

Cops and robbers with speed restrictions

by

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Abstract

The game of Cops and Robbers is a pursuit-evasion game played on graphs with two players, the cops and the robber, who take turns moving on the graph. In each turn they may move to a vertex adjacent to their current position or stay where they are. The cops' objective is to get to the same position where the robber is, which we refer to as to capture the robber, and the robber's goal is to evade capture indefinitely. The basic question is to find the minimum number of cops that can guarantee capturing the robber in a given graph. A very fruitful research area has been developed around the idea of modifying the way in which the cops or the robber move and analyzing how these changes affect the strategies and outcome of the game. In this thesis we will study the game when we impose additional speed restrictions on the players, variants of the game popularly known as "lazy-cops and robbers" and "active cops and robbers". In order to do so, we introduce the concept of the wide shadow, aiming to improve known results and obtain new tools and techniques which may provide further insight into other open problems in the area.

Keywords: wide shadow, cops and robbers, guardable graph, Helly graph, lazy cops and robbers, active cops and robbers

Dedication

To see what is in front of one's nose needs a constant struggle.

George Orwell (not in 1984)

To those who have sat next to the colossus at the edge of the world;

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Chapter 1

Introduction

A pursuit-evasion game is, broadly speaking, a game in which a set of players, the pursuers, attempt to capture another set of players, called the evaders. Depending on how we define the way in which the pursuers and the evaders move we will give rise to different pursuit evasion games, which have been extensively studied from both continuous and discrete perspectives.

Probably the best known example of a pursuit-evasion game in discrete time is the game of *cops and robbers*, a game played on graphs introduced by Nowakowski and Winkler [25], and independently by Quillot [28]. The game is played on a graph G by two players, the cop and robber. As it will often be convenient to use pronouns to refer to the players, we will assume throughout the thesis that the cop is feminine and the robber is masculine. The game starts with the cop choosing a vertex as her starting position, and after that the robber chooses his initial position. A *round* of the game consists in two turns, the first one for the cop and the second for the robber.

In each turn, a player might stay at their current position or move to a neighbour of their current position. The cop wins if he eventually occupies the same vertex as the robber, a situation we will refer to as *capturing the robber*, while the robber wins if he is able to indefinitely prevent this from happening.

While we will be using mostly standard Graph Theory definitions and notation, we refer the reader to [12] as a reference. Unless stated otherwise, we will assume that all graphs in this thesis are connected.

A graph in which a cop has a winning strategy is called a *cop-win graph*. The graphs in which the cop has a winning strategy have a very nice characterization, which is obtained from the following observation about the robber's position right before his last move: if the robber cannot prevent the cop from capturing him regardless of what his move is, that means that every neighbour of his position is also adjacent to the cop's position. If x and y are vertices corresponding to the current positions of the cop and the robber respectively, this means that $N[y] \subseteq N[x]$. In this situation, we call y a *corner* of G . A graph G is *dismantlable* if it can be reduced to K_1 by successively removing corners. A characterization of cop-win

graphs was given in [25], as the authors showed that this is the same class of graphs as the dismantlable graphs.

The game was generalized by Aigner and Fromme [1] to allow more than a single cop to play, and they defined the *cop-number* of a graph G , which we will denote by $c(G)$, to be the smallest integer k such that k cops can guarantee the robber's capture on G regardless of his strategy. Since for any graph G we have $c(G) \leq |V(G)|$, the cop-number is well defined for every finite graph. Moreover, for any graph G we have $c(G) \leq \gamma(G)$, where $\gamma(G)$ denotes the domination number.

Notice that if we play on a graph G with n vertices, using k cops, we can describe the position of every player on the graph with a vector of $\{C, R\} \times V(G)^{k+1}$, where the first entry corresponds to an indicator of whose turn it is and the next $k + 1$ entries correspond to the current position of each cop and the robber on the graph. We will refer to such vector as a *configuration* of the game. Notice that if the cops have a winning strategy to capture the robber, then they may avoid repeating any configuration. Conversely, if the robber is ever able to force the cops to repeat a previous configuration of the game, then the robber has a winning strategy. This fact will be useful to simplify the analysis of the game.

Finding bounds for the cop number of a graph (or, more accurately, a class of graphs) is the fundamental problem in the area. The most well-known question regarding the game of cops and robbers, which remains wide open, is known as Meyniel's Conjecture:

Conjecture 1.0.1 (Meyniel's Conjecture). *For any graph G with n vertices, we have that $c(G) = O(\sqrt{n})$.*

A lot of research has also been done studying the connections between a graph's topological properties and its cop-number. For a survey on this, see [7]. The classical result in this direction is due to Aigner and Fromme[1].

Theorem 1.0.1 ([1]). *The cop number of every planar graph is at most three.*

In order to prove this, Aigner and Fromme introduced one of the most useful results in the literature on cops and robbers: A theorem on using one cop to guard a path. There has been interest in generalizing this to graphs other than paths. We will address this problem in Chapter 2.

1.1 Adding speed restrictions to the game

A very fruitful research area has been developed around the idea of modifying the way in which the cops or the robber move and analyzing how these changes affect the strategies and outcome of the game. A well-known example of this was introduced by Seymour and Thomas in [29], where they define a version of cops and robbers commonly known as "helicopter cops and robber" characterizing tree-width, an idea that has been used in relation to other

width parameters (see [30]). However, this approach has also been used trying to get a better insight into Meyniel’s Conjecture. People have studied the game by adding “restrictions” either helping or hindering the players, like forcing a player to move randomly in a graph [21], forcing them to move along geodesic paths [16], allowing the cops to capture the robber at a distance [6], and many others.

In this thesis we look at the effect of restrictions on the number of agents (cops and robber) that move in each round. The earliest reference we know of in which this type of restrictions have been studied is [23], where the authors distinguished between a passive version of the game (in which all players have the option of passing) and an active one (where the robber and a non-empty set of cops must move). In [26], Offner and Ojakian introduced a variation of the game in which it is specified how many cops must move each turn, how many must remain in the same position, and how many can do either. Of these, the variant that has received more attention is the one where only one cop is allowed to move each turn, which they referred to as the “one-active-cop game”. Shortly after, this variation was introduced with different names, like *lazy cops and robbers* [2], and the “one-cop-moves” game [17].

We define the *k-cops-move* number of a graph, $c_k(G)$, as the smallest integer such that $c_k(G)$ cops guarantee the robber’s capture in G with the restriction that at most k cops can change their position each turn. We will also refer to $c_1(G)$ as the *lazy-cop number* of G .

A different way in which we can impose restrictions on both players is by forcing them to move every turn. This idea has been introduced (in slightly different ways) several times throughout the years, as can be seen in [1, 9, 18, 23]. We will refer to the variant in which all players are required to move every turn as *active cops and robbers*, and define $c_a(G)$ to be the minimum number of cops required to win in a graph G with this rules.

Probably due to its historical importance, as well as its beauty, it is frequently asked whether a result analogous to Theorem 1.0.1 holds for other variations of the game. For example, it was recently shown in [17] that there exist planar graphs with lazy-cop number more than three, and one of the authors conjectured in [33] a stronger version of Theorem 1.0.1. We provide a proof of this conjecture in Chapter 3. On the other hand, the authors of [18] provided a bound akin to Theorem 1.0.1 for the active version of the game and asked if it could be improved, a question we discuss in in Chapter 4.

Finally, in Chapter 5, we study the effect of subdividing all the edges of a graph the same number of times for both the active and the lazy version of the game.

Chapter 2

Guarding graphs

Suppose that we play the classic game of cops and robber using k cops, with $k \geq 2$, on a graph G . Given the nature of the game, we may choose to leave one cop stationary on some vertex v of the graph G on which we are playing, and then carry on by assuming that we are playing in $G - N[v]$ with $k - 1$ cops as the robber cannot enter $N[v]$ without being captured. This idea can actually be used to protect subgraphs larger than the neighbourhood of a vertex.

Let G be a graph, H a subgraph of G and k a positive integer. We say that H is *k-guardable in G* if, after finitely many moves, k cops can move on vertices of H in such a way that, if the robber moves to a vertex in H , then he will be captured in the next turn. The high-level idea is that one can *k-guard* a subgraph H if there are k cops in vertices of H and, from their positions, every vertex in H can be reached by a cop “at least as quickly as the robber”.

For a graph G and two vertices $x, y \in V(G)$, we use $d_G(x, y)$ to denote the distance in G between x and y . We say that H is an *isometric* of G if for every pair of vertices $x, y \in V(H)$, we have that $d_H(x, y) = d_G(x, y)$. We say that a graph H is *k-guardable* if H can be *k-guarded* in any graph in which it appears as an isometric subgraph. It should be noted that if a graph H is *k-guardable*, then $c(G) \leq k$: since H is an isometric subgraph of itself, then after a finite number of moves we can get k cops to a position such that if the robber moves to a vertex of H , he will be caught and, since the robber cannot move to a vertex not in H , his capture is guaranteed.

One of the most useful results in the literature on cops and robbers is a theorem on using a single cop to guard a path, which was a key element in Aigner and Fromme’s proof of Theorem 1.0.1:

Theorem 2.0.1 ([1]). *Any path is 1-guardable.*

It is natural to ask whether one can generalize Theorem 2.0.1 to other classes of graphs, and there has been recent interest in doing so. In [31], the authors showed that the result is still true if we replace “path” by “tree”, and in [22] this was further extended to a class of

graphs the authors called “vertebrate graphs” (we will say more about this in Section 2.4). This chapter is concerned with characterizing the class of graphs which we can guard using a single cop.

2.1 The Wide Shadow

In this section we will introduce the central concept of the chapter (and probably of the thesis), which we call the “Wide Shadow”. In order to do this, we need a few definitions regarding graph homomorphisms. Given two graphs G and H , a function $\phi: V(G) \rightarrow V(H)$ is a *homomorphism* if for every edge $xy \in E(G)$, we have that $\phi(x)\phi(y) \in E(H)$. We say that H is a *retract* of G if H is a subgraph of G and there exists a homomorphism $\phi: V(G) \rightarrow V(H)$ such that $\phi(x) = x$ for every $x \in V(H)$. If H is a retract of G , we call the homomorphism ϕ a *retraction* of G onto H . Retracts are useful in the study of cops and robbers for many reasons, but particularly for the following result, which allows us to find winning strategies for the robber in a graph G by focusing on some subgraph H .

Theorem 2.1.1 ([5]). *Suppose that H is a retract of G and k a positive integer. If the robber has a winning strategy in H against k cops, then he has a winning strategy in G against k cops.*

In the study of cops and robbers, the notion of a “shadow” has been of extreme importance as it serves, in some way, as a counterpart of Theorem 2.1.1, allowing the cops to play the game in a subgraph to restrict the robber’s movement in the original graph. Suppose we are playing on a graph G and ϕ is a retraction of G onto a subgraph H . If R denotes the position of the robber, the *shadow* of the robber on H is the vertex $\phi(R)$. Notice that if a cop can get on $\phi(R)$, then the cop can move in such a way that the robber will not be able to enter H without being captured. This can be achieved if the cop moves in such a way that, at the end of his turn, he is in $\phi(R)$ after the cops turn.

Notice that Theorem 2.0.1 says that if a path P in a graph G is isometric, then we can use one cop to guard P . We would like to know which graphs, other than paths, have the same property.

A family of sets \mathcal{S} has the *Helly property* if for every finite $\mathcal{T} \subseteq \mathcal{S}$ we have the following property: if $X_1 \cap X_2 \neq \emptyset$ for every $X_1, X_2 \in \mathcal{T}$, then $\bigcap \mathcal{T} \neq \emptyset$.

This notion was introduced by Helly when he proved the following theorem about convex sets of \mathbb{R}^d :

Theorem 2.1.2 ([20]). *Let X_1, \dots, X_n be a finite collection of convex subsets of \mathbb{R}^d , with $n > d$. If the intersection of every $d + 1$ of these sets is nonempty, then the whole collection has a nonempty intersection.*

This property has been studied in several areas of combinatorics and discrete mathematics, as it can be seen in [32] and [13]. Here, we focus on the classical use of the Helly property in Graph Theory, which has to do with the ball-hypergraph of a graph.

Let H be a graph, k a natural number and $N^k[v] = \{x \in V(H) : d_H(v, x) \leq k\}$. The *ball-hypergraph* of H is the family of sets $\mathcal{H} = \{N^k[v] : v \in V(H) \text{ and } k \in \mathbb{N}\}$. The graph H is a *Helly graph* if its ball-hypergraph has the Helly property. It is important to notice that different families of sets related to a graph may satisfy the Helly property, giving rise to different notions that have been studied throughout the years.

In particular, trees are Helly graphs. However, this is usually stated in the following stronger form:

Theorem 2.1.3. *Let T be a tree. If F the family of all subtrees of T , then F has the Helly property.*

Notice that the ball-hypergraph of a tree T consists of a family of subtrees of a tree (although not all subtrees of a tree correspond to balls in T).

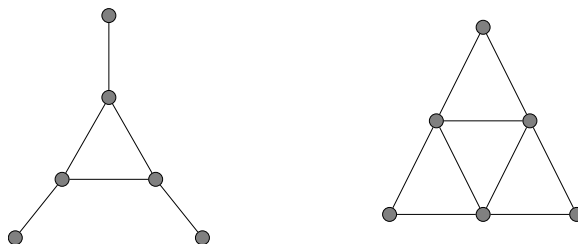


Figure 2.1: The graph on the left is a Helly graph, while the graph on the right is not. To see this, notice that the balls with radius one and centered on the vertices of degree two have pairwise non-empty intersection, but no vertex is in all three of them.

Due to their nice metric properties, Helly graphs have been studied in a variety of contexts. The Helly property is important in the study of clique graphs and the clique operator. It was shown in [4] that dismantlable clique-Helly graphs converge to the one-vertex graph by iteratively applying the clique-operator to them. They have also been used in the study of sandwich problems (a generalization of recognition problems) in [14].

There are many different known characterizations of Helly graphs. For a finite graph G , the following are equivalent:

1. G is a Helly graph.
2. G is a dismantlable clique-Helly graph (see [24]).
3. G is a pseudomodular graph in which the family of unit balls has the Helly property (see [3]).

4. For every vertex v in a diametrical pair, there exists a vertex w dominating v and the vertex-deleted subgraph $G - \{v\}$ (see [3]).
5. G is a retract of a strong product of reflexive paths. (see [24]).
6. G is an absolute retract (see [19]).

For a survey on algorithmic properties of Helly graphs, we refer the reader to [15].

2.2 Isometric Helly subgraphs

Let G be a graph, and let H be an isometric subgraph of G . For $v \in V(G)$ and $x \in V(H)$, let $H_v(x) = \{y \in V(H) : d(x, y) \leq d(x, v)\}$ be the ball with radius $k = d(x, v)$ around x in H . We define the *wide shadow* of v on H to be the set

$$S_H(v) = \bigcap_{x \in V(H)} H_v(x).$$

To define the usual shadow on a path, suppose P is an isometric path in G with a vertex x as one of its endpoints, and let $v \in V(G)$. The shadow of v with respect to x is the vertex $y \in V(P)$ such that $d(x, y) = d(x, v)$ when $d(x, v) \leq |E(P)|$ and the endpoint of P different from x otherwise. In general, the wide shadow of a vertex on an isometric subgraph H may be empty, but this is not the case when H is a Helly graph.

Using the notion of a wide shadow we provide a short self-contained proof of the following fact, which is of interest in the context of the cops and robber game.

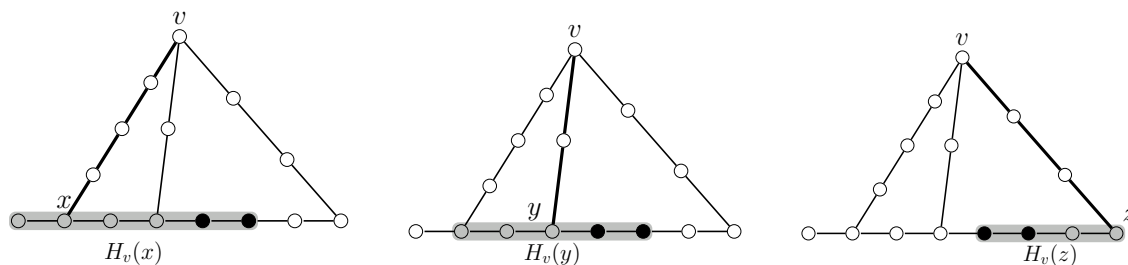


Figure 2.2: The shaded part of the paths represent some of the balls in the ball-hypergraph (the ones corresponding to the labeled vertices). The black vertices are in $S_H(v)$, the wide shadow of v on the path H .

Lemma 2.2.1. *Let G be a graph and H an isometric subgraph of G . If H is a Helly graph, then:*

- i) for every $v \in V(G)$, $S_H(v) \neq \emptyset$;

ii) for every $uv \in E(G)$, and every $x \in S_H(u)$, we have $d(x, S_H(v)) \leq 1$.

iii) for every $v \in V(G)$, $S_H(v)$ induces a connected subgraph of H .

Proof. Let $x, y \in V(H)$. Since H is isometric, $d_H(x, y) = d_G(x, y)$, so by the triangle inequality we have $d_H(x, y) \leq d_G(x, v) + d_G(v, y)$, which implies $H_v(x) \cap H_v(y) \neq \emptyset$. Since H is a Helly graph, we know that its ball-hypergraph \mathcal{H} has the Helly property, then $S_H(v) \neq \emptyset$, proving (i).

Now, let $uv \in E(G)$. Notice that for every $x \in V(H)$, we have $d(x, u) - 1 \leq d(x, v) \leq d(x, u) + 1$. This implies $H_v(x) \cap N_H[y] \neq \emptyset$ for every $x \in V(H)$ and $y \in S_H(u)$, so we have $S_H(v) \cap N_H[y] \neq \emptyset$ for every $y \in S_H(u)$. It follows that every vertex $y \in S_H(u)$ either $y \in S_H(v)$ or there exists $x \in N_H(y) \cap S_H(v)$, hence $d(y, S_H(v)) \leq 1$, completing the proof of (ii).

To show (iii), suppose the graph G_S induced by $S_H(v)$ is disconnected and let x and y be vertices in different components of G_S such that $d = d_H(x, y)$ is minimum. Notice that $N[x] \cap N^{d-1}[y] \neq \emptyset$ and, since $x, y \in S_H(v)$, we have $N[x] \cap S_H(v) \neq \emptyset$ and $N^{d-1}[y] \cap S_H(v) \neq \emptyset$. Since H is a Helly graph, we know that there exists $z \in N[x] \cap N^{d-1}[y] \cap S_H(v)$. Since $d(x, z) = 1$, then z is in the same component of G_S as x , but $d(z, y) < d$, a contradiction to the minimality of d . Hence, G_S is connected. \square

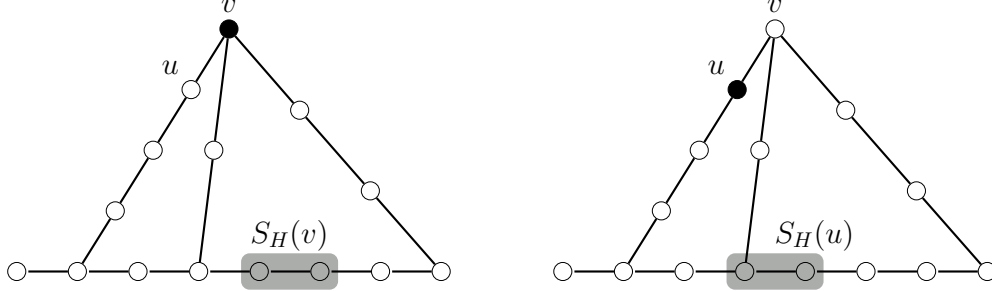


Figure 2.3: The black vertex represents the robber's position, and the shaded part of the path contains the vertices of his wide shadow.

Given the connection between guarding graphs and homomorphisms, one might ask if the wide shadow can be understood in terms of homomorphisms. Let G be a graph and H a subgraph of G . Notice that if G is a graph and H is a retract of G , then H can be 1-guarded in G if the cops can capture the robber in H .

Let \mathcal{F}_H be the set of all retractions of G onto H . We will call \mathcal{F}_H the *guarding bundle*. Let $R \in V(G)$ denote the position of the robber at some point of the game. If \mathcal{F}_H is not empty, then each $f \in \mathcal{F}_H$ provides a strategy for a cop to guard H : if there is a cop on $f(R)$, then it suffices that the cop remains in $f(R)$ to guard H . Notice that a cop on a vertex $x \in V(H)$ has a guarding strategy if there exists $f \in \mathcal{F}_H$ such that $f(R) = x$.

The usual shadow of the robber chooses one retraction and then uses the guarding strategy given by that. The wide shadow considers all retractions at the same time, and this makes it a more flexible notion.

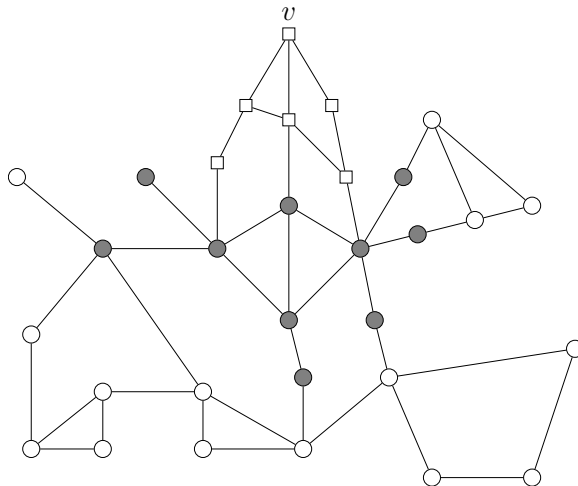


Figure 2.4: H is the subgraph induced by the round vertices. The gray vertices represent the wide shadow of vertex v on H .

2.3 Characterization of 1-guardable graphs

There has been recent interest in extending Theorem 2.0.1 to classes of graphs larger than paths. In this section, we will use the wide shadow to characterize the graphs which can be guarded by a single cop.

It was shown in [4] that Helly graphs are dismantlable. We include this result as Lemma 2.3.1. and provide a simple proof for the sake of completeness:

Lemma 2.3.1 ([4]). *If G is a Helly graph, then G is cop-win.*

Proof. Notice that if G is a Helly graph and v is a corner of G , then $G - v$ is also a Helly graph, so to prove the result it suffices to show that every Helly graph has a corner.

Let u and v be vertices of G with $d(u, v) = d$, where d is the diameter of G . Let $T_u = N^{d-1}(u)$, and for $x \in N[v]$, let $T_x = N[x]$. Clearly $T_x \cap T_y \neq \emptyset$ for all $x, y \in N[v]$, and since $d(u, v) = d$, we get $T_u \cap T_x \neq \emptyset$. Since G is a Helly graph, there exists $z \in T_u \cap \left(\bigcap_{x \in N[v]} T_x\right)$. Since $z \neq v$ and $z \in T_x$ for every $x \in N[v]$, then $N[v] \subseteq N[z]$, so G has a corner. The result follows by induction on $|V(G)|$. \square

Finally, we can prove the main result of this section.

Theorem 2.3.1. *A graph H is 1-guardable if and only if H is a Helly graph.*

Proof. Let G be a graph and H an isometric Helly subgraph of G . It follows from Lemma 2.2.1 that the wide shadow of the robber on H is not empty. Hence, we can choose one retraction of G onto H and use a cop to capture the robber's shadow given by that retraction on H (this can be done since H is dismantlable by Lemma 2.3.1). Now, since the cop is in the robber's wide shadow, by Lemma 2.2.1, she can always move in such a way that, at the end of her turn, she is in the wide shadow of the robber.

It follows from Lemma 2.3.1 and Lemma 2.2.1 that H is 1-guardable: we can capture the wide shadow of the robber by Lemma 2.3.1, and stay in it by Lemma 2.2.1. If the robber enters H , his wide shadow will consist precisely of a single vertex, so the cop in his shadow will capture him. This proves the "only if" part of the theorem.

For the "if" part, suppose H is 1-guardable but not a Helly graph. That means there exist vertices v_1, v_2, \dots, v_k and positive integers d_1, d_2, \dots, d_k such that $N_H^{d_i}[v_i] \cap N_H^{d_j}[v_j] \neq \emptyset$ for $1 \leq i < j \leq k$ but $\bigcap_{i=1}^k N_H^{d_i}(v_i) = \emptyset$. Notice that we can say positive instead of non-negative as if one of the d_i 's were zero, the vertex v_i would be in the common intersection. Let G be the graph obtained from H by adding a vertex x and internally disjoint paths $\{P_i\}_{i=1}^k$, with P_i having length d_i and joining x with v_i . Since $N_H^{d_i}[v_i] \cap N_H^{d_j}[v_j] \neq \emptyset$ for any i, j , it follows that H is an isometric subgraph of G .

Assume that the robber is in x . Since H is 1-guardable, a cop has a strategy to capture the robber if he enters H moving only in the vertices of H . Let us now move the robber to the vertex x and wait until it is the robber's turn to move. Let us assume that the cop is in a vertex u of H from which she can follow the guarding strategy. Since $\bigcap_{i=1}^k N_H^{d_i}(v_i) = \emptyset$, there exists a vertex v_j such that $d_H(u, v_j) > d_j$. Now, if the robber moves along P_j , he will be able to get to vertex v_j in exactly d_j steps. However, since $d_H(u, v_j) > d_j$, the cop will be at least at distance two from v_j when the robber enters v_j , so she cannot capture the robber in the next turn. This is a contradiction, so H must be a Helly graph. \square

In particular, this implies Theorem 2.0.1 since paths are Helly graphs.

Our result can also be interpreted as a result about absolute retracts via the following two theorems from [19] and [4]. In order to state them, we need a couple of additional definitions:

A *reflexive graph* is a graph that has a loop on every vertex. Notice that if a graph G has a vertex v with a loop on it, then there is a retraction of G to v . A reflexive graph H is an *absolute retract* if it is a retract of any graph G containing H as an isometric subgraph. As we pointed out, retracts can be used to guard graphs thanks to Theorem 2.1.1. A *hole* in a graph H is a set of vertices $\{a_i\}_{i=1}^k$ and positive integers $\{x_i\}_{i=1}^k$ such that

- H has no vertex v such that $d(v, a_i) \leq x_i$ for all $i \in \{1, \dots, k\}$ and
- for every $i, j \in \{1, \dots, k\}$ we have $d(a_i, a_j) \leq x_i + x_j$.

It turns out that holes can be used to characterize absolute retracts.

Theorem 2.3.2 ([19]). *A graph is an absolute retract if and only if it has no holes.*

Also, as mentioned before, absolute retracts coincide with Helly graphs.

Theorem 2.3.3 ([4]). *A graph is an absolute retract if and only if it is a Helly graph.*

However, this fact was never stated in terms of guarding subgraphs of a graph in the game of cops and robbers. Moreover, the use of the wide shadow allows us to look at the whole guarding bundle simultaneously instead of a single retraction.

2.4 Lu and Wang's Vertebrate Graphs

Recently, Lu and Wang extended Theorem 2.0.1 to a class of graphs which they call vertebrate graphs. They give a metric characterization of vertebrate graphs and then use it to show that they are 1-guardable (see Theorem 2.4.1). Their method makes use of two cops to guard the graph, albeit the second cop is only needed for a finite number of turns.

Using Theorem 2.3.1 we can give a shorter proof of Theorem 2.4.1 by showing that vertebrate graphs are Helly graphs; this also shows that a second cop used in the proof of Theorem 2.4.1 is not required. We begin by giving the necessary definitions from [22].

Let G be a graph and H a subgraph of G . For $x \in V(G)$ and $c \in V(H)$, we define $N_H(c, x) = \{v \in N_H(c) : d(v, x) = d(c, x) - 1\}$. A *block graph* is a connected graph G in which every block (i.e. maximal 2-connected subgraph) is complete. A connected graph G is an *extended block graph* if it can be obtained from a block graph by blowing-up the cut-vertices, i.e., replacing each cut vertex by a clique and connecting every vertex of the clique to every neighbor of the cut vertex (see Figure 2.5). Note that, since the only vertices we blow up to obtain an extended block graph from a block graph are cut vertices, each extended block graph has a unique block graph associated with it by contracting each joint block to a single vertex.

A simple connected graph G is a *vertebrate graph* if there exists B , an induced subgraph of G , such that:

- B is an extended block graph.
- For all $c \in V(B)$, $x \in V(G)$ with $d(c, x) \geq 2$, there exists $c' \in N_B(c, x)$ such that $N_G[c'] \supseteq N_G[v]$ for all $v \in N_G(c, x)$.

We will refer to the subgraph B in the previous definition as a backbone of G .

Notice the following fact about vertebrate graphs, which we state as a lemma:

Lemma 2.4.1. *Let G be a vertebrate graph with backbone B . If $x \in V(G)$, then there exists $c \in V(B)$ such that $c \in \bigcap_{v \in N_G[x]} N[v]$.*

Proof. Let c be a vertex in $N_B[x]$. If there exists $y \in N_G[x] \setminus N_G[c]$, then there exists $c' \in N_B(c, y)$ such that $N_G[c'] \supseteq N_G[v]$ for all $v \in N_G(c, y)$. In particular, $N_G[x] \subseteq N_G[c']$. \square

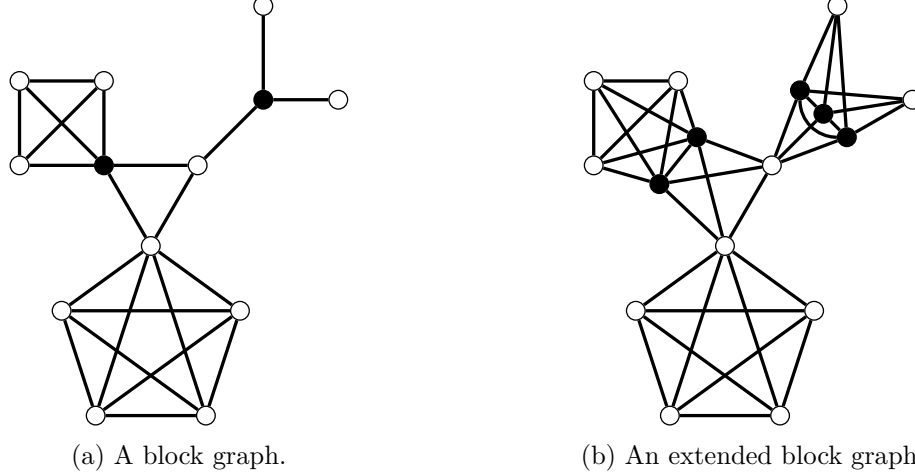


Figure 2.5: The black vertices on the block graph (a) were blown-up into cliques to obtain the extended block graph (b).

Theorem 2.4.1 ([22]). *Any vertebrate graph is 1-guardable.*

Proof. Suppose that G is a vertebrate graph, with a backbone B , which is not a Helly graph. Since G is not a Helly graph, there exists an integer $k \geq 3$, a set of vertices $\{v_i\}_{i=1}^k$ and a set of positive integers $\{d_i\}_{i=1}^k$ such that $N^{d_i}[v_i] \cap N^{d_j}[v_j] \neq \emptyset$ for all $1 \leq i < j \leq k$ but $\bigcap_{i=1}^k N^{d_i}[v_i] = \emptyset$.

Let T be the block-cut tree corresponding to the block graph from which B is obtained. Let T_i be the minimal subtree of T whose vertices correspond to the blocks and cut vertices containing vertices of $N^{d_i}[v_i]$. By Claim 2.4.1, we have $N^{d_i}[v_i] \cap N^{d_j}[v_j] \cap V(B) \neq \emptyset$, so $V(T_i) \cap V(T_j) \neq \emptyset$. Since a family of subtrees of a tree has the Helly property, then there is a subtree T' such that $V(T') \subseteq V(T_i)$ for $1 \leq i \leq k$. Let A be subset of $V(B)$ corresponding to a vertex of T' . Notice that $A \subseteq N^{d_i}(v_i)$ if $v_i \in V(B)$ or, by Lemma 2.4.1, if $d_i = 1$.

Let $C \subseteq B$ be the smallest clique such that $x \in N^{d_i}(v_i)$ if $v_i \in V(B)$ or $d_i = 1$, and such that $N[v_i] \cap C \neq \emptyset$ for $1 \leq i \leq k$. Notice that the fact that G is not Helly implies $|C| > 2$. We say that a vertex $u \in V(G) \setminus C$ is a private neighbour in C if $|N(u) \cap (C)| = 1$. We may also assume that C has the minimum possible number of private neighbours.

Since C is of minimum order, we know that for every $x \in C$, there exists $i \in \{1, \dots, k\}$ such that $d_i = 1$ and $N[v_i] \cap C = \{x\}$. Let x and y be vertices in C and assume that v_1 and v_2 are private neighbours in C such that $v_1 \in N(x)$ and $v_2 \in N(y)$.

If $N[x] \cap N[v_1] \cap N[v_2] \neq \emptyset$, then exists $c \in V(B)$ such that $(N_G[v_1] \cup N[y]) \subseteq N_G[c]$, and so by taking $C' = (C \setminus \{y\}) \cup \{c\}$ we obtain a clique C' of the same order as C but with fewer private neighbours, a contradiction. Hence, we must have that $N_G[x] \cap N_G[v_1] \cap N_G[v_2] \neq \emptyset$ and analogously that $N_G[y] \cap N_G[v_1] \cap N_G[v_2] \neq \emptyset$. In particular, this implies $d(v_1, v_2) = 2$ and, by Lemma 2.4.1, there exists $z \in N_B[v_1] \cap N_B[v_2]$.

Let u and w be vertices in B obtained by applying Lemma 2.4.1 to v_1 and v_k , respectively. Notice that x, y, x, u, w are vertices of B contained in a cycle of length five. Since B is chordal, this cycle cannot be induced, so without loss of generality we may assume that $\{x, u\} \subseteq N[w]$. Since $d(w, v_1) = 2$, there exists $c \in V(B)$ such that $(N_G[x] \cup N[z]) \subseteq N_G[c]$. However, by taking $C' = (C \setminus \{x\}) \cup \{c\}$ we obtain a clique C' of the same order as C but with fewer private neighbours.

This implies there exists $x \in C$ such that x is adjacent to every private neighbour in C . Hence, $x \in \bigcap_{i=1}^k N^{d_i}[v_i]$, contradicting the assumption that G was not Helly. \square

2.5 Other ways to guard a subgraph

While the problem of 1-guarding an isometric subgraph is relatively well understood thanks to Theorem 2.3.1, we may want to extend this notion further and guard an isometric subgraph H of G using several cops. This poses the problem of finding properties that imply that H can be k -guarded for some $k \geq 2$. The idea of finding a partition of H into subgraphs that are t -guardable for $t < k$ has been explored by Clarke in [11], where the problem of partitioning a graph into isometric paths was studied. In light of this, the following questions seems to be of interest in the area:

Question 2.5.1. *What is the minimum number of isometric Helly graphs into which we can partition a graph G of order n ?*

In particular, it would be of interest to look at the problem of partitioning a graph into isometric trees.

On the other hand, the following seems like a natural question

Question 2.5.2. *Is there a function $f(k)$ such that, if H is an isometric subgraph of G (not necessarily a Helly subgraph) and $c(H) = k$, then H can be $f(k)$ -guarded?*

Unfortunately, this fails for $k = 1$. It is not hard to check that the subgraph H in Figure 2.6 cannot be guarded by a single cop despite being isometric and dismantlable, but two suffice. In this section we provide a family of graphs, generalizing the one in Figure 2.6 which contains dismantlable isometric subgraphs that require arbitrarily many cops to be guarded.

Let $r \geq 2$ and $s \geq 1$ be integers and let $t = 2r - 1$. Let T be a complete graph with t vertices and for each $X \in \binom{V(T)}{r}$, let K_X be a copy of the complete graph on $s + r$ vertices. We define $H(r, s)$ as the graph obtained by identifying r vertices in K_X with the set X for every $X \in \binom{V(T)}{r}$. The complete graph T will be referred to as the *base* of H .

Let G be a graph and S a subset of the vertices of G . We say that S is a *dominating* set in G if for every vertex $x \in V(G)$ there exists $v \in S$ such that $x \in N[v]$.

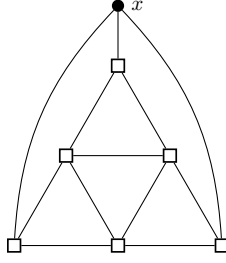


Figure 2.6: The dismantlable subgraph H induced by the square vertices is isometric in the whole graph.

Theorem 2.5.1. *For every positive integer $k \geq 3$ there exist graphs G and H such that*

- H is an isometric subgraph of G ,
- $c(H) = 1$, and
- more than k cops are required to guard H in G .

Proof. Let r and s be as above. Suppose that $s > k$ and $r > k$ and $t = 2r - 1$, and consider the graph $H = H(t, s)$.

Notice that a dominating set in H must contain at least r vertices since no smaller set can dominate a vertex in each $K_X \setminus X$, and any subset of $V(T)$ with r elements dominates H . Also, a subset A of $V(H)$ is dominating if and only if $A \cap V(K_X) \neq \emptyset$ for every X . We know that $c(H) = 1$ since H is chordal, and H has diameter two since for every $X, Y \in \mathcal{T}$, where $\mathcal{T} = \binom{V(T)}{r}$, we have $X \cap Y \neq \emptyset$.

Let \mathcal{A} be a family of subsets of $V(H) \setminus V(T)$ such that $|S \cap K_X| = 1$ for each $X \in \mathcal{T}$ and $S_1 \cap S_2 = \emptyset$ for distinct $S_1, S_2 \in \mathcal{A}$. Let G be the graph obtained from H by adding a vertex v_S adjacent to every vertex in S , for each $S \in \mathcal{A}$. Since H has diameter two, H is an isometric subgraph of G . If we have k cops in H , since $\gamma(H) > k$, there exists $X_0 \in \mathcal{T}$ such that there is no cop in K_{X_0} . Also, since $s > k$, there exists $S \in \mathcal{A}$ such that there is no cop in $N[v_S]$, so we can place the robber in v_S . Notice that regardless of the how the cops choose to move, there exists a vertex in $N(v_S)$ that no cop can reach in one move, so the robber can enter H and no cop can capture him in the following turn. \square

While Theorem 2.5.1 already shows that the Helly property is necessary in order to bound the number of cops required to guard an isometric subgraph H with $c(H) = 1$, it might seem like an artificial example as the robber cannot enter H arbitrarily many times if the cops are smart. However, we may strengthen this result to obtain a graph in which the robber may evade the cops indefinitely as well as enter H an infinite number of times. The following theorem shows that even for relatively simple dismantlable graphs, many cops

might be needed to prevent the robber from entering them an infinite number of times in the game of cops and robbers.

Theorem 2.5.2. *For every positive integer k , there exist graphs G and H such that H is a chordal graph of diameter 2 and an isometric subgraph of G , and no matter how the k cops move, the robber can step on a vertex of H arbitrarily many times without being captured.*

Proof. Let k be a fixed positive integer and let $H = H(m, 1)$ where m is a large positive integer (lower bounds for m will be given later in the proof). Notice that given any set C of vertices in the base of H , with $|C| < m$, there will be $\binom{2m-1-|C|}{m}$ subsets of m vertices of the base of H containing no vertices of C . Let X be the set of vertices of H that are not in the base and take a set Y of new vertices, with $|Y| = |X|$, and add edges between X and Y so that $X \cup Y$ induces a bipartite graph with the property that if the robber is in $X \cup Y$ and it is his turn to move, there is a vertex in its neighbourhood in $X \cup Y$ that no cop can reach in one move, which we will call a *safe neighbour*.

To show that such a bipartite graph exists, we will use a probabilistic argument. Let A_x be the event “the vertex $x \in X$ has no safe neighbour in Y ” and B_y the event “the vertex $y \in Y$ has no safe neighbour in X ”. We will show that if m is large enough, the probability that a random bipartite graph satisfies those properties is non-zero, so such a bipartite graph exists.

Let $0 < \frac{1}{4k} < p < \frac{1}{2k}$ and consider a random bipartite graph on $X \cup Y$ where the edge xy for $x \in X$ and $y \in Y$ is chosen with probability p . We claim that the robber always has a safe neighbour he can move to, regardless of the cops’ positions. Suppose that the robber is in X . In this case, since the robber wants to move to Y , any cop in the base of H will not be able to prevent the robber from moving to Y , so we may assume that all the cops are in $X \cup Y$. Also, notice that a cop in $u \in Y$ can only prevent the robber from moving to u , which she can also achieve by being on a neighbour of u in X . This implies that the worst situation for the robber is when all cops are in X , so it suffices to show that the robber can evade the cops in this case.

Suppose that the robber is at $x \in X$. Notice that the robber will be unable to move from x to y if x and y are not adjacent (which happens with probability $(1 - p)$) or if there is a cop at a vertex $u \in X$, where both xy and uy are edges of the graph, which happens with probability p^2 for every cop. We may assume that all cops are in X , as a cop in $z \in Y$ would only be able to prevent the robber from stepping on z , which is also achieved by having a cop on a neighbour of z in X .

Notice $\mathbb{P}(A_x) = \prod_{y \in Y} \mathbb{P}(y \text{ is not a safe neighbour of } x)$ due to independence. As noted before, $\mathbb{P}(y \text{ is not a safe neighbour of } x) \leq kp^2 + (1 - p)$, and we have that

$$kp^2 + (1 - p) \leq \frac{1}{4k} + 1 - p = 1 - \left(p - \frac{1}{4k}\right) < 1.$$

By taking $\varepsilon = p - \frac{1}{4k}$, we get that $\mathbb{P}(y \text{ is not a safe neighbour of } x) \leq 1 - \varepsilon$, which yields $\mathbb{P}(A_x) = (1 - \varepsilon)^{|Y|}$. We now have that

$$\begin{aligned} \mathbb{P}\left(\bigcup_{x \in X} A_x\right) &\leq |X| \mathbb{P}(A_x) \\ &\leq |X| (1 - \varepsilon)^{|Y|} \\ &\leq |X| e^{-\varepsilon|Y|} \\ &< \frac{1}{2}, \end{aligned}$$

where the strict inequality holds for large enough m .

Suppose now that the robber is on a vertex $y \in Y$. Just as in the other case, the robber will not be able to move from x to y if x and y are not adjacent or if a cop at a vertex $u \in X$ is also adjacent to y . However, we must now consider that there might be cops in the base of H guarding some of the vertices of X . However, regardless of the number of cops in the base of H or their positions, there are always at least $\binom{2m-1-k}{m}$ vertices in X that no cop in H can see, and so

$$\begin{aligned} \mathbb{P}\left(\bigcup_{y \in Y} B_y\right) &\leq \binom{2m-1}{m} \mathbb{P}(B_y) \\ &\leq \binom{2m-1}{m} (1 - \varepsilon)^{\binom{2m-1-k}{m}} \\ &\leq \binom{2m-1}{m} e^{-\varepsilon \binom{2m-1-k}{m}} \\ &< 2^{2m} e^{-\varepsilon 2^{m/2}} \\ &< \frac{1}{2}, \end{aligned}$$

where the strict inequalities hold when m is large enough. Then it follows that

$$\mathbb{P}\left(\bigcup_{x \in X} A_x \cup \bigcup_{y \in Y} B_y\right) \leq \mathbb{P}\left(\bigcup_{x \in X} A_x\right) + \mathbb{P}\left(\bigcup_{y \in Y} B_y\right) < 1.$$

We conclude that there exists m such that in G , the graph obtained from H by adding a set of vertices Y with $|Y| = \binom{2m-1}{m}$ and the corresponding bipartite graph between Y and the vertices of H , is a graph where H is an isometric subgraph and the robber can enter H arbitrarily many times without being captured by k cops. \square

Chapter 3

Planar graphs and lazy cops

Ever since Aigner and Fromme published their seminal paper about cops and robbers on planar graphs, there has been considerable interest in the study of the connections between the topological properties of a graph and the cop number. It is therefore not unexpected to see that a frequent question that arises whenever a new variation of the cop number is introduced, is whether that parameter is bounded for the class of planar graphs.

Recently, Gao and Yang [17] studied the game of lazy cops and robbers (which they refer to as “1-cop moves game”) in planar graphs. They were able to find a planar graph in which three cops cannot win the game if only one of them can move at a time, a very surprising result. Aigner and Fromme’s result tells us that if three cops are allowed to move every turn, they can capture the robber in any planar graph, while Gao and Yang showed that if only one is allowed to move, then three cops do not suffice. Recall that $c_\ell(G)$ is the minimum number of cops that can guarantee the robber’s capture in G with the restriction that at most ℓ cops can move each turn. Yang conjectured [33] the following for the case when two cops are allowed to move each turn:

Conjecture 3.0.1 ([33]). *For every planar graph G , we have $c_2(G) \leq 3$.*

In this chapter we will provide a proof of that conjecture, as well as some interesting directions for future research.

3.1 Bypaths and wide shadows

In order to describe when the wide shadow of the robber will be more than a single vertex, we need to define an additional structure which we call a bypath, and show its connection with the wide shadow of the robber.

Let G be a graph, H a subgraph of G , and $P = v_1v_2 \cdots v_k$ an isometric path in H . A path $B = b_1b_2 \cdots b_t$ ($t \geq 3$) contained in H is called a *bypath* of P in H if $B \cap P = \{b_1, b_t\}$ and the path $P_{\langle B \rangle} = Pb_1Bb_tP$ is also an isometric path in H . Note that if $b_1 = v_i$ and $b_t = v_j$ (where $i < j$), then $j - i = t - 1$ since P and $P_{\langle B \rangle}$ are both isometric. The vertices

b_1 and b_t are called the *branching vertices* of B , and the path $P_{\langle B \rangle}$ is called the bypath B of P , and we will denote it by $P_{\langle B \rangle}$. A path P is *bypath-free* in H if H contains no bypath of P . Observe that if P is bypath-free in H , then every subpath of P is bypath-free in H . Notice that, in a simple graph G , all paths of length 1 are trivially bypath-free in G . In the following, we will refer to paths of length at least 2 as non-trivial.

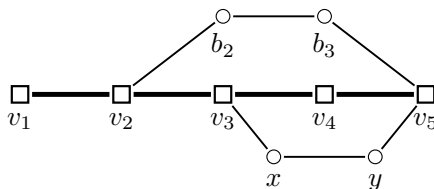


Figure 3.1: The path $B = v_2b_2b_3v_5$ is a bypath of $P = v_1v_2v_3v_4v_5$, while v_3xyv_5 is not.

Lemma 3.1.1. *Let G be a graph and P a non-trivial isometric path of G . If v is a vertex of G not in P , then $|S_P(v)| = 1$ if and only if there exists a bypath of P in G containing v .*

Proof. Notice that if B is a bypath of P with branching vertices u and w , then $d(v, u) + d(v, w) = d(u, w)$ for every $v \in V(B)$, so $|P_v(u) \cap P_v(w)| = 1$. By Lemma 2.2.1 we know that $S_P(v) \neq \emptyset$, so the fact that $S_P(v) \subseteq P_v(u) \cap P_v(w)$ implies $|S_P(v)| = 1$.

For the other direction, let $v \in V(G) - V(P)$ such that $S_P(v) = \{x\}$. Since P is a path, there exist vertices $u, w \in V(P)$ such that $P_v(u) \cap P_v(w) = \{x\}$. Let y, z be the vertices of P such that $d(y, z)$ is minimum with the property $P_v(y) \cap P_v(z) = \{x\}$.

Let B_y and B_z be shortest paths from v to y and from v to z , respectively. Notice that $V(B_y) \cap V(B_z) = \{v\}$ (otherwise, it would contradict the assumption of P being isometric), so concatenating B_y and B_z results in a path B , and take $P' = P_yBzP$. Since $d(y, v) + d(v, z) = d(y, z)$, we have $d_P(y, z) = d_{P'}(y, z)$, and so P' is also an isometric path. Hence B is a bypath of P in G . \square

3.2 The 2-cops-move number of planar graphs

The purpose of this section is to prove that $c_2(G) \leq 3$ for every connected planar graph G .

We say that an induced subgraph H is *k-leisurely-guardable* if it is *k-guardable*, and there exists a *k-guarding* strategy of H for a set of cops $\mathcal{C} = \{C_1, C_2, \dots, C_k\}$ such that, after a finite number of moves, there is a turn where either the robber enters H or at least one C_i can stay still and H is still being *k-guarded* by \mathcal{C} .

Lemma 3.2.1. *Let G be a planar graph and P an isometric path in G . If P is bypath-free in G , then P is 1-leisurely-guardable.*

Proof. After a finite number of moves, we can get one cop C to move to a vertex in the wide shadow of the robber on P . Once C is in $S_P(R)$, she will stay still if she is in $S_P(R)$

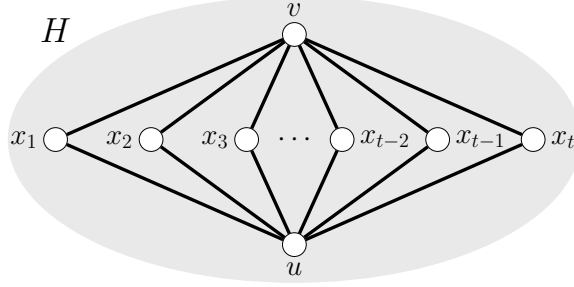


Figure 3.2: The partition induced by vertices u , v and their common neighbours.

after the robber's turn, and will move when her position after the robber's turn is not in the wide shadow of the robber. By Lemma 2.2.1, the cop can always get back in the wide shadow of R with a single move. By staying in the wide shadow of the robber, the cop can 1-guard P .

If P has length ℓ , with $\ell \leq 2$, the cop can guard it without moving by simply staying at some vertex of P , so we may assume $\ell \geq 3$. Since P is bypath-free in G , we have $|S_P(R)| \geq 2$, so the robber can only move at most ℓ consecutive times in such a way that the cop will have to move in order to stay in the wide shadow. Hence, after at most ℓ turns, the robber will either enter P or the cop can stay still and be 1-guarding P , so P is 1-leisurely-guardable. \square

Observation 3.2.1. *If H is a connected graph and $v \in V(H)$, then one of the following holds:*

- i. $N[v] = V(H)$.*
- ii. There exists a non-trivial isometric path P in H , starting at v , which is bypath-free in H .*
- iii. There exists a vertex $u \in V(H) \setminus N(v)$ such that $|N(v) \cap N(u)| \geq 2$.*

Notice that in case (iii) of Observation 3.2.1, if there is a vertex u at distance two from v having $N(u) \cap N(v) = \{x_1, x_2, \dots, x_t\}$, then the paths ux_iv partition the plane into t "faces" (see Figure 3.2).

In the sequel, we will play the game on a graph embedded in the plane.

Whenever we restrict the robber to be in a region whose boundary is a closed curve, we will assume he is in the curve's interior. By keeping this in mind, given two internally disjoint paths with the same endpoints in G , say P and Q , we will use $R(P, Q)$ to denote the set of vertices contained in the interior of the curve defined by $P \cup Q$.

The following lemma provides us with a way of forcing the robber to be inside a region whose boundary can be leisurely-guarded.

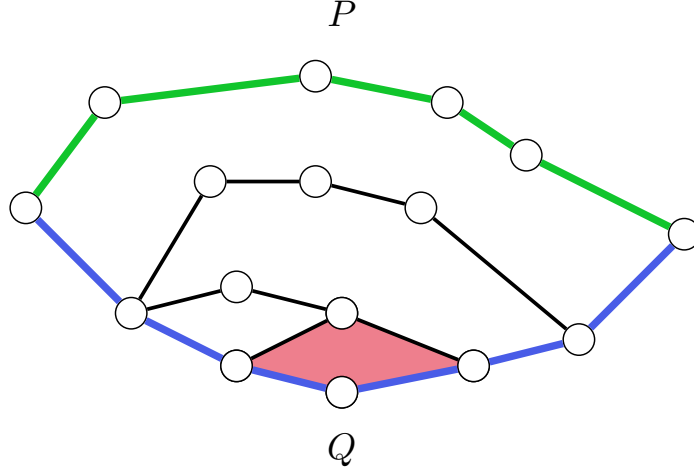


Figure 3.3: There are several bypaths for the path Q . The shaded area corresponds to the region of the plane defined by the chosen bypath of Q .

Lemma 3.2.2. *Let G be a planar graph, $u, v \in V(G)$, P and Q be internally disjoint uv -paths in G . Suppose that*

- P is isometric and bypath-free in $G[R(P, Q) \cup V(P) \cup V(Q)]$ and
- Q is isometric in $H_Q = G[R(P, Q) \cup V(Q)]$.

Then either Q is bypath-free in H_Q or there exists a bypath B of Q in H_Q such that

- i. P is isometric and bypath-free in $G[R(P, Q_{\langle B \rangle}) \cup V(P)]$.*
- ii. Q is isometric and bypath-free in $G[R(Q, Q_{\langle B \rangle}) \cup V(Q)]$.*
- iii. $Q_{\langle B \rangle}$ is isometric in $G[R(P, Q_{\langle B \rangle}) \cup V(Q_{\langle B \rangle})]$ as well as isometric and bypath-free in $G[R(Q, Q_{\langle B \rangle}) \cup V(Q_{\langle B \rangle})]$.*

Proof. Suppose that Q is not bypath-free in H_Q and let B be a bypath of Q in H_Q such that $R(Q_{\langle B' \rangle}, Q) \subseteq R(Q_{\langle B \rangle}, Q)$ implies $B' = B$ for every bypath B' of Q in H_Q (see Figure 3.3). Let us check that B satisfies the desired properties.

Clearly, (i) follows from the fact that $R(P, Q_{\langle B \rangle}) \cup V(P) \subseteq R(P, Q)$. Since Q is isometric in H_Q , then Q and $Q_{\langle B \rangle}$ have the same length, and so are isometric in $G[R(Q, Q_{\langle B \rangle}) \cup V(Q) \cup B]$. If Q has a bypath B' in $G[R(Q, Q_{\langle B \rangle}) \cup V(Q)]$, then $R(Q, Q_{\langle B \rangle}) \subseteq R(Q, Q_{\langle B' \rangle})$, which implies $B = B'$, a contradiction. If $Q_{\langle B \rangle}$ has a bypath B' in $G[R(Q, Q_{\langle B \rangle}) \cup V(Q_{\langle B \rangle})]$, take $Q' = (Q_{\langle B \rangle})_{\langle B' \rangle}$ so then $R(Q, Q') \subsetneq R(Q, Q_{\langle B \rangle})$. In this case, there exists a bypath B'' in H_Q such that $Q_{\langle B'' \rangle} = Q$, which contradicts the choice of B . This shows (ii) and (iii). \square

The proof of Theorem 3.2.1 is similar to the proof of Theorem 2.0.1 provided in [8], but the cops' strategy uses our Lemmas 3.2.1 and 3.2.2. At any point of the game, the *robber territory* is the set of vertices the robber can enter without being captured by a cop, and the rest of the vertices form the *cop territory*. Notice that if the robber territory is empty, that means the robber has been captured. The idea of the proof is to play the game using three cops in such way that we reduce the size of the robber territory until it becomes empty.

Theorem 3.2.1. $c_2(G) \leq 3$ for every planar graph G .

Proof. Let G be a planar graph and fix a drawing of G in the plane. We will distinguish three different situations the game can be in:

- a) A cop is leisurely-guarding a path P of a subgraph H of G , and every path from the robber to the cop territory includes a vertex of P .
- b) Two cops leisurely-guarding $P \cup Q$, where P and Q are internally disjoint paths joining the same two vertices, and any path from the robber to the cop territory includes a vertex of $P \cup Q$. The subgraph H is either in the internal or external region bounded by $P \cup Q$ (by possibly performing an inversion, we can always assume it is the internal region).
- c) A cop is on a vertex v of a subgraph H of G , and every path from the robber to the cop territory goes through v .

When we say that a cop guards a path, we will assume that we are using the wide shadow strategy given by Lemma 2.3.1.

Let C_1 , C_2 and C_3 be the cops and begin the game by placing them on the same vertex v . Let r be the robber's position. If v dominates G we are done. Otherwise, Observation 3.2.1 guarantees that, by taking $H = G$, and moving at most two cops, we are in case (a) or (b).

- If we get property (ii) in Observation 3.2.1, then by moving one cop to leisurely-guard P we get to case (a).
- If we get property (iii), then by moving C_1 and C_2 to v and u , we get guard $N(u) \cap N(v)$. Notice that the subgraph induced by the edges between $\{u, v\}$ and $N(u) \cap N(v)$ is isomorphic to $K_{2,n}$. Since it is drawn in the plane, this induces a quadrangulation of the plane, and the robber must be contained in the interior of one of this square, else he would be captured by C_1 or C_2 . Let $x, y \in N(u) \cap N(v)$ be the vertices such that the paths $P = uxv$ and $Q = uyv$ satisfy that the robber is in $R(P, Q)$ but $R(P, Q) \cap N(u) \cap N(v) \neq \emptyset$. If we now move C_1 to x and C_2 to y , we arrive in case (b).

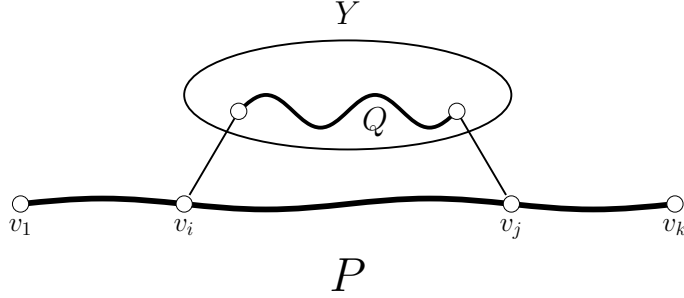


Figure 3.4: Q corresponds to the shortest $v_i v_j$ -path using vertices in Y .

We will show that starting with case (a), (b) or (c), we can move the cops and get again to one of the three cases, increasing the cop territory, moving at most two cops every turn. We will use T to denote the set of vertices in the cop territory.

Suppose we start with case (a) and C_1 is leisurely-guarding $P = v_1 v_2 \dots v_k$ with $k \geq 2$ in H . Let Y be the component of $H - T$ containing the robber. If there is a unique vertex of P with neighbours in Y , say x , then we can move C_2 to x , getting to case (c).

If P has more than one vertex with neighbours in Y , let v_i and v_j , with $i < j$, be the first and last vertices of P with neighbours in Y . Let Q be a shortest $v_i v_j$ -path whose internal vertices are in Y , and move C_2 to guard Q (see Figure 3.4). The robber cannot enter the cop territory unless he enters $v_i P v_j$, so we may assume that P is a $v_i v_j$ -path since $v_i P v_j$ is also isometric and bypath free in H , which is guaranteed by our choice of v_i and v_j . Notice that the robber is in a component of $Y - Q$ which is either in the bounded or unbounded face determined by $P \cup Q$. Without loss of generality, we may assume he is in the bounded face. If Q is being leisurely-guarded by C_2 we get case (b), adding the internal vertices of Q to the cop territory. Otherwise, P is isometric and bypath-free in $G[R(P, Q) \cup V(P) \cup V(Q)]$ and Q is isometric in $H_Q = G[R(P, Q) \cup V(Q)]$, so we can apply Lemma 3.2.2 and obtain the bypath B of Q in H_Q .

Since P is being leisurely-guarded by C_1 , we can use the turns in which C_1 stays still to move C_3 and capture the robber's wide shadow on $Q_{(B)}$. Once C_3 catches the robber's wide shadow on $Q_{(B)}$, we have two possible outcomes:

1. If $r \in G[R(Q, Q_{(B)})]$, then we get to case (b) by possibly taking subpaths of Q and $Q_{(B)}$, since both paths would be leisurely-guarded, and C_1 is free to move.
2. If $r \in G[R(P, Q_{(B)})]$, cop C_2 is free to move. Also, we can apply Lemma 3.2.2 again until the robber's territory is empty or he is in a region bounded by two leisurely-guarded paths, arriving to case (b).

In any case, we have increased the cop territory at least with the bypath B .

Now, suppose we start with case (b), with C_1 and C_2 leisurely-guarding P and Q , respectively. Let Y be the component of $H - (P \cup Q)$ containing the robber. Without loss

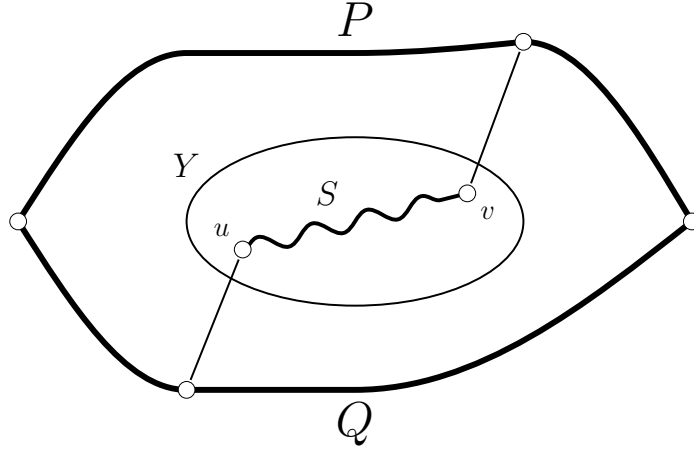


Figure 3.5: The case when $P \cup Q$ has exactly two vertices in the robber territory.

of generality, we may assume Y is contained in the interior of $P \cup Q$. If only one vertex v of Y has neighbours in $P \cup Q$, then we can use the turns when C_1 and C_2 stay still to place C_3 on v and get to case (c).

If $P \cup Q$ has exactly two neighbours in Y , namely u and v , let S be a shortest uv -path in Y (see Figure 3.5). We can move C_3 to guard S during the turns when C_1 or C_2 stay still. Once C_3 is guarding S , the robber cannot enter the cop territory since he could only do it using u or v , which are being covered by C_3 . This means C_1 and C_2 are free.

If S has length at most two, we get to case (a) by placing a cop in the middle vertex of S (or any vertex of P in the case of length one). If S has length more than two, then we can add two vertices to the graph, x and y , and make them adjacent to both u and v , and (by possibly applying an inversion) draw them one to the left and one to the right of the graph induced by $V(P) \cup V(Q) \cup Y$ (these additional vertices are included only in order to apply Lemma 3.2.2). The paths uxv , S and uyv form a theta-graph, and the robber is in a component Y of $G - S$ inside the left or the right bounded region defined by the theta graph. Without loss of generality, we may assume he is the region bounded by $P = uxv$ and S . The fact that S has length more than two implies P is isometric and bypath-free in the graph, so by applying Lemma 3.2.2 and moving at most two cops each turn, we can arrive to one of two situations:

1. The robber is in the interior of the region defined by P and a path Q in Y , which is isometric and bypath-free path in $G[Y \cap R(P, Q) \cup V(Q)]$, being leisurely-guarded.
2. There are two paths Q and Q' in Y , such that the robber is in $G[R(Q, Q') \cup V(Q) \cup V(Q')]$, Q is isometric and bypath-free in $G[Y \cap R(Q, Q') \cup V(Q)]$, and Q' is isometric and bypath-free in $G[Y \cap R(Q, Q') \cup V(Q')]$, and both Q and Q' are being leisurely-guarded.

Since we get to case (a) in the first situation, and case (b) in the later, and we add the vertices u and v to the cop territory, we make progress.

We may now assume that at least one of P and Q has at least two neighbours in Y . Without loss of generality, we may assume $P = v_1v_2 \dots v_k$, with $k \geq 2$, has two distinct neighbours in Y . Let v_i and v_j , with $i < j$, be the first and last vertices of P with neighbours in Y . Let u_1 and u_2 be neighbours of v_i and v_j in Y , respectively, such that $d_Y(u_1, u_2)$ is minimum. Let S be a shortest u_1u_2 -path in Y . In this case, we can move C_3 during the turns when C_1 or C_2 can stay still and captures the robber's wide shadow in $S' = uPv_iSv_jv$. At this point, the robber will be in one of the bounded regions in the interior of the theta graph formed by P, Q and S' .

Without loss of generality, the robber is in $R(P, S')$. Since we can leisurely-guard P and we are guarding S , cop C_2 is free. By applying Lemma 3.2.2, we can get to a situation where the robber will be in a region bounded by two paths that are being leisurely-guarded. At the end of this process, we added vertices u and v to the cop territory, so we have increased the cop territory and we get to case (b).

Finally, assume we are in case (c). By applying Observation 3.2.1 it is easy to see that properties (i), (ii), and (iii) get us to cases (a), (b), and (c), respectively, by moving at most two cops each turn. \square

3.3 Is there something between c_1 and c_2 ?

Recall that we defined $c_k(G)$ as the number of cops required to guarantee the robber's capture in G with the added restriction that at most k can move every turn. In the case of the class of planar graphs \mathcal{G} , we know that $c_1(G) \geq 4$ for some $G \in \mathcal{G}$ but $c_2(G) \leq 3$ for every $G \in \mathcal{G}$. However we might wonder if there is a way to define an "intermediate" parameter between c_1 and c_2 .

First, we might think of using "fractional cops" to move on the graph and capture the robber. Formally, we can define this as follows:

Let G be a connected graph. A *cop function* on G is a function $C: V(G) \rightarrow \mathbb{R}_{\geq 0}$. We say a cop function C has weight W_C if $\sum_{v \in V(G)} C(v) = W_C$. Let C and C' be two cop functions on G . For a positive real number r , we say that C can r -move to C' if the following three conditions are satisfied:

- $W_C = W_{C'}$.
- For every $v \in V(G)$, $C'(v) \leq \sum_{u \in N[v]} C(u)$.
- $\sum_{v \in V(G)} |C(v) - C'(v)| \leq 2r$.

If the robber is on a vertex $u \in V(G)$, we will say that a *cop function captured the robber* if $C(u) \geq 1$. For a graph G , we say that $c_r(G) \leq W$ if there exists a cop function C such

that, for every initial position of the robber on G and any strategy for the robber, there exists a sequence of cop functions $\{C_i\}_{i \in \mathbb{N}}$ with the following properties:

- $C_0 = C$.
- $W(C) = W$.
- C_j can r -move to C_{j+1} for every $j \in \mathbb{N}$.
- C_m captured the robber for some $m \in \mathbb{N}$.

While we can ask whether any result on cops and robbers extends to this variant, the following question seems interesting:

Question 3.3.1. *What is the smallest positive real number r such that for every connected planar graph G we have $c_r(G) \leq 3$?*

For the second, we will restrict the number of cops.

This first version keeps the “number of cops” that can move constant every turn and allows parts of the cops to move. The second will instead change the number of cops that can move every turn.

Let $S = \{n_i\}_{i \in \mathbb{N}}$ be a sequence of positive integers. For a graph G , let $c_S(G)$ be the minimum number of cops that can guarantee the robbers capture in G with the restriction that at most n_j cops can move during round j . If S_1 and S_2 are the constant sequences 1 and 2, respectively, we would have $c_{S_1}(G) = c_1(G)$ and $c_{S_2}(G) = c_2(G)$.

We may define an order on the set of sequences of positive integers as follows: Let $A = \{a_i\}_{i \in \mathbb{N}}$ and $B = \{b_i\}_{i \in \mathbb{N}}$ be two sequences of positive integers. We say that $A <_S B$ if $a_i < b_i$ for every $i \in \mathbb{N}$ and for every $m \in \mathbb{N}$ there exists $M \geq m$ such that $a_M < b_M$.

Here, we pose the following question:

Question 3.3.2. *Is there a sequence of positive integers A , with $A <_S S_2$ such that $c_A(G) \leq 3$ for every planar graph G ?*

Chapter 4

Active cops and robbers

We will now focus on a different way of restricting the movement of the players: no player is allowed to stay put during their turn. In contrast with the case of lazy cops and robbers, it is less obvious that this restriction actually changes the game in a significant way, so we will start by showing that, indeed, it does.

4.1 Differences between the classic and the active version

As stated in the introduction, we will use $c_a(G)$ to denote the minimum number of cops required to guarantee the robber's capture in G for the active version of the game. We will start by showing upper and lower bounds for the active cop number, a result proved in [18].

Theorem 4.1.1. *For every connected graph G , we have $c(G) - 1 \leq c_a(G) \leq 2c(G)$.*

Proof. For the upper bound, suppose that $c(G) = k$. Let S be a winning strategy for k cops P_1, P_2, \dots, P_k . To adapt S to the active version, for each $1 \leq i \leq k$, place a cop C_i on the same vertex as P_i in the usual game on G . Now, for $1 \leq i \leq k$, place a new cop C'_i on a vertex adjacent to C_i . Suppose the robber chooses his initial position on G . We may adapt S to obtain a strategy S' for the active version by doing the following: Every time a cop has to move in S , the corresponding pair of cops C_i and C'_i will move in such a way that exactly one of them is on the same vertex as the one indicated for P_i by S , and the other one is on a vertex adjacent to it. Notice that if P_i is ever required to stay put, then C_i and C'_i can simply switch positions, and so we can guarantee the robber's capture.

For the lower bound, notice that if $c_a(G) \leq c(G) - 2$, then we may capture the robber in the usual game by using one cop whose only job is to chase the robber and then $k - 1$ to follow the active strategy. Since this would give a winning strategy in the usual game using k cops, we must have that $c_a(G) \geq c(G) - 1$. \square

We might now ask if these bounds are best possible.

In [18], a graph G_k such that $c_a(G_k) = 2c(G_k) = 2k$ was constructed for every positive integer k . Additionally, it was shown that there is an infinite family of graphs \mathcal{G} such that

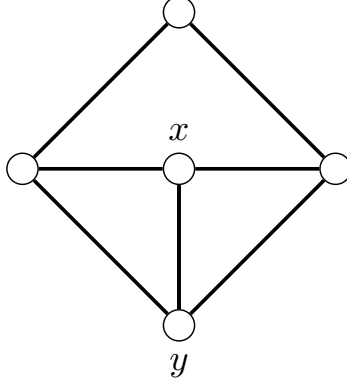


Figure 4.1: The graph H_1 .

$c_a(G) = 1$ and $c(G) = 2$ for every $G \in \mathcal{G}$. We will now build, for each positive integer k , a graph H_k such that $c_a(H_k) = c(H_k) - 1 = k$.

The following theorem is concerned with the behaviour of the active cop number with respect to the Cartesian product.

Theorem 4.1.2 ([18]). *For every connected graphs G and H , $c_a(G \square H) \leq c_a(G) + c_a(H)$.*

The graph H_1 , which has $c(H_1) = 2$ and $c_a(H_1) = 1$, is depicted in Figure 4.1. Let $H_k = H_{k-1} \square H_1$ for $k \geq 2$.

Theorem 4.1.3. $c_a(H_k) = c(H_k) - 1 = k$.

Proof. Since H_1 has C_4 as a retract, by Theorem 2.1.1 we have that $c(H_1) = 2$. Notice that H_1 has $c_a(H_1) = 1$. It follows by induction on k and Theorem 4.1.2 that $c_a(H_k) \leq k$.

In order to show that $c_a(H_k) = k$, we will show that $c(H_k) \geq k + 1$. Let $G_1 = C_4$ and for every $k > 1$ let $G_k = G_{k-1} \square C_4$. For every vertex $v \in V(H_k)$ and $1 \leq i \leq k$, let $[v]_i$ be the i -th coordinate of v . Let $v_x \in V(H_k)$ be the vertex such that $[v_x]_i = [v]_i$ if $[v]_i \neq x$ and $[v_x]_i = x$ otherwise. Take $f: V(H_k) \rightarrow V(H_k)$ such that $f(v) = v_x$.

It is easy to see that the image of f is isomorphic to G_k and that f is a homomorphism. Hence, in light of Theorem 4.1.1, it suffices to show that $c(G_k) > k$ to prove that $c(H_k) = k + 1$. Observe that for every pair of distinct vertices $x, y \in V(G_k)$, we have $|N[x] \cap N[y]| \leq 2$ and $d(x) = d(y) = 2k$. Since $|V(G_k)| = 4^k$, we have that $\gamma(G_k) > k$, so regardless of how k cops choose their initial position, there exists a vertex which the robber can choose as his initial position that is not adjacent to any cop. Since no cop can guard more than two of the robber's neighbours and, in order to force him to move, one cop must be adjacent to the robber and as such can only guard one of the robber's neighbours, we conclude that the robber has winning strategy against k cops in G_k and, by Theorem 2.1.1, he has a winning strategy against k cops in H_k . Due to Theorem 4.1.1, we get $c_a(H_k) = k$ and $c(G_k) = k + 1$. \square

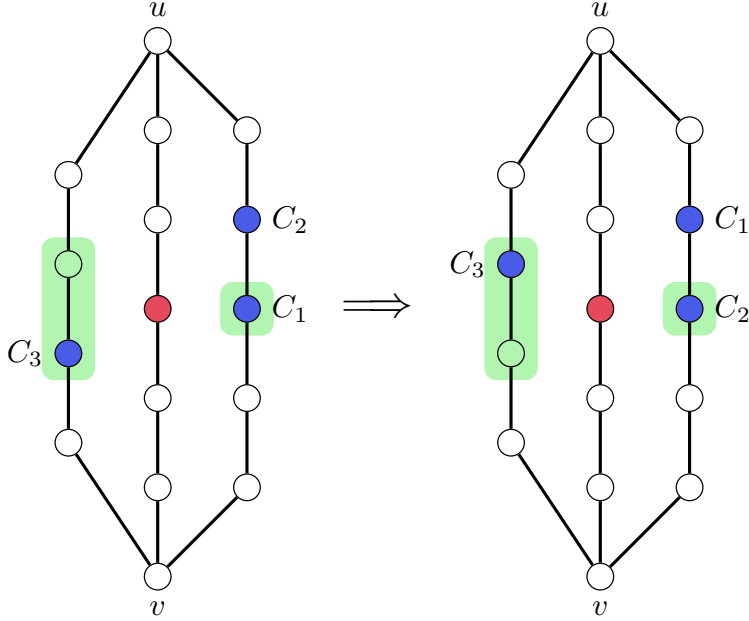


Figure 4.2: The shaded area represents the wide shadow of the robber. Cops C_1 and C_2 are tandem-guarding the right path, while C_3 is actively-guarding the left path.

Recall that Theorem 2.0.1 shows that a single cop can guard an isometric path P . This may not be possible in general for an active cop as the guarding strategy may require her to stay still during a turn, but we can use a pair of cops to do it following the strategy described in the proof of Theorem 4.1.1: we can take two cops in adjacent vertices and move them together until one of them is on the shadow of the robber on P , and follow the usual guarding strategy but, whenever the cop would be required to stand still, the two cops switch positions. We will refer to this strategy as *tandem-guarding*.

Observation 4.1.1. *A k -guardable graph can be tandem-guarded by k pairs of cops. Moreover, any winning strategy for k cops in the usual game can be translated into a winning strategy for k pairs of cops in the fully active game by moving each pair in tandem.*

We say that an induced subgraph H is *k -actively-guardable* if it is k -guardable with a strategy in which no cop stays still in any turn. For short, we will omit the integer k when $k = 1$ and, for example, say actively-guard instead of 1-actively-guard.

Now, we turn our attention to the game of active cops and robbers on planar graphs. It follows from Theorem 2.0.1 and 4.1.1 that $c_a(G) \leq 6$.

The following theorem was proved in [1].

Theorem 4.1.4. *If G is a graph with girth at least five, then $c(G) \geq \delta(G)$.*

Notice that this result also holds for the game of active cops and robbers:

Lemma 4.1.1. *If G is a connected graph with girth at least five, then $c_a(G) \geq \delta(G)$.*

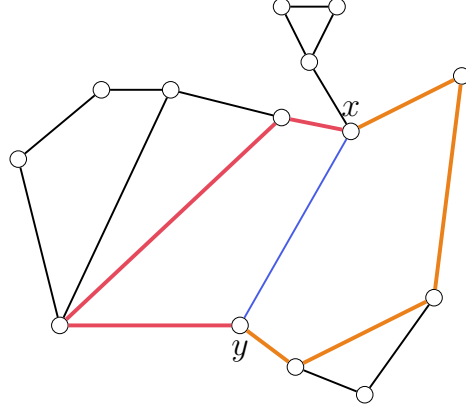


Figure 4.3: The unique xy -paths in each component of G'_0 are shown in thick lines. Notice that no such path exists in one component of G'_0 .

Proof. First, notice that under these hypotheses, we must have $\gamma(G) \geq \delta$. Suppose that S is a dominating set of G such that $u \in S$ and v is a neighbour of u not in S . Notice that no vertex in S can dominate more than one neighbour of v , and so S must have at least $d(v) \geq \delta$ vertices. This means that, at the beginning of the game in G using $\delta - 1$ cops, the robber can choose a position such that no cop is at distance less than two from the robber. After the cops' move, at least one neighbour of the robber must be at distance at least two from any cop (this is due to the girth five condition since no cop can be at distance less than two from two neighbours of v) and the robber can move to that neighbour. Since this condition is satisfied after every time the cops move, the robber always has at least one neighbour adjacent to no cop, and he can safely move there during his turn, avoiding capture indefinitely. \square

This theorem implies that the active cop number of the dodecahedron is at least three and, as a consequence of Theorem 4.1.1, we know that $c_a(G) \leq 6$ for any planar graph G . In [18], the authors mention that new ideas might be required to improve the upper bound for planar graphs. We will show that the wide shadow allows us to improve the upper bound and show that $c_a(G) \leq 4$ for every planar graph as well as to greatly simplify the proof of the following result from [18].

Theorem 4.1.5. *If G is outerplanar, then $c_a(G) \leq 2$.*

Recall that outerplanar graphs do not contain a subdivision of K_4 or $K_{3,3}$ as shown in [10]. The following lemma shows a structural property of outerplanar graphs that will be useful. Figure 4.3 illustrates the property described in the lemma.

Lemma 4.1.2. *If $e = xy$ is an edge of an outerplanar graph G and G_0 is a connected component of $G - \{x, y\}$, then there is at most one isometric xy -path in $G'_0 = G[V(G_0) \cup \{x, y\}] - xy$.*

Proof. If no xy -path exists in G'_0 we are done, so we may assume there is at least one. Take an outerplanar embedding of G and suppose that we have two such paths P_1 and P_2 . For $i \in \{1, 2\}$, let $C_i = P_i \cup e$ and notice that the outerplanarity of G implies that P_2 cannot have any vertex in the interior of the region defined by C_1 . Since C_1 must be an induced cycle, no edge of P_2 is a chord of C_1 , and so P_2 must be contained in the exterior of the curve defined by C_1 . By symmetry, we get that P_1 is in the exterior of the curve defined by C_2 . Since G_0 is connected, there is a path joining vertices of P_1 and P_2 that does not include x or y , but this implies that G contains a subdivision of K_4 , contradicting the outerplanarity of G . \square

We are now in position of giving a simple proof of Theorem 4.1.5.

Proof of Theorem 4.1.5. We will say the game is in a *simple state* if one cop is actively-guarding an edge $e = xy$ and we will use H to denote the connected component of $G - \{x, y\}$ where the robber can move without being captured. The proof is by induction on $|V(H)|$.

We begin by placing our two cops, C_1 and C_2 , on the endpoints of an edge $e = xy$ and, regardless of the robber's initial position, we can move C_1 to actively-guard $e = xy$ and let H be the connected component of $G - \{x, y\}$ containing the robber. If only one vertex in $\{x, y\}$ has neighbours in H (we may assume it is x), take $w \in V(H)$ such that $xw \in E(G)$. By moving C_2 to actively-guard xw we restrict the robber to move in a smaller connected component of G and reach again a simple state, making progress in the game. We now assume that both x and y have neighbours in H . By Lemma 4.1.2, there is a path P which is the unique isometric xy -path with all its intermediate vertices in H . The uniqueness of P implies that it is bypath-free in $G[V(H) \cup \{x, y\}] - e$, so by Lemma 2.2.1 the wide shadow of the robber on P is never a single vertex, so we can move C_2 to actively-guard P . Now, the robber is restricted to move in H' , a connected component of $H - V(P)$. The outerplanarity of G implies that H' has at most two vertices in common with P and, if it has exactly two, they must be adjacent. By moving C_1 to guard an edge containing the vertices H' and P have in common, we make progress in the game and arrive to a simple state of the game. \square

It should be noted here that no bipartite graph G such that $c_a(G) < c(G)$ is known. Moreover, no non-bipartite graph H is known such that $c_a(H) > c(H)$. This was noted by the authors in [18]. It seems to be the case that the active game behaves differently in bipartite and non-bipartite graphs. It was stated in [18] that in a bipartite graph G with $c(G) = k$, if after the initial placement by both players, all cops and the robber occupy the same partite set of G , then the cops can ensure the robber's capture in the active version of the game.

However, the statement as it appears in the paper is false (the authors have been informed). To see this, look at the following example:

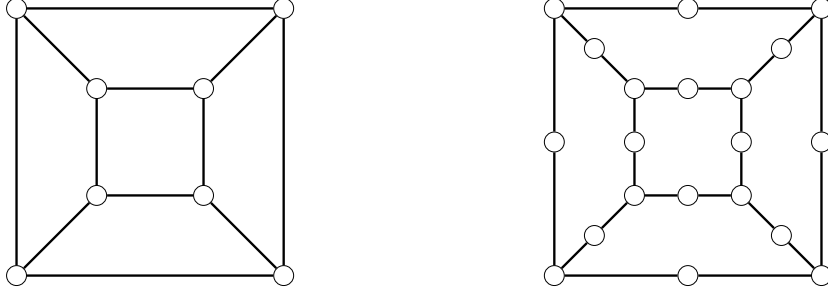


Figure 4.4: The cube and the subdivided cube.

Proposition 4.1.1. *The graph G obtained from the cube by subdividing each edge once is bipartite and $c_a(G) = 2$ but two cops cannot guarantee the capture of the robber if all the players start on the same side of the bipartition.*

Proof. Notice that for any pair of edges e_1 and e_2 in the cube, every vertex x has a neighbour y which is not an endpoint e_1 or e_2 . Also, there is an edge $e_{1,2}$ which shares no endpoint with e_1 or e_2 .

If the cops start on edges e_1 and e_2 , the robber starts at the edge $e_{1,2}$. If the cops start at vertices x and y , the robber starts at any vertex distinct from x and y . During the whole game, all players will be in the same partite set after the robber's move, which implies that the game can only end when the robber steps on top of a cop. Due to the choice of $e_{1,2}$, regardless of how the cops move, the robber can move to a vertex containing no cop, so it suffices to check what happens when all the players start at vertices. If the cops are on edges after they move, then the robber can move in the next two turns to get to a vertex x which is not an endpoint of e_1 or e_2 . Regardless of how the cops move in the meantime, they cannot be on x when the robber steps on it, so the robber can evade the cops indefinitely. \square

We can fix the result mentioned by instead requiring that the cops start the game in the opposite side of the bipartition as the robber. We will now provide a short proof of the fixed version, Lemma 4.1.3.

Lemma 4.1.3. *Let G be a bipartite graph. Let $k = c(G)$, and consider the active game on G with k cops. If, after the initial placement by both players, all cops are in the opposite side of the bipartition as the robber, then the cops can ensure the robbers capture.*

Proof. We know that the cops and the robber begin the game in opposite sides of the bipartition of G . For every cop C , we will imagine a dummy D_C to be next to C . The dummy will behave just like a cop but cannot capture the robber. By Observation 4.1.1, we can replicate a winning strategy for k cops in G in the usual game with the k cop-dummy pairs, and we can get a cop or a dummy to be on the same vertex as the robber. If it is a cop, we win, so we may assume it is a dummy D_C . Since dummies and cops are always on opposite classes of the bipartition, then the robber must be on a different class as the cops

at this point, which implies the robber was the last player to move and is now at distance one from C , which means C can capture the robber the following turn. \square

As we have seen, we can replace each cop in the usual game by a pair of cops to adapt the strategies for the fully active game. It follows from Theorem 4.1.1 and Theorem 1.0.1 that $c_a(G) \leq 6$ for any planar graph G . However, the authors in [18] posed the question of whether that upper bound can be improved. Since three active cops are necessary to catch the robber on the dodecahedron, the upper bound must be at least three. We will show that four cops suffice in any planar graph.

In light of Theorem 4.1.3 one might think that the active game behaves differently in bipartite graphs as it does in non-bipartite graphs. This seems to be the case, as any cop that starts the game in the same class of the partition as the cop cannot capture the robber unless he steps on that cop, which seems to make them weaker. Actually, we can show that capturing the robber on a bipartite graphs is harder if more cops start the game at the same partition class as the robber. When playing in a bipartite graph, a cop that start the game in the same partition class as the robber will be called *weak*, and a cop whose initial position is in the opposite partition class as the robber will be called *strong*.

Theorem 4.1.6. *Let G be a bipartite graph. If there is a winning strategy for k cops in the fully active game on G in which t cops are weak, then there is a winning strategy for k cops in which t' are weak for $0 \leq t' \leq t$.*

Proof. Suppose we have a winning strategy S as in the statement of the theorem and let t' be non-negative integer such that $t' < t$. We consider the initial position of the players given by S and replace $t - t'$ weak cops with a dummy cop D on the same vertex as the weak cop and a strong cop C_D on a neighbour of their initial position. We can now implement the strategy S using the dummy cops instead of the replaced weak cops and moving W_D making sure that she is adjacent to D after every turn. By following strategy S we either capture the robber with a strong or weak cop, or we capture the robber with a dummy D . However, since the D started in the same side of the bipartition as the robber, this means the robber moved on top of the D . Hence, the following turn we can move C_D and capture the robber. \square

Notice that if we play with k cops in a bipartite graph, we can always guarantee that, regardless of the robbers initial position, at least $\lfloor \frac{k}{2} \rfloor$ cops are strong by distributing the cops as evenly as possible in both classes of the partition.

4.2 Active cops on planar graphs

This section will be devoted to the study of Active Cops and Robbers on planar graphs. Just as in the case of Lazy Cops and Robbers, the guarding strategy of Lemma 2.0.1 does

not work if we impose speed restrictions on the cops: if the shadow of the robber does not move (which might happen even if the robber's position changes), the cop will not be able to keep guarding the path. We will show that the wide shadow is a notion that is also useful to study the active game.

The following observation, which is crucial for our proof, is a consequence of Lemma 3.1.1.

Observation 4.2.1. *Let P be an isometric path P in a graph G . If P is bypath-free in G , then P can be 1-actively-guarded.*

As mentioned before, the active game seems to behave in slightly different ways for bipartite and non-bipartite graphs. First, we will prove that two strong cops and one weak cop suffice to capture the robber in any planar bipartite graph.

Theorem 4.2.1. *Let G be a planar bipartite graph. Consider the active game on G with three cops. If the robber's initial position in G is in a different partition class as at least two cops, then three cops can guarantee the robber's capture in G .*

Proof. We will distinguish three different situations in which the game can be:

- a) A pair of cops is tandem-guarding a path P of H , and every path from the robber to the cop territory includes a vertex of P .
- b) P is being actively guarded by one cop and any path from the robber to the cop territory includes a vertex of P .
- c) P and Q are internally disjoint paths joining the same two vertices, and any path from the robber to the cop territory includes a vertex of $P \cup Q$. P and Q are bypath-free in $G[R(P, Q)]$ and are being actively guarded by two cops.

If all cops are strong the result follows directly from Theorem 4.1.3 After the initial placement of the players on G , we know we have one weak cop and two strong ones. As before, H will denote the subgraph of G induced by the vertices in the robber's territory, C_1 and C_3 will be the strong cops and C_2 , the weak one. We begin the game by choosing an edge vx and moving C_1 and C_2 to tandem-guard it, getting to case (a). We can now use Observation 3.2.1: if (i) happens, we can capture the robber using C_1 or C_2 . If (ii) happens, we can use C_3 to actively-guard P , at which point we can release C_1 and C_3 and get to case (b). If (iii) happens, we can move C_3 to u . Notice that the fact that C_1 will be on vertex v when C_3 gets to vertex u . At this point we can identify the vertices x_1 and x_2 which, together with u and v , bound the region containing the robber's territory. We can now move C_1, C_2 and C_3 to x_1, v and x_2 , respectively. By moving C_2 along the edge vx_1 and C_3 along ux_2 we can make sure the robber stays inside the region bounded by the paths $P = ux_1v$ and $Q = ux_2v$, getting to case (c) and allowing us to release C_1 . Notice that in any case we

reduce the robber's territory after a finite number of moves, so we make progress towards capturing him.

We now make two observations.

- (i) If P is an isometric bypath-free path of G , then it can be actively-guarded by any cop, regardless of whether she is weak or strong.
- (ii) If P is an isometric path of G and $S_P(x)$ is a single vertex for some $x \in V(G)$, then x and $S_P(x)$ are in the same class of the bipartition of G .

The first observation follows from the fact that the wide shadow always contains vertices of both chromatic classes. The second one is a consequence of the fact that, if $S_P(x)$ contains a single vertex, then there exists B , a bypath of P , containing x , and this implies the distance between x and $S_P(x)$ is even.

If we start from (b), we may assume that the weak cop is actively-guarding P thanks to (ii). Also, since having a single vertex in P with neighbours in H would lead us again to case (a) (by replacing the weak cop guarding P with a strong one and then tandem-guarding such vertex), we may assume that there are at least two vertices of P with neighbours in H . If $P = v_1v_2 \dots v_n$ and v_i and v_j , with $1 \leq i < j \leq n$, are the first and last vertices of P with neighbours in H respectively, then let Q to be a shortest v_iv_j -path containing vertices of H . Notice that (ii) implies that a strong cop can actively guard any isometric path and not only bypath-free ones. Hence, we can use one strong cop to actively guard Q . In virtue of Lemma 3.2.2, we may assume that we get to (c) with both P and Q bypath-free in $G[R(P, Q) \cup V(P) \cup V(Q)]$, where the robber is contained. Again, we always reach one of the other cases and reduce the size of the robber's territory in the process.

Suppose now we start from (c). We may assume that P and Q are being guarded by one weak cop and one strong cop. Having a single vertex of P or Q with neighbours in H would lead us to (a), so we may assume there are at least two such vertices. Let T be the shortest uv -path containing vertices of H and such that $V(T) \subseteq V(H) \cup V(P) \cup V(Q)$ and T' the subpath of T contained in H . Since T' is isometric in H , we can move the free strong cop to actively guard T . The robber will now be restricted to $G[R(P, T)]$ or $G[R(Q, T)]$. We may assume that the robber is restricted to $G[R(Q, T)]$, and so we can release the weak cop guarding P . Since Q is isometric and bypath-free in $G[R(Q, T)]$, we can use the weak cop to actively guard Q and then release the strong cop. Now, by repeatedly applying Lemma 3.2.2, we can use the pair of strong cops to either capture the robber or restrict him to a region bounded by two paths satisfying the conditions in case (c). Since in each case we reduce the robber's territory and get again to one of the three cases we had. \square

We are now ready to prove the main result of this section:

Theorem 4.2.2. *For any connected planar graph G , $c_a(G) \leq 4$.*

Proof. Let G be a planar graph and fix a drawing of G in the plane. We will place our four cops on the endpoints of an edge, two cops on each. This implies that, if G is bipartite, after the robber's initial placement on G there will be two strong cops and two weak ones, so the result follows from Theorem 4.2.1. We will assume now that G is not bipartite.

The proof is similar to that of Theorem 4.2.1, but we should use the extra cop to compensate for the lack of a notion of strong cops. We will distinguish three different situations the game can be in:

- a) A pair of cops is tandem-guarding a path P of H , and every path from the robber to the cop territory includes a vertex of P .
- b) P being actively guarded by one cop and any path from the robber to the cop territory includes a vertex of P .
- c) P and Q are internally disjoint paths joining the same two vertices, and any path from the robber to the cop territory includes a vertex of $P \cup Q$. P and Q are bypath-free in $G[R(P, Q)]$ are being actively guarded by two cops.

As before, H will denote the graph induced by the vertices in the robber's territory. Let C_1, C_2, C_3 and C_4 be our cops. We can choose cops C_1 and C_2 to tandem-guard v , so we get to case (a). We can now use Observation 3.2.1: if (i) happens, we can capture the robber using C_1 or C_2 . If (ii) happens, we can use C_3 to actively-guard P , at which point we can release C_1 and C_3 and get to case (b). If (iii) happens, we can move C_3 and C_4 to tandem-guard u . At this point we can identify the vertices $x_1 x_2$ which, together with u and v , bound the region to which the robber is constrained. Now, we may assume C_1 and C_3 are on v and u respectively, so they can now move around the cycle ux_1vx_2u (following that order), getting us to case (c). Notice that in any case we reduce the robber's territory after a finite number of moves, so we make progress towards capturing him.

If we start with case (b), we may assume that C_1 is actively-guarding $P = v_1v_2 \dots v_n$. If P has a single vertex v_i with neighbours in H , we can move C_2 and C_3 to tandem-guard v_i and get to case (a). If P has multiple vertices with neighbours in H , let v_i and v_j , with $1 \leq i < j \leq n$, be the first and last vertices of P with neighbours in H . Let Q be the shortest v_iv_j -path containing vertices of H . We can use C_2 and C_3 to tandem-guard Q . If Q is bypath-free in $G[R(P, Q) \cup V(Q)]$, we can move C_2 and C_3 so that, after a finite number of turns, C_2 is actively guarding Q , getting us to case (c).

Otherwise, let B the bypath of Q in $G[R(P, Q) \cup V(Q)]$ given by Lemma 3.2.2. At this point, the only free cop is C_4 , so we can move her until C_4 is on the shadow of C_3 on $Q_{\langle B \rangle}$ and, keeping this property and the adjacency of C_2 and C_3 , the cops will move until C_3 and C_4 are on the robbers wide shadow. If the robber is in $G[R(Q_{\langle B \rangle}, Q) \cup V(Q) \cup V(Q_{\langle B \rangle})]$ when we achieve this, then either he was caught or both Q and $Q_{\langle B \rangle}$ are bypath-free in $G[R(Q_{\langle B \rangle}, Q)]$. We will keep moving the cops until C_4 and C_2 are in the wide shadow of the

robber on $Q_{\langle B \rangle}$, and then we can release C_1 and C_3 , and arrive to case (c). Suppose now that the robber is in $G[R(P, Q_{\langle B \rangle}) \cup V(P)]$ instead. In this case, since C_4 is on the shadow of C_3 , we will release C_3 and move C_2 and C_4 in such a way that after every turn, either C_2 is on the shadow of the robber on Q or C_4 is on the shadow of the robber on $Q_{\langle B \rangle}$. This can be thought of as tandem-guarding the paths Q and $Q_{\langle B \rangle}$ with C_2 , C_4 and their respective shadows.

We will now move C_3 to the shadow of C_2 on $Q_{\langle B \rangle}$. If the robber enters $G[R(Q_{\langle B \rangle}, Q) \cup V(Q) \cup V(Q_{\langle B \rangle})]$ before C_3 gets there, then he will be caught or restricted to $G[R(Q_{\langle B \rangle}, Q)]$. Otherwise, we will be able to tandem-guard $Q_{\langle B \rangle}$ using C_3 and C_4 , and so we can release C_2 and repeat until we restrict the robber's movement to the region between two bypath-free paths, when we will be able to guard each with a single cop and get to case (c).

It remains only to show what happens if we start with (c), the cops can guarantee the robber's capture. If only one vertex of $P \cup Q$ has neighbours in H , we can move C_2 and C_4 to tandem guard it and arrive to case (a). Otherwise, we can find T , the shortest uv -path in $G[V(H) \cup V(P) \cup V(Q)]$ using at least one vertex of H , and tandem-guard it using C_2 and C_4 . Now, the robber must be in $G[R(P, T)]$ or $G[R(T, Q)]$ and, without loss of generality we may assume the robber is in $G[R(P, T)]$. We can now release C_3 (as Q no longer need to be guarded) and we may use C_2, C_3 and C_4 to proceed like in the last part of the previous case.

Notice that starting with cases (a) or (b) we either remove at least one vertex of the robber's territory or we get to case (a), and from this one we always make progress by removing at least one vertex from the robber's territory, finishing the proof. \square

Chapter 5

Subdivisions and speed restrictions

A *uniform subdivision* of a graph G is the graph obtained from G by replacing the edges in $E(G)$ with paths of the same length. If the paths have length $s + 1$, we denote such graph by $G^{(s)}$. Notice that $G^{(0)}$ is isomorphic to G . The effect that subdividing edges of a graph has on the cop number was studied for the first time in [5]. The classical result in this direction is the following:

Theorem 5.0.1 ([5]). *For every graph G and every integer $s \geq 0$, we have that $c(G) \leq c(G^{(s)}) \leq c(G) + 1$ for every positive integer s .*

Here we present some results analyzing this operation for the variants of active and lazy cops, which will be particularly surprising in the later case.

We can also ask what happens with $c(G^{(s)})$ when s tends to infinity. This would give rise to a game in which the cops and the robber move continuously. Moreover, we can ask what is the minimum number of cops required to guarantee the robber's capture in a compact surface. This type of problem has been studied before (see [27]). However, speed restrictions do not seem to have been considered in that context. While having a cop to be faster than the robber would make no sense as the cop would always be able to wind, when speaking of continuous movement one can consider the restriction where each cop has a maximum speed and that the total speed of the cops cannot be more than a fixed constant at any point of the game. It would be of great interest to find connections between the discrete and the continuous versions of this problem.

5.1 The effect of subdivisions in the active game

Theorem 5.1.1. *For any graph G , we have $c_a(G^{(s)}) \leq c_a(G) + 1$.*

Proof. For the upper bound, suppose $c_a(G) = k$ and take a strategy S for k active cops to win in G . Place k cops on the corresponding vertices of $G^{(s)}$ and now, if the robber chooses as his initial position a vertex of $G^{(s)}$ corresponding to a vertex of G , then place the robber on that vertex in the game in G and place the dummy robber on the same vertex.

Otherwise, place a dummy robber on a vertex of G corresponding to an endpoint of the edge whose subdivision in $G^{(s)}$ contains the robber. Place an extra cop on top of any other cop.

After the robber makes two moves in $G^{(s)}$, the dummy will choose as his new position a vertex of G which:

- Is adjacent from his previous position in G .
- If the robber begins the game on a vertex of $G^{(s)}$ corresponding to an edge of G , the dummy will choose a vertex of G corresponding to an endpoint of the edge containing the robber.
- If the robber begins the game on a vertex of $G^{(s)}$ corresponding to a vertex of G , the dummy will choose that vertex in G or a neighbour.

Notice that the dummy can always find such a vertex since, if the robber moves $s + 1$ times in $G^{(s)}$, he cannot step on vertices of $G^{(s)}$ corresponding to non-adjacent vertices of G , and so by moving like this the dummy will behave like an active robber on G . This means there is a strategy S for k active cops to capture the dummy on G . We will look at the cop's moves in G according to S and move the first k cops in $G^{(s)}$ the next $s + 1$ turns so that they reach the position corresponding to the k cops in G . By doing this, we can get a cop on top of the dummy in $G^{(s)}$. From now on, that cop will only chase the robber in $G^{(s)}$. We can use the extra cop and the remaining $k - 1$ cops and get then to the vertices of $G^{(s)}$ corresponding to the winning strategy S . Now, we can play on G using S but now with a new dummy who will be on top of the robber or on the endpoint of the edge where the robber is that does not contain the chasing cop already. By doing this, we either capture the robber, get cops on both endpoints of the edge containing the robber (which is a win for the cops), or we get the chasing robber closer to the robber (notice that every time the robber moves back to the last vertex he stepped on, the chasing robber gets one step closer to him). Since the later can only happen a finite number of times, by using this strategy we can guarantee the robber's capture, so $c_a(G^{(s)}) \leq c_a(G) + 1$. \square

As noted before, the active game behaves differently in bipartite and non-bipartite graphs.

Conjecture 5.1.1. *For any graph G , we have $c_a(G) \leq c_a(G^{(s)}) \leq c_a(G) + 1$*

5.2 The effect of subdivisions with lazy cops

The effect of subdividing all the edges of a graph the same number of times has a stranger behaviour for the game of Lazy Cops and Robbers than it does for the classic of the active version. We will see that for graphs G and $G^{(s)}$, the difference between $c_1(G)$ and $c_1(G^{(s)})$

can be arbitrarily large. However, there are some families of graphs where this numbers are not arbitrarily far away from each other, and we will mention a few of those in this section.

However, the most basic question we could ask is the following:

Question 5.2.1. *Is there a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that if G has lazy-cop number $c_1(G) = k$, then $c_1(G^{(s)}) \leq f(k)$ for every positive integer s ?*

Surprisingly, and in contrast with the classic and the active version where $f(k) = k + 1$, no such function exists for the lazy variation. To show this, it suffices to prove it for cop-win graphs, as we may always attach a cop-win graph to any graph without changing its cop number.

Theorem 5.2.1. *For every integer $k \geq 1$, there exists a uniformly subdivided cop-win graph G with $c_1(G) > k$.*

Proof. Let r and s be positive integers with $s \leq r$, and \mathcal{S} be the family of subsets of $V(K_r)$ of size s . Let $H(r, s, t)$ be the graph obtained from K_r by adding a vertex v_A and all the edges between v_A and A for every $A \in \mathcal{S}$, and then subdivide every edge t times.

Now, consider the graph $H_k = H(4k + 1, 2k + 1, k + 2)$. We will show that k cops cannot capture the robber in H_k . We will say that a vertex of degree more than two is a *main vertex*. Let k cops choose their positions in H_k . A main vertex will be called *grey* if there is a cop at distance less than $k + 3$ from it. We will use d to denote the distance between the robber and cop closest to him.

Since at any given time there can be at most $2k$ grey vertices, there exists a set $A \in \mathcal{S}$ such that $\{v_A\} \cup A$ has no grey vertices, so the robber will choose vertex v_A as his starting position. At this point, $d \geq 2(k + 3)$. We distinguish two different states of the game:

- a) The robber is at a main vertex v_X for some $X \in \mathcal{S}$ and v_X is not grey.
- b) The robber is at a main vertex v_X for some $X \in \mathcal{S}$, and v_X is grey and $d \geq 2$.

The robber will stay still as long as the game is in state (a). Notice that at most one non-grey main vertex can become grey after each turn.

Suppose that a cop moves in such a way that vertex v_A becomes grey. Since $|A| = 2k + 1$, there exists a vertex $u \in A$ which is not grey, so the robber can move towards it. If after the cops' move vertex u is not grey, then in the next $k + 2$ rounds the robber will move towards u and reach it without being captured. Now, there is a set $B \in \mathcal{S}$ such that $u \in B$ and no vertex of $v_B \cup B$ other than possibly u is grey. Since no vertex of $v_B \cup (B \setminus \{u\})$ is grey, the robber can move towards v_B the next $k + 3$ rounds and reach it without being captured. After performing this moves, no cop can get at distance less than 2 from the robber, so we get back to case (a) or case (b). Notice that at every point of this process, $d \geq 2$.

Suppose now that a main vertex v becomes grey in the turn after the robber moved towards it from another main vertex v' . In this case, the robber will move back to vertex

v . Since the cops will avoid repeating a configuration, vertex v will be grey after the cops' move. This means that, after the cops' move, d can decrease by at most 1. Also, in order for a non-grey main vertex to become grey, there must be a cop at distance $k + 3$ from it the previous round, so the cop must be on a main vertex. Hence, in order to make sure that the robber is not able to leave his current position without repeating a configuration, the number of cops in main vertices must decrease by one each time. Since this can happen at most k times, we have $d \geq 2$. Notice that each cop can turn at most 2 main vertices grey. Since $|A| = 2k + 1$, there will be a vertex $u \in A$ which is not grey after all cops have moved. The robber can now move the next $k + 3$ rounds towards u and reach it keeping $d \geq 2$. Once more, there exists $B \in \mathcal{S}$ such that $u \in B$, and no vertex of $v_B \cup B$ other than possibly u is grey. Since no vertex of $v_B \cup (B \setminus \{u\})$ is grey, the robber can move towards v_B the next $k + 3$ rounds and reach it without being captured. After performing these moves we still have that $d \geq 2$, so we get back to case (a) or case (b). \square

Notice that the graph $H(r, s, 0)$ for $r, s \geq 1$ is a chordal split graph, which means $c(H(r, s, 0)) = 1$ and $c(H(r, s, t)) \leq 2$ for all $t \geq 1$. However, $c_1(H(4k + 1, 2k + 1, k + 2)) \geq k$, showing that the gap between (c) and c_1 can be arbitrarily large.

Let G be a graph and take list $S = \{v_i\}_{i=1}^t$ such that $v_i \in V(G)$ for $i \in \{1, \dots, t\}$. We say that S is a *double-dominating multisubset* of G if for every $x \in V(G)$ there exists $1 \leq i < j \leq t$ such that $x \in N[v_i] \cap N[v_j]$. We will use $dd(G)$ to denote the minimum cardinality of a double-dominating multisubset of G .

Theorem 5.2.2. *For every graph G and every positive integer s , we have $c_1(G^{(s)}) \leq dd(G) + 1$.*

Let S be a minimum double dominating multiset of G and place a cop on each vertex of $G^{(s)}$ corresponding to a vertex of S , and choose an arbitrary vertex to place an extra cop.

Algorithm 5.2.1. *Input: $G^{(s)}$, a subset $T \subseteq V(G)$ which is a minimum double-dominating multiset of G , a cop C_v on every vertex $v \in T$, and one extra cop C on an arbitrary vertex of $G^{(s)}$. Output: Cops' strategy to capture R .*

Initialize $\lambda = C$, $B = \emptyset$, $R_{-1} = R_0$ to be the robber's current position, and $i = 1$.

Suppose we are at the beginning of round i .

- Step 1) If there is a cop adjacent to the robber, then a cop moves to capture the robber and the algorithm ends; otherwise, go to Step (2).
- Step 2) If $R_i = R_{i-1}$ or $R_i = R_{i-2}$, then move λ so that $d(\lambda, R_i) < d(\lambda, R_{i-1})$. Wait for the robber to update his position, set $i \rightarrow i + 1$ and go to Step (1). Otherwise, go to Step (3).

- Step 3) If $B = \emptyset$ and $d(R_i, u) = d(C_v, u) - 2 = s - 1$ for some $u \in V(G)$ and C_v , set $B = (C_v, u)$ and move C_v so that $d(C_v, u) - 1 = d(R_i, u)$. Wait for the robber to update his position, set $i \rightarrow i + 1$ and go to Step 1; otherwise, go to Step (4).
- Step 4) If $B = \emptyset$, move λ so that $d(\lambda, R_i) < d(\lambda, R_{i-1})$. Wait for the robber to update his position, set $i \rightarrow i + 1$ and go to Step 1; otherwise, go to Step (5).
- Step 5) If $B = (C_v, u)$ and $d(R_i, u) = d(C_v, u) - 2$, move C_v so that $d(C_v, u) - 1 = d(R_i, u)$. Wait for the robber to update his position, set $i \rightarrow i + 1$ and go to Step 1; otherwise, go to Step (6).
- Step 6) If $B = (C_v, u)$ and $d(R_i, u) = d(C_v, u) + 1$ and $d(R_i, u) \leq s$, move C_v so that $d(C_v, u) = d(R_i, u)$ and go to Step (1); otherwise, go to Step (7).
- Step 7) If $B = (C_v, u)$ and $d(R_i, u) = d(C_v, u) + 1$ and $d(R_i, u) = s + 1$, move C_v so that $d(C_v, u) = d(R_i, u)$, set $B = \emptyset$ and go to Step (1); otherwise, go to Step (8).
- Step 8) If $B = (C_v, u)$ and $d(R_i, u) = d(C_v, u)$, take x a vertex of $V(G) - u$ such that $d(R_i, x) = s$ and $d(C_v, x) = s + 1$. Take $w \in T - u$ such that $d(C_w, x) \leq s + 1$, move C_w so that $d(C_w, x) = d(R_i, x)$, set $\lambda = C_u$ and $B = (C_w, x)$. Wait for the robber to update his position, set $i \rightarrow i + 1$ and go to Step 1; otherwise, go to Step (9).
- Step 9) If $B = (C_v, u)$ and $d(R_i, u) = d(C_v, u) - 1$, move C_v so that $d(R_i, u) = d(C_v, u)$. Wait for the robber to update his position, set $i \rightarrow i + 1$ and go to Step (1).

Proof of algorithm 5.2.1. Suppose that the robber hasn't been captured, no cop is adjacent to him and it is the cops' turn. We may assume $s > 1$ since no cop is adjacent to the robber. Step 2 guarantees that the distance between cop λ decreases if the robber stays still or takes a step back to its previous position, which means that, after at most a finite number of turns, the robber will not stay still or move back.

Since T is a double-dominating multiset, Step 2 guarantees that either the robber will be captured or he will get to be at distance $s - 1$ from a vertex u which is at distance C_v at distance $s + 1$ of a cop C_v . This also implies that at least one of Step 6, Step 7 and Step 8 will be executed at least once. At this point, we will have $B = C_v \neq \emptyset$. Notice that if j and k , with $j < k$, are two distinct turns in which B is declared empty, then we have that $d(\lambda, R_j) < d(\lambda, R_k)$, which means Step 3, Step 4 and Step 7 can only be implemented a finite number of times.

The first time Step 8 is implemented, the robber will be on a vertex of $V(G^{(s)}) \setminus V(G)$. In this case, the robber will be unable to enter u or x anymore, since Step 9 guarantees that C_w and λ will be at distance one from x and u , respectively, before the robber can enter them. Hence, after at most $2s$ more steps, the robber will be captured. \square

This improves on the trivial bound of n , but it is probably far from optimal. However, it gives an analog of the domination number that works well with subdivisions in the game with lazy cops.

5.3 Bounding c_1 under subdivisions for some classes of graphs

It follows from Theorem 5.2.1 that three cops are sufficient to capture the robber in a uniform subdivision of any graph with a universal vertex (that is, a vertex adjacent to all other vertices of the graph). There are cop-win graphs whose uniform subdivisions need three cops in the lazy variation of the game, as we will show in Theorem 5.3.4. However, it came as a surprise that the class of graphs obtained from uniform subdivisions of cop-win graphs is not lazy-cop-bounded.

Lemma 5.3.1. $c_1(K_n^{(1)}) > 2$ for $n \geq 5$.

Proof. It suffices to show that $c_1(K_5^{(1)}) > 2$. We will show that two cops cannot capture the robber in $K_5^{(1)}$. We will distinguish three configurations of the game that can be achieved after the robber performs his moves:

- a) Both cops are in main vertices, the robber is in a middle vertex and no cop is at distance less than 3 from the robber.
- b) Both cops are in middle vertices, the robber is in a main vertex and no cop is at distance less than 3 from the robber.
- c) One cop is in a main vertex and another one is in a middle vertex, the robber is in a main vertex and no cop is at distance less than 2 from the robber.

Notice that, for any initial position of the cops, there exists a vertex of the graph such that, if the robber chooses it as his initial position, one of the three configurations described is achieved. We may assume that one of the cops always moves along an edge at each of the cops' turns.

If we start with configuration (a) and a cop moves, since there are five main vertices, at least one of the main vertices adjacent to the robber's position is at distance at least two from every cop. By moving to that vertex, the robber achieves configuration (c).

If we start with configuration (b) and a cop moves, then the cop would get to a main vertex. Now, of the robbers four neighbours, at least one is at distance three of either cop. By moving to this vertex, the robber achieves configuration (c).

Finally, suppose that we start with configuration (c) and a cop moves. If after the cop's move there are two cops on main vertices, then there are two vertices adjacent to the robber's position that are at distance 3 from either cop. By moving to one of them, the robber gets to configuration (a). On the other hand, if the cops are occupying middle vertices after their

move, we may assume that one of them is at distance one from the robber, for otherwise we would be in configuration (b). In this case, there exists at least one main vertex at distance three of either cop. Regardless of what the cops do, the robber will make the next two moves to get to this vertex. Regardless of which move the cops perform between the robber's moves, there will be one cop on a main vertex and one on a middle vertex, and none of them will be at distance one from the robber, thus getting configuration (c). \square

The following simple lemma will be useful when we deal with the Cartesian products of certain graphs.

Lemma 5.3.2. *Let H be a graph, $P = uv$ a path of length two, s a non-negative integer and $G = (H \square P)^{(s)}$. Let H_u be the subgraph of G induced by the vertices of $V(H) \times \{u\}$. If $x \in V(G) \setminus V(H_u)$ and $S_{H_u}(x)$ is the wide shadow of x on H_u , then $|S_{H_u}(x)| \geq d(x)$. Moreover, if $x \in V(H_v)$, then $|S_{H_u}(x)| \geq s(d(x) - 1) + d(x)$.*

Proof. For a vertex $x \in V(G) \setminus V(H_u)$, let $z \in V(H_u)$ be:

- The vertex of H_u corresponding to x if $x \in V(H_v)$.
- The vertex (y, u) if x is a vertex obtained from the subdivision of an edge of the form $(y, u)(y, v)$.

Now, let w be a vertex in $V(H_u)$. For a positive integer r , the ball with centre w and radius r is the set $B(w, r) = \{y \in V(H_u) : d(y, w) < r\}$. Notice that $d(x, w_u) \geq d(z, w_u) + 1$, which implies that $y \in \bigcap_w B(w, d(x, w))$ if $y \in V(H_u)$ and $d(z, y) \leq 1$. This shows $N_H H_u(z) \subseteq S_{H_u}(x)$. \square

It was shown in [31] that, for any positive integers r and s , with $r \geq 3$, we have $c_1(G) \leq 2$ whenever $G = C_r \square P_s$. We will give a short proof of a slightly stronger statement using Lemma 5.3.2, as well as study what happens with this class of graphs with subdivision. In the following, a vertex x in the wide shadow of the robber will be called an *inner vertex* if, regardless of what move the robber moves next, x will remain in the wide shadow of the robber.

Theorem 5.3.1. *Let T be a tree and let H be a tree or a cycle C_r . Then $c_1(H \square T) \leq 2$.*

Proof. For every $v \in V(T)$, let H_v be the subgraph induced by $V(H) \times \{v\}$. We will say that a vertex v of T has been cleared if the cops can move in such a way that the robber cannot enter H_v without being captured. We give a strategy for two cops such that, after a finite number of turns, either the robber is captured or a new vertex has been cleared. We begin the game by placing both cops on a vertex of H_v , where v is any vertex of T . Since $c_1(H_r) \leq 2$, we can capture the robber's wide shadow on H_v . Let u be the neighbour of v in T which is in the same component of $T - v$ as the robber. Now, the cops will move following the next rules:

1. If a cop can capture the robber, she captures him.
2. After the cops' turn, there must be a cop in the wide shadow of the robber on H_v .
3. If a cop is in an inner vertex of the wide shadow on H_v and both cops are in H_v , move the other cop to a vertex in H_u . In the case both cops are in an inner vertex of the wide shadow, move any one of them to H_u .
4. If a cop is in an inner vertex of the wide shadow on H_v and the cops are in H_v and H_u move the cop in H_v to H_u .

Once we have a cop in the wide shadow on H_v (which we can achieve since $c_1(H) \leq 2$), we know there will always be a move that satisfies (2). Due to Lemma 5.3.2 and the choice of initial positions of for the cops, after a finite number of turns there will be a cop in an inner vertex of the wide shadow of the robber. We now apply (3) or (4) depending on the positions of the cops. Notice that this strategy will always keep the cops in the same H_r or in two adjacent copies. After a finite number of moves, if the robber has not been captured, both cops will leave a cycle H_w that both previously visited. Condition (2) guarantees that H_w has been cleared. Since T is finite, this strategy guarantees the robber's capture. \square

Now, let's analyze this class of graphs when we subdivide the edges.

Theorem 5.3.2. *Let T be a tree and $H = C_r$ a cycle with $r \geq 3$. If $G = H \square T$, then $c_1(G^{(s)}) \leq 3$.*

Proof. For every $v \in V(T)$, let H_v be the graph induced by the set of vertices corresponding to $V(H) \times \{v\}$ and the vertices obtained by subdividing the edges between the vertices of the form (x, v) with $x \in V(H)$. Let C_1, C_2 and C_3 be the cops, which we may assume start the game some vertex x_v of H_v of the form $x_v = (x, v)$ with $x \in V(H)$. For a vertex $x \in V(T)$, let R_x be the shadow of the robber on H_x (this is the usual shadow given by Lemma 2.0.1). By Lemma 5.3.2, we have that the wide shadow of the robber on H_v is a single vertex only if the robber is on a vertex of H_v . Since $c_1(H_v) \leq 2$, we may use C_1 and C_2 to capture R_v . We may assume C_1 is in the same position as R_v .

Let $u \in N_T(v)$ be a vertex in the same component of $T - v$ as the robber and let x_u be a vertex in $V(H_u)$ that whose distance to C_2 is minimum. Since R_v has been captured by C_1 , by Lemma 5.3.2 we know that the robber must move at least $s + 2$ times in order to force C_1 out of his wide shadow. This implies that, regardless of what the robber does in the next $\lfloor \frac{s}{2} \rfloor$ turns, we can move C_2 to get to a vertex y_v of H_v of the form $y_v = (y, v)$ with $y \in V(H)$, and have $d(C_1, R_v) \leq s + 1$.

Recall that C_2 and C_3 are in (y, v) and (x, v) , respectively. Let $S_{H_v}(C_i)$ be the wide shadow of the cop C_i on H_u for $i \in \{1, 2\}$. We will give a strategy for the cops to move after the robber moves. We will begin by assuming that $S = |S_{H_v}(C_1)| + |S_{H_v}(C_2)| > 0$.

1. If C_1 is in $S_{H_v}(R_u)$ and $d(C_1, R_v) \leq s + 1$, we will move C_2 or C_3 in such a way that S decreases.
2. If $d(C_1, R_v) \geq s + 2$ and R_u did not enter $S_{H_v}(C_i)$ after his last move, we will move C_1 to decrease $d(C_1, R_v)$.
3. If $d(C_1, R_v) \geq s + 2$ and R_u entered $S_{H_v}(C_i)$ after his last move, we will move C_i to decrease S for $i \in \{2, 3\}$.

Notice that after a finite number of turns following this algorithm, we will either capture the robber or get $S = 0$ and C_1 will be in $S_{H_v}(R_u)$. We will now move C_1 keeping her in $S_{H_v}(R_u)$ until C_2 or C_3 capture R_u . Whenever C_1 is in an inner vertex of $S_{H_v}(R_u)$, we will move C_2 or C_3 until one of them is on R_u . We may assume it is C_2 . If the robber is now in a vertex which resulted from the subdivision of an edge of the form $(x, v)(x, u)$ for some $x \in H$, then we can come C_1 and C_2 to capture him. Otherwise, we can move C_2 keeping her in the robber's wide shadow on H_u . By doing so, we may release C_1 and, in the turns when there is a cop in an inner vertex of $S_{H_u}(R)$ to move C_1 to a vertex of H_u . Once this is achieved, we may exchange the names of C_1 as C_3 and apply the previous strategy again. Since this can be done at most $|V(T) - 1|$ times, it guarantees the robber's capture. \square

Theorem 5.3.3. *Let T_1 and T_2 be trees and $G = T_1 \square T_2$. For every non-negative integer s , we have $c_1(G^{(s)}) \leq 2$.*

Proof. The proof is very similar to that of Theorem 5.3.2, but the fact that trees are cop-win graphs allows us to use only two cops. Using the same notation as in Theorem 5.3.2 with $T = T_1$ and $H = T_2$, we may assume C_1 and C_2 begin the game on a vertex of H_v for $v \in V(T)$. Also, let u be the vertex of T in the same component of $T - v$ as the robber. After a finite number of turns, we may assume cop C_1 is in the wide shadow of the robber on H_v . After a finite number of turns, either the robber is captured or C_1 will be in an inner vertex of the robber's wide shadow. We will use these turns to get C_2 to H_u and capture the robber's wide shadow on H_u . If the robber has not been captured, then one of the next two things will happen:

1. The robber is on a vertex corresponding to an edge joining vertices of H_v and H_u . In this case, however he decides to move, the cops can move so that every turn the value of the sum $d(C_1, R) + d(C_2, R)$, where R denotes the position of the robber on G , decreases and the robber is not able to enter $H_v^{(s)}$ or $H_u^{(s)}$ without being captured.
2. The robber is on the side of $T - u$ not containing v . In this case, the cop on H_v can be released, and the strategy can be repeated.

By doing this, two cops can guarantee the robber's capture. \square

Besides Cartesian products, we have the following results on complete graphs and, more generally, complete multipartite graphs.

Theorem 5.3.4. *If $n \geq 5$ and $s \geq 1$, then $c(K_n^{(s)}) = 3$.*

Proof. Let $G = K_n^{(s)}$. It follows from Theorem 5.2.1 that three cops can guarantee the robber's capture, so it suffices to provide the robber with a winning strategy against two cops. In contrast with the proof of Theorem 5.3.1, in this case we will always be able to choose a main vertex as the starting position for the robber. Since $n \geq 5$ and each cop can be at distance less than $s + 1$ of at most two vertices regardless of what their initial position is, there exists a main vertex of G which is at distance at least $s + 1$ of both cops. The robber will choose that vertex as his initial position. We will use C_1 and C_2 to denote the cops and R for robber. Let $d(t)$ denote the minimum distance from the robber to a cop before the robber's turn in round t of the game. Notice that $d(1) \geq s$.

For a main vertex y not containing the robber, we will say that the robber is *running towards y* if the robber will move in such a way that $d(R, y)$ decreases, regardless of how the cops move. Recall that if any configuration of the game is repeated, that suffices to show the robber wins. At round t , the robber will move as follows:

We will give two algorithms, which we will apply depending on whether the robber is on a main vertex or not. Also, for a vertex $y \in G$, we will say that the robber is *running towards y* if the robber will move in such a way that $d(R, y)$ decreases, regardless of how the cops move. This notion will help us simplify the description of the algorithm.

Algorithm 5.3.1. *The robber is on a main vertex x and is not running towards any vertex.*

1. *If $d(t) \geq 3$, the robber stays on his current position x .*
2. *If $d(t) = 2$ and there is a main vertex y such that $d(C_1, y) > s + 1$ and $d(C_2, y) > s + 1$, then the robber will start running towards y .*
3. *If $d(t) = 2$ and $d(C_1, y) \leq s + 1$ or $d(C_2) \leq s + 1$, then there is a main vertex y such that $d(C_1, y) \geq s + 1$ and $d(C_2) \geq s + 1$. The robber will take a step towards y*
4. *If $d(t) = 1$, then there is a main vertex y such that $d(C_1, y) > s + 1$ and $d(C_2) > s + 1$. The robber will run towards y*

Notice that applying Algorithm 5.3.1 will give robber instructions to move until he reaches a main vertex unless step (4) is executed.

Algorithm 5.3.2. *The robber on a vertex which came from the subdivision of the edge $xy \in E(K_n)$ and is not running towards any vertex. Suppose further that $d(R, y) = s$.*

1. *If $d(C_1, y) \geq s + 1$ and $d(C_2, y) \geq s + 1$, then the robber will start running towards y .*
2. *If $d(t) \geq 3$ then the robber will move to x .*

Now, observe that Algorithm 5.3.2 will give robber instructions to move until he reaches a main vertex.

Since the robber chooses a main vertex x as his initial position, we will start the game with Algorithm 5.3.1. The robber will stay on x until a cop gets at distance two from him due to (1), so we may assume that cop C_1 just moved and is at distance two. If C_2 is not on a main vertex, then we will apply (2) as each cop can be at distance less than $s + 1$ of at most two vertices at any point in the game. Regardless of how the cops move, no cop can be at distance less than three from the robber when he gets to y , so he can get to y safely without being captured. Moreover, if there is a cop at distance two of y when the robber steps on y , then the other cop cannot be on a main vertex.

If C_2 is on a main vertex, then we will use step (3) of the algorithm and now we will have to apply Algorithm 5.3.2. Observe that when this step is used, it means no cop is on a main vertex of G .

Notice that we will not need to use step (4) for the first time the robber changes position, so we will leave the analysis of that step for the end of the proof.

Suppose that the cops make a move and now we apply Algorithm 5.3.2. If this algorithm is being applied we know that no cop is at distance less than two from the robber. Also, at most one cop is on a main vertex of G as mentioned before, and it would be C_2 . If step (1) is used, then the robber will get to y and no cop can be at distance less than 2 when this happens. If step (2) is used, the robber may move to x and he cannot be captured the following turn. Moreover, after the robber moves, no cop can be on a main vertex.

Finally, notice that if step 4 of Algorithm 5.3.1 is ever executed, then no cop can be on a main vertex of G . Hence, there will exist y , a main vertex of G , such that $d(C_1, y) > s + 1$ and $d(C_2) > s + 1$, so the robber can run towards y and reach it without being captured.

This shows that, regardless of the cop's moves, if the the robber follows this strategy he will always have a move available such that no cop can capture him. This shows $c(K_n^{(s)}) \geq 3$, proving the result. \square

We know that the 1-cop-moves number of the class of uniform subdivisions of chordal graphs is not bounded, but we will show that in a subclass, the uniform subdivisions of extended block graphs (which were defined in Section 2.4), this parameter is bounded.

Lemma 5.3.3. *Let H be the graph obtained from a complete graph K_n by adding a vertex u adjacent to a subset of vertices of K_n . Let $H' = H^{(s)}$ and K be the set of main vertices of H' corresponding to $V(H) - \{u\}$. If the robber is on u_1 and two cops, C_1 and C_2 , are on vertices of K , then the robber cannot move from one vertex of K to another one without guaranteeing his capture by the cops.*

Proof. Suppose R starts on u_1 . Once R is at distance exactly $s - 1$ from a main vertex $x \in K$ on which neither cop is, then C_1 will take a step towards x . From now, C_1 will move in order to keep $d(R, x) \geq d(C_1, x) \leq d(R, x) + 1$ and move only when that is not satisfied

(notice the similarity between this and the algorithm in Theorem 5.2.1). The fact that only the inequality has to be maintained guarantees that if the robber gets to x then C_1 will be at distance one from him and, if the robber does not, we will have at least one free move for another cop before he gets back to u . If the robber gets to x , then C_1 will be able to move at distance one from him, so he will be forced to move away from C_1 . However, since x is in K , if the robber moves towards a main vertex y corresponding to a neighbour of C_2 in H , C_2 can move towards y from now on, either capturing the robber at y or restricting the robber to move on vertices corresponding to the subdivided edge xy . \square

We will use this lemma to prove upper bounds for the lazy-cop number for uniform subdivisions of two families of graphs: complete multipartite graphs and extended block graphs.

Theorem 5.3.5. *If G is a uniform subdivision of a complete multipartite graph, then $c_1(G) \leq 3$.*

Proof. Let C_1 and C_2 choose as their initial positions main vertices of different colour classes of G . We can use C_3 to chase the robber until he gets to a main vertex of G . Notice that one of the cops on main vertices will be on a different colour class as the robber R . We may assume that C_1 is that cop. We can now chase the robber with C_3 to force him to move and apply Lemma 5.3.3 to a subgraph of G containing a maximal clique of G (plus maybe an additional vertex and the corresponding edges) including R , C_1 and C_2 to guarantee the robber's capture. \square

Theorem 5.3.6. *Let T be a block graph and G an extended block graph obtained from T . For every positive integer s , $c_1(G^{(s)}) \leq 3$.*

Proof. The proof is by induction on the number of maximal cliques of T . If T has a single maximal clique, then the result follows from Theorem 5.2.1. Let K_1 be a maximal clique of G that corresponds cut vertex v_1 of T . Let K'_1 be the graph obtained from G by identifying each component U of $G - K_1$ to a vertex x_U adjacent to K_1 . We can use C_3 to chase the robber to a vertex x_U in K'_1 .

Now, notice that C_1 and C_2 are on a doubly dominating set of any maximal clique of G containing K_1 which means that, after a finite number of moves, we can use C_3 to force the robber to move to a main vertex which does not correspond to any such clique. If no such main vertex exists for the robber to go to, he would be captured.

Now, there is vertex v_2 of T corresponding to a cut vertex of T separating the component where the robber is in T from the one containing v_1 . We can now get a cop to a main vertex of K_2 : We try to move C_3 to a main vertex K_2 before the robber enters K_2 and, if the robber enters K_2 before we can achieve this, the strategy that C_1 and C_2 are following will guarantee that one of them, say C_2 , will get at distance one from the robber, forcing him to move out of K_2 towards a vertex outside of K_U or be captured.

At this point, we know that the robber is not in a vertex of K_U but we have two cops in main vertices of K_U and, by Lemma 5.3.3, the robber cannot move along an edge joining main vertices of K_U . If necessary, we rename the cop in K_2 as C_1 and the cop in K_1 and C_2 , and the remaining cop will be C_3 . Again, we may use C_3 to force the robber to a main vertex of G not in K_U if such vertex exists (otherwise, the cops capture the robber). Now, notice that the robber is not able to enter K_2 without being captured by applying Lemma 5.3.3 with the graph K'_2 . This means we can get C_3 to a main vertex of K_2 without the robber entering K_2 but still using the strategy described in Lemma 5.3.3. Once we achieve this, we can rename C_3 as C_2 and continue with the strategy. The new vertex C_3 is not free to move and force the robber to a main vertex not in K_2 . We may now remove K_1 from the graph, together with all vertices of G that are not in the same component as K_2 . Since the extended block graph that we obtained has fewer maximal cliques, we get the result by induction. \square

From Theorem 4.1.4 easy to see that if G is a cop-win graph and $c_1(G^{(s)}) = 1$ for every integer $s \geq 0$, then G is a tree. However, the following problem might be of interest:

Problem 5.3.1. *For every $k > 1$, can we characterize the graphs such that $c_1(G^{(s)}) \leq k$ for every positive integer s ?*

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