solve a locus task while adhering to multiple constraints became evident in the follow-up *ellipse locus task* interview that follows.

Devon's Ellipse Locus

Devon's final sketch in response to the *ellipse locus task* demonstrated her newfound comprehension of the definition of a point, the convention of a circle passing through a point and the convention that this point does not move unless it is transformed (Figure 4-14). She was able to select a fixed point (P) and show how an inner circle could pass through this point at different positions (circles A, B, C and D) while maintaining tangency to an outer circle.

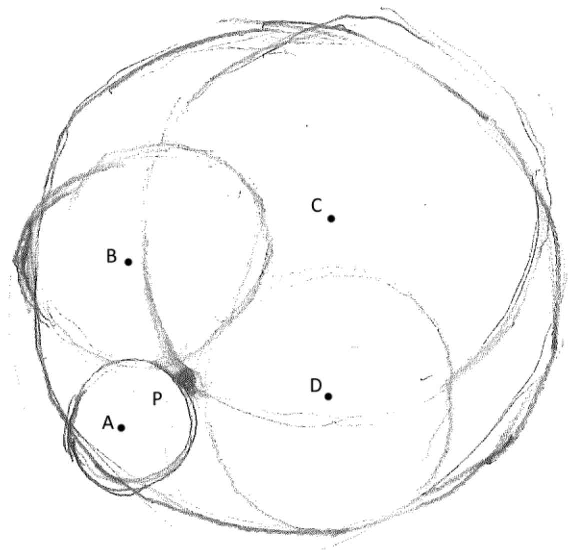


Figure 4-14 Devon's Ellipse Locus
Inner circle passes through the point and remains tangent to the outer circle in three of four positions.

Devon was able to describe her sketch verbally.

> Sure. So um, the point (P) couldn't be in the middle, but this will always have to remain tangent to the um, the point and the um circumference always moves around, through the point (P) [draws]. I have to go through the circumference of the other [outer] circle so I kind of stopped similar to that other task we did with the line where the circle rotates around and it gets bigger [through circles B, C and D] to stretch to touch to pass through the circumference of this circle but always stays tangent to the point (P) which is not in the middle.

Devon was able to transfer her learning from her collaborative work in the *parabola locus task* to her independent work on this task to find multiple ways a circle could pass through a point while remaining tangent to another circle. She developed the skill of solving the task to consider multiple constraints for the circle including passing through the point and tangency to the outer circle and was able to find multiple solutions (circles A, B, C and D).

Devon's Experience

Devon spoke positively about visualization tasks. She noted their openness. The need to listen carefully to all given criteria made these tasks challenging for Devon.

> I mean, I think that they are beneficial, but it can be a little difficult when you are not told to do something. You just have to – I would say they are very challenging because there are sort of no guidelines, you just sort of, like, re-listen to the problem and you try and visualize is like quite challenging.

Devon noted how peer collaboration allowed her to visualize a given task in different ways and this altered her own visualization.

> When I would be drawing something and I would look over and see something else someone else's drawing it would be like uh, I'm probably not doing this right because mine looks totally different from theirs, but then also when we were able to get together and discuss it you realize that you might actually be thinking the same thing you just drew it in a different way um, but I'd say, yeah, sometimes I drew my answer differently to fit more with what they told me to.

Through peer discussion, Devon was able to gain insight into multiple solutions or perspectives on a given task. Her peers' work helped her confirm her own ideas or build on others' ideas to extend her own thinking, such as in the *parabola locus task*.

When asked about what she did not enjoy about learning mathematics, Devon indicated the challenge involved when she needed to consider a number of constraints.

> It's a lot to remember and there's a lot to think about when you solve problems, there's multiple factors.

This statement could be applied to Devon's work on visualization tasks where she learned to keep track of multiple constraints in both cross-section tasks and locus tasks as described.

4.1.4. Jordan (Grade 8 Class)

Jordan joined the Mathematics 8 class in the spring, and therefore only experienced the later visualization tasks. He is a confident mathematics student who enjoys challenges, demonstrating his ability and working with his peers to solve problems. He has excelled in the course this year.

Similar to Leslie and Devon, Jordan was able to develop his understanding of how three-dimensional shapes could have cross-sections that differed in shape from their faces through engagement in visualization tasks. He developed flexibility in thinking about numerous constraints in his work on the locus task.

Jordan's Hanging Cube

Jordan's sketches showed how he had considered the cube from different vantage points (Figure 4-15). Many of his sketches showed hidden faces of the cube. In Figure 4-15a (left), he considered four edges meeting at the top vertex. He also considered three edges meet at the top vertex and a triangular cross-section that was generated in Figure 4-15 (right). He did not share either of these solutions in the class debrief.

Jordan's drawing of the hanging cube in 4-15c shows the cube accurately hanging from the string. The cross-section selected was not horizontal to the ground, and, it seems, that Jordan rotated the bottom half of the cube such that its edge was parallel to the ground in the middle sketch (4-15c). This solution is the one Jordan chose to share with his peers. He described this cross-section as square until his peers justified that it was rectangular. He was able to accept and agree with their justifications.

Ms. H.: How do you know that [all sides of the cross-section are] the same?

Devon: Because all the faces [of the cube] have the same width.

Leslie: Yeah, but [part of the cross-section is] in the middle [of the face] so it's not using any of the [sides of the] faces.

Jess: So it's like the diagonal.

Ms. H.: So, this is a face, but…

Jordan: Oh my god! [holds face]

Devon: Oh, naw… [erases square label and changes it to rectangle on her drawing in Figure 4-11]

Jordan: Oh, I'm sorry, this is a rectangle because like you know, this illustrated line [AB] is higher [longer] than, this this line [BC], so -

Ms. H.: Oh yeah - So like, these lengths are the same [AD and BC], but when I have these two to make the triangle [AB and CD], the hypotenuse is longer [AB is longer than BC].

Jordan: ‘cause like, like… these [AB and CD] are longer than the straight line [BC and AD] so it’s like this one… there….

Jordan’s discovery that this cross-section was a rectangle was a development in his own spatial reasoning.

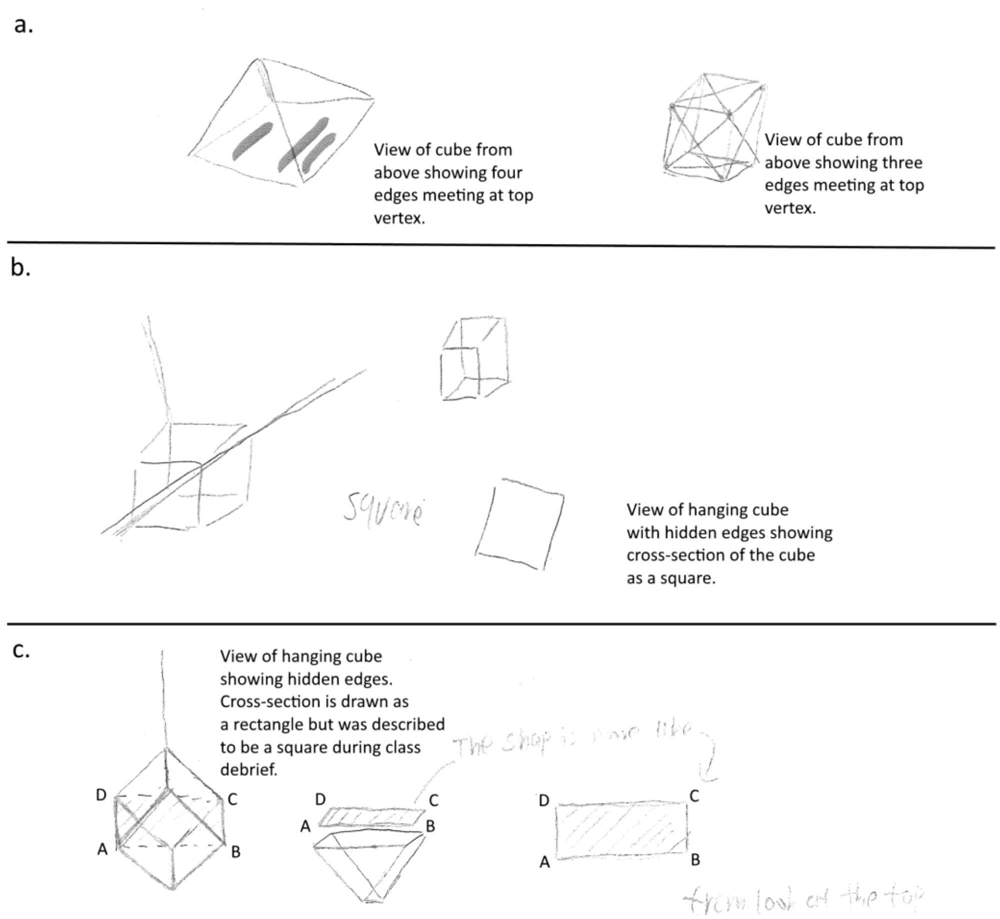


Figure 4-15 Jordan's Hanging Cube
a. Sketch with four versus three edges meeting at top vertex; b. Sketch with square cross-section; c. Sketch with rectangular cross-section

Jordan’s Tetrahedron

Jordan was able to determine that the tetrahedron had four faces, six edges and four vertices (Figure 4-16a). He was able to alter his initial drawing of a square-base pyramid to one with a triangular base without prompting or discussion (Figure 4-16a and b).

Jordan was able to describe characteristics of the divided tetrahedron to include a piece that was a similar shape to the initial tetrahedron (Figure 4-16).

> Ms. H.: Um, [inaudible], uh, can you describe how, what this [piece] looks like?
>
> Jordan: The second one [piece] would be like one, two, three, four, four faces. And one, two, three, four, five, six edges which is equal to this one.

Our discussion did not lead to his describing the faces of the larger piece. This may have been due to Jordan misunderstanding the question posed.

In the conversation about the cross-section generated when the tetrahedron was balanced on one edge, Jordan used both hands to gesture a triangular shape, referring to the front face of the tetrahedron. He was able to draw a sketch including hidden edges to suggest that the cross-section would be an acute isosceles triangle (Figure 4-16). The development of an accurately interpreted cross-section demonstrated growth in his spatial reasoning.

Similar to both Alex and Devon, Jordan provided a bold notation to highlight the edge of the tetrahedron on which the figure balanced. Jordan added this notation when I asked him about which edge the tetrahedron rested on during the debrief.

Jordan's ability to determine a cross-section that was distinct in shape from the faces of the three-dimensional figure represent developments of his spatial reasoning through the cross-section visualization tasks.

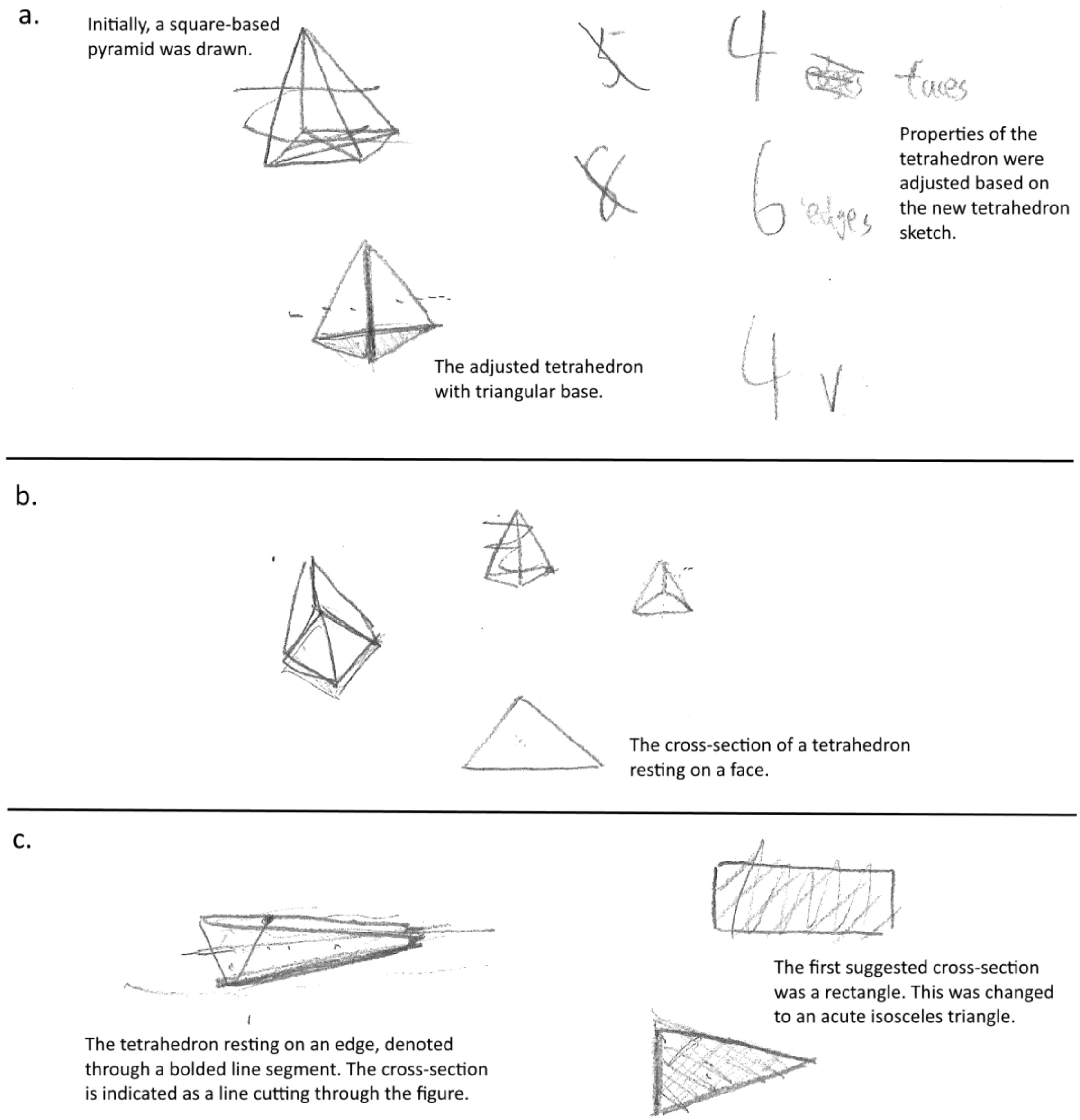


Figure 4-16 Jordan's Tetrahedron
a. Tetrahedron sketch and properties; b. Cross-section of a tetrahedron resting on a face; c. Cross-section of a tetrahedron resting on an edge

Jordan's Parabola Locus

Jordan did not participate in the *parabola locus task* due to being away from class on the day the task was explored with students in class.

Jordan's Ellipse Locus

In Jordan's initial sketch of the *ellipse locus task* (Figure 4-17a), the moving inner circle always passed through point P. If it followed the locus drawn (arrow) the circle would lose its tangency to the outer circle as it changed positions since the locus suggested that it would not change sizes.

Jordan: So this [is] the circle [A]. And the point [P] that is not in the centre. And then, it kind of then starts like in the circle. Um. Also tan…

Ms. H.: [Nods] Tangent to the outer circle.

Jordan: And it's like spinning around…. Like the midpoint [A] is moving this way. And the other one a little bit [B], it's like, upward.

Ms. H.: Oh uh, what path does the middle trace?

Jordan: This one [draws arrow locus with pencil]. I can it just goes around like this, this and then the circle you get. Yeah. [Scratches head].

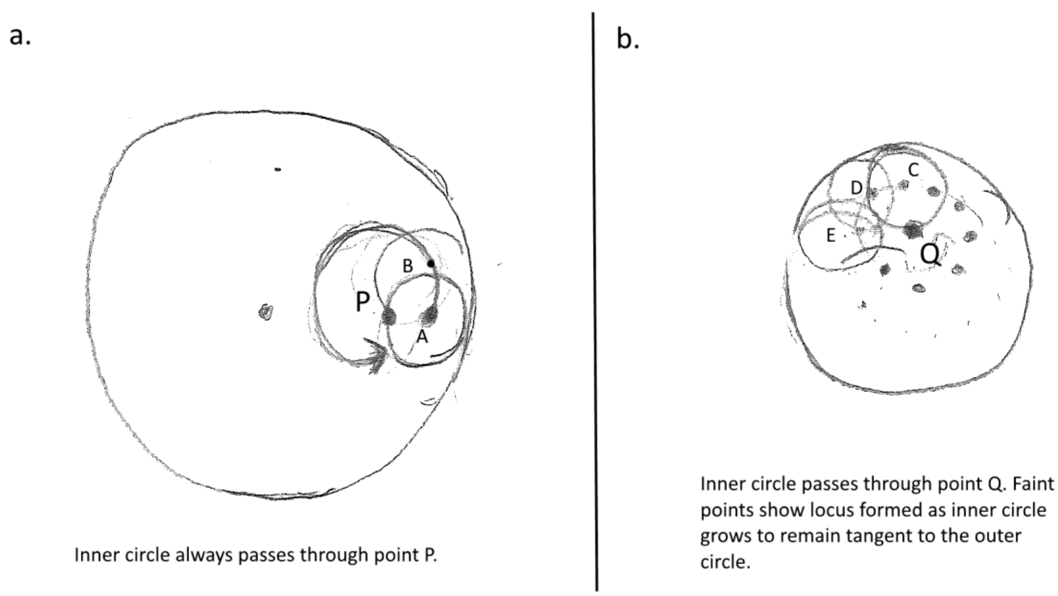


Figure 4-17 Jordan's Ellipse Locus
a. First sketch shows the locus for the inner circle always passing through the point but not remaining tangent to the outer circle; b. Second sketch shows an adjusted locus (fainter dots) to consider the possibility that the inner circle can dilate to remain tangent to the outer circle.

I drew Jordan's attention to the tangency parameter and noted that the circle could change sizes.

Ms. H.: I'll ask you one more thing. So, um, I'm wondering what'll happen if the, if this inner circle [A], like if it's moving so that it's always going through this point [P].

Jordan: Through this point [P]?

Ms. H.: Yeah, the way it is. But also, if it's [A is] also staying tangent to the other outer circle. Is there anything you would adjust?

Jordan: So moving like that, but also tangent to the…?

Ms. H.: So it's always tangent to the other circle.

Jordan: And it must be going through the circle?

Ms. H.: Um, its circumference is always cutting through the point [P].

Jordan: The point. [Sketches]. This one must be, this must be tangent to the other, at least so you calculated it, like if they both go this way this would be following and following at the same time.

Ms. H.: What if I told you the circle can change sizes?

Jordan: Then it would be… that would be…

Jordan was able to adjust his anticipated locus (Figure 4-17b). The fainter dots on the sketch to the right show his modified locus based on his consideration of the tangency criterion.

Ms. H.: Do you mind drawing it like on the side [separate from Figure 4-17a]?

Jordan: Circle [draws] like this [draws more] must go through this way. But it can also trans […] so like… One of the points [Q] is here, so the other circle would be like that… then here, oops [draws then erases] this… Maybe I sort of confused the centre of the circle would be like this point [Q], the other side or something because then mostly the centre like that must be the centre, so if, if this [circle C] need to touch this one [the outer circle], the circle [C] would be this big [draws circles C, D and E]…

Ms. H.: Ok

Jordan: … and yeah. It would follow the other circle.

Ms. H.: Uh, and where would be the path of the, uh, centres of the circle?

Jordan: [Draws fainter locus centre points] Just draw it around here, draw like it this way. Like [inaudible] this point.

Ms. H.: That's the, so it would make a path on the dots of each circle?

Jordan: Yeah. That's right. And for this one, [inaudible] you need to go through this path [gestures locus]. So yeah, I think it's just a little bigger than this one.

Drawing Jordan's attention to the tangency constraint as well as the possibility that the circle could dilate allowed him to adjust his initial visualization which might be considered to be an uncontrollable image. The flexibility that occurred within the span of the interview is evidence that his spatial reasoning developed.

Jordan's ability to verify that multiple constraints were met in a task represent development of his spatial reasoning through the locus visualization tasks.

Jordan's Experience

Jordan stated that visualization tasks were calming but also required focus.

> It calm[s] yourself down and listen to them before you sleep. It's like really interactive… You have to really, really focus on it because you'll only hear it twice and then you need to draw it. So it's a really...good thing for getting your brain focusing [inaudible] a little bit better…

Jordan shared that the tasks allowed students to learn ideas from one another.

> Oh, so maybe […] when, so when you draw down an idea and then you share with others, then you can listen to their ideas too, so you got more chance to, like, 'cause when I was doing visual tasks and I was really like good at it, so I kind like resolved it right away so when I shared with others they can learn from others' opinions…

This took place for him in the *hanging cube task* when he developed the perspective of seeing how his chosen cross-section was rectangular and not square.

4.1.5. Cameron (Grade 9/10 Class)

Cameron is a student who joined UA this year for her grade 10 year. Therefore, she has been in my class for one year. Cameron is a willing participant and confident mathematics student. She is willing to share her ideas and work positively with her peers. She has had a successful year in the course.

Cameron's Hanging Cube

Cameron sketched the hanging cube without showing its hidden edges (Figure 4-18a and b). She drew her expected cross-section to appear to be rectangular, but indicated it was a square. The orientation of the sketched cube was hanging from a top vertex and Cameron drew a rectangular cross-section parallel to where the ground would be on the sketch. However, the proportions of the sides of the cube that were drawn and precision of their angles did not allow Cameron ascertain accurately the location of the horizontal cross-section halfway down the cube. This work represents a starting point to her spatial reasoning that she would build on in the follow-up task that is described next.

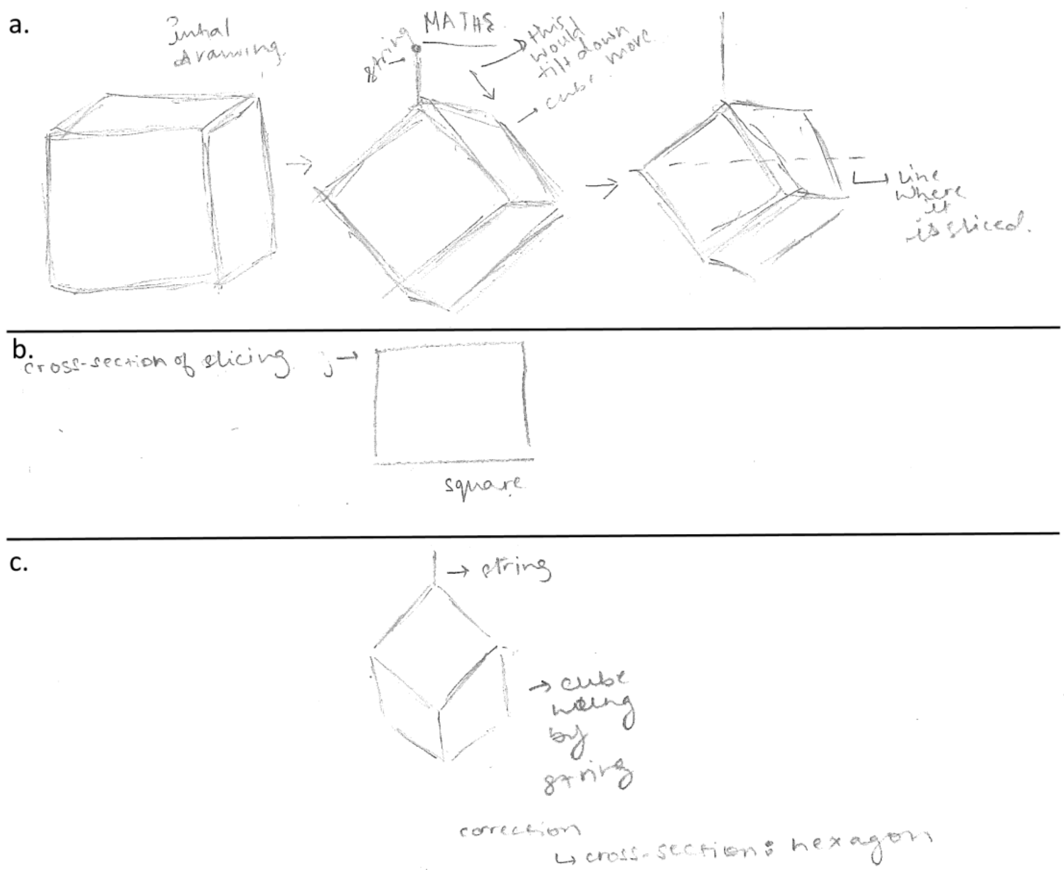


Figure 4-18 Cameron's Hanging Cube
a. Initially sketch; b. Cross-section conjecture is drawn as a rectangle but indicated that it is a square; c. Sketch after discussion/debrief suggests that a hexagon was the cross-section, but this is in not illustrated.

Following the class debrief, when the hexagonal cross-section was revealed to the group by the teacher, Cameron was able to note this on her drawing (Figure 4-18c).

Her hanging cube was drawn to hang accurately from the string without hidden edges after she had seen the physical cube during the debrief.

Cameron's Tetrahedron

Cameron's tetrahedron was accurately represented showing hidden edges, a strategy she did not present in the *hanging cube task* (Figure 4-19). Discussing her initial sketch of cutting the tetrahedron resting on its base halfway up allowed her to rethink her response to the task as she spoke about it allowing her to construe what she had imagined.

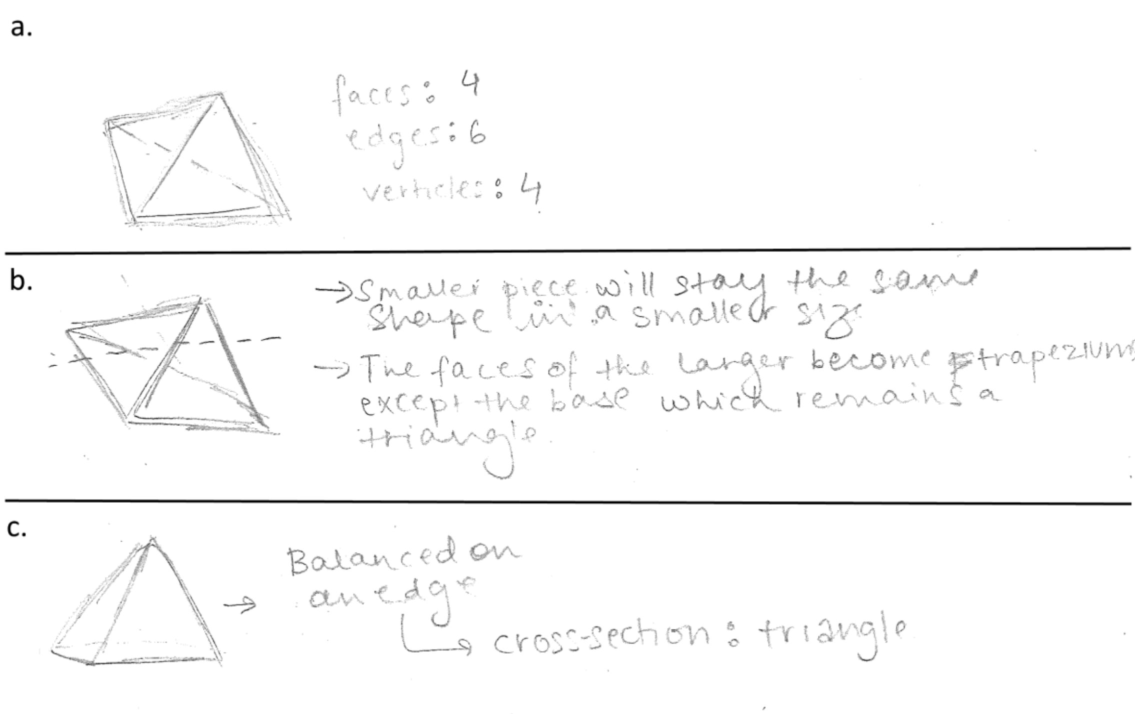


Figure 4-19 Cameron's Tetrahedron
a. Tetrahedron sketch and properties; b. Cross-section of a tetrahedron resting on a face; c. Cross-section of a tetrahedron resting on an edge

> So, um, the smaller piece which would be the top, it would become a rectangular pyramid. It'll just – oh not it won't - It'll stay the same – I didn't think this through. Um, so it'll, it'll, because it has, um, three, k, let me think, can I have another moment to think? Oh yeah, because now I discovered so many [inaudible] – like as soon as I started talking I just go on. Yeah, it will stay the same [erases].

Cameron explained a new solution but did not provide a new sketch.

> Yeah, so the smaller one, the smaller piece that's coming from the top, that would stay the same but would be just in a smaller version because that still has a triangular base […] but, um, the faces of the larger piece will become trapeziums but the base will stay the same. It would still be a triangle.

She also explained the challenge of sketching the tetrahedron balanced on its edge. Cameron decided to forego drawing this stage of the task because she believed she was not able to do this.

> If I can't draw the balancing on the edge, then can I just write about it? […] I'll try drawing it but I don't think I can do it. [Pauses and draws more] Yeah, I don't think I can do that. Yeah, because it's not a square. Yeah, it's not a square. That's okay.

To complete the task, Cameron based her solution of a triangular cross-section on properties of the tetrahedron as justification rather than on her sketch.

> Um, because, well, since all the faces, like, there's not, like, all the faces are the same shape, if you cut it, it's gonna be the same shape, 'cause you're not like changing anything, right? Even with the like trapeziums, if you look at it from the top, it's still gonna be a triangle. […] 'cause they're all equilateral triangles. The magic of the shape.

Although Cameron's sketches of the tetrahedron were disconnected with her verbal and written responses to the task, they demonstrated two distinct and accurate views of the figure (Figure 4-19 a and b) with the new development of the inclusion of hidden edges as part of her development in spatial reasoning. She was able to solve the first two tasks.

Cameron's Parabola Locus

Cameron's sketch of the *parabola locus task* (Figure 4-20a and b) demonstrated her understanding that a circle could pass through a point (P and Q) such that the point landed somewhere on the interior area of the circle. The smallest circle she drew had a circumference that passed through the point (P and Q). Cameron's drawing following collaboration (Figure 4-20) showed her new awareness of how the circle's circumference always passed through the point (R), with all circles passing through this point.

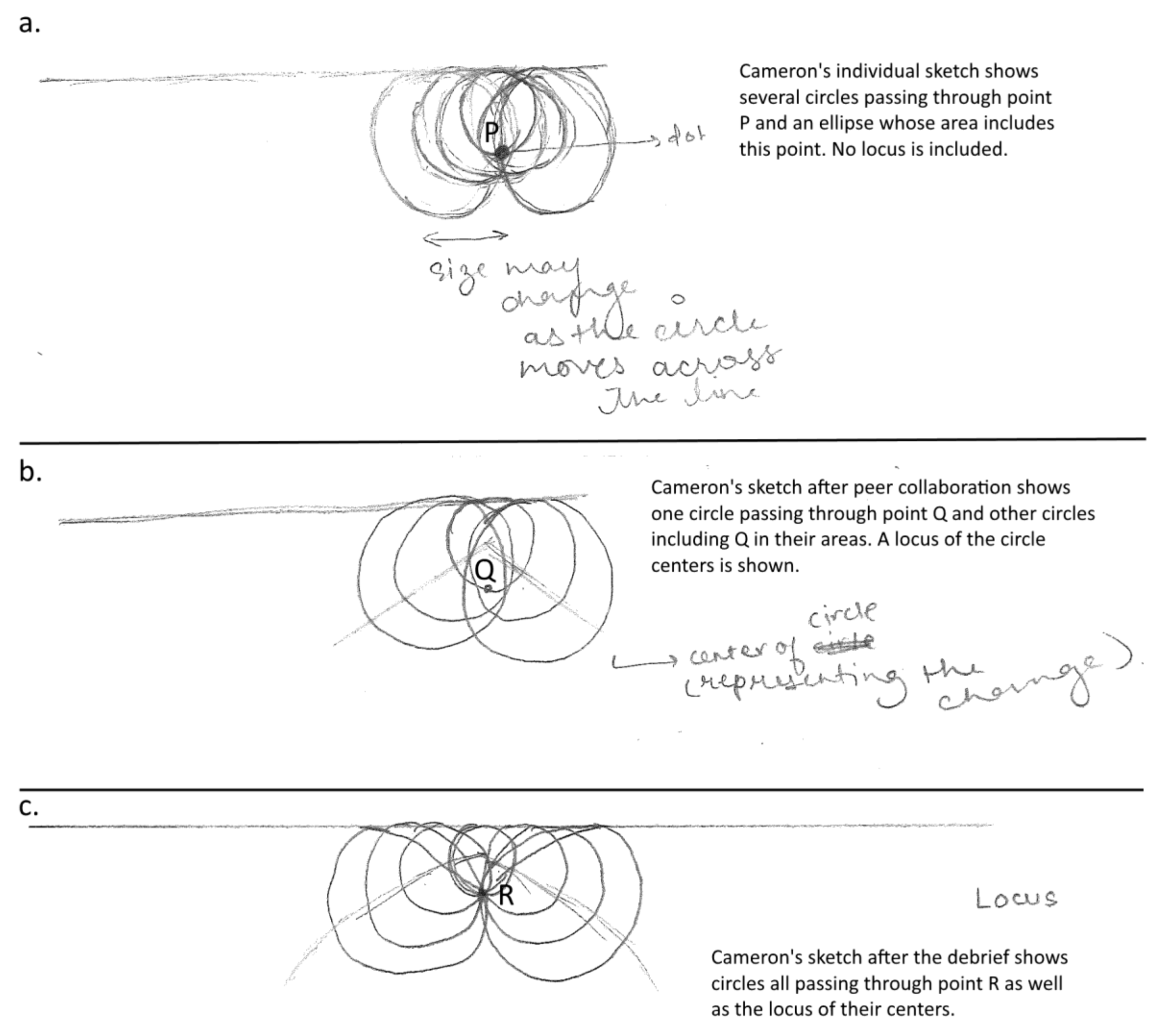


Figure 4-20 Cameron's Parabola Locus
a. Individual sketch shows several circles passing through point P and an ellipse whose area includes P. No locus is included; b. Revised sketch after peer collaboration shows one circle passing through Q and other circles whose areas include Q. A locus of the circles' centres is included; c. Sketch after class debrief shows all circles passing through point R as wells as the locus of their centres.

Cameron's Ellipse Locus

Cameron's engagement in the *ellipse locus task* demonstrated her changing awareness of constraints of the task. Initially, her sketch (Figure 4-21a) showed the locus of the inner circle as it might have been moving to maintain its tangency to the outer circle, but without always passing through the point P.

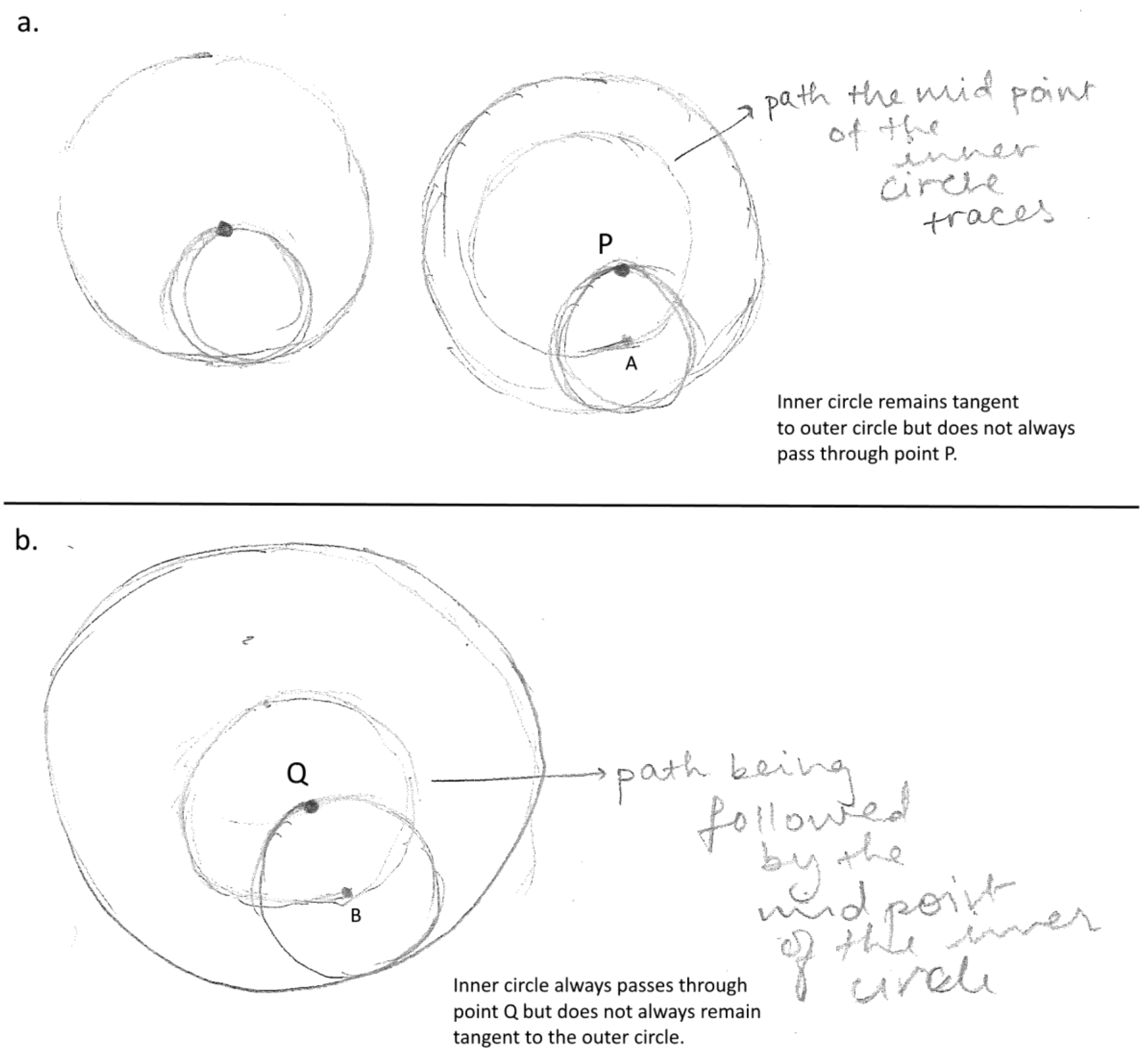


Figure 4-21 Cameron's Ellipse Locus
a. First sketch shows the inner circle remaining tangent to the outer circle not always passing through point P; b. Second sketch shows the inner circle always passing through point Q but not always maintaining tangency to the outer circle.

During Cameron's explanation of the drawing, she became aware of the constraint of passing through point P which was not always met in her initial sketch.

> So, this is the big [outer] circle, right? And then, you said that there would be a point inside the exterior circle that was not the midpoint [point P]. So, I took a point that's slightly to the side [point P] and, um, this is the inner circle [A] that's supposed to be tangent which is only touching at one point [P] and, um, uh, the circumference of the inner circle [A] is also supposed to remain on the point that you've chosen so, uh, this circumference will always remain there, I think, so, wait, oh nooo, ok. I'm going to edit something in the next part, yeah, 'cause I

> just realized something: it's always supposed to stay on [passing through point P].

She was able to elaborate upon this new awareness:

> So, when, uh, you said that the circumference of the inner circle [B] will always remain on that point [Q] right? Like it will always be somehow touching it, right? Right? Yeah. Ok. So when, this, if it's rolling, it rolls all around the circle [and remains tangent without changing sizes], it can't go in this particular direction if the thing is like, if the circumference of the inner circle would remain on the point [Q] that you'd chosen in the exterior circle. It can't follow this path [as in Figure 4-21a], it has to follow a different path [as in Figure 4-21b].

Cameron was able to explain a new locus for the circle's centre before drawing this new solution (Figure 4-21b). The new locus she drew allowed the circle B to always pass through point Q. However, this circle no longer consistently remained tangent to the exterior circle as it moved to generate the locus drawn [circle Q]. Cameron may not have noticed that the initial tangency constraint was no longer met and did not realize how the inner circle might change sizes to adhere to both constraints of tangency and passing through a point.

> Because if, ok, so if, if we get to around like here, the – ok, the, um, like here, the circle [B] is only so big so it can't like fill this place, it will probably come to about here, so the midpoint of the circle [B] would be here, so it would follow a slightly more concise path, like the midpoint of the inner circle then like what I've drawn, which I've realized just now.
>
> It's [circle B is a] more concise circle as, um, if we were to keep, like, while it's rolling, if we were to keep the circumference of the circle on this point [Q] it would always just have to be like this far away from the point, so the midpoint would like this, would be like there, so I've just made it like, more, less, more towards like this side, where like the dot actually is, like the point [Q], sorry, um, than to the other side where the point isn't [trails off…]

Cameron's flexibility in adjusting her solution to adhere to the constraint of passing through the point Q was a development in her spatial reasoning based on her new understanding of the convention of "passing through a point."

Cameron's understanding of the 'passing through a point' convention as well as her ability to verify different constraints (either tangency or passing through a point) were met in a task represent developments of her spatial reasoning through the locus visualization tasks.

Cameron's Experience

Cameron valued the class discussion around the visualization tasks that allowed her to alter her own awareness about different tasks. She highlighted that these tasks allowed her to experience many different perspectives.

> I like them. I feel like they, um, like the way that they're described in the class discussion that goes around them and then you see a lot of different um perspectives. Like, um, when we did the one where we did like a square and then like you know there was like a midpoint and the circle would go around the square and then it's tracing. Um, like, I misun- well, you can say I like misunderstood or took that in a different sense than like Jordan did and then we discussed it and I was like "that's how it's supposed to look" and then you know like the discussion around it I quite like because it's like you get to see what all these different people think about something that's [inaudible].

Cameron noted that she felt unclear about the task based on her own listening, but that working with peers extended her comprehension of the task.

> Yeah, so, uh, before you explained it, right, um, we did the group discussions amongst your peers so I think Jordan or Alex did understand it, like they did the right thing and then we were all discussing it like me, Avery, Alex and Jordan and then one of them was like you're supposed to do it like this and then that's what she mean. And I was like "oh, ok." So like I needed a little more clarification from you, like about like what you were actually trying to say like when you drew it on the board but like when they explained it to me it made a lot more sense to me than what was actually going on in my head.

Peer discussion may have allowed Cameron to move past her own uncontrollable images and develop awareness that helped her solve tasks. Her work with peers may have allowed her to develop her understanding of the 'passing through a point' convention as previously noted in the locus tasks.

4.1.6. Jess (Grade 8 Class)

Jess is a student I have taught for three years in grades 6, 7 and 8. She is a curious mathematics student who enjoys participating in class discussions and sharing her ideas. She generally achieves highly in mathematics and does a lot of preparation and studying at home. She is exceptionally persistent in assessment situations, but I notice she is not always equally persistent on tasks that are not being graded. During earlier visualization tasks when stuck, she would express frustration about not feeling she knew the solution to the task.

Jess was actively involved in discourse with peers and developed her own new insights during classroom visualization tasks. She was able to apply properties of three-dimensional shapes drawn with hidden edges to determine two-dimensional sections in the cross-section tasks. She developed insights about locus through the *parabola locus task* and was able to apply her new understanding of locus to the *ellipse locus task*.

Jess's Hanging Cube

Jess's sketch of the hanging cube included two orientations: the first cube was drawn hanging from the vertex as suggested in the task description including its hidden edges; the second showed a front view of the cube that was hanging from the middle of an edge to give a rectangular cross-section (Figure 4-22).

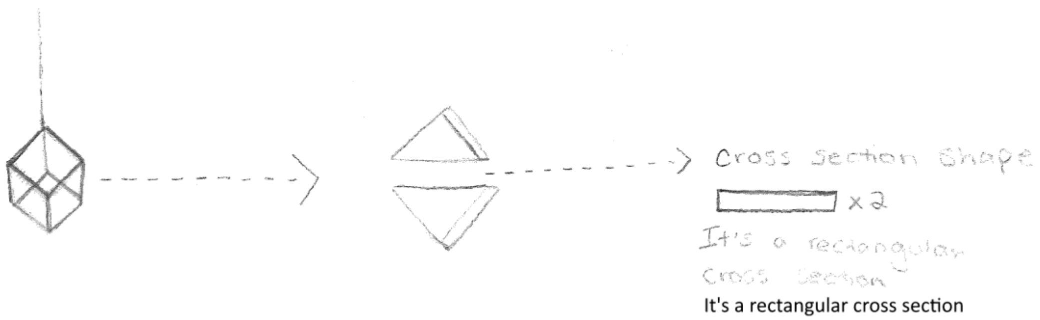


Figure 4-22 Jess's Hanging Cube

In her interview, Jess suggested her peers had influenced her choice to rotate the hanging cube in the way that she had.

> Yeah! Something like that. Um, mine was the good way. I was placing it from like the way I was hanging it from. And I looked at theirs and I was like this might make a bit more sense and I changed mine and moved the way I was cutting, and they changed theirs to mine…

Jess's Tetrahedron

Jess was able to draw a tetrahedron and identify correct numbers of faces, edges and vertices (Figure 4-23). Jess identified the cross-sectional shape of the prism balanced on a base to be triangular (Figure 4-23b).

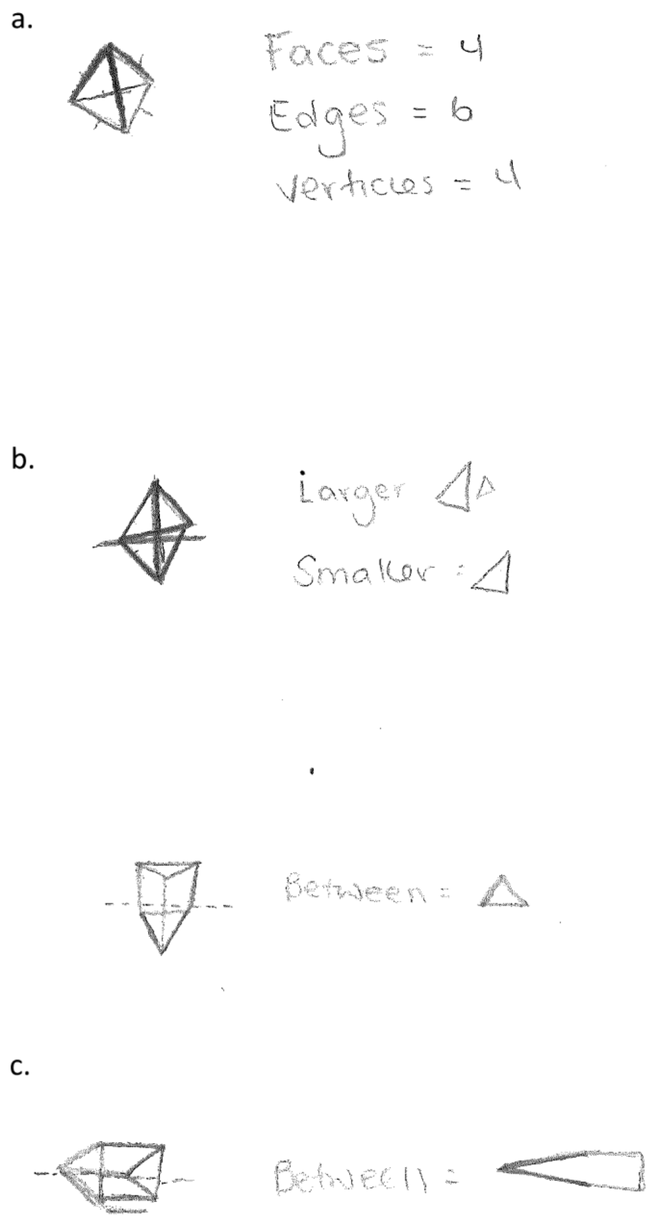


Figure 4-23 Jess's Tetrahedron
a. Sketch of the tetrahedron with 4 faces, 6 edges and 4 vertices; b. Sketch of a tetrahedron with a horizontal cross-section; c. Sketch of tetrahedron modified to a triangular prism to show horizontal cross-sections when it is balanced on a face or on an edge.

Jess: Um, so there's the prism and then it's cut in half, um, so that it looks equally smaller for the side that counts as the base shape but then the larger of the base shape has a little bit of extra.

Ms. H.: What are the shapes?

Jess: Triangles. So, its um, the way I drew it looks kind of like a right triangle but it's supposed to be an isosceles triangle so this [inaudible] but kind of like a right triangle.

Ms. H.: Can I ask like when you slice it halfway up, what shape does the cross-section…?

Jess: Um, it's a triangle.

Ms. H.: How do you know for sure.

Jess: 'cause like the base of it is a triangle and the [inaudible] is a triangle.

In her work to find a cross-section of the tetrahedron, however, she altered the drawing of the three-dimensional structure to a triangular prism and first identified the cross-section of this structure balanced on a face to be a triangle (Figure 4-23c).

Jess: So there's a point on it, it's being balanced on the bottom of facing it which is on top of the triangle shape, so then halfway up horizontally would be the triangle shape.

Jess identified the cross-sectional shape of the prism balanced on an edge to be pentagonal (Figure 4-23c bottom).

Ms. H.: What if it's balanced on like one of these edges?

Jess: So… [Draws for several minutes] Um, so like the point [inaudible] over here and then would be flat over here - like a steep triangle [EAB] and then it would be kind of flatten out [BCDE].

Jess's sketch included hidden edges of the prism. The cross-sectional shape's five sides [ABCDE] were interpreted based on this sketch. Sides AB, BC and CD were cuts through different faces on the prism. Sides DE and EA were on a single face of the prism. DE and EA were likely interpreted as two different sides of the cross-section because of the vertical segment that intersected with E.

This strategy of finding the two-dimensional cross-section based on the properties of the three-dimensional shape drawn with hidden edges was a development in her spatial reasoning.

Jess's Parabola Locus

Jess's sketch of the *parabola locus task* incorporated all constraints of the task as expected. During peer collaboration and discussion, she was able to build on her initial sketch to extend the motion of the circle and to find the parabolic locus (Figure 4-9).

Jess: [Points to her drawing (Figure 4-9)] Mine makes sense.

Ms. H.: Does hers make sense?

Devon: Um, so the circle [A] goes like this – little, then big! [Gestures to Jess's drawing]

Leslie: [Gestures to drawing as well] How does this one move [circle A] to this one [circle C]?

Ms. H.: Oh! Well you can imagine the circle can change sizes.

Devon: Yeah. So it goes little and big.

Jess: So it [circle A] goes [gestures] and then small [as circle B] to go pass through and then it's even bigger [as circle C] and then big [gestures circle getting bigger].

Leslie: Ohh, to pass through!

Jess: So technically, I have [inaudible] this. So it's like nnnnnn [sound effect and gestures circle shrinking.]

Devon: Oh! So it like gets smaller and…

Leslie: …it gets bigger, and yeah.

Jess began to consider the locus based on the three circles she had sketched (A, B and C) and was able to independently add the locus to her drawing accurately with only subtle prompting (Figure 4-9).

Ms. H.: There was another question um, another question I wanted to ask – do you remember at the end what I asked you?

Jess: A half circle – look at this! [Jess begins to draw the locus based on circle on her drawing]

Leslie: The middle of the circle does it go through.

Ms. H.: So, basically trace the whole line… [Jess traces the locus on her drawing through circles A, B and C.] Uhhh.

Jess: That's weird.

Devon: It's gorgeous.

Later, when Leslie and Devon expressed confusion about how the circle C could continue to move further along the line, Jess was able to extend her drawing by adding two additional larger circles (D and E) to the right of the first three she had drawn and extending the locus through them.

Ms. H.: Oh yeah. So if you kept moving this circle, like, to the side [gestures to the right] and it got bigger and bigger and bigger where would this path move to? Do you want to add? [moves paper toward Jess]

Leslie: Continue. Please.

Ms. H.: Ok. [Nods]

Jess: Yeah. [Jess extends the locus through the two additional circles after indicating their centres first]

Ms. H.: Oh I see. How did you solve... Can I just stop you – where are the centres of the circles?

Leslie watched Jess's work carefully as she was drawing, pointing to the drawing as Jess extended the locus (Figure 4-9).

During this task, Jess' spatial reasoning grew through her extension of her own sketch and justifying this solution in discussion with peers.

Jess's Ellipse Locus

Jess produced two sketches in response to the *ellipse locus task* (Figure 4-24). The first included the point P on the circumference of the circle (Figure 4-24a). I asked Jess to select a point not touching the circumference, and so, Jess redrew a new sketch with point Q (Figure 4-24b). Jess was able to draw a portion of the expected locus on this adjusted drawing. Her developed understanding of the concept of locus was therefore transferred from the *parabola locus task* to this task.

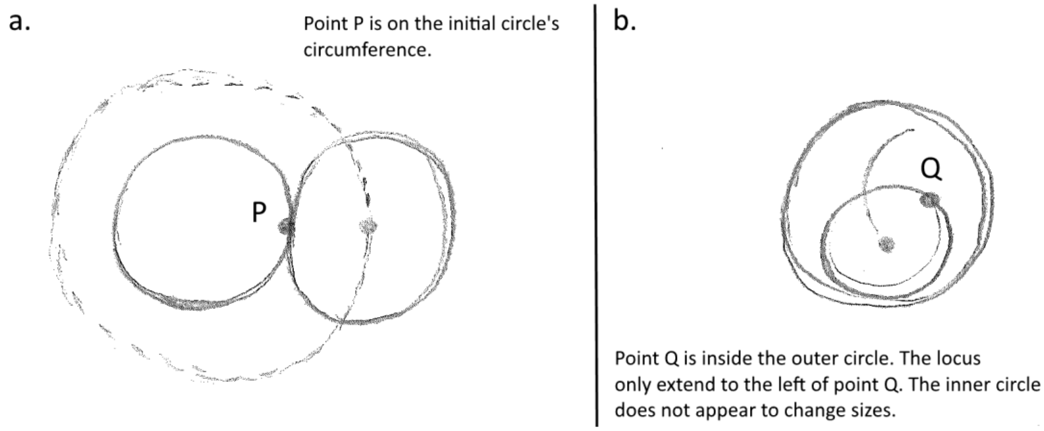


Figure 4-24 Jess's Ellipse locus
a. Initial sketch with point P on the initial circle's circumference; b. Modified sketch shows point Q inside the outer circle's area with a partial locus. The inner circle does not appear to change sizes.

Jess was able to explain her solution to the task (Figure 4-24):

> Sure. So, I have like, I would bring the circle and then the dot here [point Q] or something like that. And then the smaller circle, make it a bit bigger and it's touching [the outer circle] here [to be tangent].

She explained that the locus stopped because "then [the inner circle] would go over the circumference of the circle." If circle B did not change sizes, then the tangency constraint would not be met as the circle moved further counter-clockwise.

Jess' spatial reasoning developed as she was able to demonstrate understanding of the concept of locus in this final task without prompting. She was also able to consider multiple constraints (passing through point Q and tangent to the outer circle) simultaneously in this task. Jess' ability to verify that multiple constraints were met in a task and her understanding of locus represent development of her spatial reasoning through the locus visualization tasks.

Jess's Experience

Jess noted that visualization tasks are difficult to describe verbally without drawing. She shared her understanding about their openness and the sense that no answers are wrong.

> I like them. I think that they, 'cause like the fact that you have to figure out how that what you were thinking about it good 'cause like when like

> you imagine something in your head and try to kind of translate it with words is hard but being able draw it out and then trying to explain it especially without your hands is a little more difficult but it's still good. And the after you said that there were no wrong answers so like eventually after we would look at what it was supposed to be, we would see how close we were and then we would understand why we were further away from the like actual answer.

Jess shared that visualization tasks increased student awareness about how other students are thinking and understanding.

> I personally think it was a good idea to try visualization because um, before we just kept using work on the boards and kind of see how we thought but we never sat down to talk about it and understand how some of our fellow classmates would like see things or understand things. It's kind of cool. And then even now when we 're doing group work, like we would be able to see why someone wasn't understanding what was going on, it was helpful then to try to explain it to them.

Working with her peers on the *parabola locus task* gave Jess a sounding board for her ideas as she considered their questions and conjectures as she extended her drawing. In this task, Jess, Leslie and Devon began to develop a shared awareness which likely influenced their individual work on the later locus task.

4.1.7. Robin (Grade 9/10 Class)

Robin is a Mathematics 9 student I have taught for four years. He is an engaged student who enjoys sharing his ideas and understanding verbally. Robin seems to prefer to think and problem solve mentally before writing due to physical challenges with writing. His work on assessments is often to the point and sometimes simplifies or skips steps, perhaps for the sake to limit the need for written output. Nevertheless, Robin is a logical and competent problem-solver.

Robin did not share his ideas during group debriefs analyzed. However, his individual drawings and commentary about visualization tasks indicate his high engagement and involvement in visualization even though the tasks were challenging for him.

Robin's Hanging Cube

Robin sketched the cube several times from different vantage points (Figure 4-25). He did not share his ideas during the task debrief. However, his work shows that he took time to consider this task in different ways, rotating the cube to different orientations. His sketches did not include hidden edges of the cube. Robin was one of the only students to show how three edges met at the top vertex to suggest that the cross-sectional shape might be triangular (in the bottom right sketch). This was an accurate cross-section that was parallel to the floor for the hanging cube even though it was not halfway between top and bottom vertices of the cube.

Robin developed his own spatial reasoning through rotating the cube to different orientations and making multiple conjectures about how to position the cross-sectional cut. He did not indicate a final cross-sectional shape in his drawing.

Robin's Tetrahedron Task

Robin sketched both two and three-dimensional drawings to respond to the *tetrahedron task* (Figure 4-26). He began by drawing four separate faces, noting his familiarity with this type of shape.

> Robin: I know this because that would be the same thing to [a] four-sided die.

With some time to consider the questions posed, Robin was able to articulate that the tetrahedron possessed four faces, four vertices and six edges (Figure 4-26a).

Robin included three-dimensional drawings without hidden edges to show the shape of the smaller piece and faces of the larger piece after once the tetrahedron was cut with a horizontal cross-section (Figure 4-26b). His verbal response was clearly articulated and justified.

> Robin: [Begins to sketch] So its smaller piece would be [pause]… The smaller piece would be like a pyramid. And the larger piece would be kind of like a triangular trapezoid?

> Ms. H.: Do you know what shapes would be on the faces of the larger piece?

> Robin: Would that also be a triangle? Yeah, that would be like, um, a triangle as well, like on the side that it was cut, yeah. Yes. Yes, it would have to be. And on the, the side that you cut it from that would be the same angle, that would be a trapezoid.

> Ms. H.: Ok, um, are there, are there other faces?

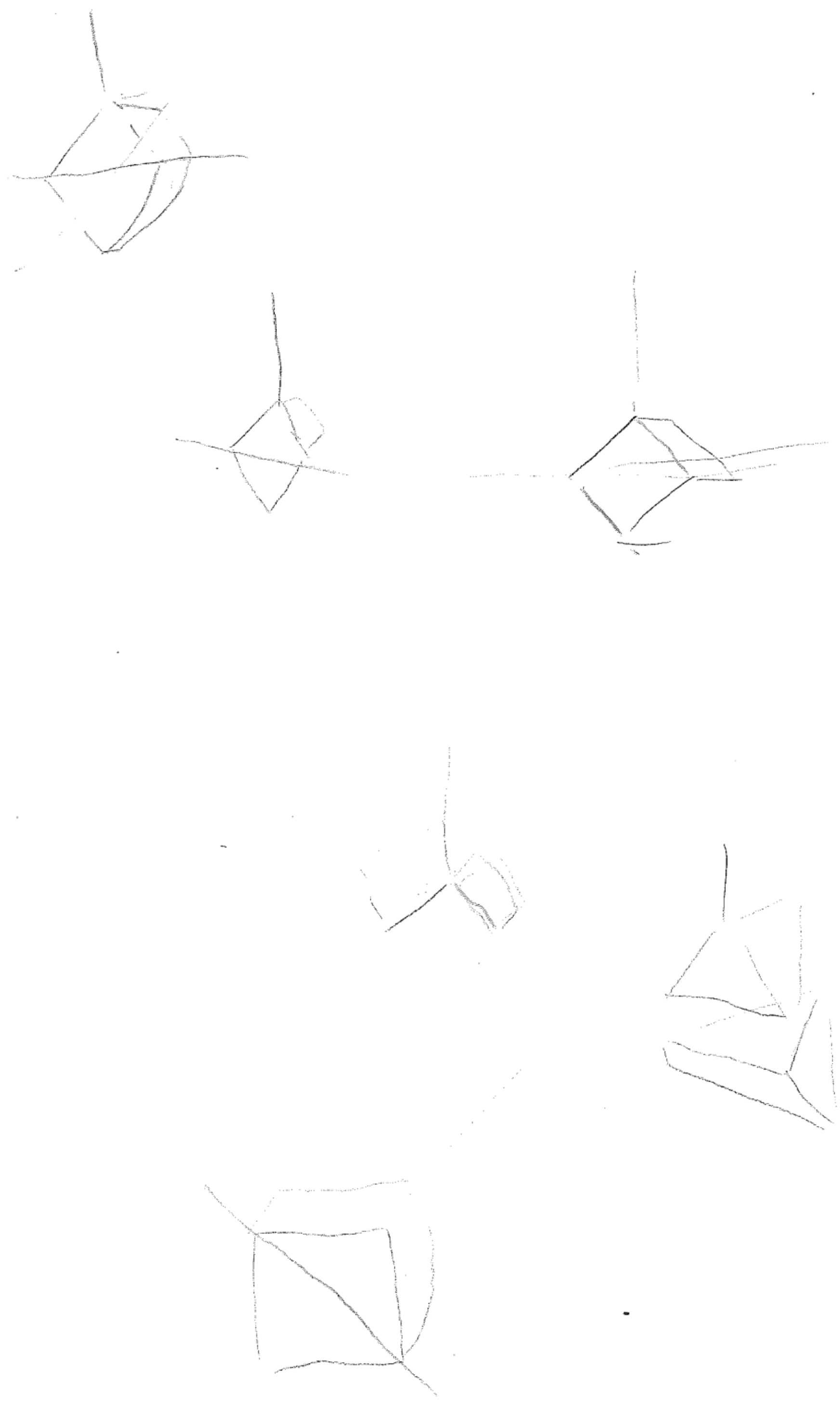


Figure 4-25 Robin's Hanging Cube
The hanging cube was sketched multiple times from different vantage points. Hidden edges are not shown.

Robin: Uh, the, there would be another one, um… There's an, there would be another one, two other ones that would both be quadrilaterals. And another one would be a triangle.

Ms. H.: Ok. You said it would have to be triangle. How do you know that?

Robin: Because if this side is a triangle and it started out like being a triangular shape, then that must mean that this one would also be a triangle for the shape to be not deformed.

The final task was challenging for Robin. He considered the task by focusing on one face connected to the balanced edge (Figure 4-26c). He articulated his response that the cross-section would be triangular before drawing the front face.

Robin: Is formed by it? Ok. That shape formed would be formed would be like a small triangle that matches the distance between the… yeah it would… no, yes… it would. Yes.

Ms. H.: Do you want to try to draw? Or it's really up to you.

Robin: Yeah, it's really… I can try. [Draws] Because the shape cut out would be connected at three points which would be connected to the shape that also connects at three points.

Ms. H.: Can you describe what's what [in your sketch]?

Robin: So this is like the front side. This is the, uh, the side that would be visible that's drawn that would have to be like…. And this is the line um, that's halfway between the bottom half and the top, so.

Robin's sketch accurately oriented the face in view to position the edge on which it would balance on the floor as well as the horizontal cross-section. However, it did not include information about the other three faces and edges that would make up the full tetrahedron. Robin's work demonstrated understanding about the orientation of both the figure and of the cross-section. Including of other edges of the tetrahedron might have helped him develop awareness that the cross-section cut through four faces rather than three.

Robin's work with three-dimensional shapes in the cross-section tasks demonstrated good awareness of properties of these shapes as he rotated the figures to consider different orientations. Awareness of all hidden edges of both the cube and the tetrahedron might have allowed him to develop his spatial reasoning even further.

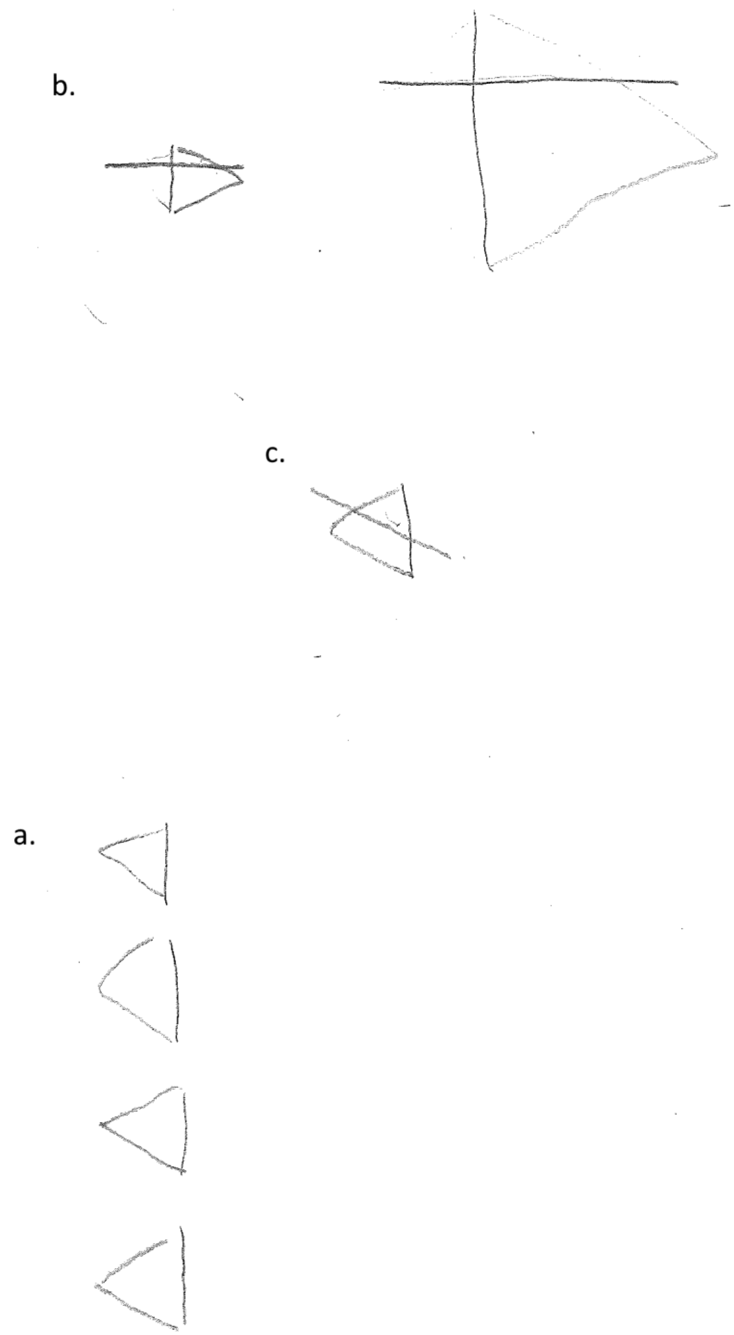


Figure 4-26 Robin's Tetrahedron
a. Sketch of four faces of a tetrahedron, drawn separately; b. Sketch of the cross-section of a tetrahedron resting on a face; c. Sketch of the cross-section of a tetrahedron resting on an edge. Only one face is shown.

Robin's Parabola Locus

Robin sketched the parabola locus with a similar understanding to Alex and Cameron (Figure 4-27). The point P was positioned within each circle's area rather than on their circumference. Robin did not include central points of each circle, nor the locus generated. His circles were not especially circular, nor were they necessarily drawn to be elliptical. Their shape may have reflected his own challenges with control in drawing.

In the follow-up interview, Robin noted that this task was the one he found to be most challenging. He stated that "I had trouble to maybe understand what exactly I was supposed to represent with what was being said." Robin was able to consider multiple parameters (tangency and his own definition of 'passing through a point') to show movement of the circle. However, this in and of itself was a lot of information to keep track of and Robin did not include the locus in this task.

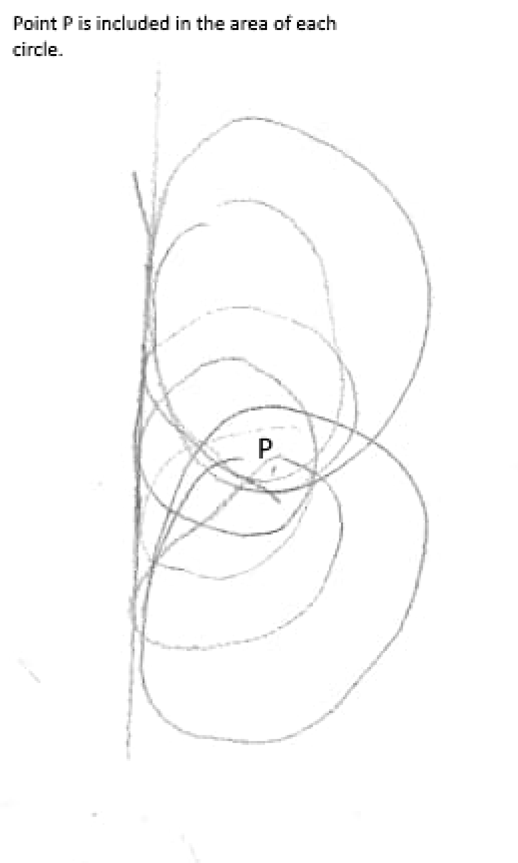


Figure 4-27 Robin's Parabola Locus
Sketch includes Point P in the area of every circle. No locus is shown.

Robin's Ellipse Locus

When solving the *ellipse locus task*, Robin considered the option of the point P being located on the initial circle's circumference. This adhered to the given criteria of the task. For further students who participated in the follow-up interview, I suggested they attempt the task by considering a point not on the circumference if this solution was given. Because Robin was the first student interviewed, I had not anticipated this occurrence and did not ask him to consider further solutions. This impacted his solution to the task significantly.

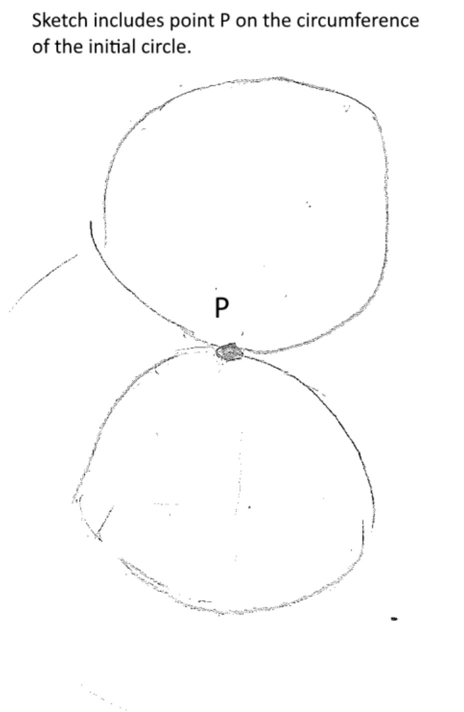


Figure 4-28 Robin's Ellipse Locus
Point P was positioned on the circumference of the initial circle.

Once Robin had positioned point P on the initial circle's circumference, he was, as might be expected, challenged to find a solution that would meet all stated constraints in the task. He asked questions to clarify his understanding of the task.

Robin: How, how would it be the inner circle if it's moving? 'cause wouldn't you consider the one not moving to be the inner circle if there's a path that you drew around it. I'm confused.

Ms. H.: Uh, can you ask me the question one more time?

Robin: So if you say the inner circle is supposed to be the one moving, wouldn't you consider the one not moving to be the inner circle because it's the one that's not being surround, so it doesn't really have a path around it?

To meet the criteria understood, Robin suggested that point P would necessarily move along with the moving circle to fulfill the description offered.

Robin: [Erases and pauses for some time] Uh, the dot [point P] would have to be [pauses] – I don't understand. I feel like with what I have the dot would have to move to always be touching the circle.

Ms. H.: Uh, where, like, where would it be?

Robin: It would just be a path just around the other circle. Is that not? Is that what I'm supposed to do?

Ms. H.: Uh, how did you [inaudible].

Robin: As to what the dot would be on the outside of this circle. As this circle moves, it would follow that path and move right along this circle.

Ms. H.: I just can't see [moves over to observe DF's sketch] So, I guess, where would the point [P] be moving?

Robin: The point [P] would be moving along this circle.

Ms. H.: Oh. Would, would one of the circles be moving?

Robin: Yeah, well the circle would have to move so that, that each point would look like this, with one point touching the sides.

Ms. H.: Oh, is, is this circle sliding around the other one?

Robin: No, it's rolling.

Ms. H.: Rolling?

Robin: This dot would be connected to this circle and [it] would move to […] all over it almost.

Ms. H.: Ok. Ok. Until it's back the circle?

Robin: Yeah. That way the trace would be the same as the circle.

His description of a rolling circle may have been based on the vocabulary choice offered in the problem of the circle "rolling." By altering the convention of the point being fixed to a set location, Robin was able to create a scenario that adhered to many constraints as he interpreted them, with exception to the information about an inner rolling circle. Although not the intention of the task, his work to find a visual that would adhere to the task criteria as he interpreted these was evidence of developing spatial reasoning for Robin.

Robin's experience led me to consider how to approach the situation where the point was placed on the initial circle by asking students to consider the case where the point was not placed on the circumference. However, I did not pause to consider how the word "rolling" might have affected student interpretation of the task. A more neutral word like "moving" may have been less misleading to Robin and to other students.

Robin's Experience

Robin articulated that visualization is a powerful tool for comprehension in mathematics.

> Uh, I think being able to visualize things can be quite powerful for comprehension and understanding of different concepts in math at least we don't usually get to do a lot before. But, I think they can sometimes make it easier to understand things.… Uh, I really am a visual learner, so maybe I can visualize things make it easier to understand the curriculum.… [It is powerful] because I think that people apply something they usually don't use, like, more about our senses would be more powerful than other, sort of to visualize to represent more than that just looking at what you're shown.

He added that collaborating and learning about his peer's ideas was valuable beyond the work done on visualization tasks.

> I think people being able to see that from another person's perspective it can help to enhance your own sort of like.

Besides influencing his work to solve the tasks, Robin appreciated the opportunity to see a different person's perspective.

4.2. Overview of Results

Data collected included drawings of seven students who participated in classroom tasks as well as individual follow-up interviews and interview tasks. Student drawings of the *tetrahedron task* were compared with the *hanging cube task* and drawings of the *ellipse locus task* were compared with those of the *parabola locus task* as shown in this chapter. Besides comparisons made between individual student drawings between analogous tasks, comparisons between drawings made by different students in response to each task were made. Through comparisons made both between students and between tasks, I found that students developed drawing strategies such as including hidden edges and rotating shapes in incremental steps. They also developed the skill of drawing a locus.

Classroom tasks debriefs for the *hanging cube* and *parabola locus tasks* were transcribed and analysed to find areas where rich discussion took place between students. Individual interviews were also transcribed. Through their discourse, the students were able to make conjectures and arguments about properties of shapes. They developed awareness of definitions, conventions, geometric properties of shapes and multiple constraints of a task. They demonstrated the ability to generalize, an essential aspect of mathematics and mathematical thinking. The geometric and mathematical learning that took place will be the focus of chapter 5.

Students who partook in visualization tasks described the experience as one that elicited thinking and understanding and that this experience led them to become aware of their own thinking as well as their peers' thinking. This related to their own thinking and even enhanced their own perspective. They demonstrated genuine enjoyment in their visualization experience as it afforded them the opportunity to express their own mathematical agency. Chapter 6 will focus on these aspects of student experience.

Discussion will take place over two chapters because I consider student awareness of geometric and mathematical properties to be fundamentally different from student experience. Through the visualization tasks, students gained awareness of geometry in particular and mathematics in general. Student description of their experience with visualization tasks showed their developing awareness of their own and of peer thinking.

Chapter 5.

Awareness of Geometric and Mathematical Properties through Drawing and Discourse

My analysis will take place through two chapters. In chapter 5, I will address four research questions that pertain to student awareness of geometric and mathematical properties that arose through drawing and discourse.

1. What types of drawing strategies do individual students develop when they engage in visualization tasks?

2. How does discourse support student awareness of geometric properties when they engage in visualization tasks?

3. What geometric properties do students develop awareness of through their (a) drawing activity and (b) discourse over time when they engage in visualization tasks?

4. What mathematical skills or competencies do students develop through their (a) drawing activity and (b) discourse over time when they engage in visualization tasks?

I leave the final research question to chapter 6 because I consider the discussion of student experience to be distinct from the geometric and mathematical learning focused on in the forthcoming chapter. While data supporting geometric and mathematical learning arose from student drawings and transcripts that took place during classroom and interview tasks, data regarding student experience arose from follow-up interviews with students

5.1. Drawing Strategies

Figuring, drawing what is seen, is one of Tahta's (1989) three abilities said to be employed when students engage in spatial reasoning. When using this power as a response to visualization tasks, students were required to access their own mental imagery (or *imagining*, seeing what is said). Drawing of diagrams, for this reason, develops student spatial reasoning (Wheatley & Reynolds, 1999).

Sinclair et al. (2019) suggest “the very act of drawing […] changes the way students see (spatially reason about) geometric images” (p. 250). A two-dimensional diagram can allow students to depict a three-dimensional image or to represent motion. However, therein lies the challenge: a three-dimensional situation or a moving situation requires strategies to be drawn in two dimensions. The types of drawing strategies utilized to address the challenges noted above may impact how students are able to solve the visualization tasks encountered. Conversely, the types of tasks encountered might influence the types of drawing strategies that arise.

By examining student drawings and their activity and discussion that took place during the lessons and interviews, I wanted to determine what kinds of drawing strategies emerged for students who engaged with the visualization tasks. I found that students developed the strategies of including hidden edges of 3D shapes, depicting multiple views of rotating shapes, and the skill of applying the concept of locus to their drawings.

5.1.1. Drawing Hidden Edges of 3D Shapes

Working through visualization tasks involving cross-sections of three-dimensional shapes resulted in student increased drawing of hidden edges and vertices that might be invisible from the front of a 3D structure with transparent faces. In the debrief of the *hanging cube task* with Mathematics 9/10, Dylan, Alex and Avery presented sketches of cubes without inclusion of hidden edges. Through the debrief discussion, both Dylan and Avery were able to add hidden vertices and/or edges and demonstrate their own new awareness of these spatial features.

During a discussion about mental imagery, a given student’s attention can be shifted by a remark made by the teacher so that she becomes “more explicitly aware of some features and less aware of others” (Mason, 1998, p. 254). During the debrief, I explicitly drew Dylan’s attention to the hidden edges of her drawing (Figure 5-1) in order to direct her awareness toward hidden edges and vertices of the figure. I did so because the information about the location of vertices and edges on all sides of the cube was necessary to determining its horizontal cross-section.

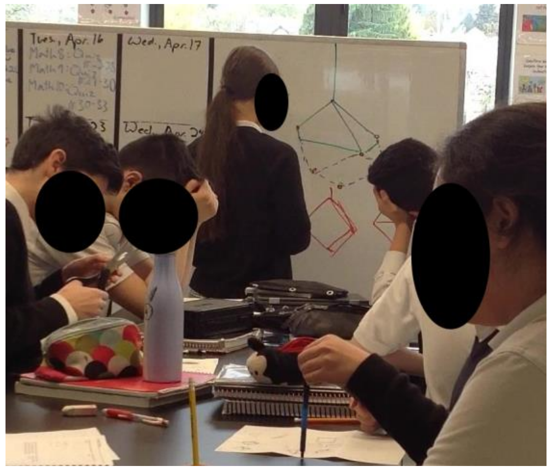


Figure 5-1 Dylan's Hanging Cube

> Ms. H.: Ok. So, now, is there like a way you could add the next vertex at the back?

My question prompted a change in Dylan's drawing strategy. She was able to include the hidden edges of the cube accurately in orange marker (Figure 5-2) showing that the structure of her attention had shifted toward features on the cube's hidden face. Dylan was able to demonstrate her own agency by extended the cross-section on her work without prompting. She took "established ideas and extend[ed] them" (Boaler, 2002, p. 45) through her drawing. She was able to extend the cross-section outline across a hidden face of the sketched cube (Figure 5-3).

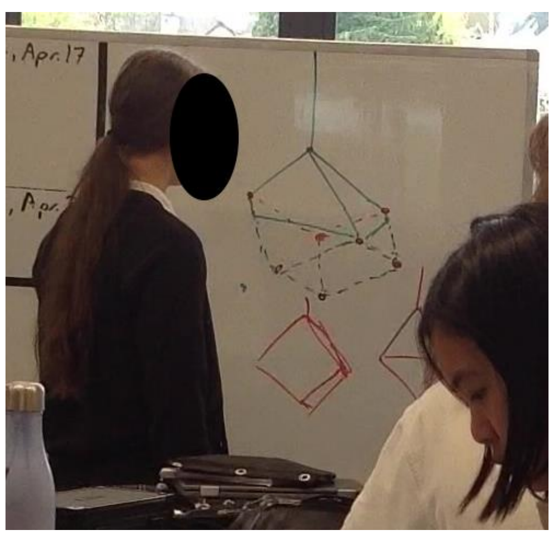


Figure 5-2 Dylan Adds Hidden Edges to her Hanging Cube

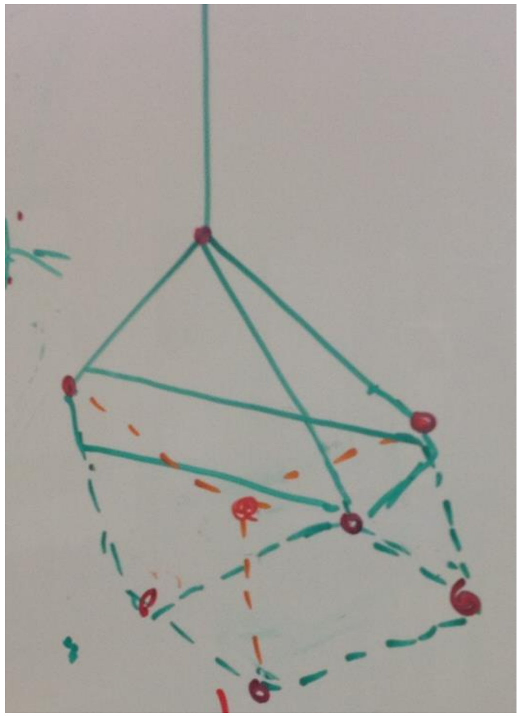


Figure 5-3 Dylan's Final Cube

Avery's proposed drawing did not include hidden edges initially (Figure 5-4).

Avery: I think it's hard to picture, but I made mine…

Ms. H.: Um – I have - My questions is: do you - can you point out how many vertices are there in the square? Four? Five? Six? Seven? Like that?

Avery: Yeah, like… [Avery marks vertices in red (Figure 5-4)].

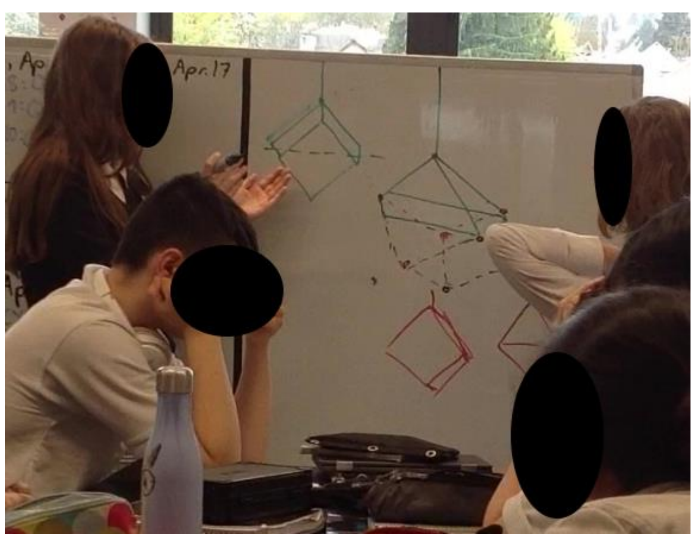


Figure 5-4 Avery's Hanging Cube

Avery was able to recognize the position of an eighth vertex behind the cube. She also recognized that the cross-sectional cut would extend behind the cube. Locating the hidden edges on the back of the cube might have helped Avery more accurately position the cut. Nevertheless, her attention was drawn to the back of the cube and she was able to begin to consider these relevant spatial features.

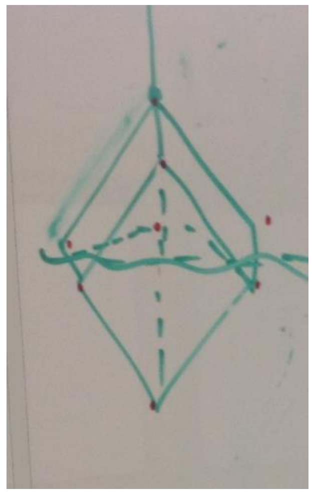


Figure 5-5 Avery's Final Cube Drawing

Teacher intervention allowed both Dylan and Avery to shift their attention towards the hidden edges and vertices of cube. Both students were as a result able to think flexibly and to alter their drawings. Through our discussion, the students were able to internalize further salient spatial features of the hanging cube including the number of vertices and edges of the cube in addition to their position. A remark by the teacher altered student awareness in this case. Mason (1998) describes that there are different degrees to our attention and someone’s remark can change where attention is most focused. This awareness was extended through their own drawing.

Hegarty and Kozhevnikov (1999) suggest that spatial visualizers “generate and process images analytically, part by part” (p. 723). Directing student attention to hidden edges and vertices supported their analytic understanding and they were therefore able to extend their spatial understanding of the cross-section task. Their work to draw the hidden edges both showed this development and supported it.

Alex had included some sketches showing hidden edges and others not including these in her work on solving the *hanging cube task* (Figure 4-1). However, all her sketches of the tetrahedron included every hidden edge. These features supported her solution to the final cross-section and determination of its quadrilateral identity (Figure 4-2). Robin did not include hidden edges in his sketches and did not accurately solve the final task (Figure 4-26).

Alex's awareness of the hidden edges of the structure allowed her to identify this cross-section as a quadrilateral in her drawing and during our debrief, demonstrating her growth. During our debrief, she pointed to each of the four sides of the cross-section depicted, explaining how these were selected based on four faces encountered, demonstrating her *awareness-in-discipline* (1998). She was aware of hidden faces of the tetrahedron based on her drawing.

Ms. H.: Uh, which one, which face is the cross-section?

Alex: Uh, this one.

Ms. H.: Um, how many sides does it have?

Alex: Um, four?

Ms. H.: How do you know?

Alex: Um, because I believe it has at least one at the first face, and after it would be here and here and after it will go back [traces each side on the drawing as this is described].

Ms. H.: Oh, ok. And I'm just trying to see the length between this drawn …Um so it has four sides. Is it, what kind of quadrilateral?

Alex: A weird rectangle? I tried to draw it here.

Translating between two dimensions and three dimensions is challenging for students and requires exposure to learned conventions (Francis & Whiteley, 2015). Inclusion of hidden edges is a convention that students may have had experience with. However, explicit instruction of these conventions has not been included in my teaching practice. Nevertheless, students in Mathematics 8 tended to include hidden edges in their work both with the cube and with the tetrahedron suggesting this was part of their experience prior to engaging with the visualization tasks.

Devon and Jordan who were able to draw accurate acute isosceles triangular cross-sections for the final task based on their interpretation of the task both included hidden edges in their sketches (Figure 4-12 and Figure 4-16). To draw these hidden edges, students needed to attend to the position and length of edges on all parts of the shape, not only those in the front-facing view. The hidden edges on their drawing provided more spatial information about the cross-section and allowed them to sketch all sides of their conjectured sections.

Interestingly, Devon and Jordan selected the same non-horizontal cross-section that passes through one vertex and two edges of the tetrahedron rather than the expected horizontal square cross-section. Their attention on these spatial features of the tetrahedron may have influenced their solution. A further teacher prompt might have involved reminding them about the need for the cross-section to be horizontal to the table on which the tetrahedron edge was resting. This might allow these students to focus on one further parameter of the task and to further develop their analytic and spatial understanding (Hegarty & Kozhevnikov, 1999).

5.1.2. Drawing Rotating Shapes

Engagement in the visualization tasks provided students with the opportunity to rotate shapes through different orientations. Mental rotation can be considered a multi-step spatial visualization task and is used in numerous studies to measure spatial reasoning (Casey et al., 2017; Lin & Chen, 2016; Linn & Petersen, 1985).

Alex and Devon both demonstrated the ability to mentally rotate the tetrahedron from resting on one face to resting on an edge based on their three-dimensional drawings (Figure 4-2, Figure 4-12). These students both drew the tetrahedron at more than one position and showed how rotation occurred through their drawing. Alex drew the tetrahedron resting on a face, resting on its edge as well as a side view of the tetrahedron resting on its edge. Devon drew the tetrahedron resting on its face and used an arrow to indicate motion involved as well as a second tetrahedron resting on its edge. Both students independently applied a similar notation to identify and followed a specific edge by 'bolding' it in drawings of the tetrahedron at different positions, tracking how the shape rotated. This strategy was implemented without any teacher guidance or prompting from a peer as it took place during individual follow-up interviews.

Alex described her strategy during the debrief interview (Figure 4-2):

Alex: Yeah, I tried to like, bold the sides [AB, BC and CD].

Ms. H.: So this [bold CD] side is moving to here [C'D']?

Alex: Yeah. Yeah.

Ms. H.: Ok. So you started with it in the position from the – exactly like before [ABCD] and then you're trying to…

Alex: Redraw it to [be balanced on] like the edge [B′C′]. Yeah.

Ms. H.: Um. I see. So you kind of like, rotated it.

Alex: Yeah so like it turned like this way, [counterclockwise]. And after [A′B′C′D′] I tried to draw it from like the side perspective [A″C″D″] which was... Similar to..

Ms. H.: Oh. Is that [A″C″D″] like looking from this side?

Alex: Yeah. And, yes, um, I'm not really sure what I imagined [inaudible questions]. Hmmm... [Draws more]

Tracking the position of individual edges was an effective analytic drawing strategy that led to Alex's developed awareness of the relationship and position of the different edges of the tetrahedron such that she was able to determine its cross-section. Her drawing allowed her to capture and attend to important features that arose in her mental imagery.

Jordan added the bold notation to his drawing after he had drawn the rotated tetrahedron as a means of clarifying which edge was resting on a surface when I asked him about this during the debrief interview.

Robin was able to rotate a single triangular face including the edge on which the tetrahedron would rest, only drawing the front face of the figure and the appropriate horizontal cross-sectional cut (Figure 4-26). Leslie and Jess were still developing their understanding of the structural identity of the tetrahedron (Figure 4-6 and Figure 4-23), therefore their ability to rotate this specific shape cannot be commented upon.

5.1.3. Drawing Locus

The concept of locus is not one my students otherwise encountered in Mathematics 8, 9 or 10. The *parabola locus task* was the first exposure students had to locus in my classes. A number of students (Leslie, Jess, Cameron and Robin) did not draw the path traced by the moving point in their initial sketches (Figure 4-7, Figure 4-9, Figure 4-20 and Figure 4-27). Leslie and Jess were able to discover the locus of the parabola through collaboration with Devon (Figure 4-9) and then applied this understanding in sketching the locus of the ellipse individually.

During the debrief to the *parabola locus task,* I prompted Jess to add the locus to her original drawing (Figure 4-9).

Ms. H.: There was another question um, another question I wanted to ask – do you remember at the end what I asked you?

Jess: A half circle – look at this! [Jess begins to draw the locus based on circles' centre locations on her drawing]

Leslie: The middle of the circle does it go through?

Ms. H.: So, basically trace the whole line… [Jess traces the locus on her drawing.] Uhhh.

Jess: That's weird.

Devon: It's gorgeous.

Jess extended her own sketch to include the locus. She also extended the sketch to include additional larger circles (D and E) and the locus continuing through their centres as justification to questions posed by Leslie about how the circle could continue to move to the right while maintaining tangency to the line, passing through point P and remaining circular in shape.

Leslie: Yeah, but then if we make [the circle] bigger, then it won't be equal on both sides.

Ms. H.: Yeah, uh…

Jess: Yeah.

Ms. H.: Well, what would…

Jess: Both circles equal both sides.

Leslie: It's gonna get bigger and bigger on this side [away from the line] but it can't get bigger on that side [close to the line].

Jess: Then it's an oval…

Ms. H.: Oh.

Leslie: So it's not a circle anymore.

Leslie observed actively as Jess was able to extend her drawing by adding two additional larger circles (D and E) to the right of the first three she had drawn and extending the locus through them (Figure 4-9). Although Jess had agreed that the circle would become an oval in the discussion, she was able to use drawing to both discover and show how it could remain circular.

Ms. H.: Oh yeah. So if you kept moving this circle, like, to the side [gestures to the right] and it got bigger and bigger and bigger where would this path move to? Do you want to add? [moves paper toward Jess]

Leslie: Continue. Please.

Ms. H.: Ok. [Nods]

Jess: Yeah. [Jess extends the locus through the two additional circles after indicating their centres first]

The shared diagram allowed for the development of a concept of locus. Jess extended the drawing to provide justification to Leslie's questions and Leslie suggested that Jess "continue" implying that seeing the drawing was helpful to her own understanding of the task.

Jess was able to transfer her newfound understanding of locus to her drawing of the *ellipse locus task* (Figure 4-24). She was prompted to include the locus as part of the solution to the task here as well.

Jess: Sure. So I have like, I would bring the circle and then the dot here or something like that. And then the smaller circle, make it a bit bigger and it's touching here.

Ms. H.: So it if was in this situation, I'm just curious to know what path would the centre trace?

Jess: [Draws]. [Inaudible] Like [the locus only traces] halfway because then it would go over the um the circumference of the circle.

Jess ended the path of her locus to allow the inner circle to remain tangent to the outer circle, remaining conscious of this criterion.

Participation in the *parabola locus task* debrief allowed Leslie to develop her own conception of locus which she was able to apply to her drawing of the Ellipse Locus (Figure 4-10) when prompted.

Ms. H.: Do you know what path the centre of the circle would trace?

Leslie: Of like here [draws] – it sort of [would] make a mini circle inside.

For Jess and for Leslie, drawing the locus was an additional step after including all other geometric components in the task, the circles and the point. Teacher prompting allowed them to add the locus to their drawing. They were able to do so successfully without redirection or help once this prompt occurred.

Devon did not include a locus in her *ellipse locus task* drawing (Figure 4-14). When asked to show the path the centre of the circle traced, she pointed to the circumference of the largest circle. Therefore, her understanding of locus was still developing.

Cameron also was able to work with peers to apply her understanding of locus to her later sketches of the parabola and also of the ellipse (Figure 4-20 and Figure 4-21). Robin did not include the locus in the *parabola locus task* (Figure 4-27) but was able to describe a possible locus for the *ellipse locus task* as the path around the initial circle.

Robin: As to what the dot would be on the outside of this circle. As this circle moves, it would follow that path and move right along this circle.

Ms. H.: I just can't see [moves over to observe Robin's sketch] So, I guess, where would the point be moving?

Robin: The point would be moving along this circle.

Alex was able to draw the locus for both *parabola locus* and *ellipse locus tasks* (Figure 4-3 and Figure 4-4) and Jordan, although absent for the *parabola locus task* was able to draw a locus for the *ellipse locus task* (Figure 4-17). His proposed loci showed motion in the task in two ways: the first with an arrowhead to suggest the direction of movement of the inner circle, the second as a dotted line indicating the trajectory of moving circles.

Interaction with the concept of locus through the *imagine-draw-talk* routine allowed many students to develop their understanding of locus through drawing and to apply it in further tasks.

5.2. Student Discourse and Agency

Construing, saying what is seen, is another one of Tahta's (1989) three abilities said to be employed when students spatially reason. When applying this power as a response to visualization tasks, students accessed either their own mental imagery (or *imagining*, seeing what is said) or their drawings (or *figuring*, drawing what is seen).

Different students imagined and drew different diagrams in response to the same visualization tasks. Therefore, their discourse about their distinct conjectures naturally led to argument about which solution was correct. In the British Columbia curriculum, students are expected to "explore […] and test [these sorts of] conjectures" (British Columbia Ministry of Education, 2016). By proposing and testing their own conjectures together, the locus of authority shifted from the teacher to the students during the task debriefs (Martinez & Ramirez, 2018). Discourse about visualization tasks provided a means of allowing for student mathematical agency to emerge.

5.2.1. Conjecture and Argument in the Hanging Cube Task

Students developed conjectures about visualization tasks which they shared during task debriefs. This occurred during the *hanging cube task* discussion in the Mathematics 8 class when the students discussed whether the cross-section Jordan had sketched was a rectangle or a square. All five students who participated in the task engaged verbally in the discussion.

Leslie: Oh, is it a rectangle?

Jess: Square.

Devon: No, it makes a square.

Leslie: No, it's a rectangle.

Ms. H.: 'cause I guess, so, this, this length…

Devon: Nah, it makes a square.

Leslie: It's a rectangle!

Blake: Woaaahh!

Ms. H.: This length [points to the diagonal] is it the same as this length [points to the cube's side length]?

Blake: Ohhh, nooo.

Jess: So we were riiiightt!!

Leslie: It's a rectangle [nodding].

Devon: No, it is the same as that point.

Leslie: No it's not.

Devon and Leslie provided justification for their conjectures after teacher-prompting.

Ms. H.: How do you know that they're the same?

Devon: Because all the faces have the same width.

Leslie: Yeah, but it's in the middle so it's not using any of the faces.

Jess: So it's like the diagonal.

Ms. H.: So, this is a face, but…

Jordan: Oh my god! [holds face]

Devon: Oh, naw… [erases square and changes it to rectangle on her paper]

Jordan: Oh, I'm sorry, this is a rectangle because like you know, this illustrated line is higher than, this this line, so -

Ms. H.: Oh yeah - So like, these lengths are the same, but when I have these two to make the triangle, the hypotenuse is longer.

Jordan: 'cause like, like… these are longer than the straight line so it's like this one… there….

This conversation allowed Devon, Jordan and Jess to come to the conclusion that the cross-section was rectangular and for all the students to consider *why* it was rectangular. Teacher intervention prompted students both to conjecture and also to justify. Students seemed to be highly involved in considering what was true and cared about the mathematical arguments taking place between one another, speaking loudly, quickly and interrupting one another to share their thoughts and ideas.

5.2.2. Conjecture and Argument in the Parabola Locus Task

Discussion also took place during the *parabola locus task*. As the students worked to extend the movement of the circles begun in Jess's sketch, Leslie inquired how the circle could move away from the point while remaining tangent to the line.

Ms. H.: So, I guess what would happen if the circle keeps moving this way [gestures on paper to the right]?

Leslie: How do you move it that way though?

Jess: You just make a big [gestures with one hand to indicate a big circle].

Ms. H.: I guess we could extend the line.

Leslie: Yeah, but then if we make it bigger, then it won't be equal on both sides.

Ms. H.: Yeah, uh…

Jess: Yeah.

Ms. H.: Well, what would…

Jess: Both circles equal both sides.

Leslie: It's gonna get bigger and bigger on this side [gestures away from the line] but it can't get bigger on that side [gestures toward the line].

Jess: Then it's an oval…

Ms. H.: Oh.

Leslie: So it's not a circle anymore. [Discussion continues. Ms. H. walks away to draw on board]

Jess: [Draws circles D and E while discussion between other students proceeds]

Ms. H.: Oh yeah. So if you kept moving this circle, like, to the side [gestures to the right] and it got bigger and bigger and bigger where this path [go]? Do you want to add? [moves paper toward Jess]

Leslie: Continue. Please.

Ms. H.: Ok. [Nods]

Jess: Yeah. [Jess begins to sketch the locus through circles D and E]

The justification provided for Leslie took place through Jess drawing and extending the drawing by addition two additional larger circles and then sketching the locus as previously described in 5.3. Leslie was able to see how the circle could change sizes to meet the criteria required in the tasks.

Secondary students show a preference for mathematics learning environments where they can “create[e] initial thought and ideas or [take] established ideas and extend them” (Boaler, 2002). The *imagine* and *draw* portions of the *imagine-draw-talk* provided students with the opportunity to create their own initial thoughts and ideas. The *talk* portion provided them with the opportunity to extend others’ ideas verbally but also through drawing during the discussion as exemplified by the discussion between Leslie and Jess and Jess’ extended drawing.

Student discourse during the visualization task routine further promoted their own agency (Bell & Pape, 2012). In both examples presented, students authored their own conjectures and developed arguments to support or refute these during discussion. The students therefore fully developed their sense of being authors of mathematical ideas through the tasks (Martinez & Ramirez, 2018).

5.3. Awareness of Geometric and Mathematical Properties

After engaging in tasks involving mental imagery, Gattegno (1965) suggested that students “talked about [distinctive imagined features] that sounded strangely like the geometrical statements we read in books as theorems” (p. 22). Engagement with visualization tasks through drawing and discourse could allow students to develop awareness of geometric properties.

For Mason (2002a), working with mental imagery is the start of generalization, the core of mathematical thinking. Wheeler (2001) further proposes that imagery itself may mediate the development of awareness.

This section seeks to explore awareness of geometric as well as mathematics properties that emerged for students through engagement in the *imagine-draw-talk* routine. Students were led to understand new definitions and geometric conventions, to become aware of properties of 3D shapes, to adhere to multiple constraints of a task and to generalize.

5.3.1. Definitions

In the British Columbia curriculum, students are expected to “use mathematical vocabulary and language to contribute to mathematical discussion (British Columbia Ministry of Education, 2016).” Visualization tasks prompted students to learn and apply new vocabulary and language.

Students independently inquired about the definitions of terms they did not understand in the scripts of several visualization tasks. When students posed these questions about definition, I chose to answer them because this information was required to complete the task.

Tangency was a concept new to students that was included in tasks 3, 10 and 13. During the in-class tasks as well as in many interviews, many students requested that I provide a definition for “tangent.” This is a concept far beyond expected Mathematics 8, 9 and 10 curricular requirements and is only explicitly part of Geometry 12 in British Columbia curricula as well as Calculus 12.

Alex was able to explain how her altered understanding of tangency allowed her to change her drawing in the *ellipse locus task* (Figure 4-4).

Alex: Um, so at first, I was thinking about the circle like the point and after I guess didn’t know what tangent mean, I thought tangent like meant intersecting…

Ms. H.: Yeah…

Alex: … so that’s why I drew that and after you explained that doesn’t. So it might be the inside of it so… this and after it was like tangent to the circle [draws] yeah. And if it would rotate around I guess it could like get bigger like that and like after smaller but it always touches the point - the circumference of the first circle.

Although Alex established an elliptical shape of the locus, she did not recall the name of this shape.

Alex: So I guess it would make an oval shape?

Ms. H.: Oh yeah. Uh, do you remember the math word for it?

Alex: Oh! I remember it! Oh I forgot!

Ms. H.: It's an – I can help you. It's an ellipse.

Alex: Oh, ok yeah. I remember now.

Alex's personal development of the locus prior to learning its name perhaps may have provided her with a stronger connection to the term ellipse.

The final interview *tetrahedron task* introduced participants to the definition of a tetrahedron. The description of a tetrahedron that I provided to students was a "triangle-based pyramid." If this did not seem sufficient, I briefly showed the interviewee a physical tetrahedron model.

Leslie understood triangle-based pyramid to signify a square-based pyramid with a triangular prism fixed to its base (Figure 4-6). This impacted how she was able to interact with the task.

Jess initially responded to the *tetrahedron task* with understanding of the tetrahedron's structure. However, in the second and third part of the task, she altered its structure to a triangular prism (Figure 4-23).

Jordan was unsure about the definition of a tetrahedron. After it was described and a physical model was shown to him, he was able to work with this structure in all parts of the task (Figure 4-16). He was able to change his initial square-based pyramid drawing to a triangular pyramid without prompting based on his new understanding of this definition.

5.3.2. Geometric Conventions

As with definitions, awareness of geometric conventions was required for students to solve each task to give the expected solution. Students unaware of certain conventions were able to learn about these and sometimes apply them to later tasks. Conventions encountered included the convention of a line or circle passing through a point and the convention of fixed points.

In the *parabola locus task*, Cameron, Alex and Robin's drawings demonstrated a similar understanding of how a circle could pass through a point. Their drawings (Figure 4-20 and Figure 4-3) both show circle areas including the point rather than their circumferences passing through it. This shared understanding may have occurred based on their sitting near one another during the lesson. Both Cameron and Alex were able to gain awareness of the convention of how a circle can pass through a point and incorporate this into the *ellipse locus task* (Figure 4-21 and Figure 4-4) in their respective follow-up interviews.

Devon was uncertain about whether the point in the *parabola locus task* moved with the circle that was being transformed or remained fixed (Figure 4-13).

> And, um, I sort of imagined it said like draw the path in the middle of the circle. So the circle moves all the way around the line but it stays, it stays matching the line at all times and there's my point. [gestures to diagram] I wasn't sure, does the point move with the circle or not?

She developed understanding about fixed points by the time she engaged with the *ellipse locus task* where she maintained a fixed point inside the outer circle through which the inner circle always passed (Figure 4-14).

5.3.3. Geometric Properties of 3D Shapes

Diagonals of Cube Faces are Longer than Cube Edges

Engagement with visualization tasks allowed students to develop awareness of properties of three-dimensional shapes. Through the exploration that took place during the *hanging cube task*, Jordan, Devon, Jess and Leslie became aware that the diagonals of the faces of a cube are longer than its edges.

> Ms. H.: How do you know that [the length and the width are] the same?
>
> Devon: Because all the faces have the same width.
>
> Leslie: Yeah, but it's in the middle so it's not using any of the faces.
>
> Jess: So it's like the diagonal.
>
> Ms. H.: So, this is a face, but…
>
> Jordan: Oh my god! [holds face]
>
> Devon: Oh, naw… [erases square and changes her written 'square' to 'rectangle's on her paper]
>
> Jordan: Oh, I'm sorry, this is a rectangle because like you know, this illustrated line is higher than, this this line, so -

Ms. H.: Oh yeah - So like, these lengths are the same, but when I have these two to make the triangle, the hypotenuse is longer.

Jordan: 'cause like, like… these are longer than the straight line so it's like this one… there….

Jordan and Jess were convinced by Leslie's conjecture and justification that the drawn cross-section was, in fact, a rectangle. Both articulated their own reasoning out loud to explain why the sides of the cross-section were not all congruent. Jess explained that one side was diagonal, implying that it was longer than one of the sides of the cube. Jordan built on this statement to note that if these two sides were not equal, then the selected cross-section would be rectangular. This analytic reasoning allowed students to gain better spatial understanding of the cube, even though the cross-section being discussed was not the horizontal cross-section intended in the task.Devon signalled agreement with her peers by erasing her own work and updating her own drawing to label her cross-section sketch as a rectangle rather than as a square as she initially indicated.The group of students was able to come to a new spatial awareness together through observation of the cube and discussion: although the cube possessed equal length edges and equal area square faces, it could possess a non-square cross-section.

Tetrahedrons have an Equilateral Cross-Section Parallel to One Face

Alex, Devon, Jordan and Robin were all able to identify the triangular equilateral horizontal cross-section of the tetrahedron when it rested on a face (Figure 4-2, Figure 4-12, Figure 4-16 and Figure 4-26).

Tetrahedrons have an Acute Isosceles Cross-Section Passing through One Vertex and Two Edges

Devon and Jordan each identified an acute isosceles cross-section that passed through one vertex and two edges of the tetrahedron (Figure 4-12 and Figure 4-16). These students determined cross-sections that differed in shape from the three-dimensional shape's faces, following the discovery that occurred during the *hanging cube tasks* debrief. Jordan applied a justification that nonetheless connected his selected cross-section, a 'long' triangle (an acute isosceles triangle with a short base and two longer equal sides) to the identity of a tetrahedron having a triangular base.

Jordan: Yeah, so [leans on paper, puts head down, draws] I'm still going to go with this one.

Ms. H.: Still the triangle?

Jordan: Yeah. But it's longer than this, uh. It's like, a long triangle

Ms. H.: Longer, yeah… Can you describe?

Jordan: This one, you cut it this way, because like the angle that cut the surface is the triangle's shape. Because it always stays the triangle, 'cause there's no like, um, what's it called? The name of the shape.

Ms. H.: Oh, the tetrahedron.

Jordan: 'cause the base is a triangle, pretty much, so it will always be a triangle, like, yeah, and you think…

Jordan's understanding of the properties observed in his drawing as distinct from the faces of the tetrahedron was beginning based on his drawing despite his assertion that all tetrahedron cross-sections would be triangular.

Tetrahedrons have a Quadrilateral (Square) Cross-Section Parallel to Two Edges

Alex and Robin both selected a cross-section of the tetrahedron resting on an edge that was horizontal and parallel to the edge on which the structure rested (Figure 4-2 and Figure 4-26) demonstrating growth in their ability to accurately position a horizontal cross-section relative to the shape. Robin did not include the three faces of the tetrahedron not facing frontward and debated whether the cross-section would be triangular.

Ms. H: Oh. What shape is the cross-section? So, like once you've sliced the tetrahedron horizontally like, what shape is formed?

Robin: Is formed by it? Ok. That shape formed would be formed would be like a small triangle that matches the distance between the… yeah it would… no, yes… it would. Yes.

Ms. H: Do you want to try to draw? Or it's really up to you.

Robin: Yeah, it's really… I can try. [Draws] Because the shape cut out would be connected at three points which would be connected to the shape that also connects at three points.

Ms. H: Can you describe what's what [on your sketch]?

Robin: So this is like the front side. This is the, uh, the side that would be visible that's drawn that would have to be like…. And this is

> the line um, that's halfway between the bottom half and the top, so.

Robin referred to three points or vertices that would be part of the cross section, but only two were drawn based on his drawing. His drawing suggested that perhaps he had not internalised salient features on the back side of the tetrahedron (Piggott & Woodham, 2008). Nevertheless, Robin's understanding of the front component of the cross-section was accurate and a step toward understanding properties of tetrahedrons.

Alex's awareness of the hidden edges of the structure allowed her to identify this square cross-section as a quadrilateral in her drawing (Figure 4-2) and during our debrief. Her drawing strategies including the bold notation during rotation as well as incorporation of hidden edges in her drawing supported her discovery of this surprising cross-section.

A number of researchers (Haciomeroglu & Lavenia, 2017; Hegarty & Kozhevnikov, 1999; Kozhevnikov et al., 2005) have noted that spatial visualizers "generate and process images analytically, part by part" in contrast with object visualizers who "encode and process objects as a single perceptual unit" (p. 723). Alex's stepwise rotation, inclusion of hidden edges and stepwise sectioning point to her processing as a spatial visualizer in this task. Drawing the situation one step at a time, first rotation and later sectioning, allowed her to focus on relationships between the edges and faces that made up the tetrahedron in different orientations. She discovered that the horizontal cross-section of the tetrahedron resting on its edge was quadrilateral and not triangular.

5.3.4. Adherence to Multiple Constraints Independently and Simultaneously in Locus Tasks

Visualization tasks involving locus required students to attend to multiple constraints. Students were able to develop their ability to imagine a solution that could either flexibly adhere to multiple constraints either independently or simultaneously.

Awareness of Task Parameters

Various students expressed that working with peers allowed them to better comprehend parameters of each visualization task, including definitions, conventions and task constraints.

For Alex, exposure to peer thinking helped her interpret the verbal aspects of the task.

> They, they would show me how they interpreted the uh, like question first. 'cause most of the times I like would like not hear what you said or like misunderstand. And after they would say "oh like the solution actually can't" – I was like "oh yeah you're right" and I changed my answer because they said that or their illustration was different and they would like hold it up and say "oh because blah blah blah" and I would say "yeah."

Cameron also relied on peers to interpret the meaning of verbally articulated task criterion.

> … The way that they're described in the class discussion that goes around them and then you see a lot of different um perspectives. Like, um, when we did the one where we did like a square and then like you know there was like a midpoint and the circle would go around the square and then it's tracing. Um, like, I misun– well, you can say I like misunderstood or took that in a different sense than like Jordan did and then we discussed it and I was like "that's how it's supposed to look" and then you know like the discussion around it I quite like because it's like you get to see what all these different people think about something that's [inaudible].

For Leslie, peer ideas helped provide her with new ideas that might result in her solving a task.

> Yeah, sometimes they heard a specific part that sort of sparked something and they did something a little differently than mine and then I started thinking about it more and then they started explaining it and then I sort of got it and I changed mine.

Students did not recount specific examples in their responses, but these can be seen in the upcoming sections. What can be garnered from their answers, however, is that they felt peer collaboration generally changed their awareness.

Passing through a Point and Tangent to a Line

Leslie initially determined a single case that allowed a circle to pass through a point and remain tangent to a line but did not see how the circle could move to continue to adhere to both criteria (Figure 4-7). She participated in discussion with Jess and Devon to better understand how the circle could dilate such that its centre would trace a parabolic locus.

Jess: [Points to her drawing (Figure 4-9)] Mine makes sense.

Ms. H.: Does hers make sense?

Devon: Um, so the circle goes like this – little, then big! [Gestures to Jess's drawing]

Leslie: [Gestures to drawing as well] How does this one move to this one?

Ms. H.: Oh! Well you can imagine the circle can change sizes.

Devon: Yeah. So it goes little and big.

Jess: So it goes [gestures] and then small to go pass through and then it's even bigger and then big [gestures circle getting bigger.]

Leslie: Ohh, to pass through!

Jess: So technically, I have [inaudible] this. So it's like nnnnnn [sound effect and gestures circle shrinking.]

Devon: Oh! So it like gets smaller and…

Leslie: … it gets bigger, and yeah.

Student collaboration allowed Leslie to build flexibility in her thinking to see how a circle could change sizes to move to different positions to remain tangent to the line while passing through the point. As Mason describes (1998), a remark by peers altered student awareness in this case. This circumvented the disadvantage of visualization noted by Presmeg (1986) and Aspinwall and Shaw (2002) where uncontrollable images arise. The *imagine-draw-talk* routine incorporated discourse and shared drawings as means of allowing students to move past uncontrollable images. Access to peer's ideas through discussion and through observation of their drawings allowed for new understanding and inclusion of task parameters.

Leslie did not have the opportunity to draw her own solution involving a circle's tangency to a line while passing through a point. However, she was able to independently find several circles that could remain tangent to a second circle while passing through a point in the *ellipse locus task* (Figure 4-10). She was able to transfer her learning from the *parabola locus task* to this task and no longer found her imagery to be stuck on finding only a single circle that could pass through the point while remaining tangent to the outer circle.

Passing through a Point or Tangent to a Circle

When approaching the *ellipse locus task*, both Jordan and Cameron were able to find alternate solutions allowing the moving circle to remain tangent to the outer circle *or* to remain passing through the point (Figure 4-17, Figure 4-21). This flexibility might be the first step of moving toward a solution adhering simultaneously to tangency and passing-through constraints. Their work represents the beginning of a shift in structure of their attention. This was supported by questions and remarks I made during the interview process.

Passing through a Point and Tangent to a Circle

Leslie and Devon were both able to transfer their understanding gained in the *parabola locus task* to their solutions to the *ellipse locus task* (Figure 4-10 and Figure 4-14) and described their work as follows:

> Leslie: This one [second drawing] I kind of just, this one it stayed the same and it's just moving around the point while this [circle]'s moving around the circumference and it's getting bigger because it doesn't just stay the same size because then it won't be touching the side [of the outer circle.]

> Devon: Sure. So um, the point couldn't be in the middle, but this will always have to remain tangent to the um, the point and the um circumference always moves around, through the point [draws]. I have to go through the circumference of the other circle so I kind of stopped similar to that other task we did with the line where the circle rotates around and it gets bigger to stretch to touch to pass through the circumference of this circle but always stays tangent to the point which is not in the middle.

Alex was able to apply the previously noted convention of a circle passing through a point to the *ellipse locus task* (Figure 4-4). She was also able to develop her understanding of tangency during this task.

Alex: Um, so at first, I was thinking about the circle like the point and after I guess didn't know what tangent mean, I thought tangent like meant intersecting…

Ms. H.: Yeah..

Alex: …so that's why I drew that and after you explained that doesn't. So [the point] might be the inside of [the circle] so…this and after it was like tangent to the circle [draws] yeah. And if it would rotate around I guess [the inner circle] could like get bigger like that and like after smaller but it always touches the point, the circumference of the first circle.

Ms. H.: Uh, if you, so the moving circle, if you focus on its centre…

Alex: Oh.

Ms. H.: Do you know? Did you draw the path?

Alex: Yeah.

Ms. H.: Oh, yeah I see.

Alex: So I guess it would make an oval shape?

Ms. H.: Oh yeah. Uh, do you remember the math word for it?

Alex: Oh! I remember it, oh I forgot!

Ms. H.: It's an, I can help you. It's an ellipse.

Alex: Oh, ok yeah. I remember now.

Alex's solution allowed the circle to remain tangent to another circle while passing through the point throughout its trajectory and therefore the task was solved accurately. By explaining her own change in understanding of the definition of tangency and explicitly re-articulating the constraints of the task, she demonstrated her own *awareness-in-discipline* (Mason, 1998) about the steps she took to solve the task.

5.3.5. Generalization

Mason (2002a) suggests that imagery allows students to look "through the particular to the general". Mason highlights that generalization is the core of mathematics and mathematical thinking.

Each drawing made by a student in response to a visualization task (Chapter 4) represented a generic case of a general statement offered in the task script. Many student drawings developed to become closer to expected solutions as described in Chapter 4.

The student who was able to most accurately solve the final interview tasks was Alex (Figure 4-2, Figure 4-4). Through her work, she was able to determine general solutions for the horizontal tetrahedron cross-section as quadrilateral and for the final local task to be elliptical. Alex was able to articulate her solutions and how she came to these and these are elaborated upon in 4.1.1. Alex's developing awareness of her own powers of mental imagery allowed the discipline of geometry to arise for her personally (Gattegno, 1965).

During his follow-up, Robin articulated that visualization was a powerful tool for comprehension in mathematics.

> Uh, I think being able to visualize things can be quite powerful for comprehension and understanding of different concepts in math at least we don't usually get to do a lot before. But, I think they can sometimes make it easier to understand things.… Uh, I really am a visual learner, so maybe I can visualize things make it easier to understand the curriculum.… [It is powerful because] because I think that people apply something they usually don't use, like, more about our senses would be more powerful than other, sort of **to visualize to represent more than that just looking at what you're shown [my emphasis]**.

His suggestion that visualization "represent[ed] more than just looking at what you're shown" may imply the power of mathematical generalization. As Mason suggests, imagery is a power that students can use to mediate generalization. Robin's recognition of this power represents the development of his *awareness-in-discipline* or awareness of his own awareness.

Chapter 6.

Student Experience

Chapters 5 described types of awareness that developed in students related to geometric and mathematical properties through drawing and discourse during visualization tasks. This chapter seeks to examine how students describe their individual experience with visualization tasks. The discussion about student experience is a distinct chapter, although brief, because student experience extended beyond geometric and mathematical disciplines that are central to the previous chapter.

The students suggested that different types of awareness developed through engagement in visualization tasks. These included both awareness of their own thinking (*awareness-in-discipline*) but also awareness of their peers' thinking. These types of awareness shaped student experience through this change in my practice.

6.1. Student Awareness of their Own Thinking

Alex described using her brain to solve tasks in a new way when engaging in a visualization task.

> I think they're really fun because I've never used my brain like that before, I guess, like to imagine like certain shapes where to go or what pattern they form so I think it's interesting outcome I can make in my brain…

This was a similar observation to Robin who also suggested that visualization was not something the students had previously had a chance to do. He also suggested that visualization is "something [people] don't usually use."

> Uh, I think being able to visualize things can be quite powerful for comprehension and understanding of different concepts in math at least we don't usually get to do a lot before. But, I think they can sometimes make it easier to understand things.… Uh, I really am a visual learner, so maybe I can visualize things make it easier to understand the curriculum.… [It is powerful because] because I think that people apply something they usually don't use, like, more about our senses would be more powerful than other, sort of **to visualize to represent more than that just looking at what you're shown [my emphasis]**.

Ryan shared that he experienced thinking and was re-energized by engaging with the visualization tasks.

Ms. H.: Can you describe your experience with the visualization tasks we've been doing so far?

Ryan: I think the visualization tasks help me kind of think better before in math class because it kind of gets your brain thinking. It's kind of like a brain exercise before you actually start the math because sometimes your brain is not that active before you get into the lesson but when you do the visualization it kind of helps you think more while you're in the actual lesson.

Ms. H.: Can you tell me more about that?

Ryan: Um – something like it kind of like gives your like brain more energy and - um – with your brain more energy - it just helps you think more – it kind of gets you to think and then when you are into the actual lesson afterwards it just kind of like [gets] your brain just thinking more

Ms. H.: Do you have any idea why?

Ryan: Because like kind of when you close your eyes and you visualize thinking in your brain it helps your brain think more slowly, in steps.

Ryan and Alex both brought up the brain and thinking with regards to the visualization tasks and routine. A variety of areas of the brain are involved in mathematics learning including specific regions involved in visual processing (Boaler et al., 2016). Neural networks in the parietal cortex play a role in both numerical and visual-spatial tasks (Hubbard et al., 2009).

"Tasks which provoke students into rehearsing or exercising skills, but which at the same time attract their attention away from the skill to be automated" can support development of *awareness-in-discipline*. Alex and Ryan articulated the use of their own brains and their own thinking during visualization tasks. Therefore, these tasks served as a catalyst of Alex and Ryan's development of higher-level awareness. I was gratified to learn that my students' experiences echoed my own in Math 604 through the novel format of visualization task and the "energizing" quality that sparked contributions by a variety of students to debrief discussions, building on their ideas.

Like students mentioned in this section, Cameron too was perhaps energized by applying her own imagining power (Dewey, 1944; Mason, 2002a) and her own mathematical agency.

> That was fun. Wracked my brain a little bit. (Cameron at the end of her final interview)

It is not every day in a mathematics classroom that a student expresses enjoyment about a challenging task as she did at the end of her interview.

6.2. Student Awareness of Peers' Thinking

Engagement in the visualization task *imagine-draw-talk* routine supported student awareness of their peers' thinking and perspective. Jess shared that visualization tasks increased student awareness about how other students are thinking and understanding.

> I personally think it was a good idea to try visualization because um, before we just kept using work on the boards and kind of see how we thought but we never sat down to talk about it and **understand how some of our fellow classmates would like see things or understand things** [my emphasis]. It's kind of cool. And then even now when we 're doing group work, like **we would be able to see why someone wasn't understanding what was going on** [my emphasis], it was helpful then to try to explain it to them.

Robin succinctly noted that these tasks promote awareness of others' perspectives.

> I think people being able to see that from another person's perspective it can help to enhance your own sort of like.

During the class debrief for the *hanging cube task*, Devon shared her interpretation of Jordan's sketch on the board as she attempted to understand his thinking.

> Devon: Oh! I thought you were looking at it from the bottom. If it were see-through and you would see it from the bottom that's what it would look like.

Later, during the final interview, she agreed that it could take time to understand how a peer's sketch could be an alternative yet correct way to represent a given task.

> When I would be drawing something and I would look over and see something else someone else's drawing it would be like uh, I'm probably not doing this right because mine looks totally different from theirs, but

> then **also when we were able to get together and discuss it you realize that you might actually be thinking the same thing** [my emphasis] you just drew it in a different way um, but I'd say, yeah, sometimes I drew my answer differently to fit more with what they told me too.

The visualization task routine offered an opportunity for students to slow down enough to understand and consider their peers' thinking and ideas.

6.3. Time to Think Individually in a Quiet Space

The *imagine-draw-talk* routine led to student engagement in visualization tasks for periods of twenty to thirty minutes. This is longer than would typically be spent on a single task in class. However, this time allowed students to think carefully and deeply about the tasks and for rich conversation and student ideas to emerge. The *imagine-draw-talk* routine intentionally required quiet and also required individual think time. There was an aspect of mindfulness and relaxation to these tasks, and students appeared to enjoy the calm involved in closing their eyes and taking time to think. Debriefs were often lengthy and often touched on many different ideas before we reached the expected solution as a group. Sometimes, I felt concerned that the duration would bore students and noticed fidgeting or conversation that I assumed to be off-task behaviour. However, upon re-watching video recordings of the lessons, I realized that students were often considering the task for the duration of the lesson, whether they were sketching new solutions, building physical structures out of scraps of paper or whispering to peers about their ideas while the debrief took place.

During her interview, Leslie described engagement with visualization tasks as a break from other mathematics curricular topics.

> It doesn't really correspond to what we are doing in class. […] I think that well, some of them, what we do doesn't really correspond to what we're learning in class so it kind of confuses you on what we're learning in class, but sometimes when we do those it **helps me kind of relax from doing other stuff** [my emphasis]. But yeah, I think that sometimes I don't understand them because it doesn't really relate to what we're learning in class sometimes so I don't know how to do it so I kind of guess.

The qualities of quiet and slow or taking a break from regular curriculum are not always emphasized in the structures I implement in my teaching practice. However, they were valuable to student learning and to the student experience.

Chapter 7.

Conclusion

Gattegno believed that the use of imagery in mathematics class would become commonplace in mathematics education.

> Once [spatial reasoning] becomes second nature in the profession it will be possible to see that rather than drop geometry from the school syllabus, it becomes natural to introduce much younger children to it and make it one of the sources of education of the powers to use one's imagery, one's imagination, one's foresight of the infinite renewal of the mind. (1965, p. 24)

Fifty-five years later, in British Columbia, geometry has lost much emphasis in the general mathematics program other than Geometry 12, an elective course. Geometric concepts are limited in many curricula and therefore implementation of tasks that support spatial reasoning are not emphasized. However, the benefits and importance of spatial reasoning has become apparent as an integral component of mathematics and mathematical thinking (Mix & Cheng, 2012). Transmission of spatial reasoning through alphanumeric modes is a known challenge (Davis et al., 2015). Classroom practices that develop this skill are beginning to be developed (Bruce et al., 2015) and introduced into certain curricula such as in Ontario (Ontario Ministry of Education, 2014) particularly at the elementary level.

I wanted to engage high school students in developing their spatial reasoning because this is highly correlated with mathematical thinking and because this skill is independently valuable for students. I hoped to provide students with an experience that promoted visualization such that students might become more familiar and comfortable with this process. Visualization tasks offered an opportunity to employ their own imagery powers. I hoped that use of this capability would be as highly engaging for my students as it had been for me during my own experience.

Through engagement with visualization tasks, the students involved developed drawing strategies including drawing hidden edges of 3D shapes, depicting multiple views of rotating shapes, and the skill of applying the concept of locus to their drawings. They demonstrated their own mathematical agency through conjecture and argument during task debriefs. They developed geometric awareness of definitions, conventions and properties of shapes as well as the mathematical ability to consider multiple constraints in a task and to generalize. Students described how they became aware of their own thinking and of their peers' thinking through visualization tasks.

Therefore, visualization tasks have a role to play in today's classroom just as Gattegno suggested. The *imagine-draw-talk* routine is one practice that can be applied to promote spatial reasoning. Knowing that this capability is difficult to transmit or assess in terms of alphanumeric modes of communication (Davis et al., 2015), visualization tasks offer one practical solution that teachers can implement in their classrooms.

Practicing the *imagine-draw-talk* routine in my classes supported my own teacher awareness of student thinking. This routine was new to me and to my students. Innumerable times, I found that students interpreted the script of a task in a way I had not intended yet was consistent with the given parameters.

Even when working to anticipate ways students might interpret a task, I was often surprised by their solutions. During the final interview, I did not anticipate so many students (Alex, Leslie, Robin and Jordan) would respond to the *ellipse locus task* by positioning the point on the initial circle's circumference rather than inside the circle. After this occurred more than once, I began to ask students to consider the case where the point was placed neither on the centre nor on the circumference.

As students sketched and described their visualizations for different tasks, I experienced the challenge of interpreting their solutions. When debriefing the *hanging cube task* with Mathematics 9/10, I was unclear that Dylan's sketch (Figure 5-1) showed a cross-section until we had discussed properties of her drawing for several minutes and several other students had contributed to the conversation.

> Ms. H.: Really soon. Oh yeah. I'm just wondering, is this, Dylan, is this the line where you would make the cross-section [gestures cross-section line B*]?
>
> Dylan: Yeah, the cube's… [inaudible]

The cross-sectional line might have been interpreted as an invisible edge making up part of the back face of the cube without Dylan's confirmation. This was a case where “what the teacher “sees” (interprets) may not have been what the student intended (Francis & Whiteley, 2015, p. 124). Taking the time to seek clarification about this was essential to my understanding of Dylan’s drawing.

As we engaged in more tasks over time, I found myself working to ask more questions to check my own interpretation of student drawings and double-checking that my assumptions about their ideas were accurate. Like the students who stated that these tasks helped them understand their peers’ thinking, I too learned about how my students understood the tasks as well as their interpretation of various mathematical definitions and conventions. I became more acutely aware of my students’ thinking.

Because this study took place in my own classroom through my own teaching, certain limitations must be considered. I often adjusted the scripts of tasks based on student questions or common misconceptions encountered. Therefore, the students who partook in the final interviews experienced adjusted visualization tasks. I provided hints as I saw fit to students during task debriefs. The way I offered these varied from class to class and from student to student. This variation might have impacted development of student drawings, discourse, geometric and mathematical awareness and experience and cannot specifically be accounted for. Although these results cannot be generalized beyond my own classroom, I believe my students benefitted from implementation of visualization tasks based on their developed awareness and described experiences.

This research project has allowed me to grow significantly as a practicing classroom teacher and as researcher. As a teacher, it required courage to make a committed change to my practice in regard to an area that focused more on current curricular competencies than on content. Through implementation, I learned to notice more about how my students were understanding a given task and how they imagined and justified their solutions to each task. I learned to listen attentively and ask questions to ensure that my interpretations of their understanding were correct. I also learned about common misunderstandings that took place for each task and to adjust my script to account for these. The ability to alter what I attend to during a lesson, even while actively teaching the lesson has allowed me to grow as a teacher. The opportunity to return to videotapes of lessons allowed me to notice additional details I could not attend to while actively teaching.

Every step of the research process of developing this project has been new to me, from developing and refining (and re-refining) a research question to leading an effective debrief with my students and conducting a research interview to analysing and writing this thesis. I have learned to notice, to ask questions and to draw connections through the lens of a researcher. I have gained a deeper and personal appreciation of educational research, in particular, action research conducted by practicing teachers.

Based on my experience, I would advocate for an emphasis on visualization in the classroom. Visualization tasks provided a practical method for connecting the alphanumeric with the spatial and offered students the opportunity to apply their own mathematical agency by engaging their own powers of imagining, figuring and construing. My students benefitted from this practice in terms of developing geometric and mathematical awareness as well as awareness of their own and their peers' thinking. These are major and important developments in terms of spatial reasoning but also in terms of student learning. Therefore, I believe that Gattegno's practice (with the opportunity for student imagining, drawing and discourse) holds in importance and relevance in today's classroom, perhaps more so than ever.

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Appendix A.

Visualization Group Tasks

Each original script is included with improvised changes when reading this task to the Mathematics 9/10 class during the first reading shown in parentheses, exemplifying the type of improvisation that occurred.

Task 1: Line

Imagine two points on a plane in space. They are not superimposed on one another. (Pause). Now, imagine all the points that are midpoints between these two points. (Pause). (The midpoint is the point halfway between the two points).

Imagine, if these (mid)points are connected, what you will see.

Task 2: Overlapping Squares

Imagine two squares of the same size with black perimeters on a two-dimension surface next to one another but not touching. Now the square on the right moves toward the square on the left such that they begin to overlap.

Move the square in your mind. Rotate the square. Imagine all the possible polygons that can be formed by the intersection of the two squares. What is the polygon with the smallest number of sides? What is the polygon with the greatest number of sides?

Task 3: Circle around a Square

Imagine a square. Now, imagine a circle tangent to the square. (Tangent means the circle touches the square at exactly one point. The circle and the square do not intersect.)

Imagine the circle rolling around the square such that it always remains tangent to the square as it traces the square’s perimeter.

Now, focus on the centre of the circle. Imagine the centre of the circle tracing a path as the circle rolls around the square. What path is formed?

Task 4: Parabola Definition

Imagine a point and a line. The point is not touching the line.

Now, imagine the set of all possible points that are equidistant between the point and the line. These points are midway between the point and the line.

Connect all these midpoints. What is formed?

Task 5: Cube Wrap-Around

Imagine a single yellow cube. This cube is covered on all its faces with a layer of red cubes to create a cube with dimensions 3 by 3 by 3 (red) cubes (surrounding the yellow cube).

Think about how many red cubes wrap around the yellow cube.

If the red cubes are rearranged (and taken away from the yellow cube),what are the dimensions of the largest possible cube of red cubes that can be made? How many cubes of this size can be made? (And then) how many smaller cubes are left over?

Task 6: Hole Punch Symmetry

I will (am going to) fold a (this) paper twice (in half and then in quarters and) then (I'm going to) hole punch it.

(My question to you will be) what will this paper look like once unfolded? (And) where will the holes be located?

Task 7: Ellipse Definition

Imagine a shoelace pinned by both of its ends onto a wooden board. The ends of the shoelace are pinned separated from one another by a distance that is smaller than the total length of the shoelace. The shoelace is loose. Now, you hold shoelace between the pins and pull it as far away as possible from the pins without pulling the pins off the board. Imagine holding the shoelace and pulling it in all possible ways to keep its distance away from the pins as far as possible. As you pull the shoelace in different directions you will trace a path. Draw a sketch of what this path looks like in relation to the pins and to the shoelace.

Task 8: Tilted Square

Imagine a square. Move the square around. Now, imagine the square being doubled on top of itself (so that there are two squares). The second square begins to rotate (it rotates left, it rotates right. Now it's going to rotate from its original position) until it has rotated 45 degrees. Then, it moves to the right such that one of its vertices touches the centre of the first square.

What shape is formed by the intersection between the two squares?

Task 9: Hanging Cube

Imagine a cube that is hanging from a string from one of its vertices (or uh, one of the corners of the cube). Halfway down between the top and the bottom vertices (corner) of the dangling cube, Ms. Herman slices the cube in half, horizontally. What is the shape that is cut?

Task 10: Parabola Locus

Imagine a line and a point near the line, but (that's) not touching the line. Now, imagine a circle tangent to the line that also passes through the point. The circle moves in a way such that it remains tangent to the line while still passing through the point; Its size might change while this happens. Imagine the circle moving across the line while staying tangent but always touching the point. It (the circle) moves back and forth.

Now, (while this is happening) focus on the centre of the circle. What is the path it traces as the circle moves as I have described?

Task 11: Quadrilateral Midpoints

A) Imagine a rectangle. Each side of the rectangle has a midpoint. (Nod if you know what a midpoint is. So it's the point in the middle of each side). Join the four consecutive (neighbouring) midpoints to form another quadrilateral. What shape is formed?

B) Imagine a parallelogram. Each side of the parallelogram has a midpoint. Join the four consecutive midpoints to form another quadrilateral. What shape is formed?

C) Imagine a trapezoid. Each side of the trapezoid has a midpoint. Join the four consecutive (neighbouring) midpoints to form another quadrilateral. What shape is formed?

D) Imagine a concave quadrilateral (so still four sides just concave). Each side of the quadrilateral has a midpoint. Join the four consecutive (neighbouring) midpoints to form another quadrilateral. What shape is formed?

E) Imagine any quadrilateral (so you can choose). Each side of the (your) quadrilateral has a midpoint. Join the four consecutive (neighbouring) midpoints to form another quadrilateral. What (image the) shape (that) is formed? Now, imagine the quadrilateral's four vertices mov(ing) around. (You can move one at a time; You can move more than one at a time. As you move those vertices around,) how does the quadrilateral formed by the four midpoints change (as) this happens? How does it stay the same?

Task 12: Tahta Loops

A) Imagine a line segment rotating about one end. Imagine another line segment rotating about the same point (in the same direction). Imagine the moving endpoints of the lines as they turn…Now imagine a point halfway between these endpoints. As the endpoint rotates, what path does the midpoint trace?

B) Imagine a line segment rotating about one end. Imagine another line segment rotating about the same point, this time (rotating) twice as fast as the first line (segment, in the same direction). Imagine the moving endpoints of the lines as they turn…Now imagine a point halfway between these endpoints. As the endpoint rotates, what path does the midpoint trace?

Appendix B.

Visualization Individual Tasks

Each original script is included. Improvised changes were different with each interviewee and these changes are not included below.

Task 13: Ellipse Locus

Imagine a circle. There is a point inside the circle that is not in the centre of the circle. Another circle has a circumference that passes through this point while remaining tangent to the initial circle. This inner circle moves in a way such that its circumference always passes through the point and always remains tangent to the exterior circle. The inner circle rolls around the inside of the outer circle.

Focus on the centre of the moving circle. What path does this point trace?

Task 14: Tetrahedron

A) A tetrahedron is a pyramid with a triangular base. How many faces does it have? How many edges? How many vertices?

B) Rest a tetrahedron on its base, and cut it halfway up, horizontally (parallel to the ground). What shape is the smaller piece? What shapes are the faces of the larger piece?

C) Rest a tetrahedron so that it is balanced on one edge and slice it horizontally halfway between its lowest edge and its highest edge. What shape is the slice?