Graph Immersions

by

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Abstract

One of the first prominent theorems in structural graph theory is the Kuratowski-Wagner theorem which characterizes planar graphs as those with no $K_{3,3}$ or K_5 minor. Numerous other classical theorems give precise descriptions of the class of graphs with no H-minor for numerous small graphs H. In particular, such classifications exist when H is W_4, W_5, Prism, $K_{3,3}$, K_5, Octahedron, and Cube. One of the most useful tools in establishing such results are splitter theorems which reduce a graph while preserving both connectivity and containment of a given minor.

In this thesis we consider analogous problems for a different containment relation: *immersion*. Although immersion is a standard containment relation, prior to this thesis there were almost no precise structure theorems for forbidden immersions. The most prominent theorems in this direction give a rough description of graphs with no W_4 immersion and those with no $K_{3,3}$ or K_5 immersion.

Our main contributions include precise structure theorems for the class of graphs with no H-immersion when H is one of K_4, W_4, Prism, and $K_{3,3}$. To assist in this exploration, we have also established two splitter theorems for graph immersions, one for $2k$-edge-connected graphs, and another for 3-edge-connected and internally 4-edge-connected graphs.

Keywords: immersion; weak immersion; edge-connectivity; splitter theorem; chain theorem; structural theorem

Dedication

To all whose presence is an inspiration for becoming a better human being;
I am indebted to so many of you for being in my life.

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Chapter 1

Introduction

1.1 Overview

The study of containment relations between graphs is a central topic in structural graph theory. A variety of containment relations may be defined when studying graphs, some of these, such as minor and topological minor are quite well-explored. A graph H is a *minor* of another graph G if a subgraph of a graph G can be transformed to H by a series of edge contraction. Graph H is a *topological minor* of G if G contains a subdivision of H as a subgraph. Arguably, the first significant structural theorem about graph minors is the Kuratowski-Wagner theorem, which characterizes planar graphs.

Theorem 1.1 (Kuratowski [27]; Wagner [47]). *A graph is planar if and only if it does not contain $K_{3,3}$ or K_5 as a minor (subdivision).*

One of the main interests about planar graphs is the 4-colour theorem (4CT) which asserts that every planar graph is 4-colourable. Thanks to the Kuratowski-Wagner theorem, the 4CT is equivalent to the assertion that graphs with no $K_{3,3}$ and K_5 minor are 4-colourable. There is a famous conjecture due to Hadwiger generalizing that.

Conjecture 1.2 (Hadwiger [25]). *Every graph with no K_{t+1}-minor is t-colourable.*

As a step toward Hadwiger's conjecture, Wagner [47] characterized graphs with no K_5-minor, and in doing so he proved that in fact Hadwiger's conjecture for $t = 4$ is equivalent to the 4CT. Wagner also found a structural theorem for graphs with no $K_{3,3}$ minor. In Table 1.1 we have included some of the graphs H for which the class of graphs which do not contain an H-minor is characterized. In fact, for any 3-connected graph H with $|E(H)| \leq 11$ the class of graphs without H-minor is characterized, see [17].

Observe that for cubic graphs, minor relation and topological minor relation are equivalent. Accordingly, the results on characterizing family of graphs without a certain cubic graph (such as K_4 and Prism) as minor can also be considered as results on excluding the graph as topological minor. Further, results regarding topological minor of W_4, W_5, W_6, and W_7 are also present in the literature, see [20, 40, 41].

H	characterization of H-minor free graphs is due to
K_5	Wagner, 1937 [47]
$K_{3,3}$	Wagner, 1937 [47]
K_4	Dirac, 1952 [18]
W_4	Tutte, 1961 [46]
Prism	Dirac, 1963 [19]
W_5	Oxley, 1989 [36]
Cube	Maharry, 2000 [33]
Octahedron	Ding, 2013 [13]
V_8	Maharry and Robertson, 2016 [32]

Table 1.1

In this thesis we are interested in a somewhat less-explored type of containment, *immersion.* A pair of distinct edges with a common neighbour xy, yz is said to *split off at* y if we delete these edges and add a new edge xz. We say a graph G *immerses* H, or alternatively H is *immersed* in G, or G *has an* H *immersion*, and write $G \succeq H$, if a subgraph of G can be transformed to a graph isomorphic to H through a series of splitting pairs of edges. If $G \succeq H$ and $G \ncong H$, we may write $G \succ H$. Equivalently, we could say the graph G has an H immersion if there is a function ϕ with domain $V(H) \cup E(H)$ satisfying the following properties:

- ϕ maps $V(H)$ injectively to $V(G)$
- ϕ assigns every $e = uv \in E(H)$ a path $\phi(e) \subseteq G$ with ends $\phi(u)$ and $\phi(v)$
- If $e, f \in E(H)$ are distinct, then $\phi(e)$ and $\phi(f)$ are edge-disjoint.

In this case, a vertex in $\phi(V(H))$ is called a *terminal* of the H immersion.

Let us pause here to comment that the above definition of immersion is generally known in the literature as *weak immersion.* There is a more restrictive notion of graph immersion, called *strong immersion*, which has the additional restriction that for every edge $e \in E(H)$ the path $\phi(e)$ is internally disjoint from the vertices in $\phi(V(H))$. This thesis only concerns weak immersion which we henceforth refer to simply as immersion.

It is worth mentioning that if H is a topological minor of G, then it is a minor of G, and it is also immersed in G (in both weak and strong senses). However, the minor and immersion relations are not comparable.

In this thesis, we have proved precise structural theorems for the class of graphs with no H immersion when H is one of K_4, W_4, Prism, and $K_{3,3}$, among others. To state our results, we will need to introduce some terminology, and notation, which is done in Section 1.2.

1.2 Basic definitions

Throughout, we will consider finite undirected graphs which may have parallel edges, but we will forbid loops (except where explicitly stated otherwise), as they contribute nothing to the theory for the graphs of interest. We will call on standard terminology for graphs as found in [12].

Let G be a graph. For $u, v \in V(G)$, we call two distinct edges with ends u, v *copies*. For $X \subset V(G)$, we use $\delta_G(X)$ to denote the edge-cut consisting of all edges of G with exactly one endpoint in X, the number of which is called the *size* of this edge-cut, and is denoted by $d_G(X)$. When G is connected we refer to both X and $X^c(= V(G) \setminus X)$ as *sides* of the edge-cut $\delta_G(X)$. (For the sake of simplicity, whenever the graph concerned is clear from the context, we may drop the subscript G.) An edge-cut is called *internal* if it has at least two vertices on either side, and *trivial* otherwise. We say G is *k-edge-connected* (*internally k-edge-connected*) if every edge-cut (internal edge-cut) in G has size at least k. The (*internal*) *edge-connectivity* of the graph G is the maximum k for which G is (internally) k-edge-connected. We denote the edge-connectivity of G by $\lambda(G)$, and the internal edge-connectivty of G by $\lambda^i(G)$.

For a subset $X \subset V(G)$, we write $G.X$ to denote the graph obtained from G by identifying X to a single vertex, followed by deleting any loops created. We call a graph G a *doubled path* (*doubled cycle*) if G is obtained from a path (cycle) by adding a second copy of every existing edge. Our structural theorems often feature a reduction which we introduce below.

Definition 1.3. Let G be a graph and let $X \subset V(G)$ satisfy $|X| = k$. We say $G[X]$ is a *chain of sausages of order k* in G if either X is a single vertex of degree four, or $G.(V(G)\setminus X)$ is a doubled cycle (of length $k+1$). We also define *sausage reduction* to be an operation which replaces every maximal chain of sausages of order ≥ 3 with a chain of sausages of order two. A graph G is said to be *sausage reduced* if it does not contain sausages of order ≥ 3.

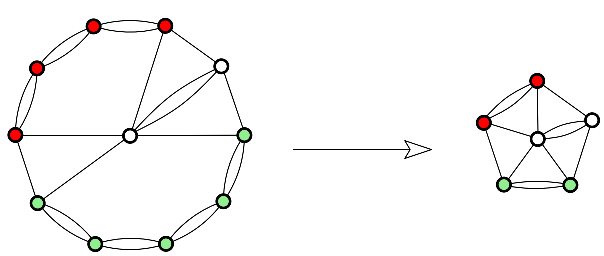


Figure 1.1: Sausage reduction

Throughout this thesis, when depicting graphs we use ... next to a chain of sausages to mean that the chain of sausages can be of any positive order.

Families of obstructions for immersion of a small graph H often turn out to feature a structure involving particular nested k-edge-cuts (for some k depending on H). This structure is captured in the next definition.

Definition 1.4. For a set X, we say there is an (a, b)-*segmentation of* X if there exist subsets $X_1 \subset X_2 \subset \ldots \subset X_t$ of X satisfying the following:

- $|X_1| = a, |X \setminus X_t| = b$,
- $|X_{i+1} \setminus X_i| = 1$, for $i = 1, \ldots, t-1$.

We refer to X_1 as *head*, and to $X \setminus X_t$ as *tail* of the segmentation, and we may simply say X has a segmentation relative to $(X_1, X \setminus X_t)$.

For a graph G, we say G has an (a, b)-*segmentation of width* k if there is an (a, b)-segmentation of $V(G)$ such that for every X_i, for $i = 1, \ldots, t$, we have $d_G(X_i) = k$. See Figure 1.2 for an example of a graph segmentation.

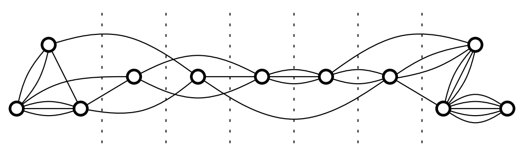


Figure 1.2: A graph with a $(3, 3)$-segmentation of width four

1.3 Structural theorems for forbidden immersions

Our results in this thesis include precise structural theorems for graphs which do not immerse H, where H is one of K_4, W_4, Prism, or $K_{3,3}$. A recurrent phenomena which appear in all our theorems is this: For a graph G to immerse a particular graph H, all that is needed is "the right edge-connectivity", "enough vertices" (depending on H), and in the case where H is nonplanar, for G to not have "obvious topological obstructions to immersing H".

However, the description of obstructions for graphs on small number of vertices is somewhat complicated. So in this chapter, for the sake of clarity and simplicity of the presentation, we are going to state corollaries of our results for "big enough" graphs. We will also state our theorems for graphs with certain edge-connectivity assumptions. However, in all instances, our results lead to a complete description of all graphs in the class (without any edge-connectivity restriction).

1.3.1 K_4 immersion

In [4], Booth et al. give a wealth of structural theorems regarding graphs with no K_4 immersion and use them to obtain a fast algorithm for testing the existence of K_4 immersion in a graph. We give a precise structural theorem for graphs without a K_4 immersion.

Theorem 1.5. *Let G be a graph with $\lambda(G) \geq 3, |V(G)| \geq 4$. Then $G \not\succ K_4$ if and only if*

- *G is a doubled cycle, or*
- *G has a $(2,2)$-segmentation of width three.*

As we will see in Chapter 3, the theorem above is in fact derived as a corollary of a stronger result on K_4 immersion in which up to two terminals of K_4 are specified in advance.

1.3.2 W_4 immersion

A *wheel graph*, denoted W_n, for every $n \geq 3$, is a simple graph obtained from C_n (a cycle of length n) by adding a new vertex (not on the cycle) which has an edge to every vertex of the cycle. Despite considerable attention in the setting of topological minor, the problem of characterizing family of graphs with no immersion of wheel graphs has been studied only for W_4. Belmonte et al. have used facts about the presence of a wall graph as a topological minor to show

Theorem 1.6 (Belmonte et al. [3])**.** *Let G be a graph with $\lambda(G) \geq 3, \lambda^i(G) \geq 4$. If $G \not\succ W_4$ then either G is cubic or has tree-width at most $2^{2^{13}.3^6.5^8.\log(2^6.3^2.5^4)}$.*

Our result on graphs excluding W_4 as immersion is in the same setting as the above theorem, except that we insist that the graph be sausage reduced.

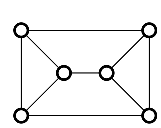
Theorem 1.7. *Let G be a graph with $\lambda(G) \geq 3, \lambda^i(G) \geq 4$ which is sausage reduced. If $|V(G)| \geq 6$ then $G \succ W_4$ unless G is cubic.*

In Chapter 4, we prove a stronger form of Theorem 1.7 in which the assumption on the order of G is relaxed to $|V(G)| \geq 5$. As a corollary we then derive that the upper bound on the tree-width in Theorem 1.6 can be reduced to three.

In fact, the above theorem is obtained as a result of a stronger theorem characterizing the graphs without an immersion of W_4 in which the terminal corresponding to the center of W_4 is specified in advance.

1.3.3 Prism immersion

As mentioned earlier, a classic theorem of Dirac characterizes the structure of graphs which do not have Prism (the graph below) as a (topological) minor. However, to our knowledge, this problem has not been looked into in the setting of graph immersions. The following is



a result of our characterization of graphs which exclude an immersion of Prism:

Theorem 1.8. *Let G be a 3-edge-connected graph so that for any subset $X \subset V(G)$ with $|X| = 2$ we have $d(X) \geq 4$. If G is sausage reduced and $|V(G)| \geq 7$, then G has a Prism immersion unless $G \cong K_{3,4}$.*

1.3.4 Immersion of Kuratowski graphs

Giannopoulou, Kamiński, and Thilikos obtained the first result on graphs which exclude Kuratowski graphs, i.e. $K_{3,3}$ and K_5 as an immersion. They used a result on the existence of a particular graph as a minor in planar graphs with branch-width ≥ 11 to prove the following:

Theorem 1.9 (Giannopoulou et al. [23])**.** *If G is a graph not containing K_5 or $K_{3,3}$ as an immersion, then G can be obtained by applying consecutive i-edge sums, for $i \leq 3$, to graphs that either are planar and sub-cubic or have branch-width at most* 10*.*

In Chapter 7, we will give a precise structural theorem for the family of graphs not containing $K_{3,3}$ as an immersion. Our theorem on excluding $K_{3,3}$ asserts the following:

Theorem 1.10. *Let G be a graph with $\lambda(G) \geq 3, \lambda^i(G) \geq 4$ which is sausage reduced. If $|V(G)| \geq 9$, then G does not immerse $K_{3,3}$ if and only if*

- *G is planar and cubic, or*
- *G has a $(3,3)$-segmentation of width four.*

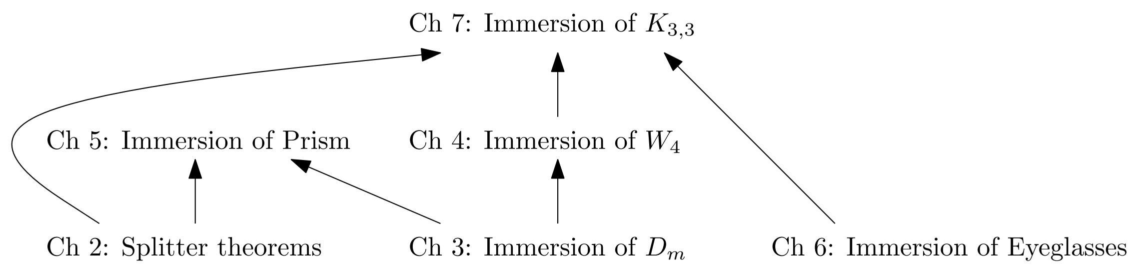
Our main result on $K_{3,3}$ immersion which appears in Chapter 7 includes graphs on six, seven, and eight vertices as well. This enables us to obtain a precise structure for graphs with no $K_{3,3}$ and K_5 immersion. As a result, we see that the upper bound on branch-width in Theorem 1.9 can be replaced by three. As in the world of graph minors, once $K_{3,3}$ is excluded, additionally excluding K_5 has little effect. Indeed, Theorem 1.10 is true with the same outcome when we forbid K_5 in addition to $K_{3,3}$.

Many of our structural theorems are proved in the setting of rooted immersions, and this allows us to use one structural theorem in proving another. The following figure shows the dependencies between our structural theorems; it also indicates that we will use another tool, called a splitter theorem, that we introduce in the next section.

1.4 Splitter Theorems

In the theory of graph minors, one of the most important tools in establishing precise structural theorems for graphs without a certain graph as minor are chain theorems and splitter theorems.

Let G be a graph with a certain connectivity. One natural question is that whether there is a way to "reduce" G while preserving the same connectivity, and possibly also the presence



of a particular graph "contained" in G. Broadly speaking, in answering such questions, two types of theorems arise. In the first type, *chain theorems*, one tries to "reduce" the graph down to some basic starting point, which is typically a particular small graph, or a small family of graphs. The other type of theorems are *splitter theorems*. Here, there is the extra information that another graph H is properly "contained" in G , and both have a certain connectivity. The idea is then to "reduce" G to a graph "one step smaller", while preserving the connectivity, and the "containment" of H.

The best known such results are the ones where the connectivity concerned is vertex-connectivity, the "reduction" is an edge-contraction or edge-deletion, and the "containment" relation is that of minor. In this realm, the first chain result is due to Tutte, who showed if a graph G is 2-connected, for every edge $e \in E(G)$, either $G \setminus e$ or G/e is 2-connected. The next result, also due to Tutte, is a classical result of a reduction theorem of chain variety.

Theorem 1.11 (Tutte [46])**.** *If G is a simple 3-connected graph, then there exists $e \in E(G)$ such that either $G \setminus e$ or G/e is simple and 3-connected, unless G is a wheel.*

Another classical result of chain type is that every simple 3-connected graph other than K_4 has an edge whose contraction results in a 3-connected graph, see [24]. There is a wide body of literature sharpening these results and extending them to other connectivity, see for instance [26, 2, 16, 34, 37], and also to the world of matroids [22, 7].

Reduction theorems of splitter variety for graph minors started with a result for 2-connected graphs which appears in a more general form in the work of Brylawski [5], and Seymour [42]. This result asserts that if G, H are 2-connected graphs, and H is a proper minor of G, then there is an edge $e \in E(G)$ such that $G \setminus e$ or G/e is 2-connected, and has H as a minor. The more famous splitter theorem is Seymour's Splitter Theorem for 3-connected graphs, which asserts:

Theorem 1.12 (Seymour [43])**.** *Let G, H be 3-connected simple graphs, where H is a proper minor of G. Also, suppose if H is a wheel, then G has no larger wheel minor. Then G has an edge e such that either $G \setminus e$ or G/e is simple and 3-connected, and contains H as a minor.*

There is an extremely wide body of literature extending these results to other connectivity and the realm of matroids, and binary matroids, see, for example, [38, 8, 9]. In this thesis, however, we are concerned with the setting of edge-connectivity and immersion. A significant chain theorem in this setting is due to Lovász ([30], Problem 6.53, see also [24]). We say a vertex $v \in V(G)$ of even degree is *completely split* if $d(v)/2$ consecutive splits are performed at v, and then the resulting isolated vertex v is deleted.

Theorem 1.13 (Lovász [30])**.** *Suppose G is $2k$-edge-connected. Then by repeatedly applying complete split, and edge-deletion it can be reduced to a graph on two vertices, with $2k$ parallel edges between them.*

This theorem was later generalized by a significant theorem of Mader that is key in our proofs in Chapter 2, and is stated there. In Chapter 2, we have established two splitter theorems for immersions, the first of which is an analogue of the aforementioned result of Lovász, and is stated below.

Theorem 1.14. *Suppose G, H are $2k$-edge-connected loopless graphs, where $k \geq 2$. If $G \succ H$ then there exists an operation taking G to G' so that G' is $2k$-edge-connected and $G' \succeq H$, where an operation is either*

- *deleting an edge,*
- *splitting at a vertex of degree $\geq 2k+2$,*
- *completely splitting a $2k$-vertex,*

each followed by iteratively deleting any loops, and suppressing vertices of degree 2.

In comparison with graph minors, the literature on splitter theorems for graph immersions is extremely sparse. Indeed, we only know of two significant papers concerning this, namely [14, 15], where Ding and Kanno have proved a handful of splitter theorems for immersion for cubic graphs, and 4-regular graphs. In particular, they have shown the following (see [15], Theorem 9):

Theorem 1.15 (Ding, Kanno[15])**.** *Suppose G, H are 4-edge-connected 4-regular graphs, and $G \succ H$. Then there exists a vertex whose complete split takes G to G' so that G' is 4-edge-connected 4-regular, and $G' \succeq H$.*

Our second theorem, stated below, is similar to the first one, but is for a different type of connectivity, and generalizes Theorem 1.15. In the statement of the theorem, Q_3 denotes the graph of the cube, and K_2^3 denotes the graph on two vertices with three parallel edges between them.

Theorem 1.16. *Let G, H be 3-edge-connected and internally 4-edge-connected graphs, where $G \succ H$. Further, assume $|V(H)| \geq 2$, and $(G, H) \ncong (Q_3, K_4), (Q_3, K_2^3)$. Then there* *exists an operation taking G to G' such that $\lambda(G') \geq 3, \lambda^i(G') \geq 4$, and $G' \succeq H$, where an operation is either*

- *deleting an edge,*
- *splitting at a vertex of degree ≥ 4,*

each followed by iteratively deleting any loops, and suppressing vertices of degree 2.

As a result of the above splitter theorem, we obtain the following chain type theorem:

Corollary 1.17. *Let G be 3-edge-connected, internally 4-edge-connected, and $|V(G)| \geq 2$. If $G \not\cong Q_3, K_2^3$, then one of the operations in the statement of Theorem 1.16 may be applied to G, where the resulting graph is 3-edge-connected, internally 4-edge-connected.*

In the world of graph minors, an immediate simple consequence of Seymour's Splitter Theorem, first observed by Wagner [47], is that every 3-connected graph on at least six vertices containing K_5 as a minor, has a $K_{3,3}$-minor. This fact is then used to obtain a precise structural description of graphs with no $K_{3,3}$-minor. In parallel to this, and as an application of Theorem 1.16, we will establish the following analogue of this result for immersions. The result will be a step towards understanding graphs with no K_5 immersion.

Corollary 1.18. *Suppose G is a 3-edge-connected, and internally 4-edge-connected graph such that $G \succ K_5$. If $|V(G)| \geq 6$ then $G \succ K_{3,3}$, or $G \cong K_{2,2,2}$.*

1.5 More known results on graph minors/immersions

1.5.1 Graph minors

Section 1.1 gave an overview mentioning some classical excluded minor structural theorems. There is a grand graph minors project of Robertson and Seymour that built upon these works and achieved some powerful consequences.

Grid theorem. A graph G either has a $g \times g$ grid as a minor or has tree-width at most $w(g)$, for some function w.

Rough structure for graphs without H-minor. Such a graph can be constructed from graphs embedded in a surface of bounded genus with bounded number of vortices of bounded width, and a bounded number of apex vertices, using a certain sum operation.

Well-Quasi-Ordering. Graphs under minor containment are well-quasi-ordered. (That is, for every infinite set of graphs, one of them contains another one as minor.)

Despite these powerful tools, Hadwiger's conjecture remains a major challenge.

1.5.2 Graph immersions

It is natural to consider the immersion analogues of the graph minors project. The analogue of Grid theorem and Well-Quasi-Ordering was proved using techniques similar to that of graph minor projects. A corresponding rough structure theorem for graphs with no K_t immersion, however, was obtained relatively simply using Gomory-Hu theorem.

Grid theorem. Chudnovsky et al. [6] proved that a 4-edge-connected graph G either has a $g \times g$ grid as an immersion or has tree-width at most $w(g)$, for some function w.

Rough structure for graphs without K_t immersion. DeVos et al. [11] and Wollan [48] proved that for every such a graph, there exists a laminar family of edge-cuts, each with size $< (t-1)^2$, so that every block of the resulting vertex partition has size less than t.

Well-Quasi-Ordering. Robertson and Seymour showed that graphs are well-quasi-ordered under weak immersion containment [39]. This confirmed a conjecture of Nash-Williams [35].

There is also an analogue of Hadwiger's conjecture in the setting of graph immersions, due to Lescure and Meyniel and, independently, to Abu-Khzam and Langston.

Conjecture 1.19 (Lescure and Meyniel [29]; Abu-Khzam and Langston [1])**.** *Every graph which does not have a weak immersion of K_{t+1} is t-colourable.*

The conjecture has been confirmed for $1 \leq t \leq 7$ thanks to the work of Abu-Khzam and Langston [1], Lescure and Meyniel [29], and DeVos et al. [10]. For large values of t, Le and Wollan [28] have recently proved that every graph with chromatic number at least $3.5t + 4$ immerses K_t.

1.6 The story of this research program

Our research started with the (unexpectedly ambitious!) problem of determining the structure of graphs with no K_5 or $K_{3,3}$ immersion. It is not hard to see that this problem naturally reduces to 3-edge-connected and internally 4-edge-connected graphs. Furthermore, Kuratowski's Theorem shows that every graph with no $K_{3,3}$ or K_5 immersion is planar, which brings us to planar graphs. Intuitively, if G is a "well-enough connected" planar graph with a vertex v having at least four distinct neighbours, performing a "non-planar" split at v should result in a non-planar graph. Accordingly, we adopted the seemingly natural approach of working with a minimum counterexample G, and trying to prove that G is "well-enough connected".

Initially we were able to prove that G has the following connectivity:

- For $X \subset V(G)$, if $d(X) = 4$ we have either $|X| \leq 2$ or $|V(G) \setminus X| \leq 2$.
- if G has a cut-vertex v, $G \setminus v$ has exactly two components, one of which is a single vertex.

We proceeded by proving that other certain forms of separation are also forbidden for G. However, taking further steps got much harder than before.

In fact, it was at this point that the need for splitter theorems was felt. We established splitter theorems, in particular for graphs of our original interest, i.e. 3-edge-connected, internally 4-edge-connected graphs. One very useful corollary of our splitter theorem (Theorem 1.16) is Corollary 1.18, which enabled us to focus on excluding $K_{3,3}$ as an immersion (rather than also excluding K_5). However, that was as far as the splitter theorem helped with our research on Kuratowski graphs. Thus, we resumed wrestling with the 2-vertex cuts in G which proved to be the hardest part of the problem. It took over a year to prove that

- G does not have a 2-vertex cut.

Finally, we had succeeded in reducing the problem to the world of 3-connected planar graphs! However, the problem was still involved. It took us another year to prove that

- G does not exist.

The whole proof of our result ran about 150 pages and consisted of massive case analysis; there were applications of Menger's theorem and uncrossing cuts arguments, as well as relying on the planarity of G which enabled us to work with a blend of arguments on vertex-bridges and certain edge-bridges (as introduced in Chapter 6), and peripheral cycles. However, the final result, the fruit of a few years of research, was yet to bear fruits of its own! Right upon the completion of work, two observations caught our attention:

- None of the sporadic graphs with no immersion of $K_{3,3}$ have a cut-vertex.
- All sporadic sausage reduced graphs with no $K_{3,3}$ immersion are on ≤ 8 vertices, most of which only on 6 vertices.

The first one, in particular, inspired another line of inquiry. It seemed to be suggesting that if G is not one of the generic obstructions to existence of $K_{3,3}$ and has a cut-vertex v, then the graph induced on the nontrivial component of $G \setminus v$ together with v (and rooted at v) has a rooted immersion of W_4. We started investigating rooted W_4 immersion with the hope that it would help dealing more efficiently with cut-vertices in the proof of $K_{3,3}$ result. This theorem was also a hard one, with the proof involving many technical lemmas. This is the proof of rooted W_4 problem which appears in Appendix B.

The second observation, on the other hand, inspired us to harness the computer to verify our $K_{3,3}$ theorem for small graphs to avoid much case analysis in the proof of $K_{3,3}$ result.

On this front, we managed to get the computer to go beyond the order of all sporadic graphs without $K_{3,3}$ immersion, thereby putting us in a very strong position to apply induction.

Yet another significant development in the proof of $K_{3,3}$ result came along; we noticed that the rooted W_4 result could in fact serve as a much more powerful tool for $K_{3,3}$ problem than originally intended. Indeed, we could take advantage of it not only to handle cut-vertices in G (a minimum counterexample to the $K_{3,3}$ result), but also to handle "deep" edge-cuts in G. To complete this approach, a structural theorem about another rooted graph, which we call eyeglasses, was proved. With all these powerful tools, we managed to get a totally different proof of our result, which is only eight pages, and avoids case analysis. More importantly, it demonstrates a new paradigm for tackling structural problems in the world of graph immersions. This is the proof that appears in Chapter 7.

On the other hand, we noticed that the approach of getting the computer to handle small graphs, and then combining structural results on certain smaller rooted graphs can also be employed for the problem of rooted W_4 immersion. (The smaller rooted graphs featuring in the rooted W_4 problem are certain four-vertex graphs with two roots, which we call D_m, with results appearing in Chapter 3.) As with the case of $K_{3,3}$, the new proof for rooted immersion of W_4 is much shorter than the original proof, and is not as nearly technical. In Chapter 4 we have included the new proof, which demonstrates another usage of our new paradigm for obtaining precise structural theorems on excluding small graphs as immersions.

Finally, we applied our new approach to the problem of immersing Prism graph. For this proof, which is one of the cleanest in this thesis, we have developed a new convenient inductive tool which takes advantage of our chain theorem.

Chapter 2

Splitter Theorems for Graph Immersions

Let G, H be nonisomorphic graphs with a certain connectivity, where G "contains" H. One may ask whether there is a way to "step down from G towards H" while preserving the same connectivity all along the way. Broadly speaking, in answering such a question, two types of theorems arise. In the first type, *chain theorems*, the starting graph H is typically a particular small graph, or is any graph from a specific small family of graphs. On the other hand there are so called *splitter theorems* in which H can be chosen arbitrarily.

The main results of this chapter provide splitter theorems in the setting where the "containment" relation is graph immersion, and the connectivity concerned is certain edge-connectivity. We will give splitter theorems for graph immersions for two families of graphs—in Section 2.1 for k-edge-connected graphs, for any even $k \geq 4$, and in Section 2.2 for 3-edge-connected, internally 4-edge-connected graphs.

The family of 3-edge-connected, internally 4-edge-connected graphs is of particular interest to us due to the fact that the problem of characterizing graphs not immersing Kuratowski graphs naturally boils down to this family of graphs. As we will see in Section 2.3, and as a result of our splitter theorem, for a graph G in this family with $|V(G)| \geq 6$ which is not isomorphic to the Octahedron, $G \succ K_5$ implies $G \succ K_{3,3}$. Our splitter theorem for 3-edge-connected, internally 4-edge-connected graphs also has a second significant corollary—it gives us a chain type theorem for this family of graphs.

The rest of this chapter is organized as follows: In Section 2.1 we state the preliminary definitions and key tools, and prove our splitter theorem for k-edge-connected graphs, for $k \geq 4$ even. In Section 2.2, we state and prove our result for 3-edge-connected, internally 4-edge-connected graphs, as well as a chain type theorem for this family. Finally in Section 2.3 we will see a useful corollary of our result in Section 2.2 for Kuratowski graphs.

2.1 k-edge-connected graphs, $k \geq 4$ even

The main result of this section is the following splitter theorem for k-edge-connected loopless graphs, where $k \geq 4$ is even.

Theorem 2.1. *Suppose G, H are k-edge-connected loopless graphs, where $k \geq 4$ is an even number. If $G \succ H$, there exists an operation taking G to G' so that G' is k-edge-connected and $G' \succeq H$, where an operation is either*

- *deleting an edge,*
- *splitting at a vertex of degree $\geq k+2$,*
- *completely splitting a k-vertex,*

each followed by iteratively deleting any loops.

Note that in order to have a splitter theorem for the family of k-edge-connected graphs, we do need to embrace completely splitting a k-vertex as one of our operations, since as soon as we do a split at a k-vertex, the graph will have a trivial $(k-2)$-edge-cut.

2.1.1 Preliminaries

Before embarking on the proof of Theorem 2.1, we fix some notation and highlight a couple facts and theorems which will feature in our proofs. Let G be a graph. For X, Y disjoint subsets of $V(G)$, we denote the set of edges between X, Y by $E_G(X, Y)$, the size of which will be denoted by $e_G(X, Y)$. For distinct vertices $x, y \in V(G)$, we simply write $e_G(x, y)$ to denote the number of edges between x and y. Also, the maximum size of a collection of pairwise edge-disjoint paths between x and y is denoted by $\lambda_G(x, y)$. For the sake of brevity, whenever the graph concerned is clear from the context, we may drop the subscript G. For a graph G, and $X \subseteq V(G)$, we use the notation X^c to denote $V(G) \setminus X$.

Observation 2.2. *Suppose G is a graph, and $X \neq Y$ are distinct nonempty subsets of $V(G)$. Then, by counting the edges contributing to the edge-cuts, we have*

$$d(X \cap Y) + d(X \cup Y) + 2e(X^c \cap Y, X \cap Y^c) = d(X) + d(Y).$$

Observe that this also implies the following inequality

$$d(X \cap Y) + d(X \cup Y) \leq d(X) + d(Y).$$

Another frequently used fact is the classical theorem of Menger. A proof may be found, for instance, in [12].

Theorem 2.3 (Menger)**.** *Let G be a graph, and x, y distinct vertices of G. Then $\lambda_G(x, y)$ equals the minimum size of an edge-cut of G separating x from y.*

The next theorem, which is a slight strengthening of a well-known theorem by Mader, is an extremely powerful tool when working with immersions, and is also a key ingredient in our proofs.

Theorem 2.4 (Mader [31], see also Frank [21])**.** *Let G be a graph and suppose for $s \in V(G)$ we have $d(s) \geq 4$, and s is not incident with a cut-edge. Then $\delta(s)$ can be partitioned into $\lfloor d(s)/2 \rfloor$ disjoint pairs (e_i, f_i), $i = 1, \ldots, d(s)/2$, so that in the graph G_i' resulting from splitting e_i, f_i, for any $x, y \in V(G_i') \setminus s$, we have $\lambda_G(x, y) = \lambda_{G_i'}(x, y)$, $i = 1, \ldots, d(s)/2$.*

Theorem 2.1 will be proved through a series of lemmas. We will begin by introducing a few convenient definitions which are motivated by the study of completely splitting a vertex u of even degree, k. We consider this operation as $\frac{k}{2}$ many splits at u (followed by deleting u), and often need to talk about graphs that are "halfway through the action". We need to keep track of the vertex that is being split, as well as the edges that have been created as a result of splits at u. In terms of edge-connectivity, we need to make sure that the intermediate graphs do not have edge-cuts of size less than k, except for possibly $\delta(u)$. The idea is then to stop doing splits at u the first time that this "halfway-through-the-action" graph is about to have an edge-cut of size $< k$ (other than $\delta(u)$), and do something else ("good operation") instead, and then "undo" the splits that was done at u. The "good operations" are defined in a way that, in particular, allow for "undoing the splits at u". All this is made precise in the following technical definitions.

Definition 2.5. Let k be an even number. A *k-enhanced graph* is a triple (G, U, F), where G is a graph, $U \subset V(G)$ and $F \subset E(G)$ such that

- U is either empty, or consists of a single vertex, where $d(U)$ is an even number,
- if U is empty, then F is also empty, and otherwise, $d(U) + 2|F| \leq k$.

If in the inequality above, equality holds we say (G, U, F) is *reversible*. If U is nonempty, we call the vertex in U *the special vertex*, and an edge in F is called a *special edge*. To *sew* a special edge xy of (G, U, F), where $U = \{u\}$, means to delete xy (from G and thus from F), and add edges xu and uy. We say splitting off xy, yz is a *bad split* if $xy \in F$ and $yz \in F$. We also say a complete split of a vertex $v \notin U$ with $d(v) = k$ is a *bad complete split* if at least one of the splits done at v is a bad split. In the case that U is empty, for the sake of brevity we may simply write G to refer to $(G, \emptyset, \emptyset)$. When talking about graph concepts (such as deleting an edge) on a k-enhanced graph (G, U, F), their corresponding in G is meant.

Suppose (G, U, F) is a k-enhanced graph. If H is a subgraph of G, then we associate H with the k-enhanced graph $(H, U, F \cap E(H))$. Also, if H is a graph obtained from G by splitting a 2-edge-path xyz (to create xz), then we associate H with the k-enhanced graph (H, U, F'), where for $e \in E(H) \setminus xz$ we have $e \in F'$ if $e \in F$, and

- if $y \in U$ then $xz \in F'$,

- if $y \notin U$ then $xz \in F'$ if $xy \in F$ or $yz \in F$.

Let O be one of the operations in the statement of Theorem 2.1. We denote by $O(G)$ the graph obtained from G by applying O, and we write $O(G,U,F)$ to denote the k-enhanced graph associated with $O(G)$ (by the above rules).

A k-enhanced graph (G,U,F) (or for brevity (G,U)) is called *nearly k-edge-connected* if either G is k-edge-connected, or U is nonempty and every edge-cut in G apart from $\delta(U)$ has size at least k.

Suppose (G,U,F) is nearly k-edge-connected, and H is a k-edge-connected graph, where $k \geq 4$ is even, and $G \succ H$. We define a *good operation* on (G,U,F) to be either

- a split at the special vertex,
- a (complete) split at a non special vertex of degree $\neq k+1$ which is not a bad (complete) split,
- deleting a non special edge,

so that the resulting graph G' immerses H and the k-enhanced graph associated with G' is nearly k-edge-connected.

Let us highlight some useful observations about the setting we work in.

Observation 2.6. *Suppose (G,U,F) is a reversible k-enhanced graph, and there is no special edge incident with u.*

1. *If O is a good operation on (G,U,F), then $O(G,U,F)$ is also reversible.*
2. *If (G,U,F) is nearly k-edge-connected, then the graph obtained from (G,U,F) by sewing all its special edges is k-edge-connected.*

Note. Throughout the rest of this section, we will assume that k $\geq$ 4 is even, and H is a k-edge-connected graph, and G is a graph where G $\succ$ H.

2.1.2 k-edge-cuts

The lemmas in this subsection concern k-edge-cuts in nearly k-edge-connected enhanced graphs. We start by establishing a technical lemma, which makes use of Mader's Theorem to provide us with a tool for dealing with k-edge-cuts, as well as $(k+1)$-edge-cuts.

Lemma 2.7. *Let (G,U,F) be nearly k-edge-connected, and let u' be a vertex of degree k where $u' \notin U$. Then there exists a complete split, O, at u' which is not bad, and moreover, $O(G,U,F)$ is nearly k-edge-connected.*

Proof. Choose O to be a complete split of u' so that subject to $O(G, U, F)$ being nearly k-edge-connected, the number of bad splits done by O is minimum. Note that it follows from Mader's Theorem (Theorem 2.4) that O is well defined. To prove the lemma, it suffices to show that O does not involve any bad splits. Otherwise, there exist special edges xu', yu' that are split under O. Since $|F| \leq \frac{k}{2}$, there exist non special edges zu', wu' that are split under O. We claim that one of the following complete splits of u'

- O_1 that splits xu' with zu' and splits yu' with wu', and agrees with O on other splits,
- O_2 that splits xu' with wu' and splits yu' with zu', and agrees with O on other splits,

contradicts the choice of O. Otherwise, since the number of bad splits done by O_1 and O_2 is one less than the number of bad splits done by O, none of $O_1(G, U, F)$ and $O_2(G, U, F)$ are nearly k-edge-connected. Then, nearly k-edge-connectivity of $O(G, U, F)$ implies that there exists an edge-cut $\delta(X)$ of size $k-1$ or $k-2$ in $G^* = O(G) - xy - zw$ that separates $\{x, z\} \subset X$ from $\{y, w\}$, and there exists an edge-cut $\delta(Y)$ of size $k-1$ or $k-2$ in G^* that separates $\{x, w\} \subset Y$ from $\{y, z\}$. Let $G' = O(G)$, so $\delta_{G'}(X)$ and $\delta_{G'}(Y)$ are edge-cuts of size $k+1$ or k in G' with $x \in X \cap Y$, $z \in X \cap Y^c$, $w \in X^c \cap Y$, $y \in X^c \cap Y^c$.

Remark. Let K be a graph with $Z \subset V(K), Z = Z_1 \cup Z_2, Z_1 \cap Z_2 = \emptyset$. Then we have

$$d(Z) = d(Z_1) + d(Z_2) - 2e(Z_1, Z_2) \qquad (*)$$

Note that $d_{G'}(X) = d_{G'}(Y) = k+1$ is impossible. Otherwise, using $(*)$, by possibly replacing Y with Y^c, we may assume that $d_{G'}(X \cap Y)$ is even and $d_{G'}(X \cap Y^c)$ is odd. Since $d_{G'}(Y) = k+1$ is odd, by $(*)$, we conclude that $d_{G'}(X^c \cap Y)$ is odd, and so both $X^c \cap Y$ and $X \cap Y^c$ contain a non special vertex, and thus $d_{G'}(X^c \cap Y), d_{G'}(X \cap Y^c) \geq k+1$. Then, since $xy \in E_{G'}(X \cap Y, X^c \cap Y^c)$, we get the contradiction

$$2k+2 = d(X) + d(Y) = d(X^c \cap Y) + d(X \cap Y^c) + 2e(X \cap Y, X^c \cap Y^c) \geq k+1+k+1+2.$$

So, without loss of generality, we may assume $d_{G'}(X) = k$; we may also assume that none of y, z, w are the special vertex, and thus $d_{G'}(X^c \cap Y), d_{G'}(X \cap Y^c) \geq k$. Now, we get

$$2k+1 \geq d(X) + d(Y) = d(X^c \cap Y) + d(X \cap Y^c) + 2e(X \cap Y, X^c \cap Y^c) \geq k+k+2$$

which is impossible. This completes the proof of the lemma. □

Lemma 2.8. *Suppose (G, U) is nearly k-edge-connected, and there exists $X \subset V(G)$ such that $d(X) = k$, and every $x \in X$ is of degree $k+1$. Then there exists $Z \subseteq X$ with $|Z| \geq 2$ such that $d(Z) = k$ and for every $e \in E(G[Z])$, $(G \setminus e, U)$ is nearly k-edge-connected.*

Proof. Choose $Z \subseteq X$ such that $d(Z) = k$, and subject to this Z is minimal. Since every $z \in Z$ has degree $k+1$, we have $|Z| \neq 1$, and $Z \cap U = \emptyset$. Let $e \in E(G[Z])$, and we will show that $(G \setminus e, U)$ is nearly k-edge-connected. For a contradiction, suppose e is in a k-edge-cut, $\delta(Y)$. Note that $d(Z^c) = d(Z) = k$ implies that Z^c contains at least one vertex v other than the special vertex. We may assume (by possibly replacing Y by Y^c) that $v \notin Y$. Since (G, U) is nearly k-edge-connected, $d(Z \cap Y), d(Z^c \cap Y^c) \geq k$. However, it follows from

$$k + k \leq d(Z \cap Y) + d(Z^c \cap Y^c) \leq d(Z) + d(Y) = k + k$$

that $d(Z \cap Y) = k$, which contradicts minimality of Z. □

Let us make an observation before proceeding:

Observation 2.9. *Suppose G is a graph with $X \subset V(G)$ such that there exists an immersion of H with all terminals in X. Then $G.X^c$ contains H as an immersion.*

The following lemma enables us to handle k-edge-cuts in nearly k-edge-connected enhanced graphs:

Lemma 2.10. *Suppose (G, U, F) is nearly k-edge-connected and G has a nontrivial k-edge-cut $\delta(X)$ such that some immersion of H has no terminal in X, then there exists a good operation.*

Proof. If the special vertex is in X, we will split it using Mader's Theorem (Theorem 2.4). Else, if there exists a vertex $x \in X$ with $d(x) \geq k+2$, we use Mader's Theorem to do a split at x which is not bad (this is possible since $|F| \leq \frac{k}{2}$). Also, if there exists a vertex $x \in X$ with $d(x) = k$, we use Lemma 2.7 to do a complete split at x. In either case, let (G', U, F') be the resulting nearly k-edge-connected enhanced graph. To see that the operation is good, observe that there remain k edge-disjoint paths in G' between any pair of non special vertices, one in X, and the other in X^c (as it was the case in G). Therefore G' immerses $G.X$, and thus immerses H.

Now suppose every vertex in X is of degree $k+1$. Note that if Z is a subset of X with $|Z| \geq 2$ and $d(Z) = k$, then the degree assumption (together with $|F| \leq \frac{k}{2}$) imply that there is a non special edge in $E(G[Z])$. It follows from this together with Lemma 2.8 that there is a non special edge e lying in $G[Z]$, for some $Z \subseteq X$ such that $(G \setminus e, U)$ is nearly k-edge-connected. Now, the same argument as above shows that $G \setminus e \succeq H$, so deleting e is indeed a good operation. □

2.1.3 $(k+1)$-edge-cuts

The next two lemmas which concern a broader family of graphs, will later be helpful dealing with $(k+1)$-edge-cuts in nearly k-edge-connected enhanced graphs.

Lemma 2.11. *Let G be an internally k-edge-connected graph in which every vertex of degree $< k$ is of even degree, and let $x \in V(G)$. If $d(x)$ is odd, then there exists $y \in V(G) \setminus x$ such that $\lambda(x, y) \geq k + 1$.*

Proof. We prove the statement by induction on $|V(G)|$. Note that by parity, there must exist another vertex of odd degree, y, in G. If every cut separating x from y is of size $\geq k+1$, by Menger's Theorem (Theorem 2.3) we are done. Otherwise, there exists a k-edge-cut $\delta(Y)$, with $y \in Y$, separating x from y.

Note that degree properties imply that $|Y| \geq 2$, so the graph $G' = G.Y$, which satisfies the lemma's hypothesis, has fewer vertices than G. Also x is of odd degree in G' as well, thus, by induction hypothesis there exists $y' \in V(G') \setminus x$ such that $\lambda_{G'}(x, y') \geq k + 1$. It follows that $\lambda_G(x, y') \geq k + 1$ as well, since $\lambda_G(x, y) = k$ implies that $G \succ G'$. $\square$

Lemma 2.12. *Let G be an internally k-edge-connected graph in which every vertex of degree $< k$ is of even degree. If $\delta(X)$ is a $(k + 1)$-edge-cut in G, there exist $x \in X, y \in X^c$ such that $\lambda(x, y) \geq k + 1$.*

Proof. Let $G_1 = G.X, G_2 = G.X^c$, with s, t being the nodes replacing X, X^c, respectively. Note that both G_1, G_2 satisfy Lemma 2.11's hypothesis. Also, s is a vertex of odd degree in G_1, so, by Lemma 2.11, there exists $y \in X^c$ such that $\lambda_{G_1}(s, y) \geq k+1$, thus $G \succeq G_2$. It can be similarly argued that there exists $x \in X$ such that $\lambda_{G_2}(x, t) \geq k+1$, which together with $G \succeq G_2$ shows that $\lambda_G(x, y) \geq k + 1$. $\square$

Having the lemma above in hand, we can now efficiently handle $(k + 1)$-edge-cuts:

Lemma 2.13. *If (G, U, F) is a nearly k-edge-connected enhanced graph with a nontrivial $(k + 1)$-edge-cut $\delta(X)$ such that some immersion of H has no terminal in X, then there exists a good operation.*

Proof. If the special vertex is in X, we will split it using Mader's Theorem. Else, if there exists a vertex $x \in X$ with $d(x) \geq k + 2$, we use Mader's Theorem to do a split at x that is not bad (this is possible since $|F| \leq \frac{k}{2}$). Also, if there exists a vertex $x \in X$ with $d(x) = k$, we use Lemma 2.7 to completely split x. We claim these operations are good. Let (G', U, F') be the nearly k-edge-connected enhanced graph resulting from applying one of these operations on (G, U, F). First, note that $\delta(X)$ remains a $(k + 1)$-edge-cut in G', since doing a split changes the size of an edge-cut by an even number, and, by the near edge-connectivity of (G', U), $d_{G'}(X) \geq k$. We may now apply Lemma 2.12 to choose $x \in X$, $y \in X^c$ with $\lambda(x, y) \geq k + 1$. Thus G' immerses $G.X$, and therefore, G' immerses H.

Now, suppose every vertex in X is of degree $k+1$. Since $d(X) = k+1$, $G[X]$ is connected, and by parity $|X| \geq 3$, and since $|F| \leq \frac{k}{2}$, there is a non special edge $e \in G[X]$. If $(G \setminus e, U)$ is nearly k-edge-connected, then the same argument as above shows that deletion of e is a good operation. So, we may now assume that e is in a k-edge-cut, $\delta(Y)$. Using $(*)$, by possibly replacing Y with Y^c, we may assume that $d(X \cap Y^c)$ is even and $d(X \cap Y)$ is odd. Since $d(Y) = k$ is even, by $(*)$, we conclude that $d(X^c \cap Y)$ is odd, and so $X^c \cap Y$ is nonempty. Thus, both $X \cap Y^c$ and $X^c \cap Y$ contain a non special vertex. We also have

$$2k + 1 = d(X) + d(Y) \geq d(X \cap Y^c) + d(X^c \cap Y) \geq 2k,$$

so by parity $d(X \cap Y^c) = k$. Therefore, $\delta(X \cap Y^c)$ is a nontrivial k-edge-cut (as every vertex in X is of degree $k+1$) with no terminal of H in $X \cap Y^c$. Applying Lemma 2.10 we may conclude that a good operation exists. $\square$

2.1.4 Finishing the proof of Theorem 2.1

The next three lemmas concern the three operations allowed in stepping from G towards H, both of which are k-edge-connected loopless graphs with $G \succ H$, and show that in each case we can take a step maintaining k-edge-connectivity and the presence of an H immersion.

Lemma 2.14. *Suppose that G, H are k-edge connected loopless graphs where $G \succ H$, and there is a complete split of a k-vertex u of G preserving an H immersion. Then a good operation exists.*

Proof. Consider the complete split of u as $\frac{k}{2}$ many splits at u, and choose a sequence of splits which, while preserving an H immersion, results in the fewest number of loops. If u could be completely split without ever creating a too small of an edge-cut other than $\delta(u)$ along the way, we are done. Otherwise, we will stop doing these splits the first time the resulting graph G' is about to have an edge-cut of size $< k$ other than $\delta(u)$. In G', therefore, there exists a subset $X \neq \{u\}, \{u\}^c$ of $V(G)$ for which $d_{G'}(X) \geq k$, doing the next split, however, makes it an edge-cut of size less than k, so $d_{G'}(X) = k$ or $k+1$. Now, let $U = \{u\}$ and let F be the set of edges of G' which are created by splitting at u. Note that since every edge in F is created by a single split at u, (G', U, F) is reversible.

Now, since completely splitting u results in $d(X) < k$ and preserves an immersion of H, there is an immersion of H with all terminals on one side of $\delta(X)$, say X^c. First, suppose $\delta(X)$ is a nontrivial cut. Then Lemma 2.10 or 2.13 applied to (G', U, F) guarantee the existence of a good operation, O, which either is a split at u, or is a (complete) split at a vertex other than u which is not bad, or is deleting a non special edge. Let $(G'', U, F') = O(G', U, F)$, so (G'', U) is nearly k-edge-connected. Note that since G is loopless, there is no special edge incident with u in G'. Now, by Observation 2.6, (G'', U, F') is reversible. If O is a split at u, we resume doing splits at u. In all the other cases, let G^* be the graph obtained from (G'', U, F') by sewing all its special edges. Then, it follows from reversibility of (G'', U, F') and Observation 2.6 that G^* is k-edge-connected. Also it follows from $G^* \succeq G'' \succeq H$ that $G^* \succeq H$, as desired.

Now suppose $\delta(X)$ is a trivial cut with $X = \{v\}$. Therefore the next split at u would create a loop at v. Let G_1' be the graph obtained from performing this split, so $G_1' \succ H$. Note that there cannot be a vertex $w \in N_{G'}(u) \setminus v$, because if there was one, then let G'_2 be the graph obtained from G' by splitting off vuw. Note that G'_2 is isomorphic to G'_1 after deleting the loop at v, and thus it follows from H being loopless and $G'_1 \succeq H$ that $G'_2 \succeq H$. Moreover, G'_2 has no loop and this contradicts the choice of the sequence of splits at u. Therefore $N_{G'}(u) = \{v\}$, implying that $d_{G'}(v) = d_{G'}(\{u, v\}) + d_{G'}(u)$. This, however, contradicts $d_{G'}(X) \in \{k, k+1\}$, as $d_{G'}(X) = d(v) = d_{G'}(\{u, v\}) + d_{G'}(u) \geq k + d_{G'}(u) \geq k + 2$, where the inequalities hold because (G', U) is nearly k-edge-connected and u is of even degree. This completes the proof. □

Lemma 2.15. *Let G, H be k-edge connected loopless graphs where $G \succ H$, and there is an edge $e \in E(G)$ such that $G \setminus e \succeq H$. Then a good operation exists.*

Proof. Suppose e is in a k-edge-cut. If it is incident with a k-vertex u, then, by the previous lemma, u could be completely split off while maintaining an H immersion and k-edge-connectivity. Otherwise, e is in a nontrivial k-edge-cut with all terminals of H on one side of the cut. Thus we can use Lemma 2.10 to find a good operation. □

Lemma 2.16. *Let G, H be k-edge connected loopless graphs where $G \succ H$. If there is a split at a vertex v preserving an H immersion, then a good operation exists.*

Proof. Suppose splitting at v makes an edge-cut $\delta(X)$ too small, then $d(X) = k$ or $k+1$. Also, all terminals of H are on one side of the cut, say X^c. If $\delta(X)$ is a nontrivial edge-cut Lemma 2.10 or 2.13 may be applied. If $|X| = 1$, with $d(X) = k$, we apply Lemma 2.14 to completely split the vertex in X, and if $d(X) = k+1$ we will apply Lemma 2.15 to delete an edge incident to v. □

The proof of Theorem 2.1 is now immediate:

Proof of Theorem 2.1. Since $H \prec G$, there is either a complete split at a vertex of degree k, or an edge deletion or a split at a vertex of degree at least $k+2$ that takes G to G' such that $G' \succeq H$. Now, apply Lemmas 2.14, 2.15, and 2.16. □

2.2 3-edge-connected, internally 4-edge-connected graphs

In this section we establish a splitter theorem for the family of 3-edge-connected, internally 4-edge-connected graphs. Later, as an application, we will also see a chain theorem for this family. The following is our main result in this section:

Theorem 2.17. *Let G, H be 3-edge-connected, and internally 4-edge connected loopless graphs, with $G \succ H$. Further, assume $|V(H)| \geq 2$, and $(G, H) \ncong (Q_3, K_4), (Q_3, K_2^3)$. Then there exists an operation taking G to G' such that G' is 3-edge connected, internally 4-edge connected, and $G' \succeq H$, where an operation is either*

- *deleting an edge,*
- *splitting at a vertex of degree ≥ 4,*

each followed by iteratively deleting any loops and isolated vertices, and suppressing vertices of degree 2.

As in the proof of Theorem 2.1, we will consider each operation separately, and the proof of the theorem will then be immediate. First, we will adjust our notion of a good operation as follows:

Definition 2.18. Suppose G, H are 3-edge-connected, and internally 4-edge connected loopless graphs, with $G \succ H$. We define a *good operation* to be either a split at a vertex of degree ≥ 4, or a deletion of an edge from G which preserves 3-edge-connectivity, internal 4-edge-connectivity, and an immersion of H in the resulting graph.

Lemma 2.19. *Suppose G, H are as in Theorem* 2.17*, and there is an edge e such that $G \setminus e$ has an H immersion. Then if $(G, H) \not\cong (Q_3, K_4), (Q_3, K_2^3)$, a good operation exists.*

Proof. Since deletion of e is followed by suppression of any resulting vertices of degree two, $G \setminus e$ is clearly 3-edge-connected. If deletion of e does not preserve internal 4-edge-connectivity, then e must be contributing to some 4-edge-cut, $\delta(X)$, in which each side has either at least three vertices, or has two vertices which are not both of degree 3. We call such a cut an *interesting cut.*

Note that H too is internally 4-edge-connected, thus all, but possibly one, of the terminals of an immersion of H lie on one side of this cut, say X. Let X' be the maximal subset of $V(G)$ containing X, such that $\delta(X')$ is interesting. Suppose there is an edge uv in X'^c not contributing to an interesting edge-cut, then deleting uv is a good operation. It is because $G \setminus uv$ is 3-edge-connected, internally 4-edge-connected. Also $G \setminus uv$ has an H immersion, because it immerses $(G \setminus e).X^c$.

We may now assume that uv is in some interesting edge-cut $\delta(Y)$. Note that maximality of X' implies that $X' \cap Y, X' \cap Y^c \neq \emptyset$. Also, we claim that there cannot be edges contributing to both $\delta(X'), \delta(Y)$. To prove the claim, suppose, to the contrary, that there are edges between, say, $X' \cap Y, X'^c \cap Y^c$, i.e. $e \neq 0$ in Figure 2.1. Then it follows from

$$8 = d(X') + d(Y) = d(X'^c \cap Y) + d(X' \cap Y^c) + 2e \geq 3 + 3 + 2e$$

that if $e \neq 0$, then $e = 1$ and, moreover, $d(X'^c \cap Y) = d(X' \cap Y^c) = 3$.

Using a similar argument, one can see that, if in addition to $e \neq 0$, there are also edges between $X' \cap Y^c, X'^c \cap Y$, then $d(X'^c \cap Y^c) = 3$. Thus, both $X'^c \cap Y^c, X'^c \cap Y$ would consist of a single vertex of degree 3, contradicting $\delta(X')$ being interesting. Therefore the number of edges contributing to both $\delta(X'), \delta(Y)$ equals e.

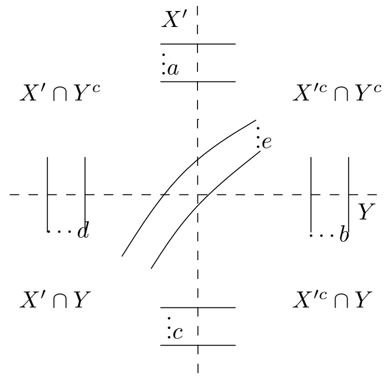


Figure 2.1: Cuts $\delta(X')$, $\delta(Y)$ relative to each other

We will now show that $e \neq 0$ results in a contradiction. Note that from $d(X'^c \cap Y) = 3$ we may conclude, without loss of generality, that $b \geq 2$. Now, by alternatively looking at the cuts $\delta(X'^c \cap Y), \delta(X'), \delta(X' \cap Y^c), \delta(Y)$, we see that if $b \geq 2$, then $c \leq 1$, so $a \geq 2$, thus $d \leq 1$. Therefore, $d(X' \cap Y) = c + d + e \leq 3$, so $X' \cap Y$ consists of a single vertex of degree three. This, however, together with the earlier conclusion of $X' \cap Y^c$ consisting of a single vertex of degree three contradicts $\delta(X')$ being interesting. Therefore $e = 0$, so there are no edges contributing to both $\delta(X')$ and $\delta(Y)$.

Now, we show that $a = b = c = d = 2$. For a contradiction, we will assume that, say $a > 2$, and, similar to the argument above, alternatively look at the cuts $\delta(X'), \delta(X' \cap Y), \delta(Y)$. It then follows that $c \leq 1$, so $d \geq 2$, thus $b \leq 2$. So, in order for $d(X'^c \cap Y) = b + c \geq 3$, we must have $b = 2, c = 1$. Also, we have $d(Y) = 4 = b + d$, so $d = 2$, thus $d(X' \cap Y) = c + d = 3$. Hence, each $X'^c \cap Y$ and $X' \cap Y$ consist of a single vertex of degree three, which contradicts $\delta(Y)$ being interesting.

Therefore, $a = b = c = d = 2$, and thus $\delta(X'^c \cap Y), \delta(X'^c \cap Y^c)$ are 4-edge-cuts. However, by maximality of X', they cannot be interesting cuts. Thus each of $X'^c \cap Y, X'^c \cap Y^c$ consists of only one vertex, or two vertices both of degree 3.

We are now ready to prove that a good operation exists unless $(G, H) \cong (Q_3, K_4)$ or $(G, H) \cong (Q_3, K_2^3)$. Consider different possibilities for $X'^c \cap Y, X'^c \cap Y^c$:

- Both sets consist of one vertex, see Fig. 2.2(a). Here, a good operation is to split off wuv. Note that the resulting graph immerses H, as it immerses $(G \setminus e).X^c$.

- Only one set consists of one vertex. Then it is easy to verify that X'^c should be as in Fig. 2.2(b). Here, deleting vw is a good operation.

- Both sets have two vertices in them, see Fig. 2.3. Here the operation will be deleting uw or vz, from which we claim at least one is a good operation unless $G \cong Q_3$. Suppose that deleting both uw and vz destroy internal 4-edge-connectivity, thus both these edges contribute to some interesting cuts.

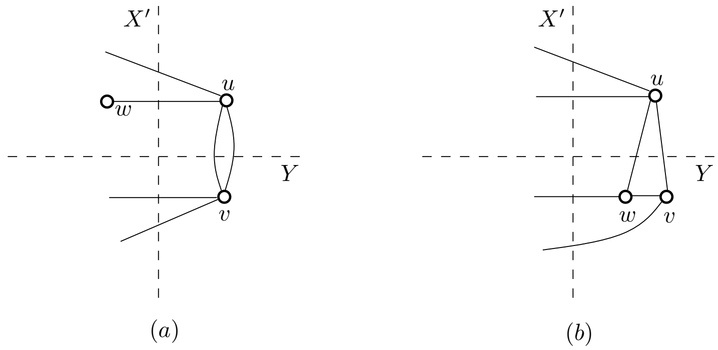


Figure 2.2: At least one of $X'^c \cap Y, X'^c \cap Y^c$ consists of only one vertex

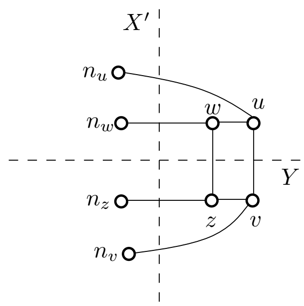


Figure 2.3: Both $X'^c \cap Y, X'^c \cap Y^c$ consist of two vertices

As before, it can be argued that the cuts look like as in Fig. 2.4 with respect to each other. Now, ignoring $\{u, v, w, z\}$ in Figures 2.3, and 2.4, we can see that there exists

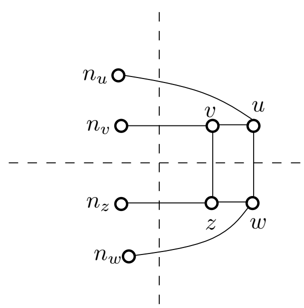


Figure 2.4: Both uw and vz are in interesting edge-cuts

a 2-edge cut separating $\{n_u, n_w\}$ from $\{n_v, n_z\}$, and another one separating $\{n_u, n_v\}$ from $\{n_w, n_z\}$, implying that n_u, n_v, n_w, n_z form a square, thus $G \cong Q_3$. It now only remains to notice that K_4, K_2^3 are the only internally 4-edge-connected graph that Q_3 immerses.

$\square$

Our next task is to deal with splits in G that preserve an H immersion, which will be done in Lemma 2.21. The following statement, which holds for a broader family of graphs than the ones we work with, features in the proof of Lemma 2.21.

Lemma 2.20. *Suppose H is a 3-edge-connected graph, and Y is a minimal subset of $V(H)$ such that $\delta(Y)$ is a nontrivial 3-edge-cut in H. Then for every edge e in $H[Y]$, $H \setminus e$ is internally 3-edge-connected.*

Proof. For a contradiction, suppose an edge $e = yz$ in $H[Y]$ contributes to some nontrivial 3-edge-cut $\delta(Z)$, where $z \in Z$. We will look into how Y, Z look like with respect to one another. Note both $Y \cap Z$ and $Y \cap Z^c$ are nonempty, as $z \in Y \cap Z, y \in Y \cap Z^c$. Also, both $Y^c \cap Z$ and $Y^c \cap Z^c$ are nonempty. It is because, if, say $Y^c \cap Z = \emptyset$, then $\delta(Y \cap Z)$ would be a nontrivial 3-edge-cut, which contradicts the choice of Y, as $Y \cap Z \subsetneq Y$.

Now, since H is 3-edge-connected, we have $d(Y \cap Z), d(Y^c \cap Z^c) \geq 3$. It now follows from $d(Y \cap Z) + d(Y^c \cap Z^c) + 2e(Y^c \cap Z, Y \cap Z^c) = d(Y) + d(Z)$ that $d(Y \cap Z) = d(Y^c \cap Z^c) = 3$ and $e(Y^c \cap Z, Y \cap Z^c) = 0$. Similarly, we obtain $d(Y \cap Z^c) = d(Y^c \cap Z) = 3$ and $e(Y \cap Z, Y^c \cap Z^c) = 0$. Now, since $d(Y) = 3 = e(Y \cap Z, Y^c \cap Z) + e(Y \cap Z^c, Y^c \cap Z^c)$, we have, say, $e(Y \cap Z, Y^c \cap Z) \leq 1$. Similarly, it follows from $d(Z) = 3$ that we have, say, $e(Y \cap Z, Y \cap Z^c) \leq 1$. Hence, $d(Y \cap Z) \leq 2$, a contradiction. □

Lemma 2.21. *Suppose G, H are as in Theorem 2.17, and there is a split at a vertex v preserving an H immersion. Then if $(G, H) \ncong (Q_3, K_4), (Q_3, K_2^3)$, a good operation exists.*

Proof. Let uvw be the 2-edge-path that is to be split. Note if $d(v) = 3$, then deleting the edge incident to v other than vu, vw preserves the H immersion. Hence, by Lemma 2.19 we are done. Also, observe that if a split is done at a vertex of degree at least four, the resulting graph is 3-edge-connected. Therefore, we only need to look into the case where splitting off uvw destroys internal 4-edge-connectivity. So, it must be the case that uv, vw contribute to some 4- or 5-edge-cut $\delta(X) = \{uv, wv, x_1y_1, x_2y_2(, x_3y_3) : u, w, x_i \in X\}$, where $|X|, |X^c| \geq 2$. We now split the analysis into cases depending on $d(X)$.

Claim. If $d(X) = 4$, then a good operation exists.

Proof of Claim. Since H is 3-edge-connected, all terminals of H lie on one side of the cut. Also, since G is 3-edge-connected, each side of the cut contains an edge completely lying in it, i.e. $E(G[X]), E(G[X^c]) \neq \emptyset$.

First, suppose all terminals of H are in X. Observe that if we can modify X^c in a way that it preserves the connectivity of y_1, y_2 in $G[X^c]$, an H immersion is present in the resulting graph. We propose to delete an edge $e \in E(G[X^c])$, and claim that deleting e preserves the H immersion. It suffices to show e is not a cut-edge in $G[X^c]$ separating y_1, y_2. For a contradiction, suppose $e = \delta(Y)$ separates y_1, y_2 in $G[X^c]$, where $y_1 \in Y$. We may also assume, without loss of generality, that $v \in Y$. Then $\delta_G(Y^c)$ would be a 2-edge-cut in G, a contradiction. Therefore, we can delete e using Lemma 2.19.

Next, suppose all terminals of H are in X^c. Similarly to the previous case, if we modify X in a way that preserves the connectivity of x_1, x_2 in $G[X]$, an H immersion is sure to exist in the resulting graph. Again, we propose to delete an edge $e \in E(G[X])$, and claim that deleting e preserves the H immersion. It suffices to show e is not a cut-edge in $G[X]$ separating x_1, x_2. For a contradiction, suppose $e = \delta(Y)$ separates x_1, x_2 in $G[X]$, where $x_1 \in Y$. Note that the 3-edge-connectivity of G implies that $\delta(Y)$ separates u, w as well. We may assume without loss of generality that $u \in Y, w \in Y^c$. Then $d_G(Y) = d_G(Y^c) = 3$, thus it follows from internal 4-edge-connectivity of G that $|Y| = |Y^c| = 1$ and $Y = \{u = x_1\}, Y^c = \{w = x_2\}$. Therefore, X consists of two vertices u, w of degree three. Thus deleting uw preserves the H immersion. This proves the claim. □

So now we may assume that $d(X) = 5$, and we will show that a good operation exists. By the internal edge-connectivity of H, all terminals of H but possibly one, lie on one side of the cut. First, suppose that most terminals of H are in X. Observe that if $G[X^c]$ is modified in a way that preserves the presence of three edge-disjoint paths form a vertex in it to X not using uv, vw, the presence of H immersion is guaranteed. Next, suppose that most terminals of H are in X^c. In this case, if we manage to modify $G[X]$ in a way that preserves the presence of three edge-disjoint paths form a vertex in X to X^c avoiding uv and vw, the presence of H immersion is guaranteed. We claim such modifications are possible.

Let G' be the graph resulting from splitting off uvw, followed by suppressing v in case $d_G(v) = 4$. We denote the edge created by splitting uvw by e'. Note, by the claim above, we may assume G' is 3-edge-connected.

Take an arbitrary nontrivial 3-edge-cut $\delta_{G'}(Y)$ in G'. Observe that $\delta_G(Y)$ must have been a 5-edge-cut in G, which both edges of the split 2-path uvw contributed to. So, in particular, e' lies either completely in Y or in Y^c. Also, there must be an edge other than e' in $G'[Y]$. It is because 3-edge-connectivity of G implies $6 \leq \sum_{v \in Y} d_G(v) = d_G(Y) + 2e_G(G[Y]) = 5 + 2e_G(G[Y])$. Thus $e_G(G[Y]) > 0$, and so there is an edge $\neq e'$ in $G'[Y]$.

Now, let Z denote the side of $\delta(X)$ containing most terminals of H (so $Z = X$ or X^c). We will show that there is an edge lying in Z^c which we could delete, while preserving an H immersion. Since $\delta_{G'}(Z)$ is a nontrivial 3-edge-cut, we may choose a minimal $Y \subseteq Z^c$ such that $\delta_{G'}(Y)$ is a nontrivial 3-edge-cut.

It is argued above that there exists an edge $e \neq e'$ in $G'[Y]$. We claim deletion of e preserves the H immersion. It is because it follows from Lemma 2.20 that $G' \setminus e$ is internally 3-edge-connected. Now, 3-edge-connectivity of G' and $d_{G'}(Y) = 3$ imply that $G'[Y] \setminus e$ has a vertex of degree at least three. Therefore, there exist in $G' \setminus e$ three edge-disjoint paths from such a vertex to Z. Observe that since these set of paths cover $\delta_{G'}(Z)$, deleting e from

G preserves the presence of an H immersion. We now can use Lemma 2.19 to find a good operation. This completes the proof of the lemma. $\square$

The proof of Theorem 2.17 is now immediate.

Proof of Theorem 2.17. Apply Lemmas 2.19, and 2.21. One of these must apply since H is immersed in G. $\square$

Having established Theorem 2.17, we will now take advantage of it to establish a chain theorem for the family of 3-edge-connected, and internally 4-edge-connected graphs.

Corollary 2.22. *Let G be* 3*-edge-connected, internally* 4*-edge-connected, and* $|V(G)| \geq 2$*. If $G \not\cong Q_3, K_2^3$, then one of the operations in the statement of Theorem* 2.17 *may be applied to G, where the resulting graph is* 3*-edge-connected, internally* 4*-edge-connected.*

Proof. Since G is 3-edge-connected, $G \succ K_2^3$. Now, apply Theorem 2.17 for $H = K_2^3$. $\square$

2.3 Implication for immersing Kuratowski graphs

In this section, we work towards establishing the following corollary of Theorem 2.17 regarding immersion of Kuratowski graphs:

Corollary 2.23. *Suppose G is* 3*-edge-connected, and internally* 4*-edge-connected, where $G \succ K_5$. If $|V(G)| \geq 6$ then $G \succ K_{3,3}$, or $G \cong K_{2,2,2}$.*

The idea is to examine 3-edge-connected, internally 4-edge-connected graphs "one step bigger", or perhaps "a few steps bigger", than K_5, and see if they immerse $K_{3,3}$. One subtlety here is that we are working with multigraphs, thus even graphs "much bigger than" K_5 may happen to be on five vertices, and thus not do possess $K_{3,3}$ as immersion. Therefore, we need some tool to limit the graphs necessary to examine. Given that K_5 itself is 4-edge-connected, Lemma 2.25 serves very well in doing so. First, however, we need the following definition.

Definition 2.24. We define a *good sequence* from G to H to be a sequence of graphs

$$G = G_l, G_{l-1}, \ldots, G_2, G_1, G_0 \cong H$$

in which each G_i, $i = 0, \ldots, l-1$, is 3-edge-connected, and internally 4-edge-connected, and G_i is obtained from G_{i+1} by either

- deleting an edge, or
- splitting at a vertex of degree at least 4,

each followed by iteratively suppressing any vertices of degree two, and deleting loops and isolated vertices. Let O_i be the operation taking G_i to G_{i-1}, for $i = 1, \ldots, l$.

Lemma 2.25. *Let G be a 3-edge-connected, internally 4-edge-connected graph, and let H be a 4-edge-connected graph, with $|V(G)| > |V(H)|$. Suppose there is a good sequence from G to H, and choose a good sequence from G to H*

$$G = G_l, G_{l-1}, \ldots, G_2, G_1, G_0 \cong H$$

such that $\min\{k : |V(G_k)| > |V(H)|\}$ is as small as possible. Then either

(a) G_1 is as in Fig. 2.5(a) and the last operation, O_1, is a complete split of u which does not result in creating any loops.

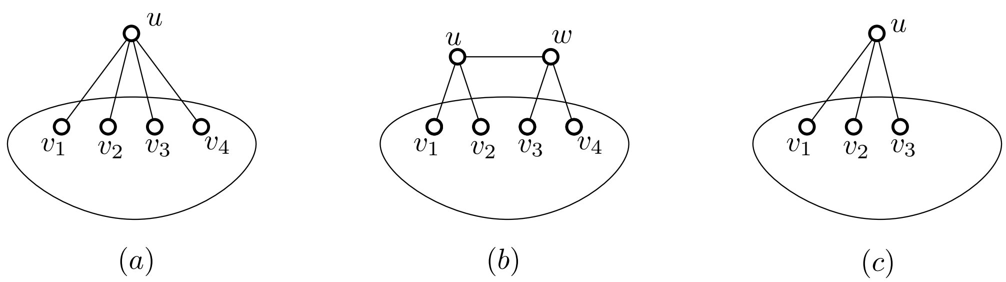


Figure 2.5: The last graphs in the sequence

(b) G_1 is as in Fig. 2.5(b), with $v_1 \neq v_2$, $v_3 \neq v_4$, and O_1 is deleting uw.

(c) G_1 is as in Fig. 2.5(c) and O_1 is deleting an edge incident with u.

(d) G_2 is as in Fig. 2.5(c) and O_2 is deleting uv_1 (and thus forming an edge v_2v_3), and O_1 is deleting v_2v_3, so $G_2 \setminus u \cong H$.

Proof. Let G_{t+1} be the graph in the sequence which attains the $\min\{k : |V(G_k)| > |V(H)|\}$, thus $|V(G_t)| = |V(H)|$. Without loss of generality we may assume $V(G_t) = V(H) = \{v_1, v_2, \ldots, v_{|H|}\}$. First, consider the case where O_{t+1} is a split. Since this split reduces the number of vertices, it must be a split at a vertex u of degree 4, see Fig. 2.5(a). Let v_1v_2, v_3v_4 be the edges resulting from splitting u. We claim that $G_t \cong H$ and $v_1 \neq v_2$ and $v_3 \neq v_4$. For a contradiction, suppose $G_t \ncong H$, and consider O_t.

- If O_t is splitting a 2-edge-path where both edges are present in G_{t+1}, or is deleting an edge present in G_{t+1}, then let $O'_{t+1} = O_t$, and $O'_t = O_{t+1}$.

- If O_t is splitting a $v_1v_2v_i$ path, $i = 1, 2, \ldots, |V(H)|$, let O'_{t+1} be splitting uv_2v_i, and let O'_t be splitting v_1uv_i (and thus splitting v_3uv_4).

- If O_t is splitting a 2-edge-path whose both edges are created by O_{t+1}, say $v_2 = v_3$, and O_t is splitting $v_1v_2v_4$, then let both O'_{t+1} and O'_t be deleting one copy of uv_2 edge (so O'_t will be followed by suppressing u).

- If either $v_1 = v_2$ or O_t is deleting one of the edges created by O_{t+1}, say v_1v_2, then let O'_{t+1} be deleting uv_1, and let O'_t be deleting uv_2.

Now, let G'_t (G'_{t-1}) be the graph obtained from G_{t+1} (G'_t) by applying O'_{t+1} (O'_t). Observe that in all cases it follows from 4-edge-connectivity of H that G'_t is indeed internally 4-edge-connected. Now, note that $|V(G'_t)| > |V(H)|$ and $G'_{t-1} \cong G_{t-1}$. So, by replacing G_t, G_{t-1} with G'_t, G'_{t-1} we obtain another good sequence from G to H which contradicts our choice of the good sequence $G_l, \ldots, G_{t+1}, G_t, G_{t-1}, \ldots, G_0$. This proves that $t = 0$, and (a) occurs.

Next, consider the case where O_{t+1} is deleting an edge. Since this deletion reduces the number of vertices, at least one of its endpoints is of degree 3. If both endpoints have degree 3 (see Fig. 2.5(b)), then a similar argument as above shows that $t = 0$, and thus (b) happens. In the only remaining case, only one endpoint of the deleted edge has degree 3 in G_{t+1}, let u be the endpoint of degree three and $\delta(u) = \{uv_1, uv_2, uv_3\}$ (see Fig. 2.5(c)). We may assume O_{t+1} is deleting uv_1. As before, O_t is neither splitting a 2-edge-path where both edges are present in G_{t+1}, nor it is deleting an edge present in G_{t+1}. Also, O_t cannot be splitting $v_2v_3v_i$, $i = 1, \ldots, |V(H)|$. Otherwise let O'_{t+1} be splitting uv_3v_i, and let O'_t be deleting uv_1(and thus followed by suppressing u). Note that $|V(G'_t)| > |V(H)|$ and $G'_{t-1} \cong G_{t-1}$, and it follows from 4-edge-connectivity of H that G'_t is internally 4-edge-connected. As before, this results in another good sequence from G to H which contradicts the choice of the good sequence $G_l, \ldots, G_t, G_{t-1}, \ldots, H$. So, either $t = 0$, and (c) occurs, or O_t is deleting v_2v_3 in which case $t = 1$, i.e. (d) occurs. □

Now, we use this lemma to establish a result on K_5 immersions discussed earlier and restated here.

Corollary 2.26. *Suppose G is 3-edge-connected, and internally 4-edge-connected, where $G \succ K_5$. If $|V(G)| \geq 6$, then $G \succ K_{3,3}$, or $G \cong$ Octahedron, where Octahedron is the graph in Fig.* 2.6*.*

Proof. Suppose $G \succ K_5$, and $|V(G)| > 5$. By Theorem 2.17, a good sequence from G to K_5 exists. Thus, we can choose a good sequence

$$G = G_l, G_{l-1}, \ldots, G_2, G_1, G_0 \cong K_5$$

such that $\min\{k : |V(G_k)| > 5\}$ is as small as possible, and apply the previous lemma. It can be easily verified that if cases (b), or (c) of the previous lemma occur, then $G_1 \succ K_{3,3}$, and if case (d) happens, $G_2 \succ K_{3,3}$, thus $G \succ K_{3,3}$.

So, suppose case (a) of the previous lemma occurs. Again, it can easily be verified that if the two edges created by o_1 share an endpoint, then $G_1 \succ K_{3,3}$, thus $G \succ K_{3,3}$. Otherwise, $K_{3,3}$ is not immersed in G_1, as G_1 would be the Octahedron, which, being planar, doesn't have $K_{3,3}$ as a subgraph. On the other hand, it has six vertices, all of degree 4, so an immersion of $K_{3,3}$ cannot be found doing splits either.

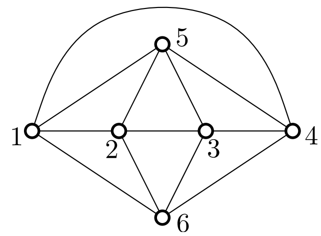


Figure 2.6: Octahedron

Therefore, if $G \cong$ Octahedron, $G \not\succeq K_{3,3}$. However, if G properly immerses Octahedron, then it immerses $K_{3,3}$ as well. To see that, note that the 6-vertex graphs from which Octahedron is obtained after deletion of an edge or splitting a 2-edge path, all immerse $K_{3,3}$. On the other hand, if $|V(G)| > 6$, we may again use Lemma 2.25 for $H =$ Octahedron, since Octahedron itself is 4-edge-connected.

To reduce the number of graphs we examine, it now helps to notice that we only need to consider the case where a 4-vertex 7 gets split to create edges $\{23, 15\}$, or $\{23, 14\}$. It is because in all other cases, the graph obtained by splitting 2-paths 163, 264 would be one of the graphs we already looked at, all of which immerse $K_{3,3}$.

If vertex 7 is split to create $\{23, 15\}$, an immersion of $K_{3,3}$ may be found after splitting 2-path 173. Also, if vertex 7 is split to create $\{23, 14\}$, then $K_{3,3}$ lies as a subgraph in G. $\square$

Chapter 3

Immersion of D_m

This chapter centres around certain rooted graphs on four vertices which we call D_m, for $m \geq 2$. The main result, appearing as Theorem 3.4, characterizes the structure of graphs which do not immerse D_m, for any $m \geq 2$. Let us first define a rooted graph, and the corresponding notion of immersion for rooted graphs.

Definition 3.1. A *rooted graph* with k roots is a connected graph G together with an ordered tuple $(x_1, \ldots, x_k)$ of distinct vertices. We call x_i the *i-th root* of G. If G and $(x_1, \ldots, x_k)$, and H and $(y_1, \ldots, y_k)$ are rooted graphs, we say G contains H as a *rooted immersion* if there is a (possibly empty) sequence of splits and deletions which transforms G into a graph isomorphic to H, where this isomorphism sends x_i to y_i, for $i = 1, \ldots, k$. We may also write $(G; x_1, \ldots, x_k) \succeq_r (H; y_1, \ldots, y_k)$ to denote that G has a rooted immersion of H. Throughout this document, when depicting rooted graphs, we show the roots of a graph by solid vertices. For the sake of simplicity, if $k = 2$ and there is an automorphism of H which sends y_1 to y_2, we simply refer to H as a rooted graph with roots y_1, y_2.

The family of the rooted graphs concerned in this chapter is introduced below.

Definition 3.2. For $m \geq 2$, let D_m denote the graph with roots x_0, x_1 where $e(x_0, x_1) = m - 2$, and $D_m \setminus E(x_0, x_1)$ is isomorphic to the rooted graph below.

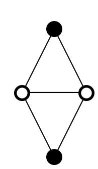


Figure 3.1: Graph D_2

Observe that D_3 is isomorphic with K_4 with two roots. Accordingly, our result on excluding a D_m immersion can be used to characterize unrooted graphs without K_4 immersions. This is done in Section 3.2.

Since in this chapter we will be dealing with graphs with two roots, we often need to distinguish between the edge-cuts which separate the two roots of the graphs and the ones which have both roots on the same side. This calls for a more refined notion of edge-connectivity, introduced below.

Definition 3.3. Let G be a connected graph with roots x_0, x_1.

- The minimum size of an (internal) edge-cut that separates x_0, x_1 will be denoted by $\lambda_s(G)$ ($\lambda_s^i(G)$).
- The minimum size of an (internal) edge-cut that does not separate x_0, x_1 will be denoted by $\lambda_n(G)$ ($\lambda_n^i(G)$).

In preparation for the main result of this chapter, we introduce two families of graphs which do not immerse D_m. Let G be a rooted graph with two roots x_0, x_1. If C is a connected component of $G \setminus \{x_0, x_1\}$, we call $G[C \cup \{x_0, x_1\}] \setminus E(x_0, x_1)$ a *lobe* of G. We also say

Type A_m. G has *type* A_m if it has a segmentation of width m relative to some (X_0, X_1) with $|X_i| \leq 2$ and $x_i \in X_i$, for $i = 0, 1$.

Type B_m. G has *type* B_m if it satisfies the following:

- Every lobe L of G is obtained from an $x_0 - x_1$-path by adding $n_L - 1$ copies of each edge for some $n_L \geq 2$ (so each parallel class has size n_L).
- If $|V(L)| \geq 4$, we have $n_L = 2$, for every lobe L.
- $e_G(x_0, x_1) + \sum_L n_L = m + 1$, where the sum is taken over all lobes L of G.

Note that if G has type A_m, then it has a $(2,2)$-segmentation of width m. We are now all set to state our main theorem concerning the structure of graphs excluding D_m immersion, for any $m \geq 2$.

Theorem 3.4. *Let $m \geq 2$, and let G be a rooted graph with roots x_0, x_1, where $|V(G)| \geq 4$. Assume further that*

- *$\lambda_n(G) \geq 3$,*
- *$\lambda_n^i(G) \geq 4$, and*
- *$\lambda_s(G) \geq m$.*

Then $G \not\succeq_r D_m$ if and only if G has type A_m or type B_m.

This result is indeed a very useful one, as it gives us the opportunity to ask for an immersion of D_m in a graph, while two of the terminals of D_m have been specified in advance. Accordingly, this result not only interesting on its own, but also can serve as a helpful tool in problems of finding the structure of graphs excluding bigger graphs as immersion. In Chapters 4, 5 we will see applications of this theorem when finding an immersion of rooted W_4, as well as prism.

The rest of this chapter is organized as follows: Section 3.1 is devoted to the proof of Theorem 3.4. In Section 3.2, we give a precise structural theorem for the family of 3-edge-connected (rooted) graphs which do not immerse (rooted) K_4 (with one or two roots).

3.1 Proof of Theorem 3.4

In this section, we prove the main result of this chapter. First, we will see the proof of the easier direction, 'if' direction.

3.1.1 'if' direction

The following simple observation often proves quite useful when arguing about the existence of an immersion of a graph, and we will call upon it in the following chapters as well.

Observation 3.5. *Let G, H be two graphs and G has an immersion of H. Suppose $v \in V(G)$ is a terminal of H, corresponding to $u \in V(H)$. If $d_G(v)$ and $d_H(u)$ have different parity then an edge (of G) incident with v can be deleted while preserving an immersion of H.*

Now, let us see why type A_m, type B_m graphs do not immerse D_m.

Proof of Theorem 3.4*,'if' direction.* Suppose G has type A_m, and suppose for a contradiction that $G \succeq_r D_m$. Let $Y_0 \subset Y_1 \subset \ldots \subset Y_t$ be a $(2,2)$-segmentation of width m of G. We may assume without loss of generality that $x_0 \in Y_0, x_1 \in V(G) \setminus Y_t$. Then since $d(Y_0) = m < m+1$, the only terminal of D_m in Y_0 is x_0. Thus, Y_1 has at most two terminals of D_m, however, it follows from $d(Y_1) = m$ that the only terminal of D_m in Y_1 is x_0. By repeating this argument, we can see that the only terminal of D_m in Y_t is x_0, and since $|V(G) \setminus Y_t| \leq 2$, D_m is not immersed in G—a contradiction.

Now suppose G is type B_m. If $m = 2$, observe that (since $|V(G)| \geq 4$) we have $e_G(x_0, x_1) = 1$ and $G \setminus E(x_0, x_1)$ is a doubled $x_0 - x_1$-path. Then Observation 3.5 implies that $G \succeq_r D_2$ if and only if a graph G' obtained from G by deleting an edge incident with x_0 immerses D_2. Note the only such graph G' for which $d_{G'}(\{x_0, x_1\}) \geq 4$ is $G' = G \setminus x_0x_1$. However, since G' has type A_2, we have $G' \not\succeq_r D_2$, and thus $G \not\succeq_r D_2$.

Now consider $m \geq 3$. If $e(x_0, x_1) > 0$, let $G' = G \setminus x_0x_1$. Observe that $G \succeq_r D_m$ if and only if $G' \succeq_r D_{m-1}$. However, since G' is type B_{m-1}, by induction we have $G' \not\succeq_r D_{m-1}$ and thus $G \not\succeq_r D_m$. On the other hand, if $e(x_0, x_1) = 0$ then G immerses D_m if and only if the graph G' resulting by splitting off an x_0x_1-path in a lobe L of G immerses D_m. However, then G' has type B_m with an edge between x_0, x_1, so G' does not immerse D_m. $\square$

3.1.2 'only if' direction

As the reader may expect, the proof of the reverse direction is more involved. Before giving the proof we begin with a couple observations which follow immediately from degree considerations.

Observation 3.6. *Suppose that G is type A_m relative to X_0, X_1 as in the above definition. If $d(x_i) = m$ we may assume $X_i = \{x_i\}$, for $i = 0, 1$.*

Observation 3.7. *Let G be a graph with a segmentation of width k relative to some (U, W). Observe that if $x \in V(G) \setminus (U \cup W)$, then $d(x)$ is even.*

Proof. Let $U = X_1 \subset X_2 \subset \ldots \subset X_t = V(G) \setminus W$ be the segmentation of G. Since $x \in V(G) \setminus (U \cup W)$, there exists $i, 0 \leq i \leq t-1$ such that $X_{i+1} = X_i \cup \{x\}$. Then since $d(X_i) = d(X_{i+1}) = k$, we have $e(x, X_i) = e(x, V \setminus X_{i+1})$, and thus $d(x) = 2e(x, X_i)$. □

We generalize the notation used in Chapter 1 to rooted graphs, and to the cases where more than one subset of vertices are identified:

Notation. Let G be a graph with a (possibly empty) set R of root vertices. Let $X_1, \ldots, X_k$ be disjoint subsets of $V(G)$. The graph obtained from G by identifying each X_i, for $i = 1, \ldots, k$, to a new vertex x_i^* will be denoted by $G.\{X_1, \ldots, X_k\}$. (Since we are assuming graphs have no loops, any loops formed by this operation are removed.) If we need the resulting graph to be rooted, we declare $(R \setminus \bigcup_i (X_i)) \cup \{x_1^*, \ldots, x_k^*\}$ to be the set of root vertices of $G.\{X_1, \ldots, X_k\}$.

Lemma 3.8. *Theorem* 3.4 *holds under the added assumption $|V(G)| = 4$.*

Proof. Assume that $V(G) = \{x_0, x_1, y_0, y_1\}$ with terminals x_0 and x_1. We may assume that G satisfies $\lambda_s^i(G) \geq m+1$, as otherwise G is type A_m relative to $\{x_0, y_i\}$, $\{x_1, y_{1-i}\}$ for some $i = 0, 1$. Let G^* be the graph obtained from the simple graph underlying G by deleting the edge x_0x_1 if it is present. First suppose that $|E(G^*)| = 5$. In this case G has a subgraph H isomorphic to D_2 in which $d_H(x_i) = 2$ for $i = 0, 1$. It follows from $\lambda_s(G) \geq m$ and $\lambda_s^i(G) \geq m+1$ that the graph $G' = G \setminus E(H)$ satisfies $\lambda_s(G') \geq m-2$, and thus G has an immersion of D_m.

Next suppose that $d_{G^*}(y_0) = d_{G^*}(y_1) = 1$. In this case G has one of the first two graphs (from the left) in Figure 3.2 as a subgraph, so $D_m \prec G$. Next suppose that $d_{G^*}(y_0) = 1$ and $d_{G^*}(y_1) \geq 2$. If y_0 is adjacent to a root, say x_0, then G immerses the middle graph in Figure 3.2; if y_0 is adjacent to y_1, then G immerses the second graph from the right in Figure 3.2. Since both of these graphs immerse D_m we may assume $d_{G^*}(y_i) \geq 2$ for $i = 0, 1$. If $d_{G^*}(x_0) = 0$, then G must immerse the rightmost graph in Figure 3.2, so again we have $D_m \prec G$.

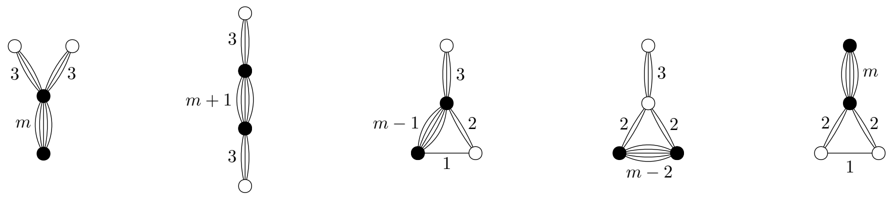


Figure 3.2: Immersions in G when $d_{G^*}(y_i) = 1$ or $d_{G^*}(x_i) = 0$

At this point, we have shown $d_{G^*}(y_i) \geq 2$ and $d_{G^*}(x_i) \geq 1$ for $i = 0, 1$ and there are just three possibilities for the graph G^* (up to interchanging the names of the roots x_0, x_1 and the names of the non-roots y_0 and y_1). They are the three graphs shown in the Figure 3.3 and will be handled in separate cases.

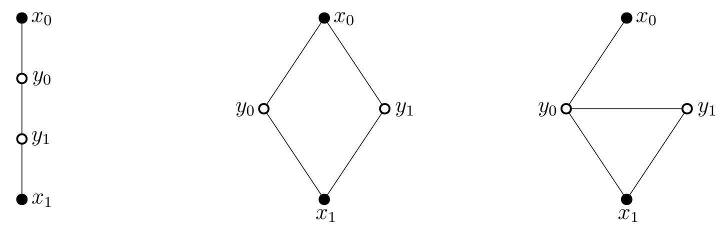


Figure 3.3: Possibilities for G^*

In all three of our cases, we shall classify the paths between x_0 and x_1 into a small number of types and these are indicated in Figure 3.4. For instance, for the rightmost graph we say that a path is type α if it has vertex sequence x_0, x_1, type β if it has vertex sequence x_0, y_0, x_1, and type γ if it has vertex sequence x_0, y_0, y_1, x_1. Now in all three cases, we choose a maximum cardinality packing of edge-disjoint paths from x_0 to x_1 say $P_1, P_2, \ldots, P_k$ and we let a (b, c) denote the number of these paths of type α (β, γ). Note that $\lambda_s(G) \geq m$ implies $k \geq m$. An edge $e \in E(G) \setminus \left(\bigcup_{i=1}^{k} E(P_i)\right)$ is called an *extra* edge. Note that there are no extra edges of type α since this would contradict the maximality of our packing.

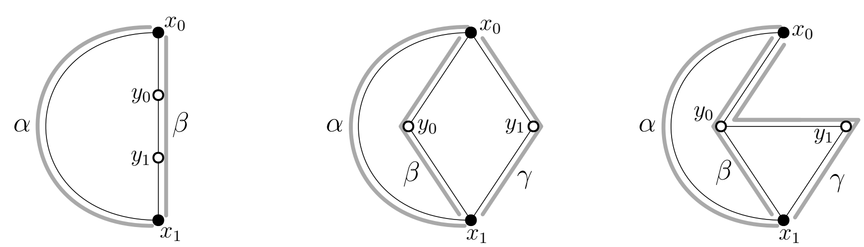


Figure 3.4: Types of Paths

Case 1: G^* is the leftmost graph in Figure 3.3

We note that $b \geq 1$ and split into subcases based on b. First suppose that $b = 1$. If $e(y_0, y_1) = 1$, then $d(y_0), d(y_1) \geq 3$ forces the existence of extra edges x_0y_0 and x_1y_1. Now $\lambda_s^i(G) \geq m + 1$ implies $a \geq m$ and we find $D_m \prec G$. If $e(y_0, y_1) \geq 2$, then the maximality of our packing implies $\min\{e(x_0, y_0), e(x_1, y_1)\} = 1$, but then $\lambda_n^i(G) \geq 4$ implies $\max\{e(x_0, y_0), e(x_1, y_1)\} \geq 3$. Since $a \geq m-1$ we again find that $D_m \prec G$. Next suppose that $b = 2$. If $a \geq m$, then we have $D_m \prec G$. If $a = m-1$, then G will be type B_m if there are no extra edges, and $D_m \prec G$ if there is an extra edge. If $a = m-2$, then $\lambda_s^i(G) \geq m+1$ implies that there is an extra edge y_0y_1 and we have $D_m \prec G$. Finally, we suppose $b \geq 3$. If $k = a + b \geq m + 1$, then $D_m \prec G$. Otherwise $a + b = m$ and $\lambda_s^i(G) \geq m + 1$ implies that there is an extra edge y_0y_1 and again we have $D_m \prec G$.

Case 2: G^* is the middle graph in Figure 3.3

First note that $b, c \geq 1$. If $k = m$, then $\lambda_s^i(G) \geq m + 1$ implies that there is an extra edge of the form x_0y_0 or of the form x_1y_1, and similarly there is an extra edge of the form x_0y_1 or of the form x_1y_0. We cannot have both x_0y_i and y_ix_1 as extra edges, since this would contradict the maximality of our packing, but then we have $D_m \prec G$. Accordingly, we may now assume $k \geq m+1$. Now we split into subcases depending on the values of b, c. If $b = c = 1$, then $d(y_i) \geq 3$ implies the existence of an extra edge incident with y_i for $i = 1, 2$ and this gives $D_m \prec G$. If $b = 1$ and $c \geq 2$, then $d(y_0) \geq 3$ implies the existence of an extra edge of the form x_0y_0 or x_1y_0 and in either case we have $D_m \prec G$. A similar argument handles the case $b \geq 2$ and $c = 1$. Finally, we consider the case $b, c \geq 2$. If $k \geq m + 2$, then $D_m \prec G$. If $m = k + 1$, then G has type B_m if there are no extra edges, and $D_m \prec G$ if there is an extra edge.

Case 3: G^* is the rightmost graph in Figure 3.3

First suppose that $e(x_0, y_0) = 1$ and note that this implies $a = e(x_0, x_1) \geq m - 1$. If $e(x_1, y_1) \geq 2$, then $D_m \prec G$. Otherwise it follows from $\lambda_n(G) \geq 3$ that $e(x_1, y_0), e(y_0, y_2) \geq 2$ and again we get $G \succ D_m$. Therefore, $e(x_0, y_0) \geq 2$ and we may assume (without loss) that in our maximum packing we have $b, c \geq 1$. First suppose that $c = 1$. If there is an extra edge y_0y_1, then (using $b \geq 1$) we find $D_m \prec G$, otherwise $d(y_1) \geq 2$ implies an extra edge of the form y_1x_1. This extra edge gives an immersion of D_m unless $k = m$ so we may now assume this. However, if $k = m$, then it follows from $d(\{y_1, x_1\}) \geq m + 1$ that there exists an extra edge which forces an immersion of D_m. So we may now assume $c \geq 2$ and then we are immediately finished if $k \geq m+1$ since a path of type γ together with a x_0x_1 path may be traded for an edge between y_0 and x_1. In the only remaining case $k = m$, and it follows from $d(\{y_1, x_1\}) \geq m + 1$ that there is an extra x_1y_0 edge which forces immersion of D_m. This completes the proof. □

With this lemma in hand, we now prove Theorem 3.4.

Proof of Theorem 3.4, *'only if' direction.* Towards a contradiction, suppose $G = (V, E)$ is a counterexample to the theorem for which $|V| + |E|$ is minimum. We will prove some properties of G which finally shows that G does not exist.

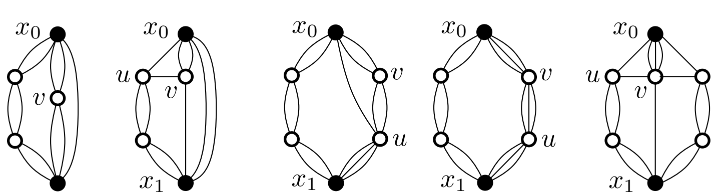
(1) $\lambda_s^i(G) \geq m+1$.

Suppose (for a contradiction) that there exists an edge-cut $\delta(U)$ of size at most m in G which separates $x_0 \in U$ from x_1, where $|U|, |V \setminus U| \geq 2$. Since $\lambda_s(G) \geq m$, we have $d(U) = m$. Define the graphs $G_0 = G.(V \setminus U), G_1 = G.U$, and let y_0, y_1 be the new vertices resulting by identifying $V \setminus U, U$ to a vertex, respectively (so $d_{G_i}(y_i) = m$). For $i = 0, 1$, let x_i, y_i be root vertices of G_i. By the edge-connectivity of G, $G \succ_r G_i$, and thus $G_i \not\succeq_r D_m$. If $|V(G_i)| \geq 4$, then (by the minimality of our counterexample G) the theorem implies that G_i is type A_m or type B_m. However, from $d_{G_i}(y_i) = m$ we deduce that G_i is type A_m. So, there exists $X_i \subseteq V(G_i)$ with $x_i \in X_i$ and $|X_i| \leq 2$ so that the graph G_i has a $(2,1)$- segmentation of width m. On the other hand, if $|V(G_i)| \leq 3$, for $X_i = V(G_i) \setminus y_i$ we have $x_i \in X_i$, $|X_i| \leq 2$, and the graph G_i has a has a $(2,1)$-segmentation (of length 1) of width m. It now follows that the original graph G has type A_m relative to X_0 and X_1, and this contradiction establishes (1).

(2) For every vertex $v \in V \setminus \{x_0, x_1\}$ we have $d(v) > 2e(v, x_i)$, for $i = 0, 1$.

Suppose (for a contradiction) that $v \in V \setminus \{x_0, x_1\}$ exists with, say, $d(v) \leq 2e(v, x_0)$. Let G' be the graph obtained from G by identifying $\{x_0, v\}$ to a new root vertex, and note that Lemma 3.8 implies that $|V(G')| \geq 4$. On the other hand, it follows from $d(v) \leq 2e(v, x_0)$ that $G \succ G'$, and thus $G' \not\succeq D_m$. Now, G being a minimum counterexample implies that the theorem holds for G', and so it is either type A_m or type B_m. It follows from (1) that G' is not type A_m. It is now straightforward to check that G satisfies the theorem.

In the picture below, we have considered a few different graphs G for which $G.\{x_0, v\}$ has type B_4. Note that the leftmost graph has type B_4, and in the other graphs there is a rooted immersion of D_4 on $\{x_0, x_1, u, v\}$. It is easy to verify that other cases also result in a graph which either has type B_4 or immerses D_4.



(3) There does not exist $i = 0, 1$ and $U \subset V \setminus \{x_0, x_1\}$ for which $|U| \geq 2$ and $d(U) = 4$, and $e(x_i, U) \geq 2$.

Towards a contradiction, suppose such U exists with, say, $e(x_0, U) \geq 2$. Choose distinct $e, e' \in E(x_0, U)$, and let $\{f, f'\} = \delta(U) \setminus \{e, e'\}$. Now let G' be the graph obtained from G by subdividing f, f' with a new vertex, and then identifying the two new vertices of degree two to a new vertex y.

Consider $H = G'[U \cup \{x_0, y\}]$, rooted at x_0, y. By construction, $d_H(x_0) = d_H(y) = 2$, and we have $|E(H)| < |E|$. If $\lambda_s(H) < 2$, it follows from $\lambda^i(G) \geq 4$ that $|U| = 2$, and that both vertices in U have degree three, and thus there exist $u \in U$ with $e_G(u, x_0) = 2$. This, however, contradicts (2), and thus $\lambda_s(H) \geq 2$. Moreover, $\lambda_n(H) \geq \lambda_n(G) \geq 3$, and $\lambda_n^i(H) \geq \lambda_n^i(G) \geq 4$. So, we can apply the theorem to H to conclude that either $H \succeq_r D_2$ or H has type A_2 or type B_2. Since the root vertices of H have degree two, H must be type A_2, and (by Observation 3.6) it is a doubled path. This, however, implies that there exists $u \in U$ with $d(u) = 4$ and $e_G(u, x_0)$, which contradicts (2). Thus, $H \succeq_r D_2$.

Next, let K be the graph obtained from $(G.U) \setminus \{e, e'\}$ by adding a new vertex z which has two edges to U and $m-2$ edges to x_0. It follows from $d(U) = 4$ and $\lambda(G) \geq m$ that there are m edge-disjoint $z - x_1$ paths in K. This, together with $H \succeq_r D_2$ implies that $G \succeq_r D_m$—a contradiction.

Before proceeding, let us introduce some helpful notation. We call an edge $e \in E$ *safe* if the graph G' obtained from $G \setminus e$, followed by suppressing any resulting degree two vertices, satisfies $|V(G')| \geq 4$, and $d_{G'}(x_0), d_{G'}(x_1) \geq m$. Observe that if e is safe, (1) implies that $\lambda_s(G') \geq m$. Moreover, it follows from $\lambda_n^i(G) \geq 4$ that $\lambda_n(G') \geq 3$. Below, we will confirm that we also have $\lambda_n^i(G') \geq 4$, which then puts us in a position to apply the theorem to it.

(4) If $e \in E$ is safe, and G' is the graph obtained from $G \setminus e$ by suppressing degree two vertcies, then $\lambda_n^i(G') \geq 4$.

Suppose (for a contradiction) that $\lambda_n^i(G') = 3$. Let $U_1, \ldots, U_k$ be the maximal subsets of $V(G') \setminus \{x_0, x_1\}$ for which $|U_i| \geq 2$ and $d_{G'}(U_i) = 3$. Note that for $1 \leq i < j \leq k$ the sets U_i and U_j are distinct. It is because otherwise $d(U_i \cap U_j) + d(U_i \cup U_j) \leq d(U_i) + d(U_j) = 6$, which together with $\lambda_n(G') \geq 3$ would imply that $d(U_i \cup U_j) = 3$—contradicting maximality of U_i, U_j.

Now, let G'' be the graph obtained from G' by identification of each set U_i to a new vertex u_i. Note that (3) implies that $|V(G'')| \geq 4$. Moreover, we have $\lambda_s(G'') \geq m, \lambda_n(G'') \geq 3$ and $\lambda_n^i(G'') \geq 4$. On the other hand, since $\lambda_n(G') \geq 3$ we have $G' \succ G''$, and thus $G'' \not\succeq_r D_m$ (else $G \succ_r G' \succeq_r D_m$). Since $|V(G'')| < |V|$, we can apply the theorem to G'' to conclude that it is either type A_m or type B_m. Now since every non-root vertex in a graph of type B_m has even degree, it must be that G'' has type A_m. Now, Observation 3.7 implies that (by possibly relabeling x_0, x_1) the graph G'' has type A_m relative to (X_0, X_1), where $X_0 = \{x_0, u_1\}$. Furthermore, it follows from $d_{G''}(x_0) \geq m, d_{G''}(u_1) = 3$ together with $d_{G''}(\{x_0, u_1\}) = m$ that $e_{G''}(x_0, u_1) \geq 2$. This, however, implies that $e_G(x_0, U_1) \geq 2$, and since $d_G(U_1) = 4$, we get a contradiction with (3). This completes the proof of (4).

(5) Every vertex $v \in V \setminus \{x_0, x_1\}$ has odd degree.

Towards a contradiction, suppose v violates (5). It follows from (2) that there is a neighbour u of v which is not a root vertex. Since $d(v) \geq 4$, uv is safe. So the graph obtained from $G \setminus uv$ by suppressing degree two vertices satisfies the hypothesis of the theorem. Since $G \succ_r G'$, we have $G' \not\succeq_r D_m$, so G' either has type A_m or type B_m. Since $d_{G'}(v)$ is odd (by Observation 3.7) G' must be type A_m relative to (X_0, X_1), with v in $X_0 \cup X_1$, say $X_0 = \{x_0, v\}$. Now $d_{G'}(x_0) \geq m$, $d_{G'}(\{x_0, v\}) = m$ together with the parity of $d_{G'}(v)$ imply that $2e_{G'}(x_0, v) \geq d_{G'}(v)$. However this implies $2e_G(x_0, v) \geq d_G(v)$, which is a contradiction with (2).

(6) We have $d(x_0) = d(x_1) = m$.

Suppose (for a contradiction) that $d(x_0) > m$. If x_0 has a neighbour v other than x_1, let $e = x_0v$; else, let $e = x_0x_1$. In either case e is safe, and consider G' which is the graph obtained from $G \setminus e$ by suppressing degree two vertcies. Since $G \succ_r G'$, we have $G' \not\succeq_r D_m$ and since $|E(G')| < |E|$, we can apply the theorem to G'. Since G' has non-root vertices of odd degree, it has type A_m relative to (X_0, X_1). As before, it follows from Observation 3.7 that either $|V(G')| = 4$, and both non-root vertices of G' have odd degree, or $|V(G')| = 5$ and there is a unique non-root vertex of G' which has even degree. In either case, we may assume $X_0 = \{x_0, u\}$. It now follows from $d_{G'}(x_0) \geq m$ and $d_{G'}(X_0) = m$ that $2e_{G'}(x_0, u) \geq d_{G'}(u)$. Now, note that e must be in $\delta(X_0)$, and thus we get $2e_G(x_0, u) \geq d_G(u)$, which is a contradiction with (2).

We can now finish the proof. Note that it follows from (5) and (6) that $|V| \geq 6$. Let $v \in V \setminus \{x_0, x_1\}$ and note that by (2), we may choose an edge $e = vv'$, where $v' \notin \{x_0, x_1\}$. The edge e is safe, and as in the proof of (5) and (6) the graph G' obtained from $G \setminus e$ by suppressing degree two vertcies has type A_m. Since $|V| \geq 6$, there exist at least two vertices $w, w' \in V \setminus \{x_0, x_1, v, v'\}$ and it follows from (5) that w, w' have odd degree (in both G and G'). Therefore we may assume G' has type A_m relative to (X_0, X_1), where $X_0 = \{x_0, w\}$. However, then we have $m = e_{G'}(X_0) = e_G(X_0)$ which contradicts (1). This final contradiction completes the proof of 3.4. □

3.2 Immersion of K_4

This section is devoted to characterizing the graphs which do not immerse K_4 with either two, one, or no roots.

3.2.1 Immersion of K_4 with two roots

Observe that D_3 is isomorphic with K_4 with two roots. So, Theorem 3.4 already provides us with a description of graphs with two roots which do not immerse K_4 (with two roots). However, as we will see in Corollary 3.10, Theorem 3.4 can be restated for $m = 3$ in a way in which we drop the assumption of $\lambda_n^i(G) \geq 4$. Before getting to that, let us make a helpful observation:

Observation 3.9. *Suppose G is a 3-edge-connected graph with two root vertices, and $|V(G)| \geq 4$. If G has type B_3 then G is a doubled cycle.*

Proof. Let x_0, x_1 be the roots of G. If $E(x_0, x_1)$ is empty, it follows from definition of type B_3 that G has exactly two (x_0, x_1)-lobes. Moreover, either lobes will be doubled-paths, and thus G is a doubled cycle. On the other hand, if $E(x_0, x_1)$ is nonempty, definition of type B_3 implies that G has exactly one (x_0, x_1)-lobe. Furthermore, the unique lobe has at least four vertices, and thus is a doubled-path. Therefore, $e(x_0, x_1) = 2$, and again G is a doubled cycle. $\square$

We can now restate Theorem 3.4 for $m = 3$, or equivalently K_4 with two roots, as follows:

Corollary 3.10. *Let G be a 3-edge-connected graph where $|V(G)| \geq 4$, with two root vertices. Then $G \not\succeq_r D_3$ if and only if either of the following occurs:*

- *G is a doubled cycle.*
- *G has a segmentation of width three relative to (X_0, X_1) in which if $|X_i| \geq 3$, then X_i does not contain a root vertex.*

Proof. Suppose $G \not\succeq_r D_3$. If $\lambda_n^i(G) \geq 4$, we apply Theorem 3.4 and Observation 3.9 to G to conclude that G is a doubled cycle. So, suppose $\lambda_n^i(G) = 3$, let $U_1, \ldots, U_k$ be the maximal subsets of $V(G) \setminus \{x_0, x_1\}$ for which $|U_i| \geq 2$ and $d_G(U_i) = 3$. A similar argument as in the proof of (4) in the proof of Theorem 3.4 shows that for $1 \leq i < j \leq k$ we have $U_i \cap U_j = \emptyset$. Now, let H be the graph obtained from identifying each U_i to a new vertex u_i, so $\lambda_n^i(H) \geq 4$. Since $\lambda(G) \geq 3$, we have $G \succ_r H$, and thus $H \not\succeq_r D_3$. Since $d_H(u_1) = 3$ Observation 3.7 implies that H must have type A_3 relative to (X_0, X_1), where (by possibly relabeling x_0, x_1) $X_0 = \{x_0, u_1\}$ and $X_1 = \{x_1, v\}$, where $v = u_2$ if u_2 exists. This implies that if U_2 exists, G has a segmentation of width three relative to either (U_1, U_2), and otherwise G has a segmentation of width three relative to (U_1, X_1), as desired. $\square$

3.2.2 Immersion of K_4 with up to one root

Corollary 3.10 gives the structure of 3-edge-connected graphs excluding a D_3 immersion, or equivalently an immersion of K_4 with two roots. Our next task is to characterize the structure of 3-edge-connected graphs excluding a rooted immersion of K_4 with one root. The next lemma is a step towards such a result, not to mention that it also proves helpful in Chapter 5, while proving our theorem on prism immersions. For the sake of brevity, in the next two statements, we say *rooted* K_4 to exclusively refer to K_4 with exactly one root.

Lemma 3.11. *Let G be a 3-edge-connected graph with $|V(G)| \geq 4$. Let G be rooted at x, where $d(x) = 3$. Then either $G \succeq_r K_4$ or there exists $Y \subset V \setminus x$ with $|Y| \leq 2$ such that G has a segmentation of width three relative to $\{x\}, Y$.*

Proof. Choose $Y \subset V \setminus x$ such that $d(Y) = 3$, $|Y| \geq 2$, and Y is minimal subject to this. If $|Y| \geq 3$, let $G' = G.(V \setminus Y)$, and note that since G is 3-edge-connected $G \succeq_r G'$. Now choose $y \in Y$, and declare $V \setminus Y, y$ to be the roots of G'. We have $|V(G')| \geq 4$ and $\lambda^i(G') \geq 4$. Since $d_{G'}(V \setminus Y) = 3$, G' is not a doubled cycle, so by Corollary 3.10 we have $G' \succeq_r D_3$, and thus $G \succeq_r K_4$.

Now, suppose $|Y| \leq 2$. Let $G' = G.Y$ where y' is the vertex resulting from identifying Y, and note that $G \succ G'$. Let x, y' be the roots of G'. It follows from Corollary 3.10 and $d_{G'}(x) = d_{G'}(y') = 3$ that either $G' \succeq_r D_3$ or G' has a segmentation of width three relative to $(\{x\}, \{y'\})$. Thus, either $G \succeq_r K_4$ or G has a segmentation of width three relative to $(\{x\}, Y)$, where $|Y| \leq 2$, as desired. □

We are now prepared to determine 3-edge-connected graphs which do not immerse rooted K_4 (with one root):

Corollary 3.12. *Let G be a 3-edge-connected graph with up to one root vertex and $|V(G)| \geq 4$. Then $G \succeq_r K_4$ unless:*

- *G is a doubled cycle.*
- *G has a $(2,2)$-segmentation of width three.*

Proof. First, suppose G has a root vertex x. If $\lambda^i(G) \geq 4$, choose an arbitrary vertex $y \neq x$, and declare G to be rooted at x, y. Then Corollary 3.10 implies that $G \succeq_r D_3$ or G is a doubled cycle. Observing that a doubled cycle fails to immerse K_4 we are done in this case. So, consider the case where G has an internal 3-edge-cut. Let us record a useful statement.

Observation. Suppose G has an internal 3-edge-cut $\delta(X)$, with $x \in X$. Then either $G \succeq_r K_4$ or there exists $Y \subset V \setminus X$ with $1 \leq |Y| \leq 2$ such that G has a segmentation of width three relative to (X, Y).
If $|V \setminus X| \leq 2$, the statement trivially holds. Otherwise, consider $G.X$ with the root vertex X, and note that $G.X \succeq_r K_4$ implies $G \succ_r K_4$. Now, apply Lemma 3.11.

Now, suppose G has an internal 3-edge-cut $\delta(Z)$, with $x \in Z$. Choose $W \subset Z$ such that $x \in W$, $d(W) = 3$, $|W| \geq 2$, and W is minimal subject to this. Then by the above observation, either $G \succeq_r K_4$ or there is $Y \subset V \setminus W$ with $1 \leq |Y| \leq 2$ such that G has a segmentation $\mathcal{S}_0$ of width three relative to (W, Y). If $|W| \leq 2$, there is nothing left to prove.

So, suppose $|W| \geq 3$, and let $G' = G.(V \setminus W)$, with y being the vertex resulting from the identification of $V \setminus W$. Note that if G' with roots x, y immerses D_3 then $G \succeq_r K_4$. Since $d_{G'}(y) = 3$ we may apply Corollary 3.10 to deduce that either $G \succeq_r K_4$ or there exists

$U \subset W$ such that G' has a segmentation $\mathcal{S}_1$ of width three relative to $(\{y\}, U)$; moreover, if $|U| \geq 3$, then $x \notin U$. So, in particular, $U \neq W$. Let $\mathcal{S}_1$ be $\{y\} \subset Y_1 \subset Y_2 \subset \ldots \subset Y_k = V(G') \setminus U$. Note that our choice of W implies that $Y_1 = \{y, x\}$. In particular, if we let $W' = W \setminus x$, then $\delta(W')$ is an internal 3-edge-cut. So by the above observation, either $G \succeq_r K_4$ or there exists $T \subset W'$ with $1 \leq |T| \leq 2$ such that G has a segmentation $\mathcal{S}_2$ of width three relative to $(T, V \setminus W')$. It now follows from the existence of $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$ that G has a $(2,2)$-segmentation of width three (relative to (T, Y)), as desired.

Observe that, perhaps surprisingly, the root vertex does not appear in the conclusion for graphs with one root. This immediately proves the corollary for graphs with no root. $\square$

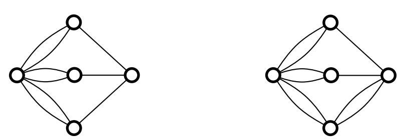
Chapter 4

Immersion of W_4

The main goal of this chapter is to give the precise structure of the graphs excluding an immersion of W_4 (with zero or one root). As mentioned in Section 1.3, the unrooted version of this problem has been studied before. Our structural theorem on the structure of graphs without a W_4 immersion is as follows:

Theorem 4.1. *Let G be 3-edge-connected, and internally 4-edge-connected which is sausage reduced. If $|V(G)| \geq 5$, then $G \not\succ W_4$ if and only if*

1. *G is cubic, or*
2. *G is isomorphic to one of the graphs below.*



Theorem 4.1 follows from a stronger theorem describing graphs without a W_4 immersion in which the image of the center of W_4 is specified in advance. Throughout this chapter, we use the term *rooted* W_4 to refer to a graph isomorphic to W_4 in which the center of W_4 is declared to be the root vertex. To state our result on rooted W_4, we need to introduce four families of rooted graphs which do not have a rooted immersion of W_4. Let G be a graph with the root vertex u. Then we say

Type 1. G is *type* 1 if it has a $(2,3)$-segmentation of width four in which u is in the head of the segmentation.

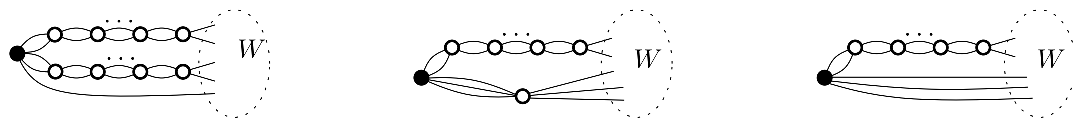
If U, W are the head and tail of such a segmentation, respectively, we may say G has type 1 relative to (U, W).

Type 2. G is *type* 2 if there exists a set $W \subseteq V(G) \setminus \{u\}$ with $|W| \leq 2$ so that the graph G^* obtained by identifying W to a single vertex w has a doubled cycle C satisfying one of the following:

(2A) u and w are not adjacent in C and $G^* = C + uw$

(2B) u and w have a common neighbour v in C and $G^* = C + uv + vw$

(2C) u and w are adjacent in C and $G^* = C + uw$



(a) Type 2A (b) Type 2B (c) Type 2C

Figure 4.1

Type 3. G is *type* 3 if after sausage reduction is isomorphic to a graph in Figure 4.2. That is G is type 3 if it can be obtained from a graph in Figure 4.2 by replacing the pair of green vertices with a chain of sausages of arbitrary order ≥ 2.

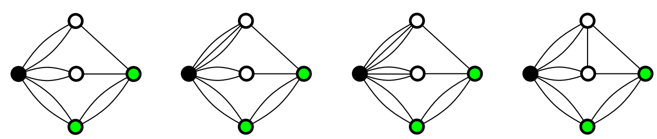


Figure 4.2: Type 3 graphs after sausage reduction

Type 4. G is *type* 4 if either

- it can be obtained from the leftmost graph in Figure 4.3 by the following process: For each vertex y, y', y'' either do nothing or add one new edge incident with this vertex and in parallel with an existing edge.
- it is isomorphic to one of the four rightmost graphs in Figure 4.3.

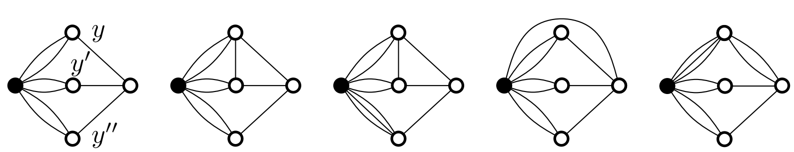


Figure 4.3: Type 4 graphs

Theorem 4.2. *Let G be a 3-edge-connected, internally 4-edge-connected graph with $|V(G)| \geq 5$ and with a root vertex x. Then G contains a rooted immersion of W_4 if and only if G does not have one of the types $1, 2, 3,$ or 4.*

The proof of the above theorem, as presented in this chapter, is computer-assisted. As we will see in Section 4.1, we have used the computer to search for an immersion of rooted W_4 on the graphs which satisfy the assumptions of Theorem 4.2, and have at most eight vertices. However, this result was first proved independent of the computer, with the proof appearing in Appendix B. We have favored the new computer-assisted proof mostly due to the fact that the argument for the existence of a rooted immersion of W_4 is considerably more suggestive for 'big enough' graphs. Indeed, the current argument for existence of rooted W_4 in (sausage reduced) graphs on at least eight vertices, shows a paradigm for finding an immersion of a bigger graph using structural theorems on immersion of smaller rooted graphs. In the proof of Theorem 4.2 that is presented below, we will see how our results in Chapter 3 on the existence of D_2- and D_4 immersion play a central role in finding an immersion of rooted W_4.

4.1 Rooted W_4 immersions

4.1.1 Proof of the 'if' direction of Theorem 4.2

Before proving the easier part of the theorem, we make a couple simple observations.

Observation 4.3. *Suppose that G is a graph, rooted at x, which has a rooted immersion of W_4 with T being the set of terminals.*

1. *If G has a segmentation $X_0 \subset X_1 \subset \ldots \subset X_k$ of width four with $x \in X_0$ and $|X_0| \leq 2$, then $T \cap X_k = \{x\}$.*
2. *If $v \in T \setminus x$ has $d(v)$ even, then there is an edge e incident with v so that $G - e \succeq_r W_4$.*
3. *The graph obtained from G by sausage reducing it has a rooted immersion of W_4.*

Proof. The first part follows from the fact that for every set $X \subset V(W_4)$ with $|X| = 2$ which contains the center we have $d(X) = 5$. The second part is immediate from our definitions, and the last follows from part 2 and the edge-connectivity of W_4. □

Proof of the 'if' direction of Theorem 4.2. We show that graphs of types 1, 2, 3, or 4 do not immerse rooted W_4. For graphs of type 1, this is immediate from part 1 of the previous observation. To verify this for graphs of type 3, by part 3 of the previous observation we only need to show that graphs in Figure 4.2 do not have a rooted immersion of W_4. This, as well as verifying the statement for graphs of type 4, is easy enough to do by hand, but we have used a computer to do so.

So let G, rooted at x, be type 2 relative to W, and suppose (for a contradiction) that $G \succ_r W_4$ with T being the set of terminals. By part 3 of the above observation, we may assume G is sausage reduced. Suppose that there is a chain of sausages $G[\{y, z\}]$ in $V(G) \setminus (W \cup x)$. Note that $|\{y, z\} \cap T| \leq 1$, otherwise it follows from part 2 of the previous observation and the internal 4-edge-connectivity of W_4 that the graph G' obtained from G by deleting one copy of the edge yz contains a rooted immersion of W_4. This, however, is impossible since G' has type 1. Therefore, if we let G'' be the graph obtained from G by splitting off any chain of sausages disjoint from $W \cup x$ (if present) down to only one vertex, then $G'' \succeq_r W_4$. This, immediately gives a contradiction in the cases where G has type 2C, or $|W| = 1$. In other cases, $|W| = 2$, and $|V(G'')| = 5$, so every vertex in G'' is a terminal of W_4. Then, by part 2 of the above observation, an edge incident to each vertex in $V(G'') \setminus (W \cup x)$ may be removed while an immersion of W_4 is preserved. However, in the resulting graph either $d(x) < 4$, or there is an internal 3-edge-cut, so this is impossible. □

4.1.2 Proof of the 'only if' direction of Theorem 4.2

The goal for this section is to prove Theorem 4.2 which gives the structure of graphs with no rooted W_4 immersion. To begin the proof of this theorem, assume (for a contradiction) that it is false, and choose a graph $G = (V, E)$ with root vertex x so that G is a counterexample to Theorem 4.2 with $|V| + |E|$ minimum.

(1) For every $y \in N(x)$ we have $d(\{x, y\}) \geq 5$.

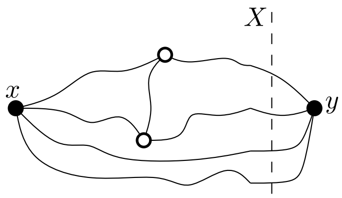
Suppose (for a contradiction) this is false and choose $y \in N(x)$ so that $d(\{x, y\}) < 5$. Let $X = \{x, y\}$. Note that the internal 4-edge-connectivity of G implies $d(X) = 4$. If $|V(G)| = 5$, then G has a $(2, 3)$-segmentation of width four relative to $(X, V \setminus X)$, and thus G has type 1. So we must have $|V(G)| \geq 6$. If $G.X$ has a rooted W_4 immersion, then it follows from internal 4-edge-connectivity that G also has a rooted W_4 immersion, giving us a contradiction. So, the minimality of the counterexample G implies that the theorem holds for $G.X$, so it must have type $1, 2, 3$, or 4. Let x^* be the root vertex of $G.X$, where x^* is the vertex obtained from identifying X. Since x^* has degree four, $G.X$ can only be type 1, and moreover we may assume $G.X$ has in fact a $(1, 3)$-segmentation of width four relative to $(\{x^*\}, W)$, for some $W \subset V(G.X)$. Now the graph G has a $(2, 3)$-segmentation of width four relative to (X, W), and thus has type 1. This contradiction completes the proof.

(2) Let $\delta(X)$ be a 4-edge-cut in G with $x \in X$ and $|X| \geq 3$. Then $G.(V \setminus X) \succeq_r D_4$.

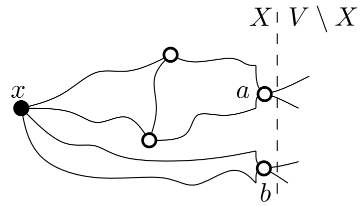
Let y denote the vertex in $G' = G.(V \setminus X)$ resulting from identifying $V \setminus X$. Note that G' is internally 4-edge-connected, and $|V(G')| \geq 4$. So, if $G' \not\succeq_r D_4$, it follows from Theorem 3.4 that G' has type A_4 or type B_4. However, since $d_{G'}(y) = 4$, G' must be type A_4. So there exists $U \subset V(G')$ such that $x \in U$, $|U| \leq 2$ and G' has type A_4 relative to $U, \{y\}$. This in particular implies that $d_G(U) = d_{G'}(U) = 4$ which contradicts (1).

(3) Let $\delta(X)$ be a 4-edge-cut in G with $x \in X$ and $|X|, |V \setminus X| \geq 3$. Then $G[V \setminus X]$ is a chain of sausages.

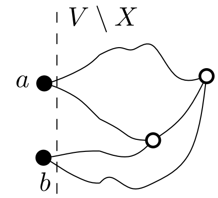
Let $G' = G.(V \setminus X)$, and denote the vertex resulting from identifying $V \setminus X$ by y. By (2), we know that G' with roots x, y has a rooted immersion of D_4. Consider the graph H which is isomorphic to D_4 on vertex set $\{x, y, v_0, v_1\}$, and with x, y as roots. Let P, P' (Q, Q') be the two paths in G' which correspond to the two xy edges (yv_0, yv_1 edges) in the H immersion. Let $\{e, e'\} = \delta_{G'}(y) \cap (P \cup P')$, and let $\{f, f'\} = \delta_{G'}(y) \cap (Q \cup Q')$. Now let G'' be the graph obtained from G by subdividing e, e' (f, f') with a new vertex, and then identifying the two-vertices to a new vertex, a (b).



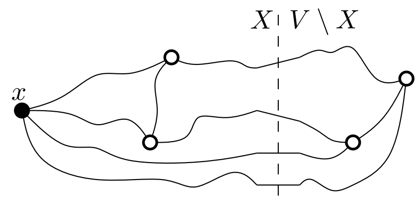
Immersion of D_4 in G'



G''



Immersion of D_2 in G^*



$G' \succ_r D_4, G^* \succ_r D_2$ implies $G \succ_r W_4$

We define $G^* = G''[V \setminus X \cup \{a, b\}]$, with a, b as its root vertices. Observe that (since $G' \succeq_r D_4$) if there is a rooted immersion of D_2 in G^*, then $G \succeq_r W_4$. So $G^* \not\succeq_r D_2$. Note it follows from $|V \setminus X| \geq 3$ and the internal edge-connectivity of G that G^* satisfies the hypothesis of Theorem 3.4 for $m = 2$. So, G^* is type A_2 or type B_2, and since $d_{G^*}(a) = 2$, G^* must be type A_2. Now since $d_{G^*}(a) = d_{G^*}(b) = 2$, G^* is in fact a chain of sausages, as desired.

(4) Every $X \subseteq V$ with $|X| \geq 2, |V \setminus X| \geq 3$ with $x \in X$ satisfies $d(X) \geq 5$.

By (1) we may assume $|X| \geq 3$. Since G is internally 4-edge-connected, we have $d(X) \geq 4$. Suppose $d(X) = 4$, then using (3), we know that $G[V \setminus X]$ is a chain of sausages of order at least three in G. Since W_4 is simple and 3-edge-connected, it follows from Observation 4.8 that if we let G'' be the graph obtained from G by sausage reducing it, we have $G'' \not\succeq_r W_4$. Note that G'' is 3-edge-connected, and internally 4-edge-connected as G is. Since $|V(G'')| < |V(G)|$, G'' satisfies the theorem. Note G'' is sausage reduced and since G has a chain of sausages of order ≥ 3, G'' has at least one pair of neighbours each of degree four, with two edges between them. So, in particular, G'' is not type 4.

So G'' has one of the types 1, 2, or 3. However, if G'' is type 1 relative to some (U, W), then $\delta_G(U)$ would be a 4-edge-cut in G which contradicts (1), so G'' is not type 1.

Suppose G'' is type 2 relative to W. We will assume that W is minimal subject to this. Let $G''[Y]$ be a chain of sausages in G'' (so, is of order at most two). Now, since $d_{G''}(Y) = 4$, and $d_{G''}(W) = 5$, the minimality of W implies that $W \cap Y = \emptyset$. Therefore, G is also type 2 – a contradiction. Note that G'' is not isomorphic to one of the graphs in Figure 4.2 either, else G would be type 3. This final contradiction completes the proof of (4).

Let us pause to make a helpful observation which will be called upon in later chapters as well.

Observation 4.4. *Let G, H be (rooted) graphs, and let $u, v \in V(G)$ satisfy $e(u,v) > |E(H)|$. Then $G \succ H$ ($G \succ_r H$) if and only if the graph G' obtained from G by deleting one copy of uv edge satisfies $G' \succeq H$ ($G' \succeq_r H$).*

Proof. It is because the maximum contribution of uv to any set of paths $\mathcal{P}$ corresponding to the edge-set of H occurs when uv appears in every path in $\mathcal{P}$. So the maximum contribution of uv to $\mathcal{P}$ is $|E(H)|$ times. □

We now continue by establishing more properties of G.

(5) $|V(G)| \geq 9$.

Suppose G is a graph with a root vertex x, where $d(x) \geq 4$ and $|V(G)| \leq 8$. Suppose further that G is 3-edge-connected, internally 4-edge-connected, and satisfies (4). We will show that G satisfies Theorem 4.2, i.e. if $G \not\succeq_r W_4$, it is either type 2 or type 4, or it is isomorphic to one of the graphs in Figures 4.2.

Note that thanks to Observation 4.4, (5) can get verified through a finite calculation. In fact it follows that to verify (5) it suffices to check the finite number of rooted 3-edge-connected, internally 4-edge-connected graphs, with edge-multiplicity at most $|E(W_4)| = 8$, for which the root vertex x has degree at least four, and for any set $Y \subset V(G) \setminus \{x\}$ with $|Y| \geq 3$ we have $d(Y) \geq 5$. This calculation is done in Sagemath, and here is a high-level description of the algorithm.

Let $5 \leq n \leq 8$.

Step 1. We take the list of connected simple graphs on n vertices, and use it to generate all rooted graphs on n vertices. Then, the rooted graphs which have a rooted immersion of W_4 are filtered out.

Step 2. For any rooted graph G surviving from Step 1, `repair(G)` generates a list consisting of all edge-minimal rooted multigraphs G' such that:

- the underlying simple graph of G' is G,
- $d_{G'}(x) \geq 4$, where x is the root vertex of G',

- $\delta(G') \geq 3$,
- G' is internally 4-edge-connected,
- for any internal edge-cut $\delta(Y)$ with $x \notin Y$ and $|Y| \geq 3$ we have $d(Y) \geq 5$,
- G' does not have a rooted immersion of W_4.

Step 3. Suppose the simple rooted (connected) graph G is such that $\mathcal{G}_1 =$ `repair(G)` is nonempty. Then, using $\mathcal{G}_1$, we generate $\mathcal{G}_2 =$ `obstruction(G)` which consists of all rooted multigraphs whose underlying simple rooted graph is G, meet the edge-connectivity conditions that the graphs in $\mathcal{G}_1$ satisfy, have edge-multiplicity at most eight, and do not immerse rooted W_4.

Step 4. Every graph in $\mathcal{G}_2$ is tested if it has type 2, or is isomorphic to one of the graphs in Figures 4.2 or 4.3.

The calculation is done rather fast. The calculation for every $n \in \{5, 6, 7\}$ took a desktop computer less than a minute. However, the calculation for $n = 8$ took much longer– almost 20 minutes. It took the computer one minute to carry out step 1, i.e. to check the nearly 72,500 connected simple rooted graphs on eight vertices for a W_4 immersion, thereby giving a list `N8` of almost 40,000 simple rooted connected graphs on eight vertices which do not immerse rooted W_4. Then 20 minutes was spent on carrying out steps 2, 3 for every graph in `N8`. Since no obstruction is found for $n = 8$, step 4 is not performed for this case.

(6) There do not exist $v, w \in V \setminus x$ such that $e(v, w) \geq \frac{1}{2}d(w)$.

Suppose for a contradiction that such v, w exist. Let $G' = G.\{v, w\}$, rooted at x. Note that $|V(G')| = |V(G)| - 1$, and since $e(v, w) \geq \frac{1}{2}d(w)$, we have $G \succ_r G'$. Therefore $G' \not\succeq_r W_4$, and Theorem 4.2 holds for G'. Since $|V(G')| \geq 8$, G' has one of the types 1, 2, or 3. However, G' being type 1 implies that there is an internal 4-edge-cut in G with at least three vertices on opposite side of x, contradicting (4). So G' is not type 1. In a similar manner we conclude that G' is sausage reduced, and since $|V(G')| \geq 8$, G' is not type 2 or 3 either, a contradiction. (Observe that after sausage reduction a graph of type 2 or 3 has at most seven vertices).

(7) Suppose $\delta(Y)$ is an internal 4-edge-cut in G, with $|Y| \leq |V \setminus Y|$. Then we have $x \notin Y, |Y| = 2$, and both vertices in Y have degree three.

It follows from (4) that $|Y| = 2$, and $x \notin Y$. Let $Y = \{u, v\}$. Since $d(Y) = 4$, we have $e(u, v) > 0$. It follows from (6) that exactly two edges of $\delta(Y)$ is incident with each u, v, and that $e(u, v) = 1$, as desired.

(8) Let $\delta(X)$ be a 5-edge-cut in G with $x \in X$ and $|X| \geq 3$. Then $G.(V \setminus X) \succ_r D_4$.

Let y denote the vertex in $G' = G.(V \setminus X)$ resulting from identifying $V \setminus X$. If $G' \not\succeq_r D_4$, it follows from Theorem 3.4 that G' has type A_4 or type B_4. If G' has type A_4, we get a contradiction with (1), so G' must be type B_4. Then, however, it is straightforward to check that there will be a neighbour u of x with $e(x, u) = 2$ and $d(u) = 4$. This contradicts (6), and this contradiction proves (8).

(9) There does not exist a 5-edge-cut $\delta(X)$ in G such that $x \in X, |X| \geq 3, |V \setminus X| \geq 4$.

Let $Y = V \setminus X$. By (8) we know that $G.Y \succ_r D_4$. Thus there exists $e \in \delta(X)$ such that $(G \setminus e).Y \succeq_r D_4$ as well. Let $H = G \setminus e$, in which vertices of degree two (resulting from the deletion of e) are suppressed, and let Y' denote the set corresponding to Y in H. Note that $|Y'| \geq 3$, and an argument similar to that of (3) then shows that $H[Y']$ is a chain of sausages (of order at least three). However, it is easy to check that whether the endpoint of e in G is a vertex present in H or it got suppressed, there is a vertex of degree four incident with parallel edges in Y, which contradicts (6).

(10) G is simple.

For a contradiction suppose adjacent vertices u, v exist such that $e(u, v) \geq 2$. Let G' be the graph obtained from G by deleting one copy of uv. Note that (5) implies that $|V(G')| \geq 9$ and (6) implies that G' is 3-edge-connected. It also follows from (7) that G' is internally 4-edge-connected. So by minimality of G, Theorem 4.2 holds for G'. If G' is type 1, then there exists $X \subset V$ with $|X|, |V \setminus X| \geq 4$ such that $d_{G'}(X) = 4$, and this contradicts (4) or (9). Now note that it follows from (6) that G' is sausage reduced. So $|V(G')| \geq 9$ implies that G' is not type 2 or 3 either, a contradiction.

With this last item in place, we are now ready to complete the proof of Theorem 4.2. Choose an edge e which is not incident with x. Let G' be the graph obtained from G by deleting e, and suppressing any degree two vertices. So $|V(G')| \geq |V(G)| - 2 \geq 7$, and G' is 3-edge-connected. It follows from (7) that G' is also internally 4-edge-connected. So by minimality of G, Theorem 4.2 holds for G'. As in the proof of (10), G' cannot have type 1, otherwise G would contradict either (4) or (9). Finally, note that G' is sausage reduced. It is because if $G'[X']$ is a chain of sausages of order ≥ 3 in G', then (4) implies that $e \in \delta_G(X)$, where X is the set in G which corresponds to X' in G'. But then $G[X]$ would contain parallel edges, contradicting (10). This rules out the possibility of G' having any type other than 2A. Moreover, it implies that if G' has type 2A relative to $\{x\}, W$, for some $W \subset V$ with $|W| = 2$, then both chain of sausages between $\{x\}, W$ are of order exactly two. A quick check shows that then G contradicts (10). This establishes Theorem 4.2.

4.1.3 A note about the code

The code used in the proof of Theorem 4.2 is written in Sagemath, however its use of the graph package is limited to generating simple rooted graphs. The computation first ran using the graph package of Sagemath more heavily. However, certain issues arose which made a transition from the package advantageous.

The first issue was that with the rooted W_4 problem, we needed to check for isomorphism of rooted graphs. However, the isomorphism tester in Sagemath works with non-rooted graphs. Of course, this issue is solvable by putting a loop on the root vertex whenever checking for isomorphism (since our original graphs are loopless). However, as this addition and removal of loops was done many times, it slowed down the code. The other issue is not particular to the rooted graphs, and has to do with the slowness of copying the graph object of Sagemath. Since any simple graph in our computation goes through a few operations (for instance to increase the multiplicity of edges so that the resulting graph meets the minimum required size of various edge-cuts), the slowness of copying graphs was a rather significant factor in the computation time.

So, it seemed beneficial to try running the code not using the graph package. Once simple graphs on a given number of vertices was generated, we continued the rest of the code with a new representation of graphs. This new representation uses dictionary objects of Sagemath. An issue here was that the isomorphism tester that we wrote for this new representation of graphs is not as nearly sophisticated as Sagemath's graph isomorphism tester. A comparison of the two testers over a sample of 2000 simple non-rooted graphs on eight vertices showed a slow-down by a factor of three. Despite this, the new code was considerably faster than the one using the graph package. The new code was run in 25 minutes, whereas the first code took over 2.5 hours to finish.

4.2 Unrooted W_4 immersions

In this section we will use our result on rooted immersions of W_4 to prove our theorem on unrooted immersions of W_4. For this purpose we will frequently need to call on the type 1 obstruction to the existence of a rooted W_4 immersion. Recall that we say a graph G with root u has type 1 if it has a $(2,3)$-segmentation of width four relative to some (U, W) with $u \in U$. In other words if there exist $U = U_0 \subset U_1 \subset \dots U_t = V(G) \setminus W$, where

- $u \in U$,
- $|U| \leq 2, |W| \leq 3$,
- $d(U_j) = 4$, for $0 \leq j \leq t$,
- $|U_{j+1} \setminus U_j| = 1$, for $0 \leq j \leq t-1$.

The theorem we prove here has a slightly different form than Theorem 4.1. Here we have relaxed the assumption that the graph should be sausage reduced. However, it is straightforward to see that the two results are in fact equivalent.

Theorem 4.5. *Let G be 3-edge-connected, and internally 4-edge-connected. If $|V(G)| \geq 5$, then $G \not\succeq W_4$ if and only if*

1. *G is cubic, or*

2. *G is isomorphic to the left graph in Figure 4.4.*

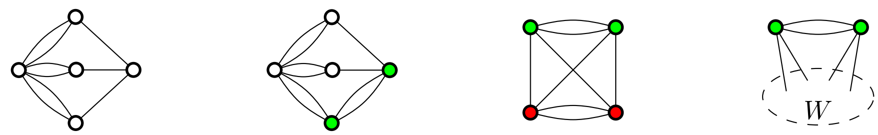


Figure 4.4: Non-cubic sausage reduced graphs without a W_4 immersion

3. *G after sausage reduction is isomorphic with the second left graph above. That is, G can be obtained from the second left graph above by replacing the pair of green vertices with an arbitrary chain of sausages of order at least two.*

4. *G after sausage reduction is isomorphic with the second right graph above. That is, G can be constructed from the second right graph above by replacing each pair of same-colored vertices with an arbitrary chain of sausages of order at least three.*

5. *There exists $W \subset V(G)$ with $1 \leq |W| \leq 2$ so that the graph obtained from G by identifying W to a single vertex is a doubled cycle (as in the rightmost graph above).*

Proof. Let $G = (V, E)$ be a minimal counterexample to the theorem. We will establish a sequence of properties of G eventually proving it cannot exist.

(1) There is a vertex of even degree in G.

If G is cubic the theorem holds, so we may choose $u \in V(G)$ with $d(u) \geq 4$. If we treat u as a root vertex, there cannot be a rooted immersion of W_4, so by Theorem 4.2 this rooted graph must have type $1, 2, 3$, or 4. All graphs of types 2, 3, and 4 have a vertex of even degree, so we are done unless our rooted graph has type 1 relative to some (U, W). If $|V(G)| = 5$, then G has a vertex of even degree by parity, and else, by Observation 3.7 any vertex in $V \setminus (U \cup W)$ has even degree.

(2) If $u \in V(G)$ has even degree, there is a $(2, 3)$-segmentation of width four of G relative to some (U, W), where $u \in U$.

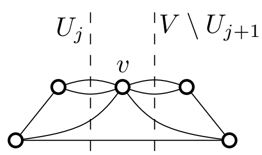
Treat u as a root vertex of G and apply Theorem 4.2. Since G does not have an immersion of W_4 (and $d(u)$ is even), this rooted graph must have type 1, 3, or 4. For types 3 or 4 a straightforward check shows that G has an immersion of W_4 unless either case 2 or 3 occur.

(3) If $u \in V(G)$ has even degree, there exists $v \in N(u)$ with $e(u,v) \geq 2$ so that $d(\{u,v\}) = 4$. (Note that v must also have even degree.)

Apply (2) to u and let U be the head of the corresponding $(2,3)$-segmentation of width four. Then setting $U = \{u, v\}$ we have $4 = d(\{u,v\}) = d(u) + d(v) - e(u,v)$ and it follows that v has the desired properties.

(4) If $U_0, \ldots, U_t$ is a $(2,3)$-segmentation of width four of G and $U_{j+1} \setminus U_j = \{v\}$, then $e(v, U_j) = 2 = e(v, V \setminus U_{j+1})$.

Choose an edge $xy \in E(G[U_j])$ $(E(G[V \setminus U_{j+1}]))$. Then it follows from the internal 4-edge-connectivity of G that there are four edge-disjoint paths, two starting at each x, y which end at $\{v, V \setminus U_{j+1}\}$ $(\{v, U_j\})$. On the other hand, it follows from $d(U_j) = 4 = d(U_{j+1})$ that $e(v, U_j) = e(v, V \setminus U_{j+1})$. So, if either $e(v, U_j) = 3 = e(v, V \setminus U_{j+1})$ or $e(v, U_j) = 4 = e(v, V \setminus U_{j+1})$, then G immerses the graph below, and thus has a W_4 immersion, a contradiction. So, we must have $e(v, U_j) = e(v, V \setminus U_{j+1}) \leq 2$ and since



$d(v) \geq 3$ this completes the proof of (4).

(5) If $U_0, \ldots, U_t$ is a $(2,3)$-segmentation of width four of G with $t \geq 1$ and $0 \leq j \leq t-1$, then either $G.U_j$ or $G.(V \setminus U_{j+1})$ is a doubled cycle.

Suppose for a contradiction that the above is violated. Let $U_{j+1} \setminus U_j = \{x\}$ and construct a new graph G' (G'') from G by identifying U_j $(V \setminus U_{j+1})$ to a vertex y and deleting all edges between x and y. Declare x and y to be root vertices of both G' and G'' and note that x and y both have degree 2 in both G' and G''. It now follows from our assumptions and Theorem 3.4 that G' and G'' have rooted immersions of D_2. Therefore G has an immersion of W_4, giving a contradiction.

(6) If $|V| \geq 6$ there exists $X \subset V$ with $|X| \leq 3$ so that $G.X$ is a doubled cycle.

It follows from (1) and (2) that we may choose a $(2,3)$-segmentation of width four $U_0, \ldots, U_t$. Since $|V| \geq 6$ we must have $t \geq 1$. If either $G.U_0$ or $G.(V \setminus U_t)$ is a doubled cycle we are done. Otherwise, there exists $1 \leq j \leq t$ so that both $G.U_j$ and $G.(V \setminus U_j)$ are doubled cycles. It is because since $G.U_0$ is not a doubled cycle, (5) implies that $G.(V \setminus U_1)$ is a doubled cycle. Now, if $G.U_1$ is also a doubled cycle we are done. Else, it follows from (5) that $G.(V \setminus U_2)$ is a doubled cycle. By repeating this argument, it follows that $G.(V \setminus U_t)$ is a doubled cycle– a contradiction.

However, in this case the original graph G is either a doubled cycle or may be obtained from K_4 by replacing a pair of opposite edges by doubled paths (i.e either case 4 or 5 happens).

With the last property in place we are ready to complete the proof. First we will resolve the case when $|V| \geq 6$. In this case we may apply (6) to choose a nonempty set X with $|X| \leq 3$ so that $G.X$ is a doubled cycle. If $|X| \leq 2$ then we are finished, so we may assume $|X| = 3$. The graph $G.(V \setminus X)$ has four vertices, and the vertex $V \setminus X$ has degree 4 in this graph. It follows that there must exist $y \in X$ with $d_G(y)$ even. Now (3) implies that there exists a set $Y \subseteq V$ with $y \in Y$ and $|Y| = 2$ so that $d(Y) = 4$ and the graph induced on Y has at least 2 edges. If $Y \not\subseteq X$ then $G.(X \setminus y)$ is a doubled cycle and the proof is complete. Let $X \setminus Y = \{z\}$ and note that $d_G(z)$ must be even since $d(Y)$ and $d(X)$ are both even. As before, we may apply (3) to choose a set $Z \subseteq V$ with $z \in Z$ and $|Z| = 2$ so that $d(Z) = 4$ and the graph induced on Z has at least 2 edges. Also as before, if $Z \not\subseteq X$ then $G.(X \setminus z)$ is a doubled cycle and we are done. So $Z \subseteq X$ and we have $Y \cup Z = X$. Since $d(y)$ is even and $d(Y)$ is even the unique vertex in $Y \setminus y$ must also have even degree. A similar argument for Z shows that all vertices in X have even degree. Now uncrossing arguments give us

$$d(Y \cap Z) + d(Y \cup Z) \leq d(Y) + d(Z) = 8$$
$$d(Y \setminus Z) + d(Z \setminus Y) \leq d(Y) + d(Z) = 8$$

It follows from these inequalities and the above parity considerations that every vertex in X has degree 4. Furthermore, the graph $G.(V \setminus X)$ must be a doubled cycle of length 4. We conclude that G is either a doubled cycle, or a graph obtained from K_4 by replacing two non-adjacent edges by doubled paths.

It remains to prove the theorem when $|V| = 5$. It follows from (1) and (3) that we may choose a 2 element subset $X \subseteq V$ so that $d(X) = 4$, both vertices in X have even degree, and the graph induced on X has at least two edges. Now $G.X$ is a graph on four vertices and the vertex replacing X has even degree, so there exists $y \in V \setminus X$ with $d_G(y)$ even. Applying (3) we may choose a set $Y \subseteq V$ with $y \in Y$ and $d(Y) = 4$ so that the graph induced on Y has at least 2 edges. If $X \cap Y \neq \emptyset$ then by arguments as in the case when $|V| \geq 6$ we deduce that the graph $G.(V \setminus (X \cup Y))$ is a doubled cycle and the proof is done. If $Y \cap X = \emptyset$ then define $\{z\} = V \setminus (X \cup Y)$. Since $d(X)$ and $d(Y)$ are even, it must be that $d(z)$ is even, and now we may apply (3) to z to choose a set Z with $z \in Z$ and $|Z| = 2$ so that the graph induced on Z has at least 2 edges. Now either $X \cap Z \neq \emptyset$ or $Y \cap Z \neq \emptyset$ and an argument similar to the previous case completes the proof. $\square$

4.3 Additional tools

In this section, we will see a couple easy observations about some reductions that enable us to assume a "good enough edge-connectivity" when trying to characterize the family of graphs which do not immerse a "well edge-connected graph". We will also see a couple observations about tree-width and branch-width that are helpful in finding these parameters for graphs which arise as obstructions to the existence of, for instance, W_4 immersion. These statements will be called upon later in this chapter, as well as in the following chapters.

4.3.1 Reductions

Let H be a 3-edge-connected (and internally 4-edge-connected) graph. The next two observations explain why the problem of finding the structure of graphs which exclude an immersion of H boils down to the family of 3-edge-connected (and internally 4-edge-connected) graphs.

Observation 4.6. *Let H be a 3-edge-connected graph. A graph G does not immerse H if and only if G can be reduced to a graph where every component does not immerse H by one of the following operations*

- *If e is a cut-edge in G, then modify G to $G \setminus e$.*
- *If K is a 2-edge-connected component of G, and $X \subset V(K)$ has $\delta_K(X) = \{x_1y_1, x_2y_2\}$, where $x_1, x_2 \in X$ then modify G to $G \setminus \delta_K(X) + x_1x_2 + y_1y_2$.*

Proof. Observe that a graph G immerses H if and only if one of its components does. Suppose G has a cut-edge e. Clearly $G \setminus e \succeq H$ implies $G \succ H$. On the other hand, $\lambda(H) \geq 3$ implies that if $G \succeq H$ then $G \setminus e \succeq H$.

Now, suppose K is a 2-edge-connected component of G, which has a 2-edge-cut $\delta_K(X) = \{x_1y_1, x_2y_2\}$, where $x_1, x_2 \in X \subset V(K)$. Let $K_1 = K[X] + x_1x_2$ and $K_2 = K[X] + y_1y_2$. Then it follows from $\lambda(H) \geq 3$ that if K immerses H, either $K_1 \succeq H$ or $K_2 \succeq H$. Also it follows from the edge-connectivity of K that K immerses both K_1 and K_2. So, $G \succeq H$ if and only if $G \setminus \delta_K(X) + x_1x_2 + y_1y_2 \succeq H$. □

Observation 4.7. *Let H be a graph with $\lambda^i(H) \geq 4$. Suppose K is a 3-edge-connected component of a graph G, and $\delta_K(X)$ is a 3-edge-cut in K. Then G immerses H if and only if the graph obtained from G by replacing K with the disjoint union of $K.X$ and $K.(V(K) \setminus X)$ immerses H.*

Proof. Let $K_1 = K.X$, and $K_2 = K.(V(K) \setminus X)$. Note that $\lambda^i(H) \geq 4$ implies that if $K \succ H$, one of the graphs K_1 or K_2 immerse H. On the other hand, it follows from the edge-connectivity of K that $K \succ K_1, K_2$. So, G immerses H if and only if the graph obtained from G by replacing K with disjoint union of K_1 and K_2 immerses H. □

The observation below shows that for many graphs H, characterizing the graphs without an immersion of H boils down to sausage reduced graphs.

Observation 4.8. *Let H be a simple graph with $\delta(H) \geq 3$. Then G immerses H if and only if G after sausage reduction immerses H.*

Proof. Suppose $G[X]$ is a chain of sausages of order three in G, for some $X \subset V(G)$. Let G' be the graph obtained from G by completely splitting a vertex in X. Since $G \succ G'$, it is clear that $G' \succ H$ implies $G \succ H$. Now, suppose $G \succ H$, and we will show that $G' \succ H$. Note that it follows from $\delta(H) \geq 3$ and H being simple that X contains at most two terminals of H. So, in an immersion of H at least one vertex in X gets completely split. By symmetry, we may assume the split vertex is any of the vertices in X. Now, if G has a chain of sausages $G[Z]$ of order ≥ 4, by repeatedly applying this argument we can decrease the order of the chain of sausages one at a time, while preserving the immersion of H, until the resulting graph is sausage reduced. □

4.3.2 Tree-width; Branch-width

We start by giving the definitions of tree-width, and branch-width, and some variants of them. A *tree-decomposition* of a graph G is a pair $(T, (W_t : t \in V(T)))$ such that the followings hold:

- T is a tree
- $W_t \subseteq V(G)$ for each $t \in V(T)$
- $V(G) = \bigcup(W_t : t \in V(T))$
- for every edge uv of G, there exists $t \in V(T)$ with $u, v \in W_t$
- for $t_1, t_2, t_3 \in V(T)$, if t_2 is in the path of T between t_1, t_3, then $W_{t_1} \cap W_{t_3} \subseteq W_{t_2}$.

The $\max\{|W_t| - 1 : t \in V(T)\}$ is called the *width* of the tree-decomposition. We call a tree-decomposition a *path-decomposition* if the tree T is a path. A graph G is said to have *tree-width* (*path-width*) k, if k is the minimum such that G has a tree-decomposition (path-decomposition) of width k. The tree-width of G will be denoted by $tw(G)$, and the path-width of G by $pw(G)$, respectively.

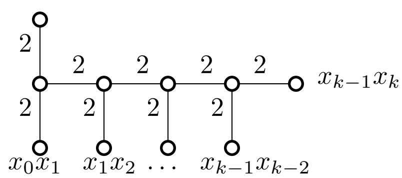
A *branch-decomposition* of a graph G is a cubic tree (a tree where every vertex either has degree three or one) T together with an injective mapping from $E(G)$ to leaves of T. Let $e \in E(T)$ be an edge, and let T_e, T'_e be the two subtrees of $T \setminus e$. Let F, F' correspond to the subset of edges of G which are a leaf in T_e, T'_e, respectively. Assign to e a label $w(e)$ which is the number of vertices shared by F, F'. The *width* of the decomposition is $\max\{w(e) : e \in E(T)\}$. In the spacial case where the graph obtained from T by deleting leaf vertices is a path, the branch-decomposition is called *caterpillar-decomposition*. The *branch-width* (*caterpillar-width*) of G is defined to be the minimum k such that G has a branch-decomposition (caterpillar-decomposition) of width k. The branch-width and caterpillar-width of G is denoted by $bw(G), cw(G)$, respectively.

Observation 4.9. *Let G be a graph, and H a subdivision of G. Then*

- *if $tw(G) \geq 2$, we have $tw(G) = tw(H)$.*
- *If $bw(G) \geq 2$, we have $bw(G) = bw(H)$*

Proof. Let $x = x_0, x_1, \ldots, x_k = y$ be the path in H which corresponds to an edge $e = xy \in E(G)$. Clearly we have $tw(H) \geq tw(G)$ and $bw(H) \geq bw(G)$. First, we will show $tw(H) \leq tw(G)$. For a tree decomposition $(T, (W_t : t \in V(T)))$ of G, there is a vertex $t \in V(T)$ such that $\{x, y\} \subseteq W_t$. Modify T by appending a path $t, t_1, \ldots, t_{k-1}$ to t, where the set associated with each t_i is $\{x, x_{k-i+1}, x_{k-i}\}$, for $i = 1, \ldots, k-1$. Since $|\{x, x_{k-i+1}, x_{k-i}\}| = 3$, the width of the new tree-decomposition does not exceed $tw(G)$.

Next, we will show $bw(H) \leq bw(G)$. Consider a branch-decomposition T of H, and replace the leaf in T which corresponds to xy with

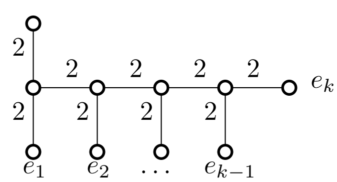


□

Observation 4.10. *Let G be a graph, and H the underlying simple graph of G. Then*

- *$tw(G) = tw(H)$*
- *$bw(G) = bw(H)$*

Proof. Clearly, we have $tw(G) = tw(H)$, and $bw(G) \geq bw(H)$. To verify $bw(G) \leq bw(H)$, consider a branch-decomposition T of H. If $e_1, \ldots, e_k$ in $E(G)$ are copies of $e \in E(H)$, replace the leaf corresponding to e in T with



□

The following is an immediate corollary of Observations 4.9 and 4.10.

Corollary 4.11. *Let H be a graph. If G is a graph which is obtained from H by sausage reducing it, then $tw(G) = tw(H)$ and $bw(G) = bw(H)$.*

4.4 Graphs with arbitrary edge-connectivity without a W_4 immersion

In this section, we will use Theorem 4.5 to identify graphs with arbitrary edge-connectivity which do not immerse W_4. Since W_4 is 3-edge-connected, and internally 4-edge-connected, we can use Observations 4.6, 4.7 to get the following as an immediate corollary of Theorem 4.5.

Corollary 4.12. *A graph G has no W_4 immersion if and only if G can be reduced to a graph where every component is either*

1. *of order at most four, or*
2. *is one of the types in the statement of Theorem* 4.5

by the operations

- *If e is a cut-edge in G, then modify G to $G \setminus e$.*
- *If H is a 2-edge-connected component of G, and $X \subset V(H)$ has $\delta(X) = \{x_1y_1, x_2y_2\}$, where $x_1, x_2 \in X$ then modify G to $G \setminus \delta(X) + x_1x_2 + y_1y_2$.*
- *If H is a 3-edge-connected component of G, and and $X \subset V(H)$ exists such that $\delta(X)$ is an internal 3-edge-cut in H, then replace H with the disjoint union of $H.X$ and $H.(V(H) \setminus X)$.*

We also get the following corollary of Theorem 4.5, which is a significant strengthening of previously known result about immersion of W_4 (Theorem 1.6).

Corollary 4.13. *Let G be a graph which does not immerse W_4. Then G can be constructed from i-edge-sums, for $i = 1, 2, 3$ from cubic graphs and graphs with path-width (caterpillar-width) at most* 3*.*

Proof. Thanks to Observations 4.9, 4.10 it suffices to note that $K_{2,3}$ and any graph on at most four vertices has path-width (caterpillar-width) at most three. □

Chapter 5

Immersion of Prism

In this chapter we study the problem of Prism immersion in graphs, and find a precise description of those who do not contain Prism as immersion. As Observation 4.8 implies, inserting long chain of sausages does not change the presence of a Prism immersion in a graph. However, as with the case of W_4, once the long chain of sausages are reduced, every graph with "the right edge-connectivity" and "enough vertices" immerses Prism. In the case of Prism, "enough number of vertices" turns out to be quite small—with only one exception, it is seven!

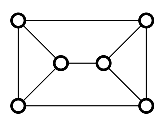


Figure 5.1: Prism graph

In preparation for our theorem, we introduce four families of graphs which do not immerse Prism. Consider the graph $K_{2,3}$ with bipartition $(\{u_1, u_2\}, \{v_1, v_2, v_3\})$. Let $J_{2,3}$ be the graph obtained from $K_{2,3}$ by adding a second copy of every existing edge except for u_2v_1 and u_2v_2. We declare v_3 to be the root of $J_{2,3}$. Also, in the description below, we let C_4^2 denote the doubled cycle on four vertices, with two opposite vertices being the roots. Finally, Consider K_4 on the vertex set $\{u_1, u_2, v_1, v_2\}$. Let J_4 be the graph obtained from subdividing u_1u_2 with a new vertex s, and adding a second copy of v_1v_2, and each edge incident with s. Declare J_4 to be rooted at s. Let G be a graph. Then we say

Type 1. G is *type* 1 if there exists a set $W \subset V(G)$ with $|W| = 2$ so that one of the following holds:

- The graph $G.W$, with root W, is isomorphic with $J_{2,3}$, see the structure on the left in Figure 5.2.

- There exist a set $W' \subset V(G)$ disjoint from W such that $|W'| = 2$ and $G.\{W, W'\}$, with roots W, W' is isomorphic to C_4^2, see the structure on the right in Figure 5.2.

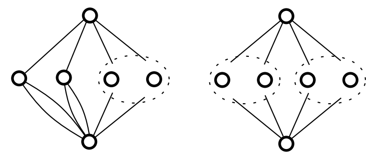


Figure 5.2: Graphs of type 1

Type 2. G is *type* 2 if there exists a set $W \subset V(G)$ with $|W| = 2$ so that the graph $G.W$ with root W after sausage reduction is isomorphic with J_4, see the structure below.

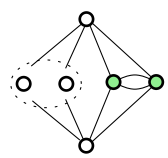


Figure 5.3: Graphs of type 2 after sausage reduction

Type 3. G is *type* 3 if G after sausage reduction is isomorphic to one of the graphs in Figure 5.4. That is G is type 3 if it can be obtained from a graph in Figure 5.4 by replacing any pair of same-colored (not white) vertices with a chain of sausages of arbitrary order ≥ 2.

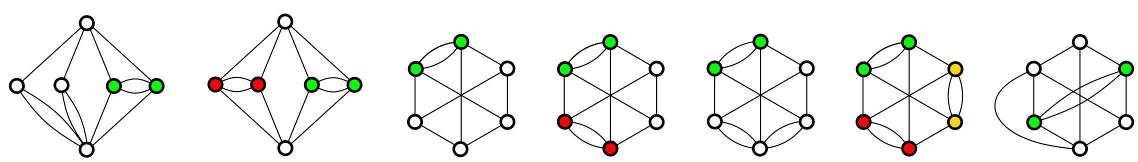


Figure 5.4: Graphs of type 3 after sausage reduction

Type 4. G is *type* 4 if it is isomorphic to one of the eight graphs in Figure 5.5.

Type 5. G is *type* 5 if it is obtained from a graph in Figure 5.6, by replacing each pair of same-colored vertices by a chain of sausages of order at least two. In the leftmost graph, W is a subset of vertices of G with $|W| \leq 3$.

We can now state our result on Prism immersion as follows:

Theorem 5.1. *Let G be a 3-edge-connected graph with $|V(G)| \geq 6$ so that for every $X \subset V(G)$ with $|X| = 2$ we have $d(X) \geq 4$. Then G has a Prism immersion if and only if G does not have one of the types* 1, 2, 3, 4, *or* 5.

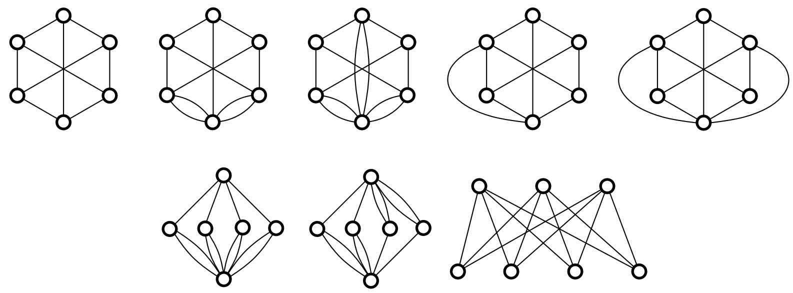


Figure 5.5: Graphs of type 4

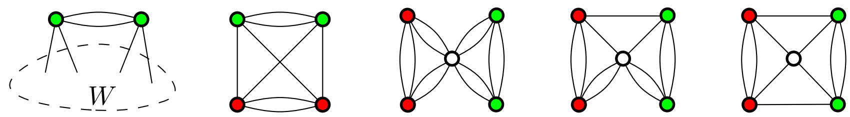


Figure 5.6: Graphs of type 5

Observe that the theorem above implies that under the assumptions of the theorem, if a graph G is sausage reduced, and has more than six vertices, then it either immerses Prism or it is isomorphic to $K_{3,4}$. So, we get Theorem 1.8 as a corollary.

In Section 5.1 we revisit our chain result on 3-edge-connected, internally 4-edge-connected graphs to get a new tool for the proof of our main theorem. Section 5.2 is devoted to the proof of Theorem 5.1. The proof, as with the proof of rooted W_4 is computer-assisted. We will also call on a result established in Section 5.1, as well as our result from Chapter 3 on immersion of K_4 with one root.

5.1 A corollary of Theorem 2.17

Recall that in Chapter 2 we established the following corollary, which we have restated below for convenience.

Corollary 5.2. *Let G be a graph with $\lambda(G) \geq 3, \lambda^i(G) \geq 4$, and $|V(G)| \geq 2$. If $G \not\cong Q_3, K_2^3$, there exists an operation taking G to G' such that $\lambda(G') \geq 3, \lambda^i(G') \geq 4$, where an operation is either*

- *deleting an edge,*
- *splitting at a vertex of degree ≥ 4,*

each followed by iteratively deleting any loops, and suppressing vertices of degree 2.

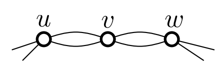
The Corollary above can be improved upon if we equip G with the additional assumption that G does not have a four-vertex u with $|N(u)| \leq 3$ (where $N(u)$ is the set of neighbours of u). This assumption may not look so natural to consider at first, and thus Lemma 5.3 may look like a bit unmotivated to the reader at this point. However, as we will see later in Section 5.2 in the heart of the proof of Theorem 5.1, the following lemma is indeed helpful.

Lemma 5.3. *Under the assumptions of Corollary* 5.2*, suppose further that*

($*$) *There does not exist a vertex u of degree four in G for which $|N(u)| \leq 3$.*

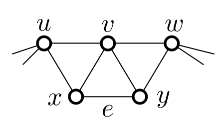
Then one of the operations in the statement of Corollary 5.2 *may be applied to G, so that the resulting graph G' is sausage reduced, and also $\lambda(G') \geq 3, \lambda^i(G') \geq 4$.*

Proof. By Corollary 5.2, we know that one of the operations in the statement of Corollary 5.2 may be applied to G, so that the resulting graph G' has $\lambda(G') \geq 3$ and $\lambda^i(G') \geq 4$. If G' is sausage reduced we have nothing left to prove. So, we may assume that G' has a chain of sausages $G'[X]$ of order three. As shown in the figure below we will assume that $X = \{u, v, w\}$, where u and w are nonadjacent. Note that it follows from ($*$) that G itself



is sausage reduced. So it must be the case that the chain of sausages $G'[X]$ is created only after performing one of the operations in Corollary 5.2. We can assume that $|X| = 3$. Let o be the operation taking G to G'. First, suppose that o is a complete split at (a vertex of degree four) x. In this case $d_G(u) = d_G(w) = 4$, so it follows from ($*$) that exactly one copy of either uv, vw is created after o. This implies that $e_G(x, v) = 2$, which together with $d_G(x) = 4$ contradicts ($*$). Next, suppose that o is a split at a vertex of degree at least five. Here, the implication of ($*$) for u, w is that one copy of, say, uv is created by doing a split at w. Then, however, ($*$) is violated for v.

In the remaining case, o is deletion of an edge e. Suppose $e_G(u, v) \geq 2$. Then ($*$) implies that e is not a uv edge. So either $d_G(u) = d_{G'}(u) = 4$ or $d_G(v) = d_{G'}(v) = 4$ and both cases contradict ($*$). Thus $e_G(u, v), e_G(v, w) \leq 1$. It follows that G may be obtained from G' by subdividing uv with a new vertex x, subdividing vw with a new vertex y, and then either identifying x, y or adding xy edge. In the former case, v violates ($*$), so the later case occurs, i.e. G is as in the figure below. We will see that we could perform an alternative



operation at G to get G'', where G'' is sausage reduced, and $\lambda(G'') \geq 3, \lambda^i(G'') \geq 4$.

Here, instead of o we can delete xv from G to get G''. Note that as we saw above, deletion of an edge which is incident with exactly one three-vertex does not create a chain of sausages of order more than two. So, G'' is sausage reduced. Clearly $\lambda(G'') \geq 3$. We will show that $\lambda^i(G'') \geq 4$. Else, suppose an internal 4-edge-cut $\delta_G(X)$ exists that separates x, v.

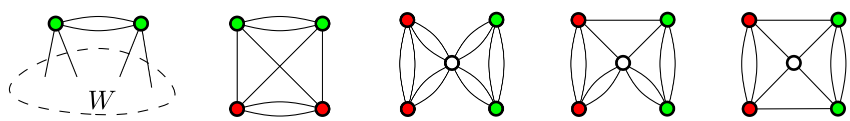
Without loss of generality, we may assume $x \in X$. Observe that the graph obtained from G'' by identifying y and v is isomorphic to G', which is internally 4-edge-connected. So, y must be in X. If $X = \{x, y\}$, there is nothing to verify. Otherwise, it follows from edge-connectivity of G and $e_G(\{x, y\}, \{v\}) = 2$ that $G[X]$ is connected. So $\{u, w\} \cap X \neq \emptyset$, say $u \in X$. Then since $\delta_{G''}(X)$ is an internal edge-cut, we get $w \notin X$. This, however, implies $d_G(X \cup v) = 2$—a contradiction. So, G'' is internally 4-edge-connected, as desired. This completes the proof of Lemma 5.3. □

5.2 Proof of Theorem 5.1

Before starting the proof of Theorem 5.1, we will see that the graphs which sausage reduction reduces them to less than six vertices are exactly the ones appearing as type 5 graphs in this chapter. As the following lemma will get called upon in other chapters as well, we have included the type 5 definition in the statement for convenience.

Lemma 5.4. *Suppose G is 3-edge-connected with $|V(G)| \geq 6$. Let G' be the graph obtained from G by sausage reducing it. If $|V(G')| < 6$, then G is type 5.*

That is, G is obtained from a graph in the picture below, by replacing each pair of same-colored vertices by a chain of sausages of order at least three. In the leftmost graph, W is a subset of vertices of G with $|W| \leq 3$.



Proof. First, suppose there is only one maximal chain of sausages in G getting reduced. Since any maximal chain of sausages gets reduced to exactly two vertices, if $|V(G')| \leq 5$, it must be that there are at most three more vertices in G', hence G' is as in the leftmost graph above. So, suppose there are at least two disjoint maximal chain of sausages in G which got reduced in G'. Since $|V(G')| \leq 5$, there must be exactly two disjoint maximal chain of sausages in G of order ≥ 3. Let x_1, x_2 (y_1, y_2) be the vertices in G' resulting from reducing the first (second) maximal chain of sausages. If $V(G') = \{x_1, x_2, y_1, y_2\}$, then since the original chain of sausages were maximal, we have $\{y_1, y_2\} \subset N(x_1) \cap N(x_2)$, that is G' is as the second left graph in the above picture.

So, suppose $|V(G')| = 5$, and let z be the vertex in G which is not in any chain of sausages. Since every vertex other than z has degree four, parity implies that $d_{G'}(z)$ is also even, and by edge-connectivity we have $d_{G'}(z) \geq 3$. Therefore $d_{G'}(z) \in \{4, 6, 8\}$. If $d_{G'}(z) = 8$, since $d_{G'}(\{x_1, x_2\}) + d_{G'}(\{y_1, y_2\}) = 8$ we have $\delta_{G'}(z) = \delta_{G'}(\{x_1, x_2\}) \cup \delta_{G'}(\{y_1, y_2\})$—that is G' is as in the middle graph above. If $d_{G'}(z) = 6$, then $e(\{x_1, x_2\}, \{y_1, y_2\})$ must equal one (note that by definition the graph induced on a chain of sausages is a doubled path). Thus, in this case G' is as in the second right graph above. Finally, suppose $d_{G'}(z) = 4$. Since z is not part of the chain of sausages that was replaced by $\{x_1, x_2\}$ we have $e(z, x_1), e(z, x_2) \leq 1$. Similarly, $e(z, y_1), e(z, y_2) \leq 1$, and since $d_{G'}(z) = 4$, we have $e(z, x_i) = e(z, y_i) = 1$, for $i = 1, 2$. That is, G' is as in the rightmost graph above. □

We are now prepared to prove our main theorem on Prism immersion.

Proof of Theorem 5.1. Assume (for a contradiction) that $G = (V, E)$ is a counterexample to the theorem for which $|V| + |E|$ is minimum. We will establish a sequence of properties of G.

(1) $\lambda^i(G) \geq 4$.

Suppose (for a contradiction) there exists an edge-cut $\delta(X)$ which violates (1). It then follows from the assumptions that $|X|, |V \setminus X| \geq 3$. We first claim that $G' = G.(V \setminus X) \succeq_r K_4$. Let y be the vertex in G' obtained from identifying $V \setminus X$, and note that $d_{G'}(y) = 3$, and $|V(G')| \geq 4$. We now apply Lemma 3.11 to G', rooted at y. If G' does not have a rooted immersion of K_4, there would be a 3-edge-cut in G' with exactly two vertices of $V(G') \setminus \{y\}$ on a side. Given that $V(G') \setminus \{y\} \subset V(G)$, the existence of such an edge-cut contradicts the assumptions of Theorem 5.1.

A similar argument shows that $G.X$ has a rooted immersion of K_4 as well. It is now straightforward to see that linking together the set of paths which give an immersion of rooted K_4 in $G.(V \setminus X)$ and in $G.X$ give an immersion of Prism in G—a contradiction.

(2) G is sausage reduced.

Suppose not, and let G' be the graph obtained from G by sausage reducing it. By Observation 4.8, G' does not immerse Prism. It follows from (1) that G' is 3-edge-connected, and internally 4-edge-connected. Since $|V(G')| < |V(G)|$, G' satisfies the theorem. If $|V(G')| < 6$, by Lemma 5.4, G is type 5, a contradiction. So, we must have $|V(G')| \geq 6$. Since G has a chain of sausages of order ≥ 3, G' has at least one pair of neighbours each of degree four with two edges between them. So G' either has one of the structures in Figures 5.2, 5.3 or it is isomorphic to a graph in Figure 5.4. It is easy to check that in any case, G has type 2, or 3, a contradiction.

(3) $|V(G)| \geq 10$.

Suppose for a graph G we have $6 \leq |V(G)| \leq 9$. Further suppose that G is 3-edge-connected, internally 4-edge-connected, and does not have any chain of sausages of order more than two. We will show that G satisfies Theorem 5.1, i.e. if G does not immerse Prism, it is either type 1, or has the structure in Figure 5.3, or it is isomorphic to one of the graphs in Figures 5.4 or 5.5. By Observation 4.4, to verify (3) it suffices to check the finitely many 3-edge-connected graphs with edge-multiplicity at most nine satisfying (1) and (2). This calculation is done in Sagemath, with the code appearing in the Appendix. Here is a high-level description of the algorithm.

Let $6 \leq n \leq 9$.

Step 1. We take the list of connected simple graphs on n vertices, and filter out the ones which immerse Prism.

Step 2. For any graph G surviving from Step 1, `repair(G)` generates a list consisting of all edge-minimal multigraphs G' such that:

- the underlying simple graph of G' is G,
- $\delta(G') \geq 3$,
- G' is internally 4-edge-connected,
- G' does not have chain of sausages of order more than two,
- G' does not immerse Prism.

Step 3. Suppose the simple connected graph G is such that `repair(G)` is nonempty. Let $\mathcal{G}_1 =$ `repair(G)`. Then, using $\mathcal{G}_1$, we generate $\mathcal{G}_2 =$ `obstruction(G)` which is the list consisting of all multigraphs whose underlying simple graph is G, meet the edge-connectivity conditions that the graphs in $\mathcal{G}_1$ satisfy, have edge-multiplicity at most nine, and do not immerse Prism.

Step 4. Every graph in $\mathcal{G}_2$ is tested if it has type 1, or has the structure in Figure 5.3, or it is isomorphic to one of the graphs in Figures 5.4 or 5.5.

The calculation is done rather fast. It took a desktop computer 2 minutes to do the calculation for every $n \in \{6, 7, 8\}$. However, the time spent on $n = 9$ was more. It took the computer 6 minutes to carry out step 1, i.e. to check the nearly 262,000 connected simple graphs on nine vertices for a Prism immersion, thereby giving a list `N9` of almost 24,800 simple connected graphs on nine vertices without a Prism immersion. Then 12 minutes was spent on carrying out steps 2, 3 for every graph in `N9`. Since no obstruction is found for $n = 9$, step 4 is not performed for this case.

(4) There does not exist $u \in V$ which has a neighbour v such that $e(u, v) \geq \frac{1}{2}d(u)$.

Suppose for a contradiction that such u, v exist, and let $G' = G.\{u, v\}$. Note $|V(G')| = |V(G)| - 1 \geq 9$, and since $e(u, v) \geq \frac{1}{2}d(u)$, we have $G \succ G'$. Therefore $G' \not\succeq$ Prism , and Theorem 5.1 holds for G'. Note, however, that since G is sausage reduced, so is G', and thus G' immerses Prism—a contradiction.

With the help of items (1) - (4), and a couple results established earlier, we will now complete the proof of Theorem 5.1. Since $|V(G)| \geq 10$, we have $G \not\cong Q_3, K_2^3$. On the other hand, (4) implies that every degree four vertex of G has four distinct neighbours. So, by Lemma 5.3, one of the operations in the statement of Corollary 5.2 may be performed to G to get G' where G' is 3-edge-connected, internally 4-edge-connected, and is sausage reduced. Since $|V(G')| + |E(G')| < |V| + |E|$, Theorem 5.1 holds for G'. On the other hand,

$|V(G')| \geq |V(G)| - 2 \geq 10 - 2 = 8$. Since all graphs of type 1, 2, 3, or 4 after sausage reduction have at most seven vertices, G' is not one of the types. So, G' immerses Prism, and thus G immerses Prism—a contradiction which completes the proof of Theorem 5.1. $\square$

As our last task in this chapter, we use Theorem 5.1 to characterize graphs with arbitrary edge-connectivity excluding an immersion of Prism. Since Prism is 3-edge-connected, we can use Observation 4.6 for the case where H is the Prism graph to deduce how we can break a graph into 3-edge-connected pieces via operations mentioned in Observation 4.6. Moreover, suppose G is 3-edge-connected which has a 3-edge-cut $\delta(X)$ with $|X| = 2$. Observe that G immerses Prism if and only if $G.X$ does (by edge-connectivity of G we have $G \succ G.X$). From all this, we get the following as an immediate corollary of Theorem 5.1:

Corollary 5.5. *A loopless graph G does not immerse the Prism graph if and only if G can be reduced to a graph where every component is either*

1. *of order at most five, or*

2. *is one of the types* 1, 2, 3, 4, *or* 5.

by the operations

- *If e is a cut-edge in G, then modify G to $G \setminus e$.*

- *If H is* 2*-edge-connected component of G, and $X \subset V(H)$ has $\delta(X) = \{x_1y_1, x_2y_2\}$, where $x_1, x_2 \in X$ then modify G to $G \setminus \delta(X) + x_1x_2 + y_1y_2$.*

- *If H is a* 3*-edge-connected component of G and $\delta(X)$ is a* 3*-edge-cut in H with $|X| = 2$, then replace H by $H.X$.*

Chapter 6

Immersion of Eyeglasses

Generally speaking, precise structural theorems for rooted graphs are extremely useful tools in finding immersion of bigger (rooted) graphs. They enable us to break the bigger graph into smaller rooted pieces whose 'tying together in the right way' give an immersion of the desired bigger graph. This idea was instantiated in Chapter 4 where we saw how 'tying' D_2, D_4 together gives a rooted immersion of W_4.

In light of this, and originally motivated by the study of $K_{3,3}$ immersion, in this chapter we study the immersion of a rooted graph which we refer to as *Eyeglasses*. An Eyeglasses is a graph obtained from a path of length three in which the edges incident with the two ends of the path are doubled, and the two ends of the paths are the roots of the graph, see Figure 6.1.

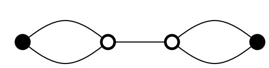


Figure 6.1: Eyeglasses

As with our results on D_m immersion, the structural theorem for Eyeglasses will also prove to be quite useful in finding immersion of certain bigger graphs. In this thesis, we will see an application of this only in Chapter 7, where an Eyeglasses 'tied in the right way' with rooted W_4 helps in finding an immersion of $K_{3,3}$. We expect that this result has applications elsewhere as well.

Our sole task in this chapter is to establish a structural theorems for graphs with two degree-two roots which exclude Eyeglasses as an immersion. Let us begin by introducing a convenient definition, and then three classes of graphs which do not immerse Eyeglasses.

Definition 6.1. We say a rooted graph H with root x is *mostly cubic* if for every vertex $v \in V(H) \setminus x$ we have $d(v) = 3$, and $d(x) = 4$.

Now, suppose $G = (V, E)$ has two root vertices x_0, x_1, where $d(x_0) = d(x_1) = 2$.

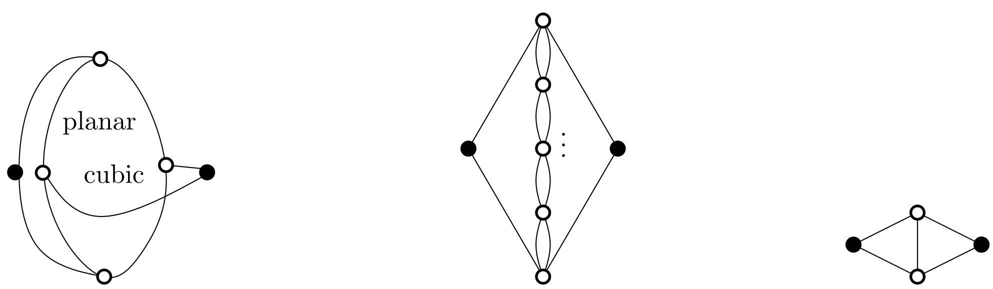
Type 1 G is *type* 1 if

- $G.\{x_0, x_1\}$ with root $\{x_0, x_1\}$ is planar, and mostly cubic.
- There are four distinct vertices $s_0, s_1, t_0, t_1 \in V \setminus \{x_0, x_1\}$ such that $N(x_0) = \{s_0, t_0\}, N(x_1) = \{s_1, t_1\}$,
- There is a planar embedding of $G \setminus \{x_0, x_1\}$ in which s_0, s_1, t_0, t_1 appear on the boundary of the outer face, in this cyclic order (see Figure 6.2a).

Type 2 G is *type* 2 if there exist two distinct vertices $s, t \in V \setminus \{x_0, x_1\}$ such that

- $N(x_0) = N(x_1) = \{s, t\}$,
- $G \setminus \{x_0, x_1\}$ is isomorphic to an $s - t$-doubled path (see Figure 6.2b).

Type 3 G is *type* 3 if it is isomorphic to the graph in Figure 6.2c (also known as D_2).



(a) Type 1 (b) Type 2 (c) Type 3

Figure 6.2: Obstructions to an immersion of Eyeglasses

Theorem 6.2. *Let G be a graph with $|V(G)| \geq 4$ and two root vertices x_0, x_1, where $d(x_0) = d(x_1) = 2$. Assume further that $\lambda_s(G) \geq 2$, $\lambda_n(G) \geq 3$, and $\lambda_n^i(G) \geq 4$. Then G has a rooted immersion of Eyeglasses if and only if G is not of one of the types $1, 2$, or 3.*

In our proof of the above theorem we call on a well-known problem, appearing in the literature as 2-linkage problem [45], or two-disjoint paths problem [44]. Let (s_0, t_0, s_1, t_1) be an ordered set of distinct vertices of a simple graph G. If G has two vertex-disjoint paths P_0, P_1 so that P_i is an $s_i - t_i$-path, for $i = 0, 1$, we say G has a 2-linkage. The following characterization of graphs without a 2-linkage follows from Theorem 1 in [45]:

Theorem 6.3 (Thomassen [45]). *Let (s_0, t_0, s_1, t_1) be an ordered set of distinct vertices of a graph G, let $T = \{s_0, t_0, s_1, t_1\}$. Suppose that for any set $X \subset V(G) \setminus T$ with $|X| = 3$, $G \setminus X$ has at most one component disjoint from T, and if such a component exists, it is trivial. Then either of the following holds:*

- *G has a 2-linkage, or*

- *G has a planar embedding in which s_0, s_1, t_0, t_1 appear on the boundary of the infinite face in this cyclic order.*

We are now well-prepared to prove the main result of this chapter.

Proof of Theorem 6.2. It is straightforward to see that the graphs of type 1, 2 and 3 do not have an immersion of Eyeglasses. For the reverse direction, suppose (for a contradiction) that $G = (V, E)$ is a counterexample to the theorem. We will establish a sequence of properties of G, eventually proving that it does not exist. For the sake of exposition, if there is an immersion of Eyeglasses on $\{x_0, v_0, v_1, x_1\}$ in which there is an immersion of two edges between x_i, v_i, for $i = 0, 1$ and a v_0v_1 edge, we write there is an immersion of Eyeglasses on (x_0, v_0, v_1, x_1).

First we will show that every vertex in G other than x_0, x_1 has degree three. For the items (1)- (8) assume (for a contradiction) that G has a vertex v with $d(v) \geq 4$. Then we may choose four edge-disjoint paths P_1, P_2, P_3, P_4 starting at v, where two of these paths end at x_0 and the other two end at x_1. (To see this, add a new vertex s which has two edges to each x_i, for $i = 0, 1$, and then apply Menger's Theorem to get four edge-disjoint $v - s$ paths.) Suppose P_1, P_2 end at x_0 and P_3, P_4 end at x_1. Let V_i denote the internal vertices of P_i, and $H = G \setminus \bigcup_i E(P_i)$. Note since $d_G(x_0) = d_G(x_1) = 2$, we have $x_0, x_1 \notin \bigcup_i V_i$, and $d_H(x_0) = d_H(x_1) = 0$.

(1) There does not exist a nontrivial path (containing at least one edge) in H between V_i and V_j, for $1 \leq i \neq j \leq 4$.

    Else, let Q be a nontrivial path in H from $x \in V_i$ to $y \in V_j$. If, say, $\{i, j\} = \{1, 2\}$, then $P_1 \cup \ldots \cup P_4 \cup Q$ has a rooted immersion of Eyeglasses on (x_0, x, v, x_1), a contradiction. Also, if $\{i, j\} = \{1, 3\}$, then $P_1 \cup \ldots \cup P_4 \cup Q$ gives a rooted immersion of Eyeglasses on (x_0, x, y, x_1)– again a contradiction.

(2) $V_1 \cap V_2 = \emptyset$.

    It is because if there existed $x \in V_1 \cap V_2$, then $P_1 \cup \ldots \cup P_4$ would have a rooted immersion of Eyeglasses on (x_0, x, v, x_1).

(3) If $V_1 \cap V_3$ is nonempty, then $V_1 \cap V_4$ is empty.

    Suppose (for a contradiction) that $x \in V_1 \cap V_3$ and $y \in V_1 \cap V_4$. Note by (2) we have $V_3 \cap V_4 = \emptyset$, and thus $x \neq y$. Without loss of generality, we may assume x appears before y on P_1. However, then $P_1 \cup \ldots \cup P_4$ has a rooted immersion of Eyeglasses on (x_0, v, x, x_1), a contradiction.

(4) If $V_1 \cap V_3$ is nonempty, then the order of appearing the vertices in $V_1 \cap V_3$ on P_1, P_3 is the same.

Otherwise, there exist $x, y \in V_1 \cap V_3$ so that x appears before y on P_1, but after y on P_3. Then $P_1 \cup \ldots \cup P_4$ would contain an Eyeglasses immersion on (x_0, y, x, x_1), a contradiction.

Let us pause to introduce a convenient notion.

Definition 6.4. Let G be a graph, and H a subgraph of G. We define an *edge-bridge* of H to be a nontrivial connected component of $G \setminus E(H)$. For B an edge-bridge of H, we also call a vertex in $V(B) \cap V(H)$ an *attachment* of B on H.

For the next four items, suppose B is an edge-bridge of $\bigcup_{i=1}^4 E(P_i)$.

(5) $v \notin V(B) \cap (\bigcup_i V(P_i))$.

For a contradiction, suppose B has an attachment to v. If B had attachments only at v there would exist three edge-disjoint paths Q_1, Q_2, Q_3 from a vertex w in $V(B) \setminus \bigcup_i E(P_i)$ to v. Then $\left(\bigcup_{i=1}^4 P_i\right) \cup \left(\bigcup_{j=1}^3 Q_j\right)$ would have a rooted immersion of Eyeglasses on (x_0, v, w, x_1), a contradiction. So, suppose B has attachments to v, and say $x \in V_1$. Then there would exist a path Q in B from v to x. Then $P_1 \cup \ldots \cup P_4 \cup Q$ gives an immersion of Eyeglasses on (x_0, x, v, x_1) – a contradiction.

(6) Suppose B has attachments to V_1 at two distinct vertices w, w'. If P_1' is the $w - w'$-subpath of P_1, then $V(P_1') \cap \left(\bigcup_{i=2}^4 V_i\right) = \emptyset$.

Suppose (for a contradiction) that there exists $x \in V(P_1') \cap \left(\bigcup_{i=2}^4 V_i\right)$, and let Q be a path in B from w to w'. Note that $x \notin \{w, w'\}$, otherwise Q would be a nontrivial path from V_1 to $V_2 \cup V_3 \cup V_4$, contradicting (1). Note (2) implies that $x \in V_3 \cup V_4$, say $x \in V_3$. Then $P_1 \cup \ldots \cup P_4 \cup Q$ has an immersion of Eyeglasses on (x_0, w, x, x_1), a contradiction.

(7) $|V(B) \cap V(P_1)| \leq 1$.

Suppose (for a contradiction) that $|V(B) \cap V(P_1)| \geq 2$. Let w, w' be the first and last attachments of B on P_1. By (6), there exist y, y' such that y is the last vertex before w which is in $V(P_1) \cap (V(P_3) \cup V(P_4))$, and y' is the first vertex after w' which is in $V(P_1) \cap (V(P_3) \cup V(P_4))$. Now let P_1' be the path obtained from the $y - y'$-subpath of P_1 by deleting y, y'. Let $\mathcal{B}$ denote all edge-bridges of $\bigcup_i E(P_i)$ that have some attachment in P_1'. It then follows from (6) together with (5) that $d\left(V(P_1') \cup (\bigcup_{B \in \mathcal{B}} V(B))\right) < 3$, contradicting $\lambda_n(G) \geq 3$.

(8) $B = \emptyset$, and thus $E(G) = \bigcup_i E(P_i)$.

Else, it follows from (7) that $|V(B) \cap V(P_1)| = 1$. Let $V(B) \cap V(P_1) = \{w\}$, where by (5) we have $w \in V_1$. Since G is loopless, there must exist $x \in V(B) \setminus V_1$. Therefore there are three edge-disjoint $x - w$-paths Q_1, Q_2, Q_3 in B. It follows from $d_G(B \cup \{w\}) \geq 3$ and

(1) that $w \in V_3 \cup V_4$. Then $\left(\bigcup_{i=1}^{4} P_i\right) \cup \left(\bigcup_{j=1}^{3} Q_j\right)$ has a rooted immersion of Eyeglasses on (x_0, x, w, x_1).

We are now ready to prove that G, except for x_0, x_1, is cubic. Else, it follows from (8) and edge-connectivity that every vertex on $V(P_i) \setminus \{v\}$ appears on at least two V_i's. Now (2) together with (3) and (4) imply that G has type 2– a contradiction.

So G except for x_0, x_1 is cubic, let $N(x_0) = \{s_0, t_0\}$ and $N(x_1) = \{s_1, t_1\}$. If, say, $s_0 = t_0$ then G immerses Eyeglasses. To see why take a vertex $y \in G \setminus \{x_0\}$, where $y \neq x_1$. It follows from $\lambda_s(G) \geq 2$ that there exist in G' three edge-disjoint paths, starting at y, two ending at x_1, and one ending at s_0. Therefore, G immerses Eyeglasses as claimed, a contradiction. If, say, $s_0 = s_1$ then since $d(s_0) = 3$, we have $d_G(V(G) \setminus \{x_0, x_1, s_0\}) = 3$. Therefore, by edge-connectivity of G we have $|V(G) \setminus \{x_0, x_1, s_0\}| = 1$, so G is isomorphic to the graph in Fig. 6.2c, a contradiction.

Therefore $|\{s_0, t_0, s_1, t_1\}| = 4$, and we will consider $G' = G \setminus \{x_0, x_1\}$. If there existed two vertex-disjoint paths P_0, P_1 in G', where P_0 is an $s_0 - t_0$ path, and P_1 is an $s_1 - t_1$ path then we would get a contradiction. It is because then connectivity of G implies that there exists a path Q from $V(P_1)$ to $V(P_2)$, which is enough for G to immerse Eyeglasses. So, suppose such paths do not exist. In other words, G' does not have a 2-linkage.

In order to be able to take advantage of Theorem 6.3 we need to verify one more technical condition. Let $T = \{s_0, s_1, t_0, t_1\}$. We need to verify that if for $X \subseteq V(G') \setminus T$ we have $|X| = 3$, then $G' \setminus X$ has at most one component not intersecting T, and if such a component exists, it is trivial. Let C be a component of $G' \setminus X$ so that $C \cap T = \emptyset$. We define $X' = \{x \in X : e(\{x\}, C) \geq 2\}$. Then $d_{G'}(X' \cup C) = 3$, so it follows from $\lambda_n^i(G) \geq 4$ that $|X' \cup C| \leq 1$, in particular, $|C| \leq 1$. Also, if $C' \neq C$ is a component of $G' \setminus X$ so that $C' \cap T = \emptyset$, we would have $d_G(C \cup C' \cup X) \leq 3$, contradicting $\lambda_n^i(G) \geq 4$.

So we can now apply Theorem 6.3 to deduce that the desired paths in G' exist unless if G' is planar, and has an embedding with s_0, s_1, t_0, t_1 appearing on the boundary of the outer face with this cyclic order. However, then G is type 1– a contradiction. $\square$

Chapter 7

Immersion of $K_{3,3}$

This chapter concerns immersion of the Kuratowski graphs, $K_{3,3}$ and K_5. It is not hard to see that the problem naturally reduces to the family of 3-edge-connected and internally 4-edge-connected graphs. Moreover, as we saw in Chapter 2, the problem on the this family of graphs essentially boils down to the problem of $K_{3,3}$ immersion, which is resolved in this chapter. The result on excluding Kuratowski graphs as immersion is in fact the analogue of Kuratowski-Wagner Theorem in the setting of graph immersions. Let us start by introducing four families of graphs which do not immerse $K_{3,3}$.

Type 0. G is *type* 0 if it is planar and cubic.

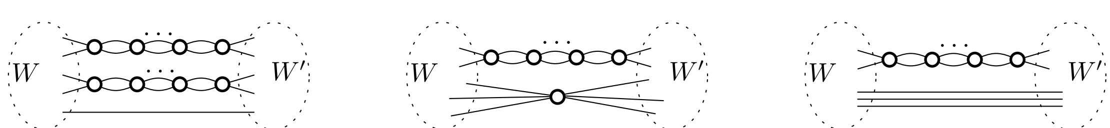
Type 1. G is *type* 1 if it has a $(3,3)$-segmentation of width four.

Type 2. G is *type* 2 if there exist disjoint sets $W, W' \subseteq V(G)$ with $|W|, |W'| \leq 2$ such that the graph G^* obtained by identifying W (W') to a single vertex w (w') has a doubled cycle C containing w, w' satisfying one of the following:

(2A) w and w' are not adjacent in C and $G^* = C + ww'$ (see Fig. 7.1a)

(2B) w and w' have a common neighbour v in C and $G^* = C + wv + vw'$ (see Fig. 7.1b)

(2C) w and w' are adjacent in C and $G^* = C + ww'$ (see Fig. 7.1c)



(a) Type $2A$ (b) Type $2B$ (c) Type $2C$

Figure 7.1: Type 2 graphs

Type 3. G is *type* 3 if after sausage reduction it is isomorphic to one of the 20 graphs in Figure 7.2. That is G is type 3 if it can be obtained from a graph in Figure 7.2 by replacing any pair of same-colored (not white) vertices with a chain of sausages of arbitrary order ≥ 2.

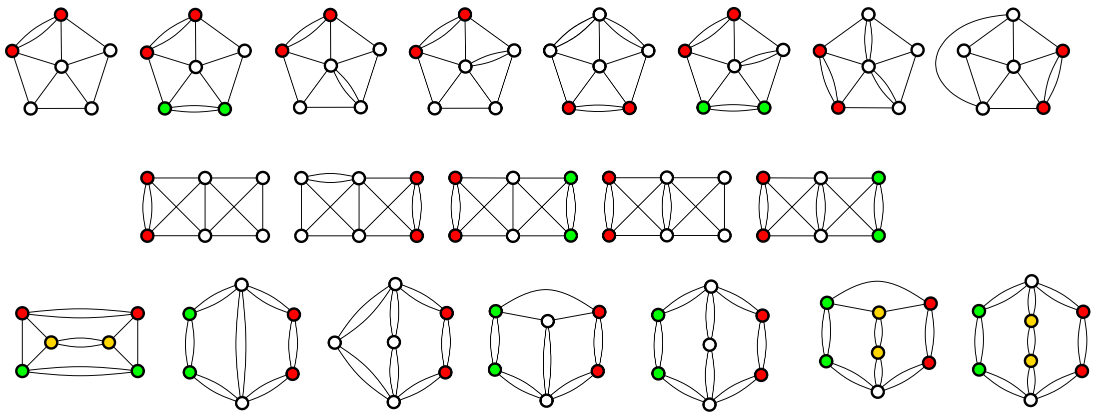


Figure 7.2: Graphs of Type 3 after sausage reduction

Type 4. G is *type* 4 if it is isomorphic to one of the 14 graphs in Figure 7.3.

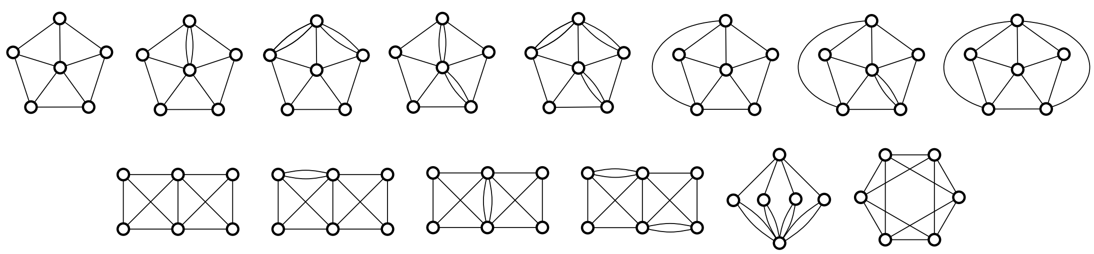


Figure 7.3: Type 4 graphs

With these definitions of types, we can now state our characterization of graphs with no $K_{3,3}$ immersion as follows:

Theorem 7.1. *Let G be a graph with $\lambda(G) \geq 3$, $\lambda^i(G) \geq 4$ and $|V(G)| \geq 6$. Then G does not immerse $K_{3,3}$ if and only if G is one of the types* 0, 1, 2, 3, *or* 4.

Suppose G has one of the types $0, 1, 2, 3$, or 4. Observe that if G is sausage reduced and $|V(G)| \geq 9$, then G is either type 0, or type 1. This shows that Theorem 1.10 is in fact a corollary of the above theorem. On another note, recall that in Chapter 2 we showed that

Theorem 7.2. *Suppose G is a graph with $|V(G)| \geq 6$ such that $\lambda(G) \geq 3$, $\lambda^i(G) \geq 4$. If $G \succ K_5$ we have $G \succ K_{3,3}$ unless G is isomorphic to the octahedron (the last graph in Figure 7.3).*

Therefore, as an immediate corollary of Theorems 7.1, 7.2 we have the following structural theorem identifying the graphs which exclude both Kuratowski graphs as immersion:

Corollary 7.3. *Let G be a graph with $|V(G)| \geq 6$, and $\lambda(G) \geq 3$, $\lambda^i(G) \geq 4$. Then G does not immerse $K_{3,3}$ or K_5 if and only if G is one of the types* 0, 1, 2, 3, *or* 4 *except for the octahedron.*

7.1 Helpful tools for proof of Theorem 7.1

Since our proof of Theorem 7.1 relies on our results on the structure of graphs excluding rooted W_4 and Eyeglasses, we are going to restate the parts of these results that are needed in this chapter for convenience. The following is part of our result on immersion of rooted W_4, established in Chapter 4:

Theorem 7.4. *Let G be a graph with $|V(G)| \geq 5$ and with a root vertex x, where $d(x) \in \{4, 5\}$. Suppose $\lambda(G) \geq 3, \lambda^i(G) \geq 4$. Then G contains a rooted immersion of W_4 if and only if G does not have one of the following types:*

Type I. *G is type I if it has a $(2, 3)$-segmentation of width four in which x is in the head of the segmentation.*

Type II. *G is type II if there exists a set $W \subseteq V(G) \setminus \{x\}$ with $|W| \leq 2$ such that the graph G^* obtained by identifying W to a single vertex w has a doubled cycle C satisfying one of the following:*

(II A) *x and w are not adjacent in C and $G^* = C + xw$*

(II B) *x and w have a common neighbour v in C and $G^* = C + xv + vw$*

(II C) *x and w are adjacent in C and $G^* = C + xw$. Moreover we have $|W| = 2$.*

It is worth noting that for type II graphs we have $d(x) = 5$. The following result on the immersion of Eyeglasses is proved in Chapter 6. Recall that we call a rooted graph H with root x is called *mostly cubic* if for every vertex $v \in V(H) \setminus x$ we have $d(v) = 3$, and $d(x) = 4$.

Theorem 7.5. *Let $G = (V, E)$ be a graph with $|V| \geq 5$ and two root vertices x_0, x_1, where $d(x_0) = d(x_1) = 2$. Assume further that $\lambda_s(G) \geq 2, \lambda_n(G) \geq 3$, and $\lambda_n^i(G) \geq 4$. Then G has a rooted immersion of Eyeglasses if and only if G does not have one of the following types:*

Type III *G is type III if*

- *$G.\{x_0, x_1\}$ with root $\{x_0, x_1\}$ is planar, and mostly cubic.*
- *There are four distinct vertices $s_0, s_1, t_0, t_1 \in V \setminus \{x_0, x_1\}$ such that $N(x_0) = \{s_0, t_0\}, N(x_1) = \{s_1, t_1\}$,*
- *There is a planar embedding of $G \setminus \{x_0, x_1\}$ in which s_0, s_1, t_0, t_1 appear on the boundary of the outer face, in this cyclic order.*

Type IV *G is type IV if there exist two distinct vertices $s, t \in V \setminus \{x_0, x_1\}$ such that*

- $N(x_0) = N(x_1) = \{s, t\}$,
- $G \setminus \{x_0, x_1\}$ *is isomorphic to an $s - t$-doubled path.*

Let us continue with an easy, yet helpful observation.

Observation 7.6. *Suppose G is a graph which immerses $K_{3,3}$. Moreover, suppose G has a segmentation of width four $U_0 \subset U_1 \subset U_2 \ldots \subset U_t$ with $|U_0| \leq 3$. If T is the set of terminals of an immersion of $K_{3,3}$ in G, then $|T \cap U_t| \leq 2$.*

Proof. Note since the edge-boundary of any triple of vertices of $K_{3,3}$ has size at least five, $|T \cap U_0| \leq 2$. Therefore, $|T \cap U_1| \leq |T \cap U_0| + |U_1 \setminus U_0| \leq 3$. However, since $d(U_1) = 4 < 5$, we have $|T \cap U_1| \leq 2$. By repeating this argument we conclude that $|T \cap U_t| \leq 2$. □

Lemma 7.7. *Suppose $G = (V, E)$ is a graph with $\lambda(G) \geq 3, \lambda^i(G) \geq 4$, where $G \not\succeq K_{3,3}$. Suppose further that $X \subset V$ exists such that*

- $|X|, |V \setminus X| \geq 3$,
- $d(X) = 4$, *and*
- $G.X$ *(with root X) has a rooted immersion of W_4.*

Then one of the following occurs:

1. *$G[X]$ is a chain of sausages, or*
2. *for every vertex $v \in X$ we have $d_G(v) = 3$, and $G.(V \setminus X)$ has a rooted immersion of W_4.*

Proof. Denote the root vertex of $G' = G.X$ by a (resulting from identification of X). Let the terminals of a rooted immersion of W_4 in G' be $\{a, v_1, v_2, v_3, v_4\}$, where there is an immersion of C_4 on $v_1v_2v_3v_4v_1$ in this cyclic order. Let P_{av_i} be the path in G' corresponding to the av_i edge of W_4, and let $e_i = E(P_{av_i}) \cap \delta_{G'}(a)$, for $1 \leq i \leq 4$. Now we define G'' to be the rooted graph obtained from G by subdividing e_1, e_3 (e_2, e_4) with a new vertex, and then identifying the two-degree vertices to a new vertex b (c). Let $G^* = G''[X \cup \{b, c\}]$ with roots b, c.

Observe that (since $G' \succeq_r W_4$) if there is a rooted immersion of Eyeglasses (Fig. 6.1) in G^*, then $G \succeq K_{3,3}$. Note that $|V(G^*)| \geq 5$, and it follows from edge-connectivity of G that G^* satisfies the hypothesis of Theorem 7.5. So we may conclude that G^* is either type III or IV. In the former case for every $v \in X$ we have $d_G(v) = 3$ and $G.(V \setminus X)$ has a rooted immersion of W_4; in the latter case $G[X]$ is a chain of sausages, as desired. □

7.2 Proof of Theorem 7.1

We now embark on the proof of Theorem 7.1. First, we see why type 0, 1, 2, 3 and 4 graphs do not immerse $K_{3,3}$. Suppose G is type 0. Since G is cubic, it has a $K_{3,3}$ immersion if and only if it has a $K_{3,3}$-subdivision. However, since G is planar, G does not have a $K_{3,3}$-subdivision, so $G \not\succeq K_{3,3}$. Now suppose G has type 1, relative to $U_0, \ldots, U_t$. If G has an immersion of $K_{3,3}$, let T be the set of terminals of $K_{3,3}$ in G. It follows from Observation 7.6 that $|T \cap U_t| \leq 2$. However, then since $|V \setminus U_t| \leq 3$, we get $|T| \leq 5$– a contradiction.

Next, suppose G is type 2 relative to W, W'. For a contradiction suppose $G \succ K_{3,3}$, and let T be terminals of an immersion of $K_{3,3}$ in G. Note it follows from Observation 7.6 that at most two vertices of a chain of sausages are in T. Then we may split all but two vertices of any chain of sausages while preserving a $K_{3,3}$ immersion. So, we may assume G is sausage reduced. Now, if a chain of sausages has two vertices u, v of T, Observation 3.5 implies that we may delete an edge incident to each u, v while preserving a $K_{3,3}$ immersion. The only way to do so without $d(\{u, v\})$ dropping to < 4 is to delete one copy of the edge uv. Call the resulting graph G'. However, G' is type 1, so does not immerse $K_{3,3}$, a contradiction. Therefore at most one terminal of $K_{3,3}$ lies on any chain of sausages between W, W'.

So, if we let G'' be the graph obtained from G by splitting off each chain of sausages down to only one vertex, then $G'' \succ K_{3,3}$. This immediately gives a contradiction in the cases where either G has type 2C, or $|W| = 1$, or $|W'| = 1$, as then $|V(G'')| \leq 5$. So, suppose G has type 2A or 2B, and $|W| = |W'| = 2$. So we have $|V(G'')| = 6$, and thus every vertex of G'' should be in T. Then again by Observation 3.5, we may delete an edge incident to each vertex not in $W \cup W'$, while preserving an immersion of $K_{3,3}$. However, then the resulting graph would have an internal 3-edge-cut, and thus cannot immerse $K_{3,3}$– a contradiction.

Finally, suppose G is type 3 or 4. Observe that using Observation 7.6, it only suffices to show that the graphs in Figure 7.2 and 7.3 do not immerse $K_{3,3}$. This verification is indeed straightforward enough to do by hand, although we also got the computer to do it (the computer code appears in the Appendix).

Lemma 7.8. *If $G = (V, E)$ is a counterexample to Theorem* 7.1*, then G is planar and is not cubic.*

Proof. First, suppose (for a contradiction) that G is nonplanar. Then by Kuratowski's Theorem, it has a subdivision of $K_{3,3}$ or K_5. Since G does not immerse $K_{3,3}$, it must have a subdivision, and thus an immersion of K_5. However, this contradicts Theorem 7.2.

Next, suppose (for a contradiction) that G is cubic. In this case, the edge-connectivity of G implies that G is simple. Then by Lemma 7.8, G is planar. However, then G would be type 0, a contradiction. □

For the reverse direction, suppose (for a contradiction) that $G = (V, E)$ is a counterexample to the theorem with minimum $|V| + |E|$. We will establish a sequence of properties of G, eventually proving that it does not exist.

7.2.1 4-edge-cuts

The main result of this subsection is Lemma 7.11 which essentially asserts that one side of every 4-edge-cut in a minimum counterexample to Theorem 7.1 has at most two vertices. Let us first make an easy observation.

Observation 7.9. *Let the rooted graph H with root u be type I relative to (U, W). If $d_H(u) = 4$, we may assume $U = \{u\}$.*

We now proceed by establishing the following lemma which is a helpful tool in proving Lemma 7.11.

Lemma 7.10. *Suppose $G = (V, E)$ is a counterexample to Theorem 7.1. Let $\delta(X)$ be a 4-edge-cut of G with $|X|, |V \setminus X| \geq 3$. Then either $G[X]$ or $G[V \setminus X]$ is a chain of sausages.*

Proof. If $|X| = |V \setminus X| = 3$, G has type 1, a contradiction. So without loss of generality, we may assume $|V \setminus X| \geq 4$. Let $G' = G.X$, with the root vertex a (resulting from identification of X). Note that $d_{G'}(a) = 4$ and $\lambda(G') \geq 3, \lambda^i(G') \geq 4$. First, suppose $G' \succeq_r W_4$, and apply Lemma 7.7. If $G[X]$ is a chain of sausages we are done. Else, $G.(V \setminus X)$ has a rooted immersion of W_4 and every vertex in X is of degree three. Now we replace X with $V \setminus X$ in the statement of Lemma 7.7. Note that Lemma 7.8 implies that $G[V \setminus X]$ must be a chain of sausages– as desired.

Next, suppose $G' \not\succeq_r W_4$. Since $d_{G'}(a) = 4$, it follows from Theorem 7.4 that G' has type I structure relative to some (U, W). Since $d_{G'}(a) = 4$, we may assume $U = \{a\}$. Thus, if $|X| \leq 3$ then G has type 1, a contradiction. Therefore, $|X| \geq 4$, and let $G'' = G.(V \setminus X)$ with the root vertex b. Observe that $G'' \succeq_r W_4$. Otherwise, since $d_{G''}(b) = 4$, G'' would have type I relative to some (U', W'). Then since we may assume $U = \{a\}, U' = \{b\}$, G would have type 1, a contradiction.

So, $G'' = G.(V \setminus X) \succeq_r W_4$. On the other hand, note that since $G' = G.X$ has type I and $|V(G')| \geq 5$, there are vertices of degree even (and thus not equal three) in $G[V \setminus X]$. So, Lemma 7.7 implies that $G[V \setminus X]$ is a chain of sausages, as desired. □

Lemma 7.11. *If $G = (V, E)$ is a counterexample to Theorem 7.1 with $|V| + |E|$ minimum, then every $X \subset V$ with $|X|, |V \setminus X| \geq 3$ satisfies $d(X) \geq 5$.*

Proof. Suppose (for a contradiction) that the lemma does not hold, and let G' be the graph obtained by sausage reducing G. By Lemma 7.10, $|V(G')| < |V(G)|$, so G' satisfies the theorem. If $G' \succ K_{3,3}$ then Observation 4.8 implies $G \succ K_{3,3}$, which is a contradiction. So either $|V(G')| < 6$, or G' has one of types 0, 1, 2, 3 or 4. If $|V(G')| < 6$, apply Lemma 5.4, and a straightforward check reveals that in this case G has type 1, a contradiction.

So, suppose $|V(G')| \geq 6$. Since sausage reduction has been applied to G nontrivially to obtain G', G' has at least one pair of neighbours each of degree four, with two edges between them. So, in particular, G' is not simple (so is not type 0), and is not type 4 either. So G' has one of the types 1, 2, or 3. However, Lemma 7.10 implies that G' is not type 1. Note that G' is not isomorphic to one of the 20 graphs in Figure 7.2 either, else G would be type 3. Suppose G' is type 2 relative to W, W'. Let $G'[Y]$ be a chain of sausages in G'. Since $d_{G'}(W) = 5$, we have $|W \cap Y|, |W' \cap Y| \leq 1$. It is now straightforward to see that if G' has type 2A or 2B, G also has the same type—a contradiction. In the remaining case, G' has type 2C. As before, if, say $|W \cap Y| = \emptyset$, we find that G also has type 2C—a contradiction. So, we must have $|W \cap Y|, |W' \cap Y| = 1$, in which case G has type 2A, again a contradiction. This completes the proof of the lemma. □

7.2.2 Computation

In this subsection, we state the result of the computation done by computer.

Lemma 7.12. *If $G = (V, E)$ is a counterexample to Theorem 7.1 with $|V| + |E|$ minimum, then $|V(G)| \geq 10$.*

Proof. Suppose $|V(G)| \leq 9$, and G is 3-edge-connected, internally 4-edge-connected, and in which one side of every edge-cut of size four has at most two vertices. We will show that G satisfies Theorem 7.1, i.e. if $G \not\succeq K_{3,3}$, it is either type 0, 2, or is isomorphic to one of the graphs in Figure 7.2, or 7.3. Using Observation 4.4, we see that to verify the lemma it suffices to check the finite number of 3-edge-connected, internally 4-edge-connected graphs, with edge-multiplicity at most $|E(K_{3,3})| = 9$, for which every $X \subset V(G)$ with $2 < |X| < |V(G)| - 2$ satisfies $d(X) \geq 5$. This calculation is done in Sagemath, with the code appearing in the Appendix. Here is a high-level description of the algorithm.

Let $6 \leq n \leq 9$.

Step 1. We take the list of all connected simple graphs on n vertices, and filter out the ones which immerse $K_{3,3}$.

Step 2. For any graph G surviving from Step 1, `repair(G)` generates a list consisting of all edge-minimal multigraphs G' such that:

- the underlying simple graph of G' is G,
- $\delta(G') \geq 3$,
- $\lambda^i(G') \geq 4$,
- for any set $X \subset V(G)$ where $3 \leq |X| \leq \lfloor \frac{n}{2} \rfloor$ we have $d_{G'}(X) \geq 5$,
- G' does not immerse $K_{3,3}$.

Step 3. Suppose the simple connected graph G is such that `repair(G)` is nonempty. Let $\mathcal{G}_1 = $ `repair(G)`. Then, using $\mathcal{G}_1$, we generate $\mathcal{G}_2 = $ `obstruction(G)` which is the list consisting of all multigraphs whose underlying simple graph is G, meet the edge-connectivity conditions that the graphs in $\mathcal{G}_1$ satisfy, have edge-multiplicity at most nine, and do not immerse $K_{3,3}$.

Step 4. Every graph in $\mathcal{G}_2$ is tested if it has one of the types 0, 2, or is isomorphic to one of the graphs in Fig. 7.2 or 7.3.

The calculation is done rather fast. It took a desktop computer 25 minutes to do the calculation for every $n \in \{6, 7, 8\}$. However, the time spent on $n = 9$ was considerably more. It took the computer 1 hour to carry out step 1, i.e. to check the nearly 262,000 connected simple graphs on nine vertices for a $K_{3,3}$ immersion, thereby giving a list `N9` of almost 34,100 simple connected graphs on nine vertices without a $K_{3,3}$ immersion. Then a total of 4 hours was spent on carrying out steps 2, 3 for every graph in `N9`. Since no obstruction is found for $n = 9$, step 4 is not performed for this case. $\square$

7.2.3 Two local properties

We continue by recording two local properties of a minimal counterexample, which get frequently called upon.

Lemma 7.13. *If $G = (V, E)$ is a counterexample to Theorem 7.1 with $|V| + |E|$ minimum, then:*

(1) There does not exist $u \in V$ which has a neighbour v such that $e(u, v) \geq \frac{1}{2}d(u)$.

(2) If $\delta(X)$ is an internal 4-edge-cut in G, with $|X| \leq |V \setminus X|$, we have $|X| = 2$, and both vertices in X have degree three.

Proof. For part (1), suppose for a contradiction that such u, v exist, and let $U = \{u, v\}$. Let $G' = G.U$, where a denotes the vertex resulting from identifying u, v. Note $e(u, v) \geq \frac{1}{2}d(u)$ implies $G \succ G'$. Therefore, $G' \not\succ K_{3,3}$, and since $|V(G')| = |V(G)| - 1$, Theorem 7.1 holds for G'. Since $|V(G')| \geq 9$, G' has one of the types 0-3. However, since $d_{G'}(a) = d_G(U) \geq 4$, G' is not type 0. Also, G' being type 1 implies that there is a 4-edge-cut in G with at least three vertices on either side, contradicting Lemma 7.11. So G' is not type 1. In a similar manner we conclude that G' is sausage reduced, and since $|V(G')| \geq 9$, G' is not type 2 or 3 either, a contradiction. (Observe that after sausage reduction a graph of type 2 or 3 has at most eight vertices).

For part (2), note that it follows from Lemma 7.11 that $|X| = 2$, let $X = \{u, v\}$. Since $d(X) = 4$, we have $e(u, v) > 0$. It follows from part (1) that exactly two edges of $\delta(X)$ is incident with each u, v, and that $e(u, v) = 1$. $\square$

7.2.4 5-edge-cuts

In this subsection we study 5-edge-cuts in a minimal counterexample to Theorem 7.1. The main result is Lemma 7.19, which tells us that one side of every 5-edge-cut in a minimal counterexample has at most three vertices. The lemma is a powerful tool in carrying out the inductive step in the proof of Theorem 7.1. In this subsection, we see a couple of lemmas which eventually lead to establish the result. First, we take note of a useful fact about planar graphs:

Proposition 7.14. *If G is a 2-connected graph embedded in the plane, every face of G is bounded by a cycle.*

Let us proceed by introducing some convenient terminology.

Definition 7.15. Suppose G is a graph, and $X \subset V(G)$. Let $e \in \delta(X)$, with x being the endpoint of e in X. Then we say X is *almost cubic relative to* e if $d(x) \in \{3, 4\}$, and for every vertex $u \in X \setminus \{x\}$ we have $d(u) = 3$.

We now start looking into 5-edge-cuts. In the lemmas throughout this subsection, we assume G is a minimum counterexample to Theorem 7.1 with $|V| + |E|$ minimum. The next lemma motivates studying rooted immersion of W_4 in a minimum counterexample to Theorem 7.1.

Lemma 7.16. *Let $\delta(X)$ be a 5-edge-cut in G, where $|X|, |V \setminus X| \geq 4$. Then either of $G.X$ or $G.(V \setminus X)$ has a rooted immersion of W_4.*

Proof. For a contradiction, suppose $G' = G.X \not\succeq_r W_4$ and $G.(V \setminus X) \not\succeq_r W_4$. Denote the root vertex of G' by a. Since $d_{G'}(a) = d_G(X) = 5$, by Theorem 7.4, G' is type I or II. However, it follows form Lemma 7.11 that G' is not type I, and does not have any chain of sausages of order > 1. So G' must be type II A or II B, in which any chain of sausages has order one, and therefore $|V(G.(V \setminus X))| \leq 5$. Since $|V(G.(V \setminus X))| = |X| + 1$, we get $|X| \leq 4$. Then, a similar argument for $G.(V \setminus X)$ shows that $|V \setminus X| \leq 4$, and thus $|V(G)| \leq 8$. However, then we get a contradiction with Lemma 7.12. □

Lemma 7.17. *Let $\delta(X)$ be a 5-edge-cut such that $|X|, |V \setminus X| \geq 4$. Suppose $e \in \delta(X)$ exists such that $(G \setminus e).X \succeq_r W_4$, then:*

- *$(G \setminus e).(V \setminus X) \succeq_r W_4$,*
- *X is almost cubic relative to e.*

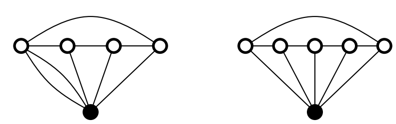
Proof. Let H be the graph obtained from G by deleting e in which any vertices of degree two (resulting from deletion of e) are suppressed. Note that $\lambda(H) \geq 3$, and Lemma 7.13(2) implies that $\lambda^i(H) \geq 4$. Also, clearly G immerses H, so $H \not\succeq K_{3,3}$. On the other hand, let $X' \subset V(H)$ denote the set corresponding to $X \subset V(G)$. Since $|X'| \geq |X| - 1$ and

$|V(H) \setminus X'| \geq |V(G) \setminus X|$, we have $|X'|, |V(H) \setminus X'| \geq 3$. We now apply Lemma 7.7 to H to conclude that either cases below occur:

- $H[X']$ is a chain of sausages, or
- for every vertex $v \in X'$ we have $d_H(v) = 3$, and $H.(V \setminus X')$ has a rooted immersion of W_4.

If the latter case happens, we have nothing left to prove. In the former case, G must contain a vertex of degree four in X which is incident with parallel edges– contradicting Lemma 7.13(1). $\square$

The next lemma features two new rooted graphs, W_4^+ and W_5. We define W_4^+ to be the rooted graph appearing on the left below, and rooted W_5 to be the graph obtained from W_5, whose center is treated as the root vertex, see the graph on the right below.



Lemma 7.18. *Let $\delta(X)$ be a 5-edge-cut such that $|X|, |V \setminus X| \geq 4$, and assume $G' = G.(V \setminus X) \succ_r W_4$. Then $G.X \succ_r W_4$, and moreover, one of the following holds:*

- *$G' \succeq_r W_5$, and every vertex in X has degree three, or*
- *$G' \succeq_r W_4^+$, and there exists a vertex of degree four $x \in X$ such that every vertex in $X \setminus x$ has degree three.*

Proof. Since $G' \succ_r W_4$ (and $d(X) = 5$), there exists $e \in \delta(X)$ such that $(G \setminus e).(V \setminus X) \succeq_r W_4$. So Lemma 7.17 implies that $(G \setminus e).X \succeq_r W_4$ (and thus $G.X \succ_r W_4$) and that X is almost cubic relative to e. Let $e = xy$, where $x \in X$, so we have $d(x) \in \{3, 4\}$ and for every vertex $u \in X \setminus x$ we have $d(u) = 3$. So, to prove the lemma, it remains to show that all neighbours of the root vertex of G' lie on a common cycle. Denote the root vertex of G' by a. Note that it follows from Lemma 7.8 that G' is planar, so by Proposition 7.14, it suffices to show that $G' \setminus a$ is 2-connected. Since $G' \setminus a = G[X]$ is subcubic, it suffices to prove that it is 2-edge-connected. This follows from $|X| \geq 4$, $\lambda^i(G) \geq 4$ and Lemma 7.13(2). $\square$

With the lemmas above in hand, the proof of our main result on 5-edge-cuts in G, stated below, is straightforward.

Lemma 7.19. *There does not exist a 5-edge-cut $\delta(X)$ in G, where $|X|, |V \setminus X| \geq 4$.*

Proof. Towards a contradiction, let $\delta(X)$ be a 5-edge-cut in G with $|X|, |V \setminus X| \geq 4$. Applying Lemma 7.16, without loss of generality, we may assume $G.X \succ_r W_4$. Then Lemma

7.18 implies that $G.(V \setminus X) \succ_r W_4$ as well. It also implies that if we let S, T denote the set of endpoints of $\delta_G(X)$ in $X, V \setminus X$, respectively, then up to symmetry one of the following holds:

- $|S| = |T| = 5$, and every vertex in $X \cup (V \setminus X)$ has degree three.

  Here G will be cubic, a contradiction with Lemma 7.8.

- $|S| = 5, |T| = 4$, and there exists $t \in T$ such that $d(t) = 4$, and for every $v \in V \setminus t$ we have $d(v) = 3$.

  In this case, choose $e \in \delta(X)$ which is not incident with t. Then $(G \setminus e).(V \setminus X)$ immerses W_4. On the other hand, $d_{(G \setminus e).X}(t) = 4$, so X is not almost cubic relative to e– a contradiction with Lemma 7.17.

- $|S| = |T| = 4$, and there exist $s \in S, t \in T$ such that for every $v \in V \setminus \{s, t\}$ we have $d(v) = 3$, and $d(t) = d(s) = 4$.

  Note it follows from Lemma 7.13(1) that $e(s, t) \leq 1$, so we can choose $e \in \delta(X)$ incident with s which is not incident with t. Then $(G \setminus e).(V \setminus X)$ immerses W_4. However, $d_{(G \setminus e).X}(t) = 4$, so X is not almost cubic relative to e– again a contradiction with Lemma 7.17.

□

7.2.5 Finishing the proof

In this subsection, we use Lemmas 7.12, 7.13, and 7.19 to prove Theorem 7.1.

Proof of Theorem 7.1. Towards a contradiction, let $G = (V, E)$ be a counterexample to the theorem with $|V| + |E|$ minimum. We first show that G is simple. Suppose (for a contradiction) that adjacent vertices u, v exist such that $e(u, v) \geq 2$. Let G' be the graph obtained from G by deleting one copy of uv. Note that Lemma 7.13(1) implies that $\lambda(G') \geq 3$ and thus $|V(G')| = |V(G)| \geq 10$. It also follows from Lemma 7.13(2) that $\lambda^i(G') \geq 4$. So by minimality of G, Theorem 7.1 holds for G'. If G' is type 0, it is cubic and simple, and thus G contradicts Lemma 7.13(2). If G' is type 1, we get a contradiction with either Lemma 7.13(2) or 7.19. Now note that it follows from Lemma 7.13(1) that G' is sausage reduced. So $|V(G')| \geq 10$ implies that G' is not type 2 or 3 either– a contradiction.

We are now ready to complete the proof of Theorem 7.1. Since G is not cubic, it has a vertex v of degree ≥ 4. Choose an edge e which is not incident with v. Let G' be the graph obtained from G by deleting e, and suppressing any degree two vertices. So $|V(G')| \geq |V(G)| - 2 \geq 8$, and $\lambda(G') \geq 3$. It follows from Lemma 7.13(2) that $\lambda^i(G') \geq 4$. So by minimality of G, Theorem 7.1 holds for G'. Since G' is not cubic, it is not type 0. As in the last paragraph, G' cannot have type 1 either, otherwise G would contradict either Lemma 7.13(2) or 7.19.

Finally, note that G' is sausage reduced. It is because if $G'[X']$ is a chain of sausages of order ≥ 3 in G', then Lemma 7.13(2) implies that $e \in \delta_G(X)$, where X is the set in G which corresponds to X' in G'. But then $G[X]$ would contain parallel edges, a contradiction. So, if G' has type 2, it is type 2A relative to some W, W', where $|W| = |W'| = 2$. However then (since G is simple) G would contradict 7.19. If G' is type 3, since $|V(G')| \geq 8$, it must be isomorphic to the last graph in Figure 7.2. Then, however, G is not simple– a contradiction. This contradiction completes the proof of Theorem 7.1. □

7.3 Corollaries of Theorem 7.1

In this section, we use Theorem 7.1 to identify all graphs which do not immerse $K_{3,3}$. We will also see a strengthening of the only previously known result about graphs which do not immerse $K_{3,3}$ or K_5.

7.3.1 Graphs of arbitrary edge-connectivity excluding a $K_{3,3}$ immersion

We will use Theorem 7.1 to characterize graphs with arbitrary edge-connectivity which do not immerse $K_{3,3}$. Since $K_{3,3}$ is 3-edge-connected, internally 4-edge-connected, we can use Observations 4.6, 4.7 to get the following as an immediate corollary of Theorem 7.1.

Corollary 7.20. *A graph G has no $K_{3,3}$ immersion if and only if G can be reduced to a graph where every component is either*

1. *of order at most five, or*
2. *is one of the types $0, 1, 2, 3,$ or 4.*

by the operations

- *If e is a cut-edge in G, then modify G to $G \setminus e$.*
- *If H is a 2-edge-connected component of G, and $X \subset V(H)$ has $\delta(X) = \{x_1y_1, x_2y_2\}$, where $x_1, x_2 \in X$ then modify G to $G \setminus \delta(X) + x_1x_2 + y_1y_2$.*
- *If H is a 3-edge-connected component of G, and $\delta(X)$ is an internal 3-edge-cut in H, then replace H by the disjoint union of $H.X$ and $H.(V(H) \setminus X)$.*

7.3.2 Tree-width and branch-width of graphs without $K_{3,3}$ immersion

As we saw earlier in this chapter, as an immediate corollary of Theorems 7.1, 7.2 we have the following structural theorem identifying the graphs which exclude both Kuratowski graphs as immersion:

Corollary 7.21. *Let G be a with $|V(G)| \geq 6$, and $\lambda(G) \geq 3$, $\lambda^i(G) \geq 4$. Then G does not immerse $K_{3,3}$ or K_5 if and only if G is one of the types $0, 1, 2, 3,$ or 4 except for the Octahedron.*

It follows from Observations 4.10 and 4.11 that in order to find tree-width and branch-width of graphs of types 2, 3, or 4, it suffices to consider only the ones which are sausage reduced, and have distinct underlying simple graphs. It is then easy to check that all such graphs have treewidth at most three, except for octahedron, which is known to have treewidth four. Below, we will find that if G a type 1 graph, then $cw(G) \leq 3$ and $pw(G) \leq 3$. We begin with an easy yet helpful observation.

Observation 7.22. *Let G be a 3-edge-connected graph with a segmentation of width four relative to $U_0 \subset U_1 \subset U_2 \subset \ldots \subset U_t$, where $|U_0|, |V \setminus U_t| \leq 3$. Let $U_i \setminus U_{i-1} = \{x_i\}$, and let $Z_i = \{v \in U_i, \delta(v) \cap \delta(U_i) \neq \emptyset\}$, for $i = 1, \ldots, t$. Then for every $1 \leq i \leq t$ we have:*

(1) $e(x_i, U_{i-1}) = e(x_i, V \setminus U_i) \geq 2$

(2) $x_i \in Z_i$ *and* $|Z_i| \leq 3$.

Proof. (1) Since $d(U_{i-1}) = d(V \setminus U_i) = 4$, we have $e(x_i, U_{i-1}) = e(x_i, V \setminus U_i)$. Since G is 3-edge-connected, we have $d(x_i) \geq 3$, and thus $e(x_i, U_{i-1}) \geq 2$.

(2) It follows from part (1) that $x_i \in Z_i$, and since $d(U_i) = e(U_{i-1}, V \setminus U_i) + e(x_i, V \setminus U_i) = 4$, we have $|U_{i-1} \cap Z_i| \leq e(U_{i-1}, V \setminus U_i) \leq 2$, and thus $|Z_i| \leq 3$.

□

Let G be a graph. For u and v adjacent vertices in G, we call the set of edges $E(u, v)$ a *parallel class* of G. For $F \subseteq E(G)$, let $G[F]$ denote the subgraph of G on all vertices that are incident with an edge in F, and with the edge-set F.

Lemma 7.23. *Let G be a graph with a segmentation of width four relative to some (U, W), where $|U|, |W| \leq 3$. Then there exists a partition of $E(G)$ into $\{E_0, E_1, \ldots, E_k\}$ such that*

- *Every E_i with $1 \leq i \leq k-1$ is a parallel class*
- $|V(G[E_0])|, |V(G[E_k])| \leq 3$
- $\left|V\left(G\left[\bigcup_{i=0}^{j} E_i\right]\right) \cap V\left(G\left[\bigcup_{i=j+1}^{k} E_i\right]\right)\right| \leq 3$, *for* $0 \leq i \leq k-1$.

Proof. Let $U = U_0 \subset U_1 \subset U_2 \subset \ldots \subset U_t = V \setminus W$ be a segmentation of width four of G. Let S be a union of parallel classes of G. We say S has a *good ordering* if there exists a partition of S into parallel classes $\{E_1, \ldots, E_k\}$ such that

> if we let G_j, G_j^c be the subgraph of G induced by $\bigcup_{i=0}^{j} E_i$ and $E \setminus E(G_j)$, respectively, then $|V(G_j) \cap V(G_j^c)| \leq 3$.

Note that in order to prove the lemma, it suffices to show that $E(G)$ has a good ordering. Clearly, there is a good ordering of $E[G(U_0)]$. If $t = 0$, it is straightforward to see that this good ordering can be extended to a good ordering of $E(G)$. So, suppose $t \geq 1$. Let

$U_i \setminus U_{i-1} = \{x_i\}$, for $i = 1, \dots, t$, and let $Z_i = \{v \in U_i, \delta(v) \cap \delta(U_i) \neq \emptyset\}$, for $i = 0, \dots, t$. Suppose there exists a good ordering of $E(G[U_{i-1}])$, and we extend this to a good ordering for $E(G[U_i])$ by appending edges in $E(x_i, U_{i-1})$ to it in a certain order. Note that by Observation 7.22(2) we have $|Z_i| \leq 3$. If $|Z_{i-1}| \leq 2$, adding the edges in $E(x_i, U_{i-1})$ in any order to $(E_1, \dots, E_k)$ results in a good ordering. Suppose $|Z_{i-1}| = 3$. Then it follows from Observation 7.22(1) that there exists $v \in Z_{i-1} \setminus Z_i$. Now, we extend $(E_1, \dots, E_k)$ by first adding the parallel class of the edge vx_i, and then adding other edges between x_i, U_{i-1} (if any) in an arbitrary order. This gives a good ordering of $E(G[U_i])$. □

Immediately from the above lemma we find that:

Observation 7.24. *Let G be a graph with a segmentation of width four relative to some (U, W), where $|U|, |W| \leq 3$. Then caterpillar-width of the underlying simple graph of G is at most three.*

Lemma 7.25. *Let G be a graph with a segmentation of width four relative to some (U, W) where $|U|, |W| \leq 3$. Then path-width of G is at most three.*

Proof. Let $U = U_0 \subset U_1 \subset U_2 \subset \dots \subset U_t = V \setminus W$ be a segmentation of width four of G. We show that G has a path-decomposition of width three in which U_0 is the set of vertices of G which is associated to one endpoint of the path. First, suppose $t = 0$. Then it is straightforward to see that such a path-decomposition exists.

Now, suppose $t \geq 1$. Let $U_i \setminus U_{i-1} = \{x_i\}$, and let $Z_i = \{v \in U_i, \delta(v) \cap \delta(U_i) \neq \emptyset\}$, for $i = 1, \dots, t$. It follows from Observation 7.22(2) that there is a vertex $u \in U_0 \setminus Z_1$. Now consider the graph $G \setminus u$, which has a $(3,3)$-segmentation of width four (relative to $(Z_1, V \setminus U_t)$. Let P be a path-decomposition of width three for $G \setminus u$ in which the set associated with an endpoint s of the path is Z_1. Now extend this path by one vertex whose only neighbour is s, and associate to this vertex U_1. Since $|U_1| \leq 4$, the width of this path-decomposition is at most three. Inductively, we get a path-decomposition of width at most three for G. □

As a result of Corollary 7.21, and Lemmas 7.23, 7.25 we get the followings, one of which is a sharpening of the previously known result (Theorem 1.9) about the structure of graphs which exclude $K_{3,3}$ and K_5 as immersion.

Corollary 7.26. *Let G be a connected graph with $|V(G)| \geq 6$. If $G \not\succeq K_{3,3}$ or K_5, then G can be constructed via 1-, 2-, and 3-edge-sum of simple planar cubic graphs and graphs with tree-width (branch-width) at most three.*

Corollary 7.27. *Let G be a graph which does not immerse $K_{3,3}$. Then G can be constructed from i-edge-sums, for $i = 1, 2, 3$ from cubic graphs, octahedron, and graphs with tree-width (branch-width) at most 3.*

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Appendix A

Code used in the proof of Theorem 7.1

Here, we have included the code written in Sagemath whose result features in the proof of Theorem 7.1.

K_{3, 3}-immersion

```
# Here is the function which takes a graph H, and a vertex v and returns
nonisomorphic graphs resulted by splitting off different pair of edges at v (one
stage) (First draft by Stefan Hannie)

def split_off (H, v):
    S = []
    N = H.edges_incident (v, labels = False)
    for pair in Combinations (N, 2):
        for w in pair [0]:
            if not w == v: x = w
        for w in pair [1]:
            if not w == v: y = w
        G = H.copy ()
        G.allow_multiple_edges (True)
        if not x == y: G.add_edge (x, y)
        G.delete_edges ([(v, x), (v, y)])
        if G.degree (v) < 2: G.delete_vertex (v)
        if is_new (S, G):
            yield G
            S.append (G)
```

```
# Checks whether x is a "new" graph to L, meaning that no graph isomorphic to x is
already in L
def is_new (L, x):
    if L == []: return True
    else:
        T = True
        for y in L:
            if x.is_isomorphic (y): return False
        return True
```

```
# Takes a graph G, a vertex v, and a positive integer k, and returns all graphs
obtained by doing k splits at v in all possible ways
def split_off_k (G, v, k):
    if k == 1:
        for g in split_off (G, v): yield g
    if k >= 2:
        Lt = []
        for g in split_off_k (G, v, k-1):
            for g_split in split_off (g, v):
                if is_new (Lt, g_split):
                    yield g_split
                    Lt.append (g_split)
```

```
# This function takes a list of graphs, and cuts off all the isomorphic ones
def cutIso (L):
    S = []
    for g in L:
        if is_new (S, g): S.append(g)
    return S
```

```
# This function computes the size of minimum edge-cut separating tuples of
vertices with at most six vertices from the rest of the graph for multigraphs

def mind_k_tuple (G, k, min_needed = None):
    if min_needed == None:
        if k >= G.order(): return 0
        a = G.size()
        for tuple in Combinations (G.vertices(), k):
            l = len (G.edge_boundary (tuple))
            if l < a:
                [a, cert] = [l, tuple]
        return [a, cert]
    else:
        if k >= G.order(): return False
        for tuple in Combinations (G.vertices(), k):
            if len (G.edge_boundary (tuple)) < min_needed: return False
        return True
```

```
def has_trivial_econ (G):
    if not min (G.degree()) >= 3: return False
    for e in Set (G.edge_iterator(labels = False)):
        if len (G.edge_boundary (e)) < 4: return False
    return True
```

```
# Takes a multigraph on at most 9 vertices, and checks if it has min degree three,
is int 4-ec, and there is no (<=4)-edge-cut separating at least three vertices
opposite others

def well_edge_connected (G, reduced = False):
    if G.order() < 6: return False
    if not reduced and not has_trivial_econ (G): return False
    if not mind_k_tuple (G, 3, 5): return False
    if G.order() < 8: return True
    else: return mind_k_tuple (G, 4, 5)
```

```
# Computes the number of edges between two vertices
def e_mult (G, u, v):
    if not u in G.neighbors(v): return 0
    m = 0
    for e in G.edges_incident(v):
        if u in e: m = m+1
    return m
```

```
# Computes the minimum and maximum edge-multiplicity in a graph G
def min_max_mult (G):
    [m, M] = [100, 0]
    for e in Set (G.edges()):
        k = e_mult (G, e[0], e[1])
        if k > M: M = k
        if k < m: m = k
    return [m, M]
```

```
# Decides if G is a doubled cycle
def is_doubled_cycle (G, four_regular= False):
    if not four_regular:
        if not G.is_regular(4) : return False
    if not min_max_mult (G) == [2, 2]: return False
    H = G.to_simple ()
    return H.is_isomorphic (graphs.CycleGraph (G.order()))
```

```
# Tells if G has type 2A with respect to u, v
def is_Type2A (G, u, v):
    if not G.degree (u)== 5 == G.degree (v) or not e_mult (G, u, v) == 1: return
False
    #  or not min_max_mult(G) == [1, 2]
    for x in G.vertex_iterator():
        if not x in [u, v]:
            if not G.degree (x) == 4: return False
    H = G.copy ()
    H.delete_edge (u, v)
    return is_doubled_cycle (H, True)
```

```
# Tells if G has type 2B with respect to u, v
def is_Type2B (G, u, v):
    if u in G.neighbors (v) or not G.degree (u)== 5 == G.degree (v): return False
    # or not min_max_mult(G) == [2, 3]
    [candidate, deg4] = [[], []]
    for x in G.vertex_iterator():
        if not x in [u, v]:
            if G.degree(x) == 6:
                if e_mult (G, u, x) == 3 and e_mult (G, x, v) == 3:
candidate.append (x)
                if len (candidate) > 1 : return False
            elif G.degree(x) == 4: deg4.append (x)
    if len (candidate) == 0 or not len (deg4) == G.order() -3: return False
    H = G.copy()
    H.delete_edges ([(candidate[0], u), (candidate[0], v)])
    return is_doubled_cycle (H, True)
```

```
# Tells if G has type 2C with respect to u, v
def is_Type2C (G, u, v):
    if not G.degree (u)== G.degree (v) == 5 or not e_mult (G, u, v) == 3: return
False
    # if not min_max_mult (G) == [2, 3]
    for x in G.vertex_iterator():
        if not x in [u, v]:
            if not G.degree (x) == 4: return False
    H = G.copy()
    H.delete_edge (u, v)
    return is_doubled_cycle (H, True)
```

```
# Takes a multigraph H and outputs a list, whose 0-th entry tells if H has type
2-sub. If it is type 2-sub, the 1-st and 2-nd entry shows the "bags"
def is_Type2sub (H, sub_type):
    [deg4, deg5] = [[], []]
    for x in H.vertex_iterator():
```

```
        if H.degree (x) == 4: deg4.append (x)
        elif H.degree (x) == 5: deg5.append (x)
    k = H.order()- len (deg4)
    if sub_type == 'A' or sub_type == 'C':
        if k > 4: return [False]
    elif sub_type == 'B' and k > 5: return [False]
    T = False
    if len (deg5) == 2:
        if sub_type == 'A' or sub_type == 'C':
            if k == 2: T = True
        elif k == 3: T= True
    if T:
        if sub_type == 'A':
            if is_Type2A (H, deg5[0], deg5 [1]): return [True, deg5]
        if sub_type == 'B':
            if is_Type2B (H, deg5[0], deg5 [1]): return [True, deg5]
        if sub_type == 'C':
            if is_Type2C (H, deg5[0], deg5 [1]): return [True, deg5]
    Eset = sorted (Set(H.edges(labels = False)), key=lambda e: e_mult (H, e[0],
e[1]), reverse = True)
    CE = Combinations (Eset, 2)
    S = False
    if len (deg5) > 0:
        if sub_type == 'A' or sub_type == 'C':
            if k == 3: S = True
        elif k == 4: S = True
    if S:
        for v in deg5:
            for e in Eset:
                if not v in e:
                    K = H.copy()
                    K.allow_multiple_edges(True)
                    K.merge_vertices (e)
                    if sub_type == 'A':
                        if is_Type2A (K, e[0], v): return [True, v, e]
                    if sub_type == 'B':
                        if is_Type2B (K, e[0], v): return [True, v, e]
                    if sub_type == 'C':
                        if is_Type2C (K, e[0], v): return [True, v, e]
    for pair in CE:
        K = H.copy()
        K.allow_multiple_edges(True)
        [e, f] = [pair [0], pair [1]]
        if not e[0] in f and not e[1] in f:
            K.merge_vertices (pair [0])
            K.merge_vertices (pair [1])
            if sub_type == 'A':
                if is_Type2A (K, e[0], f[0]): return [True, e ,f]
            if sub_type == 'B':
                if is_Type2B (K, e[0], f[0]): return [True, e ,f]
            if sub_type == 'C':
                if is_Type2C (K, e[0], f[0]): return [True, e ,f]
    return [False]
```

```
# Tells if H has type 2
def is_Type2 (H, well_econ = False):
```

```
    k = H.order()
    if not well_econ or k == 6:
        return is_Type2sub (H, 'A')[0] or is_Type2sub (H, 'B')[0] or is_Type2sub
(H, 'C')[0]
    else:
        if k == 7: return is_Type2sub (H, 'A')[0] or is_Type2sub (H, 'B')[0]
        elif k == 8: return is_Type2sub (H, 'A')[0]
        else: return False
```

```
def two_ecut_rdn_X (G, X):
    H = G.copy()
    H.allow_multiple_edges (True)
    E_X = H.edge_boundary (X, labels = False)
    for vx in E_X[0]:
        if not vx in X: u = vx
    for vx in E_X[1]:
        if not vx in X: v = vx
    if not u == v : H.add_edge (u, v)
    H.delete_vertices (X)
    return H
```

```
# Takes a graph G and reduces its 1, 2- edge-cuts, and internal 3-edge-cuts
def red (G):
    small_deg = []
    for v in G.vertices():
        if G.degree(v) <3:
            small_deg.append(v)
    if G.order()- (len (small_deg)) < 6: return graphs.CompleteGraph (3)
    H = G.copy()
    H.allow_multiple_edges (True)

    # We reduce 1, 2-edge-cuts
    for j in [1, 2]:
        for k in [4, 3, 2, 1]:
            while H.order () >= max ([6, 2*k]):
                L = mind_k_tuple (H, k)
                if L[0] == j:
                    if j == 1: H.delete_vertices (L[1])
                    else: H = two_ecut_rdn_X (H, L[1])
                else: break
            if H.order() < 6: break
        if H.order() < 6: break

    # We reduce internal 3-edge-cuts
    for k in [4, 3, 2]:
        while H.order () >= max ([6, 2*k]):
            L = mind_k_tuple (H, k)
            if L[0] == 3:
                H.merge_vertices (L[1])
            else: break
        if H.order () < 6: break
    return H
```

```
# This function takes a multigraph on six vertices as input and determines whether
it has a subgraph of K_{3,3}
```

```
def has_K33_sbg_six (G):
    # H is the underlying simple graph of G
    H = G.copy()
    H.allow_multiple_edges(False)
    # counti is the number of connceted components of H of size i
    [count1, count2, count3] = [0, 0, 0]
    for x in (H.complement()).connected_components_sizes():
        if x > 3: return False
        else:
            if x == 1: count1 = count1 + 1
            elif x == 2: count2 = count2 + 1
            elif x == 3: count3 = count3 + 1
    if count3 == 0 and count1 == 0: return False
    return True
```

```
# Takes a graph H on six vertices and decides whether it has an immersion of
K_{3,3}. The optional arguments tell if we already know H is reduced, or well-
edge-connected, or not type 2
def has_K33_im_six (H, reduced = False, econ = 'undecided', type2 = 'undecided'):
    if econ == 'poor': return False
    if not econ == 'well':
        if not reduced:
            if not red(H).is_isomorphic (H): return False
        if not mind_k_tuple (H, 3, 5): return False
    if type2 == 'undcided':
        if is_Type2 (H, True): return False
    if has_K33_sbg_six (H): return True

    #hideg consits of all vertices of degree at least five
    hideg = [ v for v in H.vertex_iterator() if H.degree(v) >= 5]
    if hideg == []: return False

    Lv_1_less_split = [H]
    for v in hideg:
        Lv = []
        # Lv is a list consisting of graphs obtained by doing at most (d_H(v)-
3)// 2 splits at v at all graphs in L
        for j in range ((H.degree (v)- 3)// 2):
            Lv_j_split = []
            for g in Lv_1_less_split:
                for g_split in split_off (g, v):
                    if min (g_split.degree()) >= 3:
                        T = True
                        for e in Set (g_split.edge_iterator(labels = False)):
                            if len (g_split.edge_boundary (e)) < 4:
                                T = False
                                break
                        if T and mind_k_tuple (g_split, 3, 5) and is_new
(Lv_j_split, g_split):
                            if has_K33_sbg_six (g_split): return True
                            else:
                                Lv_j_split.append (g_split)
            Lv_1_less_split = Lv_j_split
            Lv += Lv_j_split
        L = Lv
    return False
```

```
# Takes G and two vertices of it, and decides if there is an arrangement x_1x_2
... x_{|V|-2} of V(G) \{u, v} for which d(u) = d({u, x_1, \ldots, x_i}) =4, for
every i<= |V|-2
def is_nested_4ecut (G, u, v, rest =[], checked_deg_uv = False, checked_deg_rest =
False):
    if not checked_deg_uv:
        if not G.degree (u)== G.degree (v) == 4: return [False]
    if not checked_deg_rest:
        for x in G.vertex_iterator():
            if not G.degree (x) in [2, 4, 6, 8]: return [False]
    if rest == []:
        rest = [x for x in G.vertex_iterator() if not x in [u, v]]
    for perm in itertools.permutations (rest):
        chunk = [u]
        for i in range (G.order() -2):
            chunk += [perm[i]]
            if not len (G.edge_boundary(chunk)) == 4: break
            elif i == G.order()-3: return [True, perm]
    return [False]
```

```
#The 0-th entry tells if G has type 1. If it is type 1, then we get three other
entries, the first triple of vertices, the permutation of other vertices in
between, and lastly the second triple of vertices
def is_Type1 (G):
    even_deg = [v for v in G.vertex_iterator() if G.degree (v) % 2 == 0]
    if len (even_deg) < G.order () - 6: return [False]
    for rest in Combinations (even_deg, G.order()-6):
        comb_6 = [v for v in G.vertices() if not v in rest]
        for triple1 in Combinations (comb_6, 3):
            if len (G.edge_boundary (triple1)) == 4:
                triple2 = [v for v in comb_6 if not v in triple1]
                if len (G.edge_boundary (triple2)) == 4:
                    [u, v] = [triple1[0], triple2 [0]]
                    H = G.copy()
                    H.allow_multiple_edges(True)
                    H.merge_vertices(triple1)
                    H.merge_vertices(triple2)
                    L = is_nested_4ecut (H, u, v, rest, True, True)
                    if L[0]: return [True, triple1, L[1], triple2]
    return [False]
```

```
# determines if a graph on at least seven vertices immerses K_{3,3} by splitting
its vertices completely (one at a time), and check if the resulting graph on fewer
vertices immerse K_{3,3}
def has_im_split (G, global_info):
    for v in sorted (G.vertex_iterator(), key=lambda v: G.degree(v)):
        for g in split_off_k (G, v, (G.degree (v)//2)):
            if has_K33_im (g, global_info): return True
    return False
```

```
# The G_info tells us what we already know about G. It tells us if G is (3-e-cut)
reduced, is well-edge-connected.
# global_info[0] gives the helper for reduced graphs on seven and eight vertices
```

```
that are not well_edge_con or type 1, 2. global_info[1] gives the list of the
known well-edge-connected exceptions, and global_info[2] indicates the largest n
for which all the exceptions of order <=n are known is.
# The econ can be either 'well', 'poor', or 'undecided'-- meaning that G is /is
not/ we don't know if is well-edge-connected, respectively.

def has_K33_im (G, global_info = [[{}, {}], [], 5], G_info = [False,
'undecided']):
    [reduced, econ] = G_info
    if econ == 'well' or reduced: H = G
    else: H = red (G)
    k = H.order ()
    if k < 6: return False
    if econ == 'well': well_econ = True
    else:
        if econ == 'poor' and reduced: well_econ = False
        else: well_econ = well_edge_connected (H, True)
    if k == 6:
        if not well_econ: return False
        elif is_Type2 (H, True) : return False
    else:
        if min_max_mult (H)[1] > 1 and is_Type2 (H, well_econ) : return False
    [helper, exception, accuracy] = global_info

    if accuracy >= k and well_econ:
        for g in exception:
            if H.is_isomorphic (g):
                return False
        return True
    if not well_econ:
        if is_Type1 (H)[0]: return False
        elif k in [7, 8]:
            H = H.canonical_label()
            H_dict = H.copy (immutable = True)
            if H_dict in helper [k -7]: return helper [k -7] [H_dict]
            T = has_im_split (H, global_info)
            helper [k -7] [H_dict] = T
            return T
    if k == 6: return has_K33_im_six (H, True, 'well', 'no')
    return has_im_split (H, global_info)
```

```
def edges_multiplicity (G):
    L = {}
    for e in Set (G.edges (labels = False)):
        L[e] = e_mult (G, e[0], e[1])
    return L
```

```
def has_less_multiplicity (L, M):
    for x in L:
        if not x in M or L[x] > M[x]: return False
    return True
```

```
# L is a list of multigraphs with the same underlying simple graph. The output is
the minimal ones
def get_minimal (L):
```

```
    if len(L) == 0: return []
    maxnumberofedges = 30
    [L_mid, L_final, minimal_comb]= [[], [], []]
    [m, M] = [500, 0]
    for i in range (maxnumberofedges +1 ):
        L_mid.append ([])
    # L_mid[k] consists of all graphs in L that have k number of edges
    for k in range (maxnumberofedges +1):
        L_mid [k] = [graph_2 for graph_2 in L if graph_2.size () == k ]
        if len (L_mid [k]) > 0:
            if k < m: m = k
            if k > M: M = k
    # every graph in L_mid[m] is minimal
    L_final += L_mid [m]
    for minimal in L_mid [m]:
        minimal_comb.append (edges_multiplicity (minimal))
    for extra in range (m+ 1, M+ 1):
        # L_t is the minimal graphs found in this round, and comb_t is the
combination of graphs in L_t
        comb_t = []
        L_t = []
        # every graph in L_mid [extra], extra > m, is compared against all graphs
already in L_final from previous rounds.
        for graph_1 in L_mid [extra]:
            # T will tell us if h is a supergraph of a graph in L_final
            T = False
            graph_1_comb = edges_multiplicity (graph_1)
            for comb in minimal_comb:
                if has_less_multiplicity ( comb, graph_1_comb ):
                    T = True
                    break
            if not T:
                L_t.append (graph_1)
                comb_t.append (graph_1_comb)
        L_final += L_t
        minimal_comb += comb_t
    return L_final
```

```
# Adds k edges to the boundary of X in parallel to the existing edges in all
possible ways
def add_dX (G, X, k):
    L = []
    E_X = G.edge_boundary (X)
    Set_E_X = Set (E_X)
    for added in itertools.combinations_with_replacement (Set_E_X, k):
        Gplus = G.copy ()
        Gplus.allow_multiple_edges (True)
        Gplus.add_edges (added)
        L.append (Gplus)
    return L
```

```
# Determines if a graph immerses K_{3,3}
def is_loser (G, global_info = [[{}, {}], [], 5], info_underlying = [[], [], [],
[]], G_info = [False, 'undecided']):
    [wincomb, failcomb, non_iso_winners, non_iso_losers] = info_underlying
```

```
    [helper, exception, accuracy] = global_info
    if accuracy >= G.order():
        if G_info [1] == 'undecided':
            if well_edge_connected (G): G_info = [True, 'well']
            else: G_info [1] = 'poor'
        if G_info [1] == 'well': return not has_K33_im (G, global_info, G_info)

    G_comb = edges_multiplicity (G)
    for comb in wincomb:
        if has_less_multiplicity (comb, G_comb):
            return False
    for comb in failcomb:
        if has_less_multiplicity (G_comb, comb):
            return True
    for x in non_iso_winners:
        if G.is_isomorphic (x):
            wincomb.append (G_comb)
            return False
    for x in non_iso_losers:
        if G.is_isomorphic (x):
            failcomb.append (G_comb)
            return True

    if has_K33_im (G, global_info, G_info):
        non_iso_winners.append (G)
        wincomb.append (G_comb)
        return False
    else:
        non_iso_losers.append (G)
        failcomb.append (G_comb)
        return True
```

```
# Adds edges in parallel to the existing edges in the edge boundary of m-tuples in
G, so that if m=1, d(tuple) >=3, if m=2, d(tuple)>= 4, and if m>=3, d(tuple) >=5
# info_underlying is the information we have so far about what combination of
edges make G immerse or not immerse K_{3,3}

def repair_d_m_tuple (G, m, global_info, info_underlying):
    H = G.copy()
    H.allow_multiple_edges (True)
    L = [H]
    if m == 1: needed = 3
    elif m == 2: needed = 4
    elif m >= 3: needed = 5
    for m_tuple in Combinations (H.vertices() , m):
        Lt = []
        for g in L:
            k = len (g.edge_boundary (m_tuple))
            if k >= needed: Lt.append (g)
            else:
                for h in add_dX (g, m_tuple, needed- k):
                    if not h in Lt:
                        if H.order()== 6: Lt.append (h)
                        elif is_loser (h, global_info, info_underlying): Lt.append
(h)
        L = Lt
```

```
    return L
```

```
# Takes a simple graph H as input, and outputs all edge-minimal graphs G up to
isomorphism whose undelying simple graph is H, are 3-edge-connected, internally
4-edge-connected, and d(X) >= 5 for X\subset V(G) with |X|, |V(G) \X| >= 3, and
have edge-multiplicity up to nine, and do not immerse K_{3, 3}
def repair (H, global_info, info_underlying):
    L = [H]
    #every graph in L, has the right number of edges in the edge-boundary of any
<=m set of vertices
    for m in range (1, H.order()//2 + 1):
        Lt = []
        for minimal in L:
            for g in repair_d_m_tuple (minimal, m, global_info, info_underlying):
                if not g in Lt: Lt.append (g)
        if Lt == []: return []
        L = get_minimal (Lt)
    if H.order () == 6: L = [x for x in L if is_loser (x, global_info,
info_underlying, [True, 'well'])]
    return L
```

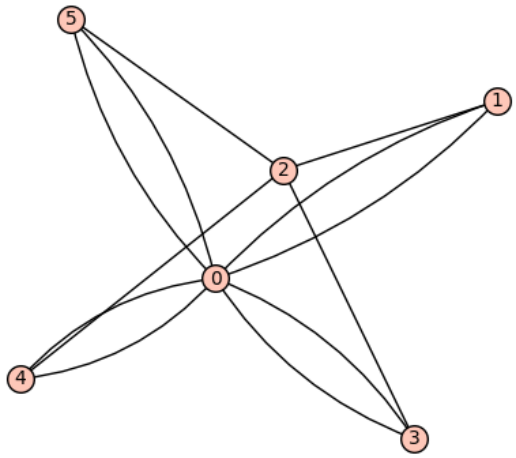
```
# Takes a simple graph G as input, and outputs all graphs H up to isomorphism
whose undelying simple graph is G, are 3-edge-connected, internally 4-edge-
connected, and d(X) >= 5 for X\subset V(H) with |X|, |V(H) \X| >= 3, and have
edge-multiplicity up to nine, and do not immerse K_{3, 3}
def Obstruction (G, global_info = [[{}, {}], [], 5]):
    H = G.copy()
    H.allow_multiple_edges (True)
    info_underlying = [[], [], [], []]

    losers = repair (H, global_info, info_underlying)
    if losers == []: return []
    #losers = [x for x in L if is_loser (x, global_info, info_underlying, [True,
'well'])]
    obstruction = cutIso (losers)
    for step in range (8* H.size()):
        Lt = []
        for g in losers:
            for e in Set (g.edges()):
                if e_mult (g, e[0], e[1]) < 9 :
                    gplus = g.copy()
                    gplus.allow_multiple_edges(True)
                    gplus.add_edge(e)
                    Lt.append (gplus)
        losers_step = [x for x in Lt if is_loser (x, global_info, info_underlying,
[True, 'well'])]
        if losers_step == []: break
        losers = cutIso (losers_step)
        for x in losers:
            if not x in obstruction: obstruction.append(x)
    return obstruction
```

```
import datetime
import itertools
```

```
# L[i-6] consists of all graphs of order i, M [i-6] contains all connected graphs
of order i, N[i-6] contains all graphs in M[i-6] which don't immerse K_{3,3}, and
O[i-6] consists of all "obstructions" arising from the graphs in N[i-6]
global_info = [[{}, {}], [], 5]
[L, M, N, O] = [[], [], [], []]
exception = []
for i in [6, 7, 8, 9]:
    L.append (list(graphs(i)))
    M.append ([g for g in L[i -6] if g.is_connected()])
    print 'Size of L_', i, len (L[i -6]), 'Size of M_',i, len (M[i -6])
    now = datetime.datetime.now()
    N.append ([g for g in M [i -6] if not has_K33_im (g, global_info)])
    then = datetime.datetime.now()
    print 'N_', i, 'has size', len (N[i -6]), 'is calculated in', then - now
    now = datetime.datetime.now()
    O_i = []
    for g in N[i -6]:
        O_g = Obstruction (g, global_info)
        if not O_g == []: O_i += O_g
    then = datetime.datetime.now()
    print 'O_', i, 'has size', len (O_i), 'is calculated in', then - now
    now = datetime.datetime.now()
    for g in O_i:
        if not is_Type2 (g, True):
            exception.append (g)
    then = datetime.datetime.now()
    print 'The number of not type 2 obstructions until now is', len (exception),
'Filtering out type 2 graphs for n =', i, 'took', then - now
    [global_info [1], global_info [2]]= [exception, i]
    O.append (O_i)
```

```
Size of L_ 6 156 Size of M_ 6 112
N_ 6 has size 102 is calculated in 0:00:00.207323
O_ 6 has size 4016 is calculated in 0:00:45.566772
The number of not type 2 obstructions until now is 31 Filtering out
type 2 graphs for n = 6 took 0:00:08.609543
Size of L_ 7 1044 Size of M_ 7 853
N_ 7 has size 605 is calculated in 0:00:04.361521
O_ 7 has size 3709 is calculated in 0:01:28.735165
The number of not type 2 obstructions until now is 33 Filtering out
type 2 graphs for n = 7 took 0:00:03.575169
Size of L_ 8 12346 Size of M_ 8 11117
N_ 8 has size 4363 is calculated in 0:01:30.946276
O_ 8 has size 1405 is calculated in 0:18:34.866261
The number of not type 2 obstructions until now is 34 Filtering out
type 2 graphs for n = 8 took 0:00:01.157978
Size of L_ 9 274668 Size of M_ 9 261080
N_ 9 has size 34101 is calculated in 0:59:15.704713
O_ 9 has size 0 is calculated in 3:51:32.119485
The number of not type 2 obstructions until now is 34 Filtering out
type 2 graphs for n = 9 took 0:00:00.000004
```

```
for g in exception:
    g.show()
```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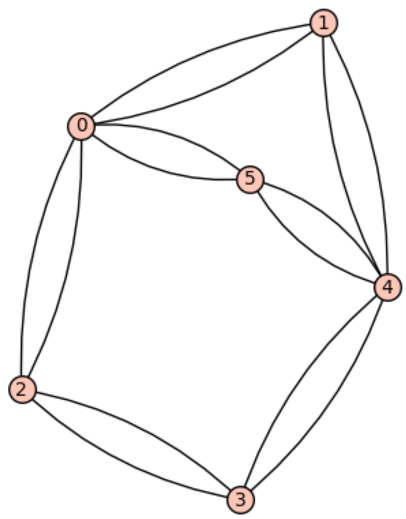
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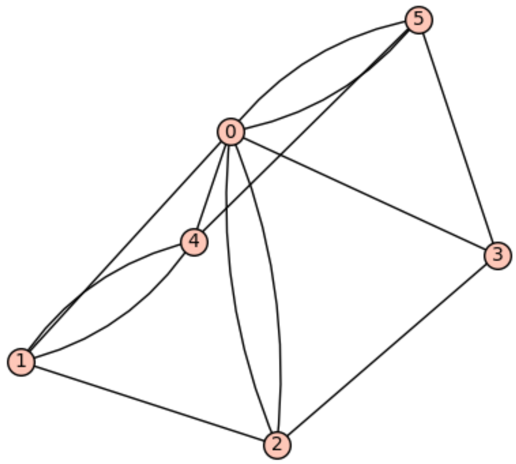
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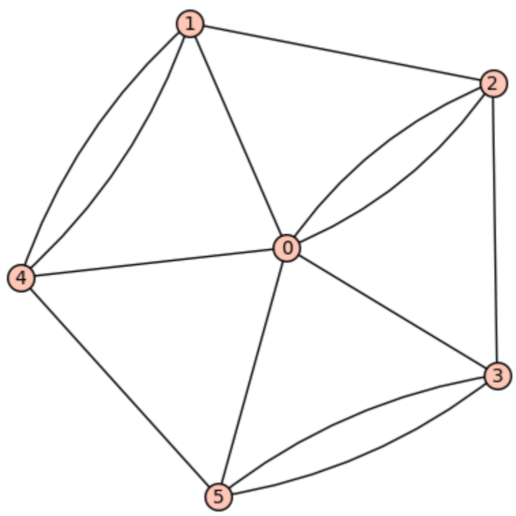
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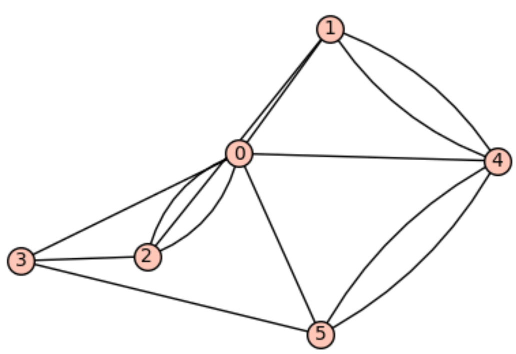
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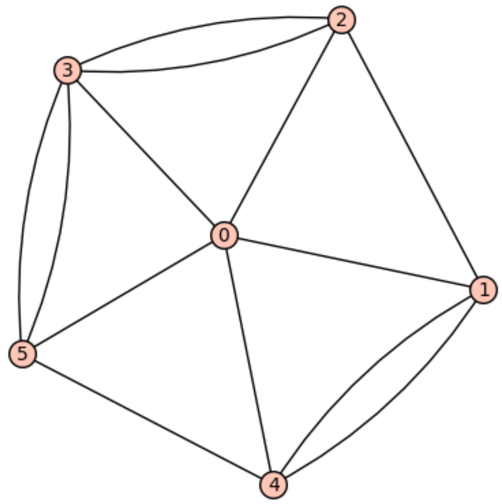

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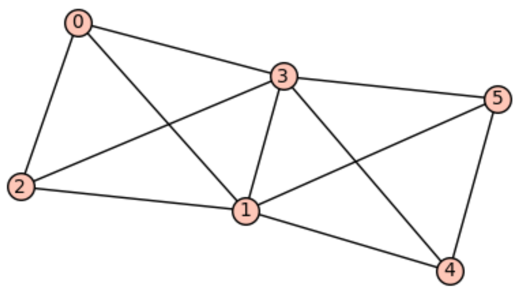

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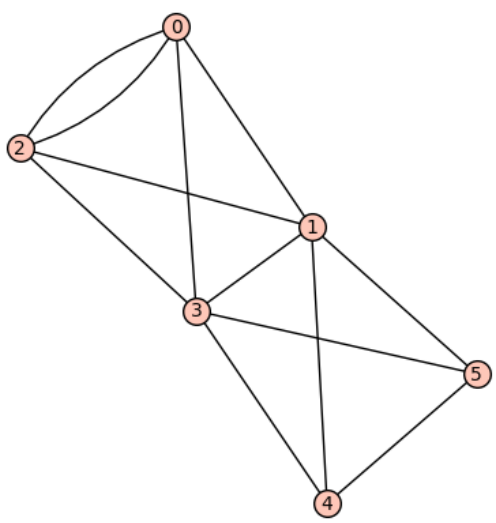

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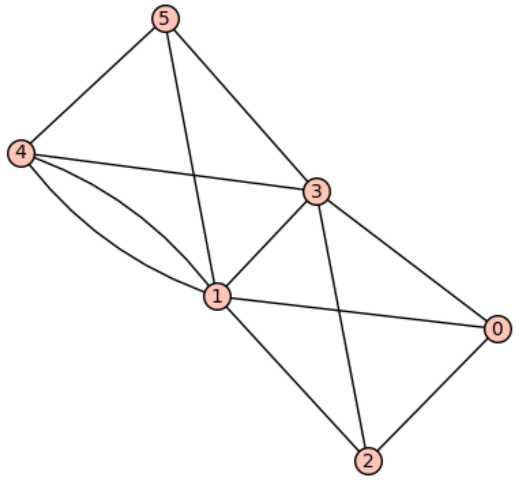
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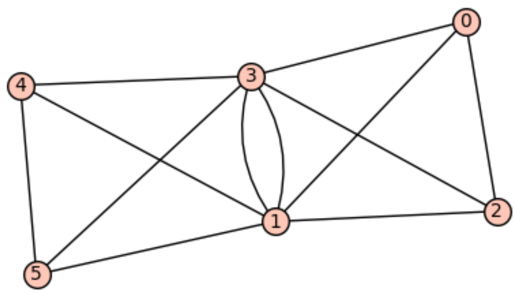
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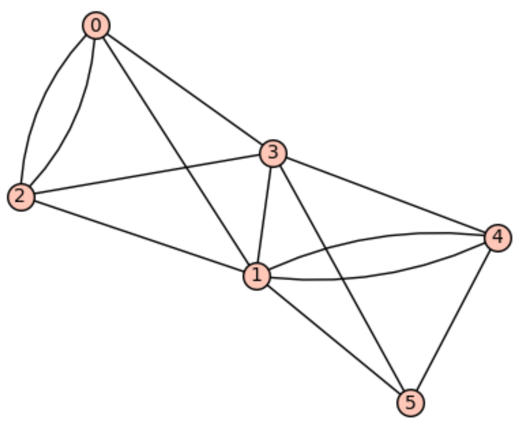
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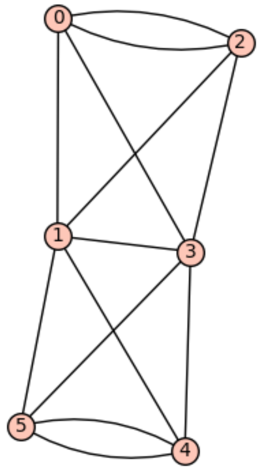
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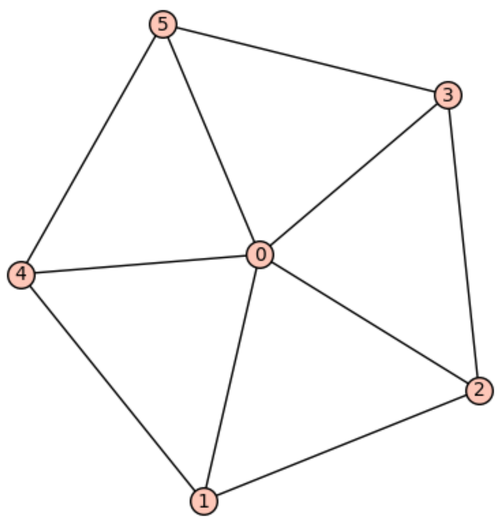
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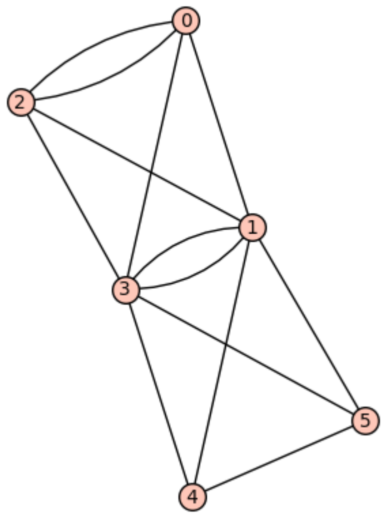
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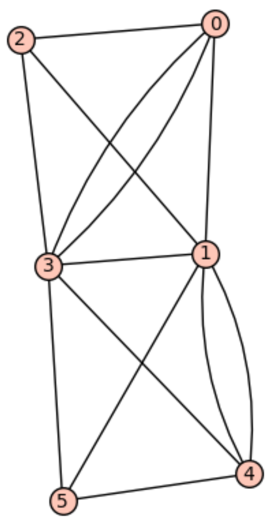
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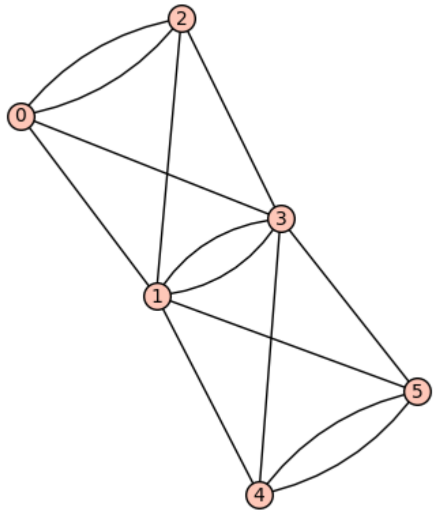
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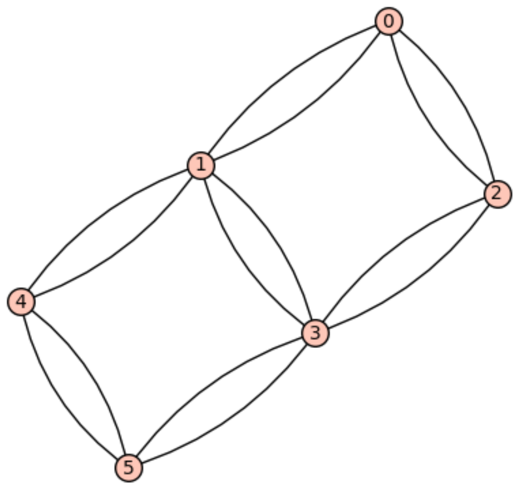
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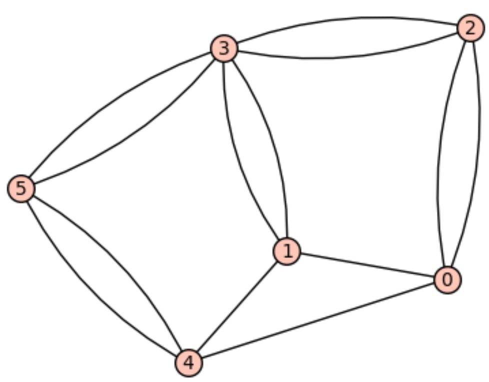
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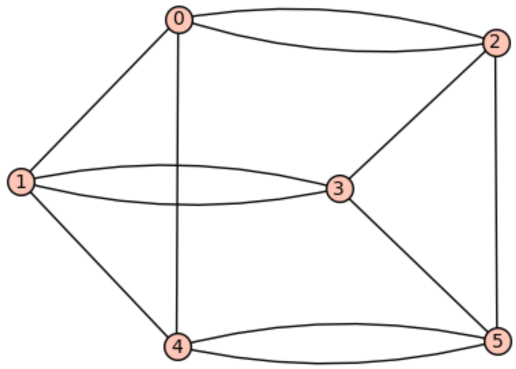
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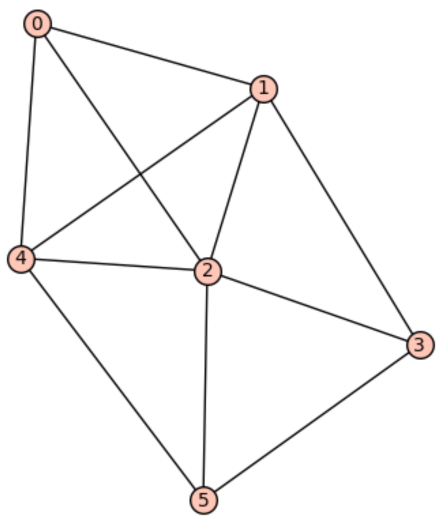
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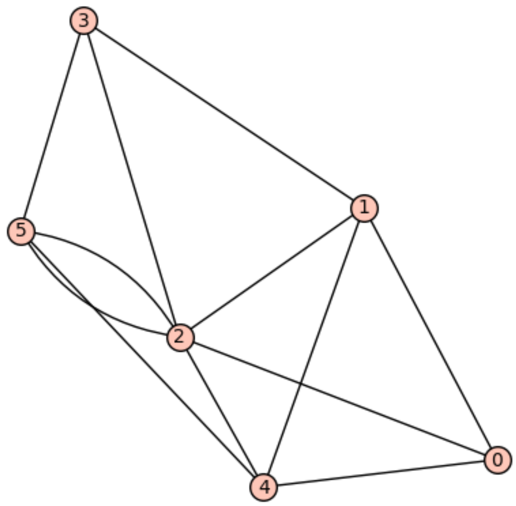
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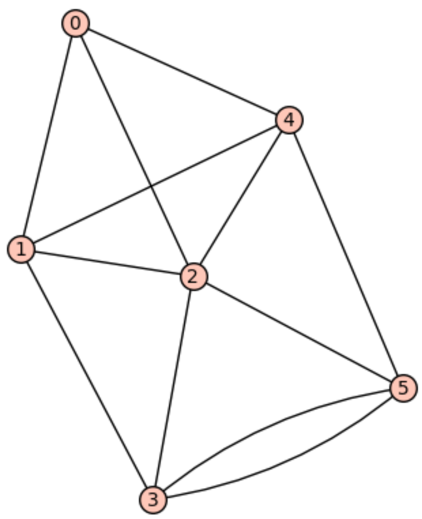
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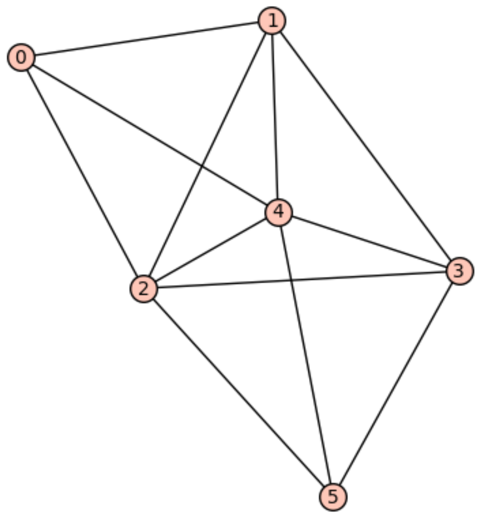
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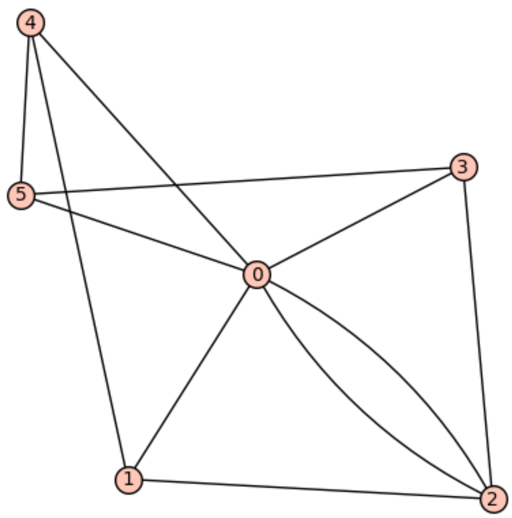
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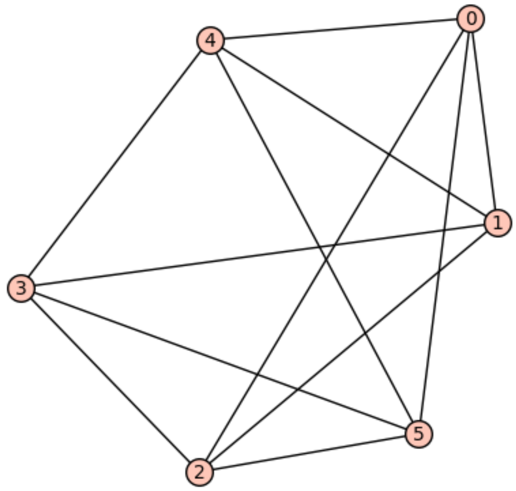
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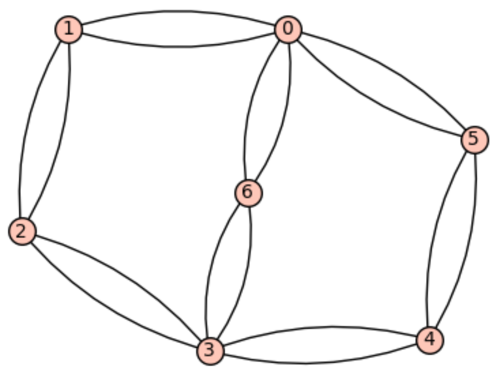
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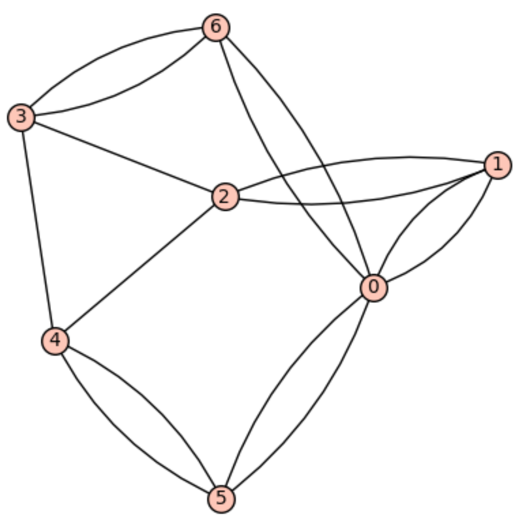
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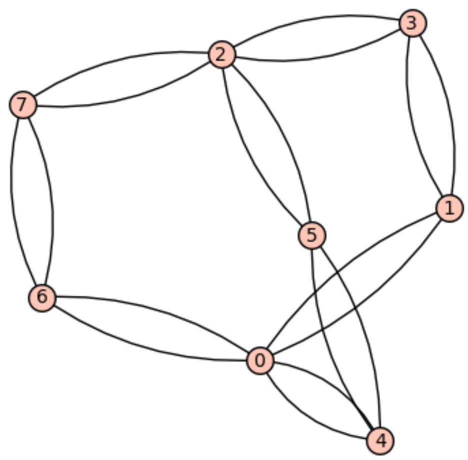
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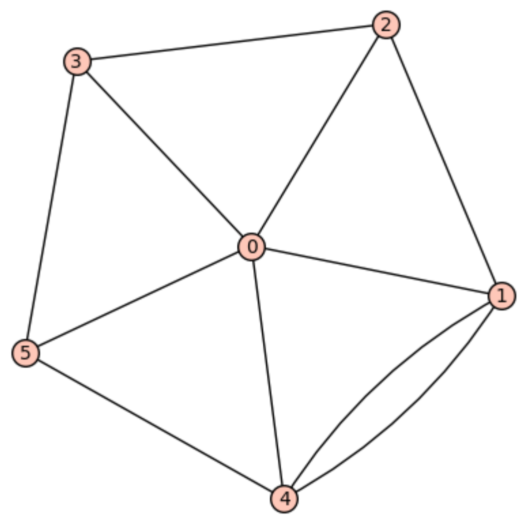
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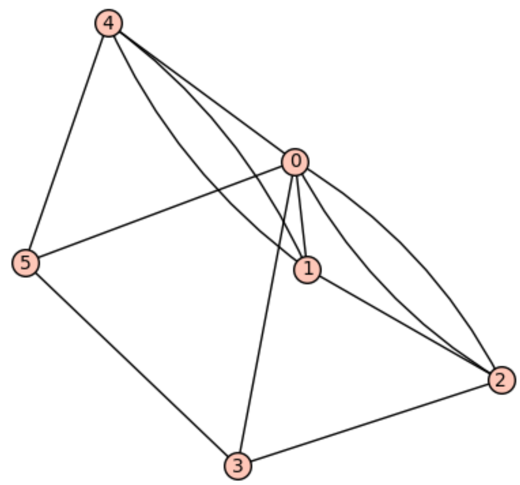
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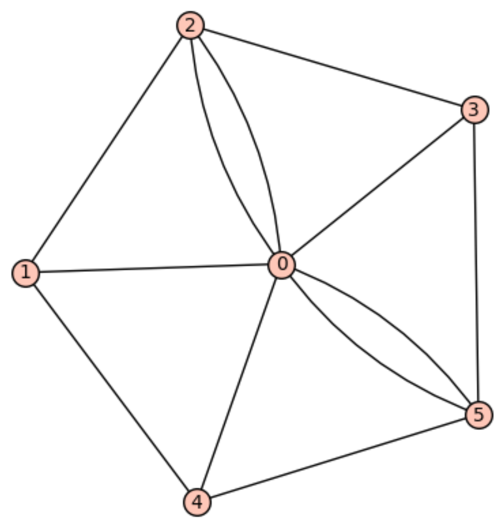
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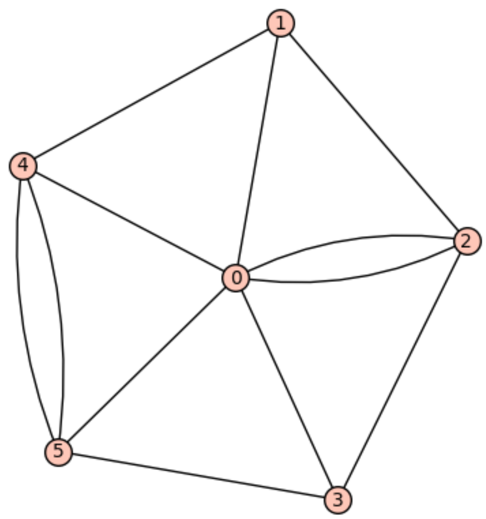
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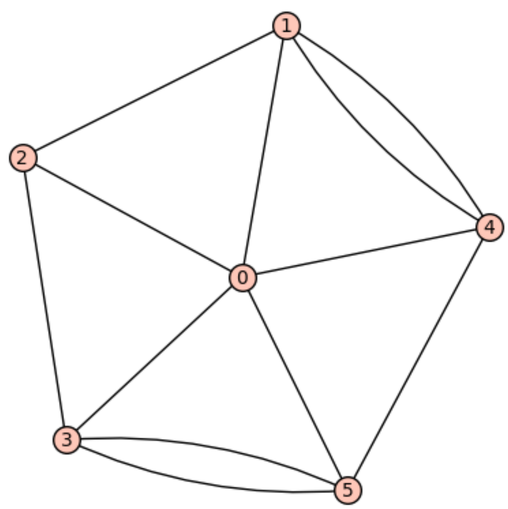
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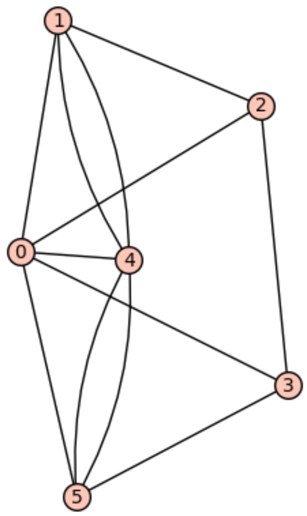
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Appendix B

A second proof of Theorem 4.2

In this chapter, we present a proof for Theorem 4.2 which is not computer-assisted. For convenience, we restate the theorem here, with a slightly different naming of graphs of types 3 or 4. Before stating this theorem we will need to introduce four families of graphs which do not have the rooted immersion of W_4. In preparation for that let us now introduce two particular little graphs which will be helpful in defining our families. The rooted graph $J_{2,3}$ is obtained from a graph isomorphic to $K_{2,3}$ with bipartition $(\{x, z\}, \{y, y', y''\})$ by adding a second copy of each edge incident with x and then declaring x to be the root vertex. We define the rooted graph $J_{2,2} = J_{2,3} - y''$.

Theorem B.1. *Let G be a 3-edge-connected, internally 4-edge-connected graph with $|V(G)| \geq 5$ and with a root vertex x. Then G contains a rooted immersion of W_4 if and only if G does not have one of the following types:*

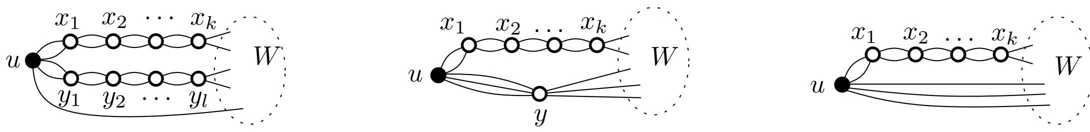
Type 1. *G is type 1 if it has a $(2, 3)$-segmentation of width four in which u is in the head of the segmentation.*

Type 2. *G is type 2 if there exists a set $W \subseteq V(G) \setminus \{u\}$ with $|W| \leq 2$ so that the graph G^* obtained by identifying W to a single vertex w has a doubled cycle C satisfying one of the following:*

- **(2A)** *u and w are not adjacent in C and $G^* = C + uw$*
- **(2B)** *u and w have a common neighbour v in C and $G^* = C + uv + vw$*
- **(2C)** *u and w are adjacent in C and $G^* = C + uw$*

Type 3. *G is type 3 if it is isomorphic to a rooted graph obtained from $J_{2,2}$ in one of the following ways:*

- **(3A)**
  - *Add a doubled path from x to z internally disjoint from $\{y, y'\}$, and*
  - *for each vertex in $\{y, y'\}$ either add another copy of an edge with this vertex and x or do nothing.*
- **(3B)** *Add a doubled path from x to z internally disjoint from $\{y, y'\}$ and add an edge with ends yy'*



(a) Type 2A (b) Type 2B (c) Type 2C

Figure B.1: Type 2 graphs

(a) Type 3A (b) Type 3B

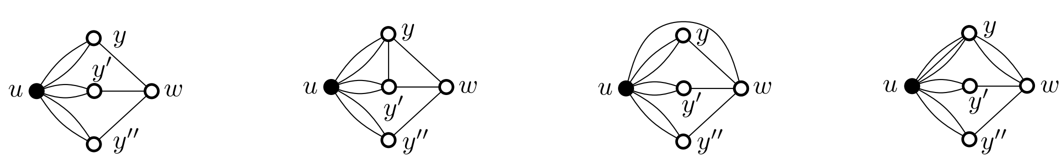
Figure B.2: Type 3 graphs

Type 4. *G is type 4 if it is isomorphic to a rooted graph obtained from $J_{2,3}$ in one of the following ways:*

- **(4A)** *For each vertex in $\{y, y', y''\}$ either add another copy of an edge incident to this vertex or do nothing.*
- **(4B)**
  - *Add the edge yy', and*
  - *either add another copy of the edge xy'' or do nothing.*
- **(4C)** *Add the edge xz*
- **(4D)** *Add the edges xy and yz*

For type 3A we may add up to one more copy of uy and/or uy' edge. For type 4A we may also add up to one more edge incident to each of y, y', y'' in parallel to an existing edge. For type 4B we may add up to one more copy of the edge uy''. For convenience, throughout this section, we will use the same labeling of vertices as in Figures B.2 and B.3 when graphs of type 3 or 4 are dealt with.

To begin the proof of this theorem, assume (for a contradiction) that it is false, and choose a graph $G = (V, E)$ with root vertex x so that G is a counterexample to Theorem B.1 with $|V| + |E|$ minimum. Our proof will involve numerous lemmas establishing properties of this minimum counterexample G. This argument is divided over 5 subsections.



(a) Type 4A (b) Type 4B (c) Type 4C (d) Type 4D

Figure B.3: Type 4 graphs

B.1 Connectivity I

For the purposes of determining if a given rooted graph has a rooted W_4 immersion, the problem reduces naturally to the case when the graph is 3-edge-connected and internally 4-edge-connected as we have seen (in Observations 4.6, 4.7). However, there is another type of connectivity which will be useful for us. Say that a graph H with a root vertex u is *near-root k-edge-connected* if $d(\{u, v\}) \geq k$ for every $v \in N(u)$. For brevity, we say that H is *nicely edge-connected* if H is 3-edge-connected, internally 4-edge-connected, and near-root 5-edge-connected.

Lemma B.2. *The graph G is nicely edge-connected.*

Proof. By assumption G is 3-edge-connected and internally 4-edge-connected, so we only need to show that it is near-root 5-edge-connected. Suppose (for a contradiction) this is false and choose $X \subseteq V(G)$ with $x \in X$ and $|X| = 2$ so that $d(X) < 5$. Note that the internal 4-edge-connectivity of G implies $d(X) = 4$. If $|V(G)| = 5$, then G has type 1 relative to the sequence of subsets X, so we must have $|V(G)| \geq 6$. If G_X has a rooted W_4 immersion, then it follows from internal 4-edge-connectivity that G also has a rooted W_4 immersion, giving us a contradiction. Otherwise, the minimality of the counterexample G implies that the theorem holds for $G.X$, so it must have type 1, 2, 3, or 4. However, since the root vertex of $G.X$ has degree 4, it can only be type 1, so we may choose a nested sequence of sets $U_0 \subseteq U_1 \ldots \subseteq U_t$ in $G.X$ in accordance with this type. We may also assume that U_0 consists only of the root vertex (since it has degree four). For every $0 \leq i \leq t$ let U'_i be the subset of $V(G)$ obtained from U_i by deleting X (the root vertex of $G.X$) and then adding both vertices (of G) contained in X. Now the graph G has type 1 relative to $U'_0 \subseteq U'_1 \ldots \subseteq U'_t$ and this contradiction completes the proof. □

We define the graph *hat* to be a graph obtained from K_3 by choosing a vertex u and adding a second copy of each edge incident with u. We call u the *top* of hat.

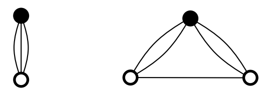
Lemma B.3. *Let H be a 3-edge-connected and internally 4-edge-connected graph with $|V(H)| \geq 3$ and let $w \in V(H)$ satisfy $d(w) \geq 4$. Then H immerses a graph H' isomorphic to hat where w is the top of the hat H'.*

Proof. If $H - w$ has no edges, choose $u, v \in V(H) \setminus \{w\}$ and note that $e(u, w), e(w, v) \geq 3$ (by 3-edge-connectivity) so the graph $H[\{u, v, w\}]$ has the desired immersion. Otherwise, choose $e = uv \in E(H)$ with $w \neq u, v$ and form a new graph H^* from H as follows: Add a new edge e' with ends u, v, then subdivide the edges e and e', and then identifying the two resulting vertices of degree 2 to a new vertex z. It follows from the internal 4-edge-connectivity of G that the graph H^* has 4 edge-disjoint paths $P_1, \ldots, P_4$ from w to z. Now the paths $P_1 - \{z\}, \ldots, P_4 - \{z\}$ together with the edge e form a rooted hat immersion in H as desired. □

Lemma B.4. *The graph $G - x$ is connected.*

Proof. Assume (for a contradiction) that $G - x$ is disconnected. Define two new rooted graphs *hook* and *hat* with the figure below. As usual, we say that an arbitrary rooted graph

H has a rooted hook (hat) immersion if there is an immersion of hook (hat) in H with the added property that the root of hook (hat) corresponds to the root of H. Our argument leans on finding rooted hook and hat immersions in subgraphs of G (note that we treat x as a root vertex in any subgraph of G containing it).



Claim: Let H be a component of $G - x$ and let $H^+ = G[\{x\} \cup V(H)]$.

1. If H is trivial, then H^+ has a rooted hook immersion.
2. If H is nontrivial, then H^+ has a rooted hat immersion.

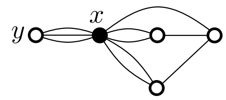
To prove the first part of the Claim, simply choose $v \in V(H)$ and then apply 3-edge-connectivity of G to choose 3 edge-disjoint paths from v to x. For the second part of the claim, begin by choosing an edge $e = uv \in E(H)$. Form a new graph H^* from H^+ by adding a new edge e' with ends u, v, then subdividing the edges e and e', and then identifying the two resulting vertices of degree 2 to a new vertex w. It follows from the internal 4-edge-connectivity of G that the graph H^* has 4 edge-disjoint paths $P_1, \ldots, P_4$ from w to x. Now the paths $P_1 - \{w\}, \ldots, P_4 - \{w\}$ together with the edge e form a rooted hat immersion in H^+ as desired.

It follows from a straightforward application of the above claim that G contains a rooted W_4 immersion in all of the following cases:

- $G - x$ has at least four components
- $G - x$ has at least two nontrivial components
- $G - x$ has one nontrivial component and at least two trivial components

Since G is a counterexample to Theorem B.1 it cannot have a rooted W_4 immersion. Therefore we have just one remaining possibility: $G - x$ has exactly one nontrivial component and exactly one trivial component ($G - x$ cannot have only trivial components thanks to $|V(G)| \geq 5$).

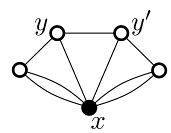
Let y be the vertex contained in the unique trivial component of $G - x$ and note that $e(x, y) \geq 3$. Choose $z \in N(x) \setminus \{y\}$ and form a new graph G' from G by splitting off yx with xz. Note that if $\{X, Y\}$ is a partition of $V(G)$ with $x \in X$ for which $e_{G'}(X, Y) < e_G(X, Y)$ then it must be that $z, y \in Y$. Using this it is straightforward to verify that the graph G' is nicely edge-connected. If G' contains a rooted W_4 immersion, then G also has a rooted W_4 immersion and this is a contradiction. Therefore, by the minimality of the counterexample G it must be that G' has type 1, 2, 3, or 4. Now, G' cannot be type 1 since it is near-root 5-edge-connected. The graph G' cannot be type 2 since $d_{G'}(x) = d_G(x) - 2 = e_G(x, y) + d_G(\{x, y\}) - 2 \geq 6$ (here $d_G(\{x, y\}) \geq 5$ by the near-root 5-edge-connectivity of G). To prepare for the remaining cases, observe that in the graph G' the vertex y satisfies $N(y) = \{x, z\}$ and $e(y, z) = 1$. By our analysis, G' must have type 3 or 4, but then by reversing the split used to obtain G' we find that G has a rooted immersion of the following graph.



It is straightforward to check that this graph has a rooted W_4 immersion, and this final contradiction completes the proof. $\square$

Lemma B.5. *The graph $G - x$ does not have a cut-edge e so that both components of $(G - x) - e$ have at least two vertices.*

Proof. Suppose (for a contradiction) that the above scenario is realized. Let $e = yy'$, let H be the component of $(G - x) - e$ containing the vertex y, and define $H^+ = G[\{x\} \cup V(H)]$. Choose an edge $f \in E(H^+)$ incident with y, say $f = yz$, and form a new graph H^* from H^+ by subdividing the edge f with a new vertex w and then adding a new edge between w and z. It follows from the 3-edge-connectivity and internal 4-edge-connectivity of G that the graph H^* is 3-edge-connected. Choose 3 edge-disjoint paths P_1, P_2, P_3 from x to w in H^*. Now $P_1 - w, P_2 - w, P_3 - w$ are edge-disjoint paths in H^+ (none of which contain the edge e) all starting at x with two ending at z and one ending at y. It now follows from a similar argument applied to the other component of $(G - x) - e$ that the original graph G has a rooted immersion of the following graph.



It is straightforward to check that this graph has a rooted W_4 immersion, and this contradiction completes the proof. $\square$

B.2 Four neighbours

We continue to work on our rooted graph G which is a minimum counterexample to Theorem B.1. In this section we call on the connectivity results from the previous section to prove that the root vertex x of G must have at most 3 neighbours. First we require a basic lemma concerning 2-edge-connected graphs. A graph F is *Eulerian* if it is connected and every vertex has even degree.

Lemma B.6. *Let H be a 2-edge-connected graph and let $z, y_1, y_2 \in V(H)$ be distinct. Then there exists an Eulerian subgraph $F \subseteq H$ with $z \in V(F)$ and (possibly trivial) pairwise vertex disjoint paths P_1, P_2 so that P_i is a path from y_i to $V(F)$.*

Proof. Choose two edge-disjoint paths from z to y_1 and let F be the union of these two paths. If $y_2 \in V(F)$ the proof is complete. Otherwise, choose two edge-disjoint paths Q_1, Q_2 from y_2 to $V(F)$ and assume Q_1, Q_2 are internally disjoint from $V(F)$. If Q_1 and Q_2 have both endpoints in common, then $F \cup Q_1 \cup Q_2$ is Eulerian and satisfies the lemma, so we may assume otherwise. If one of Q_1, Q_2 has z as an endpoint and the other has y_1 as an endpoint, then we may choose a path $R \subseteq F$ from z to y_1 and then $Q_1 \cup Q_2 \cup R$ is an Eulerian graph satisfying the lemma. Otherwise, we may choose $i \in \{1, 2\}$ so that Q_i has an endpoint in $V(F) \setminus \{z, y_1\}$. Now the Eulerian graph F and the path $P_2 = Q_i$ satisfy the lemma, thus completing the proof. □

Lemma B.7. *Let H be a 2-edge-connected graph and let $y_1, \ldots, y_4 \in V(H)$ be distinct. Then there exists an Eulerian subgraph $F \subseteq H$ and (possibly trivial) pairwise vertex disjoint paths $P_1, \ldots, P_4$ so that P_i is a path from y_i to $V(F)$.*

Proof. We proceed by induction on $|V(H)|$. First suppose that the graph H has a cut-vertex z. If there is a component K of $H - z$ with no vertex in $\{y_1, \ldots, y_4\}$, then the result follows by applying induction to the graph $H - V(K)$. Next suppose that $z \in \{y_1, \ldots, y_4\}$ and observe that for every component K of $H - z$ with $V(K) \cap \{y_1, \ldots, y_4\} = \{y_i\}$, the graph $H[K \cup \{z\}]$ contains an Eulerian subgraph spanning z and y_i (this follows from the existence of two edge-disjoint paths from z to y_i). Similarly, if K is a component of $H-z$ with $V(K) \cap \{y_1, \ldots, y_4\} = \{y_i, y_j\}$ then we may apply the previous lemma to choose an Eulerian subgraph F of $H[K \cup \{z\}]$ containing z. Combining the Eulerian subgraphs from each such component gives a solution to the problem. Therefore, we may assume $z \notin \{y_1, \ldots, y_4\}$. Suppose that K is a component of $H - z$ with $V(K) \cap \{y_1, \ldots, y_4\} = \{y_i\}$. In this case we may apply induction to the graph $H' = H - V(K)$ with the vertex z in place of y_i. Now we may modify the structure from this solution on H' by appending a path from y_i to z to get a solution to the original problem. In the only remaining case, $H - z$ has exactly two components, say K, K', each containing two vertices of $\{y_1, \ldots, y_4\}$. Now the result follows by applying the previous lemma to $H[K \cap \{z\}]$ and $H[K' \cup \{z\}]$. This completes the proof in the case H has a cut vertex, so we may now assume H is 2-connected.

Suppose there exists a cycle C with $|V(C) \cap \{y_1, \ldots, y_4\}| \geq 3$. If C contains all of $y_1, \ldots, y_4$ there is nothing left to prove. Otherwise, choose $i \in \{1, \ldots, 4\}$ so that $y_i \notin V(C)$ and choose a path P containing y_i in the interior so that both ends of P are in $V(C)$ but P is internally disjoint from $V(C)$. If both ends of P are in $\{y_1, \ldots, y_4\}$ then G has a cycle containing $y_1, \ldots, y_4$ and the proof is complete. Otherwise the cycle C together with a suitable subpath of P satisfy the lemma. So, every cycle of H contains at most two of $y_1, \ldots, y_4$.

Choose a cycle C containing y_1 and y_2 (note that $y_3, y_4 \notin V(C)$). Choose a path P with y_3 in the interior so that P has both ends in $V(C)$ but is internally disjoint from C. If $y_4 \in V(P)$ there is a cycle containing at least three of $y_1, \ldots, y_4$, contradicting our assumptions. Similarly, if P has an endpoint in $\{y_1, y_2\}$ there is a cycle containing y_1, y_2, and y_3 which is a contradiction. So, if P has ends z_1, z_2, the graph C is the disjoint union of two internally disjoint paths, P', P'' from z_1 to z_2 each of which contains one of y_1, y_2 in its interior. Finally, choose a path Q with y_4 in the interior so that Q has both ends in $C \cup P$ but is internally disjoint from $V(C) \cup V(P)$. If Q does not have both ends in one of $V(P)$, $V(P')$, or $V(P'')$ there is a cycle containing three of $y_1, \ldots, y_4$ which is contradictory. So, we may assume (without loss) that both ends of Q are in $V(P)$. If $Q \cup P$ does not contain a cycle C' with $y_3, y_4 \in V(C')$ we again get a contradiction as G has a cycle containing three of $y_1, \ldots, y_4$. So there must exist a cycle $C' \subseteq P \cup Q$ containing y_3 and y_4 and now the graph $C \cup P \cup Q$ contains two vertex disjoint paths from $\{y_1, y_2\}$ to $V(C')$ and these paths together with the subgraph C' satisfy the lemma. □

Lemma B.8. *The root vertex x of G satisfies $|N(x)| \leq 3$.*

Proof. Assume (for a contradiction) that $|N(x)| \geq 4$. Let L be the set of leaf vertices in $G - x$ and note that Lemma B.5 implies that the graph $H = G - x - L$ is 2-edge-connected. Next we establish a sequence of properties concerning L and H.

(1) H does not have a vertex which is adjacent to three vertices in L.

Suppose for a contradiction that $z \in V(H)$ is adjacent to the distinct vertices $y, y', y'' \in L$. Note that the graph $G' = G - \{y, y', y''\}$ must be connected (this follows from the internal 4-edge-connectivity of G). If G' contains two edge-disjoint paths from x to z, then G has a rooted W_4 immersion. So, we may assume that G' has a cut-edge e separating x and z. Note that the 3-edge-connectivity of G implies that e is the only cut-edge in G'. Let $e = uv$, let G_x (G_z) be the component of $G' - e$ containing x (z) and assume that $u \in V(G_x)$ and $v \in V(G_z)$. If G_x is nontrivial, then $x \neq u$ (otherwise x would be a cut-vertex) and G_x has two edge-disjoint paths from x to u. By combining these paths together with the edge e, a path in G_z from v to z and the edges incident with y, y', y'' we obtain a rooted W_4 immersion. Thus G_x must be trivial. If $v \neq z$ then there are two edge-disjoint paths from v to z and combining these with e and the edges incident to y, y', y'' we obtain a W_4 immersion. So, we must have $v = z$. If G_z is also trivial, then G is type 3. Otherwise, we may choose another vertex $w \in V(G_z) \setminus \{z\}$ and three edge-disjoint paths between w and z. Now these three paths together with the edge e and the edges incident to y, y', y'' form a rooted W_4 immersion.

(2) H does not have two vertices each adjacent to a vertex in L.

Suppose for a contradiction that $z, z' \in V(H)$ are distinct, $y, y' \in L$ and that $yz, y'z' \in E(G)$. Consider the graph $G' = G - \{y, y'\}$ and note that our assumptions imply that G' is 2-edge-connected. Choose two edge-disjoint paths P_1, P_2 in G' starting at z and ending at z'. If either path contains x, then these paths and the edges incident to y, y' form a rooted W_4 immersion. Otherwise, there exist two edge-disjoint paths in G' starting at x and ending at a vertex of $V(P_1) \cup V(P_2)$ and we may choose such paths Q_1, Q_2 edge-disjoint from P_1, P_2. If both Q_1 and Q_2 end in $V(P_1)$ or both end in $V(P_2)$ then we have a rooted W_4 as before. Otherwise, we may assume that Q_1 ends in $V(P_1) \setminus V(P_2)$ and Q_2 ends in $V(P_2) \setminus V(P_1)$ and now P_1, P_2, Q_1, Q_2 together with the edges incident to y, y' form a rooted W_4 immersion.

(3) $|L| \leq 1$

If this condition is violated, then by the previous two properties we may assume that $L = \{y, y'\}$ and that $z \in V(H)$ is adjacent to both y and y'. First suppose that there exist distinct vertices $z', z'' \in V(H) \setminus \{z\}$ so that $xz', xz'' \in E$. Choose two edge-disjoint paths P_1, P_2 in the graph H from z to z'. If z'' is contained in one of these paths, then these paths together with the edges xz', xz'' and the edges incident with y, y' form a rooted W_4 immersion. Otherwise we may choose two edge-disjoint paths Q_1, Q_2 starting at z'' and ending at a vertex of $V(P_1) \cup V(P_2)$ and we may further assume that Q_1, Q_2

are edge-disjoint from P_1, P_2. If both Q_1 and Q_2 end in $V(P_1)$ or both end in $V(P_2)$ then we have a rooted W_4 immersion as before. Otherwise the path Q_1 does not end in $\{z, z'\}$ and now the paths P_1, P_2, Q_1 together with the edges xz', xz'' and the edges incident with y, y' form a rooted W_4 immersion.

If there do not exist two vertices in $V(H) \setminus \{z\}$ which are incident with x, then z must be adjacent to x and there must be one vertex $z' \in V(H) \setminus \{z\}$ adjacent to x (since $|N(x)| \geq 4$). In this case, choose two edge-disjoint paths P_1, P_2 in the graph H from z to z'. Now P_1, P_2 together with the edges xz, xz' and the edges incident with y, y' form a rooted W_4 immersion.

(4) If $|N(x)| = 4$ and $L \neq \emptyset$, the unique vertex in $V(H)$ with a neighbour in L is not adjacent to x.

Suppose for a contradiction that the stated condition is violated. So we may assume that $L = \{y\}$ and that $z \in V(H)$ is adjacent to both x and y. Since $|N(x)| = 4$, there are exactly two other neighbours of x, say $z', z'' \in V(H) \setminus \{z\}$. First suppose that $e(x, z') > 1$. In this case we may choose two edge-disjoint paths P_1, P_2 in H starting at z'' with one ending at z and the other ending at z'. Now the paths P_1, P_2 together with the edges between x and $\{z, z', z''\}$ and the edges incident with y form a W_4 immersion. So, we must have $e(x, z') = 1$ and by a similar argument $e(x, z'') = 1$. By assumption we must have $d(\{x, y\}) \geq 5$ and this implies that $e(x, z) > 1$. Choose two edge-disjoint paths P_1, P_2 in H from z to z'. If one of these paths contains z'', then together with the edges between x and $\{z, z', z''\}$ and the edges incident with y we have a W_4 immersion. Otherwise we may choose two edge-disjoint paths Q_1, Q_2 in H starting at z and ending in $V(P_1) \cup V(P_2)$ and we may assume Q_1, Q_2 are edge-disjoint from P_1, P_2. Now the paths P_1, P_2, Q_1, Q_2 together with the edges between x and $\{z, z', z''\}$ and the edges incident with y form a W_4 immersion.

With this last item in place, we are now ready to complete the proof. If $|N(x) \cap V(H)| \geq 4$ then choose a set $Y \subseteq N(x) \cap V(H)$ with $|Y| = 4$. Otherwise the assumption $|N(x)| \geq 4$ and the numbered properties imply that $|N(x) \cap V(H)| = 3$ and $|L| = 1$. In this case we let Y be the set consisting of $N(x) \cap V(H)$ together with the unique vertex in $V(H)$ adjacent to the vertex in L. Note that by the previous property we have $|Y| = 4$. Now we may apply Lemma B.7 to choose an Eulerian subgraph K of H and paths $P_1, \ldots, P_4$ from Y to $V(K)$ as indicated. Now we may perform splits to transform the graph K into a cycle and it follows that G has a rooted immersion of W_4, a contradiction which completes the proof. □

B.3 One step above an obstruction

The purpose of this subsection is to prove the following lemma, showing that G does not have one of a few particular local operations taking it to a smaller rooted graph which is still nicely edge-connected (and immersed in G).

Lemma B.9. *The graph G does not have a vertex $v \in N(u)$ and an operation matching one of the descriptions below taking G to a nicely edge-connected graph G'.*

1. *If $|N(u)| \leq 3$, split a vertex v of degree 4, where $e(u,v) = 2$.*
2. *If $e(u,v) \geq 2$, and $d(v) \geq 5$, and there exists $z \in N(v) \setminus N(u)$: Split uv and vz.*
3. *If $e(u,v) \geq 2$, and $d(v) \geq 5$, and $N(v) \subseteq \{u\} \cup N(u)$: Split uv and vz for $z \in N(u)$.*
4. *If $v \in N(u)$ satisfies $e(u,v) = 3$ and $d(v) = 4$, delete one copy of the edge uv.*

Before starting the proof of the above lemma, we record below an immediate corollary of Theorem 3.4 for $m = 2$ that, perhaps surprisingly, is useful for graphs on four vertices.

Corollary B.10. *Let H be a graph with $V(H) = \{x_1, x_2, y_1, y_2\}$ and root vertices x_1, x_2. Assume that x_1 and x_2 are not adjacent and $d(x_i) \geq 2$ and $d(y_i) \geq 3$ for $i = 1, 2$. Then H has a rooted D_2 immersion if and only if there does not exist $i, j \in \{1, 2\}$ so that $d(\{x_i, y_j\}) \leq 2$.*

Proof. Since G is a minimum counterexample, Theorem 4.2 holds for G'. If G' has a rooted immersion of W_4, then G also has a rooted immersion of W_4 which is a contradiction. The graph G' cannot have type 1 since it is near-root 5-edge-connected. Therefore G' must have type 2, 3, or 4. Next we establish some conventions for working with these possibilities.

When G' is type 2, we will assume that it is type 2 relative to the set W as appearing in the definition. Furthermore, we will assume that W is minimal subject to this. So if $|W| = 2$, then G' is not type 2 relative to any singleton contained in W. If $|W| = 2$ we let $W = \{w, w'\}$ and if $|W| = 1$ we let $W = \{w\}$. As in the definition of type 2, let G^* denote the graph obtained from G' by identifying W to a single vertex w^*. We assume the following

- If G' is type 2A, the two internally disjoint $u - w^*$ doubled paths in G^* have vertex sequences $u, x_1, x_2 \ldots, x_k, w^*$ and $u, y_1, y_2 \ldots, y_l, w^*$ (see Fig. B.1a). We will also assume $N_{G'}(u) \cap W = \{w\}$.
- If G' is type 2B, the doubled path from u to w^* in G^* has vertex sequence $u, x_1, x_2 \ldots, x_k, w^*$, and the tripled 2-path from u to w^* has vertex sequence u, y, w^*(see Fig. B.1b).
- If G' is type 2C, the doubled path from u to w^* in G^* has vertex sequence $u, x_1, x_2 \ldots, x_k, w^*$ (see Fig. B.1c).

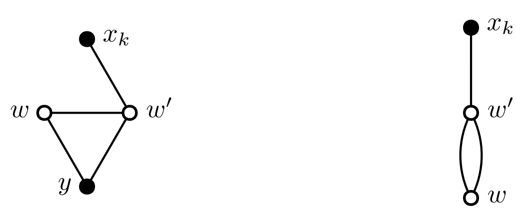


Figure B.4

Claim: If G' has type 2 and $|W| = 2$ we have the following immersions:

1. If G' is type 2A and $N(u) \cap W = \{w\}$, then $H = G'[W \cup \{x_k, y_l\}]$ satisfies

   (a) H with roots x_k and y_l has a rooted immersion of D_2 unless it is isomorphic to one of the two graphs on the left in Figure B.5.

   (b) H has an immersion of the graph on the left in Figure B.4 unless it is isomorphic to the leftmost graph in Figure B.5 where x_k is the top vertex and y_l the bottom.

2. If G' is Type 2B, then $H = G'[W \cup \{x_k, y\}]$ with roots x_k and y has a rooted immersion of D_2 unless it is isomorphic to the graph on the right in Figure B.5.

3. If G' is type 2C and does not have type 2A or 2B, then $H = G'[W \cup \{x_k\}]$ satisfies

   (a) H immerses a triangle.

   (b) Suppose $e(u, w) = 1$ (and thus $e(u, w') = 2$). Then H immerses the graph on the right in Figure B.4.

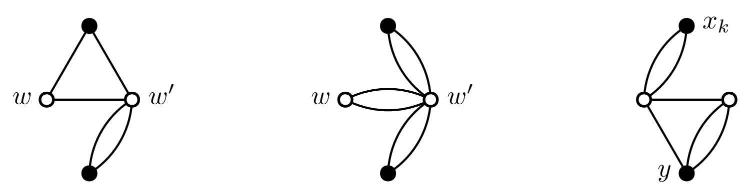


Figure B.5

Proof of Claim: For (a) of the first part, observe that the depicted graphs are the only possible cases in which the vertex w has degree at most two in the graph H. Otherwise Corollary B.10 implies that H has the desired rooted immersion unless there exists $U \subseteq V(H)$ with $|U| = 2$ and $d_H(U) \leq 2$. If such a set U exists, degree considerations imply that the underlying simple graph of H must be a path of length three from x_k to y_l where the middle edge has multiplicity either 1 or 2 in H. In the former case we have a contradiction to the internal 4-edge-connectivity of G; in the latter we have a contradiction to our minimality assumption on W. The proof of (b) follows immediately from (a).

For the second part, Corollary B.10 implies the existence of a rooted D_2 immersion in H unless there exists $U \subseteq V(H)$ with $|U| = 2$ and $d_H(U) \leq 2$. If such a set U exists, then either H is as depicted on the right in the figure, or the underlying simple graph of H is a path of length three from x_k to y where the middle edge has multiplicity either 1 or 2 in H. The former case contradicts the internal 4-edge-connectivity of G and the latter contradicts the minimality assumption on W.

For (a) of the third part, note that the desired immersion exists if x_k has two neighbours in H or if there are two edges with both ends in W. Otherwise we may assume (without loss) that $e(x_k, w) = e(y, w') = 2$ and $e(w, w') = e(y, w) = 1$, but then $d_G(\{y, w'\}) = 4$ contradicting the near-root 5-edge-connectivity of G. For (b) we must have $d_H(w') \geq 2$ or we get a similar contradiction to the near-root 5-edge-connectivity. If $d_H(w') = 2$, then G' is type 2A relative to $\{x_k, w\}$ which is a contradiction. So we must have $d_H(w') \geq 3$ and the result follows easily from this. This completes the proof of the claim.

The remainder of the proof of the lemma will be broken up into cases depending on the operation taking G to G' and then subcases depending on the type of the resulting graph.

For each such subcase we will then have to consider all of the possibilities for the vertex v (and possibly z). Our first case is where the operation taking G to G' is Operation 1.

- If G' is type 2A:

  If $N_G(v) = \{u, x_1\}$ or $\{u, y_1\}$, then G is type 2A (a contradiction). Now if $w \notin N_G(v)$, then $|N_G(u)| > 3$, a contradiction. So, $w \in N_G(v)$ and without loss we have $N_G(v) = \{u, w, x_1\}$. If $|W| = 1$, and $k \geq 2$, then G has a rooted W_4-immersion on $\{u, v, x_1, x_k, w\}$. However, if $k = 1$, then G is type 2A (relative to $\{x_1, w\}$).

  Next suppose $|W| = 2$, and let $H = G'[W \cup \{x_k, y_l\}] = G[W \cup \{x_k, y_l\}]$. If H immerses the graph on the left in Figure B.4, then G has a rooted W_4-immersion. Otherwise part 1 of the claim implies that G is type 2A relative to $\{w, x_1\}$.

- If G' is type 2B:

  It is straightforward to verify that if $N(v) = \{u, x_1\}$, then G has type 2B. Also, one could easily see that if $N(v) = \{u, x_1, y\}$ then G immerses the graph in Fig. B.6a, thus immerses W_4. In the remaining cases, we have $N(v) = \{u, y\}$. If $|W| = 1$, then G is type 2A (relative to $\{y, w\}$). Suppose $|W| = 2$, and let $H = G[W \cup \{x_k, y\}] = G'[W \cup \{x_k, y\}]$. Observe that if there is an immersion of D_2 in H with roots x_k, y, then G has a W_4-immersion on $\{u, v, y, w, w'\}$. Otherwise part 2 of the claim implies that G has type 2A relative to the doubleton consisting of y and the vertex in W joined to y by two edges.

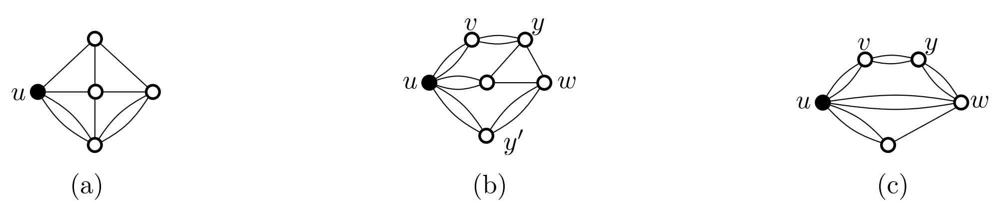


Figure B.6

- If G' is type 2C (and not 2A or 2B):

  If $N_G(v) = \{u, x_1\}$, then G has type 2C. We now split the analysis into cases depending on $|N_{G'}(u) \cap W|$. First suppose $N(u) \cap W = \{w\}$. If $N(v) = \{u, w\}$, then G is type 2A (relative to $\{w, w'\}$). Also, it is easy to see that if $N(v) = \{u, x_1, w\}$ then G immerses the graph in Fig. B.6a, thus immerses W_4.

  Next suppose $N_{G'}(u) \cap W = \{w, w'\}$, where $e_{G'}(u, w) = 1$ (, and so $e_{G'}(u, w') = 2$). Note v is adjacent to w, otherwise $|N_G(u)| > 3$. There are now two possibilities: $N_G(v) = \{u, w, x_1\}$, or $N_G(v) = \{u, w, w'\}$. In the latter case, G is type 2A (relative to $\{w, w'\}$). In the former case, part 3 of the claim implies that $G'[W \cup \{x_k\}] = G[W \cup \{x_k\}]$ immerses a triangle. Therefore, G has a W_4-immersion on $\{u, v, x_1, x_k, w\}$.

- G' is type 3 or 4:

  Observe that $|N_{G'}(u)| \geq 3$. So, in order for $|N_G(u)| \leq 3$ to hold, we must have $|N_G(v)| = 2$. Let $N_G(v) = \{u, z\}$, where z is a neighbour of u in G', and thus $e_{G'}(u, z) \geq 2$. It could then immediately be seen that in all type 3, and 4 graphs $z \neq w$. If G' has type 3 and $z = x_1$ then G has the same type as G'. So, suppose

$z \in \{y, y', y''\}$ (, see Fig. B.2 and B.3). If G' has type 3B or 4B, and $z = y$, then G immerses the graph in Fig. B.6b, which has a W_4-immersion on $\{u, v, y, y', w\}$.

For the remaining cases, we have $N_{G'}(z) = \{u, w\}$. Note it follows from the internal 4-edge-connectivity of G that $d_G(\{v, z\}) \geq 4$. So $d_{G'}(z) \geq 4$, so either $e_{G'}(u, z) \geq 3$, or $e_{G'}(z, w) \geq 2$. If $e_{G'}(u, z) \geq 3$, G has a W_4-immersion. Observe that it suffices to verify this for G' being type 4A, and $z = y$. In this case, G has a W_4-immersion on $\{u, v, y, y', w\}$. On the other hand, if $e_{G'}(u, z) < 3$, we have $e_{G'}(z, w) \geq 2$. So, in particular G' has type 4A or type 4B with $z = y''$. In the latter case, G has type 3B. In the former case, suppose $z = y$. Then if $e(y', w) = e(y'', w) = 1$, G has type 3A. Else, G immerses the graph in Fig. B.6c, which has a W_4-immersion.

Next, we consider the case where Operation 2 applied to G gives G'. Note that in this case the vertex z must satisfy $e_{G'}(u, z) = 1$, so G' cannot be type 2B. Below we resolve the other subcases.

- If G' is type 2A:

  Since $e_{G'}(u, z) = 1$ we may assume (without loss) that $z = w$ and $v = x_1$. If $|W| = 1$, and $k < 3$, then G has type 2B. However if $k \geq 3$, there is a W_4-immersion in G on $\{u, x_1, x_2, x_k, w\}$. So, suppose $|W| = 2$. If $k = 1$, then G is of type 2B. If $k \geq 2$ and $G[W \cup \{x_k, y_l\}] = G'[W \cup \{x_k, y_l\}]$ has an immersion of the graph on the left in Figure B.4, then G has a W_4-immersion on $\{u, x_1, x_k, w, w'\}$. Otherwise, by applying part 1 of the claim we deduce that for $k = 2$ the graph G is type 2B (relative to $\{x_2, w\}$) and for $k \geq 3$ the graph G has a rooted W_4 immersion on $\{u, x_1, x_2, x_k, w\}$.

- If G' is type 2C (and not 2A or 2B):

  Since $e_{G'}(u, z) = 1$ it must be that $|N_{G'}(u) \cap W| = 2$. Let $W = \{w, w'\}$, where $e_{G'}(u, w) = 1$ (, and so $e_{G'}(u, w') = 2$), so $z = w$. There are two possibilities for v, it could be x_1 or w'. If $v = w'$, then G is type 2C. So suppose $v = x_1$. Now by part 3 of the claim we find that $H = G[W \cup \{x_k\}]$ immerses the graph on the right in Figure B.4. It follows from this that G has a rooted W_4-immersion on $\{u, x_1, x_k, w, w'\}$.

- If G' is type 3 or 4:

  Since $e_{G'}(u, z) = 1$ it must be that G' is type 4C and $z = w$. However, then G is type 4D.

The next step is to consider the cases where the operation on G which takes it to G' is Operation 3. Note that in this case the vertex z must satisfy $e_{G'}(u, z) \geq 2$ and v must satisfy $N(v) \setminus N(u) = \{u\}$. So, in particular, the root vertex u of G' must have a neighbour which has no other neighbour outside $N(u) \cup \{u\}$. It follows that G' cannot be type 2B. Below, we consider the remaining cases:

- If G' is type 2A:

  First suppose that $v = w$. If $|W| = 1$, in order for $N_G(v) \subset \{u\} \cup N(u)$ to hold, we must have $k = 1$ and $l = 1$, thus $|V(G)| < 5$, a contradiction. If $|W| = 2$, then the 3-edge-connectivity of G implies that $G[W]$ is connected and we have a violation of $N_G(v) \subset \{u\} \cup N(u)$. Therefore, $v \neq w$.

Without loss we may now assume $v = x_1$. Since $e_{G'}(u, z) \geq 2$ it must be that $z = y_1$. Now observe in order for $N_G(x_1) \subset \{u\} \cup N(u)$ to hold we must have $k = 1$ and $N_G(x_1) = \{u, y_1, w\}$. If $|W| = 1$, since $|V(G)| \geq 5$, we have $l \geq 2$. Then it is easy to verify that G has a W_4-immersion on $\{u, x_1, w, y_1, y_l\}$. And if $|W| = 2$, part 1 of the claim implies that there is an immersion of D_2 with roots $x_k = x_1$ and y_l (since w is incident with at least three edges of $\delta(W)$). Therefore, we get an immersion of W_4 on $\{u, x_1, w, w', y_1\}$.

- If G' is type 2C (and not 2A or 2B):

  Note $x_2 \in N_{G'}(x_1) \setminus (\{u\} \cup N_{G'}(u))$, so we must have $v \neq x_1$ (since $N(v) \subseteq N(u) \cup \{u\}$), and thus $v \in W$. If $z \in W$, then G too has type 2C. In the only remaining case $z \notin W$ and $v \in W$. So $z = x_1$. Now part 3 of the claim implies that $G[\{x_k\} \cup W] = G'[\{x_k\} \cup W]$ immerses a triangle. It follows that G has a rooted W_4-immersion on $\{u, x_1, x_k, w, w'\}$.

- If G' is type 3 or 4:

  Observe that in all type 3 and 4 graphs, except for type 4C, if $v \in N(u)$, then $N(v) \setminus (\{u\} \cup N(u))$ is nonempty. So the only case we need to check is when G' is type 4C, and $e_{G'}(u, z) \geq 2$. We may assume without loss of generality that $z = y$. If $v = y'$, by splitting $y'u$ and uy'' we get an immersion of W_4 in G. And if $v = w$, we split yw, wy' and split $y'u, uy''$, and delete one copy of edge uw to get a W_4-immersion in G.

Lastly, we consider the cases where the operation on G which takes it to G' is Operation 4. It is easy to verify that G' cannot be type 2, or 3B. If G' is type 3A (with $v \in \{y, y'\}$) or 4A or 4B (with $v = y''$), then G also has the same type. Finally, if G' is type 4C or 4D, then G immerses the graph in Fig. B.7, which has a W_4-immersion (split yu, uy', and $y''u, uy'$, and delete $y'w$). $\square$

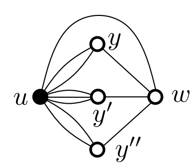


Figure B.7

B.4 Connectivity II

Thanks to Lemma B.8, we know that our minimum counterexample G has a root vertex x with at most 3 neighbours. It follows that there is at least one vertex $y \in N(x)$ with $e(x, y) \geq 2$. The main goal in this section is to prove a lemma which gives us an additional edge-connectivity property which applies whenever such a vertex y also satisfies $d(y) \geq 5$. Before proving this, we need a lemma to resolve one rather special case.

Lemma B.11. *The graph G does not contain distinct vertices $y, z, z' \in V \setminus \{x\}$ satisfying:*

- $N(y) = \{x, z, z'\}$
- $e(y, x) = e(y, z) = e(y, z') = 2$
- $d(z) = d(z') = 3$

Proof. Assume for a contradiction that the lemma is false. Note that z cannot be adjacent to z', as otherwise $\delta(\{y, z, z'\}) = E(x, y)$, contradicting 3-edge-connectivity. Let $N(z) = \{w, y\}$, and $N(z') = \{w', y\}$.

First suppose that $w = x$ and note that we must have $w' \neq x$ (otherwise x would be a cut vertex). It is not possible for $V(G) = \{x, y, z, z', w'\}$ since in this case w' has to be a neighbour of x, however $d(\{x, w'\}) < 5$. It follows from this and the internal 4-edge-connectivity of G that $d(x) \geq 6$. Since $e(x, \{y, z\}) = 3$, x has a neighbour $v \neq y, z$. By Lemma B.8 we may assume $N(x) = \{y, z, v\}$. Therefore, $e(x, v) \geq 3$. Consider $H = G - \{y, z, z'\}$. It follows from 3-edge-connectivity of G that H is 2-edge-connected. So, if $v \neq w'$ we may choose two edge-disjoint $v - w'$ paths in H. These paths together with the edges incident with x, y, z, z' in G give a rooted W_4 immersion. If $v = w'$, then choose $v' \in V(G) \setminus \{x, w'\}$. Since w' is a cut-vertex in G separating v' from $\{x, y, z, z'\}$, there exist three edge-disjoint $v' - w'$ paths in G (not using xw' edges). Now these paths together with the edges incident with x, y, z, z', w' in G give a rooted W_4 immersion.

Next suppose that $w = w'$. Let G^* be the graph obtained from $G - y$ by identifying $\{z, z'\}$ to a new vertex z^* and consider x and z^* to be root vertices of G^*. Note that $|V(G^*)| \geq 4$ as otherwise we would have $w \in N(x)$ and $d_G(\{x, w\}) = 4$ contradicting the near-root 5-edge-connectivity of G. If G^* has a rooted immersion of D_2 then the original graph G has a rooted immersion of W_4 and we are done. Otherwise, it follows from Theorem 3.4 for $m = 2$ (and $d_{G^*}(z^*) = 2$) that G^* has a set of vertices X with $x \in X$ and $|X| \leq 2$ so that the graph obtained from G^* by identifying X to a single vertex is a doubled path. In this case, we find that the original graph G has type 1, and this is a contradiction.

In the only remaining case we have that x, w, w' are distinct and it follows from this that the graph $H = G - \{y, z, z'\}$ is 2-edge-connected. Choose two edge-disjoint paths P_1, P_2 in H from x to w. If w' is contained in one of these paths then P_1, P_2 together with the edges incident to y, z, z' give a rooted W_4 immersion. Otherwise we may choose 2-edge-disjoint paths Q_1, Q_2 from w' to $V(P_1) \cup V(P_2)$ and we may assume that Q_1, Q_2 are edge-disjoint from P_1, P_2. If either Q_1 or Q_2 ends at a vertex in $(V(P_1) \cup V(P_2)) \setminus \{x, w\}$ then this path together with P_1, P_2 and the edges incident with y, z, z' give a rooted W_4 immersion. Otherwise Q_1 and Q_2 both end in $\{x, w\}$ and the paths P_1, P_2, Q_1, Q_2 together with the edges incident with y, z, z' give a rooted W_4 immersion. □

Lemma B.12. *If the graph G contains a vertex $y \in N(x)$ which satisfies $d(y) \geq 5$ and $e(x, y) \geq 2$, then every internal edge-cut separating x and y has size at least 6.*

Proof. Suppose for a contradiction that the lemma is false, and choose a minimal set $Y \subseteq V$ satisfying the following properties:

- $x \notin Y$ and $y \in Y$,

- $|Y|, |V \setminus Y| \geq 2$
- $d(Y) \leq 5$

Note that the 3-edge-connectivity of G implies that the subgraph induced on Y is connected. Now we will choose a neighbour of y according to the following rule: If $(N(y) \cap Y) \setminus N(x)$ is nonempty, let $z \in (N(y) \cap Y) \setminus N(x)$. Otherwise, let $z \in N(y) \cap Y$. We proceed to prove a series of properties concerning Y and z.

(1) There does not exist $Z \subseteq V$ satisfying all of the following properties:

- $x, z \in Z$ and $y \notin Z$,
- $|Z| \geq 3$ and $|V \setminus Z| \geq 2$
- $d(Z) \leq 5$

Suppose (for a contradiction) that Z satisfies all of the above. If $Y \cup Z = V$ then $Y' = V \setminus Z$ satisfies $Y' \subseteq Y$ and $|Y'| \geq 2$ and $d(Y') \leq 5$ so it contradicts the choice of Y. So all four of the sets $Y \setminus Z$, $Z \setminus Y$, $Y \cap Z$, and $V \setminus (Y \cup Z)$ are nonempty. The edges between x and y are contained in $E(Y \setminus Z, Z \setminus Y)$ so by uncrossing we have

$$d(Y \cap Z) + d(Y \cup Z) = d(Y) + d(Z) - 2e(Y \setminus Z, Z \setminus Y) \leq 5 + 5 - 4 = 6.$$

It now follows from 3-edge-connectivity that $d(Y \cap Z) = 3 = d(Y \cup Z)$ and from internal 4-edge-connectivity that $Y \cap Z = \{z\}$ and $V \setminus (Y \cup Z) = \{z'\}$ for some vertex z'. Furthermore, we must have equality in the above equation, so $d(Y) = d(Z) = 5$. The number $d(Y \setminus Z)$ must be even since $d(Y) = d(Y \setminus Z) + d(Y \cap Z) - 2e(Y \setminus Z, Y \cap Z)$. If $d(Y \setminus Z) = 4$ then $Y \setminus Z$ cannot equal $\{y\}$ by the assumption $d(y) \geq 5$, but then $Y \setminus Z$ contradicts the choice of Y. Therefore we must have $d(Y \setminus Z) \geq 6$. It follows from a parity argument (similar to that for $d(Y \setminus Z)$) that $d(Z \setminus Y)$ is even. Uncrossing gives us

$$d(Z \setminus Y) + d(Y \setminus Z) \leq d(Y) + d(Z) = 10$$

from which it follows that $d(Z \setminus Y) = 4$ and $d(Y \setminus Z) = 6$. Furthermore, we must have $d(z, Z \setminus Y) = 1 = d(w, Z \setminus Y)$ and $d(z, Y \setminus Z) = 2 = d(w, Y \setminus Z)$. Now form a new graph H from G by identifying $Z \setminus Y$ to a new root vertex (of degree 4). Note that $|V(H)| < |V(G)$ follows from the assumption $|Z| \geq 3$. The graph H is 3-edge-connected, internally 4-edge-connected, and near-root 5-edge-connected. If $|V(H)| = 4$ then Lemma B.11 gives a contradiction. Otherwise $|V(H)| \geq 5$ and Theorem 4.2 applies nontrivially to H. Since the root vertex in H has degree 4 and H is near-root 5-edge-connected, H must contain a rooted W_4 immersion. Now the internal 4-edge-connectivity of G implies that G also contains a rooted W_4 immersion.

(2) $N(y) \cap Y \subseteq N(x)$ (in particular $z \in N(x)$).

Suppose (for a contradiction) the above condition is violated. It then follows from our choice of z that $z \notin N(x)$. Form a new rooted graph G' from G by splitting off the edges xy and yz. Note that $d_{G'}(\{x, z\}) = d_G(x) + d_G(z) - 2 \geq 5$. It now follows from (1) that the graph G' is nicely edge-connected, but this contradicts Lemma B.9.

(3) $d(\{x,z\}) > 6$

Suppose (for a contradiction) that $d(\{x,z\}) \leq 6$. Note that $Y \cup \{x\} \neq V$, so all four of the sets $\{x\}$, $\{z\}$, $Y \setminus \{z\}$ and $V \setminus (Y \cup \{x\})$ are nonempty. Suppose (for a contradiction) that $Y \setminus \{z\}$ contains a neighbour z' of y. It follows from uncrossing that

$$d(Y \cup \{x\}) + d(z) = d(\{x,z\}) + d(Y) - 2e(x, Y \setminus \{z\}) \leq 11 - 2e(x, Y \setminus \{z\}). \quad \text{(B.1)}$$

Since $d(Y \cup \{x\}), d(z) \geq 3$ it must be that the only edges between x and $Y \setminus \{z\}$ are two edges with endpoints x, y. However in this case the vertex z' contradicts (2). So y is an isolated vertex in the graph induced on $Y \setminus \{z\}$. Now uncrossing yields

$$4 + d(y) + d(Y \setminus \{y,z\}) \leq d(x) + d(Y \setminus \{z\}) \leq d(\{x,z\}) + d(Y) \leq 11.$$

Since $d(y) \geq 5$ it follows that $Y \setminus \{y,z\} = \emptyset$. The entire graph G must have at least 5 vertices, so the set $W = V \setminus (\{x\} \cup Y)$ has size at least 2. Now internal 4-edge-connectivity implies $d(W) = d(Y \cup \{x\}) \geq 4$ so equation (B.1) implies $d(z) = 3$ and $d(W) = 4$. If $e(y,z) = 1$ then $d(Y) = d(\{y,z\}) = d(y) + d(z) - 2 \geq 6$ which is a contradiction. Therefore, $e(y,z) = 2$ and now the constraints $d(W) = 4$, $d(\{x,z\}) \leq 6$, and $d(Y) = d(\{y,z\}) \leq 5$ imply that $d(y,W) = 2 = d(x,W)$.

Now we will consider the graph H obtained from the graph induced on $V \setminus \{z\}$ by deleting the edges between x and y. First suppose that H has a cut-edge e separating x and y and let U be the vertex set of the component of $H - e$ that contains x. Since $d_H(x) = 2$ the set $U \setminus \{x\}$ is nonempty and $d_G(U \setminus \{x\}) \leq 3$. It follows from this that $U = \{x, y'\}$ for some $y' \in N(x)$. However, we now have $d_G(U) = 4$ and this contradicts Lemma B.2. So the graph H has no cut-edge separating x and y and therefore H is 2-edge-connected. If H has a rooted D_2 immersion with the roots x, y then the original graph G has a W_4-immersion. Otherwise it follows from Theorem 3.4 that H may be obtained by taking a path from x to y and then adding one additional copy of each edge. In this case G is type 2 giving us a contradiction.

With this last item in place we are ready to complete the proof of the lemma. Let G' be the graph obtained from G by splitting the $x - y - z$. It follows from (3) and (1) that G' is nicely edge-connected. Now we get a contradiction as in the proof of (2). □

B.5 Proof of main theorem

In this section we will complete our proof of our theorem on rooted W_4 immersions. We require just one additional lemma before we commence with this argument.

Lemma B.13. *The graph G does not have a vertex $y \in N(x)$ satisfying the following:*

- $e(x,y) \geq 2$
- $|N(y)| = 3$
- $N(y) \setminus \{x\} \subseteq N(x)$

Proof. Let $N(y) = \{x, y', y''\}$, so $y', y'' \in N(x)$. Note by Lemma B.8 we may assume $N(x) = \{y, y', y''\}$.

First, suppose $e(y, y') \geq 2$ and $e(y, y'') \geq 2$. Then it follows from 3-edge-connectivity that $d(\{x, y, y', y''\}) \geq 3$. So we have, say, $e(y', V \setminus \{x, y, y', y''\}) \geq 2$, and thus $d_G(\{x, y'\}) \geq 7$. Now we split off edges xy and yy' to get a new graph G'. Note $|V(G')| \geq 5$, and it follows from Lemma B.12 and $d_G(\{x, y'\}) \geq 7$ that G' is nicely-edge-connected. However this gives a contradiction to Lemma B.9.

Next, suppose $e(y, y') \geq 2$ and $e(y, y'') = 1$. Then it follows from $d_G(\{x, y, y'\}) \geq 4$ that $d_G(\{x, y'\}) \geq 7$. Now we split off edges xy and yy' to get a new graph G'. As in the previous case, G' is nicely-edge-connected, and we get a contradiction.

Finally, suppose $e(y, y') = e(y, y'') = 1$. By near-root 5-edge-connectivity of G we have $d(\{x, y\}) \geq 5$, so either $e(x, y') \geq 2$ or $e(x, y'') \geq 2$. Suppose (for a contradiction) that both $e(x, y') \geq 2$ and $e(x, y'') \geq 2$. Then $d(\{x, y, y', y''\}) \geq 3$ implies that, say $e(y', V \setminus \{x, y, y', y''\}) \geq 2$. Thus $d(y') \geq 5$, and y' has a neighbour $z \notin \{x\} \cup N(x)$. Now, we form a new graph G' by splitting off xy' and $y'z$. Lemma B.12 implies that G' is nicely-edge-connected and we will get a contradiction as before.

Therefore, suppose $e(x, y') \geq 2$, and $e(x, y'') = 1$. It then follows from $d(\{x, y, y'\}) \geq 4$ that $d(y') \geq 5$. If y' has a neighbour $z \in V \setminus \{x, y, y''\}$, we will split off xy' and $y'z$, and as before, Lemma B.12 gives us a contradiction. Also, if $N(y') \subseteq \{x\} \cup N(x)$, we would have $e(y', y'') \geq 2$, and using a similar argument as above for y' (instead of y) we get a contradiction. □

Proof of Theorem B.1. We will establish a sequence of properties of G eventually proving it cannot exist.

(1) There does not exist $y \in N(x)$ so that $e(x, y) \geq 2$ and $d(y) \geq 5$.

Suppose (for a contradiction) that such a vertex y exists. If there exists a vertex $z \in N(y) \setminus \{x\}$ which is not a neighbour of x then we may form a new graph G' by splitting off the edges xy and yz. It follows from Lemma B.12 that G' is nicely edge-connected, but this contradicts Lemma B.9. It follows that $N(y) \setminus \{x\} \subseteq N(x)$. Next suppose that $N(y) = \{x, z\}$ for some $z \in V$. As before, we form the graph G' by splitting off the edges xy with yz. Lemma B.12 implies that G' is 3-edge-connected and internally 4-edge-connected. In addition, it follows from $d_G(\{x, y, z\}) \geq 4$ that $d_{G'}(\{x, z\}) \geq 5$ and this implies that G' is nicely edge-connected. Now we get a contradiction by the same argument as above. The only remaining case is resolved by Lemma B.13.

(2) There does not exist $y \in N(x)$ so that $e(x, y) \geq 3$.

Suppose (for a contradiction) that such a vertex y exists. The set $N(y) \setminus \{x\}$ must be nonempty by Lemma B.4. However $d(y) \leq 4$ by (1), so the only possibility is $e(x, y) = 3$ and $d(y) = 4$. In this case we form a new graph G' from G by deleting one edge with ends x, y. It follows from the internal 4-edge-connectivity of G that $d_{G'}(\{x, y'\}) \geq 5$ for every $y' \in N(x) \setminus \{y\}$ and thus G' must be nicely edge-connected. However this contradicts Lemma B.9.

(3) If $|V(G)| > 5$, there does not exist $y \in N(x)$ satisfying $e(x, y) = 2$ and $d(y) = 4$ and $E(y, N(x) \setminus \{x\}) = \emptyset$.

Suppose (for a contradiction) that such a vertex y exists. Let e, e' be the edges incident with y and not x and assume $e = yz$ and $e' = yz'$. Construct a new graph G' from G by splitting the vertex y so as to form a new edge with ends x, z and another new edge with ends x, z'. It follows from our assumptions that G' is nicely edge-connected, but this contradicts Lemma B.9.

(4) We have $|N(x)| = 3$ and $5 \leq d(x) \leq 6$.

It follows from Lemma B.8 that $|N(x)| \leq 3$. First suppose that $d(x) = 4$ and choose $y \in N(x)$ so that $e(x, y) \geq 2$. Note that (2) implies $e(x, y) = 2$. Now $d(\{x, y\}) \geq 5$ (by the near-root 5-edge-connectivity of G), but this implies $d(y) \geq 5$ which contradicts (1). So it must be that $d(x) \geq 5$. It follows from this and (2) that $|N(x)| = 3$ and $5 \leq d(x) \leq 6$ as claimed.

(5) There is at most one pair of adjacent vertices in $N(x)$.

Suppose (for a contradiction) that $N(x) = \{y, y', y''\}$ where y is adjacent to both y' and y''. If $e(x, y) > 1$ then (1) and Lemma B.13 give us a contradiction. Now (4) and (2) imply that $d(x, y) = 1$ and $e(x, y') = 2 = e(x, y'')$. It follows from the near-root 5-edge-connectivity of G and (1) that $d(y') = d(y'') = 4$. (Note that Lemma B.13 implies that y' and y'' are not adjacent.) If $|V(G)| = 5$ then G has type 2 and we have a contradiction. Otherwise, note that the internal 4-edge-connectivity of G implies that $e(y, V \setminus \{x, y, y', y''\}) \geq 2$. Let $N(y') = \{x, y, z\}$ (, and note z may be equal to y). Now, form a graph G' by splitting the vertex y' so as to form one new edge with ends x, y and another with ends x, z. The graph G' is 3-edge-connected and internally 4-edge-connected as a result of G having these properties. Furthermore, the near-root 5-edge-connectivity of G and the observation $e_G(y, V \setminus \{x, y, y', y''\}) \geq 2$ imply that G' is nicely edge-connected. Now Lemma B.9 gives us a contradiction.

(6) $d(x) = 6$.

Suppose (for a contradiction) that this condition fails. Then by (4) and (2) we may assume $N(x) = \{y, y', y''\}$ where $e(x, y) = e(x, y') = 2$ and $e(x, y'') = 1$. Now (1) and the near-root 5-edge-connectivity of G imply $d(y) = d(y') = 4$. The vertices y and y' must be non-adjacent as otherwise we would have $d(\{x, y, y'\}) \leq 3$ contradicting the internal 4-edge-connectivity of G. If $|V(G)| = 5$ then G has type 2, which is also a contradiction, so $|V(G)| \geq 6$. Now by (5) one of the vertices y, y' must have no neighbour in $N(x) \setminus \{x\}$ and this gives a contradiction to (3).

(7) There does not exist a pair of adjacent vertices in $N(x)$.

Suppose (for a contradiction) that $N(x) = \{y, y', y''\}$ and that $e(y', y'') > 0$. Note that by (6) and (2) we must have $e(x, y) = e(x, y') = e(x, y'') = 2$. It follows from this and (1) that $e(y', V \setminus \{x, y''\}), e(y'', V \setminus \{x, y'\}) \leq 1$. However, since G is internally 4-edge-connected we must have $e(y', V \setminus \{x, y''\}), e(y'', V \setminus \{x, y'\}) = 1$. Note that this condition together with (1) implies $e(y', y'') = 1$. If $d(y) = 3$ then $d(\{x, y, y', y''\}) = 3$ and the internal 4-edge-connectivity of G implies that G is type 4B. Otherwise (1) implies that

$d(y) = 4$. If $|V(G)| > 5$ then y gives us a contradiction to (3); otherwise G has type 3B which again gives us a contradiction.

With this last property in place, the proof is nearly done. Every $y \in N(x)$ satisfies $e(x, y) = 2$ by (6) and (2) and then y must satisfy $d(y) \leq 4$ by (1). If $|V(G)| = 5$ then (7) implies that G has type 3A or 4A. Otherwise the internal 4-edge-connectivity of G implies that there exists $y \in N(x)$ with $d(y) = 4$. Now this vertex y gives a contradiction to (3) and this completes the proof. $\square$