The Psychometric Confound:
A Perennial Gremlin of the Psychopathology Literature

by

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Abstract

Chapman and Chapman (1973) identified an issue in psychopathology research that has
since come to be known as the Psychometric Confound (MacDonald, 2008). They
claimed, essentially, that various traditional inferential methods for drawing conclusions
regarding ability deficits in a population with some particular pathology were flawed.
The work of the Chapmans has since been cited frequently in the psychopathology field,
with most citing authors echoing their concerns, and some applying their proposed
solutions. However, the precise nature of the phenomenon remains in question. The goal
of the current work is to elucidate, in mathematics, the issues raised by Chapman and
Chapman, and their commentators, to a level which allows for an adjudication of the core
claims of these authors. We begin by providing a clear and concise description of
Chapman and Chapman's account of the Psychometric Confound, including a description
of the research context; an articulation of the general inferential problem; an itemization
of claims, including claims regarding methodological solutions; and a description of
problems inherent in Chapman and Chapman's account. We then consider the influence
of the Chapmans' discussion regarding the Psychometric Confound on the
psychopathology literature as a whole, including a summary of the alternative accounts of
the problem that have emerged in response to the work of Chapman and Chapman. A full
mathematization, and consequent adjudication, of the claims of Chapman and Chapman,
is then provided. Fundamentally, this involves an elucidation and formalization of the
test theory, both classical and modern, nascent in all work regarding the Psychometric
Confound from Chapman and Chapman on. A mathematization and adjudication of the
claims of the alternative accounts follows. Finally, we determine if valid methodological
solutions for the quantities of interest are possible, given the technical, test-theory based
framework established. We show that a structural equation model consistent with the
proto-framework implied by Chapman and Chapman provides a basis for valid inference
regarding the quantity of interest.

Acknowledgements

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Chapter 1.   Introduction

Beginning in 1973, Loren J. Chapman and Jean P. Chapman published a series of papers claiming that methods commonly employed in the psychopathology literature were seriously flawed, and consequently, that the conclusions drawn in numerous past studies conducted in the psychopathology field were erroneous. They published voluminously on this issue, which has since come to be known as the issue of the *Psychometric Confound* (MacDonald, 2008), and offered to the research community a multitude of methods by which it could be overcome.

Soon, other psychopathy researchers joined the fray, providing commentary on the account of Chapman and Chapman, and, oft-times, offering up their own, distinct, characterizations of the problem, as well as their own novel solutions. Despite such attempts at clarification of the issue of the Psychometric Confound, it is viewed as "largely unresolved" (Palmer, Dawes & Heaton, 2009, p.368), and "a perennial gremlin in the experimental psychopathology literature" (MacDonald, 2008, p. 621). Nevertheless, acknowledgment of the issue can, now, be found not only within a broad selection of psychopathology research areas[1], but also areas of research beyond psychopathology, including memory research (Lum, Kidd, Davis, & Conti-Ramsden,

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[1] Examples include major depressive disorder (Joorman & Gotlib, 2008), attention-deficit / hyperactivity disorder (Huang-Pollock & Karalunas, 2010), learning disorders (Savage, Lavers, & Pillay, 2007), autistic disorder (Pellicano, 2010), and bipolar disorders (Kurtz & Garrety, 2009).

2009), research into the psychological effects of alcohol (Moberg & Curtin, 2009), and research into aging (e.g., Salthouse & Coon, 1994).

Critically, despite widespread belief that the Chapmans identified a science-undermining problem referred to by the term "Psychometric Confound", the nontechnical approach to the problem taken by both Chapman and Chapman, and by those who have followed in their footsteps, has prevented agreement regarding its precise nature, or even proof of its existence. Consequently, virtually all major claims surrounding the Psychometric Confound, including the efficacies of the multitude of solutions to the problem that have been proposed, remain in doubt.

The goal of the thesis is to elucidate, in mathematics, the issues raised by Chapman and Chapman, and their commentators, under the heading "Psychometric Confound" to a level sufficient to allow for an adjudication of the core claims of the Chapmans and their commentators.

The thesis will be organized as follows: In chapter 2, a clear and concise description of Chapman and Chapman's account of the Psychometric Confound will be provided, including a description of the research context; an articulation of the general inferential problem; an itemization of claims, including claims regarding methodological solutions; and a description of problems inherent in Chapman and Chapman's account. In Chapter 3, the influence of the Chapmans' discussion regarding the Psychometric Confound on the psychopathology literature as a whole will be examined, including a summary of the alternative accounts of the problem that have emerged in response to the

work of Chapman and Chapman. These alternative accounts, and associated claims made, will be outlined. In Chapter 4, a full mathematization, and consequent adjudication, of the claims of Chapman and Chapman, is undertaken. Fundamentally, this will involve an elucidation and formalization of the test theory, both classical and modern[2], nascent in all work regarding the Psychometric Confound from Chapman and Chapman on. In Chapter 5, a mathematization and adjudication of the claims of the alternative accounts will be completed. In Chapter 6, we determine if valid methodological solutions for the quantities of interest are possible, given the technical, test-theory based framework established. In Chapter 7, we summarize the key points of the work.

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[2] While divisions between classical and modern test theory differ by author (McDonald, 1999), we, in accord with, e.g. Borsboom, 2006, take as central to classical test theory the concept of the true score. In contrast, modern test theory is characterized by references to latent variables.

Chapter 2.

The Psychometric Confound According to the Chapmans

Discussion of the issues housed under the term "Psychometric Confound" originates in the work of Loren and Jean Chapman, and, in particular, papers that they wrote in 1973 and 1978, published in *Psychological Bulletin* and *The Journal of Psychiatric Research*, respectively. In this chapter, we provide: a) a clear and concise description of Chapman and Chapman's account of the Psychometric Confound, including an elucidation of the central claims they made apropos the nature and existence of the issue and the methods they put forth as solutions to the problem; b) an overview of the problems of the account provided by the Chapmans.

The structure of the chapter is as follows. Firstly (section 2.1), we consider the research context implied by Chapman and Chapman, including the types of tests, populations, abilities, and quantities under consideration. Secondly (section 2.2), we describe the general inferential problem that they considered. Thirdly (section 2.3), we describe two purported flaws ("confounds") in traditional methods meant to draw inferences regarding the quantities of interest, isolating, for each, the core claim. We label these purported flaws the "First Confound of Chapman and Chapman" (C&C Confound1 for short) and the "Second Confound of Chapman and Chapman" (C&C Confound2 for short). It is the general inferential problem, C&C Confound1, and C&C

Confound2, that constitute the issue of the Psychometric Confound, according to the Chapmans[3]. Fourthly (section 2.4), we outline Chapman and Chapman's proposed inferential solutions for overcoming C&C Confound2, isolating the core claim on which each method is based in section 2.5. Lastly (section 2.6), we highlight problems in Chapman and Chapman's account.

2.1 **Research Context**

The psychopathology research context in which is grounded the concern for the Psychometric Confound, and as was implied by Chapman and Chapman (1973, 1978a, 1978b), is as follows. There exist:

    a) two populations of individuals, a population C of healthy controls and a population P of individuals suffering from some particular psychopathology;

    b) a set of k abilities $\{s_1, s_2,\ldots,s_k\}$, each individual having a score (which is unknown to the researcher) on each ability;

    c) k sets of tests $(T_1\ldots T_k)$, all tests t_{lj}, $l=1..p_j$, contained within the jth set, invented for the purpose of scaling individuals on ability s_j;

    d) composites of the scores of the items contained in a test (known as "test composites"), $c_{lj} = \sum_{d=1}^{plj} w_{dlj} I_{dlj}$, wherein test t_{lj} is comprised of p_{lj} items, I_{dlj}, $d=1\ldots p_{lj}$, and w_d, $d=1\ldots p_{lj}$, is the weight for item I_{dlj}. Unless otherwise specified, the typical, or default, circumstance should be assumed, in which $w_d = 1$, $d=1\ldots p_{lj}$, the test composite c_{lj} being, then, a *unit-weighted* composite.

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[3] Note that Chapman and Chapman did not, to our knowledge, employ the term "Psychometric Confound". However, the general inferential problem and confounds listed are now considered to fall under this term (e.g. Kang and MacDonald, 2010). Further note that the term "Psychometric Confound" refers not only to issues raised by Chapman and Chapman, but also those raised by their commentators, as reviewed in Chapter 3.

The interest of the Chapmans, consistent with the interest of many researchers in the psychopathology field, was in the determination of whether a "specific cognitive deficit" exists in group *P*; essentially, whether individuals in *P* suffer from a deficit, relative to those individuals in *C*, in respect to some particular ability s_j. In other words, when ruling on a "specific cognitive deficit", the aim is to determine whether the conditional distribution $s_j|P$ is stochastically lower than the conditional distribution $s_j|C$ (meaning that, in general, scores on ability s_j are lower in population *P*, than in population *C*). Though interest is in stochastic differences, in general, it cannot be denied that virtually all of the attention has been given to differences in the means, $\mu_{s_j|P}$ and $\mu_{s_j|C}$, of the *C* and *P* distributions. We will call the state of affairs in which $s_j|P$ is stochastically lower than $s_j|C$, and, in particular, $\mu_{s_j|P}$ is less than $\mu_{s_j|C}$, a *Specific Ability Deficit* (in reference to ability s_j and populations *P* and *C*; hereafter, an "*S-deficit*" in reference to ability s_j and populations *P* and *C*).

Of secondary interest to the Chapmans, and many other psychopathology researchers, was what we shall call a *Differential Ability Deficit* (hereafter, *D-deficit*)[4]. A D-deficit, in reference to abilities s_j and s_v, and populations *P* and *C*, exists when the difference $s_j|P$ - $s_j|C$ is stochastically lower than the difference $s_v|P - s_v|C$. Under certain

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[4] The Chapmans employed the synonymous but less descriptive term "differential deficit".

conditions, the existence of a D-Deficit can be shown to imply an S-Deficit[5]. Once again, apropos D-deficits, the focus has been almost exclusively upon mean differences.

2.2 General Inferential Problem

According to the Chapmans, considering the research context described above, the researcher whose aim it is to determine whether there exists an S-deficit in respect to an ability s_j faces an inferential problem (the general inferential problem): he does not have available to him the distributions $s_j|P$ and $s_j|C$ (it is held, in fact, that he cannot know the scores of individuals apropos an ability such as s_j, i.e. the ability is unobservable), but only inferential information about the distributions $c_{lj}|P$ and $c_{lj}|C$ obtained through the application of one or more of the tests in set T_j to samples drawn from populations P and C. Hence, he must devise, and then employ, a strategy that takes, as input, inferential information about the distributions $c_{lj}|P$ and $c_{lj}|C$, and yields, as output, a decision as to whether there exists an S-deficit in respect to s_j. In the words of Chapman and Chapman, "the central problem is how to move from statements about…deficit in performance on specific tests to statements about…deficit in ability" (Chapman & Chapman, 1978a, p.303).

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[5] A D-Deficit in s_j with respect to s_v implies an S-Deficit in reference to s_j under the condition of an existing S-Deficit in reference to s_v. The logic is as follows: Suppose a D-Deficit as indicated by $(\mu_{s_j/P} - \mu_{s_j/C}) - (\mu_{s_v/P} - \mu_{s_v/C}) < 0$. Furthermore, suppose an S-Deficit with respect to ability s_v as indicated by the following: $(\mu_{s_v/P} - \mu_{s_v/C}) = q < 0$, where q is some quantity. Taken together, $(\mu_{s_j/P} - \mu_{s_j/C}) - (q) < 0$, so that $(\mu_{s_j/P} - \mu_{s_j/C}) < q < 0$, implying an S-Deficit with respect to s_j.

Traditional inferential strategies meant to overcome the general inferential problem, however, according to Chapman and Chapman, were flawed. Below, we describe the *First-* and *Second- Confound of Chapman and Chapman*, which are flaws, or confounds[6], associated with, respectively, traditional inferential methods which we refer to as the *First-* and *Second Strategy of Chapman and Chapman*.

2.3 Flaws in Traditional Inferential Methods

2.3.1 First Confound of Chapman and Chapman

2.3.1.1 *First Strategy of Chapman and Chapman*

The *First Confound of Chapman and Chapman* (hereafter, *C&C Confound1*) can be described with reference to the following traditional strategy (which we call the *First Strategy of Chapman and Chapman*, or *C&C Strategy1*), still commonly employed by psychopathology researchers: a) acquired inferential information about distributions $c_{lj}|P$ and $c_{lj}|C$[7] is employed to make an inferential decision as to which of H_0: $\mu_{c_{lj}/P} - \mu_{c_{lj}/C} \geq 0$ and H_1: $\mu_{c_{lj}/P} - \mu_{c_{lj}/C} < 0$ is the case (say, through employment of the standard, two independent samples t-test); b) if the decision made is in favour of H_1, it is concluded that $\mu_{s_j/P} < \mu_{s_j/C}$, i.e., that there exists an S-deficit in respect to s_j.

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[6] We employ the term "confound" due to the practice of speaking of the broader issue as the "Psychometric Confound". Note, however, that we are now pointing out that, rather than there being one "confound" associated with the broader issue, there are multiple, distinct claimed confounds, including the two articulated by the Chapmans.

[7] Obtained, in this case, by the application of a single test.

2.3.1.2 *Core Claim*

In light of this strategy, C&C Confound1 can, then, be described as follows: a) test t_{lj} scales individuals with respect to not only ability s_j, but, also, a *general ability*, g; b) the distributions $c_{lj}|P$ and $c_{lj}|C$, therefore, depend (in ways largely unknown) upon both specific ability s_j and general ability g; c) the distribution $g|P$ can be expected to be stochastically lower (to a degree, largely unknown) than the distribution $g|C$ [8]; d) thus, differences in the distributions $c_{lj}|P$ and $c_{lj}|C$, hence, *inferred* differences in the distributions $c_{lj}|P$ and $c_{lj}|C$ (notably, inferences about the parameter ($\mu_{c_{lj}|P} - \mu_{c_{lj}|C}$)), are confounded, as bases for making decisions about S-deficits (notably about the parameter ($\mu_{s_j|P} - \mu_{s_j|C}$)), by differences in $g|P$ and $g|C$ (notably, as reflected in the parameter ($\mu_{g|P} - \mu_{g|C}$)).

The state of affairs $\mu_{c_{lj}|P} - \mu_{c_{lj}|C} < 0$ could, for example, be produced entirely by differences between P and C in respect to general ability g. In the words of Chapman and Chapman, because of the phenomenon of "…generalized performance deficit..", "…a lower than normal score on any single task cannot be interpreted as a deficit of special importance.."(p. 303).

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[8] The thinking, here, is that it is an empirical fact that individuals who are lesser in respect any ability s_j (belong to P) tend to be lesser in respect *other* abilities. These individuals can be thought of as suffering, to a greater or lesser degree, from a global psychopathology, or decrement, in respect g.

2.3.2 **Second Confound of Chapman and Chapman**

2.3.2.1 *Second Strategy of Chapman and Chapman*

The *Second Confound of Chapman and Chapman* (hereafter, *C&C Confound2*) may be described with reference to the following, commonly employed, strategy (we will call this the *Second Strategy of Chapman and Chapman*, or *C&C Strategy2*) which, interestingly, was invented by psychopathy researchers expressly with the aim of overcoming what we refer to as C&C Confound1 (see, e.g. Maher, 1974; Chan, Li, & Cheung, 2010): a) two tests, $t_{lj} \in T_j$ and $t_{uv} \in T_v$, are employed; b) acquired inferential information about the distributions $c_{lj}|P$, $c_{lj}|C$, $c_{uv}|P$, and $c_{uv}|C$ is employed to make an inferential decision as to which of H$_0$: $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C}) \geq 0$ and H$_1$: $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C}) < 0$ is the case; c) if the decision made is in favour of H$_1$, it is concluded that there exists a D-deficit in respect to s_j and s_v[9].

Now, the reasoning on which this strategy rests requires some clarification, and seems to be as follows: All of the parameters $\mu_{c_{lj}/P}$, $\mu_{c_{lj}/C}$, $\mu_{c_{uv}/P}$, and $\mu_{c_{uv}/C}$ are functions of their respective test score distributions, and are thus determined, in part, by the ability score distributions, $g|P$, and $g|C$, as implied in section 2.3.1.2 above. It is thought that, because both $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C})$ and $(\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$ are influenced by g, the subtraction of the second of these quantities, from the first, i.e., $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$, has the

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[9] As noted above, D-Deficits are of interest as they imply S-Deficits under certain conditions (as detailed in section 2.2). Thus, with respect to the strategy described, Chapman and Chapman comment that "…meaningful statements about specific deficit must be made in terms of differential deficit.." (1973, p. 380).

effect of cancelling out the influence of g; hence, freeing from the influence of g the

difference parameter $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$, and, thereby, making it solely a

function of the specific abilities s_j and s_v. The difference parameter would, then,

constitute a basis for valid inference regarding the existence of a D-Deficit in respect s_j

and s_v, and, under the conditions described on page 7 (footnote 5), a basis for the making

of a valid inference regarding the existence of an S-deficit in respect to s_j.

2.3.2.2 *Core Claim*

In light of C&C Strategy2, C&C Confound2 can be explicated as follows:

Because the sensitivities of test composites c_{lj} and c_{uv} to changes in g (i.e., the

discriminating power of these composites in respect to the scaling of g^{10}), are

characterized by parameters $\gamma_{clj/g}$ and $\gamma_{cuv/g}$, respectively, if it is not the case that

$\gamma_{clj/g} = \gamma_{cuv/g}$, then the influence of g on $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C})$ and $(\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$, respectively, is

not equal. In that case, the difference parameter $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$ remains

influenced by g, and hence is confounded as a basis for making inferences about D-

deficits, by differences in the unknown discriminating powers of the test composites

employed. In the words of Chapman and Chapman, "…differential deficit in

performance does not necessarily indicate a differential deficit in ability…", and

differential deficit in performance may result from "…generalized performance deficit

coupled with the fact that one of the two tasks measures generalized deficit better than

the other." (1978a, p. 303).

[10] These parameters are held to be invariant over populations P and C.

2.4 **Proposed Inferential Solutions**

In response to C&C Confound2, the Chapmans proposed a variety of methods by which, they claimed, this confound could be overcome. The methods, reviewed below, involved either: a) identification of "equivalent" test composites, b) the construction of "matched" test composites, c) comparisons of test composite "true score variances".

2.4.1 **Equivalent Test Composites**

Chapman and Chapman (1978a, 2001) proposed a method by which, they claimed, test composites equivalent in discriminating power in respect to ability g could be identified, thus enabling, in cases of equivalence, the use of the C&C Strategy2 to identify D-Deficits. The method proposed was rooted in the core claim summarized below.

2.4.1.1 *Core Claim*

Suppose C is split into two sub-populations of individuals by the following procedure: a) expected scores for each individual in population C on a test composite $c_{lj}{}^{11}$, i.e., $E(c_{lji})$, are considered, b) the median of expected scores, $m_{c_{lj}}$, is calculated, c) two sub-populations, $C_{H_{clj}}$ and $C_{L_{clj}}$ are formed, in accordance with the rule: if $E(c_{lji}) \geq m_{c_{lj}}$, $i \in$

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[11] Say the application of a test yields a score for an individual on the variable x. Over an infinity of replications of this procedure, there is a probability density function for this individual, where the value of the probability density at score x is $f(x)$. Then the expected value for the individual on the variable x, the "expected score", is $\int_{-\infty}^{\infty} x \cdot f(x)$.

$C_{H_{clj}}$; else, $i \in C_{L_{clj}}$ (the particulars of this procedure are implied in Chapman and

Chapman, 1978).

Now, considering the means $\mu_{c_{lj}}\big|C_{H_{clj}}$ and $\mu_{c_{lj}}\big|C_{L_{clj}}$, it was claimed that a) If

$(\mu_{c_{lj}}\big|C_{H_{clj}} - \mu_{c_{lj}}\big|C_{L_{clj}}) = (\mu_{c_{uv}}\big|C_{H_{cuv}} - \mu_{c_{uv}}\big|C_{L_{cuv}})$ then $\gamma_{clj/g} = \gamma_{cuv/g}$; b) If $(\mu_{c_{lj}}\big|C_{H_{clj}} - \mu_{c_{lj}}\big|C_{L_{clj}}) >$

$(\mu_{c_{uv}}\big|C_{H_{cuv}} - \mu_{c_{uv}}\big|C_{L_{cuv}})$ then $\gamma_{clj/g} > \gamma_{cuv/g}$.

2.4.1.2 *Control High-Low Scorers Comparison Method*

Given the above claim, a method for overcoming C&C Confound2 was proposed:

For two tests t_{lj} and t_{uv}, an inference about the relative magnitude of $(\mu_{c_{lj}}\big|C_{H_{clj}} - \mu_{c_{lj}}\big|C_{L_{clj}})$

and $(\mu_{c_{uv}}\big|C_{H_{cuv}} - \mu_{c_{uv}}\big|C_{L_{cuv}})$ is drawn. If it is inferred that $(\mu_{c_{lj}}\big|C_{H_{clj}} - \mu_{c_{lj}}\big|C_{L_{clj}}) =$

$(\mu_{c_{uv}}\big|C_{H_{tuv}} - \mu_{c_{uv}}\big|C_{L_{cuv}})$ then it is concluded that $\gamma_{clj/g} = \gamma_{cuv/g}$. If it is concluded that $\gamma_{clj/g} =$

$\gamma_{cuv/g}$, then it is concluded that the influence of g on $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C})$ and $(\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$,

respectively, is equal, and therefore, under this condition there exists a basis for the

drawing of valid conclusions regarding D-Deficit by employment of the C&C Strategy2.

In the words of Chapman and Chapman (2001), test composites for which the

discriminating power parameters in respect to g are identical "will yield the same size

mean difference between groups…" of "better and poorer performance normal subjects"

(p.33); and, with regard to use of the above comparison method, "this cross-validation of

matching is the gold standard for equivalence…". At least one attempt to apply the

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Control High-Low Scorers Comparison Method, Hanlon et al. (2005), exists in the literature.

2.4.2 **Test Matching Method 1**

A second means by which, it was claimed, C&C Confound2 could be overcome was by the construction of two tests composites matched, item-by-item, in terms of particular psychometric characteristics. Suppose that 2 tests, t_{lj} and t_{uv}, are each composed of dichotomous items (each taking, as values, 0 and 1), that $p_{lj}=p_{uv}$, and that, for each of $d=1..p_{lj}$, there exists an item of t_{uv}, say, I_{dMuv}, $d=1..p_{uv}$, such that both of the following hold: $\beta_{Idlj} = \beta_{IdMuv}$ and $\rho_{clj,dlj}= \rho_{cuv,dMuv}$, in which β_{Idlj} is the difficulty of item I_{dlj} (difficulty will be formally defined in Chapter 4, but is, in the case of a dichotomous [0,1] item, the item's mean), and $\rho_{clj,dlj}$ is the item-total correlation of item I_{dlj} (the correlation between the item I_{dlj} and the unit-weighted composite, c_{lj}, of the items of t_{lj}). Tests t_{lj} and t_{uv} are, then, said to be matched with respect their items (or, simply, *matched*), and it is claimed that, as a consequence of their being matched, $\gamma_{clj/g} = \gamma_{cuv/g}$. Conclusions about D-Deficits can then be drawn, it is asserted, through use of the C&C Strategy2.

In proposing this method, the Chapmans were inspired by Gullikson's (1950) method of constructing parallel tests. In the words of Chapman and Chapman (2001), parallel tests are "designed to be psychometrically identical, including comparable content…" and by extension they envisioned pairs of tests that were "…psychometrically identical but of differing content" (p.33).

If a pair of tests is constructed in the above fashion, Chapman and Chapman held that not only would it be the case that the discriminating power parameters would be matched, but, also, other test properties would be identical, namely, "reliability, shape of the distribution of scores… mean, variance, and shape of the distribution of item difficulty… mean item covariance" (Chapman and Chapman, 1973, p.380).

2.4.2.1 *Core Claim*

In light of Test Matching Method 1, a third Core Claim of Chapman and Chapman may be explicated as follows: For tests t_{uv} and t_{lj} composed of dichotomous items, if, $\forall\, d$, $\beta_{Idlj} = \beta_{IdMuv}$, and $\rho_{clj,dlj} = \rho_{cuv,dMuv}$, then $\gamma_{clj/g} = \gamma_{cuv/g}$.

2.4.3 **Test Matching Method 2**

An alternative approach to Test Matching Method 1 called, herein, Test Matching Method 2, operates as follows: Suppose that 2 tests, t_{lj} and t_{uv}, are each composed of dichotomous items (each taking, as values, 0 and 1), that $p_{lj}=p_{uv}$, and that, for each of $d=1..p_{lj}$, there exists an item of t_{uv}, say, I_{dMuv}, $d=1..p_{uv}$, such that both of the following hold: $\beta_{Idlj} = 1 - \beta_{IdMuv}$ and $\rho_{clj,dlj} = \rho_{cuv,dMuv}$, Tests t_{uv} and t_{lj} are, then, said to be Method 2 Matched in respect to their items, and it is claimed that, as a consequence of their being matched, $\gamma_{clj/g} = \gamma_{cuv/g}$. Accordingly, conclusions about D-Deficits can then be drawn through the use of C&C Strategy 2.

Test Matching Method 2 was created in response to the belief that it was not only very difficult to construct two tests comprised of items, pairwise matched in respect

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difficulty and item-total correlation, as required by Test Matching Method 1, but

sometimes *undesirable* to do so. Miller, Chapman, Chapman, and Collins (1995) stated

that in some cases, a match on difficulties is obtainable only at the expense of the

adequacy of scaling of one of the constructed tests. In particular, they were concerned

that such a match would require one of the tests to scale in respect to an additional ability.

Recall that, under the claims of C&C Confound1 (section 2.3.1), the two tests employed,

$t_{lj} \in T_j$ and $t_{uv} \in T_v$, scale individuals in respect to s_j and g, and s_v and g, respectively. Now,

suppose test t_{lj} is modified by the selection of new test items to produce a test, $t_{lj}*$, the

constituent items of which are matched in terms of difficulty to items in t_{uv}. The concern

is that the test $t_{lj}*$ might now scale individuals not only in respect to s_j and g, but also in

respect to an additional ability, s_m[12]. Hence, the difference parameter of the C&C

Strategy2, $(\mu_{c_{lj}*/P} - \mu_{c_{lj}*/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$ might be influenced not only by specific

abilities s_j and s_v, but also by the additional ability s_m, being therefore confounded as a

basis for making decisions regarding D-Deficits. In the words of Miller et al. (1995),

speaking in reference to schizophrenia studies, "When one type of task is inherently more

difficult than the other, the investigator who matches tasks may do so by introducing a

second variable that raises the difficulty level of the less difficult task… The problem…is

that one does not know which of the two variables accounts for the schizophrenic

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[12] The idea seems to be that for any test t_{lj} scaling only in respect to s_j and g, the constituent test items must fall within a particular range of difficulty, as appropriate to ability s_j. For example, a test of "working memory ability" may be composed of more difficult items than a test of "basic attention". For a modified test $t_{lj}*$, it follows that if test items do not fall within the appropriate range, the scaling of this new test must be understood differently; as one possibility, perhaps the new test scales in respect not only to s_j and g but also an additional ability.

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differential deficit" (p. 253)[13]. Test Matching Method 2, then, was thought to be of value

in that it purportedly produced two test composites of equivalent discriminating power

without the need for pairs of test items of identical difficulty[14].

2.4.3.1 *Core Claim*

In light of Test Matching Method 2, a fourth Core Claim of Chapman and

Chapman may be explicated as follows: For tests t_{uv} and t_{lj} composed of dichotomous

items, if, $\forall\, d$, $\beta_{Idlj} = 1 - \beta_{IdMuv}$, and $\rho_{clj,dlj} = \rho_{cuv,dMuv}$, then $\gamma_{clj/g} = \gamma_{cuv/g}$.

2.4.4 **True Score Variance Comparison**

Another means by which, it was claimed, C&C Confound2 could be overcome

(in addition to the Control High-Low Scorers Comparison Method and test matching

methods) was through the comparison of test composite true-score variances. The method

follows from the core claim below.

[13] In this quote, "variable" is taken as equivalent to "ability". There exists, in the psychometric and psychopathology literatures, a wide-spread practice of equating concepts such as "construct", "factor", "latent variable", and "ability" (Maraun and Gabriel, 2013).

[14] The motivation for this method seems to be that since items of a test that scale only in respect to s_j and g must fall within a particular range of difficulty (as articulated in footnote 12), it is desirable to increase the likelihood that this particular range of difficulty is appropriate for matching. The alternative item matching method increases this likelihood by enabling matching of items of complementary difficulty when matching by identical difficulty is not appropriate.

2.4.4.1 *Core Claim*

Under classical test theory, each test composite c_{lj} within a population P has a property called "true score variance", denoted $\sigma^2_{\tau_{lj}}$ [15]. Now, given a second test composite, c_{uv}, it is claimed that if $\sigma^2_{\tau_{lj}} < \sigma^2_{\tau_{uv}}$ then $\gamma_{clj/g} < \gamma_{cuv/g}$.

2.4.4.2 *True Score Variance Comparison Method*

As a consequence of the above claim, a *True Score Variance Comparison Method* was recommended, in which, a) an inference is made regarding whether $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C}) < 0$, and, if it is inferred that this is the case, b) true score variances for c_{lj} ($\sigma^2_{\tau_{lj}}$) and c_{uv} ($\sigma^2_{\tau_{uv}}$) are compared. If it is determined that $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C}) < 0$ and $\sigma^2_{\tau_{lj}} < \sigma^2_{\tau_{uv}}$, then it is concluded that there exists a D-deficit in respect to s_j and s_v. In the words of Chapman and Chapman, "a task's discriminating power is indexed by its true score variance" (2001, p.33), and an exception to the need to match is in the circumstance in which "the… schizophrenic participants respond to two unmatched tasks with a greater performance deficit on the task of lesser discriminating power" (2001, p.32).

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[15] The term "true score variance" is precisely defined in Chapter 4. For now, note that the true score variance of a test composite is determined with regard to a particular populations of individuals, in this case, the population of healthy control participants, C. In the remainder of the chapter, all references to composite test true score variance can be assumed to imply reference to C.

2.5 **Summary of Core Claims**

In summary, the claims of Chapman and Chapman apropos the Psychometric Confound and possible solutions are the following:

CC1: Differences in the distributions $c_{lj}|P$ and $c_{lj}|C$, hence, *inferred* differences in the distributions $c_{lj}|P$ and $c_{lj}|C$ (notably, inferences about the parameter ($\mu_{c_{lj}/P} - \mu_{c_{lj}/C}$)), are confounded, as bases for making decisions about S-deficits (notably about the parameter ($\mu_{s_j/P} - \mu_{s_j/C}$)), by differences in $g|P$ and $g|C$ (notably, as reflected in the parameter ($\mu_{g/P} - \mu_{g/C}$) (this is the claim of the existence of C&C Confound1).

CC2: Inferences about the parameter ($\mu_{c_{lj}/P} - \mu_{c_{lj}/C}$) $-$ ($\mu_{c_{uv}/P} - \mu_{c_{uv}/C}$) are confounded, as bases for making decisions about D-Deficits (notably, inferences about the parameter ($\mu_{s_j/P} - \mu_{s_j/C}$) $-$ ($\mu_{s_v/P} - \mu_{s_v/C}$)), by differences in $\gamma_{clj/g}$ and $\gamma_{cuv/g}$ (this is the claim of the existence of C&C Confound2).

CC3: For tests $t_{lj} \in T_j$ and $t_{uv} \in T_v$, a) If ($\mu_{c_{lj}}\left|C_{H_{clj}}\right. - \mu_{c_{lj}}\left|C_{L_{clj}}\right.$) = ($\mu_{c_{uv}}\left|C_{H_{cuv}}\right. - \mu_{c_{uv}}\left|C_{L_{cuv}}\right.$) then $\gamma_{clj/g} = \gamma_{cuv/g}$; b) If ($\mu_{c_{lj}}\left|C_{H_{clj}}\right. - \mu_{c_{lj}}\left|C_{L_{clj}}\right.$) > ($\mu_{c_{uv}}\left|C_{H_{cuv}}\right. - \mu_{c_{uv}}\left|C_{L_{cuv}}\right.$) then $\gamma_{clj/g} > \gamma_{cuv/g}$ (this is the basis for the Control High-Low Scorers Comparison Method).

CC4: If two tests t_{lj} and t_{uv} are Method 1 Matched, i.e. $p_{lj}=p_{uv}$ and, for each of $d=1..p_{uv}$, $\beta_{Idlj} = \beta_{IdMuv}$, and $\rho_{clj,dlj}= \rho_{cuv,dMuv}$, then $\gamma_{clj/g} = \gamma_{cuv/g}$.

CC5: If two tests t_{lj} and t_{uv} are Method 2 Matched, i.e. $p_{lj}=p_{uv}$ and, for each of $d=1..p_{uv}$, $\beta_{Idlj} = 1 - \beta_{IdMuv}$, and $\rho_{clj,dlj} = \rho_{cuv,dMuv}$, then $\gamma_{clj/g} = \gamma_{cuv/g}$.

CC6: For two test composites c_{lj} and c_{uv}, if $\sigma^2_{\tau_{lj}} < \sigma^2_{\tau_{uv}}$ then $\gamma_{clj/g} < \gamma_{cuv/g}$ (this claim is the basis for the True Score Variance Comparison Method).

2.6 **Problems with the Account**

2.6.1 **Unsubstantiated Claims**

The most notable ommission in the work of Chapman and Chapman (and, as will be seen, those who have commented upon their work) is simply that the existence of the Psychometric Confound, as constituted by the general inferential problem, C&C Confound1 and C&C Confound2, and as articulated in core claims *CC1* and *CC2*, was never proven. Where there was an attempt to support these claims, this effort consisted of the use of illustrative examples. For example, Chapman and Chapman (1973) provided a data set in which the reliability (which was loosely linked to test composite true score variance, and hence to discriminating power through the claim *CC6*) of various test composites appeared to be related systematically to mean differences (in particular the quantity ($\mu_{c_{lj}/P} - \mu_{c_{lj}/C}$), as introduced in section 2.3.1), a finding that was implied to be relevant to the truth status of *CC2*. In general, however, *CC1* and *CC2* enjoyed only brief consideration by the Chapmans, who tended, in their articles, to quickly move to discussion of claims *CC3-CC6*.

Claims *CC3-CC6*, however, were also unproven. Where support for these claims was given, it consisted of passing references to works on classical test theory or, again, on illustrative examples. The entirety of the supporting evidence provided for these claims is as follows: a) with regard to *CC4* and *CC5*, the classical test theorist Gulliksen (1950) was credited with inspiring Test Matching Methods 1 and 2, although it would be inaccurate to say that Gulliksen's work bore on the validity of this approach; b) for *CC6*, invented frequency distributions supporting a relationship between true score variance and mean differences were provided (Chapman and Chapman, 1978a), which would certainly not constitute proof of the claim; c) in a linking of *CC4* and *CC5*/*CC6*, data from a sample was used to argue for a correspondence between item difficulties and test composite true score variances (Chapman & Chapman, 1973), which, again, would not constitute proof of any of *CC4-CC6*. Overall, then, the support offered for *CC3-CC6* was not adequate.

2.6.2 Absence of a Technical Treatment

The claims of Chapman and Chapman tie together a variety of terms in a way that suggests that there exists an underlying technical framework by which the relevant quantitative concepts (i.e. ability scores, discriminating power, item difficulty, test true score variance, etc.) are mathematically related. However, such a framework is not recoverable from their work, with the root of this deficiency being a lack of definition of the quantitative concepts themselves. We elaborate upon the absence of a technical framework and definitions of key terms below.

2.6.2.1 *Lack of a Technical Framework*

What is recoverable from the Chapmans' articles in terms of a technical

framework is only a proto-framework describable as follows (we refer to this as the

Chapman and Chapman Proto-Framework, or *C&CP-f*) [16]:

a) each test t_{lj} scales individuals, in a manner that is largely unknown to the researcher, with respect to both specific ability s_j and a general ability g;

b) each test composite c_{lj} is characterized by two parameters, say, $\gamma_{clj/g}$ and $\gamma_{clj/sj}$, the former which characterizes the *discriminating power* in respect to the scaling of g (i.e., its sensitivity to changes in g), and the latter which characterizes the sensitivity to changes in s_j[17];

c) the parameters $\gamma_{clj/sj}$[18] and $\gamma_{clj/g}$ determine, in part, the observed-score distributions $c_{lj}|P$ and $c_{lj}|C$.

This proto-framework leaves key mathematical relationships, that is, between

abilities, test composite scores and discriminating power terms, unspecified.

It should further be noted that while the above proto-framework is consistent

with Chapman and Chapman's (1973 and elsewhere) description of their core claims,

they elsewhere equivocate with regard to the number of abilities associated with a test.

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[16] The following supports this summary: i) the contrasting of "specific deficit" with "generalized cognitive deficit" (Chapman & Chapman, 1973); ii) the contrasting of "deficit in performance" with "deficit in ability" (Chapman & Chapman, 1978a); implying, in combination with point i, the possibility of deficit with regard to specific ability and general ability; iii) discriminating power defined as "the extent to which the [test] score differentiates the more able from the less able subjects.." (Chapman & Chapman, 1973, p.380), implying the discriminating power parameters in reference to the specific and general abilities.

[17] These parameters are held to be invariant over populations *P* and *C*.

[18] Note that we have not previously spoken of the parameter $\gamma_{clj/sj}$, as the Chapmans' account of the second confound was most appropriately summarized by reference only to discriminating power parameters linked to g. Nevertheless, as expressed in point 12 above, a broader reading of their work, within the context of relevant test theory, suggests also the existence of $\gamma_{clj/sj}$, a parameter whose role we will need to consider further. However, note that we take the Chapmans' use of the term "discriminating power" to be synonymous, in most contexts, with parameters related to g.

Chapman and Chapman stated that, "Discriminating power refers here to the extent to which the score… differentiates two groups that differ in *the ability* [emphasis added] measured by the test" (1973, p.380). Similarly, they state, "The term 'true score' may be misleading… it does not refer to *the ability* [emphasis added] which the subject truly has" (1978a, p.304). These quotes imply that a test scales with respect to one ability, in conflict with the two-ability-per-test proto-framework. Such equivocation prevents confident extraction of the proto-framework described above.

2.6.2.2   *Failure to Define Key Quantitative Concepts*

2.6.2.2.1   **"Ability"**

The general inferential problem considered by the Chapmans (section 2.2) depends upon a distinction between test composite scores and abilities. However, no definition of the key concept "ability" is to be found in the work of the Chapmans. Furthermore, their usage of the term suggests multiple meanings. In one sense, "ability" is linked with "ability scores", and accordingly there is talk of "ability levels" (e.g. Chapman & Chapman, 1975, p. 45) and "mean ability level" (Chapman & Chapman, 1973, p. 382). Such talk is often accompanied by a reference to Lord's (1952) work *A Theory of Test Scores*, in which ability is conceptualized as "a function of the item scores" (p.1), and is itself representable by a score. However, the Chapmans also speak of abilities as if these entities, rather than following mathematically from item scores, cause such scores and exist independently of such scores. The implication of causality is present in the quote, "the extent of the inferiority of score of the pathological subjects depends not only on their deficit in ability…" (Chapman and Chapman, 1973, p. 380).

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The implication of independent existence of abilities is present in the Chapmans' suggestion that tests be matched through administration of the tests to "normal group with a wide range of *ability* [emphasis added]" (2001, p.33), in which it is implied that the abilities of individuals within a group exist and can somehow be known previous to test administration. Depending on the meaning of "processes", the belief in an independent existence and causal (with respect to item/composite scores) role of abilities may also be reflected in their statement, "we infer that processes are, in many cases, either a subset of what we call abilities or are a superordinate term for a number of abilities" (Chapman & Chapman, 2001, p.36).

2.6.2.2.2 **"Discriminating Power"**

Also of importance to the Psychometric Confound, through its bearing on C&C Confound2, is the concept of discriminating power, which, as seen in section 2.3.2.2, is characterized by parameters linking test composite scores to abilities. The Chapmans' inconsistent usage of "ability", then, would obviously threaten the coherence of this concept. Further, however, the Chapmans speak of two conflicting "indices" of discriminating power, stating "reliability is the index of discriminating power" (1973, p.382), but also that discriminating power "is indexed by the true score variance" (2001, p.33). Elsewhere, they implied that both indices were inadequate by emphasizing the importance of the item matching and Control-High Low Scorers Comparison methods. Inadequate definition of "discriminating power" was acknowledged by Chapman and Chapman (2001), who stated, "It appears from our reading of the Knight and Silverstein

(2001) article that we have failed to distinguish clearly our use of the term *discriminating power* from the more familiar *statistical power*" (p.33).

2.6.2.2.3 **"Difficulty"**

A key claim of the Chapmans involves a match of item difficulties (see *CC4* and *CC5*). As we will detail in coming sections, item difficulty is a technical property of test items. However, Chapman and Chapman equivocated in respect to the meaning of "difficulty". Chapman and Chapman (1973) were consistent in characterizing *item difficulty* by the mean of all item scores and the *level of test difficulty* to the mean item difficulty. However, despite acknowledging that there is no clear definition of item difficulty in the case in which items are not dichotomously scored (e.g., in the case in which scores represent reaction times), Chapman and Chapman claimed that, in such cases, "one may nevertheless be sure that differences in difficulty produce differential discriminating of groups" (1973, p.382). As we will see in Chapter 3, the absence of universal definitions of test and item difficulty (across dichotomous and non-dichotomous cases) in the work of Chapman was a likely cause of continued confusion in respect to these concepts in the literature on the Psychometric Confound.

2.7 **Summary**

This chapter provides a review of the account of the Psychometric Confound put forth by Chapman and Chapman. The Psychometric Confound, according to Chapman and Chapman, is constituted by a general inferential problem, as well as two flaws in methods meant to overcome the general inferential problem (these flaws abbreviated

C&C Confound1 and C&C Confound2, and which are purported to exist as a consequence of C&C Strategy1 and C&C Strategy2). The Chapmans proposed three means by which C&C Confound2 could be overcome: the detection of "equivalent" tests (the Control High-Low Scorers Comparison Method), the construction of "matched" tests (Test Matching Method 1 and 2), or the comparison of test composite true score variances (the True Score Variance Comparison Method). We have extracted, from the work of Chapman and Chapman, six core claims apropos the Psychometric Confound and possible solutions (*CC1-CC6*). The Chapmans did not prove their key claims, this being a result of the nontechnical approach taken. It is only possible to detect what we call a proto-framework, which we have summarized. In addition, there is an unfortunate absence, in the work of the Chapmans, of definitions of quantitative concepts central to their account, including "ability", "discriminating power", and "difficulty".

We have established, in this chapter, the beginnings of a technical account of the Psychometric Confound according to Chapman and Chapman. Before further developing this account, which will require grounding of the issue in test theory, we turn to an overview of work on the Psychometric Confound in the literature more broadly.

Chapter 3.

The Psychometric Confound in the Literature

In this chapter, we survey the impact of the Chapmans' account of the Psychometric Confound on the psychopathology literature as a whole. We outline the range of influence of their work, and summarize attempts to apply their recommended methods. Over the years, a number of accounts alternative to that of the Chapmans have emerged. These accounts were distinct in terms of claims made, and often contradicted the account of Chapman and Chapman. We describe several of these alternative accounts. As we shall see, the proliferation of alternative accounts has further confused the issues housed under the term "Psychometric Confound", demonstrating the extant need for clarification of the problem.

3.1 Range of Citations

Although of greatest influence in the schizophrenia literature (e.g., Palmer, Dawes, & Heaton, 2009), the Chapmans' work on the issue of the Psychometric Confound has been cited frequently across a wide variety of psychopathology research areas, including major depressive disorder (Joorman & Gotlib, 2008), attention-deficit / hyperactivity disorder (Huang-Pollock & Karalunas, 2010), learning disorders (Savage, Lavers, & Pillay, 2007), autistic disorder (Pellicano, 2010), and bipolar disorders (Kurtz

& Garrety, 2009). Concern regarding the issue has also spread to areas outside of psychopathology, including research on memory in children (Lum, Kidd, Davis, & Conti-Ramsden, 2009), the psychological effects of alcohol (Moberg & Curtin, 2009), and aging (e.g., Salthouse & Coon, 1994). The majority of citing authors have treated the Chapmans as authorities on the matter, echoing their warnings and, in some cases, as noted below, attempting to apply their suggested methods.

3.2 Application of Methods

The method suggested by Chapman and Chapman that has been most frequently applied in the literature is the True Score Variance Comparison Method. Several investigators have faithfully applied the method as outlined in 2.4.3.2 (e.g., Chang & Lenzenweger, 2001; Delevoye-Turrell, Giersch, Wing, & Danion, 2007; Fuller et al., 2006; Kerns & Becker, 2008). A variation on the method, that rests on a first step in which is tested the hypothesis of cross-test *equality* in test composite true score variance, has also been frequently employed. Horan et al. (2008) produced a pair of memory tests for use in research, the use of which was intended to yield similar true score variances; others (Van Erp et al., 2008; Hauer, Wessel, Geraerts, Merckelbach, and Dalgleish, 2008; Kerns & Berenbaum, 2003) have employed post-hoc comparisons to test the hypothesis that true score variance for their test composites was equivalent[19]. In studies in which true score variance equality was inferred, it was claimed (explicitly or implicitly) that such

[19] These authors estimated true score variance by multiplication of reliability and observed variance, a practice that can be justified by inter-relationships of properties of test composite scores, as further elaborated upon in Chapter 4.

equivalence enabled the use of C&C Strategy2 to draw conclusions regarding D-Deficits[20].

There have been few attempts to faithfully apply the Chapmans' item matching methods, perhaps due to the perceived onerousness of such procedures (as expressed in MacDonald & Carter, 2002). Examples of studies that have attempted such matching include Corrigan, Silverman, Stephenson, Nugent-Hirschbeck, and Buican (1996), and Kagan and Oltmanns (1981).

As noted above, Hanlon et al. (2005) attempted to employ the Control High-Low Scorers Comparison Method (as outlined in 2.4.1.2). These authors compared scores of groups presumably drawn from C_H and C_L populations on three tests (by comparison of effect sizes) and inferred that the tests were equivalent in terms of discriminating power. Magnetoencephalography (MEG) was then used to contrast activation across these tests in schizophrenic and control subjects, and conclusions regarding D-Deficits were then drawn according to C&C Strategy2[21].

Some authors, though endorsing the Chapmans' account of the Psychometric Confound, have applied idiosyncratic solutions, the justifications for which are unclear or absent[22], for example, comparing test composites in terms of their reliabilities (Lencz et

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[20] It is unclear whether the Chapmans would consider this variation appropriate.

[21] Problematically, *behavioral* test data was used to argue that associated MEG tests (*brain activation-based tests*) were equivalent, an inferential leap likely inconsistent with what the Chapmans would suggest.

[22] The frequency of use of idiosyncratic methods is notable, and suggests confusion regarding the basic elements of the issue of the Psychometric Confound.

al., 2006), or claiming that the addition of a control group resolves the issue of the Psychometric Confound (Chan, Cheung, & Gong, 2010). Other endorsers of the Chapmans' views have assumed a simple monotonic relationship between test difficulty and discriminating power, a view that is at odds with Chapman and Chapman (1978a), and have drawn, as a consequence, the conclusion that C&C Confound2 can be addressed through the making of certain inferences in respect test difficulty contrasts. For example, Joorman and Gotlib (2008) argued, using reaction time data, that the task yielding the larger mean difference between groups was of lesser difficulty (as determined by shorter reaction times), and therefore lower in discriminating power, and that conclusions regarding D-Deficits in line with C&C Strategy2 were therefore justified (for similar examples, see also Chambon et al., 2008; Green, Nuechterlein, Breitmeyer, & Mintz, 1999; Pickup & Frith, 2001).

Application of the Chapmans' methods, then, has occurred with some frequency, but a variety of ill-justified idiosyncratic methods meant to address Chapman and Chapman's concerns have also been employed.

3.3 **Alternative Accounts**

Distinct from the ill-justified idiosyncratic methods noted above are the more extensive alternative accounts of the issues housed under the term "Psychometric Confound". In general, these alternative accounts of the issue accord, in respect the research context considered and the general inferential problem to be overcome, with the account of Chapman and Chapman. They deviate from the treatment of Chapman and

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Chapman in respect the flaws seen to exist in traditional inferential methods, the inferential solutions proposed, and/or the proto-frameworks seen as being implied. With regard to the issue of inferential solution, some accounts explicitly contest the correctness of what the Chapmans have put forth (in particular, core claims *CC4-CC6*). Four main alternative accounts, those of Baron and Treiman (1980), Salthouse and Coon (1994), Knight and Silverstein (2001), and Kang and MacDonald (2010), are outlined below. The distinct elements of each account are articulated and summarized in a list of core claims.

3.3.1 **Baron and Treiman (1980)**

Baron and Treiman (1980), though claiming to "argue that the methodological problems discussed by Chapman and Chapman arise in a wide range of studies of group differences and individual differences…" (p. 313), in fact articulated an account distinct from that of the Chapmans. They described, in respect to two strategies, C&C Strategy1, and a strategy which we call the *Baron and Treiman Strategy*, two distinct flaws, or confounds; they proposed a unique inferential solution; and their work suggests a proto-framework alternative to that recoverable from the Chapmans' articles.

3.3.1.1 *Perceived Flaws in Traditional Inferential Methods*

3.3.1.1.1 **The First Confound of Baron and Treiman**

In light of C&C Strategy1, the first of Baron and Treiman's (1980) confounds (referred to, herein, as B&T Confound1) can be described as follows: a) "experimental test" t_{lj} scales individuals in respect to not only ability s_j, but, also, other abilities $s_1…s_k$; b) the test composite distributions $c_{lj}|P$ and $c_{lj}|C$, therefore, depend (in ways largely

unknown) upon both specific ability s_j and other of the abilities $s_1...s_k$; c) thus,

differences in the distributions $c_{lj}|P$ and $c_{lj}|C$, hence, *inferred* differences in the

distributions $c_{lj}|P$ and $c_{lj}|C$ (notably, inferences about the parameter ($\mu_{c_{lj}|P} - \mu_{c_{lj}|C}$)), are

confounded, as bases for making decisions about S-deficits (notably about the parameter

($\mu_{s_j|P} - \mu_{s_j|C}$)), by differences between distributions $s_1|P$ and $s_1|C$, $s_2|P$ and $s_2|C$, ... , $s_k|P$

and $s_k|C$.

The position put forth by Baron and Treiman in respect to C&C Strategy1,

therefore, can be considered a *variant* of *CC1*, in that, though these authors accepted that

inferred differences in the distributions $c_{lj}|P$ and $c_{lj}|C$ are confounded as bases for making

decisions about S-deficits, they appeared to disagree with Chapman and Chapman as to

the source of this confounding: Chapman and Chapman claiming that the source of the

confounding lies in differences in $g|P$ and $g|C$; Baron and Treiman claiming, in contrast,

that the source lies in differences between distributions $s_1|P$ and $s_1|C$, $s_2|P$ and $s_2|C$, ... ,

$s_k|P$ and $s_k|C$. In the words of Baron and Treiman, there are "other influences on the

experimental task" (p.313) besides the "ability of interest". They provided an example in

which there was a single test for which the ability of interest is labelled "distractibility",

but the test also scaled in respect to an ability labelled "choice". The authors stated, "We

might be tempted to compare the performance of the two groups on the experimental

task… but we must admit that [group C may]…perform better simply because they are

better at the choice..." (p. 314).

3.3.1.1.2 The Second Confound of Baron and Treiman

3.3.1.1.2.1 Baron and Treiman Strategy

Baron and Treiman (1980) summarized a commonly employed strategy meant to overcome B&T Confound1, described as follows (we will call this the *Baron and Treiman Strategy,* or *B&T Strategy* for short): a) two tests, $t_{lj} \in T_j$ and $t_{uv} \in T_v$, are employed (called, in this case, the "experimental test" and the "control test", respectively); b) there exist composites of the scores of the items in tests t_{lj} and t_{uv}, called c_{lj} and c_{uv}, respectively; c) acquired inferential information about the distributions $c_{lj}|P$, $c_{lj}|C$, $c_{uv}|P$, and $c_{uv}|C$ is employed to make an inferential decision as to which of H_0: (

$$\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C}) \geq 0 \text{ and } H_1: (\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C}) < 0 \text{ is the case;}$$

d) if the decision made is in favour of H_1, it is concluded that there exists a S-deficit in respect to s_j. Note that this strategy differs from C&C Strategy2 in that the conclusion drawn is in respect to an S-Deficit, rather than a D-Deficit.

Now, the reasoning on which this strategy rests requires some clarification, and seems to be as follows: a) as above, the "experimental test" t_{lj}, as well as its corresponding composite c_{lj}, scales individuals in respect to not only ability s_j, but, also, the other abilities of $s_1...s_k$; b) it is thought that the "control test" t_{uv}, as well as its corresponding composite c_{uv}, scales individuals in respect to $s_1...s_k$; c) because both (

$$\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) \text{ and } (\mu_{c_{uv}/P} - \mu_{c_{uc}/C}) \text{ scale in respect to } s_1...s_k, \text{ it is thought that the subtraction}$$

of the second of these quantities, from the first, i.e., $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$, has the effect of cancelling out the influence of $s_1...s_k$; hence, freeing from the influence of

$s_1...s_k$, the difference parameter ($\mu_{c_{lj}/P} - \mu_{c_{lj}/C}$) - ($\mu_{c_{uv}/P} - \mu_{c_{uv}/C}$), thereby, making it solely

a function of the specific ability s_j. In the words of Baron and Treiman (1980), the control

test "measures other influences on the experimental task" (p. 313) and the experimental

and control test composites are contrasted to decide whether the "groups differ on the

ability of interest" (p. 313).

3.3.1.1.2.2 **Core Claim**

In light of the B&T Strategy, the second of Baron and Treiman's (1980)

confounds (*B&T Confound2*) may be described as follows: *i*) test t_{lj} scales individuals in

respect to not only ability s_j, but, also, abilities $s_1...s_k$, while test t_{uv} scales in respect to

abilities $s_1...s_k$; *ii*) test t_{lj} is characterized by parameters, say $\gamma_{clj/s1} ... \gamma_{clj/sk}$, that

characterize the sensitivities of the corresponding test composite c_{lj} to changes in each of

$s_1...s_k$, while test t_{uv} is characterized by parameters, say $\gamma_{cuv/s1} ... \gamma_{cuv/sk}$, that characterize

the sensitivities of the corresponding test composite c_{uv} to changes in each of $s_1...s_k$ (these

parameters characterize a test's discriminating power in respect to each of the abilities);

iii) the difference parameter ($\mu_{c_{lj}/P} - \mu_{c_{lj}/C}$) - ($\mu_{c_{uv}/P} - \mu_{c_{uv}/C}$) is influenced by abilities

$s_1...s_k$, and hence is confounded as a basis for making inferences about S-deficits (in

particular, with respect to s_j), by differences in the unknown discriminating powers of the

tests employed. In the words of Baron and Treiman (1980), "When an interaction

between groups and tasks is found, it is possible that both tasks measure the same

individual difference variables, but that Task E (the experimental task) is more sensitive

to them than task C (the control task)." (p. 314).

3.3.1.2 *Proposed Inferential Solution*

In response to the perceived existence of B&T Confound1 and B&T Confound2, Baron and Treiman (1980) suggested an alternative method for drawing conclusions regarding S-Deficits. The method was rooted in the Core Claim and sub-claims summarized below.

3.3.1.2.1 **Core Claim**

Let: A be a dichotomous [0,1] indicator variable, with the property that, if $i \in C$, $A=1$, and if $i \in P$, $A=0$;, $\rho_{Ac_{lj}}$ and $\rho_{Ac_{uv}}$ be the correlations of A with c_{lj} and c_{uv}, respectively; and $\rho_{c_{lj}c_{lj'}}$ and $\rho_{c_{uv}c_{uv'}}$ be the reliabilities of c_{lj} and c_{uv}, respectively.

It was claimed by Baron and Treiman, then, that if both $\rho_{Ac_{lj}} > \rho_{Ac_{uv}}$ and $\rho_{c_{lj}c_{lj'}} < \rho_{c_{uv}c_{uv'}}$ hold, then $(\mu_{s_j/P} - \mu_{s_j/C}) < 0$, in other words, that there exists an S-Deficit in respect to ability s_j and populations P and C.

3.3.1.2.1.1 **Sub-Claims**

The Core Claim above is justified by several implied sub-claims, these labelled *sC1-sC8* below:

    sC1. $\rho_{Ac_{lj}}$ is an increasing function of $\mu_{s_j/P} - \mu_{s_j/C}$.

    sC2. $\rho_{Ac_{lj}}$ is a function of the distributions $c_{lj}\big|C$ and $c_{lj}\big|P$ and is thus determined (as implied in section 3.3.1.1.2.2), in part, by ability score distributions $s_j\big|P$ and $s_j\big|C$, as well as $s_l\big|P...s_k\big|P$ and $s_l\big|C...s_k\big|C$.

    sC3. $\rho_{Ac_{lj}}$ is not a function of the parameters $\gamma_{clj/s1}$ ... $\gamma_{clj/sk}$ (introduced in section 3.3.1.1.2.2 above).

sC4. $\rho_{Ac_{lj}}$ is a function of $\rho_{c_{lj}c_{lj}'}$.

sC5. $\rho_{Ac_{uv}}$ is a function of the distributions $c_{uv}|C$ and $c_{uv}|P$ and is thus determined (as implied in section 3.3.1.1.2.2), in part, by ability score distributions $s_1|P...s_k|P$ and $s_1|C...s_k|C$.

sC6. $\rho_{Ac_{uv}}$ is not a function of $\gamma_{cuv/s1}$ ... $\gamma_{cuv/sk}$ (introduced in section 3.3.1.1.2.2 above).

sC7. $\rho_{Ac_{uv}}$ is a function of $\rho_{c_{uv}c_{uv}'}$.

sC8. The functions referred to in *sC7* and *sC4* are identical.

Now, the following appears to be the reasoning of Baron and Treiman: because a) given that *sC2* and *sC5* are true, both $\rho_{Ac_{lj}}$ and $\rho_{Ac_{uv}}$ are influenced by $s_1|P...s_k|P$ and

$s_1|C...s_k|C$, but, given that *sC3* and *sC6* are true, not by by $\gamma_{clj/s1}$ ... $\gamma_{clj/sk}$ and $\gamma_{cuv/s1}$ ...

$\gamma_{cuv/sk}$, the subtraction of $\rho_{Ac_{uv}}$ from $\rho_{Ac_{lj}}$ [23] has the effect of cancelling out the influence of

$s_1|P...s_k|P$ and $s_1|C...s_k|C$; b) by *sC8*, $\rho_{Ac_{uv}}$ and $\rho_{Ac_{lj}}$ are influenced to the same degree

by their reliabilities; when it occurs that both $\rho_{c_{lj}c_{lj}'} < \rho_{c_{uv}c_{uv}'}$ and $\rho_{Ac_{lj}} > \rho_{Ac_{uv}}$, the latter

state of affairs cannot have arisen in consequence of a difference in reliabilities, hence, must have arisen as a consequence of *sC1*, *sC1* implying, in turn, the existence of an S-Deficit.

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[23] A decision on whether it is the case that $\rho_{Ac_{lj}} > \rho_{Ac_{uv}}$ is equivalent to a decision as to whether $\rho_{Ac_{lj}} - \rho_{Ac_{uv}} > 0$, the latter illustrating the subtraction referred to herein.

3.3.1.2.2 **Baron and Treiman Method**

As a consequence of the Core Claim above, Baron and Treiman (1980) proposed a method for drawing conclusions about S-Deficits as follows:

Step 1. Acquired information about the joint distribution of $[A, c_{lj}, c_{uv}]$ is employed to make an inferential decision as to whether H_o: $\rho_{Ac_{lj}} \leq \rho_{Ac_{uv}}$ or H_1: $\rho_{Ac_{lj}} > \rho_{Ac_{uv}}$ (the quantities defined in section 3.3.1.2.1) If a decision is made in favour of H_1, proceed to step 2. If not, draw no conclusion regarding S-Deficits.

Step 2. Acquired information about the test score distributions $c_{lj}\big|C$, and $c_{uv}\big|C$ is employed to make an inferential decision as to whether H_o: $\rho_{c_{lj}c_{lj}'} \geq \rho_{c_{uv}c_{uv}'}$ or H_1:

$\rho_{c_{lj}c_{lj}'} < \rho_{c_{uv}c_{uv}'}$. If a decision is made in favour of H_1, it is concluded that there exists an S-Deficit in respect to s_j.

3.3.1.3 *Summary of Core Claims*

In summary, the claims of Baron and Treiman are as follows:

BTC1. Baron and Treiman asserted that: a) *inferred* differences in the distributions $c_{lj}|P$ and $c_{lj}|C$ (notably, inferences about the parameter $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C})$), are, indeed, confounded as bases for making decisions about S-deficits (notably about the parameter $(\mu_{s_j/P} - \mu_{s_j/C})$; b) the source of this confounding is not, as claimed by Chapman and Chapman, differences in $g|P$ and $g|C$ (notably, as reflected in the parameter $(\mu_{g/P} - \mu_{g/C})$), but, rather

differences between distributions $s_1|P$ and $s_1|C$, $s_2|P$ and $s_1|C$,…, $s_k|P$ and $s_k|C$ (this is the claim of existence of B&T Confound1).

BTC2. Baron and Treiman claimed that there exists a confound in respect to the B&T Strategy, wherein it is asserted that: a) inferences about the parameter $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$ are confounded as bases for making decisions about S-Deficits; b) the source of this confounding is differences in the unknown discriminating powers of the tests employed, i.e., differences between $\gamma_{clj/s1} \dots \gamma_{clj/sk}$ and $\gamma_{cuv/s1} \dots \gamma_{cuv/sk}$ (this is the claim of existence of B&T Confound2).

BTC3. It was claimed that if $\rho_{Ac_{lj}} > \rho_{Ac_{uv}}$ and $\rho_{c_{lj}c_{lj}'} < \rho_{c_{uv}c_{uv}'}$ then there exists an S-Deficit in respect to s_j and populations P and C (this is the basis for the Baron and Treiman Method).

3.3.1.4 *Proto-framework*

The above summary of the position of Baron and Treiman (1980), suggests a proto-framework alternative to that recoverable from the Chapmans' work, this describable as follows (we shall call this the *Baron and Treiman Proto-Framework*, hereafter, *B&TP-f*):

a) "control test" t_{uv} scales individuals, in a manner that is largely unknown to the researcher, with respect to abilities $s_1\dots s_k$;
b) an "experimental test" t_{lj} scales individuals with respect to the ability s_j, as well as abilities $s_1\dots s_k$;
c) test t_{uv} is, for each ability $s_1\dots s_k$, characterized by a parameter that quantifies discriminating power in respect to the scaling of that ability,

and which determines, in part, the observed-score distributions $c_{uv}|P$
and $c_{uv}|C$;

d) test t_{lj}, is, for each ability $s_1\ldots s_k$, characterized by a parameter that quantifies discriminating power in respect to the scaling of that ability, and which determines, in part, the observed-score distributions $c_{lj}|P$ and $c_{lj}|C$.

3.3.2 **Salthouse and Coon (1994)**

Salthouse and Coon (1994), though, apparently, agreeing with Chapman and Chapman regarding the research context and general inferential problem in play, as well as that C&C Confound2 exists, did not acknowledge the inferential solutions proposed by the Chapmans, instead, putting forth a distinct method.

3.3.2.1 *Proposed Inferential Solution*

In response to C&C Confound2, Salthouse and Coon (1994) suggested an alternative method for drawing conclusions regarding D-Deficits (note that their method involves a ruling on S-Deficits as well, but only in the service of drawing conclusions regarding D-Deficits, as explained in the "Summary of Logic" section below). The method was rooted in the following Core Claims.

3.3.2.1.1 **Core Claims**

Let: a) A be a dichotomous [0,1] indicator variable, with the property that, if $i \in C$, $A=1$, and if $i \in P$, $A=0$; b) c_{lj} and c_{uv} be, as above, test composites of items in t_{lj} and t_{uv} , tests drawn from T_j and T_v; c) $\rho_{A(c_{lj}.c_{uv})}$ be the semi-partial correlation between c_{lj} and A, holding constant c_{uv}, d) B be a nominal variable, the values {1,2} of which stand for test

composite (1=c_{lj}, 2=c_{uv}); e) the *A X B* interaction variance component (σ_{AB}^2) be defined

under the two-factor mixed model (A, between subject, B, within subject).

The core claims are, then, as follows:

SC1. If $\rho_{A(c_{lj}.c_{uv})} < 0$, then there exists an S-Deficit in respect to s_j and populations *P* and

C.

SC2. If $\rho_{A(c_{lj}.c_{uv})} < 0$ and $\sigma_{AB}^2 \neq 0$ then there exists a D-Deficit in respect abilities s_j and s_v

and populations *P* and *C*.

3.3.2.1.2 **Salthouse and Coon Method**

Following from the above claims, a two-step method for drawing conclusions

regarding a D-Deficit was outlined:

Step 1. Acquired information about the test score distributions $c_{lj}|C$, $c_{lj}|P$, $c_{uv}|C$ and

$c_{uv}|P$ is employed to make an inferential decision as to whether H$_0$: $\sigma_{AB}^2 = 0$ or

H$_1$: $\sigma_{AB}^2 \neq 0$. If a decision is made in favour of H$_1$, proceed to step 2. If not, draw no

conclusions regarding D-Deficits.

Step 2. Acquired information about the test score distributions $c_{lj}|C$, $c_{lj}|P$, $c_{uv}|C$ and

$c_{uv}|P$ is employed to make an inferential decision as to whether H$_o$: $\rho_{A(c_{lj}.c_{uv})} \geq 0$ or

H$_1$: $\rho_{A(c_{lj}.c_{uv})} < 0$ (the quantities in H$_0$ and H$_1$ being semi-partial correlations). If a decision

is made in favour of H_1, it is concluded that there exists an S-Deficit in respect to s_j and populations P and C, as well as a D-Deficit in respect to s_j and s_v and populations P and C.

3.3.2.1.2.1 **Summary of Logic**

The logic of the above solution requires explanation, and seems to be as follows. There exist two possible explanations for the state of affairs that $\sigma_{AB}^2 \neq 0$, which is equivalent, in fact, to the condition that $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C}) \neq 0$: a) a D-Deficit, in respect to abilities s_j and s_v, does not exist, i.e., $(\mu_{s_j|P} - \mu_{s_j|P}) - (\mu_{s_v|P} - \mu_{s_v|P}) = 0$ and, hence, the state of affairs $\sigma_{AB}^2 \neq 0$ is solely a consequence of the existence of differences in $\gamma_{clj/g}$ and $\gamma_{cuv/g}$ (the parameters referenced in *CC2*); b) a D-Deficit, in respect to abilities s_j and s_v, does, in fact, exist, and is responsible, to an unknown degree, for the state of affairs $\sigma_{AB}^2 \neq 0$.

In the case in which a decision is made, at Step 1, in favour of H_1, it is concluded that one of these two explanations is correct. The decision as to which of these explanations is correct, is made at Step 2, the decision-making, therein, relying on the following assumption: If there exists an S-Deficit in respect to s_j (i.e. $\mu_{s_j/P} < \mu_{s_j/C}$), then $(\mu_{s_j|P} - \mu_{s_j|P}) - (\mu_{s_v|P} - \mu_{s_v|P}) \neq 0$. Thus, at Step 2, a conclusion is drawn, via Core Claim *SC1*, as to whether there is an S-Deficit in respect s_j. If the answer is in the affirmative, it is concluded that there exists a D-Deficit in respect to abilities s_j and s_v.

3.3.2.2 *Summary of Core Claims*

To review, the claims of Salthouse and Coon (1994) apropos the Psychometric

Confound and possible solutions are the following:

SC1. If $\rho_{A(c_{lj} \cdot c_{uv})} < 0$ then there exists an S-Deficit in respect to s_j and populations P and C.

SC2. If $\rho_{A(c_{lj} \cdot c_{uv})} < 0$ and $\sigma^2_{AB} \neq 0$ then there exists a D-Deficit in respect abilities s_j and s_v

and populations P and C.

3.3.2.3 *Proto-Framework*

Salthouse and Coon (1994) appeared to employ the proto-framework recoverable

from the Chapmans' work, and summarized in section 2.6.2.1.

3.3.3 **Knight and Silverstein (2001)**

Knight and Silverstein (2001) offered an account distinct in several respects from

that of the Chapmans. They described a distinct flaw, or confound; they proposed unique

inferential methods; they contested the validity of certain of the Chapmans' key claims;

and their work suggests a proto-framework alternative to that recoverable from the work

of Chapman and Chapman.

3.3.3.1 *Flaws in Traditional Inferential Methods*

Although Knight and Silverstein (2001) made reference to C&C Confound2, and

claimed to present a "solution" to this confound, their explanation of the flaw inherent in

C&C Strategy2, in fact, implied a distinct confound. This claimed confound, which we

call the *Knight and Silverstein Confound* and abbreviate *K&S Confound*, is described in the Core Claim below (as this is the first of several core claims they put forth, we label it *KSC1*).

3.3.3.1.1 **Core Claim**

KSC1: a) test t_{lj} scales individuals in respect to not only ability s_j, but, also, other abilities $s_1 \ldots s_k$, and, similarly, t_{uv} scales individuals in respect to not only ability s_v, but, also, other abilities $s_1 \ldots s_k$; b) test t_{lj} is associated with discriminating power parameters, $\gamma_{clj/s1} \ldots \gamma_{clj/sk}$, that characterize the sensitivity of the test composite c_{lj} to changes in $s_1 \ldots s_k$, and test t_{uv} is associated with parameters, $\gamma_{cuv/s1} \ldots \gamma_{cuv/sk}$, that characterize the sensitivity of test composite c_{uv} to changes in $s_1 \ldots s_k$; c) the difference parameter $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C})$ - $(\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$ is influenced by abilities $s_1 \ldots s_k$, and hence is confounded as a basis for making inferences about D-deficits by differences in the unknown discriminating powers of the tests employed.[24] In the words of Knight and Silverstein (2001), "The differential performance deficit could simply be an artifact of the differential discriminatory power of the tasks used." (p. 15).

3.3.3.2 *Proposed Inferential Solutions*

While not contradicting the research context and general inferential problem of the Chapmans, and, in particular, the desirability of identifying S- and D-Deficits, Knight

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[24] The following supports this summary: *i*) Knight and Silverstein's (2001) reference to "differential discriminatory power", *ii*) talk of the problem inadvertently scaling in respect to additional abilities (similar to the discussion found in Miller et. al., 1995, and reviewed about in section 2.4.3), and, most explicitly, *iii*) Silverstein's (2008) follow-up work, in which he writes formula for a "typical neuropsychological test" based the "common factor model" containing the equivalent of s_j and other abilities $s_1 \ldots s_k$, as well as terms for the test's scaling in respect to each.

and Silverstein (2001) proposed, as solutions to the K&S Confound, several methods that side-stepped the problem of identifying S- and/or D-Deficits by focusing on other mean differences. The methods were rooted in the Core Claims summarized below.

3.3.3.2.1 **Core Claims**

For the following, let: a) A be a dichotomous [0,1] variable, wherein $0 = C$ and $1 = P$; b) B be a nominal variable, the values {1,2} of which stand for test composite ($1=c_{lj}$, $2=c_{uv}$); c) σ^2_{AB} be the $A \ X \ B$ interaction variance component defined under the two-factor mixed model (A, between subject, B, within subject).

KSC2: If $\sigma^2_{AB} = 0$ then $(\mu_{s_1/P} - \mu_{s_1/C}) = (\mu_{s_2/P} - \mu_{s_2/C}) = \ldots = (\mu_{s_k/P} - \mu_{s_k/C})$ and $\gamma_{clj/s1} = \gamma_{cuv/s1}$, $\gamma_{clj/s2} = \gamma_{cuv/s2} \ldots \gamma_{clj/sk} = \gamma_{cuv/sk}$ (these quantities defined in KSC1).

KSC3: If $\sigma^2_{AB} = 0$, $\mu_{c_{lj}|C} = \mu_{c_{uv}|C}$ and $\mu_{c_{lj}|P} = \mu_{c_{uv}|P}$, then $\mu_{s_j|C} = \mu_{s_v|C}$ and $\mu_{s_j|P} = \mu_{s_v|P}$.

KSC4: If $\sigma^2_{AB} = 0$, $\mu_{c_{lj}|C} > \mu_{c_{uv}|C}$ and $\mu_{c_{lj}|P} > \mu_{c_{uv}|P}$, then $\mu_{s_j|C} > \mu_{s_v|C}$ and $\mu_{s_j|P} > \mu_{s_v|P}$.

KSC5: If $\sigma^2_{AB} = 0$, $\mu_{c_{lj}|C} < \mu_{c_{uv}|C}$ and $\mu_{c_{lj}|P} < \mu_{c_{uv}|P}$, then $\mu_{s_j|C} < \mu_{s_v|C}$ and $\mu_{s_j|P} < \mu_{s_v|P}$.

KSC6: If $\mu_{c_{lj}/P} - \mu_{c_{lj}/C} > 0$ then $\mu_{s_j/P} - \mu_{s_j/C} > 0$.

KSC7: If $\mu_{c_{lj}|C} > \mu_{c_{uv}|C}$ and $\mu_{c_{lj}|P} < \mu_{c_{uv}|P}$ then $\mu_{s_j|C} > \mu_{s_v|C}$ and $\mu_{s_j|P} < \mu_{s_v|P}$.

KSC8: If $\mu_{c_{lj}|C} < \mu_{c_{uv}|C}$ and $\mu_{c_{lj}|P} > \mu_{c_{uv}|P}$ then $\mu_{s_j|C} < \mu_{s_v|C}$ and $\mu_{s_j|P} > \mu_{s_v|P}$.

3.3.3.2.2 **Methods**

3.3.3.2.2.1 **Disconfirmation Method**

According to Knight and Silverstein (2001), the *Disconfirmation Method* involves the demonstration of a "predicted… pattern of differences among conditions or levels of a task that parallels the same pattern for controls" (p.18). Examination of applications of this method (e.g., Knight, Manoach, Elliott, & Hershenson, 2000) reveals the two steps involved in this method:

Step 1. Acquired information about the test composite distributions $c_{lj}|C$, $c_{lj}|P$, $c_{uv}|C$ and $c_{uv}|P$ is employed to make an inferential decision as to whether H_0: $\sigma^2_{AB} = 0$ or H_1: $\sigma^2_{AB} \neq 0$ is true. Given that *KSC2* is true, if a decision is made in favour of H_0, it is concluded that the following is the case:

$$(\mu_{s_1/P} - \mu_{s_1/C}) = (\mu_{s_2/P} - \mu_{s_2/C}) = \ldots = (\mu_{s_k/P} - \mu_{s_k/C}). \tag{3.1}$$

If a decision is made in favour of H_1, no conclusion is drawn regarding stochastic differences in ability scores.

Step 2. If, in Step 1, a decision is made that H_0 is true, hypothesis sets pre-specified by the researcher concerning each pair of tests are then considered. For each pair of test composite c_{lj} and c_{uv}, there are three possible sets of hypotheses (one set is pre-specified by the researcher for each pair of tests):

Possible Hypothesis Set 1:

$$\text{H}_0|C: \mu_{c_{lj}|C} = \mu_{c_{uv}|C} \quad \text{and} \quad \text{H}_0|P: \mu_{c_{lj}|P} = \mu_{c_{uv}|P} \tag{3.2}$$

Given that *KSC3* is true, if an inferential decision is made in favour of $\text{H}_0|C$ and $\text{H}_0|P$,

the researcher concludes that $\mu_{s_j|C} = \mu_{s_v|C}$ and $\mu_{s_j|P} = \mu_{s_v|P}$.

Possible Hypothesis Set 2:

$$\text{H}_0|C: \mu_{c_{lj}|C} \leq \mu_{c_{uv}|C} \quad \text{and} \quad \text{H}_0|P: \mu_{c_{lj}|P} \leq \mu_{c_{uv}|P} \tag{3.3}$$

Given that *KSC4* is true, if an inferential decision is made to *reject* $\text{H}_0|C$ and $\text{H}_0|P$, the

researcher concludes that $\mu_{s_j|C} > \mu_{s_v|C}$ and $\mu_{s_j|P} > \mu_{s_v|P}$.

Possible Hypothesis Set 3:

$$\text{H}_0|C: \mu_{c_{lj}|C} \geq \mu_{c_{uv}|C} \quad \text{and} \quad \text{H}_0|P: \mu_{c_{lj}|P} \geq \mu_{c_{uv}|P} \tag{3.4}$$

Given that *KSC5* is true, if an inferential decision is made to *reject* $\text{H}_0|C$ and $\text{H}_0|P$, the

researcher concludes that $\mu_{s_j|C} < \mu_{s_v|C}$ and $\mu_{s_j|P} < \mu_{s_v|P}$.

3.3.3.2.2.1.1 Summary of Logic

For the Disconfirmation Method, the logic seems to be as follows: Given that

KSC2 is true, in the case in which $\sigma_{AB}^2 = 0$, the influence of $s_1...s_k$ (the "other abilities"

referred to in KSC1) on the means $\mu_{c_{lj}|C}$ and $\mu_{c_{uv}|C}$ is equivalent, and the testing of

hypotheses regarding the equality or inequality of these quantities (or, equivalently, the

subtraction of these quantities) has the effect of cancelling out the influence of $s_1...s_k$, and

bears upon the relative values of $\mu_{s_j|C}$ and $\mu_{s_v|C}$.

3.3.3.2.2.2 **Superiority Method**

According to Knight and Silverstein (2001), the *Superiority Method* involves

inferring a "performance advantage" for population *P* on a single test. The strategy is as

follows: a) a single test $t_{lj} \in T_j$, is employed; b) acquired inferential information about the

distributions $c_{lj}|P$ and $c_{lj}|C$ is employed to make an inferential decision as to whether H$_0$:

$\mu_{c_{lj}|P} - \mu_{c_{lj}|C} \le 0$ or H$_1$: $\mu_{c_{lj}|P} - \mu_{c_{lj}|C} > 0$ is the case; c) given that *KSC6* is true, if a decision

is made in favour of H$_1$, it is concluded that $\mu_{s_j|P} - \mu_{s_j|C} > 0$.

3.3.3.2.2.2.1 Summary of Logic

The Superiority Method has as a basis *KSC6*, which states that if $\mu_{c_{lj}|P} - \mu_{c_{lj}|C} > 0$

then $\mu_{s_j|P} - \mu_{s_j|C} > 0$. The logic of *KSC6* seems to be as follows: For abilities $s_1...s_k$, the

distribution $s_1|P$ is expected to be stochastically lower than $s_1|C$, $s_2|P$ is expected to be

stochastically lower than $s_2|C$,... , $s_k|P$ is expected to be stochastically lower than $s_k|C$.

Furthermore, the sensitivity of test composite c_{lj} to changes in scores on abilities $s_1...s_k$ is

believed to be such that the above state of affairs would cause $c_{lj}|P$ to be stochastically

lower than $c_{lj}|C$, and, in particular, $\mu_{c_{lj}|P} - \mu_{c_{lj}|C} < 0$. Therefore, if this is not the case, in

particular, if $\mu_{c_{lj}/P} - \mu_{c_{lj}/C} > 0$, it is thought that this cannot be due to the influence of

$s_1 \ldots s_k$ (the "other abilities" referred to in *KSC1*) and, instead, must be due to

$\mu_{s_j/P} - \mu_{s_j/C} > 0$.

3.3.3.2.2.3 Relative Superiority Method

According to Knight and Silverstein (2001) the *Relative Superiority Method*

involves a "specific reversal, compared to controls in the relative performance of at least

two tasks" (p. 19). Like the Disconfirmation Method, there are two steps involved, as

follows:

Step 1. Acquired information about the composite test score distributions $c_{lj} | C$, $c_{lj} | P$,

$c_{uv} | C$ and $c_{uv} | P$ is employed to make an inferential decision as to whether H$_0$: $\sigma^2_{AB} = 0$ or

H$_1$: $\sigma^2_{AB} \neq 0$. If a decision is made in favour of H$_1$, it is concluded that $(\mu_{s_j/P} - \mu_{s_j/C}) \neq ($

$\mu_{s_v/P} - \mu_{s_v/C})$. If the decision is in favour of H$_0$, no conclusion is drawn regarding

stochastic differences in ability scores.

Step 2. If, in step 1, a decision is made in favour of H$_1$, hypothesis sets pre-specified by

the researcher are then considered. For the pair of tests $\{t_{lj}, t_{uv}\}$, there are two possible

sets of hypotheses (one set is pre-specified by the researcher):

Possible Hypothesis Set 1

$$\text{H}_0 | C : \mu_{c_{lj}|C} \leq \mu_{c_{uv}|C} \quad \text{and} \quad \text{H}_0 | P : \mu_{c_{lj}|P} \geq \mu_{c_{uv}|P} \tag{3.5}$$

Given that *KSC7* is true, if an inferential decision is made that these hypotheses are *false*, the researcher concludes that $\mu_{s_j|C} > \mu_{s_v|C}$ and $\mu_{s_j|P} < \mu_{s_v|P}$.

Possible Hypothesis Set 2

$$H_0 | C : \mu_{c_{lj}|C} \geq \mu_{c_{uv}|C} \text{ and } H_0 | P : \mu_{c_{lj}|P} \leq \mu_{c_{uv}|P} \tag{3.6}$$

Given that *KSC8* is true, if an inferential decision is made that these hypotheses are *false*, the researcher concludes that $\mu_{s_j|C} < \mu_{s_v|C}$ and $\mu_{s_j|P} > \mu_{s_v|P}$.

3.3.3.2.2.3.1 Summary of Logic

The Relative Superiority Method has as bases *KSC7* and *KSC8*, which state that if $\mu_{c_{lj}|C} > \mu_{c_{uv}|C}$ and $\mu_{c_{lj}|P} < \mu_{c_{uv}|P}$ then $\mu_{s_j|C} > \mu_{s_v|C}$ and $\mu_{s_j|P} < \mu_{s_v|P}$ (*KSC7*), and if $\mu_{c_{lj}|C} < \mu_{c_{uv}|C}$ and $\mu_{c_{lj}|P} > \mu_{c_{uv}|P}$ then $\mu_{s_j|C} < \mu_{s_v|C}$ and $\mu_{s_j|P} > \mu_{s_v|P}$ (*KSC8*). The logic of these claims seems to be as follows: For all abilities $s_1...s_k$ (the "other abilities" referred to in *KSC1*) scores tend to be stochastically equal to or lower than 0 for both populations *C* and *P*. Say test composite c_{lj} is more sensitive to changes in $s_1...s_k$ than test composite c_{uv}. Then, it would be expected that $\mu_{c_{lj}|C} < \mu_{c_{uv}|C}$ and $\mu_{c_{lj}|P} < \mu_{c_{uv}|P}$. However, if it is the case (as in *KSC7*) that $\mu_{c_{lj}|C} < \mu_{c_{uv}|C}$ and $\mu_{c_{lj}|P} > \mu_{c_{uv}|P}$ then this must be due to the additional dependence of the distributions $c_{lj}|P$ and $c_{uv}|P$ on the distributions $s_j|P$ and $s_v|P$, and furthermore, it must be the case that that $\mu_{s_j|C} < \mu_{s_v|C}$ and $\mu_{s_j|P} > \mu_{s_v|P}$. Similarly, if it is the case (as in *KSC8*) that $\mu_{c_{lj}|C} > \mu_{c_{uv}|C}$ and $\mu_{c_{lj}|P} < \mu_{c_{uv}|P}$ then this must be due to the

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additional dependence of the distributions $c_{lj}|C$ and $c_{uv}|C$ on the distributions $s_j|C$ and $s_v|C$,

and furthermore, it must be the case that that $\mu_{s_j|C} > \mu_{s_v|C}$ and $\mu_{s_j|P} < \mu_{s_v|P}$.

3.3.3.3 *Criticisms of the Chapmans' Account*

Knight and Silverstein (2001) expressed direct objections to the account provided

by Chapman and Chapman, and rooted in the Core Claims below.

3.3.3.3.1 **Core Claims**

KSC9: It is not the case that if, for all item pairs I_{dlj} and I_{dMuv}, $d=1..p_{uv}$, $\rho_{clj,dlj} = \rho_{cuv,dMuv}$,

and either: *i)* $\beta_{Idlj} = \beta_{IdMuv}$, or *ii)* $\beta_{Idlj} = 1 - \beta_{IdMuv}$ then conclusions about D-Deficits may

be drawn through use of C&C Strategy2 (in contradiction to the implication of *CC4* and

CC5).

KSC10: It is *not the case* that if, for two tests c_{lj} and c_{uv}, if $\sigma^2_{\tau_{lj}} < \sigma^2_{\tau_{uv}}$ then the C&C

Strategy2 may be employed to draw conclusions in regard D-Deficits (in contradiction to

the implication of *CC6*).

3.3.3.3.1.1 **Summary of Logic**

In regard to *KSC9*, Knight and Silverstein (2001) stated that "matching tasks on

psychometric characteristics, particularly difficulty level, often can only be achieved at

the expense of confounding the hypothetical constructs being compared." (p.15). This is

interpreted as follows: the selection of item pairs such that $\rho_{clj,dlj} = \rho_{cuv,dMuv}$ and either: *i)*

$\beta_{Idlj} = \beta_{IdMuv}$, or *ii)* $\beta_{Idlj} = 1 - \beta_{IdMuv}$ has the effect of changing the values of the parameters

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$\gamma_{clj/s1} \dots \gamma_{clj/sk}$ and $\gamma_{cuv/s1} \dots \gamma_{cuv/sk}$ in unknown ways, and, hence, *KSC1* remains the case,

i.e., the difference parameter $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C})$ - $(\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$ is influenced by abilities

$s_1 \dots s_k$, and hence is confounded as a basis for making inferences about D-deficits.

In regard to *KSC10*, *CC6* implies that comparison of test true score variances may

remove from consideration the influence of abilities other than s_j and s_v upon the

difference parameter $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C})$ - $(\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$, allowing conclusions regarding D-

Deficits. Recall that, as stated in *KSC1*, Knight and Silverstein considered the case in

which test t_{lj} scales individuals in respect to not only ability s_j, but, also, other abilities

$s_1 \dots s_k$, and, similarly, t_{uv} scales individuals in respect to not only ability s_v, but, also,

other abilities $s_1 \dots s_k$. In this case, then, the implication would be that comparison of test

composite true score variances may remove from consideration the influence of abilities

$s_1 \dots s_k$. However, according to Knight and Silverstein, the true score variance of

composite c_{lj} reflects the variances of $s_1 \dots s_k$ as well as parameters, $\gamma_{clj/s1} \dots \gamma_{clj/sk}$, whereas

the contribution of composite c_{lj} to the difference parameter (the contribution being

$\mu_{c_{lj}/P} - \mu_{c_{lj}/C}$) reflects the parameters $\gamma_{clj/s1} \dots \gamma_{clj/sk}$ as well as the differences

$(\mu_{s_1/P} - \mu_{s_1/C}),(\mu_{s_2/P} - \mu_{s_2/C}),\dots,(\mu_{s_k/P} - \mu_{s_k/C})$. It is thought, then, that the comparison of

test composite true score variances cannot remove from consideration the influence of

abilities $s_1 \dots s_k$, as composite true score variance does not reflect differences

$(\mu_{s_1/P} - \mu_{s_1/C}),(\mu_{s_2/P} - \mu_{s_2/C}),\dots,(\mu_{s_k/P} - \mu_{s_k/C})$. In the words of Knight and Silverstein

(2001), "power reductions occur when the increase in true score variance does not

increase group discrimination (i.e., treatment effect or numerator in the F-ratio)" (p. 17).

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Silverstein (2008) elaborated on this point, stating, "For the purposes of maximizing effect sizes between groups, the sources of variance that must be eliminated are those that do not discriminate between groups. To do this, we need to… eliminate all 'nonspecific' sources of true score variance…"(p.645). These quotes vaguely illustrate the components of *KSC10*.

3.3.3.4 *Summary of Core Claims*

In summary, the claims of Knight and Silverstein (2001) apropos the Psychometric Confound and possible solutions are the following:

KSC1: a) test t_{lj} scales individuals in respect to ability s_j, and other abilities $s_1…s_k$, and, similarly, t_{uv} scales individuals in respect to s_v and other abilities $s_1…s_k$; b) test t_{lj} is associated with parameters, $\gamma_{clj/s1} … \gamma_{clj/sk}$, that characterize the sensitivities of composite c_{lj} to changes in $s_1…s_k$, and test t_{uv} is associated with parameters, $\gamma_{cuv/s1} … \gamma_{cuv/sk}$, that characterize the sensitivities of composite c_{uv} to changes in $s_1…s_k$ (these known as discriminating power parameters); c) the difference parameter $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$ is a function of abilities $s_1…s_k$, and hence is confounded as a basis for making inferences about D-deficits by differences in the unknown discriminating powers of the tests employed.

KSC2: If $\sigma^2_{AB} = 0$ then $(\mu_{s_1/P} - \mu_{s_1/C}) = (\mu_{s_2/P} - \mu_{s_2/C}) = … = (\mu_{s_k/P} - \mu_{s_k/C})$ and $\gamma_{clj/s1} = \gamma_{cuv/s1}, \gamma_{clj/s2} = \gamma_{cuv/s2} … \gamma_{clj/sk} = \gamma_{cuv/sk}$.

KSC3: If $\sigma^2_{AB} = 0$, $\mu_{c_{lj}|C} = \mu_{c_{uv}|C}$ and $\mu_{c_{lj}|P} = \mu_{c_{uv}|P}$, then $\mu_{s_j|C} = \mu_{s_v|C}$ and $\mu_{s_j|P} = \mu_{s_v|P}$.

KSC4: If $\sigma^2_{AB} = 0$, $\mu_{c_{lj}|C} > \mu_{c_{uv}|C}$ and $\mu_{t_{lj}|P} > \mu_{t_{uv}|P}$, then $\mu_{s_j|C} > \mu_{s_v|C}$ and $\mu_{s_j|P} > \mu_{s_v|P}$.

KSC5: If $\sigma^2_{AB} = 0$, $\mu_{c_{lj}|C} < \mu_{c_{uv}|C}$ and $\mu_{c_{lj}|P} < \mu_{c_{uv}|P}$, then $\mu_{s_j|C} < \mu_{s_v|C}$ and $\mu_{s_j|P} < \mu_{s_v|P}$.

KSC6: If $\mu_{c_{lj}|P} - \mu_{c_{lj}|C} > 0$ then $\mu_{s_j|P} - \mu_{s_j|C} > 0$.

KSC7: If $\mu_{c_{lj}|C} > \mu_{c_{uv}|C}$ and $\mu_{c_{lj}|P} < \mu_{c_{uv}|P}$ then $\mu_{s_j|C} > \mu_{s_v|C}$ and $\mu_{s_j|P} < \mu_{s_v|P}$.

KSC8: If $\mu_{c_{lj}|C} < \mu_{c_{uv}|C}$ and $\mu_{c_{lj}|P} > \mu_{c_{uv}|P}$ then $\mu_{s_j|C} < \mu_{s_v|C}$ and $\mu_{s_j|P} > \mu_{s_v|P}$.

KSC9: If, for all item pairs I_{dlj} and I_{dMuv}, $d=1..p_{uv}$, $\left|0.5 - \beta_{Idlj}\right| = \left|0.5 - \beta_{IdMuv}\right|$ and $\rho_{clj,dlj} = \rho_{cuv,dMuv}$, it is *not the case* that conclusions about D-Deficits may be drawn through use of C&C Strategy2 (in contradiction to an implications of *CC4* and *CC5*).

KSC10: It is *not the case* that if, for two tests c_{lj} and c_{uv}, if $\sigma^2_{\tau_{lj}} < \sigma^2_{\tau_{uv}}$ then the C&C Strategy2 may be employed to draw conclusions in regard D-Deficits (in contradiction to an implication of *CC6*).

3.3.3.5 *Proto-Framework*

The above summary of the position of Knight and Silverstein (2001), suggests a proto-framework alternative to that recoverable from the Chapmans' work, as follows (we shall call this the "*Knight and Silverstein*" proto-framework, hereafter, *K&SP-f*) :

   a) each test t_{lj} scales individuals, in a manner that is largely unknown to the researcher, in respect to ability s_j as well as other abilities $s_1...s_k$.

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b) test t_{lj} is associated with parameters, $\gamma_{clj/s1} \ldots \gamma_{clj/sk}$, that characterize the sensitivity of associated composite c_{lj} to changes in $s_1 \ldots s_k$, (these known as discriminating power parameters)

c) The parameters $\gamma_{clj/s1} \ldots \gamma_{clj/sk}$ determine, in part, the observed score distributions $c_{lj}|P$ and $c_{lj}|C$.

3.3.4 **Kang and MacDonald (2010)**

Kang and MacDonald (2010) provided an account distinct from that of the Chapmans, articulating a unique confound and contesting the validity of certain of the Chapmans' key claims.

3.3.4.1 *Flaws in Traditional Inferential Methods*

Although Kang and MacDonald (2010), like Knight and Silverstein (2001), made reference to C&C Confound2, and sought to evaluate proposed solutions to this confound, their explanation of the flaw inherent in C&C Strategy2, in fact, implied a distinct confound. We call this new claimed confound the *Kang and MacDonald Confound* (abbreviated *K&M Confound*), and describe it as a core claim of Kang and MacDonald below.

3.3.4.1.1 **Core Claim**

In light of C&C Strategy2, Kang and MacDonald's (2010) confound may be described as follows:

KMC1: a) test t_{lj} scales individuals in respect to not only ability s_j, but, also, other abilities $s_1 \ldots s_k$, as well as a *general ability*, g, and, similarly, t_{uv} scales individuals in respect to not only ability s_v, but, also, g and other abilities $s_1 \ldots s_k$; b) test t_{lj} is associated

with parameters, $\gamma_{clj/s1} \dots \gamma_{clj/sk}$ and $\gamma_{clj/g}$, that characterize the sensitivities of associated

composite c_{lj} to changes in $s_1 \dots s_k$ and g, and test t_{uv} is associated with parameters,

$\gamma_{cuv/s1} \dots \gamma_{cuv/sk}$ and $\gamma_{cuv/g}$, that characterize the sensitivities of associated composite c_{uv} to

changes in $s_1 \dots s_k$ and g; c) the difference parameter $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$ is

influenced by g and abilities $s_1 \dots s_k$, and hence is confounded as a basis for making

inferences about D-deficits by differences in the unknown discriminating powers. In the

words of Kang and MacDonald (2010), "Although cognitive tests are intended to be

sensitive to a particular cognitive ability, their measurement inevitably includes variance

from other common cognitive and noncognitive factors... In addition... patients show

generalized performance deficits." (p. 300).

3.3.4.2 *Criticisms of the Chapmans' Account*

In addition to claiming the existence of a distinct confound from those articulated

by the Chapmans, Kang and MacDonald (2010) expressed direct objections to the

account provided by Chapman and Chapman, rooted in their Core Claim below.

3.3.4.2.1 **Core Claim**

KMC2: It is not the case that $\sigma^2_{\tau_{lj}}$ (true score variance for composite c_{lj}) provides an

estimate of $\gamma_{clj/g}$, and therefore it is not the case that if $\sigma^2_{\tau_{lj}} < \sigma^2_{\tau_{uv}}$ then $\gamma_{clj/g} < \gamma_{cuv/g}$ (in

contrast to *CC6*).

3.3.4.2.1.1 **Summary of Logic**

In support of KMC2, Kang and MacDonald (2010) conducted a simulation study in which composite scores were constructed by the following procedure: First, each individual was assigned what they called a "replicable ability score" (RAS); second, an "item ability score" was constructed for each item by summing the RAS and an error score; third, this "item ability score" was dichotomized using a particular threshold; lastly, for each individual, groups of dichotomized item scores were summed to produce composite scores. After constructing simulated test composite scores in this manner, Kang and MacDonald employed a "direct index of discriminating power" for each composite, this being the correlation of ability scores (RAS scores) and composite scores. The authors then generated an estimated true score variance (ETSV) for each composite by multiplying observed-score variance by estimated reliability. It was found that ETSV and the index of discriminating power were only weakly related.

3.3.4.3 *Summary of Core Claims*

In summary, the core claims of Kang and MacDonald (2010) are the following:

KMC1: Inferences about the parameter $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$ are confounded, as bases for making decisions about D-Deficits (notably, inferences about the parameter $(\mu_{s_j/P} - \mu_{s_j/C}) - (\mu_{s_v/P} - \mu_{s_v/C})$), by differences in between the groups of parameters $(\gamma_{clj/s1} \ldots \gamma_{clj/sk}, \gamma_{clj/g})$ and $(\gamma_{cuv/s1} \ldots \gamma_{cuv/sk}, \gamma_{cuv/g})$.

KMC2: It is not the case that $\sigma^2_{\tau_{lj}}$ provides an estimate of $\gamma_{clj/g}$, and therefore it is not the case that if $\sigma^2_{\tau_{lj}} < \sigma^2_{\tau_{uv}}$ then $\gamma_{clj/g} < \gamma_{cuv/g}$ (in contrast to *CC6*).

3.3.4.4 *Proto-Framework*

The above summary of the position of Kang and MacDonald (2010), suggests a proto-framework alternative to that recoverable from the Chapmans' work, as follows (we shall call this the "*Kang and MacDonald*" proto-framework, hereafter, *K&MP-f*) :

a) each test t_{lj} scales individuals, in a manner that is largely unknown to the researcher, in respect to ability s_j as well as other abilities $s_1 \dots s_k$ and "general ability" g;

b) associated composite c_{lj} is associated with parameters, $\gamma_{clj/s1} \dots \gamma_{clj/sk}$, that characterize its sensitivities to changes in $s_1 \dots s_k$, and a parameter, $\gamma_{clj/g}$, that characterizes its sensitivity to change in g (these known as discriminating power parameters);

c) the parameters $\gamma_{clj/s1} \dots \gamma_{clj/sk}$ and $\gamma_{clj/g}$ determine, in part, the observed score distributions $c_{lj}|P$ and $c_{lj}|C$.

3.3.5 **Problems with Alternative Accounts**

Many of the problems noted in the Chapmans' work (as reviewed in section 2.6) are also apparent in the works of their critics. In particular, these authors did not specify a technical framework, and therefore their claims were unproven. Only alternative and mathematically incomplete proto-frameworks, as summarized above, are recoverable from these works. As should be apparent from the above summary of accounts, the critics of the Chapmans have, in general, continued to employ nontechnical language to describe the issues housed under the term "Psychometric Confound", and have, as a result, left the meanings of key terms, including "discriminating power" and "ability", in question

3.4 **Summary**

Chapman and Chapman's work on the Psychometric Confound has been widely referenced, and their proposed solutions have been applied occasionally. More often, endorsers of Chapman and Chapman's account have employed idiosyncratic methods, the justifications for which are unclear, in an attempt to deal with C&C Confound1, C&C Confound2, and similar confounds. Notwithstanding the popularity of the Chapmans' account of the Psychometric Confound, a number of alternative accounts have emerged, with additional and often competing claims, as summarized above. However, the alternative accounts have, in general, failed to resolve the problems identified. There is therefore a pressing need for a definitive technical articulation of the issues surrounding the term "Psychometric Confound", which would enable adjudication both the claims made by the Chapmans and those made in the alternative accounts. We take up this task in Chapter 4.

Chapter 4.

Mathematization and Evaluation of Claims: Chapman and Chapman

Running tacitly through the accounts of both Chapman and Chapman and their commentators is test theory, both of the classical variety, as when Chapman and Chapman, 1978a, decompose test composite scores into "true" and "error" components, and of the modern sort, as when (see section 2.6.2.1) a distinction between *specific* and *general* ability is implied.

In this chapter, we: a) make explicit the test theory that is nascent within the work of Chapman and Chapman, b) complete, employing this test theory, the proto-framework that, in Chapter 2, was extracted from this work; c) elucidate, given this test theory and completed proto-framework, the claims of Chapman and Chapman by mathematizing them; d) adjudicate, given this mathematization, the Chapmans' claims (an analogous treatment of the various alternative accounts authored by the Chapmans' commentators, and described in Chapter 3, is undertaken in Chapter 5).

We begin by providing a general review of the classical test theory (in particular the multivariate classical true-score model) and the modern test theory (in particular the

multi-population linear factor model) that is nascent in all work carried out, to date, on the issue of the psychometric confound.

4.1 A Review of the Classical Test Theory

4.1.1 Properties of an item

An item *I* is a rule that assigns to any individual in a population a score on a corresponding variable *X*. A single application of *I* to an individual is called a trial. Because *I* can, in theory, be applied to an individual an infinity of times, associated with each individual is an infinite universe of trials, hence, an infinite population of scores on *X*, a notion that is the very foundation of CTT (see, e.g., Lord & Novick, 1968).

Let us formalize these ideas. Consider a particular individual *i* belonging to a population *Δ*, and a particular item *I* yielding scores on a variable *X*. Then the population of scores on *X* yielded by an infinity of trials on individual *i* is called the *propensity distribution of i* in respect *I*, and is represented as follows:

$$X \mid i \sim (\tau_i, \sigma_{\varepsilon i}^2). \tag{4.1}$$

In expression 4.1, $\tau_i = \mathrm{E}(X/i)$ is called the *true score of person i in respect item I*, and $\sigma_{\varepsilon i}^2 = \mathrm{V}(X \mid i) = \mathrm{E}\,[(X - \tau_i)^2 \mid i]$, the *error variance of i in respect item I* (sometimes, also, the *person-specific error variance*).

It follows from expression 4.1 that the unconditional distribution of X (over all individuals belonging to Δ) is

$$X \sim (u_X, \sigma_X^2), \qquad (4.2)$$

in which

$$\mu_X = E(X) = EE(X \mid i) = E(\tau_i) = \mu_\tau \qquad (4.3)$$

and

$$\sigma_X^2 = V(X) = E[V(X \mid i)] + V[E(X \mid i)] = E(\sigma_{\varepsilon i}^2) + V(\tau_i) = \sigma_E^2 + \sigma_\tau^2. \qquad (4.4)$$

Expression 4.4 shows, importantly, that the variation in X, over trials and individuals, is decomposable into two fundamental variance parameters: a) σ_E^2, which is the average, over all $i \in \Delta$, of the person-specific error variances, and is called the *error variance of item I*; b) σ_τ^2, which is the variance, over $i \in \Delta$, of the true-scores, and is called the *true-score variance of item I*.

The *reliability of item I*, in population Δ, is, then, by definition

$$\rho_{XX'} = \frac{\sigma_\tau^2}{\sigma_X^2}, \qquad (4.5)$$

and the *reliability index* (the Pearson Product Moment Correlation, over all $i \in \Delta$, between the true score variable and X), the square root of (4.5).

4.1.2 **Special case of a dichotomous [0,1] item**

A dichotomous [0,1] item is a rule that assigns to any $i \in \Delta$ either a 0 or a 1 on an associated variable X. As a consequence,

$$E(X/i) = \tau_i = P(X=1|i), \tag{4.6}$$

i.e., the true-score of individual i is equal to the proportion of unities he yields on the item over an infinity of trials, and

$$V(X/i) = P(X=1|i)(1 - P(X=1|i)) = \tau_i(1-\tau_i). \tag{4.7}$$

The unconditional mean of X is, then,

$$E(X) = E(\tau_i) = E(P(X=1|i)) = \beta, \tag{4.8}$$

a parameter that has, traditionally, been called the *difficulty* of the item. The unconditional variance assumes, in this case, the rather unusual form

$$V(X) = \beta(1-\beta), \tag{4.9}$$

by which it is deduced that $V(X)$: a) is a quadratic function of item difficulty; b) has a maximum of .25, attained when β is equal to .5 (i.e., when *half* the total number of scores on the item, over trials and individuals, are equal to 1, and half to 0).

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4.1.3 **Properties of a set of items**

Consider, now, a set of p items $\{I_1,I_2,...,I_p\}$ yielding scores on a p-element variable X, and a particular individual $i \in \Delta$. An infinity of trials of $\{I_1,I_2,...,I_p\}$ on i, induces, now, a p-dimensional propensity distribution for i:

$$X \mid i \sim (\tau_i, \Sigma_{\varepsilon i}). \tag{4.10}$$

In expression 4.10, $\tau_i = E(X \mid i)$ and $\Sigma_{\varepsilon i}$ is the $p \times p$ covariance matrix of the errors, $(X - \tau_i) \mid i$.[25] It follows from 4.10 that the unconditional distribution of X (over all $i \in \Delta$) is:

$$X \sim (\mu_X, \Sigma_X) \tag{4.11}$$

in which

$$\mu_X = EE(X \mid i) = E(\tau_i) \tag{4.12}$$

and

$$\Sigma_X = C(E(X \mid i)) + E(C(X \mid i) = C(\tau_i) + E(\Sigma_{\varepsilon i}) = \Sigma_\tau + \Sigma_E. \tag{4.13}$$

The matrix Σ_τ is sometimes called the *true-score covariance matrix*; Σ_E, the *error covariance matrix*.

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[25] In CTT, this matrix is taken, as an axiom of CTT, to be diagonal, meaning that, conditional on any individual i, the items are uncorrelated.

4.1.4 **Properties of a composite of items**

We have already (see 2.1) introduced the notion of a linear composite of test items, and, in fact, have employed the notation c_{lj} to stand for a linear composite of the items of a test t_{lj}, t_{lj} being the lth test of the jth set, the jth set containing tests invented for the purpose of scaling individuals on ability s_j. More generally, a composite of items (popularly, a *test composite*) is, simply, a function $f(X)$ of X, of type either linear or non-linear. Once again, in the special case of a linear composite,

$$c_{lj} = \sum_{y=1}^{p} w_y X_y = w'X , \qquad (4.14)$$

in which w_y is the *item weight* for item y, and w is the vector of item weights. As is well know, it is only for the class of linear composites that exact, general, results apropos moments can be derived. In consequence of these results, the propensity distribution of i in respect a linear composite $w'X$ of items is

$$w'X \,|\, i \sim (w'\tau_i, \, w'\Sigma_{\varepsilon i} w) . \qquad (4.15)$$

In consequence of (4.15), the unconditional distribution (over all $i \in \Delta$) of $w'X$ is

$$w'X \sim (\mu_{w'X}, \sigma^2_{w'X}), \qquad (4.16)$$

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in which $\mu_{w'X} = w'\mu_X$, called the *linear composite mean*, $\sigma^2_{w'X} = w'\Sigma_\tau w + w'\Sigma_E w$, the

linear composite variance, and in which $w'\Sigma_E w$ is the *error variance of the linear*

composite, and $w'\Sigma_\tau w$, the *true score variance of the composite*.

Thus, the reliability of the linear composite $w'X$ is

$$\rho_{w'Xw'X'} = \frac{w'\Sigma_\tau w}{w'\Sigma_X w} \,. \tag{4.17}$$

4.1.5 Special case of a composite of dichotomous [0,1] items

For the case of a linear composite of p dichotomous [0,1] items, it follows from

expressions 4.7 and 4.14 that

$$w'X \mid i \sim (w'\tau_i, \sum_{y=1}^{p} w_y^2 \tau_{iy}(1-\tau_{iy})) \,, \tag{4.18}$$

in which τ_{iy} is the score of $i \in \Delta$ of item y . The unconditional distribution of $w'X$ is, then,

$$w'X \sim (\mu_{w'X}, \sigma^2_{w'X}) \,, \tag{4.19}$$

in which $\mu_{w'X} = w'\beta$, the *yth* element of β, β_y, and in which, as usual,

$\sigma^2_{w'X} = w'\Sigma_\tau w + w'\Sigma_E w$.

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If the linear composite happens to be the *average* of the p items, i.e., all of the elements

of w are equal to $\dfrac{1}{p}$, then the parameter

$$d = \mu_{w'X} = w'\beta = \frac{1}{p}\mathbf{1}'\beta \tag{4.20}$$

is called the difficulty of the composite.

4.2 A Review of the Multi-Population Linear Factor Model

Consider the research context in which random p-vectors, X_r, $r = 1..s$, are

distributed over each of s populations $\varDelta_r$ of individuals. Define the $p \times p$ covariance

matrices $\Sigma_r = \mathrm{E}(X_r - \mu_r)(X_r - \mu_r)'$ and mean vectors $\mu_r = \mathrm{E}(X_r)$, $r = 1..s$. The X_r,

$r = 1..s$, are said to be representable by the m-dimensional, multi-population, linear factor

model[26] if

$$X_r = \gamma_r + \Lambda_r \xi_r + \delta_r \, , \, r = 1..s, \tag{4.21}$$

wherein ξ_r is a vector of m common factor random variables and δ_r is a p-vector of

random uniquenesses. The parameters of the representation are: Λ_r, $r = 1..s$, each Λ_r a

$p \times m$ matrix of regression coefficients (factor loadings); γ_r, $r = 1..s$, each γ_r a p-vector

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[26] In the above section, X was a p-element variable, the elements of which were item variables. As the multi-population linear factor model may be applied to items *or* tests, X, in this section, is simply a p-element variable, without specification as to whether the p variables are item variables or test variables.

of intercepts; $\boldsymbol{\Phi}_r$, $r = 1..s$, $\boldsymbol{\Phi}_r$, the $m \times m$ covariance matrix of $\boldsymbol{\xi}_r$; $\boldsymbol{\Theta}_{\delta r}$, $r = 1..s$, $\boldsymbol{\Theta}_{\delta r}$ the p

$\times$ p, diagonal, positive definite, covariance matrix of $\boldsymbol{\delta}_r$; and $\boldsymbol{\kappa}_r$, $r = 1..s$, $\boldsymbol{\kappa}_r$, the mean

vector of $\boldsymbol{\xi}_r$.

It is fundamental to linear factor analytic representations that $E(\boldsymbol{\delta}_r) = \mathbf{0}$ and

$C(\boldsymbol{\xi}_r, \boldsymbol{\delta}_r) = \mathbf{0}$, $r = 1..s$. A first identification constraint is that, for $r = 1..s$, $\boldsymbol{\Phi}_r$ is a

correlation matrix, implying that the variances of the common factors are set to unity[27].

If each $\boldsymbol{X}_r$ is representable as in expression 4.21, i.e., the multi-population, linear

factor model holds for $\varDelta_r$, $r = 1..s$, then the following *factor-* and *mean structures* hold in

these populations:

$$\boldsymbol{\Sigma}_r = \boldsymbol{\Lambda}_r \boldsymbol{\Phi}_r \boldsymbol{\Lambda}_r{}' + \boldsymbol{\Theta}_{\delta r} \tag{4.22}$$

$$\boldsymbol{\mu}_r = \boldsymbol{\gamma}_r + \boldsymbol{\Lambda}_r \boldsymbol{\kappa}_r . \tag{4.23}$$

In respect to the parameters $\boldsymbol{\Lambda}_r$, $\boldsymbol{\gamma}_r$, and $\boldsymbol{\Theta}_{\delta r}$, there are various possibilities with

regard to cross-population invariance, including the constituents of the following, oft-

tested, nested hierarchy: i) *configural invariance.* $\boldsymbol{\Lambda}_r$, $r = 1..s$, have their null elements in

the same locations (Thurstone, 1947); ii) *weak or pattern invariance.* $\boldsymbol{\Lambda}_r = \boldsymbol{\Lambda}$, $r = 1..s$

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[27] In the absence of additional constraints, the model will not be identified, and even if enough constraints are placed to identify the model, certain parameters may remain unrecoverable, unless further, specific, restrictions are imposed.

(Thurstone, 1947); iii) *strong factorial invariance.* $\gamma_r = \gamma$, $\Lambda_r = \Lambda$, $r = 1..s$; iv) *strict*

factorial invariance. $\gamma_r = \gamma$, $\Lambda_r = \Lambda$, $\Theta_\delta = \Theta_{\delta r}$, $r = 1..s$. Joreskog's (1971) invention of

Multigroup Confirmatory Factor Analysis (MGCFA) provided researchers with easily

implementable statistical tools by which they could carry out tests of a wide variety of

invariance hypotheses.

4.3 **Completion of Proto-Framework**

The reader will recall that the research context and elements of the proto-

framework extracted in Chapter 2 are as follows:

a) there are two populations of individuals, a population C of healthy controls and a population P of individuals suffering from some particular psychopathology;

b) there is, in play, a set of k abilities $\{s_1, s_2,…,s_k\}$, each individual having a score (which is unknown to the researcher) on each ability;

c) there are k sets of tests $(T_1…T_k)$, all tests t_{lj}, $l=1..p_j$, contained with the jth set, invented for the purpose of scaling individuals on ability s_j;

d) there are linear composites of the items of tests, $c_{lj} = \sum_{d=1}^{plj} w_{dlj} I_{dlj}$,

wherein test t_{lj} is comprised of p_{lj} items, I_{dlj}, $d=1…p_{lj}$, and w_{dlj}, $d=1…p_{lj}$, is the weight for item I_{dlj}. Unless otherwise specified, it should be assumed that $w_{dlj}=1$, $d=1…p_{lj}$, the test composite c_{lj} being, then, a *unit-weighted* composite;

e) each test composite c_{lj} scales individuals, in a manner that is largely unknown to the researcher, with respect to both specific ability s_j and a general ability g;

f) each test composite c_{lj} is characterized by two parameters, say, $\gamma_{clj/sj}$ and $\gamma_{clj/g}$, that quantify its discriminating power in respect to the scaling of each of s_j and g (i.e., its sensitivity to changes in s_j and g, respectively);

g) as is clear from the fact that they are not subscripted in respect C and P, the $\gamma_{clj/sj}$ and $\gamma_{clj/g}$ are invariant over populations C and P;

h) the parameters $\gamma_{clj/sj}$ and $\gamma_{clj/g}$ determine, in part, the observed-score distributions $c_{lj}|P$ and $c_{lj}|C$.

We, now, complete this proto-framework. Let: a) once again, i stand for individual; b) $r=\{C,P\}$; and c) X be the (p_j+p_v)-element random vector $\begin{bmatrix} X_j \\ X_v \end{bmatrix}$, the first p_j elements of which are the test composites of the tests in T_j (these, it will be recalled, invented to scale individuals with respect ability s_j), and the final p_v, the test composites of the tests of T_v (which scale in respect, s_v)[28].

From their frequent references to classical test theorists such as Gullikson (1950) and Lord (1952), as well as their use of classical test theory terms such as "true score variance", "reliability", etc., it is clear that Chapman and Chapman invoke (4.10). That is to say,

$$X \mid i,r \sim (\boldsymbol{\tau}_{ir}, \boldsymbol{\Sigma}_{\varepsilon ir}), \ \boldsymbol{\Sigma}_{\varepsilon ir}, \text{ diagonal}, \ i \in r, \ r=\{C,P\}. \tag{4.24}$$

From the references to general ability, specific abilities, discriminating power, and the multi-population context in which concern for the psychometric confound is situated, we deduce that (4.21) is invoked as a representation, or model, of the (p_j+p_v) vector $\boldsymbol{\tau}_{ir}$, i.e.,

$$\boldsymbol{\tau}_{ir} = \boldsymbol{\gamma}_r + \boldsymbol{\Lambda}_r \boldsymbol{\xi}_{ir} + \boldsymbol{\zeta}_{ir}, i \in \Delta_r, r=1...s, \tag{4.25}$$

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[28] In this section, unlike section 4.1, X takes on the specific meaning of a vector of *test composite* variables.

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with the following particularizations, restrictions, and clarifications being made:

a) $s=2$, i.e., $r = \{C,P\}$; $m=3$, in which

$$\boldsymbol{\xi}_{ir} = \begin{bmatrix} g_{ir} \\ s_{j_{ir}} \\ s_{v_{ir}} \end{bmatrix};$$

(4.26)

b) $\boldsymbol{\Lambda}_r = \boldsymbol{\Lambda}$, $(p_j + p_v)$ by 3, and structured as follows:

$$\begin{pmatrix} \boldsymbol{\lambda}_{jg} & \boldsymbol{\lambda}_{js_j} & \mathbf{0}_{p_j} \\ \boldsymbol{\lambda}_{vg} & \mathbf{0}_{p_v} & \boldsymbol{\lambda}_{vs_v} \end{pmatrix},$$

(4.27)

in which the vector $\boldsymbol{\lambda}_{jg}$ contains the loadings of the p_j elements of $\boldsymbol{X}_j$ on ability g, $\boldsymbol{\lambda}_{js_j}$, the loadings of the p_j elements of $\boldsymbol{X}_j$ on ability s_j, etc.[29];

c) over $i \in \varDelta_r$, and for $r = \{C,P\}$,

$$\underset{i}{C}(\boldsymbol{\xi}_{ir}) = \boldsymbol{\Phi}_r = \mathbf{I},$$

(4.28)

$$\underset{i}{C}(\boldsymbol{\zeta}_{ir}) = \boldsymbol{\Psi}_r,$$

(4.29)

[29] The $\mathbf{0}$ vectors express the fact that the elements of $\boldsymbol{X}_j$ do not load on s_v, and those of $\boldsymbol{X}_v$ do not load on s_j.

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in which $\boldsymbol{\Psi}_r$ is an (pj+pv) by (pj+pv), diagonal, positive definite matrix, structured as

$$\boldsymbol{\Psi}_r = \begin{bmatrix} \boldsymbol{\Psi}_{j|r} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Psi}_{v|r} \end{bmatrix} \text{ and}$$

$$\underset{i}{C}(\boldsymbol{\xi}_{ir}, \boldsymbol{\zeta}_{ir}) = \mathbf{0}, \tag{4.30}$$

a 3 by (p_j+p_v) null matrix.

The reader should note that, under this mathematization: a) over populations C and P, weak or pattern invariance holds; b) discriminating power parameters $\gamma_{clj|sj}$ and $\gamma_{clj|g}$ are taken to be factor loadings, i.e., elements of $\boldsymbol{\Lambda}$. This implies, in particular, that $\gamma_{clj|sj}$ is the lth element of $\boldsymbol{\lambda}_{js_j}$ (this element notated $\boldsymbol{\lambda}_{js_j}[l]$) and $\gamma_{clj|g}$, the lth element of $\boldsymbol{\lambda}_{jg}$ (notated $\boldsymbol{\lambda}_{jg}[l]$). By mathematizing the concept of *discriminating power* of, say, test composite c_{lj} in respect to ability s_j, as a factor loading, we quantitatively paraphrase it as "the number of units change in the lth element of $\boldsymbol{X}_j$ associated with a one standard deviation increase in s_j (the second element of $\boldsymbol{\xi}_r$)".

All told, we have, thus, the following, fully articulated, mathematical framework:

$$\mathbf{X}|i,r \sim (\boldsymbol{\gamma}_r + \boldsymbol{\Lambda}_r\boldsymbol{\xi}_{ir} + \boldsymbol{\zeta}_{ir}, \boldsymbol{\Sigma}_{\varepsilon ir}), \text{ } i \in \varDelta, \text{ } r, \text{ } r=\{C,P\} \tag{4.31}$$

with consequence that

$$\mathbf{X}|r \sim (\boldsymbol{\gamma}_r + \boldsymbol{\Lambda}\boldsymbol{\kappa}_r, \boldsymbol{\Lambda}\boldsymbol{\Lambda}' + \boldsymbol{\Theta}_{\delta r}), \text{ } r = \{C,P\}, \tag{4.32}$$

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in which $\Theta_{\delta r} = \Sigma_{Er} + \Psi_r$, diagonal, $r=\{C,P\}$, with $\Sigma_{Er} = \mathrm{E}(\Sigma_{\varepsilon ir})$ [30].

4.4 **Mathematization of Claims**

We are, now, ready to mathematize the core claims of Chapman and Chapman, i.e., *CC1* to *CC5*.

4.4.1 **Mathematization of Core Claims (CC1-CC5)**

CC1: As will be recalled (see Section 2.5), *CC1* asserts that differences in the distributions $c_{lj}|P$ and $c_{lj}|C$, hence, *inferred* differences in the distributions $c_{lj}|P$ and $c_{lj}|C$ (notably, inferences about the parameter ($\mu_{c_{lj}|P} - \mu_{c_{lj}|C}$)), are confounded, as bases for making decisions about S-deficits (notably about the parameter ($\mu_{s_j|P} - \mu_{s_j|C}$)), by differences in $g|P$ and $g|C$ (notably, as reflected in the parameter ($\mu_{g|P} - \mu_{g|C}$)). Now, under the proto-framework of section 4.3, $\mu_{c_{lj}|P}$ is the lth element of the vector $\boldsymbol{\mu}_{X_j|P}$ (notated $\boldsymbol{\mu}_{X_j|P}[l]$ [31]), and $\mu_{c_{lj}|P}$, the lth element of $\boldsymbol{\mu}_{X_j|P}$. Presupposing, then, that expressions (4.31)-(4.32) hold, *CC1* can be interpreted as follows: The bias in estimating

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[30] Note that $\mathrm{E}(\Sigma_{\varepsilon ir})$ is the expectation of $\Sigma_{\varepsilon ir}$.

[31] Note that square brackets denote an element of the vector or matrix preceding the brackets.

$\kappa_P[2] - \kappa_C[2]$ [32] by $\mu_{X_j|P}[l] - \mu_{X_j|C}[l]$, i.e., $\left| (\mu_{X_j|P}[l] - \mu_{X_j|C}[l]) - (\kappa_P[2] - \kappa_C[2]) \right|$, is a

function of the quantity $\kappa_P[1] - \kappa_C[1]$.

CC2: As will be recalled, *CC2* asserts that inferences about the parameter $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C})$

$-(\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$ are confounded, as bases for making decisions about D-Deficits

(notably, inferences about the parameter $(\mu_{s_j/P} - \mu_{s_j/C}) - (\mu_{s_v/P} - \mu_{s_v/C})$), by differences in

$\gamma_{clj/g}$ and $\gamma_{cuv/g}$. Under the proto-framework of section 4.3, and presupposing that (4.31)-

(4.32) hold, *CC2* can be interpreted as follows: The bias in estimating

$(\kappa_P[2] - \kappa_C[2]) - (\kappa_P[3] - \kappa_C[3])$ by $(\mu_{X_j|P}[l] - \mu_{X_j|C}[l]) - (\mu_{X_v|P}[u] - \mu_{X_v|C}[u])$, i.e.,

$\left| ((\mu_{X_j|P}[l] - \mu_{X_j|C}[l]) - (\mu_{X_v|P}[u] - \mu_{X_v|C}[u])) - ((\kappa_P[2] - \kappa_C[2]) - (\kappa_P[3] - \kappa_C[3])) \right|$, is a

function of the quantity $\lambda_{jg}[l] - \lambda_{vg}[u]$.

CC3: As will be recalled (see section 2.5), *CC3* asserts that for tests $t_{lj} \in T_j$ and $t_{uv} \in T_v$, a)

If $(\mu_{c_{lj}} \left| C_{H_{clj}} \right. - \mu_{c_{lj}} \left| C_{L_{clj}} \right.) = (\mu_{c_{uv}} \left| C_{H_{cuv}} \right. - \mu_{c_{uv}} \left| C_{L_{cuv}} \right.)$ then $\gamma_{clj/g} = \gamma_{cuv/g}$; b) If $(\mu_{c_{lj}} \left| C_{H_{clj}} \right. - \mu_{c_{lj}} \left| C_{L_{clj}} \right.)$

$> (\mu_{c_{uv}} \left| C_{H_{cuv}} \right. - \mu_{c_{uv}} \left| C_{L_{cuv}} \right.)$ then $\gamma_{clj/g} > \gamma_{cuv/g}$.

Now, under the proto-framework of section 4.3, and considering the p_j-element

random vector X_j , there is a corresponding p_j-element vector $(\mu_{X_j \left| C_{H_{clj}} \right.} - \mu_{X_j \left| C_{L_{clj}} \right.})$, in

---

[32] $\kappa_P[2]$ and $\kappa_C[2]$ represent the second element of each of κ_P and κ_C , which are the mean vectors of ξ_{iP} and ξ_{iC} .

which $\mu_{X_j|C_{H_{clj}}}$ is a vector of means for sub-populations of C defined, for each test

composite c_{lj}, by $\forall i \in C_{H_{clj}}, \tau_{ij}[l] > m_j[l]$, in which $m_j[l]$ is the lth element of a vector of

medians, and $\tau_{ij}[l]$ is the true score of person i in respect test composite c_{lj}. Similarly,

$\mu_{X_j|C_{L_{clj}}}$ is a vector of means for sub-populations of C defined, for each test composite c_{lj},

by $\forall i \in C_{L_{clj}}, \tau_{ij}[l] < m_j[l]$. The quantities $(\mu_{c_{lj}}\big|C_{H_{clj}} - \mu_{c_{lj}}\big|C_{L_{clj}})$ and $(\mu_{c_{uv}}\big|C_{H_{cuv}} - \mu_{c_{uv}}\big|C_{L_{cuv}})$

are simply elements of $(\mu_{X_j|C_{H_{clj}}} - \mu_{X_j|C_{L_{clj}}})$ and $(\mu_{X_v|C_{H_{cuv}}} - \mu_{X_v|C_{L_{cuv}}})$. Presupposing that

expressions 4.31-4.32 hold, we, then, interpret *CC3* as follows: a) If

$(\mu_{X_j|C_{H_{clj}}} - \mu_{X_j|C_{L_{clj}}})[l] = (\mu_{X_v|C_{H_{cuv}}} - \mu_{X_v|C_{L_{cuv}}})[u]$ then $\lambda_{jg}[l] = \lambda_{vg}[u]$; b) If

$(\mu_{X_j|C_{H_{clj}}} - \mu_{X_j|C_{L_{clj}}})[l] > (\mu_{X_v|C_{H_{cuv}}} - \mu_{X_v|C_{L_{cuv}}})[u]$ then $\lambda_{jg}[l] > \lambda_{vg}[u]$.

CC4: As will be recalled, *CC4* asserts that if two tests t_{lj} and t_{uv} are method 1 matched,

i.e., $p_{lj}=p_{uv}$, and that, for each of $d=1..p_{lj}, \beta_{Idlj} = \beta_{IdMuv}$ and $\rho_{clj,dlj}= \rho_{cuv,dMuv}$, then

$\gamma_{clj/g} = \gamma_{cuv/g}$. Let it be the case, then, that X_j and X_v are method 1 matched, and $X =$

$\begin{bmatrix} X_j \\ X_v \end{bmatrix}$, the first p_j elements of which, now, are *items* contributing to a unit-weighted test

composite $c_j = 1'X_j$ (wherein, $1'$ is a p_j-element vector of 1s), and the second p_v elements

of which are *items* contributing to a unit-weighted test composite $\qquad\qquad c_v = 1'X_v$

[33](wherein, $\boldsymbol{1}'$ is a p_v-element vector of 1s). Note that, since $\boldsymbol{X}_j$ and $\boldsymbol{X}_v$ are method 1 matched, $p_j = p_v$. Let, further, $\boldsymbol{X}$ be describable as per section 4.3. Then, it can be shown that the expression for the mean of population Δ_r is

$$\mu_{c_j|r} = \boldsymbol{1}'\mu_{X_j|r} = \boldsymbol{1}'(\gamma_{j|r} + \lambda_{jg}(\kappa_r[1]) + \lambda_{js_j}(\kappa_r[2])) \tag{4.33}$$

wherein $\boldsymbol{1}'$ is a p_j- element vector of 1s. This shows that the discriminating power term $\gamma_{clj|g}$ referenced in *CC4* is translated as $\boldsymbol{1}'\lambda_{jg}$, and the term $\gamma_{cuv|g}$ is translated as $\boldsymbol{1}'\lambda_{vg}$.

Now, the claim above is translated as follows: If, $\forall l$, $\boldsymbol{\beta}_j[l] = \boldsymbol{\beta}_v[l]$, and

$\rho_{X_j[l],\boldsymbol{1}'X_j} = \rho_{X_v[l],\boldsymbol{1}'X_v}$, then $\boldsymbol{1}'\lambda_{jg} = \boldsymbol{1}'\lambda_{vg}$, in which $\boldsymbol{\beta}_j$ is the p_j-vector of difficulties

associated with $\boldsymbol{X}_j$, $\boldsymbol{\beta}_v$ is the p_j-vector of difficulties associated with $\boldsymbol{X}_v$, and $\rho_{X_j[l],\boldsymbol{1}'X_j}$,

the correlation of $\boldsymbol{1}'X_j$ with $\boldsymbol{X}_j[l]$.

CC5. As will be recalled, *CC5* asserts that if two tests t_{lj} and t_{uv} are method 2 matched, i.e., $p_{lj} = p_{uv}$, and that, for each of $d = 1..p_{lj}$, $\beta_{Idlj} = 1 - \beta_{IdMuv}$ and $\rho_{clj,dlj} = \rho_{cuv,dMuv}$, then

$\gamma_{clj|g} = \gamma_{cuv|g}$. Let it be the case that $\boldsymbol{X}_j$ and $\boldsymbol{X}_v$ are method 2 matched and describable as per the mathematization of *CC4* above. Then the claim is translated as follows: If, $\forall l$,

$\boldsymbol{\beta}_j[l] = 1 - \boldsymbol{\beta}_v[l]$, and $\rho_{X_j[l],\boldsymbol{1}'X_j} = \rho_{X_v[l],\boldsymbol{1}'X_v}$, then $\boldsymbol{1}'\lambda_{jg} = \boldsymbol{1}'\lambda_{vg}$, in which $\boldsymbol{\beta}_j$ is the p_j-vector

of difficulties associated with $\boldsymbol{X}_j$, $\rho_{X_j[l],\boldsymbol{1}'X_j}$, the correlation of $\boldsymbol{1}'X_j$ with $\boldsymbol{X}_j[l]$.

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[33] Note that, in this section, X takes on the specific meaning of a vector of items, rather than, as introduced in section 4.3, a vector of test composite variables.

CC6. As will be recalled, *CC6* states that for two tests c_{lj} and c_{uv}, if $\sigma^2_{\tau_{lj}} < \sigma^2_{\tau_{uv}}$ then

$\gamma_{clj/g} < \gamma_{cuv/g}$. Under the proto-framework outlined in section 4.3, the claim is translated as follows: if $\boldsymbol{\Sigma}_{\tau_j}[l,l] < \boldsymbol{\Sigma}_{\tau_v}[u,u]$ then $\boldsymbol{\lambda}_{jg}[l] < \boldsymbol{\lambda}_{vg}[u]$, in which $\boldsymbol{\Sigma}_{\tau_j}$ is, consistent with expression 4.13, the true score covariance matrix in respect $\boldsymbol{X}_j$.

4.5 Adjudication of Core Claims

CC1: From 4.23,

$$\boldsymbol{\mu}_r = \boldsymbol{\gamma}_r + \boldsymbol{\Lambda}_r \boldsymbol{\kappa}_r. \tag{4.34}$$

Since, under the proto-framework of section 4.3, $\boldsymbol{\Lambda}_P = \boldsymbol{\Lambda}_C = \boldsymbol{\Lambda}$,

$$\boldsymbol{\mu}_P - \boldsymbol{\mu}_C = \boldsymbol{\gamma}_P - \boldsymbol{\gamma}_C + \boldsymbol{\Lambda}(\boldsymbol{\kappa}_P - \boldsymbol{\kappa}_C). \tag{4.35}$$

Thus, for the first p_j test composites (each formed of the items of a particular test contained in T_j), it follows from expression 4.35 that,

$$\boldsymbol{\mu}_{X_j|P} - \boldsymbol{\mu}_{X_j|C} = \boldsymbol{\gamma}_{j|P} - \boldsymbol{\gamma}_{j|C} + \boldsymbol{\lambda}_{jg}(\boldsymbol{\kappa}_P[1] - \boldsymbol{\kappa}_C[1]) + \boldsymbol{\lambda}_{js_j}(\boldsymbol{\kappa}_P[2] - \boldsymbol{\kappa}_C[2]), \tag{4.36}$$

in which $\boldsymbol{\gamma}_{j|P}$ and $\boldsymbol{\gamma}_{j|C}$ are p_j-vectors of intercepts. Considering the lth test composite among the p_j composites, then, the difference in means may, from expression 4.36, be expressed as follows:

$$\mu_{X_j|P}[l] - \mu_{X_j|C}[l] = \gamma_{j|P}[l] - \gamma_{j|C}[l] + \lambda_{jg}[l](\kappa_P[1] - \kappa_C[1]) + \lambda_{js_j}[l](\kappa_P[2] - \kappa_C[2]) \,. (4.37)$$

By rearrangement of expression 4.37,

$$(\kappa_P[2] - \kappa_C[2]) =$$

$$((\mu_{X_j|P}[l] - \mu_{X_j|C}[l]) - (\gamma_{j|P}[l] - \gamma_{j|C}[l] + \lambda_{jg}[l](\kappa_P[1] - \kappa_C[1])) / \lambda_{js_j}[l] \,. \qquad (4.38)$$

It follows that *CC1* is true, i.e., $\left| (\mu_{X_j|P}[l] - \mu_{X_j|C}[l]) - (\kappa_P[2] - \kappa_C[2]) \right|$ is a function of the

quantity $\kappa_P[1] - \kappa_C[1]$, except in the special case in which $\lambda_{jg}[l] = 0$. Therefore, under

the proto-framework of section 4.3, *CC1* is true, and C&C Confound1 exists.

CC2: *CC2* is interpreted under the proto-framework articulated in section 4.3. Therefore,

to repeat expression 4.37,

$$\mu_{X_j|P}[l] - \mu_{X_j|C}[l] = \gamma_{j|P}[l] - \gamma_{j|C}[l] + \lambda_{jg}[l](\kappa_P[1] - \kappa_C[1]) + \lambda_{js_j}[l](\kappa_P[2] - \kappa_C[2]) \,. \qquad (4.39)$$

Analogously, for the uth composite of the p_v composites contained in X_v,

$$\mu_{X_v|P}[u] - \mu_{X_v|C}[u] = \gamma_{v|P}[u] - \gamma_{v|C}[u] + \lambda_{vg}[u](\kappa_P[1] - \kappa_C[1]) + \lambda_{vs_v}[u](\kappa_P[3] - \kappa_C[3]) \,.(4.40)$$

It follows that

$$(\mu_{X_j|P}[l] - \mu_{X_j|C}[l]) - (\mu_{X_v|P}[u] - \mu_{X_v|C}[u]) = (\gamma_{j|P}[l] - \gamma_{j|C}[l]) - (\gamma_{v|P}[u] - \gamma_{v|C}[u])$$

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$$+(\lambda_{jg}[l]-\lambda_{vg}[u])(\kappa_P[1]-\kappa_C[1])+\lambda_{js_j}[l](\kappa_P[2]-\kappa_C[2])-\lambda_{vs_v}[u](\kappa_P[3]-\kappa_C[3]) \,.(4.41)$$

From this expression, it is apparent that

$$\left|((\mu_{X_j|P}[l]-\mu_{X_j|C}[l])-(\mu_{X_v|P}[u]-\mu_{X_v|C}[u]))-((\kappa_P[2]-\kappa_C[2])-(\kappa_P[3]-\kappa_C[3]))\right|$$, is, in

part, a function of the quantity $\lambda_{jg}[l]-\lambda_{vg}[u]$, unless $\kappa_P[1]-\kappa_C[1]=0$. Therefore, *CC2* is

true and C&C Confound2 exists under the proto-framework of section 4.3.

CC3: To repeat, *CC3* is interpreted as: a) If $(\mu_{X_j|C_{H_{clj}}}-\mu_{X_j|C_{L_{clj}}})[l]=(\mu_{X_v|C_{H_{cuv}}}-\mu_{X_v|C_{L_{cuv}}})[u]$

then $\lambda_{jg}[l]=\lambda_{vg}[u]$; b) If $(\mu_{X_j|C_{H_{clj}}}-\mu_{X_j|C_{L_{clj}}})[l]>(\mu_{X_v|C_{H_{cuv}}}-\mu_{X_v|C_{L_{cuv}}})[u]$ then

$\lambda_{jg}[l]>\lambda_{vg}[u]$. From expression (4.25), the vector of true scores for the distribution of

$X|i,r$ (the distribution of all composites for individual *i* in population *r*) may be

expressed as follows:

$$\tau_{ir}=\gamma_r+\Lambda_r\xi_{ir}+\zeta_{ir},i\in\varDelta_r,r=1...s\,.\tag{4.42}$$

Taking an expectation over all $i\in C_{H_{clj}}$, the expression for the vector of means of the test

composites in X_j is

$$\mu_{X_j|C_{H_{clj}}}=\gamma_{j|C}+\lambda_{jg}(\kappa_{C_{H_{clj}}}[1])+\lambda_{js_j}(\kappa_{C_{H_{clj}}}[2])+\mathrm{E}(\zeta_{iC_{H_{clj}}}),i\in C_{H_{clj}},\tag{4.43}$$

in which $\kappa_{C_{H_{clj}}} = \mathrm{E}(\xi_{iC_{H_{clj}}})$, the expectation over all $i \in C_{H_{clj}}$. The expressions for $\mu_{X_j|C_{L_{clj}}}$, $\mu_{X_v|C_{H_{cuv}}}$, and $\mu_{X_v|C_{L_{cuv}}}$ are of a form, analogous. Considering the lth element of the vector X_j, then (composite c_{lj}), the mean over all $i \in C_{H_{clj}}$ is expressed as follows:

$$\mu_{X_j|C_{H_{clj}}}[l] = \gamma_{j|C}[l] + \lambda_{jg}[l](\kappa_{C_{H_{clj}}}[1]) + \lambda_{js_j}[l](\kappa_{C_{H_{clj}}}[2]) + \mathrm{E}(\zeta_{iC_{H_{clj}}}[l]). \qquad (4.44)$$

Similarly, for the uth element of the vector, X_v (composite c_{uv}),

$$\mu_{X_v|C_{H_{cuv}}}[u] = \gamma_{v|C}[u] + \lambda_{vg}[u](\kappa_{C_{H_{cuv}}}[1]) + \lambda_{vs_v}[u](\kappa_{C_{H_{cuv}}}[3]) + \mathrm{E}(\zeta_{iC_{H_{cuv}}}[u]). \qquad (4.45)$$

The expressions for $\mu_{X_j|C_{L_{clj}}}[l]$ and $\mu_{X_v|C_{L_{cuv}}}[u]$ are, in form, analogous.

It follows, then, that

$$(\mu_{X_j|C_{H_{clj}}} - \mu_{X_j|C_{L_{clj}}})[l] = (\lambda_{jg}[l])(\kappa_{C_{H_{clj}}}[1] - \kappa_{C_{L_{clj}}}[1]) + (\lambda_{js_j}[l])(\kappa_{C_{H_{clj}}}[2] - \kappa_{C_{H_{clj}}}[2]) +$$

$$\mathrm{E}(\zeta_{iC_{H_{clj}}}[l]) - \mathrm{E}(\zeta_{iC_{L_{clj}}}[l]), \qquad (4.46)$$

and

$$(\mu_{X_v|C_{H_{cuv}}} - \mu_{X_v|C_{L_{cuv}}})[u] = (\lambda_{vg}[u])(\kappa_{C_{H_{cuv}}}[1] - \kappa_{C_{L_{cuv}}}[1]) + (\lambda_{vs_v}[u])(\kappa_{C_{H_{cuv}}}[3] - \kappa_{C_{H_{cuv}}}[3]) +$$

$$\mathrm{E}(\zeta_{iC_{H_{cuv}}}[u]) - \mathrm{E}(\zeta_{iC_{L_{cuv}}}[u]), \qquad (4.47)$$

from which it follows that

$$(\mu_{X_j|C_{H_{clj}}} - \mu_{X_j|C_{L_{clj}}})[l] - (\mu_{X_v|C_{H_{cuv}}} - \mu_{X_v|C_{L_{cuv}}})[u] =$$

$$(\lambda_{jg}[l])(\kappa_{C_{H_{clj}}}[1] - \kappa_{C_{L_{clj}}}[1]) - (\lambda_{vg}[u])(\kappa_{C_{H_{cuv}}}[1] - \kappa_{C_{L_{cuv}}}[1]) +$$

$$(\lambda_{js_j}[l])(\kappa_{C_{H_{clj}}}[2] - \kappa_{C_{H_{clj}}}[2]) - (\lambda_{vs_v}[u])(\kappa_{C_{H_{cuv}}}[3] - \kappa_{C_{H_{cuv}}}[3]) +$$

$$(E(\zeta_{iC_{H_{clj}}}[l]) - E(\zeta_{iC_{L_{clj}}}[l])) - (E(\zeta_{iC_{H_{cuv}}}[u]) - E(\zeta_{iC_{L_{cuv}}}[u])) . \qquad (4.48)$$

By inspection of 4.48, it is evident that $(\mu_{X_j|C_{H_{clj}}} - \mu_{X_j|C_{L_{clj}}})[l] - (\mu_{X_v|C_{H_{cuv}}} - \mu_{X_v|C_{L_{cuv}}})[u]$ is a

function of a number of quantities besides $\lambda_{jg}[l]$ and $\lambda_{vg}[u]$. CC3 is therefore false, for:

a) If $(\mu_{X_j|C_{H_{clj}}} - \mu_{X_j|C_{L_{clj}}})[l] = (\mu_{X_v|C_{H_{cuv}}} - \mu_{X_v|C_{L_{cuv}}})[u]$, and hence the left side of expression

4.48 is equal to 0, it need not be the case that $\lambda_{jg}[l] = \lambda_{vg}[u]$; b) If

$(\mu_{X_j|C_{H_{clj}}} - \mu_{X_j|C_{L_{clj}}})[l] > (\mu_{X_v|C_{H_{cuv}}} - \mu_{X_v|C_{L_{cuv}}})[u]$, and hence the left side of expression 4.48 is

a positive quantity, it need not be the case that $\lambda_{jg}[l] > \lambda_{vg}[u]$.

CC4: To repeat, the claim is that, if, $\forall l$, $\beta_j[l] = \beta_v[l]$, and $\rho_{X_j[l],I'X_j} = \rho_{X_v[l],I'X_v}$, then

$I'\lambda_{jg} = I'\lambda_{vg}$. Note that, if $\beta_j[l] = \beta_v[l]$, then $\beta_j[l](1 - \beta_j[l]) = \beta_v[l](1 - \beta_v[l])$. Now, from

equations (4.9) and (4.32),

$$\beta_j[l](1 - \beta_j[l]) = (\Sigma_{X_j}[l,l]) = (\lambda_{js_j}[l])^2 + (\lambda_{jg}[l])^2 + \Theta_{\delta_j}[l,l], \qquad (4.49)$$

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wherein Θ_{δ_j} is the $p_j \times p_j$ covariance matrix of δ_j. Similarly,

$$\beta_v[l](1-\beta_v[l]) = (\Sigma_{X_v}[l,l]) = (\lambda_{vs_v}[l])^2 + (\lambda_{vg}[l])^2 + \Theta_{\delta_v}[l,l]. \tag{4.50}$$

Therefore,

$$\beta_j[l](1-\beta_j[l]) - \beta_v[l](1-\beta_v[l]) = ((\lambda_{js_j}[l])^2 + (\lambda_{jg}[l])^2 + \Theta_{\delta_j}[l,l]) -$$

$$((\lambda_{vs_v}[l])^2 + (\lambda_{vg}[l])^2 + \Theta_{\delta_v}[l,l]). \tag{4.51}$$

The above expression shows that if $\beta_j[l] = \beta_v[l]$, and hence

$\beta_j[l](1-\beta_j[l]) = \beta_v[l](1-\beta_v[l])$, then the elements of λ_{jg} and λ_{vg} remain free to vary.

Therefore, it is not that case that if, $\forall l$, $\beta_j[l] = \beta_v[l]$, alone, then $\mathbf{1}'\lambda_{jg} = \mathbf{1}'\lambda_{vg}$. However,

it remains to evaluate the case in which both $\beta_j[l] = \beta_v[l]$, and $\rho_{X_j[l],\mathbf{1}'X_j} = \rho_{X_v[l],\mathbf{1}'X_v}$.

Now, it can be shown that

$$\rho_{X_j[l],\mathbf{1}'X_j} = \frac{\sum_{y=1}^{p_j} \Sigma_{X_j}[y,l]}{\sigma_{\mathbf{1}'X_j} \sigma_{X_j[l]}}, \tag{4.52}$$

the constituent parts of which are worked out as follows:

[from 4.22] $\quad \sum_{y=1}^{p_j} \Sigma_{X_j}[y,l] = \sum_{y=1}^{p_j} (\lambda_{js_j}[y]\lambda_{js_j}[l] + \lambda_{jg}[y]\lambda_{jg}[l] + \Theta_{\delta_j}[y,l]); \tag{4.53}$

[from 4.22] $$\sigma^2_{I'X_j} = I'\Sigma_{X_j} I = I'(\Lambda_j\Lambda_j{}' + \Theta_{\delta_j})I \ {}^{34};$$ (4.54)

[from 4.22] $$\sigma_{X_{j[l]}}{}^2 = (\Sigma_{X_j}[l,l]).$$ (4.55)

Now, in the case in which, $\forall l$, $\beta_j[l] = \beta_v[l]$, then, by expression 4.49,

$\Sigma_{X_j}[l,l] = \Sigma_{X_v}[l,l]$. Therefore, under condition that $\rho_{X_{j[l]},I'X_j} = \rho_{X_{v[l]},I'X_v}$, and, $\forall l$,

$\beta_j[l] = \beta_v[l]$,

$$\frac{\sum_{y=1}^{p_j}(\lambda_{js_j}[y]\lambda_{js_j}[l] + \lambda_{jg}[y]\lambda_{jg}[l] + \Theta_{\delta_j}[y,l])}{I'(\Lambda_j\Lambda_j{}' + \Theta_{\delta_j})I} =$$

$$\frac{\sum_{y=1}^{p_j}(\lambda_{vs_v}[y]\lambda_{vs_v}[l] + \lambda_{vg}[y]\lambda_{vg}[l] + \Theta_{\delta_v}[y,l])}{I'(\Lambda_v\Lambda_v{}' + \Theta_{\delta_v})I}.$$ (4.56)

By inspection of expression 4.56, it is evident that the elements of λ_{jg} and λ_{vg} remain

free to vary, and, consequently, $I'\lambda_{jg}$ and $I'\lambda_{vg}$ are free to vary. Therefore,

$\rho_{X_{j[l]},I'X_j} = \rho_{X_{v[l]},I'X_v}$, and, $\forall l$, $\beta_j[l] = \beta_v[l]$, do not jointly imply that $I'\lambda_{jg} = I'\lambda_{vg}$. Under

the the proto-framework in play, then, *CC4* is false.

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[34] wherein $\Lambda_j = \begin{bmatrix} \lambda_{jg} & \lambda_{js_j} & \mathbf{0}_{p_j} \end{bmatrix}$

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CC5. To repeat, the claim is that if, $\forall l$, $\beta_j[l]=1-\beta_v[l]$, and $\rho_{X_j[l],I'X_j}=\rho_{X_v[l],I'X_v}$, then

$I'\lambda_{jg}=I'\lambda_{vg}$. Note that if $\beta_j[l]=1-\beta_v[l]$, then $\beta_j[l](1-\beta_j[l])=\beta_v[l](1-\beta_v[l])$. By the same logic as in the adjudication of *CC4*, then, *CC5* is false under the proto-framework in play.

CC6. Under the proto-framework outlined in section 4.3, the claim is translated as follows: If $\Sigma_{\tau_j}[l,l]<\Sigma_{\tau_v}[u,u]$ then $\lambda_{jg}[l]<\lambda_{vg}[u]$. Now, From 4.32 and 4.13,

$$\Sigma_\tau=\Lambda\Lambda'+\Psi_r,\tag{4.57}$$

hence,

$$\Sigma_{\tau_j}[l,l]=(\lambda_{js_j}[l])^2+(\lambda_{jg}[l])^2+\Psi_{j|C}[l,l],\tag{4.58}$$

and

$$\Sigma_{\tau_v}[u,u]=(\lambda_{vs_v}[u])^2+(\lambda_{vg}[u])^2+\Psi_{v|C}[u,u],\tag{4.59}$$

Therefore, if $\Sigma_{\tau_j}[l,l]<\Sigma_{\tau_v}[u,u]$, then

$$((\lambda_{js_j}[l])^2+(\lambda_{jg}[l])^2+\Psi_{j|C}[l,l])<((\lambda_{js_j}[l])^2+(\lambda_{jg}[l])^2+\Psi_{j|C}[l,l]),\tag{4.60}$$

by which it is evident that it is not the case that if $\Sigma_{\tau_j}[l,l]<\Sigma_{\tau_v}[u,u]$ then $\lambda_{jg}[l]<\lambda_{vg}[u]$, as $\lambda_{jg}[l]$ and $\lambda_{vg}[u]$ remain free to vary. Consequently, *CC5* is false.

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4.6 **Summary**

In this chapter we have, 1) described the multivariate classical true score model and multi-population linear factor model nascent within the work of Chapman and Chapman; 2) mathematized Chapman and Chapman's focal claims (*CC1-CC6*); 3) adjudicated claims *CC1-CC6*. To review, we concluded that: a) *CC1* and *CC2*, asserted under the proto-framework of section 4.3, are true. This implies that, under this proto-framework, it is true that there are flaws inherent in C&C Strategies 1 and 2, these flaws called C&C Confound1 and C&C Confound2; b) *CC3*, asserted under the proto-framework of section 4.3, is false, implying that conclusions drawn regarding D-Deficits via the Control High-Low Scorers Comparison Method are faulty; c) *CC4* and *CC5*, asserted under a modification of the proto-framework of 4.3, are false, such that conclusions drawn regarding D-Deficits via the test matching methods 1 and 2 are faulty; d) *CC6*, asserted under the proto-framework of section 4.3, is false, implying that conclusions drawn regarding D-Deficits via the True Score Variance Comparison method are faulty.

In the next chapter, we turn to a mathematization and adjudication of the claims of the alternative accounts outlined in Chapter 3.

Chapter 5.

Mathematization and Evaluation of Claims of Alternative Accounts

In Chapter 3, accounts of the Psychometric Confound alternative to that put forward by Chapman and Chapman were reviewed, and the Core Claims of these account summarized. We now turn to an adjudication of these claims, account by account. For each alternative account, we: *i*) complete the proto-framework that was implied by each author or group of authors, the beginnings of which were extracted in Chapter 3 (note that all of the proto-frameworks completed below are extensions of the general Multi-Population Linear Factor Model of section 4.2); *ii*) elucidate, given the completed proto-framework and the test theory of section 4.1, the claims of the alternative account by mathematizing them, *iii*) adjudicate, given this mathematization, the claims.

5.1 Baron and Treiman

5.1.1 Completion of Proto-Framework

As will be recalled (see section 3.3.1.4) the elements of *B&TP-f* extracted in Chapter 3 are as follows:

a) a "control test" t_{uv} scales individuals, in a manner that is largely unknown to the researcher, with respect to abilities $s_1 \ldots s_k$;

b) an "experimental test" t_{lj} scales individuals with respect to the ability s_j, as well as abilities $s_1...s_k$;

c) test t_{uv} is, for each ability $s_1...s_k$, characterized by a parameter that quantifies discriminating power in respect to the scaling of that ability, and which determines, in part, the observed-score distributions $c_{uv}|P$ and $c_{uv}|C$;

d) test t_{lj}, is, for each ability $s_1...s_k$, characterized by a parameter that quantifies discriminating power in respect to the scaling of that ability, and which determines, in part, the observed-score distributions $c_{lj}|P$ and $c_{lj}|C$.

We, now, complete this proto-framework. Let: a) once again, i stand for individual; b) $r=\{C,P\}$; and c) X be the (p_j+p_v)-element random vector $\begin{bmatrix} X_j \\ X_v \end{bmatrix}$, the first p_j elements of which are the *items* contributing to a unit-weighted test composite c_{lj} , in which $c_{lj}=\mathbf{1'X}_j$, $\mathbf{1'}$ being a p_j-element vector of 1s, and the second p_v items of which are items contributing to a unit-weighted test composite c_{uv}, in which $c_{uv}=\mathbf{1'X}_v$, $\mathbf{1'}$ being, in this case, a p_v-element vector of 1s.

From references to the classical test theory concept "reliability", etc., it is clear that (4.10) is invoked, that is to say,

$$X|i,r \sim (\boldsymbol{\tau}_{ir}, \boldsymbol{\Sigma}_{\varepsilon ir}), \ \boldsymbol{\Sigma}_{\varepsilon ir}, \ \text{diagonal}, \ i \in r, \ r=\{C,P\}. \tag{5.1}$$

From the references to multiple abilities, and the multi-population context in which concern for the psychometric confound is situated, we deduce that (4.21) is invoked as a representation, or model, of the (p_j+p_v) vector $\boldsymbol{\tau}_{ir}$, i.e.,

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$$\tau_{ir} = \gamma_r + \Lambda_r \xi_{ir} + \zeta_{ir}, i \in \Delta_r, r = 1...s, \qquad (5.2)$$

with the following particularizations, restrictions, and clarifications being made:

a) $s=2$, i.e., $r = \{C,P\}$; $m=k+1$, in which

$$\xi_{ir} = \begin{bmatrix} s_{ij} \\ s_{i1} \\ \vdots \\ s_{ik} \end{bmatrix}; \qquad (5.3)$$

b) $\Lambda_r = \Lambda$, (p_j+p_v) by $(k+1)$, and structured as follows:

$$\begin{pmatrix} \lambda_{js_j} & \lambda_{js_1} & \cdots & \lambda_{js_k} \\ \mathbf{0}_{p_j} & \lambda_{vs_1} & \cdots & \lambda_{vs_k} \end{pmatrix}, \qquad (5.4)$$

in which the vector λ_{js_j} contains the loadings of the p_j elements of X_j on ability s_j, λ_{vs_1},

the loadings of the p_v elements of X_v on ability s_1, etc.[35];

c) over $i \in \Delta_r$, and for $r = \{C,P\}$,

$$\underset{i}{C}(\xi_{ir}) = \Phi_r = \mathbf{I}, \qquad (5.5)$$

[35] The $\mathbf{0}$ vector express the fact that the elements of X_v do not load on s_j.

$$C(\boldsymbol{\zeta}_{ir}) = \boldsymbol{\Psi}_r, \tag{5.6}$$

in which $\boldsymbol{\Psi}_r$ is an (p_j+p_v) by (p_j+p_v), diagonal, positive definite matrix, structured as

$$\boldsymbol{\Psi}_r = \begin{bmatrix} \boldsymbol{\Psi}_{j|r} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Psi}_{v|r} \end{bmatrix} \text{ and}$$

$$C(\boldsymbol{\xi}_{ir}, \boldsymbol{\zeta}_{ir}) = \mathbf{0}, \tag{5.7}$$

a $(k+1)$ by (p_j+p_v) null matrix.

Under this mathematization: a) over populations C and P, weak or pattern invariance holds; b) for composite c_{lj} there exists discriminating power parameters in respect s_j and $s_1...s_k$ that are sums of the factor loadings, i.e. elements of $\boldsymbol{\Lambda}$. The concept *discriminating power* of say, test composite c_{lj} in respect to ability s_j, is, in this case quantitatively paraphrased as "the number of units change in the test composite c_{lj} associated with a one standard deviation increase in s_j (the first element of $\boldsymbol{\xi}_r$)".

All told, we have, thus, the following, fully articulated, mathematical framework:

$$\mathbf{X}|i, r \sim (\boldsymbol{\gamma}_r + \boldsymbol{\Lambda}_r\boldsymbol{\xi}_{ir} + \boldsymbol{\zeta}_{ir}, \boldsymbol{\Sigma}_{\varepsilon ir}), \, i \in \varDelta, r, r=\{C,P\}, \tag{5.8}$$

with consequence that

$$\mathbf{X}|r \sim (\boldsymbol{\gamma}_r + \boldsymbol{\Lambda}\boldsymbol{\kappa}_r, \boldsymbol{\Lambda}\boldsymbol{\Lambda}' + \boldsymbol{\Theta}_{\delta r}), \, r = \{C,P\}, \tag{5.9}$$

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in which $\Theta_{\delta r} = \Sigma_{Er} + \Psi_r$, diagonal, $r=\{C,P\}$, with $\Sigma_{Er} = \mathrm{E}(\Sigma_{\varepsilon ir})$.

5.1.2 **Mathematization of Claims**

BTC1: As will be recalled (section 3.3.1.3), Baron and Treiman asserted that: a) *inferred* differences in the distributions $c_{lj}|P$ and $c_{lj}|C$ (notably, inferences about the parameter $(\mu_{c_{lj}|P} - \mu_{c_{lj}|C})$), are, indeed, confounded as bases for making decisions about S-deficits (notably about the parameter $(\mu_{s_j|P} - \mu_{s_j|C})$; b) the source of this confounding is not, as claimed by Chapman and Chapman, differences in $g|P$ and $g|C$ (notably, as reflected in the parameter $(\mu_{g|P} - \mu_{g|C})$), but, rather differences between distributions $s_1|P$ and $s_1|C$, $s_2|P$ and $s_1|C$,…, $s_k|P$ and $s_k|C$ (this is the claim of existence of B&T Confound1).

Now, under the proto-framework of section 5.1.1, the expression for the mean of the composite c_{lj} in population P is:

$$\mu_{c_{lj}|P} = \boldsymbol{1}'\boldsymbol{\mu}_{X_j|P}, \tag{5.10}$$

wherein $\boldsymbol{1}'$ is a p_j-*element* vector of 1s. Similarly, the expression for the mean of the composite c_{lj} in population C is:

$$\mu_{c_{lj}|C} = \boldsymbol{1}'\boldsymbol{\mu}_{X_j|C}, \tag{5.11}$$

wherein $\boldsymbol{1}'$ is a p_v-*element* vector of 1s. The parameter $(\mu_{s_j|P} - \mu_{s_j|C})$ is translated as $\boldsymbol{\kappa}_P[1] - \boldsymbol{\kappa}_C[1]$. Analogous to the mathematization of *CC1* in section 4.4.1, then, *BTC1* is

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translated as follows: The bias in estimating $\kappa_P[1] - \kappa_C[1]$ by $\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C}$ is a function of the quantities $(\kappa_P[2] - \kappa_C[2]), (\kappa_P[3] - \kappa_C[3]), ..., (\kappa_P[k+1] - \kappa_C[k+1])$.

BTC2: As will be recalled (see section 3.3.1.3) Baron and Treiman claimed that there exists a confound in respect to the B&T Strategy, wherein it is asserted that: a) inferences about the parameter $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$ are confounded as bases for making decisions about S-Deficits; b) the source of this confounding is differences in the unknown discriminating powers of the tests employed, i.e., differences between $\gamma_{clj/s1}$ ... $\gamma_{clj/sk}$ and $\gamma_{cuv/s1}$ ... $\gamma_{cuv/sk}$ (this is the claim of existence of B&T Confound2).

Now, the existence of an S-Deficit is defined by the quantity $(\mu_{s_j/P} - \mu_{s_j/C})$, which is translated under the proto-framework above as $\kappa_P[1] - \kappa_C[1]$. The parameter $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C})$ is, from equations (5.10) and (5.11), written as $\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C}$, wherein, as above, $\mathbf{1}'$ is a p_j-*element* vector of 1s. Similarly, the parameter $(\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$ is written as $\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C}$, wherein, in this case, $\mathbf{1}'$ is a p_v-*element* vector of 1s. The parameter $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$ is, therefore, written as $(\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C}) - (\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C})$. The discriminating powers are sums of elements of Λ, i.e. $\mathbf{1}'\boldsymbol{\lambda}_{js_1}$ ... $\mathbf{1}'\boldsymbol{\lambda}_{js_k}$ (in which $\mathbf{1}'$ is a p_j-*element* vector of 1s) and $\mathbf{1}'\boldsymbol{\lambda}_{vs_1}$ ... $\mathbf{1}'\boldsymbol{\lambda}_{vs_k}$ (in which $\mathbf{1}'$ is a p_v-*element* vector of 1s).

The claim is therefore translated as follows: The bias in estimating $\kappa_P[1] - \kappa_C[1]$

by $(\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C}) - (\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C})$, i.e.

$\left| (\kappa_P[1] - \kappa_C[1]) - ((\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C}) - (\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C})) \right|$ is a function of differences

between $\mathbf{1}'\boldsymbol{\lambda}_{js_1} \dots \mathbf{1}'\boldsymbol{\lambda}_{js_k}$ and $\mathbf{1}'\boldsymbol{\lambda}_{vs_1} \dots \mathbf{1}'\boldsymbol{\lambda}_{vs_k}$.

BTC3: As will be recalled (see section 3.3.1.3) It was claimed that if $\rho_{Ac_{lj}} > \rho_{Ac_{uv}}$ and

$\rho_{c_{lj}c_{lj}'} < \rho_{c_{uv}c_{uv}'}$, then there exists an S-Deficit in respect to s_j and populations P and C (this

is the basis for the Baron and Treiman Method).

Under the proto-framework of 5.1.1, and the definition of reliability (4.17),

$$\rho_{c_{lj}c_{lj}'} = \rho_{w'X_j w'X_j'} = \frac{\mathbf{1}'\boldsymbol{\Sigma}_{\tau_j}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}_{X_j}\mathbf{1}} = \frac{\mathbf{1}'(\boldsymbol{\Lambda}_j\boldsymbol{\Lambda}_j' + \boldsymbol{\Psi}_{j|r})\mathbf{1}}{\mathbf{1}'(\boldsymbol{\Lambda}_j\boldsymbol{\Lambda}_j' + \boldsymbol{\Sigma}_{E_j|r} + \boldsymbol{\Psi}_{j|r})\mathbf{1}}, \tag{5.12}$$

in which $\boldsymbol{\Lambda}_j = \begin{bmatrix} \boldsymbol{\lambda}_{js_j} & \boldsymbol{\lambda}_{js_1} & \dots & \boldsymbol{\lambda}_{js_k} \end{bmatrix}$, and

$$\boldsymbol{\Sigma}_{Er} = \begin{bmatrix} \boldsymbol{\Sigma}_{E_j|r} & 0 \\ 0 & \boldsymbol{\Sigma}_{E_v|r} \end{bmatrix}. \tag{5.13}$$

Furthermore, it can be shown that

$$\rho_{Ac_{lj}} = \mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C} / 2\sqrt{\sigma_{c_{lj}}^2}, \tag{5.14}$$

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in which $\sigma^2_{c_{lj}}$ is calculated over a combined population of P and C. The claim is therefore translated as follows: If

$$\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C} / 2\sqrt{\sigma^2_{c_{lj}}} > \mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C} / 2\sqrt{\sigma^2_{c_{uv}}} \qquad (5.15)$$

and

$$\frac{\mathbf{1}'(\boldsymbol{\Lambda}_j\boldsymbol{\Lambda}_j' + \boldsymbol{\Psi}_{j|r})\mathbf{1}}{\mathbf{1}'(\boldsymbol{\Lambda}_j\boldsymbol{\Lambda}_j' + \boldsymbol{\Sigma}_{E_{j|r}} + \boldsymbol{\Psi}_{j|r})\mathbf{1}} < \frac{\mathbf{1}'(\boldsymbol{\Lambda}_v\boldsymbol{\Lambda}_v' + \boldsymbol{\Psi}_{v|r})\mathbf{1}}{\mathbf{1}'(\boldsymbol{\Lambda}_v\boldsymbol{\Lambda}_v' + \boldsymbol{\Sigma}_{E_{v|r}} + \boldsymbol{\Psi}_{v|r})\mathbf{1}} \qquad (5.16)$$

then $\boldsymbol{\kappa}_P[1] - \boldsymbol{\kappa}_C[1] < 0$.

5.1.3 Adjudication of Claims

BTC1. As will be recalled, BTC1 states that the bias in estimating $\boldsymbol{\kappa}_P[1] - \boldsymbol{\kappa}_C[1]$ by $\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C}$ (the means of composite c_{lj} in populations P and C) is a function of the quantities $(\boldsymbol{\kappa}_P[2] - \boldsymbol{\kappa}_C[2]), (\boldsymbol{\kappa}_P[3] - \boldsymbol{\kappa}_C[3]), ..., (\boldsymbol{\kappa}_P[k+1] - \boldsymbol{\kappa}_C[k+1])$.

From 4.23,

$$\boldsymbol{\mu}_r = \boldsymbol{\gamma}_r + \boldsymbol{\Lambda}_r\boldsymbol{\kappa}_r. \qquad (5.17)$$

Since, under the proto-framework, $\boldsymbol{\Lambda}_P = \boldsymbol{\Lambda}_C = \boldsymbol{\Lambda}$,

$$\boldsymbol{\mu}_P - \boldsymbol{\mu}_C = \boldsymbol{\gamma}_P - \boldsymbol{\gamma}_C + \boldsymbol{\Lambda}(\boldsymbol{\kappa}_P - \boldsymbol{\kappa}_C). \qquad (5.18)$$

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It follows that

$$\mathbf{1'}\boldsymbol{\mu}_{X_j|P} - \mathbf{1'}\boldsymbol{\mu}_{X_j|C} = \mathbf{1'}\boldsymbol{\gamma}_{j|P} - \mathbf{1'}\boldsymbol{\gamma}_{j|C} + \mathbf{1'}(\boldsymbol{\Lambda}_j(\boldsymbol{\kappa}_P - \boldsymbol{\kappa}_C)), \tag{5.19}$$

in which

$$\mathbf{1'}(\boldsymbol{\Lambda}_j(\boldsymbol{\kappa}_P - \boldsymbol{\kappa}_C)) =$$

$$\mathbf{1'}\boldsymbol{\lambda}_{js_j}(\boldsymbol{\kappa}_P[1] - \boldsymbol{\kappa}_C[1]) + \mathbf{1'}\boldsymbol{\lambda}_{js_1}(\boldsymbol{\kappa}_P[2] - \boldsymbol{\kappa}_C[2]) + \ldots + \mathbf{1'}\boldsymbol{\lambda}_{js_k}(\boldsymbol{\kappa}_P[k+1] - \boldsymbol{\kappa}_C[k+1]) . \tag{5.20}$$

This shows that bias in estimating $\boldsymbol{\kappa}_P[1] - \boldsymbol{\kappa}_C[1]$ by $\mathbf{1'}\boldsymbol{\mu}_{X_j|P} - \mathbf{1'}\boldsymbol{\mu}_{X_j|C}$ is, under the proto-framework employed, a function of the quantities

$(\boldsymbol{\kappa}_P[2] - \boldsymbol{\kappa}_C[2]), (\boldsymbol{\kappa}_P[3] - \boldsymbol{\kappa}_C[3]), \ldots, (\boldsymbol{\kappa}_P[k+1] - \boldsymbol{\kappa}_C[k+1])$, as claimed by Baron and

Treiman. Hence, we conclude that *BTC1* is, under the proto-framework, correct.

BTC2. As will be recalled, *BTC2* states that

$\left| (\boldsymbol{\kappa}_P[1] - \boldsymbol{\kappa}_C[1]) - ((\mathbf{1'}\boldsymbol{\mu}_{X_j|P} - \mathbf{1'}\boldsymbol{\mu}_{X_j|C}) - (\mathbf{1'}\boldsymbol{\mu}_{X_v|P} - \mathbf{1'}\boldsymbol{\mu}_{X_v|C})) \right|$ is a function of differences

between $\mathbf{1'}\boldsymbol{\lambda}_{js_1} \ldots \mathbf{1'}\boldsymbol{\lambda}_{js_k}$ and $\mathbf{1'}\boldsymbol{\lambda}_{vs_1} \ldots \mathbf{1'}\boldsymbol{\lambda}_{vs_k}$.

From expression 5.18, under the proto-framework implied,

$$\mathbf{1'}\boldsymbol{\mu}_{X_j|P} - \mathbf{1'}\boldsymbol{\mu}_{X_j|C} = \mathbf{1'}\boldsymbol{\gamma}_{j|P} - \mathbf{1'}\boldsymbol{\gamma}_{j|C} + \mathbf{1'}(\boldsymbol{\Lambda}_j(\boldsymbol{\kappa}_P - \boldsymbol{\kappa}_C)), \tag{5.21}$$

and

$$\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C} = \mathbf{1}'\boldsymbol{\gamma}_{v|P} - \mathbf{1}'\boldsymbol{\gamma}_{v|C} + \mathbf{1}'(\boldsymbol{\Lambda}_v(\boldsymbol{\kappa}_P - \boldsymbol{\kappa}_C)), \tag{5.22}$$

wherein, again, $\mathbf{1}'$ is a *p_j-element* vector of 1s in (5.21), and a *p_j-element* vector of 1s in (5.22). Now,

$$\mathbf{1}'(\boldsymbol{\Lambda}_j(\boldsymbol{\kappa}_P - \boldsymbol{\kappa}_C)) =$$

$$\mathbf{1}'\boldsymbol{\lambda}_{js_j}(\boldsymbol{\kappa}_P[1] - \boldsymbol{\kappa}_C[1]) + \mathbf{1}'\boldsymbol{\lambda}_{js_1}(\boldsymbol{\kappa}_P[2] - \boldsymbol{\kappa}_C[2]) + ... + \mathbf{1}'\boldsymbol{\lambda}_{js_k}(\boldsymbol{\kappa}_P[k+1] - \boldsymbol{\kappa}_C[k+1]), \tag{5.23}$$

and

$$\mathbf{1}'(\boldsymbol{\Lambda}_v(\boldsymbol{\kappa}_P - \boldsymbol{\kappa}_C)) = \mathbf{1}'\boldsymbol{\lambda}_{vs_1}(\boldsymbol{\kappa}_P[2] - \boldsymbol{\kappa}_C[2]) + ... + \mathbf{1}'\boldsymbol{\lambda}_{vs_k}(\boldsymbol{\kappa}_P[k+1] - \boldsymbol{\kappa}_C[k+1]). \tag{5.24}$$

Therefore,

$$(\boldsymbol{\kappa}_P[1] - \boldsymbol{\kappa}_C[1]) - ((\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C}) - (\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C})) =$$

$$(\boldsymbol{\kappa}_P[1] - \boldsymbol{\kappa}_C[1]) - (\mathbf{1}'\boldsymbol{\lambda}_{js_j}(\boldsymbol{\kappa}_P[1] - \boldsymbol{\kappa}_C[1]) + (\mathbf{1}'\boldsymbol{\lambda}_{js_1} - \mathbf{1}'\boldsymbol{\lambda}_{vs_1})(\boldsymbol{\kappa}_P[2] - \boldsymbol{\kappa}_C[2]) + ...$$

$$+ (\mathbf{1}'\boldsymbol{\lambda}_{js_k} - \mathbf{1}'\boldsymbol{\lambda}_{vs_k})(\boldsymbol{\kappa}_P[k+1] - \boldsymbol{\kappa}_C[k+1])). \tag{5.25}$$

This establishes that $\left|(\boldsymbol{\kappa}_P[1] - \boldsymbol{\kappa}_C[1]) - ((\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C}) - (\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C}))\right|$ is, in part, a

function of differences between $\mathbf{1}'\boldsymbol{\lambda}_{js_1} ... \mathbf{1}'\boldsymbol{\lambda}_{js_k}$ and $\mathbf{1}'\boldsymbol{\lambda}_{vs_1} ... \mathbf{1}'\boldsymbol{\lambda}_{jvs_k}$. Therefore, it may be

concluded that *BTC2* is, under the proto-framework, correct, i.e., the bias in estimating

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$\kappa_P[1] - \kappa_C[1]$ by $(\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C}) - (\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C})$, is a function of differences

between $\mathbf{1}'\boldsymbol{\lambda}_{js_1} \dots \mathbf{1}'\boldsymbol{\lambda}_{js_k}$ and $\mathbf{1}'\boldsymbol{\lambda}_{vs_1} \dots \mathbf{1}'\boldsymbol{\lambda}_{vs_k}$.

BTC3. As will be recalled, *BTC3* states that if

$$\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C} / 2\sqrt{\sigma^2_{c_{lj}}} > \mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C} / 2\sqrt{\sigma^2_{c_{uv}}} \tag{5.26}$$

and

$$\frac{\mathbf{1}'(\boldsymbol{\Lambda}_j\boldsymbol{\Lambda}_j' + \boldsymbol{\Psi}_{j|r})\mathbf{1}}{\mathbf{1}'(\boldsymbol{\Lambda}_j\boldsymbol{\Lambda}_j' + \boldsymbol{\Sigma}_{E_{j|r}} + \boldsymbol{\Psi}_{j|r})\mathbf{1}} < \frac{\mathbf{1}'(\boldsymbol{\Lambda}_v\boldsymbol{\Lambda}_v' + \boldsymbol{\Psi}_{v|r})\mathbf{1}}{\mathbf{1}'(\boldsymbol{\Lambda}_v\boldsymbol{\Lambda}_v' + \boldsymbol{\Sigma}_{E_{v|r}} + \boldsymbol{\Psi}_{v|r})\mathbf{1}} \tag{5.27}$$

then $\kappa_P[1] - \kappa_C[1] < 0$.

Now, note that under expression 5.27, $\boldsymbol{\Lambda}_j\boldsymbol{\Lambda}_j', \boldsymbol{\Lambda}_v\boldsymbol{\Lambda}_v', \boldsymbol{\Psi}_{j|r}, \boldsymbol{\Psi}_{v|r}, \boldsymbol{\Sigma}_{E_{j|r}}$, and $\boldsymbol{\Psi}_{E_{v|r}}$

remain free to vary. Let us now consider expression 5.26. Firstly, it can be shown that

$$\sigma^2_{c_{lj}} = 0.5\sigma^2_{c_{lj}|C} + 0.5\sigma^2_{c_{lj}|P} + 0.25(\mu_{c_{lj}|P} - \mu_{c_{lj}|C})^2. \tag{5.28}$$

By the proto-framework of section 5.1.1, then

$$\sigma^2_{c_{lj}} = 0.5(\mathbf{1}'(\boldsymbol{\Lambda}_j\boldsymbol{\Lambda}_j' + \boldsymbol{\Sigma}_{E_{j|C}} + \boldsymbol{\Psi}_{j|C})\mathbf{1} + \mathbf{1}'(\boldsymbol{\Lambda}_j\boldsymbol{\Lambda}_j' + \boldsymbol{\Sigma}_{E_{j|P}} + \boldsymbol{\Psi}_{j|P})\mathbf{1})$$

$$+0.25(\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C})^2. \tag{5.29}$$

Secondly, note that

$$\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C} = \mathbf{1}'\boldsymbol{\gamma}_{j|P} - \mathbf{1}'\boldsymbol{\gamma}_{j|C} + \mathbf{1}'(\boldsymbol{\Lambda}_j(\boldsymbol{\kappa}_P - \boldsymbol{\kappa}_C)). \tag{5.30}$$

Similarly,

$$\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C} = \mathbf{1}'\boldsymbol{\gamma}_{v|P} - \mathbf{1}'\boldsymbol{\gamma}_{v|C} + \mathbf{1}'(\boldsymbol{\Lambda}_v(\boldsymbol{\kappa}_P - \boldsymbol{\kappa}_C)). \tag{5.31}$$

Therefore,

$$(\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C} / 2\sqrt{\sigma^2_{c_{lj}}}) - (\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C} / 2\sqrt{\sigma^2_{c_{uv}}})$$

$$= \frac{\mathbf{1}'\boldsymbol{\gamma}_{j|P} - \mathbf{1}'\boldsymbol{\gamma}_{j|C} + \mathbf{1}'(\boldsymbol{\Lambda}_j(\boldsymbol{\kappa}_P - \boldsymbol{\kappa}_C)))}{2\sqrt{0.5(\mathbf{1}'(\boldsymbol{\Lambda}_j\boldsymbol{\Lambda}_j' + \boldsymbol{\Sigma}_{E_{j|C}} + \boldsymbol{\Psi}_{j|C})\mathbf{1} + \mathbf{1}'(\boldsymbol{\Lambda}_j\boldsymbol{\Lambda}_j' + \boldsymbol{\Sigma}_{E_{j|P}} + \boldsymbol{\Psi}_{j|P})\mathbf{1}) + 0.25(\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C})^2}}$$

$$- \frac{\mathbf{1}'\boldsymbol{\gamma}_{v|P} - \mathbf{1}'\boldsymbol{\gamma}_{v|C} + \mathbf{1}'(\boldsymbol{\Lambda}_v(\boldsymbol{\kappa}_P - \boldsymbol{\kappa}_C)))}{2\sqrt{0.5(\mathbf{1}'(\boldsymbol{\Lambda}_v\boldsymbol{\Lambda}_v' + \boldsymbol{\Sigma}_{E_{v|C}} + \boldsymbol{\Psi}_{v|C})\mathbf{1} + \mathbf{1}'(\boldsymbol{\Lambda}_v\boldsymbol{\Lambda}_v' + \boldsymbol{\Sigma}_{E_{v|P}} + \boldsymbol{\Psi}_{v|P})\mathbf{1}) + 0.25(\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C})^2}} .$$

$$\tag{5.32}$$

It is evident, by inspection of this equation, that if this quantity is positive as in expression 5.26, then several quantities, including $\boldsymbol{\kappa}_P - \boldsymbol{\kappa}_C$, are free to vary. Furthermore, as noted above, expression 5.27 does not cause the quantities

$\boldsymbol{\Lambda}_j\boldsymbol{\Lambda}_j', \boldsymbol{\Lambda}_v\boldsymbol{\Lambda}_v', \boldsymbol{\Psi}_{j|r}, \boldsymbol{\Psi}_{v|r}, \boldsymbol{\Sigma}_{E_{j|r}}$, and $\boldsymbol{\Psi}_{E_{v|r}}$ to be further restricted. Hence, the elements of

$\kappa_P - \kappa_C$, including $\kappa_P[1] - \kappa_C[1]$, remain free to vary, under the inequalities expressed in

5.26 and 5.27. Therefore, *BTC3* is false.

5.2 Salthouse and Coon

5.2.1 Completion of Proto-framework

As noted in section 3.3.2.3, Salthouse and Coon (1994) appeared to employ the

proto-framework recoverable from the Chapmans' work, *C&CP-f*, which was completed

in section 4.3 above.

5.2.2 Mathematization of Claims

SC1: As will be recalled (section 3.3.2.2), Salthouse and Coon asserted that if

$\rho_{A(c_{lj}.c_{uv})} < 0$, then there exists an S-Deficit in respect to s_j and populations *P* and *C*. Now

in the proto-framework of section 4.3, c_{lj} is translated as $X_j[l]$ and c_{uv} is translated as

$X_v[u]$. Therefore, the claim is translated as: If $\rho_{A(X_j[l].X_v[u])} < 0$, in which $\rho_{A(X_j[l].X_v[u])}$ is

calculated over the combined *C* and *P* populations, then $\kappa_P[2] - \kappa_C[2] < 0$.

SC2: As will be recalled (section 3.3.2.2) Salthouse and Coon asserted that if $\rho_{A(c_{lj}.c_{uv})} < 0$

and $\sigma_{AB}^2 \neq 0$ then there exists a D-Deficit in respect abilities s_j and s_v and populations *P*

and *C*. As noted in section 3.3.2.1.1, the condition of $\sigma_{AB}^2 \neq 0$, is equivalent, in fact, to the

condition that $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C}) \neq 0$, which is translated, under the proto-

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framework of section 4.3, as $(\mu_{X_j|P}[l] - \mu_{X_j|C}[l]) - (\mu_{X_v|P}[u] - \mu_{X_v|C}[u]) \neq 0$. The claim is,

then, translated as follows: If $\rho_{A(X_j[l].X_v[u])} < 0$ and

$(\mu_{X_j|P}[l] - \mu_{X_j|C}[l]) - (\mu_{X_v|P}[u] - \mu_{X_v|C}[u]) \neq 0$, then $(\kappa_P[2] - \kappa_C[2]) - (\kappa_P[3] - \kappa_C[3]) < 0$.

5.2.3 **Adjudication of Claims**

SC1: To repeat, *SC1* states that if $\rho_{A(X_j[l].X_v[u])} < 0$, in which $\rho_{A(X_j[l].X_v[u])}$ is calculated over

the combined C and P populations, then $\kappa_P[2] - \kappa_C[2] < 0$.

From the general part correlation formula:

$$\rho_{A(X_j[l].X_v[u])} = \frac{\rho_{AX_j[l]} - \rho_{AX_v[u]}\rho_{X_j[l]X_v[u]}}{\sqrt{1 - \rho_{X_j[l]X_v[u]}^2}} \,. \tag{5.33}$$

Now, the components of expression 5.32 can be expressed as follows:

i)   It can be shown that

$$\rho_{A(X_j[l])} = (\mu_{X_j|C}[l] - \mu_{X_j|P}[l]) / 2\sqrt{\sigma_{X_j[l]}^2} \,, \tag{5.34}$$

in which

$$\sigma_{X_j[l]}^2 = 0.5\sigma_{X_j[l]|C}^2 + 0.5\sigma_{X_j[l]|P}^2 + 0.25(\mu_{X_j|P}[l] - \mu_{X_j|C}[l])^2 \,, \tag{5.35}$$

and, from 4.36,

$$\mu_{X_j|P}[l] - \mu_{X_j|C}[l] = \gamma_{j|P}[l] - \gamma_{j|C}[l] + \lambda_{jg}[l](\kappa_P[1] - \kappa_C[1]) + \lambda_{js_j}[l](\kappa_P[2] - \kappa_C[2]) ;(5.36)$$

ii) It can be shown that

$$\rho_{A(X_v[u])} = (\mu_{X_v|C}[u] - \mu_{X_v|P}[u]) / 2\sqrt{\sigma^2_{X_v[u]}} , \tag{5.37}$$

in which

$$\sigma^2_{X_v[u]} = 0.5\sigma^2_{X_v[u]|C} + 0.5\sigma^2_{X_v[u]|P} + 0.25(\mu_{X_v|P}[u] - \mu_{X_v|C}[u])^2 , \tag{5.38}$$

and, from 4.36,

$$\mu_{X_v|P}[u] - \mu_{X_v|C}[u] =$$

$$\gamma_{v|P}[u] - \gamma_{v|C}[u] + \lambda_{vg}[u](\kappa_P[1] - \kappa_C[1]) + \lambda_{vs_v}[u](\kappa_P[3] - \kappa_C[3]) ; \tag{5.39}$$

iii) From equation 4.32,

$$\rho_{X_v[u]X_j[l]} = \frac{\lambda_{jg}[l]\lambda_{vg}[u]}{\sigma_{X_j[l]}\sigma_{X_v[u]}} , \tag{5.40}$$

in which the expressions for $\sigma_{X_j[l]}$ and $\sigma_{X_v[u]}$ are presented as in 5.35 and 5.38. Now,

from equations 5.34, 5.37, and 5.40,

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$$\rho_{A(X_j[l].X_v[u])} = \frac{\dfrac{\mu_{X_j|C}[l] - \mu_{X_j|P}[l]}{2\sqrt{\sigma^2_{X_j[l]}}} - \dfrac{(\mu_{X_v|C}[u] - \mu_{X_v|P}[u])(\lambda_{jg}[l]\lambda_{vg}[u])}{(\sigma_{X_j[l]}\sigma_{X_v[u]})2\sqrt{\sigma^2_{X_v[u]}}}}{\sqrt{1 - (\dfrac{\lambda_{jg}[l]\lambda_{vg}[u]}{\sigma_{X_j[l]}\sigma_{X_v[u]}})^2}}. \qquad (5.41)$$

Substituting in equations 5.36 and 5.39 causes $\kappa_P[2] - \kappa_C[2]$ to remain free to vary, as

terms remain for the unknowns $\gamma_{v|P}[u], \gamma_{v|C}[u], \lambda_{vg}[u], \lambda_{vs_v}[u], \lambda_{jg}[l], \lambda_{js_j}[l], \kappa_P[1] - \kappa_C[1]$

, and $\kappa_P[3] - \kappa_C[3]$. Therefore it is not the case that when $\rho_{A(X_j[l].X_v[u])} < 0$ then

$\kappa_P[2] - \kappa_C[2] < 0$. We conclude that *SC1* is false.

SC2: To repeat, *SC2* states that if *i)* $\rho_{A(X_j[l].X_v[u])} < 0$ and *ii)*

$(\mu_{X_j|P}[l] - \mu_{X_j|C}[l]) - (\mu_{X_v|P}[u] - \mu_{X_v|C}[u]) \neq 0$, then $(\kappa_P[2] - \kappa_C[2]) - (\kappa_P[3] - \kappa_C[3]) < 0$.

Firstly, note it has been shown above that, in the case in which condition *i* is true, then

$\kappa_P[2] - \kappa_C[2]$ remains free to vary, as quantities $\gamma_{v|P}[u], \gamma_{v|C}[u], \lambda_{vg}[u], \lambda_{vs_v}[u], \lambda_{jg}[l]$,

$\lambda_{js_j}[l]$, and $\kappa_P[1] - \kappa_C[1]$ are unknown. By inspection of equation 5.41, in conjunction

with equations 5.36 and 5.39, it evident that $\kappa_P[3] - \kappa_C[3]$, as well, remains free to vary

in this case. Therefore, $(\kappa_P[2] - \kappa_C[2]) - (\kappa_P[3] - \kappa_C[3])$, is free to vary under condition

i. Secondly, in considering condition *ii*, note that, by equations 5.36 and 5.39,

$$(\mu_{X_j|P}[l] - \mu_{X_j|C}[l]) - (\mu_{X_v|P}[u] - \mu_{X_v|C}[u]) =$$

$$(\gamma_{j|P}[l] - \gamma_{j|C}[l] + \lambda_{jg}[l](\kappa_P[1] - \kappa_C[1]) + \lambda_{js_j}[l](\kappa_P[2] - \kappa_C[2]))$$

$$-(\gamma_{v|P}[u]-\gamma_{v|C}[u]+\lambda_{vg}[u](\kappa_P[1]-\kappa_C[1])+\lambda_{vs_v}[u](\kappa_P[3]-\kappa_C[3]))\ . \qquad (5.42)$$

Inspection of the above equation reveals that when this quantity is non-zero, $(\kappa_P[2]-\kappa_C[2])-(\kappa_P[3]-\kappa_C[3])$ remains free to vary due to the presence of unknowns $\gamma_{v|P}[u],\gamma_{v|C}[u],\lambda_{vg}[u],\lambda_{vs_v}[u],\lambda_{jg}[l],\ \lambda_{js_j}[l]$, and $(\kappa_P[1]-\kappa_C[1])$. Jointly, then, under conditions i and ii, $(\kappa_P[2]-\kappa_C[2])-(\kappa_P[3]-\kappa_C[3])$ remains free to vary as quantities $\gamma_{v|P}[u],\gamma_{v|C}[u],\lambda_{vg}[u],\lambda_{vs_v}[u],\lambda_{jg}[l],\ \lambda_{js_j}[l]$, and $\kappa_P[1]-\kappa_C[1]$ are unknown. Therefore, it is not the case that if $i)\rho_{A(X_j[l].X_v[u])}<0$ and

$ii)(\mu_{X_j|P}[l]-\mu_{X_j|C}[l])-(\mu_{X_v|P}[u]-\mu_{X_v|C}[u])\neq 0$, then $(\kappa_P[2]-\kappa_C[2])-(\kappa_P[3]-\kappa_C[3])<0$.

We conclude that *SC2* is false.

5.3 **Knight and Silverstein**

5.3.1 **Completion of Proto-framework**

As will be recalled (section 3.3.3.5) the elements of *K&SP-f* extracted in Chapter 3 are as follows:

a) each test t_{lj} scales individuals, in a manner that is largely unknown to the researcher, in respect to ability s_j as well as other abilities $s_1...s_k$;

b) test t_{lj} is associated with parameters, $\gamma_{clj/s1}\ ...\ \gamma_{clj/sk}$, that characterize the sensitivity of associated composite c_{lj} to changes in $s_1...s_k$, (these known as discriminating power parameters);

c) The parameters $\gamma_{clj/s1}\ ...\ \gamma_{clj/sk}$ determine, in part, the observed score distributions $c_{lj}|P$ and $c_{lj}|C$.

We, now, complete this proto-framework. Let: a) once again, i stand for individual; b) $r=\{C,P\}$; and c) X be the (p_j+p_v)-element random vector $\begin{bmatrix} X_j \\ X_v \end{bmatrix}$, the first p_j elements of which are the *items* contributing to a unit-weighted test composite c_{lj}, the second p_v items of which are items contributing to a unit-weighted test composite c_{uv}.

From references to the classical test theory concept "reliability", etc., it is clear that (4.10) is invoked, that is to say,

$$X \mid i,r \sim (\tau_{ir}, \Sigma_{\varepsilon ir}),\ \Sigma_{\varepsilon ir},\ \text{diagonal},\ i \in r,\ r=\{C,P\}. \tag{5.43}$$

From the references to multiple abilities, and the multi-population context in which concern for the psychometric confound is situated, we deduce that (4.21) is invoked as a representation, or model, of the (p_j+p_v) vector τ_{ir}, i.e.,

$$\tau_{ir} = \gamma_r + \Lambda_r \xi_{ir} + \zeta_{ir}, i \in \Delta_r, r=1...s, \tag{5.44}$$

with the following particularizations, restrictions, and clarifications being made:

a) $s=2$, i.e., $r=\{C,P\}$; $m=k+2$, in which

$$\xi_{ir} = \begin{bmatrix} s_{ij} \\ s_{iv} \\ s_{i1} \\ \vdots \\ s_{ik} \end{bmatrix} ; \tag{5.45}$$

b) $\Lambda_r = \Lambda$, (p_j+p_v) by $(k+2)$, and structured as follows:

$$\begin{pmatrix} \lambda_{js_j} & \mathbf{0}_{p_v} & \lambda_{js_1} & \cdots & \lambda_{js_k} \\ \mathbf{0}_{p_j} & \lambda_{vs_v} & \lambda_{vs_1} & \cdots & \lambda_{vs_k} \end{pmatrix}, \tag{5.46}$$

in which the vector λ_{js_j} contains the loadings of the p_j elements of X_j on ability s_j, λ_{vs_v} ,

the loadings of the p_v elements of X_v on ability s_v, etc.;

c) over $i \in \Delta_r$, and for $r = \{C,P\}$,

$$\underset{i}{C}(\xi_{ir}) = \Phi_r = \mathbf{I}, \tag{5.47}$$

$$\underset{i}{C}(\zeta_{ir}) = \Psi_r, \tag{5.48}$$

in which Ψ_r is an (p_j+p_v) by (p_j+p_v), diagonal, positive definite matrix, structured as

$$\Psi_r = \begin{bmatrix} \Psi_{j|r} & \mathbf{0} \\ \mathbf{0} & \Psi_{v|r} \end{bmatrix}, \text{ and}$$

$$\underset{i}{C}(\xi_{ir}, \zeta_{ir}) = \mathbf{0}, \tag{5.49}$$

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a $(k+2)$ by (p_j+p_v) null matrix.

The reader should note that, under this mathematization: a) over populations C and P, weak or pattern invariance holds; b) for composite c_{lj} there exists discriminating power parameters in respect s_j and $s_1 \ldots s_k$ that are sums of the factor loadings, i.e. elements of Λ. The concept *discriminating power* of say, test composite c_{lj} in respect to ability s_j, is, in this case quantitatively paraphrased as "the number of units change in the test composite c_{lj} associated with a one standard deviation increase in s_j (the first element of ξ_r)".

All told, we have, thus, the following, fully articulated, mathematical framework:

$$\mathbf{X}|i,r \sim (\gamma_r + \Lambda_r \xi_{ir} + \zeta_{ir}, \Sigma_{\varepsilon ir}), \; i \in \Delta_r r, \; r=\{C,P\}, \tag{5.50}$$

with consequence that

$$\mathbf{X}|r \sim (\gamma_r + \Lambda \kappa_r, \Lambda\Lambda' + \Theta_{\delta r}), \; r = \{C,P\}, \tag{5.51}$$

in which $\Theta_{\delta r} = \Sigma_{Er} + \Psi_r$, diagonal, $r=\{C,P\}$, with $\Sigma_{Er} = \mathrm{E}(\Sigma_{\varepsilon ir})$.

5.3.2 **Mathematization of Claims**

KSC1: As will be recalled, Knight and Silverstein claimed that the difference parameter ($\mu_{c_{lj}/P} - \mu_{c_{lj}/C}$) - ($\mu_{c_{uv}/P} - \mu_{c_{uv}/C}$) is a function of abilities $s_1 \ldots s_k$, and hence is confounded as

a basis for making inferences about D-deficits by differences in the unknown

discriminating powers of the tests employed.

Analogous to *BTC2*, the claim is translated as follows: The bias in estimating

$(\kappa_P[1] - \kappa_C[1]) - (\kappa_P[2] - \kappa_C[2])$ by $(\mathbf{1}'\mu_{X_j|P} - \mathbf{1}'\mu_{X_j|C}) - (\mathbf{1}'\mu_{X_v|P} - \mathbf{1}'\mu_{X_v|C})$, i.e.

$\left| ((\kappa_P[1] - \kappa_C[1]) - (\kappa_P[2] - \kappa_C[2])) - ((\mathbf{1}'\mu_{X_j|P} - \mathbf{1}'\mu_{X_j|C}) - (\mathbf{1}'\mu_{X_v|P} - \mathbf{1}'\mu_{X_v|C})) \right|$ is a function

of differences between $\mathbf{1}'\lambda_{js_1} \dots \mathbf{1}'\lambda_{js_k}$ (in which $\mathbf{1}'$ is a p_j-element vector of 1s) and

$\mathbf{1}'\lambda_{vs_1} \dots \mathbf{1}'\lambda_{vs_k}$ (in which $\mathbf{1}'$ is a p_v-element vector of 1s).

KSC2: As will be recalled, this claim states that if $\sigma_{AB}^2 = 0$ then $(\mu_{s_1|P} - \mu_{s_1|C}) = (\mu_{s_2|P} - $

$\mu_{s_2|C}) = \dots = (\mu_{s_k|P} - \mu_{s_k|C})$ and $\gamma_{clj/s1} = \gamma_{cuv/s1}, \gamma_{clj/s2} = \gamma_{cuv/s2} \dots \gamma_{clj/sk} = \gamma_{cuv/sk}$. Now, note that

$\sigma_{AB}^2 = 0$ is equivalent to $(\mu_{c_{lj}|P} - \mu_{c_{lj}|C}) - (\mu_{c_{uv}|P} - \mu_{c_{uv}|C}) = 0$. Under the current proto-

framework, the claim is therefore mathematized as follows: If

$(\mathbf{1}'\mu_{X_j|P} - \mathbf{1}'\mu_{X_j|C}) - (\mathbf{1}'\mu_{X_v|P} - \mathbf{1}'\mu_{X_v|C}) = 0$, then

$(\kappa_P[3] - \kappa_C[3]) = (\kappa_P[4] - \kappa_C[4]) = \dots = (\kappa_P[k+2] - \kappa_C[k+2])$, and

$\mathbf{1}'\lambda_{js_1} = \mathbf{1}'\lambda_{vs_1}, \mathbf{1}'\lambda_{js_2} = \mathbf{1}'\lambda_{vs_2}, \dots, \mathbf{1}'\lambda_{js_k} = \mathbf{1}'\lambda_{vs_k}$.

KSC3: The claim states that if $\sigma_{AB}^2 = 0$, $\mu_{c_{lj}|C} = \mu_{c_{uv}|C}$ and $\mu_{c_{lj}|P} = \mu_{c_{uv}|P}$, then $\mu_{s_j|C} = \mu_{s_v|C}$

and $\mu_{s_j|P} = \mu_{s_v|P}$. This claim is translated as follows: If

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$(\boldsymbol{1}'\boldsymbol{\mu}_{X_j|P} - \boldsymbol{1}'\boldsymbol{\mu}_{X_j|C}) - (\boldsymbol{1}'\boldsymbol{\mu}_{X_v|P} - \boldsymbol{1}'\boldsymbol{\mu}_{X_v|C}) = 0$, $\boldsymbol{1}'\boldsymbol{\mu}_{X_j|C} = \boldsymbol{1}'\boldsymbol{\mu}_{X_v|C}$, and $\boldsymbol{1}'\boldsymbol{\mu}_{X_j|P} = \boldsymbol{1}'\boldsymbol{\mu}_{X_v|P}$ then

$\boldsymbol{\kappa}_C[1] = \boldsymbol{\kappa}_C[2]$, $\boldsymbol{\kappa}_P[1] = \boldsymbol{\kappa}_P[2]$.

KSC4: The claim states that if $\sigma_{AB}^2 = 0$, $\mu_{c_{ij}|C} > \mu_{c_{uv}|C}$ and $\mu_{c_{ij}|P} > \mu_{c_{uv}|P}$, then $\mu_{s_j|C} > \mu_{s_v|C}$

and $\mu_{s_j|P} > \mu_{s_v|P}$. This claim is translated as follows: If

$(\boldsymbol{1}'\boldsymbol{\mu}_{X_j|P} - \boldsymbol{1}'\boldsymbol{\mu}_{X_j|C}) - (\boldsymbol{1}'\boldsymbol{\mu}_{X_v|P} - \boldsymbol{1}'\boldsymbol{\mu}_{X_v|C}) = 0$, $\boldsymbol{1}'\boldsymbol{\mu}_{X_j|C} > \boldsymbol{1}'\boldsymbol{\mu}_{X_v|C}$, and $\boldsymbol{1}'\boldsymbol{\mu}_{X_j|P} > \boldsymbol{1}'\boldsymbol{\mu}_{X_v|P}$ then

$\boldsymbol{\kappa}_C[1] > \boldsymbol{\kappa}_C[2]$, $\boldsymbol{\kappa}_P[1] > \boldsymbol{\kappa}_P[2]$.

KSC5: The claim states that if $\sigma_{AB}^2 = 0$, $\mu_{c_{ij}|C} < \mu_{c_{uv}|C}$ and $\mu_{c_{ij}|P} < \mu_{c_{uv}|P}$, then $\mu_{s_j|C} < \mu_{s_v|C}$

and $\mu_{s_j|P} < \mu_{s_v|P}$. It is translated as follows: If $(\boldsymbol{1}'\boldsymbol{\mu}_{X_j|P} - \boldsymbol{1}'\boldsymbol{\mu}_{X_j|C}) - (\boldsymbol{1}'\boldsymbol{\mu}_{X_v|P} - \boldsymbol{1}'\boldsymbol{\mu}_{X_v|C}) = 0$,

$\boldsymbol{1}'\boldsymbol{\mu}_{X_j|C} < \boldsymbol{1}'\boldsymbol{\mu}_{X_v|C}$, and $\boldsymbol{1}'\boldsymbol{\mu}_{X_j|P} < \boldsymbol{1}'\boldsymbol{\mu}_{X_v|P}$ then $\boldsymbol{\kappa}_C[1] < \boldsymbol{\kappa}_C[2]$, $\boldsymbol{\kappa}_P[1] < \boldsymbol{\kappa}_P[2]$.

KSC6: The claim states that if $\mu_{c_{ij}/P} - \mu_{c_{ij}/C} > 0$ then $\mu_{s_j/P} - \mu_{s_j/C} > 0$. It is translated as

follows: If $\boldsymbol{1}'\boldsymbol{\mu}_{X_j|P} - \boldsymbol{1}'\boldsymbol{\mu}_{X_j|C} > 0$ then $\boldsymbol{\kappa}_P[1] - \boldsymbol{\kappa}_C[1] > 0$.

KSC7: The claim states that if $\mu_{c_{ij}|C} > \mu_{c_{uv}|C}$ and $\mu_{c_{ij}|P} < \mu_{c_{uv}|P}$ then $\mu_{s_j|C} > \mu_{s_v|C}$ and

$\mu_{s_j|P} < \mu_{s_v|P}$. It is translated as follows: If $\boldsymbol{1}'\boldsymbol{\mu}_{X_j|C} > \boldsymbol{1}'\boldsymbol{\mu}_{X_v|C}$ and $\boldsymbol{1}'\boldsymbol{\mu}_{X_j|P} < \boldsymbol{1}'\boldsymbol{\mu}_{X_v|P}$ then

$\boldsymbol{\kappa}_C[1] > \boldsymbol{\kappa}_C[2]$ and $\boldsymbol{\kappa}_P[1] < \boldsymbol{\kappa}_P[2]$.

KSC8: The claim states that if $\mu_{c_{lj}|C} < \mu_{c_{uv}|C}$ and $\mu_{c_{lj}|P} > \mu_{c_{uv}|P}$ then $\mu_{s_j|C} < \mu_{s_v|C}$ and

$\mu_{s_j|P} > \mu_{s_v|P}$. It is translated as follows: If $\mathbf{1}'\mu_{X_j|C} < \mathbf{1}'\mu_{X_v|C}$ and $\mathbf{1}'\mu_{X_j|P} > \mathbf{1}'\mu_{X_v|P}$ then

$\kappa_C[1] < \kappa_C[2]$ and $\kappa_P[1] > \kappa_P[2]$.

KSC9: The claim states that if, for all item pairs I_{dlj} and I_{dMuv}, $d=1..p_{uv}$,

$|0.5 - \beta_{Idlj}| = |0.5 - \beta_{IdMuv}|$ and $\rho_{clj,dlj} = \rho_{cuv,dMuv}$, it is *not the case* that conclusions about D-

Deficits may be drawn through use of C&C Strategy2 (in contradiction to an implication

of CC4). It is translated as: If, $\forall l$, $|0.5 - \beta_j[l]| = |0.5 - \beta_v[l]|$, and $\rho_{X_j[l],\mathbf{1}'X_j} = \rho_{X_v[l],\mathbf{1}'X_v}$, in

which β_j is the p_j-vector of difficulties associated with X_j, $\rho_{X_j[l],\mathbf{1}'X_j}$, the correlation of

$\mathbf{1}'X_j$ with $X_j[l]$, it remains the case that:

a) $\mathbf{1}'\lambda_{js_1}$ ... $\mathbf{1}'\lambda_{js_k}$ and $\mathbf{1}'\lambda_{vs_1}$ ... $\mathbf{1}'\lambda_{vs_k}$ are free to vary,

b) $\left|((\kappa_P[1] - \kappa_C[1]) - (\kappa_P[2] - \kappa_C[2])) - ((\mathbf{1}'\mu_{X_j|P} - \mathbf{1}'\mu_{X_j|C}) - (\mathbf{1}'\mu_{X_v|P} - \mathbf{1}'\mu_{X_v|C}))\right|$ is a

function of these quantities.

KSC10: The claim states that it is *not the case* that if, for two tests c_{lj} and c_{uv}, if $\sigma_{\tau_{lj}}^2 < \sigma_{\tau_{uv}}^2$

then the C&C Strategy2 may be employed to draw conclusions in regard D-Deficits. This

is translated as follows: If $\mathbf{1}'(\Lambda_j\Lambda_j{}' + \Psi_{j|r})\mathbf{1} < \mathbf{1}'(\Lambda_v\Lambda_v{}' + \Psi_{v|r})\mathbf{1}$, in which

$\Lambda_j = \begin{bmatrix} \lambda_{js_j} & \lambda_{js_1} & ... & \lambda_{js_k} \end{bmatrix}$, then

$\left| ((\mathbf{\kappa}_P[1] - \mathbf{\kappa}_C[1]) - (\mathbf{\kappa}_P[2] - \mathbf{\kappa}_C[2])) - ((\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C}) - (\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C})) \right|$ remains a

function of differences between $\mathbf{1}'\boldsymbol{\lambda}_{js_1} \dots \mathbf{1}'\boldsymbol{\lambda}_{js_k}$ and $\mathbf{1}'\boldsymbol{\lambda}_{vs_1} \dots \mathbf{1}'\boldsymbol{\lambda}_{vs_k}$.

5.3.3 Adjudication of Claims

KSC1: To repeat, *KSC1* states that

$\left| ((\mathbf{\kappa}_P[1] - \mathbf{\kappa}_C[1]) - (\mathbf{\kappa}_P[2] - \mathbf{\kappa}_C[2])) - ((\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C}) - (\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C})) \right|$ is a function

of differences between $\mathbf{1}'\boldsymbol{\lambda}_{js_1} \dots \mathbf{1}'\boldsymbol{\lambda}_{js_k}$ and $\mathbf{1}'\boldsymbol{\lambda}_{vs_1} \dots \mathbf{1}'\boldsymbol{\lambda}_{vs_k}$.

Under the proto-framework completed above,

$$\boldsymbol{\mu}_P - \boldsymbol{\mu}_C = \boldsymbol{\gamma}_P - \boldsymbol{\gamma}_C + \boldsymbol{\Lambda}(\mathbf{\kappa}_P - \mathbf{\kappa}_C). \tag{5.52}$$

It follows that

$$\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C} = \mathbf{1}'\boldsymbol{\gamma}_{j|P} - \mathbf{1}'\boldsymbol{\gamma}_{j|C} + \mathbf{1}'(\boldsymbol{\Lambda}_j(\mathbf{\kappa}_P - \mathbf{\kappa}_C)), \tag{5.53}$$

in which $\mathbf{1}'$ is a p_j-element vector of 1s, and

$$\mathbf{1}'(\boldsymbol{\Lambda}_j(\mathbf{\kappa}_P - \mathbf{\kappa}_C)) =$$

$$\mathbf{1}'\boldsymbol{\lambda}_{js_j}(\mathbf{\kappa}_P[1] - \mathbf{\kappa}_C[1]) + \mathbf{1}'\boldsymbol{\lambda}_{js_1}(\mathbf{\kappa}_P[3] - \mathbf{\kappa}_C[3]) + \dots + \mathbf{1}'\boldsymbol{\lambda}_{js_k}(\mathbf{\kappa}_P[k+2] - \mathbf{\kappa}_C[k+2]). \tag{5.54}$$

Similarly,

$$\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C} = \mathbf{1}'\boldsymbol{\gamma}_{v|P} - \mathbf{1}'\boldsymbol{\gamma}_{v|C} + \mathbf{1}'(\boldsymbol{\Lambda}_v(\boldsymbol{\kappa}_P - \boldsymbol{\kappa}_C)), \tag{5.55}$$

in which $\mathbf{1}'$ is a p_v-element vector of 1s, and

$$\mathbf{1}'(\boldsymbol{\Lambda}_v(\boldsymbol{\kappa}_P - \boldsymbol{\kappa}_C)) =$$

$$\mathbf{1}'\boldsymbol{\lambda}_{vs_v}(\boldsymbol{\kappa}_P[2] - \boldsymbol{\kappa}_C[2]) + \mathbf{1}'\boldsymbol{\lambda}_{vs_1}(\boldsymbol{\kappa}_P[3] - \boldsymbol{\kappa}_C[3]) + \dots + \mathbf{1}'\boldsymbol{\lambda}_{vs_k}(\boldsymbol{\kappa}_P[k+2] - \boldsymbol{\kappa}_C[k+2]) . \tag{5.56}$$

Now, by expressions 5.53 and 5.55,

$$((\boldsymbol{\kappa}_P[1] - \boldsymbol{\kappa}_C[1]) - (\boldsymbol{\kappa}_P[2] - \boldsymbol{\kappa}_C[2])) - ((\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C}) - (\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C}))$$

$$= ((\boldsymbol{\kappa}_P[1] - \boldsymbol{\kappa}_C[1]) - (\boldsymbol{\kappa}_P[2] - \boldsymbol{\kappa}_C[2])) - (\mathbf{1}'\boldsymbol{\gamma}_{j|P} - \mathbf{1}'\boldsymbol{\gamma}_{j|C} + \mathbf{1}'(\boldsymbol{\Lambda}_j(\boldsymbol{\kappa}_P - \boldsymbol{\kappa}_C)))$$

$$+ (\mathbf{1}'\boldsymbol{\gamma}_{v|P} - \mathbf{1}'\boldsymbol{\gamma}_{v|C} + \mathbf{1}'(\boldsymbol{\Lambda}_v(\boldsymbol{\kappa}_P - \boldsymbol{\kappa}_C))) . \tag{5.57}$$

By inspection of this equation as well as equations 5.54 and 5.56 it is evident that the

quantity $\left| ((\boldsymbol{\kappa}_P[1] - \boldsymbol{\kappa}_C[1]) - (\boldsymbol{\kappa}_P[2] - \boldsymbol{\kappa}_C[2])) - ((\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C}) - (\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C})) \right|$ is, in

fact, in part a function of differences between

$\mathbf{1}'\boldsymbol{\lambda}_{js_1} \dots \mathbf{1}'\boldsymbol{\lambda}_{js_k}$ and $\mathbf{1}'\boldsymbol{\lambda}_{vs_1} \dots \mathbf{1}'\boldsymbol{\lambda}_{vs_k}$. We conclude that *KSC1* is correct.

KSC2: To repeat, *KSC2* states that if

$(\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C}) - (\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C}) = 0$, then

$(\kappa_P[3]-\kappa_C[3])=(\kappa_P[4]-\kappa_C[4])=...=(\kappa_P[k+2]-\kappa_C[k+2])$, and

$$\mathbf{1}'\boldsymbol{\lambda}_{js_1}=\mathbf{1}'\boldsymbol{\lambda}_{vs_1},\mathbf{1}'\boldsymbol{\lambda}_{js_2}=\mathbf{1}'\boldsymbol{\lambda}_{vs_2},...,\mathbf{1}'\boldsymbol{\lambda}_{js_k}=\mathbf{1}'\boldsymbol{\lambda}_{vs_k}.$$

From equations 5.53 and 5.55,

$$\mathbf{1}'\boldsymbol{\mu}_{X_j|P}-\mathbf{1}'\boldsymbol{\mu}_{X_j|C}=\mathbf{1}'\boldsymbol{\gamma}_P-\mathbf{1}'\boldsymbol{\gamma}_C+\mathbf{1}'(\boldsymbol{\Lambda}_{lj}(\boldsymbol{\kappa}_P-\boldsymbol{\kappa}_C)),\text{ and} \tag{5.58}$$

$$\mathbf{1}'\boldsymbol{\mu}_{X_v|P}-\mathbf{1}'\boldsymbol{\mu}_{X_v|C}=\mathbf{1}'\boldsymbol{\gamma}_P-\mathbf{1}'\boldsymbol{\gamma}_C+\mathbf{1}'(\boldsymbol{\Lambda}_{uv}(\boldsymbol{\kappa}_P-\boldsymbol{\kappa}_C)). \tag{5.59}$$

By inspection of these equations as well as 5.54 and 5.56, it is evident that, under the case

in which $(\mathbf{1}'\boldsymbol{\mu}_{X_j|P}-\mathbf{1}'\boldsymbol{\mu}_{X_j|C})-(\mathbf{1}'\boldsymbol{\mu}_{X_v|P}-\mathbf{1}'\boldsymbol{\mu}_{X_v|C})=0$, the quantities

$(\kappa_P[3]-\kappa_C[3]),(\kappa_P[4]-\kappa_C[4]),...,(\kappa_P[k+2]-\kappa_C[k+2])$ and $\mathbf{1}'\boldsymbol{\lambda}_{js_1},\mathbf{1}'\boldsymbol{\lambda}_{js_2},...,\mathbf{1}'\boldsymbol{\lambda}_{js_k}$ and

$\mathbf{1}'\boldsymbol{\lambda}_{vs_1},\mathbf{1}'\boldsymbol{\lambda}_{vs_2},...,\mathbf{1}'\boldsymbol{\lambda}_{vs_k}$ remain free to vary. Accordingly, it is not the case that that if

$(\mathbf{1}'\boldsymbol{\mu}_{X_j|P}-\mathbf{1}'\boldsymbol{\mu}_{X_j|C})-(\mathbf{1}'\boldsymbol{\mu}_{X_v|P}-\mathbf{1}'\boldsymbol{\mu}_{X_v|C})=0$, then

$(\kappa_P[3]-\kappa_C[3])=(\kappa_P[4]-\kappa_C[4])=...=(\kappa_P[k+2]-\kappa_C[k+2])$, and

$\mathbf{1}'\boldsymbol{\lambda}_{js_1}=\mathbf{1}'\boldsymbol{\lambda}_{vs_1},\mathbf{1}'\boldsymbol{\lambda}_{js_2}=\mathbf{1}'\boldsymbol{\lambda}_{vs_2},...,\mathbf{1}'\boldsymbol{\lambda}_{js_k}=\mathbf{1}'\boldsymbol{\lambda}_{vs_k}$. We conclude that *KSC2* is false.

KSC3: To repeat, *KSC3* states that if $(\mathbf{1}'\boldsymbol{\mu}_{X_j|P}-\mathbf{1}'\boldsymbol{\mu}_{X_j|C})-(\mathbf{1}'\boldsymbol{\mu}_{X_v|P}-\mathbf{1}'\boldsymbol{\mu}_{X_v|C})=0$,

$\mathbf{1}'\boldsymbol{\mu}_{X_j|C}=\mathbf{1}'\boldsymbol{\mu}_{X_v|C}$, and $\mathbf{1}'\boldsymbol{\mu}_{X_j|P}=\mathbf{1}'\boldsymbol{\mu}_{X_v|P}$ then $\kappa_C[1]=\kappa_C[2]$, $\kappa_P[1]=\kappa_P[2]$.

From equations 5.46 and 5.51,

$$\mathbf{1}'\boldsymbol{\mu}_{X_j|C} = \mathbf{1}'\boldsymbol{\gamma}_{j|C} + \mathbf{1}'(\boldsymbol{\lambda}_{js_j}(\boldsymbol{\kappa}_C[1]) + \boldsymbol{\lambda}_{js_1}(\boldsymbol{\kappa}_C[3]) + ... + \boldsymbol{\lambda}_{js_k}(\boldsymbol{\kappa}_C[k+2])), \qquad (5.60)$$

$$\mathbf{1}'\boldsymbol{\mu}_{X_v|C} = \mathbf{1}'\boldsymbol{\gamma}_{v|C} + \mathbf{1}'(\boldsymbol{\lambda}_{vs_v}(\boldsymbol{\kappa}_C[2]) + \boldsymbol{\lambda}_{vs_1}(\boldsymbol{\kappa}_C[3]) + ... + \boldsymbol{\lambda}_{vs_k}(\boldsymbol{\kappa}_C[k+2])), \qquad (5.61)$$

$$\mathbf{1}'\boldsymbol{\mu}_{X_j|P} = \mathbf{1}'\boldsymbol{\gamma}_{j|P} + \mathbf{1}'(\boldsymbol{\lambda}_{js_j}(\boldsymbol{\kappa}_P[1]) + \boldsymbol{\lambda}_{js_1}(\boldsymbol{\kappa}_P[3]) + ... + \boldsymbol{\lambda}_{js_k}(\boldsymbol{\kappa}_P[k+2])), \text{ and} \qquad (5.62)$$

$$\mathbf{1}'\boldsymbol{\mu}_{X_v|P} = \mathbf{1}'\boldsymbol{\gamma}_{j|P} + \mathbf{1}'(\boldsymbol{\lambda}_{vs_v}(\boldsymbol{\kappa}_P[2]) + \boldsymbol{\lambda}_{vs_1}(\boldsymbol{\kappa}_P[3]) + ... + \boldsymbol{\lambda}_{vs_k}(\boldsymbol{\kappa}_P[k+2])). \qquad (5.63)$$

By inspection of these equations, it is apparent that: *i*) If $\mathbf{1}'\boldsymbol{\mu}_{X_j|C} = \mathbf{1}'\boldsymbol{\mu}_{X_v|C}$, then $\boldsymbol{\kappa}_C[1]$ and

$\boldsymbol{\kappa}_C[2]$ remain free to vary due to the presence of unknowns

$\boldsymbol{\gamma}_{j|C}, \boldsymbol{\lambda}_{js_j}, \boldsymbol{\lambda}_{js_1}...\boldsymbol{\lambda}_{js_k}, \boldsymbol{\kappa}_C[3]...\boldsymbol{\kappa}_C[k+2], \boldsymbol{\gamma}_{v|C}, \boldsymbol{\lambda}_{vs_v}$, and $\boldsymbol{\lambda}_{vs_1}...\boldsymbol{\lambda}_{vs_k}$; *ii*) If $\mathbf{1}'\boldsymbol{\mu}_{X_j|P} = \mathbf{1}'\boldsymbol{\mu}_{X_v|P}$, then

$\boldsymbol{\kappa}_P[1]$ and $\boldsymbol{\kappa}_P[2]$ remain free to vary due to the presence of unknowns

$\boldsymbol{\gamma}_{j|P}, \boldsymbol{\lambda}_{js_j}, \boldsymbol{\lambda}_{js_1}...\boldsymbol{\lambda}_{js_k}, \boldsymbol{\kappa}_P[3]...\boldsymbol{\kappa}_P[k+2], \boldsymbol{\gamma}_{v|P}, \boldsymbol{\lambda}_{vs_v}$, and $\boldsymbol{\lambda}_{vs_1}...\boldsymbol{\lambda}_{vs_k}$; *iii*) If

$(\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C}) - (\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C}) = 0$, $\boldsymbol{\kappa}_C[1], \boldsymbol{\kappa}_C[2], \boldsymbol{\kappa}_P[1]$ and $\boldsymbol{\kappa}_P[2]$ remain free to

vary due to the presence of all unknowns listed in *i* and *ii*; *iv*) Jointly, if

$(\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C}) - (\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C}) = 0$, $\mathbf{1}'\boldsymbol{\mu}_{X_j|C} = \mathbf{1}'\boldsymbol{\mu}_{X_v|C}$, and $\mathbf{1}'\boldsymbol{\mu}_{X_j|P} = \mathbf{1}'\boldsymbol{\mu}_{X_v|P}$, $\boldsymbol{\kappa}_C[1]$,

$\boldsymbol{\kappa}_C[2], \boldsymbol{\kappa}_P[1]$ and $\boldsymbol{\kappa}_P[2]$ remain free to vary. Therefore, it is not the case that that if

$(\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C}) - (\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C}) = 0$, $\mathbf{1}'\boldsymbol{\mu}_{X_j|C} = \mathbf{1}'\boldsymbol{\mu}_{X_v|C}$, and $\mathbf{1}'\boldsymbol{\mu}_{X_j|P} = \mathbf{1}'\boldsymbol{\mu}_{X_v|P}$ then

$\boldsymbol{\kappa}_C[1] = \boldsymbol{\kappa}_C[2]$, $\boldsymbol{\kappa}_P[1] = \boldsymbol{\kappa}_P[2]$. We conclude that *KSC3* is false.

KSC4: To repeat, *KSC4* states that if $(1'\mu_{X_j|P} - 1'\mu_{X_j|C}) - (1'\mu_{X_v|P} - 1'\mu_{X_v|C}) = 0$,

$1'\mu_{X_j|C} > 1'\mu_{X_v|C}$, and $1'\mu_{X_j|P} > 1'\mu_{X_v|P}$ then $\kappa_C[1] > \kappa_C[2]$, $\kappa_P[1] > \kappa_P[2]$. By the same

logic employed in our adjudication of *KSC3* (including equations 5.60 to 5.63) it is

apparent that: *i*) If $1'\mu_{X_j|C} > 1'\mu_{X_v|C}$, then $\kappa_C[1]$ and $\kappa_C[2]$ remain free to vary due to the

presence of unknowns $\gamma_{j|C}, \lambda_{js_j}, \lambda_{js_1}...\lambda_{js_k}, \kappa_C[3]...\kappa_C[k+2], \gamma_{v|C}, \lambda_{vs_v}$, and $\lambda_{vs_1}...\lambda_{vs_k}$; *ii*) If

$1'\mu_{X_j|P} > 1'\mu_{X_v|P}$, then $\kappa_P[1]$ and $\kappa_P[2]$ remain free to vary due to the presence of

unknowns $\gamma_{j|P}, \lambda_{js_j}, \lambda_{js_1}...\lambda_{js_k}, \kappa_P[3]...\kappa_P[k+2], \gamma_{v|P}, \lambda_{vs_v}$, and $\lambda_{vs_1}...\lambda_{vs_k}$; *iii*) If

$(1'\mu_{X_j|P} - 1'\mu_{X_j|C}) - (1'\mu_{X_v|P} - 1'\mu_{X_v|C}) > 0$, $\kappa_C[1], \kappa_C[2], \kappa_P[1]$ and $\kappa_P[2]$ remain free to

vary due to the presence of all unknowns listed in *i* and *ii*; *iv*) Jointly, if

$(1'\mu_{X_j|P} - 1'\mu_{X_j|C}) - (1'\mu_{X_v|P} - 1'\mu_{X_v|C}) > 0$, $1'\mu_{X_j|C} > 1'\mu_{X_v|C}$, and $1'\mu_{X_j|P} > 1'\mu_{X_v|P}$, $\kappa_C[1]$,

$\kappa_C[2], \kappa_P[1]$ and $\kappa_P[2]$ remain free to vary. It is therefore not the case that if

$(1'\mu_{X_j|P} - 1'\mu_{X_j|C}) - (1'\mu_{X_v|P} - 1'\mu_{X_v|C}) = 0$, $1'\mu_{X_j|C} > 1'\mu_{X_v|C}$, and $1'\mu_{X_j|P} > 1'\mu_{X_v|P}$ then

$\kappa_C[1] > \kappa_C[2]$, $\kappa_P[1] > \kappa_P[2]$. We conclude that *KSC4* is false.

KSC5: To repeat, *KSC5* states that if $(1'\mu_{X_j|P} - 1'\mu_{X_j|C}) - (1'\mu_{X_v|P} - 1'\mu_{X_v|C}) = 0$,

$1'\mu_{X_j|C} < 1'\mu_{X_v|C}$, and $1'\mu_{X_j|P} < 1'\mu_{X_v|P}$ then $\kappa_C[1] < \kappa_C[2]$, $\kappa_P[1] < \kappa_P[2]$. By the same

logic employed in our adjudication of *KSC3* (including equations 5.60 to 5.63) it is

apparent that: *i*) If $1'\mu_{X_j|C} < 1'\mu_{X_v|C}$, then $\kappa_C[1]$ and $\kappa_C[2]$ remain free to vary due to the

presence of unknowns $\gamma_{j|C}, \lambda_{js_j}, \lambda_{js_1}...\lambda_{js_k}, \kappa_C[3]...\kappa_C[k+2], \gamma_{v|C}, \lambda_{vs_v}$, and $\lambda_{vs_1}...\lambda_{vs_k}$; *ii*) If

$\mathbf{1'\mu_{X_j|P} < 1'\mu_{X_v|P}}$, then $\mathbf{\kappa}_P[1]$ and $\mathbf{\kappa}_P[2]$ remain free to vary due to the presence of

unknowns $\mathbf{\gamma}_{j|P}, \mathbf{\lambda}_{js_j}, \mathbf{\lambda}_{js_1} ... \mathbf{\lambda}_{js_k}, \mathbf{\kappa}_P[3]...\mathbf{\kappa}_P[k+2], \mathbf{\gamma}_{v|P}, \mathbf{\lambda}_{vs_v},$ and $\mathbf{\lambda}_{vs_1}...\mathbf{\lambda}_{vs_k}$; *iii)* If

$(\mathbf{1'\mu_{X_j|P}} - \mathbf{1'\mu_{X_j|C}}) - (\mathbf{1'\mu_{X_v|P}} - \mathbf{1'\mu_{X_v|C}}) = 0$, $\mathbf{\kappa}_C[1], \mathbf{\kappa}_C[2], \mathbf{\kappa}_P[1]$ and $\mathbf{\kappa}_P[2]$ remain free to

vary due to the presence of all unknowns listed in *i* and *ii*; *iv)* Jointly, if

$(\mathbf{1'\mu_{X_j|P}} - \mathbf{1'\mu_{X_j|C}}) - (\mathbf{1'\mu_{X_v|P}} - \mathbf{1'\mu_{X_v|C}}) = 0$, $\mathbf{1'\mu_{X_j|C}} < \mathbf{1'\mu_{X_v|C}}$, and $\mathbf{1'\mu_{X_j|P}} < \mathbf{1'\mu_{X_v|P}}$, $\mathbf{\kappa}_C[1]$,

$\mathbf{\kappa}_C[2]$, $\mathbf{\kappa}_P[1]$ and $\mathbf{\kappa}_P[2]$ remain free to vary. Therefore, it is not the case that if

$(\mathbf{1'\mu_{X_j|P}} - \mathbf{1'\mu_{X_j|C}}) - (\mathbf{1'\mu_{X_v|P}} - \mathbf{1'\mu_{X_v|C}}) = 0$, $\mathbf{1'\mu_{X_j|C}} < \mathbf{1'\mu_{X_v|C}}$, and $\mathbf{1'\mu_{X_j|P}} < \mathbf{1'\mu_{X_v|P}}$ then

$\mathbf{\kappa}_C[1] < \mathbf{\kappa}_C[2]$, $\mathbf{\kappa}_P[1] < \mathbf{\kappa}_P[2]$. We conclude that *KSC5* is false.

KSC6: To repeat, *KSC6* states that If $\mathbf{1'\mu_{X_j|P}} - \mathbf{1'\mu_{X_j|C}} > 0$ then $\mathbf{\kappa}_P[1] - \mathbf{\kappa}_C[1] > 0$. Now, to

repeat equation 5.58,

$$\mathbf{1'\mu_{X_j|P}} - \mathbf{1'\mu_{X_j|C}} = \mathbf{1'\gamma}_P - \mathbf{1'\gamma}_C + \mathbf{1'}(\Lambda_{lj}(\mathbf{\kappa}_P - \mathbf{\kappa}_C)), \tag{5.64}$$

in which

$$\mathbf{1'}(\Lambda_j(\mathbf{\kappa}_P - \mathbf{\kappa}_C)) =$$

$$\mathbf{1'\lambda}_{js_j}(\mathbf{\kappa}_P[1] - \mathbf{\kappa}_C[1]) + \mathbf{1'\lambda}_{js_1}(\mathbf{\kappa}_P[3] - \mathbf{\kappa}_C[3]) + ... + \mathbf{1'\lambda}_{js_k}(\mathbf{\kappa}_P[k+2] - \mathbf{\kappa}_C[k+2]). \tag{5.65}$$

It is clear then, that when $\mathbf{1'\mu_{X_j|P}} - \mathbf{1'\mu_{X_j|C}} > 0$, $\mathbf{\kappa}_P[1]$ and $\mathbf{\kappa}_C[1]$ remain free to vary due

to presence of unknowns $\mathbf{\gamma}_{j|P}, \mathbf{\gamma}_{j|C}, \mathbf{\lambda}_{js_j}, \mathbf{\lambda}_{js_1}...\mathbf{\lambda}_{js_k}, \mathbf{\kappa}_P[3]...\mathbf{\kappa}_P[k+2],$ and $\mathbf{\kappa}_C[3]...\mathbf{\kappa}_C[k+2]$.

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Therefore, it is not the case that if $1'\boldsymbol{\mu}_{X_j|P} - 1'\boldsymbol{\mu}_{X_j|C} > 0$ then $\kappa_P[1] - \kappa_C[1] > 0$. We

conclude that *KSC6* is false.

KSC7: To review, *KSC7* states that if $1'\boldsymbol{\mu}_{X_j|C} > 1'\boldsymbol{\mu}_{X_v|C}$ and $1'\boldsymbol{\mu}_{X_j|P} < 1'\boldsymbol{\mu}_{X_v|P}$ then

$\kappa_C[1] > \kappa_C[2]$ and $\kappa_P[1] < \kappa_P[2]$. By the same logic employed in our adjudication of

KSC3 (including equations 5.60 to 5.63) it is apparent that: *i*) If $1'\boldsymbol{\mu}_{X_j|C} > 1'\boldsymbol{\mu}_{X_v|C}$, then

$\kappa_C[1]$ and $\kappa_C[2]$ remain free to vary due to the presence of unknowns

$\gamma_{j|C}, \lambda_{js_j}, \lambda_{js_1}...\lambda_{js_k}, \kappa_C[3]...\kappa_C[k+2], \gamma_{v|C}, \lambda_{vs_v}$, and $\lambda_{vs_1}...\lambda_{vs_k}$; *ii*) If $1'\boldsymbol{\mu}_{X_j|P} < 1'\boldsymbol{\mu}_{X_v|P}$, then

$\kappa_P[1]$ and $\kappa_P[2]$ remain free to vary due to the presence of unknowns

$\gamma_{j|P}, \lambda_{js_j}, \lambda_{js_1}...\lambda_{js_k}, \kappa_P[3]...\kappa_P[k+2], \gamma_{v|P}, \lambda_{vs_v}$, and $\lambda_{vs_1}...\lambda_{vs_k}$; *iii*) Jointly, in the case in

which $1'\boldsymbol{\mu}_{X_j|C} > 1'\boldsymbol{\mu}_{X_v|C}$ and $1'\boldsymbol{\mu}_{X_j|P} < 1'\boldsymbol{\mu}_{X_v|P}$, $\kappa_C[1]$, $\kappa_C[2]$, $\kappa_P[1]$ and $\kappa_P[2]$ remain free to

vary due to the presence of the unknowns listed in points *i* and *ii*. Therefore, it is not the

case that if $1'\boldsymbol{\mu}_{X_j|C} > 1'\boldsymbol{\mu}_{X_v|C}$ and $1'\boldsymbol{\mu}_{X_j|P} < 1'\boldsymbol{\mu}_{X_v|P}$ then $\kappa_C[1] > \kappa_C[2]$ and $\kappa_P[1] < \kappa_P[2]$.

We conclude that *KSC7* is false.

KSC8: To repeat, *KSC8* states that if $1'\boldsymbol{\mu}_{X_j|C} < 1'\boldsymbol{\mu}_{X_v|C}$ and $1'\boldsymbol{\mu}_{X_j|P} > 1'\boldsymbol{\mu}_{X_v|P}$ then

$\kappa_C[1] < \kappa_C[2]$ and $\kappa_P[1] > \kappa_P[2]$. By the same logic employed in our adjudication of

KSC3 (including equations 5.60 to 5.63) it is apparent that: *i*) If $1'\boldsymbol{\mu}_{X_j|C} < 1'\boldsymbol{\mu}_{X_v|C}$, then

$\kappa_C[1]$ and $\kappa_C[2]$ remain free to vary due to the presence of unknowns

$\gamma_{j|C}, \lambda_{js_j}, \lambda_{js_1}...\lambda_{js_k}, \kappa_C[3]...\kappa_C[k+2], \gamma_{v|C}, \lambda_{vs_v}$, and $\lambda_{vs_1}...\lambda_{vs_k}$; *ii*) If $1'\boldsymbol{\mu}_{X_j|P} > 1'\boldsymbol{\mu}_{X_v|P}$, then

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$\kappa_P[1]$ and $\kappa_P[2]$ remain free to vary due to the presence of unknowns

$\gamma_{j|P}, \lambda_{js_j}, \lambda_{js_1}...\lambda_{js_k}, \kappa_P[3]...\kappa_P[k+2], \gamma_{v|P}, \lambda_{vs_v}$, and $\lambda_{vs_1}...\lambda_{vs_k}$; *iii*) Jointly, in the case in

which $\mathbf{1}'\boldsymbol{\mu}_{X_j|C} < \mathbf{1}'\boldsymbol{\mu}_{X_v|C}$ and $\mathbf{1}'\boldsymbol{\mu}_{X_j|P} > \mathbf{1}'\boldsymbol{\mu}_{X_v|P}$, $\kappa_C[1]$, $\kappa_C[2]$, $\kappa_P[1]$ and $\kappa_P[2]$ remain free to

vary due to the presence of the unknowns listed in points *i* and *ii*. Therefore, it is not the

case that if $\mathbf{1}'\boldsymbol{\mu}_{X_j|C} < \mathbf{1}'\boldsymbol{\mu}_{X_v|C}$ and $\mathbf{1}'\boldsymbol{\mu}_{X_j|P} > \mathbf{1}'\boldsymbol{\mu}_{X_v|P}$ then $\kappa_C[1] < \kappa_C[2]$ and $\kappa_P[1] > \kappa_P[2]$.

We conclude that *KSC8* is false.

KSC9: To repeat, KSC9 states that if, $\forall l$, $\left|0.5 - \boldsymbol{\beta}_j[l]\right| = \left|0.5 - \boldsymbol{\beta}_v[l]\right|$, and

$\rho_{X_j[l],\mathbf{1}'X_j} = \rho_{X_v[l],\mathbf{1}'X_v}$, it remains the case that:

a) there exist differences between $\mathbf{1}'\lambda_{js_1} \; ... \; \mathbf{1}'\lambda_{js_k}$ and $\mathbf{1}'\lambda_{vs_1} \; ... \mathbf{1}'\lambda_{vs_k}$,

b) $\left|((\kappa_P[1] - \kappa_C[1]) - (\kappa_P[2] - \kappa_C[2])) - ((\mathbf{1}'\boldsymbol{\mu}_{X_j|P} - \mathbf{1}'\boldsymbol{\mu}_{X_j|C}) - (\mathbf{1}'\boldsymbol{\mu}_{X_v|P} - \mathbf{1}'\boldsymbol{\mu}_{X_v|C}))\right|$ is a

function of these differences.

Now, firstly, note that, when $\left|0.5 - \boldsymbol{\beta}_j[l]\right| = \left|0.5 - \boldsymbol{\beta}_v[l]\right|$,

$\boldsymbol{\beta}_j[l](1 - \boldsymbol{\beta}_j[l]) = \boldsymbol{\beta}_v[l](1 - \boldsymbol{\beta}_v[l])$. From equations 4.9 and 5.51,

$$\boldsymbol{\beta}_j[l](1 - \boldsymbol{\beta}_j[l]) = (\Sigma_{X_j}[l,l]) = (\lambda_{js_j}[l])^2 + (\lambda_{js_1}[l])^2 + ... + (\lambda_{js_k}[l])^2 + \Theta_{\delta_j C}[l,l], \quad (5.66)$$

and

$$\boldsymbol{\beta}_v[l](1 - \boldsymbol{\beta}_v[l]) = (\Sigma_{X_v}[l,l]) = (\lambda_{vs_v}[l])^2 + (\lambda_{vs_1}[l])^2 + ... + (\lambda_{vs_k}[l])^2 + \Theta_{\delta_v C}[l,l]. \quad (5.67)$$

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By inspection of these equations, it is apparent that if $\forall l$,

$\left|0.5-\boldsymbol{\beta}_j[l]\right| = \left|0.5-\boldsymbol{\beta}_v[l]\right|$, then the quantities $\boldsymbol{1'\lambda}_{js_1} \dots \boldsymbol{1'\lambda}_{js_k}$ and $\boldsymbol{1'\lambda}_{vs_1} \dots \boldsymbol{1'\lambda}_{vs_k}$ remain

free to vary. Secondly, it can be shown that

$$\rho_{X_j[l],\boldsymbol{1'X}_j} = \frac{\displaystyle\sum_{y=1}^{p_j}\boldsymbol{\Sigma}_{X_j}[y,l]}{\sigma_{\boldsymbol{1'X}_j}\sigma_{X_j[l]}}, \qquad (5.68)$$

in which[36], from expression 5.51,

$$\sum_{y=1}^{p_j}\boldsymbol{\Sigma}_{X_j}[y,l] = \sum_{y=1}^{p_j}(\boldsymbol{\lambda}_{js_j}[y]\boldsymbol{\lambda}_{js_j}[l] + \boldsymbol{\lambda}_{js_1}[y]\boldsymbol{\lambda}_{js_1}[l] + \dots + \boldsymbol{\lambda}_{js_k}[y]\boldsymbol{\lambda}_{js_k}[l] + \boldsymbol{\Theta}_{\delta_jC}[y,l]), \quad (5.69)$$

$$\sigma_{\boldsymbol{1'X}_j}^2 = \boldsymbol{1'\Sigma}_{X_j}\boldsymbol{1} = \boldsymbol{1'}(\boldsymbol{\Lambda}_j\boldsymbol{\Lambda}_j' + \boldsymbol{\Theta}_{\delta_jC})\boldsymbol{1}, \qquad (5.70)$$

and $\sigma_{X_j[l]}^2 = (\boldsymbol{\Sigma}_{X_j}[l,l])$, for which equation 5.56 applies.

Similarly, it can be shown that

$$\rho_{X_v[l],\boldsymbol{1'X}_v} = \frac{\displaystyle\sum_{y=1}^{p_j}\boldsymbol{\Sigma}_{X_v}[y,l]}{\sigma_{\boldsymbol{1'X}_v}\sigma_{X_v[l]}}, \qquad (5.71)$$

in which, from expression 5.51,

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[36] Note, here, the two meanings of Σ : as a summation sign and as a matrix.

$$\sum_{y=1}^{p_j} \Sigma_{X_v}[y,l] = \sum_{y=1}^{p_v} (\lambda_{vs_v}[y]\lambda_{vs_v}[l] + \lambda_{vs_1}[y]\lambda_{vs_1}[l] + ... + \lambda_{vs_k}[y]\lambda_{vs_k}[l] + \Theta_{\delta_v C}[y,l]),$$

$$\sigma^2_{I'X_v} = I'\Sigma_{X_v}I = I'(\Lambda_v \Lambda_v{'} + \Theta_{\delta_v C})I, \tag{5.72}$$

and $\sigma_{X_v[l]}{}^2 = (\Sigma_{X_v}[l,l])$, for which equation 5.57 applies.

From these equations, it is apparent that $\forall l$, the quantities $\lambda_{js_1}[l] ... \lambda_{js_k}[l]$ and

$\lambda_{vs_1}[l] ... \lambda_{vs_k}[l]$ remain free to vary when $\rho_{X_j[l],I'X_j} = \rho_{X_v[l],I'X_v}$, and therefore the

quantities $I'\lambda_{js_1} ... I'\lambda_{js_k}$ and $I'\lambda_{vs_1} ... I'\lambda_{vs_k}$ remain free to vary in this case. Jointly, if,

$\forall l$, $|0.5 - \beta_j[l]| = |0.5 - \beta_v[l]|$, and $\rho_{X_j[l],I'X_j} = \rho_{X_v[l],I'X_v}$, it remains the case that quantities

$I'\lambda_{js_1} ... I'\lambda_{js_k}$ and $I'\lambda_{vs_1} ... I'\lambda_{vs_k}$ remain free to vary. Furthermore, from our evaluation

of *KSC1*, we know that

$\left| ((\kappa_P[1] - \kappa_C[1]) - (\kappa_P[2] - \kappa_C[2])) - ((I'\mu_{X_j|P} - I'\mu_{X_j|C}) - (I'\mu_{X_v|P} - I'\mu_{X_v|C})) \right|$ is, in part, a

function of differences between $I'\lambda_{js_1} ... I'\lambda_{js_k}$ and $I'\lambda_{vs_1} ... I'\lambda_{vs_k}$. Therefore, we

conclude that *KSC9* is true.

KSC10: To repeat, *KSC10* states that if $I'(\Lambda_j \Lambda_j{'} + \Psi_{j|r})I < I'(\Lambda_v \Lambda_v{'} + \Psi_{v|r})I$, in which

$\Lambda_j = \begin{bmatrix} \lambda_{js_j} & \mathbf{0}_{p_j} & \lambda_{js_1} & ... & \lambda_{js_k} \end{bmatrix}$, and $\Lambda_v = \begin{bmatrix} \mathbf{0}_{p_v} & \lambda_{vs_v} & \lambda_{vs_1} & ... & \lambda_{vs_k} \end{bmatrix}$ then

$\left| ((\kappa_P[1] - \kappa_C[1]) - (\kappa_P[2] - \kappa_C[2])) - ((I'\mu_{X_j|P} - I'\mu_{X_j|C}) - (I'\mu_{X_v|P} - I'\mu_{X_v|C})) \right|$ remains a

function of differences between $I'\lambda_{js_1} ... I'\lambda_{js_k}$ and $I'\lambda_{vs_1} ... I'\lambda_{vs_k}$.

Now, it is apparent that under the condition of

$$\mathbf{1}'(\Lambda_j\Lambda_j{}' + \Psi_{j|r})\mathbf{1} < \mathbf{1}'(\Lambda_v\Lambda_v{}' + \Psi_{v|r})\mathbf{1}, \ \Lambda_j\Lambda_j{}' \text{ and } \Lambda_v\Lambda_v{}' \text{ remain free to vary, and,}$$

therefore, the elements of $\lambda_{js_1} \dots \lambda_{js_k}$ and $\lambda_{vs_1} \dots \lambda_{vs_k}$, and hence, sums of elements $\mathbf{1}'\lambda_{js_1}$

$\dots \mathbf{1}'\lambda_{js_k}$ and $\mathbf{1}'\lambda_{vs_1} \dots \mathbf{1}'\lambda_{vs_k}$ are free to vary. As we know through our evaluation of *KSC1*

that $\left| ((\kappa_P[1] - \kappa_C[1]) - (\kappa_P[2] - \kappa_C[2])) - ((\mathbf{1}'\mu_{X_j|P} - \mathbf{1}'\mu_{X_j|C}) - (\mathbf{1}'\mu_{X_v|P} - \mathbf{1}'\mu_{X_v|C})) \right|$ is, in part,

a function of differences between $\mathbf{1}'\lambda_{js_1} \dots \mathbf{1}'\lambda_{js_k}$ and $\mathbf{1}'\lambda_{vs_1} \dots \mathbf{1}'\lambda_{vs_k}$, this therefore

remains the case if $\mathbf{1}'(\Lambda_j\Lambda_j{}' + \Psi_{j|r})\mathbf{1} < \mathbf{1}'(\Lambda_v\Lambda_v{}' + \Psi_{v|r})\mathbf{1}$. Accordingly, we conclude that

KSC10 is true.

5.4 **Kang and MacDonald**

5.4.1 **Completion of Proto-Framework**

As will be recalled, the elements of *K&MP-f* extracted in Chapter 3 are as

follows:

a) each test t_{lj} scales individuals, in a manner that is largely unknown to the researcher, in respect to ability s_j as well as other abilities $s_1 \dots s_k$ and "general ability" g;

b) composite c_{lj} is associated with parameters, $\gamma_{clj/s1} \dots \gamma_{clj/sk}$, that characterize its sensitivities to changes in $s_1 \dots s_k$, and a parameter, $\gamma_{clj/g}$, that characterizes its sensitivity to change in g (these known as discriminating power parameters);

c) The parameters $\gamma_{clj/s1} \dots \gamma_{clj/sk}$ and $\gamma_{clj/g}$ determine, in part, the observed score distributions $c_{lj}|P$ and $c_{lj}|C$.

We, now, complete this proto-framework. Let: a) once again, i stand for individual; b)

$r=\{C,P\}$; and c) X be the (p_j+p_v)-element random vector $\begin{bmatrix} X_j \\ X_v \end{bmatrix}$, the first p_j elements of

which are the test composites of the tests in T_j (these, it will be recalled, invented to scale

individuals with respect ability s_j), and the final p_v, the test composites of the tests of T_v

(which scale in respect, s_v).

From their frequent references to classical test theory, it is clear that Kang and

MacDonald invoke (4.10). That is to say,

$$X \mid i,r \sim (\boldsymbol{\tau}_{ir}, \boldsymbol{\Sigma}_{\varepsilon ir}), \ \boldsymbol{\Sigma}_{\varepsilon ir}, \text{ diagonal, } i \in r, \ r=\{C,P\}. \tag{5.73}$$

From the references to cognitive abilities, and the multi-population context in

which concern for the psychometric confound is situated, we deduce that (4.21) is

invoked as a representation, or model, of the (p_j+p_v) vector $\boldsymbol{\tau}_{ir}$, i.e.,

$$\boldsymbol{\tau}_{ir} = \boldsymbol{\gamma}_r + \boldsymbol{\Lambda}_r \boldsymbol{\xi}_{ir} + \boldsymbol{\zeta}_{ir}, i \in \mathcal{A}_r, r=1...s, \tag{5.74}$$

with the following particularizations, restrictions, and clarifications being made:

a) $s=2$, i.e., $r=\{C,P\}$; $m=k+3$, in which

$$\xi_{ir} = \begin{bmatrix} g_{ir} \\ s_{j_{ir}} \\ s_{v_{ir}} \\ s_{1_{ir}} \\ \vdots \\ s_{k_{ir}} \end{bmatrix}; \qquad (5.75)$$

b) $\Lambda_r = \Lambda$, (p_j+p_v) by $k+3$, and structured as follows:

$$\begin{bmatrix} \lambda_{jg} & \lambda_{js_j} & \mathbf{0}_{p_v} & \lambda_{js_1} & \cdots & \lambda_{js_k} \\ \lambda_{jg} & \mathbf{0}_{p_j} & \lambda_{vs_v} & \lambda_{vs_1} & \cdots & \lambda_{vs_k} \end{bmatrix}, \qquad (5.76)$$

in which the vector λ_{jg} contains the loadings of the p_j elements of X_j on ability g, λ_{js_j},

the loadings of the p_j elements of X_j on ability s_j, etc.[37];

c) over $i \in \Delta_r$, and for $r = \{C, P\}$,

$$\underset{i}{C}(\xi_{ir}) = \Phi_r = \mathbf{I}, \qquad (5.77)$$

$$\underset{i}{C}(\zeta_{ir}) = \Psi_r, \qquad (5.78)$$

in which Ψ_r is an (p_j+p_v) by (p_j+p_v), diagonal, positive definite matrix, structured as

$$\Psi_r = \begin{bmatrix} \Psi_{j|r} & \mathbf{0} \\ \mathbf{0} & \Psi_{v|r} \end{bmatrix} \text{ and}$$

[37] The $\mathbf{0}$ vectors express the fact that the elements of X_j do not load on s_v, and those of X_v do not load on s_j.

$$\underset{i}{C}(\boldsymbol{\xi}_{ir},\boldsymbol{\zeta}_{ir})=\mathbf{0}, \tag{5.79}$$

a 3 by (p_j+p_v) null matrix.

The reader should note that, under this mathematization: a) over populations C and P, weak or pattern invariance holds; b) discriminating power parameters $\gamma_{clj/sj}$ and $\gamma_{clj/g}$ are taken to be factor loadings, i.e., elements of $\boldsymbol{\Lambda}$. This implies, in particular, that $\gamma_{clj/sj}$ is the lth element of $\boldsymbol{\lambda}_{js_j}$ (this element notated $\boldsymbol{\lambda}_{js_j}[l]$) and $\gamma_{tlj/g}$, the lth element of $\boldsymbol{\lambda}_{jg}$ (notated $\boldsymbol{\lambda}_{jg}[l]$). By mathematizing the concept of *discriminating power* of, say, test composite c_{lj} in respect to ability s_j, as a factor loading, we quantitatively paraphrase it as "the number of units change in the lth element of $\boldsymbol{X}_j$ associated with a one standard deviation increase in s_j (the second element of $\boldsymbol{\xi}_r$)".

All told, we have, thus, the following, fully articulated, mathematical framework:

$$\mathbf{X}|i,r \sim (\boldsymbol{\gamma}_r + \boldsymbol{\Lambda}_r\boldsymbol{\xi}_{ir} + \boldsymbol{\zeta}_{ir}, \boldsymbol{\Sigma}_{\varepsilon ir}), \, i\in \varDelta_r r, \, r=\{C,P\}, \tag{5.80}$$

with consequence that

$$\mathbf{X}|r \sim (\boldsymbol{\gamma}_r + \boldsymbol{\Lambda}\boldsymbol{\kappa}_r, \boldsymbol{\Lambda}\boldsymbol{\Lambda}' + \boldsymbol{\Theta}_{\delta r}), \, r=\{C,P\}, \tag{5.81}$$

in which $\boldsymbol{\Theta}_{\delta r} = \boldsymbol{\Sigma}_{Er} + \boldsymbol{\Psi}_r$, diagonal, $r=\{C,P\}$, with $\boldsymbol{\Sigma}_{Er} = \mathrm{E}(\boldsymbol{\Sigma}_{\varepsilon ir})$.

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5.4.2 **Mathematization of Claims**

KMC1: As will be recalled, *KMC1* states that inferences about the parameter ($\mu_{c_{lj}/P} -$

$\mu_{c_{lj}/C}$) $-$ ($\mu_{c_{uv}/P} - \mu_{c_{uv}/C}$) are confounded, as bases for making decisions about D-Deficits

(notably, inferences about the parameter ($\mu_{s_j/P} - \mu_{s_j/C}$) $-$ ($\mu_{s_v/P} - \mu_{s_v/C}$)), by differences in

between the groups of parameters ($\gamma_{clj/s1} \dots \gamma_{clj/sk}$, $\gamma_{clj/g}$) and ($\gamma_{cuv/s1} \dots \gamma_{cuv/sk}$, $\gamma_{cuv/g}$). The

claim is translated as follows: The bias in estimating ($\kappa_P[2] - \kappa_C[2]$) $-$ ($\kappa_P[3] - \kappa_C[3]$) by

($\mu_{X_j|P}[l] - \mu_{X_j|C}[l]$) $-$ ($\mu_{X_v|P}[u] - \mu_{X_v|C}[u]$) , i.e.,

$\left| ((\mu_{X_j|P}[l] - \mu_{X_j|C}[l]) - (\mu_{X_v|P}[u] - \mu_{X_v|C}[u])) - ((\kappa_P[2] - \kappa_C[2]) - (\kappa_P[3] - \kappa_C[3])) \right|$, is a

function of differences between $\lambda_{js_I}[l] \dots \lambda_{js_k}[l]$ and $\lambda_{vs_I}[u] \dots \lambda_{vs_k}[u]$.

KMC2: As will be recalled, *KMC2* states that it is not the case that $\sigma^2_{\tau_{lj}}$ provides an

estimate of $\gamma_{clj/g}$, and therefore it is not the case that if $\sigma^2_{\tau_{lj}} < \sigma^2_{\tau_{uv}}$ then $\gamma_{clj/g} < \gamma_{cuv/g}$ (in

contrast to *CC6*). The claim is translated as follows: It is not the case that if

$\Sigma_{\tau_j}[l,l] < \Sigma_{\tau_v}[u,u]$ then $\lambda_{jg}[l] < \lambda_{vg}[u]$, in which Σ_{τ_j} is, consistent with expression 4.13,

the true score covariance matrix in respect X_j.

5.4.3 **Adjudication of Claims**

KMC1: To repeat, *KMC1* states that the bias in estimating

($\kappa_P[2] - \kappa_C[2]$) $-$ ($\kappa_P[3] - \kappa_C[3]$) by ($\mu_{X_j|P}[l] - \mu_{X_j|C}[l]$) $-$ ($\mu_{X_v|P}[u] - \mu_{X_v|C}[u]$) , i.e.,

$$\left|((\mu_{X_j|P}[l]-\mu_{X_j|C}[l])-(\mu_{X_v|P}[u]-\mu_{X_v|C}[u]))-((\kappa_P[2]-\kappa_C[2])-(\kappa_P[3]-\kappa_C[3]))\right| \text{ , is a}$$

function of differences between $\lambda_{js_1}[l]\dots\lambda_{js_k}[l]$ and $\lambda_{vs_1}[u]\dots\lambda_{vs_k}[u]$.

From expression 5.81,

$$\mu_{X_j|C}[l]=\gamma_{j|C}[l]+(\lambda_{jg}[l])(\kappa_C[1])+(\lambda_{js_j}[l])(\kappa_C[2])+$$

$$(\lambda_{js_1}[l])(\kappa_C[4])+\dots+(\lambda_{js_k}[l])(\kappa_C[k+3]), \tag{5.82}$$

$$\mu_{X_v|C}[u]=\gamma_{v|C}[u]+(\lambda_{vg}[u])(\kappa_C[1])+(\lambda_{vs_v}[u])(\kappa_C[3])+$$

$$(\lambda_{vs_1}[u])(\kappa_C[4])+\dots+(\lambda_{vs_k}[u])(\kappa_C[k+3]), \tag{5.83}$$

$$\mu_{X_j|P}[l]=\gamma_{j|P}[l]+(\lambda_{jg}[l])(\kappa_P[1])+(\lambda_{js_j}[l])(\kappa_P[2])+$$

$$(\lambda_{js_1}[l])(\kappa_P[4])+\dots+(\lambda_{js_k}[l])(\kappa_P[k+3]), \text{ and} \tag{5.84}$$

$$\mu_{X_v|P}[u]=\gamma_{v|P}[u]+(\lambda_{vg}[u])(\kappa_P[1])+(\lambda_{vs_v}[u])(\kappa_P[3])+$$

$$(\lambda_{vs_1}[u])(\kappa_P[4])+\dots+(\lambda_{vs_k}[u])(\kappa_P[k+3]). \tag{5.85}$$

From these expressions,

$$((\mu_{X_j|P}[l]-\mu_{X_j|C}[l])-(\mu_{X_v|P}[u]-\mu_{X_v|C}[u]))-((\kappa_P[2]-\kappa_C[2])-(\kappa_P[3]-\kappa_C[3]))$$

$$= (\gamma_{j|P}[l] - \gamma_{j|C}[l] + (\lambda_{jg}[l] - \lambda_{vg}[u])(\kappa_P[1] - \kappa_C[1]) + (\lambda_{js_j}[l])(\kappa_P[2] - \kappa_C[2])$$

$$+ (\lambda_{js_1}[l] - \lambda_{vs_1}[u])(\kappa_P[4] - \kappa_C[4]) + ... + (\lambda_{js_k}[l] - \lambda_{vs_k}[u])(\kappa_P[k+3] - \kappa_C[k+3]))$$

$$- (\gamma_{v|P}[u] - \gamma_{v|C}[u] + (\lambda_{vs_v}[u])(\kappa_P[3] - \kappa_C[3])) - ((\kappa_P[2] - \kappa_C[2]) - (\kappa_P[3] - \kappa_C[3])) \, . \quad (5.86)$$

By inspection of expression 5.86, it is apparent that

$$\left| ((\mu_{X_j|P}[l] - \mu_{X_j|C}[l]) - (\mu_{X_v|P}[u] - \mu_{X_v|C}[u])) - ((\kappa_P[2] - \kappa_C[2]) - (\kappa_P[3] - \kappa_C[3])) \right| \text{ is, in}$$

part, a function of differences between $\lambda_{js_1}[l] ... \lambda_{js_k}[l]$ and $\lambda_{vs_1}[u] ... \lambda_{vs_k}[u]$.

Accordingly, KMC1 is true.

KMC2: To repeat, *KMC2* states that it is not the case that if $\Sigma_{\tau_j}[l,l] < \Sigma_{\tau_v}[u,u]$ then

$\lambda_{jg}[l] < \lambda_{vg}[u]$.

From expression 5.81,

$$\Sigma_{\tau_j}[l,l] = (\lambda_{js_j}[l])^2 + (\lambda_{jg}[l])^2 + (\lambda_{js_1}[l])^2 + \cdots + (\lambda_{js_k}[l])^2 + \Psi_{j|C}[l,l], \quad (5.87)$$

and

$$\Sigma_{\tau_v}[u,u] = (\lambda_{vs_v}[u])^2 + (\lambda_{vg}[u])^2 + (\lambda_{vs_1}[u])^2 + \cdots + (\lambda_{vs_k}[u])^2 + \Psi_{v|C}[u,u] \, , \quad (5.88)$$

from which it is apparent that if $\Sigma_{\tau_j}[l,l] < \Sigma_{\tau_v}[u,u]$, then $\lambda_{jg}[l]$ and $\lambda_{vg}[u]$ remain free to vary. Therefore, it is not the case that if $\Sigma_{\tau_j}[l,l] < \Sigma_{\tau_v}[u,u]$ then $\lambda_{jg}[l] < \lambda_{vg}[u]$.

Accordingly, we conclude that *KMC2* is true.

5.5 **Summary**

In this chapter we have: 1) Mathematized the claims of the alternative account of the Psychometric Confound reviewed in Chapter 3; 2) Adjudicated these claims, given the proto-frameworks implied by the authors of the various claims. A review of our conclusions follows (this information is summarized in Table 1).

Under the proto-framework *B&TPf*, we concluded that: a) *BTC1* is true, i.e. that there exists a confound in the use of *inferred* differences in the distributions $c_{lj}|P$ and $c_{lj}|C$ as bases for making decisions about S-deficits, b) *BTC2* is true, i.e., that inferences about the parameter $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$ are confounded as bases for making decisions about S-Deficits; c) BTC3, which forms the basis for the Baron and Treiman method, is false.

Under the proto-framework employed by Salthouse and Coon, we concluded that: a) *SC1*, which is the basis for a method to rule on the existence of an S-Deficit, is false; b) *SC2*, which is the basis for a method to rule on the existence of a D-Deficit, is false.

Under the proto-framework *K&SP-f* we concluded that: a) *KSC1* is true, i.e. that there exists a confound in the use of the difference parameter $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} -$

$\mu_{c_{uv}/C}$) as a basis for making inferences about D-deficits; b) *KSC2-KSC8*, which form the bases for a variety of methods that side-stepped the problem of identifying S- and/or D-Deficits by focusing on other mean differences, are false; c) KSC9 and KSC10, which provided direct objections to the account of Chapman and Chapman by contradicting implications of CC4-CC6, are true.

Under the proto-framework *K&MP-f* we concluded that: a) *KMC1*, states that the difference parameter ($\mu_{c_{lj}/P} - \mu_{c_{lj}/C}$) - ($\mu_{c_{uv}/P} - \mu_{c_{uv}/C}$) is confounded as a basis for making inferences about D-deficits, is true; b) *KMC2*, which provided a direct objection to the Chapmans' claim CC6, is true.

Overall, then, our evaluation reveals that the alternative accounts correctly identified confounds in proposed methods, but have, in response, suggested methods that are faulty, under the proto-frameworks implied. By all extant articulations of the Psychometric Confound, then, the problem remains unresolved. In the next chapter, we take up the task of evaluating whether solutions to the problems falling under the heading "Psychometric Confound" are possible.

Chapter 6.

Solutions

In Chapter 2 we elucidated, at a non-technical level, the claims and proposed solutions of Chapman and Chapman apropos the issue of the psychometric confound. The same was done, in Chapter 3, for various of the prominent accounts, alternative to, and following in the wake of, Chapman and Chapman. In Chapter 4, the test theory nascent within the work of Chapman and Chapman and their commentators was given full expression within the context of the multi-population linear factor model. Therein, it was established that, although there does exist a "Psychometric Confound", roughly speaking, as described by Chapman and Chapman, each and every one of the solutions proposed by Chapman and Chapman are faulty. Chapter 5 contained demonstrations that the solutions proposed under the various alternative accounts are, likewise, faulty.

It should be noted, however, that even if it had been the case that one or more of the solutions proposed by Chapman and Chapman and their commentators, had turned out to be, technically, not incorrect, these solutions would, nevertheless, remain sub-optimal. For, without exception, they were in their construction *ad hoc* and heuristic, and the basic logic on which they rested was, to some extent, non-transparent. Consequently, the inferential performance that they were capable of delivering would have been virtually impossible to ascertain. If the scientist is to take seriously the latent variable

thinking that, as has been shown, permeates the work of both Chapman and Chapman, and their commentators, solutions must be engineered within the inferentially transparent, well-understood, context of structural equation modelling, this being the most general quantitative-representational framework for (linear) classical and modern test theory. That is to say, this is the context within which: *a*) the quantities of relevance to researchers in psychopathology, at least within the settings in which the Psychometric Confound is seen as being problematic, can be expressed, unambiguously, as parameters; *b*) hypothesis testing, and estimation, tasks, as bearing on these parameters, can be clearly worked out; and *c*) a wide variety of estimation and hypothesis testing strategies, the inferential performances of which are well understood, are available (within programs such as LISREL, Joreskog & Sorbom, 1993).

In this chapter, we do precisely this. Fundamentally, the parameter of value which is of interest to the psychopathology researcher, faced with the confounding effects of the psychometric confound, is, depending upon the proto-framework at play, either $\kappa_P[1] - \kappa_C[1]$ or $\kappa_P[2] - \kappa_C[2]$. For each proto-framework, estimating the value of one or the other of these parameters provides the basis for drawing an inferential conclusion as to whether an S-Deficit, in respect to ability s_j and populations *P* and *C*, in fact, exists. While, as reviewed above, the Chapmans and the authors of alternative accounts have proposed methods for the drawing of conclusions regarding other parameters, we take as patent, within the entire field, e.g., in Chapman's and Chapman's (2001) references to "specific cognitive differential deficits", Baron's and Treiman's (1980) statement that the question to be addressed is whether "groups differ on the ability of interest" (p. 313),

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Salthouse's and Coon's (1994) stated aim as being to draw a "strong conclusion of selective impairment" (p. 1172)[38], Knight's and Silverstein's (2001) expressed desire to "isolate specific cognitive deficiencies" (p. 15), and Kang's and MacDonald's (2010) declared interest in demonstrating "specific cognitive deficit" (p. 300), that the primary goal of research in the area is to draw conclusions regarding S-Deficits. In this chapter of the thesis, then, we determine, within the context of structural equation modelling, whether a direct approach to inference, apropos the central parameter (either $\kappa_P[1] - \kappa_C[1]$ or $\kappa_P[2] - \kappa_C[2]$), is possible. Specifically, under each of the quantitative frameworks elucidated in Chapters 4 and 5 we: *i*) specify a well-defined model, of structural equation sort, under which the parameter of interest is defined; and *ii*) determine whether this parameter, under the model in question, is identified, and, hence, *estimable*.

For any model in which is shown to be identified, the parameter (either, as appropriate, $\kappa_P[1] - \kappa_C[1]$ or $\kappa_P[2] - \kappa_C[2]$), estimates of the parameter may be obtained via computer programs such as LISREL (Joreskog & Sorbom, 1993). Alternatively, this machinery can be employed to test the appropriate hypothesis pair, either [H_0: $\kappa_P[1] - \kappa_C[1] \geq 0$, H_1: $\kappa_P[1] - \kappa_C[1] < 0$] or [$H_0$: $\kappa_P[2] - \kappa_C[2] \geq 0$, H_1: $\kappa_P[2] - \kappa_C[2] < 0$].

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[38] In the work of Salthouse and Coon, as reviewed in Chapter 3, a method was proposed in which the drawing of conclusions regarding S-Deficit was a prerequisite step to the determination as to whether there exists a D-Deficit. This method would seem to be nonsensical, if, as claimed herein, the authors' primary interest was in the determination of the existence of an S-Deficit. Much of Salthouse and Coon's language, however, clearly does indicate the latter, in agreement with the other authors listed. We therefore take Salthouse and Coon's primary interest to be the drawing of conclusions regarding S-Deficits, and assume that their proposal of a conflicting method is a symptom of the lack of conceptual clarity around the issue of the psychometric confound, as discussed in sections 2.6 and 3.3.5.

Though beyond the scope of this thesis, surveys of the inferential strategies (of type, both

estimation and hypothesis testing) on offer from programs such as LISREL are readily

available (see, e.g., Bollen,1989).

It should be noted, of course, that within a particular empirical setting, an estimate

of $\kappa_P[1] - \kappa_C[1]$ or $\kappa_P[2] - \kappa_C[2]$ (or meaningful test of one the appropriate hypothesis

pairs listed in the last paragraph) will, in fact, be obtainable, only if the model that is the

vehicle for generating this inference is shown to fit acceptably. This may seem a

disadvantage of the current approach, but, in fact, if it appears thus, is mere illusion. For

although, in some cases in which estimates of $\kappa_P[1] - \kappa_C[1]$ or $\kappa_P[2] - \kappa_C[2]$ are not

obtainable, traditional, ad-hoc, methods do yield inferences, the inferences that they

yield, are, as established in Chapters 4 and 5, faulty. Moreover, as already noted, even if,

in broad terms, the yield of these methods was "correct", the inferential performance of

these methods would stand as unknown (and be, essentially, unknowable).

6.1 **Chapman and Chapman; Salthouse and Coon**

Under the proto-framework of Chapman and Chapman (also, that of Salthouse

and Coon), the fundamental problem was to make a decision as to whether or not, was

less than zero, the parameter $\mu_{c_{lj}/P} - \mu_{c_{lj}/C}$. As reviewed in Chapters 2 and 3, sundry ad

hoc strategies were proposed by these authors for the drawing of conclusion regarding

linked parameter $(\mu_{c_{lj}/P} - \mu_{c_{lj}/C}) - (\mu_{c_{uv}/P} - \mu_{c_{uv}/C})$ (the link between the parameters is

articulated in section 2.2). In Chapter 4, we provided a technical, modern-test theoretic,

foundation to the work of Chapman and Chapman (and Salthouse and Coon), under

which the fundamental problem is reinterpreted as a one of making an inference as to

whether the parameter $\kappa_P[2] - \kappa_C[2]$ is less than 0. In this section, we complete the

project, by addressing, formally, the issue as to how to make inferential decisions about

this parameter, using the tools of structural equation modeling, as potentially

implementable by a program such as LISREL. In particular, we specify a structural

equation model through: *i*) an extension of the quantitative framework in play, i.e., the

proto-framework of Chapman and Chapman; and, *ii*) the specification of general and

identifying constraints on the model. We then show, regarding the covariance structure,

that the model is identified. A demonstration that the parameter of interest,

$\kappa_P[2] - \kappa_C[2]$, is identified under the model follows.

6.1.1 **Extension of Quantitative Framework**

Let *X* now be a $(p_j + p_v + p_z)$-element random vector $\begin{bmatrix} \boldsymbol{X}_j \\ \boldsymbol{X}_v \\ \boldsymbol{X}_z \end{bmatrix}$, the first p_j elements of

which scale in respect ability s_j, the second p_v elements of which scale in respect ability

s_v, and the third p_z elements of which scale in respect the additional ability s_z. Let the

particulars of the Multi-Population Linear Factor Model, as described in section 4.2,

apply, and let the model constraints be as summarized as in section 6.1.2 (note that Figure

1 illustrates the covariance structure portion of the model).

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6.1.2 Constraints: General and Identifying

1) $p_j \geq 3$; $p_v \geq 3$; $p_z \geq 3$;

2) $\boldsymbol{\xi}_{ir} = \begin{bmatrix} g_{ir} \\ s_{j_{ir}} \\ s_{v_{ir}} \\ s_{z_{ir}} \end{bmatrix}$;

3) $\boldsymbol{\Lambda}_r = \boldsymbol{\Lambda}$, $(p_j+p_v+p_z)$ by 4, nonnegative, and structured as follows:

$$\begin{pmatrix} \boldsymbol{\lambda}_{jg} & \boldsymbol{\lambda}_{js_j} & \mathbf{0}_{p_j} & \mathbf{0}_{p_j} \\ \boldsymbol{\lambda}_{vg} & \mathbf{0}_{p_v} & \boldsymbol{\lambda}_{vs_v} & \mathbf{0}_{p_v} \\ \boldsymbol{\lambda}_{zg} & \mathbf{0}_{p_z} & \mathbf{0}_{p_z} & \boldsymbol{\lambda}_{zs_z} \end{pmatrix};$$

4) $\boldsymbol{\Theta}_{\delta r} = \boldsymbol{\Sigma}_{Er} + \boldsymbol{\Psi}_r$, diagonal, $r=\{C,P\}$;

5) $E(\boldsymbol{\delta}_r) = \mathbf{0}$ and $C(\boldsymbol{\xi}_r, \boldsymbol{\delta}_r) = \mathbf{0}$, $r = 1..s$;

6) $\boldsymbol{\Phi}_r$ is a correlation matrix, implying that the variances of the common factors are set to unity;

7) $\boldsymbol{\Phi}_r$ diagonal;

8) $\kappa_C[1] = \kappa_C[2] = \kappa_C[3] = \kappa_C[4] = 0;$[39]

9) $\gamma_C = \gamma_P$.

6.1.3 **Proof of Identification of Covariance Structure Model**

Recall, as written in equation 4.22, that the covariance structure of the model is defined by

$$\Sigma_r = \Lambda_r \Phi_r \Lambda_r{}' + \Theta_{\delta r} \tag{6.1}$$

In order to show that the model, in respect this covariance structure, is identified, it must be shown that all unconstrained elements of matrices Λ_r and $\Theta_{\delta r}$ are identified (note that Φ_r is completely constrained, by constraints 6 and 7 above). In total, given the $(p_j + p_v + p_z)$ elements in X, there are $(p_j + p_v + p_z)(p_j + p_v + p_z + 1)/2$ independent covariance equations; and, in terms of independent, unconstrained parameters, $2(p_j + p_v + p_z)$ in Λ_r (by constraint 3), and $(p_j + p_v + p_z)$ in $\Theta_{\delta r}$ (by constraint 4). Under constraint 1, the number of covariance equations is greater than the number of independent, unconstrained parameters, meeting a necessary but not sufficient condition for identification (Long, 1983). Therefore, we now move to a proof of identification of the elements of each matrix in turn.

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[39] Though this constraint is not necessary for purposes of identification, it, without loss of generality, effects a simplification in the estimation of the quantity of interest, when using computer programs such as LISREL.

6.1.3.1 *6.1.3.1 Proof of Identification of* Λ_r

We first solve for $\lambda_{jg}, \lambda_{vg}$, and λ_{zg}. Consider the lth element of $\boldsymbol{X}_j$, the uth element of $\boldsymbol{X}_v$, and the hth element of $\boldsymbol{X}_z$, corresponding to loadings $\lambda_{jg}[l], \lambda_{vg}[u]$, and $\lambda_{zg}[h]$. From equation 6.1, and constraints 4-7, then,

$$\Sigma_r[l,(p_j+u)] = (\lambda_{jg}[l])(\lambda_{vg}[u]); \tag{6.2}$$

$$\Sigma_r[(p_j+u),(p_j+p_v+h)] = (\lambda_{vg}[u])(\lambda_{zg}[h]); \tag{6.3}$$

$$\Sigma_r[l,(p_j+p_v+h)] = (\lambda_{jg}[l])(\lambda_{zg}[h]). \tag{6.4}$$

It is now possible to solve for each of $\lambda_{jg}[l], \lambda_{vg}[u]$, and $\lambda_{zg}[h]$:

$$(\lambda_{jg}[l])^2 = \frac{(\lambda_{jg}[l])(\lambda_{zg}[h])}{(\lambda_{vg}[u])(\lambda_{zg}[h])}(\lambda_{jg}[l])(\lambda_{vg}[u]) =$$

$$\frac{(\Sigma_r[l,(p_j+p_v+h)])(\Sigma_r[l,(p_j+u)])}{\Sigma_r[(p_j+u),(p_j+p_v+h)]}; \tag{6.5}$$

$$(\lambda_{vg}[u])^2 = \frac{(\lambda_{jg}[l])(\lambda_{vg}[u])}{(\lambda_{jg}[l])(\lambda_{zg}[h])}(\lambda_{vg}[u])(\lambda_{zg}[h]) =$$

$$\frac{(\Sigma_r[l,(p_j+u)])(\Sigma_r[(p_j+u),(p_j+p_v+h)])}{\Sigma_r[l,(p_j+p_v+h)]}; \tag{6.6}$$

$$(\lambda_{zg}[h])^2 = \frac{(\lambda_{vg}[u])(\lambda_{zg}[h])}{(\lambda_{jg}[l])(\lambda_{vg}[u])}(\lambda_{jg}[l])(\lambda_{zg}[h]) =$$

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$$\frac{(\Sigma_r[(p_j+u),(p_j+p_v+h)])(\Sigma_r[l,(p_j+p_v+h)])}{\Sigma_r[l,(p_j+u)]} \qquad (6.7)$$

As the lth element of $\boldsymbol{X}_j$, the uth element of $\boldsymbol{X}_v$, and the hth element of $\boldsymbol{X}_z$ may represent any of the elements in $\boldsymbol{X}_j$, $\boldsymbol{X}_v$, and $\boldsymbol{X}_z$, this procedure demonstrates that all loadings in $\lambda_{jg}, \lambda_{vg}$, and λ_{zg} are identified.

Consider, now, three elements of $\boldsymbol{X}_j$: $\boldsymbol{X}_j[l], \boldsymbol{X}_j[l+1], \boldsymbol{X}_j[l+2]$. We now solve for corresponding parameters $\lambda_{js_j}[l]$, $\lambda_{js_j}[l+1]$ and $\lambda_{js_j}[l+2]$. Note that the covariances of these three elements of $\boldsymbol{X}_j$, by equation 6.1 and constraints 4-7, are expressed as follows:

$$\Sigma_r[l,l+1] = (\lambda_{jg}[l])(\lambda_{jg}[l+1]) + (\lambda_{js_j}[l])(\lambda_{js_j}[l+1]) \ ; \qquad (6.8)$$

$$\Sigma_r[l,l+2] = (\lambda_{jg}[l])(\lambda_{jg}[l+2]) + (\lambda_{js_j}[l])(\lambda_{js_j}[l+2]) \ ; \qquad (6.9)$$

$$\Sigma_r[l+1,l+2] = (\lambda_{jg}[l+1])(\lambda_{jg}[l+2]) + (\lambda_{js_j}[l+1])(\lambda_{js_j}[l+2]) . \qquad (6.10)$$

Hence,

$$(\lambda_{js_j}[l])(\lambda_{js_j}[l+1]) = \Sigma_r[l,l+1] - (\lambda_{jg}[l])(\lambda_{jg}[l+1]) \ ; \qquad (6.11)$$

$$(\lambda_{js_j}[l])(\lambda_{js_j}[l+2]) = \Sigma_r[l,l+2] - (\lambda_{jg}[l])(\lambda_{jg}[l+2]) ; \qquad (6.12)$$

$$(\lambda_{js_j}[l+1])(\lambda_{js_j}[l+2]) = \Sigma_r[l+1,l+2] - (\lambda_{jg}[l+1])(\lambda_{jg}[l+2]) . \qquad (6.13)$$

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Now, λ_{jg} is known from equation 6.5. Hence, $\lambda_{js_j}[l]$, $\lambda_{js_j}[l+1]$ and $\lambda_{js_j}[l+2]$ are known by expressions 6.11-6.13 and the following equations:

$$(\lambda_{js_j}[l])^2 = \frac{(\lambda_{js_j}[l])(\lambda_{js_j}[l+1])}{(\lambda_{js_j}[l+1])(\lambda_{js_j}[l+2])}(\lambda_{js_j}[l])(\lambda_{js_j}[l+2]) ; \qquad (6.14)$$

$$(\lambda_{js_j}[l+1])^2 = \frac{(\lambda_{js_j}[l])(\lambda_{js_j}[l+1])}{(\lambda_{js_j}[l])(\lambda_{js_j}[l+2])}(\lambda_{js_j}[l+1])(\lambda_{js_j}[l+2]) ; \qquad (6.15)$$

$$(\lambda_{js_j}[l+2])^2 = \frac{(\lambda_{js_j}[l])(\lambda_{js_j}[l+2])}{(\lambda_{js_j}[l])(\lambda_{js_j}[l+1])}(\lambda_{js_j}[l+1])(\lambda_{js_j}[l+2]) . \qquad (6.16)$$

The production of such triads of equations solves for the entirety of λ_{js_j}. An identical procedure may be employed to solve for all elements of λ_{vs_v} and λ_{zs_z}. This shows that Λ_r is identified, as apart from λ_{js_j}, λ_{vs_v}, λ_{zs_z}, λ_{jg}, λ_{vg}, and λ_{zg} (all which we have shown are identified), Λ_r is constrained (by constraint 3).

6.1.3.2 *Proof of Identification of $\Theta_{\delta r}$*

To repeat equation 6.1,

$$\Sigma_r = \Lambda_r \Phi_r \Lambda_r' + \Theta_{\delta r} , \qquad (6.17)$$

in which Φ_r is a correlation matrix, and is diagonal (constraints 5 and 6). As shown in

section 6.1.3.1, Λ_r is identified. Therefore, $\Theta_{\delta r}$ is identified. For example, to take

$\Theta_{\delta r}[l,l]$,

$$\Sigma_r[l,l] = (\lambda_{jg}[l])^2 + (\lambda_{js_j}[l])^2 + \Theta_{\delta r}[l,l], \qquad (6.18)$$

which solves for $\Theta_{\delta r}[l,l]$, as $\lambda_{jg}[l]$ and $\lambda_{js_j}[l]$ are identified. This completes the

identification, in respect the covariance structure, of the model.

6.1.4 **Proof of Identification of** $\kappa_P[2] - \kappa_C[2]$

As Λ_r is, by section 6.1.3, identified, the parameter of interest, $\kappa_P[2] - \kappa_C[2]$, is

identified, according to the following demonstration:

From 4.23 and constraint 8,

$$\mu_{X_j|P}[l] - \mu_{X_j|C}[l] = \lambda_{jg}[l](\kappa_P[1] - \kappa_C[1]) + \lambda_{js_j}[l](\kappa_P[2] - \kappa_C[2]); \qquad (6.19)$$

$$\mu_{X_j|P}[l+1] - \mu_{X_j|C}[l+1] = \lambda_{jg}[l+1](\kappa_P[1] - \kappa_C[1]) + \lambda_{js_j}[l+1](\kappa_P[2] - \kappa_C[2]). \quad (6.20)$$

Through algebraic manipulation,

$$(\kappa_P[2] - \kappa_C[2]) = \frac{(\lambda_{jg}[l]/\lambda_{jg}[l+1])(\mu_{X_j|P}[l+1] - \mu_{X_j|C}[l+1]) - (\mu_{X_j|P}[l] - \mu_{X_j|C}[l])}{((\lambda_{jg}[l])(\lambda_{js_j}[l+1])/\lambda_{jg}[l+1]) - (\lambda_{js_j}[l])}. \quad (6.21)$$

The parameter is, in fact, over-identified, as any pair of means from $\mu_{X_j|P}$ and $\mu_{X_j|C}$ can be employed to generate equations analogous, in form, to 6.19 and 6.20.

6.1.5 **Model Fit and Parameter Estimation**

The model defined in sections 6.1.1 and 6.1.2 can be fit by a program such as LISREL. Presuming that the fit of the model is acceptable, then the estimate yielded by LISREL of $\kappa_P[2] - \kappa_C[2]$ may be taken seriously. Alternatively, the researcher may test the hypothesis pair [H$_0$: $\kappa_P[2] - \kappa_C[2] \geq 0$, H$_1$: $\kappa_P[2] - \kappa_C[2] < 0$].

6.2 **Baron and Treiman**

Under the proto-framework of Baron and Treiman, the fundamental problem was to make a decision as to whether or not, was less than zero, the parameter $\mu_{c_{lj}/P} - \mu_{c_{lj}/C}$. As reviewed in Chapters 3, these authors proposed an ad hoc strategy for the drawing of conclusions regarding this parameter. Building on the modern test-theoretic framework of Chapter 4, we, in Chapter 5, articulated, in technical terms, the proto-framework of Baron and Treiman, under which the fundamental problem is reinterpreted as a one of making an inference as to whether the parameter $\kappa_P[1] - \kappa_C[1]$ is less than 0. In this section, we articulate a structural equation model consistent with the relevant quantitative framework described in Chapter 5 (the proto-framework of Baron and Treiman), by a review of the model's general constraints. When then show that this model is not identified, and

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therefore conclude that estimation and hypothesis testing strategies for parameter of

interest, $\kappa_P[1] - \kappa_C[1]$, are not, under this model, available.

6.2.1 Constraints: General

1) $\Lambda_r = \Lambda$, $(p_j + p_v)$ by $(k + 1)$, nonnegative, and structured as follows:

$$
\begin{pmatrix}
\lambda_{js_j} & \lambda_{js_1} & \cdots & \lambda_{js_k} \\
\mathbf{0}_{p_j} & \lambda_{vs_1} & \cdots & \lambda_{vs_k}
\end{pmatrix};
$$

2) $\Theta_{\delta r} = \Sigma_{Er} + \Psi_r$, diagonal, $r = \{C, P\}$;

3) $E(\delta_r) = \mathbf{0}$ and $C(\xi_r, \delta_r) = \mathbf{0}$, $r = 1..s$;

4) Φ_r is a correlation matrix, implying that the variances of the common factors are

   set to unity;

5) Φ_r diagonal;

6.2.2 Evaluation of Identification

Recall, as written in equation 4.22, that the covariance structure of the model is

defined by

$$
\Sigma_r = \Lambda_r \Phi_r \Lambda_r' + \Theta_{\delta r} . \tag{6.22}
$$

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In order to show that the model, in respect this covariance structure, is identified, it must

be shown that all unconstrained elements of matrices Λ_r and $\Theta_{\delta r}$ are identified (note that

Φ_r is completely constrained, by constraints 4 and 5). Now, as noted in section 6.1.3, a

necessary condition of identification is that the number of covariance equations is greater

than or equal to the number of independent, unconstrained parameters. In total, given the

$(p_j + p_v)$ elements in X, there are $(p_j + p_v)(p_j + p_v + 1)/2$ independent covariance

equations; and, in terms of independent, unconstrained parameters, $(k+1)(p_j) + k(p_v)$ in

Λ_r, and $(p_j + p_v)$ in $\Theta_{\delta r}$. For conditions in which $(p_j + p_v)$ is large, and k is relatively

smaller, the necessary condition can be met. For example, if $p_j = p_v = 5$ and $k = 3$, then

there are 55 covariance equations and 45 unconstrained parameters. We therefore move

to a consideration of whether Λ_r can be identified.

6.2.2.1   ***6.2.2.1 Evaluation of Identification of Λ_r***

There are five general forms of the covariance equations: two for covariances of

elements of X_j, two for covariances of elements of X_v, and one for covariances of

elements, one from X_j, one from X_v.

For items *l* and *l+1* of X_j, it follows from equation 4.22 and the constraints

above that:

$$\Sigma_r[l,l] = (\lambda_{js_j}[l])^2 + (\lambda_{js_1}[l])^2 + (\lambda_{js_2}[l])^2 + ... + (\lambda_{js_k}[l])^2 + \Theta_{\delta C}[l,l]; \qquad (6.23)$$

$$\Sigma_r[l, l+1] = (\lambda_{js_j}[l])(\lambda_{js_j}[l+1]) + (\lambda_{js_1}[l])(\lambda_{js_1}[l+1]) + \ldots + (\lambda_{js_k}[l])(\lambda_{js_k}[l+1]) . \quad (6.24)$$

For items u and $u+1$ of $\boldsymbol{X}_v$, it follows from equation 4.22 and the constraints above that:

$$\Sigma_r[p_j + u, p_j + u] = (\lambda_{vs_1}[u])^2 + \ldots + (\lambda_{vs_k}[u])^2 + \Theta_{\delta C}[p_j + u, p_j + u] ; \quad (6.25)$$

$$\Sigma_r[p_j + u, p_j + u + 1] = (\lambda_{vs_1}[u])(\lambda_{vs_1}[u+1]) + \ldots + (\lambda_{vs_k}[u])(\lambda_{vs_k}[u+1]) . \quad (6.26)$$

For items l of $\boldsymbol{X}_j$ and u of $\boldsymbol{X}_v$,

$$\Sigma_r[l, p_j + u] = (\lambda_{js_1}[l])(\lambda_{vs_1}[u]) + \ldots + (\lambda_{js_k}[l])(\lambda_{vs_k}[u]) . \quad (6.27)$$

Now, consider the two sets of loadings $\lambda_{js_1}[l]\ldots\lambda_{js_k}[l]$ and $\lambda_{js_1}[l+1]\ldots\lambda_{js_k}[l+1]$. Equation 6.24 is the only equation in which these two sets of loadings co-occur. However, note that even if $\lambda_{js_j}[l+1]$ and $\lambda_{js_1}[l+1]\ldots\lambda_{js_k}[l+1]$ were known, there remain $k+1$ unknowns in this equation, i.e., $\lambda_{js_j}[l]$ and $\lambda_{js_1}[l]\ldots\lambda_{js_k}[l]$. This equation would not, therefore, solve for $\lambda_{js_1}[l]\ldots\lambda_{js_k}[l]$. This is true for any two elements of $\boldsymbol{X}_j$. Furthermore, because of the structure of equations 6.23-6.27, there is no manner in which such equations for elements of Σ_r may be combined to isolate elements of $\lambda_{js_1}\ldots\lambda_{js_k}$, regardless of the values of k and p_j (the reader is encouraged to verify this). It follows that $\lambda_{js_1}\ldots\lambda_{js_k}$ are not identified. By equation 6.26 and the same logic, $\lambda_{vs_1}\ldots\lambda_{vs_k}$ are not identified. Therefore, neither Λ_r nor the model as a whole are identified.

6.2.3 **Model Fit and Parameter Estimation**

A prerequisite for valid determination of model fit and estimation of parameters is model identification. However, the model is not identified, as shown in section 6.2.2. Furthermore, there are no additional, reasonable constraints that can be placed on the model that would cause the model to be identified. Therefore, under the structural equation model implied by Baron and Treiman, procedures for estimating, or performing hypothesis testing in relation to, $\kappa_P[1] - \kappa_C[1]$ are not available.

6.3 **Knight and Silverstein**

Under the proto-framework of Knight and Silverstein, the fundamental problem was to make a decision as to whether or not, was less than zero, the parameter $\mu_{c_{lj}/P} - \mu_{c_{lj}/C}$. As reviewed in Chapters 3, these authors proposed several ad hoc methods that side-stepped the problem, focusing on other mean differences. Building on the modern test-theoretic framework of Chapter 4, we, in Chapter 5, articulated, in technical terms, the proto-framework of Knight and Silverstein, under which the fundamental problem is reinterpreted as a one of making an inference as to whether the parameter $\kappa_P[1] - \kappa_C[1]$ is less than 0. In this section, we articulate a structural equation model consistent with the relevant quantitative framework described in Chapter 5 (the proto-framework of Knight and Silverstein), by a review of the model's general constraints. When then show that this model is not identified, and therefore conclude estimation and

hypothesis testing strategies for parameter of interest, $\kappa_P[1] - \kappa_C[1]$, are not, under this

model, available.

6.3.1 **Constraints: General**

1) $\Lambda_r = \Lambda$, $(p_j + p_v)$ by $(k + 2)$, nonnegative, and structured as follows:

$$
\begin{pmatrix}
\lambda_{js_j} & \mathbf{0}_{p_v} & \lambda_{js_1} & \cdots & \lambda_{js_k} \\
\mathbf{0}_{p_j} & \lambda_{vs_v} & \lambda_{vs_1} & \cdots & \lambda_{vs_k}
\end{pmatrix} ;
$$

2) $\Theta_{\delta r} = \Sigma_{Er} + \Psi_r$, diagonal, $r = \{C, P\}$;

3) $E(\delta_r) = \mathbf{0}$ and $C(\xi_r, \delta_r) = \mathbf{0}$, $r = 1..s$;

4) Φ_r is a correlation matrix, implying that the variances of the common factors are

set to unity;

5) Φ_r diagonal;

6.3.2 **Evaluation of Identification**

Recall, as written in equation 4.22, that the covariance structure of the model is

defined by

$$
\Sigma_r = \Lambda_r \Phi_r \Lambda_r{}' + \Theta_{\delta r} . \tag{6.28}
$$

In order to show that the model, in respect this covariance structure, is identified, it must be shown that all unconstrained elements of matrices Λ_r and $\Theta_{\delta r}$ are identified (note that Φ_r is completely constrained, by constraints 4 and 5). Now, as noted in section 6.1.3, a necessary condition of identification is that the number of covariance equations is greater than or equal to the number of independent, unconstrained parameters. In total, given the $(p_j + p_v)$ elements in X, there are $(p_j + p_v)(p_j + p_v + 1)/2$ independent covariance equations; and, in terms of independent, unconstrained parameters, $(k+1)(p_j + p_v)$ in Λ_r, and $(p_j + p_v)$ in $\Theta_{\delta r}$. In total, then, there are $(k+2)(p_j + p_v)$ unconstrained parameters.

For conditions in which $(p_j + p_v)$ is large, and k is relatively smaller, the necessary condition can be met. For example, if $p_j = p_v = 5$ and $k = 3$, then there are 55 covariance equations and 50 unconstrained parameters. We therefore move to a consideration as to whether Λ_r can be identified.

6.3.2.1 *Evaluation of Identification of Λ_r*

There are five general forms of the covariance equations: two for covariances of elements of X_j, two for covariances of elements of X_v, and one for covariances of elements, one from X_j, one from X_v.

For items l and $l+1$ of X_j, it follows from equation 4.22 and the constraints above that:

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$$\Sigma_r[l,l] = (\lambda_{js_j}[l])^2 + (\lambda_{js_1}[l])^2 + (\lambda_{js_2}[l])^2 + ... + (\lambda_{js_k}[l])^2 + \Theta_{\delta C}[l,l]; \qquad (6.29)$$

$$\Sigma_r[l,l+1] = (\lambda_{js_j}[l])(\lambda_{js_j}[l+1]) + (\lambda_{js_1}[l])(\lambda_{js_1}[l+1]) + ... + (\lambda_{js_k}[l])(\lambda_{js_k}[l+1]). \,(6.30)$$

For items u and $u+1$ of $\boldsymbol{X}_v$, it follows from equation 4.22 and the constraints above that:

$$\Sigma_r[p_j+u, p_j+u] = (\lambda_{vs_v}[u])^2 + (\lambda_{vs_1}[u])^2 + ... + (\lambda_{vs_k}[u])^2 + \Theta_{\delta C}[p_j+u, p_j+u]; (6.31)$$

$$\Sigma_r[p_j+u, p_j+u+1] = (\lambda_{vs_v}[u])(\lambda_{vs_v}[u+1]) +$$

$$(\lambda_{vs_1}[u])(\lambda_{vs_1}[u+1]) + ... + (\lambda_{vs_k}[u])(\lambda_{vs_k}[u+1]). \qquad (6.32)$$

For items l of $\boldsymbol{X}_j$ and u of $\boldsymbol{X}_v$,

$$\Sigma_r[l, p_j+1] = (\lambda_{js_1}[l])(\lambda_{vs_1}[u]) + ... + (\lambda_{js_k}[l])(\lambda_{vs_k}[u]). \qquad (6.33)$$

Now, consider the two sets of loadings $\lambda_{js_1}[l]...\lambda_{js_k}[l]$ and $\lambda_{js_1}[l+1]...\lambda_{js_k}[l+1]$. Equation 6.30 is the only equation in which these two sets of loadings co-occur. However, note that even if $\lambda_{js_j}[l+1]$ and $\lambda_{js_1}[l+1]...\lambda_{js_k}[l+1]$ were known, there remain $k+1$ unknowns in this equation, i.e., $\lambda_{js_j}[l]$ and $\lambda_{js_1}[l]...\lambda_{js_k}[l]$. This equation would not, therefore, solve for $\lambda_{js_1}[l]...\lambda_{js_k}[l]$. This is true for any two elements of $\boldsymbol{X}_j$. Furthermore, because of the structure of equations 6.29-6.33, there is no manner in which such equations for elements of Σ_r may be combined to isolate elements of $\lambda_{js_1}...\lambda_{js_k}$, regardless of the values of k and

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p_j (the reader is encouraged to verify this). It follows that $\lambda_{js_1}...\lambda_{js_k}$ are not identified. By

equation 6.32 and the same logic, $\lambda_{vs_1}...\lambda_{vs_k}$ are not identified. Therefore, neither Λ_r nor

the model as a whole are identified.

6.3.3 **Model Fit and Parameter Estimation**

A prerequisite for valid determination of model fit and estimation of parameters is

model identification. However, the model is not identified, as shown in 6.3.2.

Furthermore, there are no additional, reasonable constraints that could be placed on the

model that would cause the model to be identified. Therefore, under the structural

equation model implied by Knight and Silverstein, procedures for estimating, or

performing hypothesis testing in relation to $\kappa_P[1] - \kappa_C[1]$ are not available.

6.4 **Kang and MacDonald**

Under the proto-framework of Kang and MacDonald, the fundamental problem

was to make a decision as to whether or not, was less than zero, the parameter

$\mu_{c_{ij}/P} - \mu_{c_{ij}/C}$. Building on the modern test-theoretic framework of Chapter 4, we, in

Chapter 5, articulated, in technical terms, the proto-framework of Kang and MacDonald,

under which the fundamental problem is reinterpreted as a one of making an inference as

to whether the parameter $\kappa_P[2] - \kappa_C[2]$ is less than 0. In this section, we articulate a

structural equation model consistent with the relevant quantitative framework described

in Chapter 5 (the proto-framework of Kang and MacDonald), by a review of the model's

general constraints. When then show that this model is not identified, and therefore

conclude estimation and hypothesis testing strategies for parameter of interest,

$\kappa_P[2] - \kappa_C[2]$, are not, under this model, available.

6.4.1 Constraints: General

1) $\Lambda_r = \Lambda$, $(p_j + p_v)$ by $(k + 3)$, nonnegative, and structured as follows:

$$\begin{bmatrix} \lambda_{jg} & \lambda_{js_j} & \mathbf{0}_{p_v} & \lambda_{js_1} & \cdots & \lambda_{js_k} \\ \lambda_{jg} & \mathbf{0}_{p_j} & \lambda_{vs_v} & \lambda_{vs_1} & \cdots & \lambda_{vs_k} \end{bmatrix};$$

2) $\Theta_{\delta r} = \Sigma_{Er} + \Psi_r$, diagonal, $r = \{C, P\}$;

3) $\mathrm{E}(\delta_r) = \mathbf{0}$ and $\mathrm{C}(\xi_r, \delta_r) = \mathbf{0}$, $r = 1..s$;

4) Φ_r is a correlation matrix, implying that the variances of the common factors are

set to unity;

5) Φ_r diagonal.

6.4.2 Evaluation of Identification

Recall, as written in equation 4.22, that the covariance structure of the model is

defined by

$$\Sigma_r = \Lambda_r \Phi_r \Lambda_r' + \Theta_{\delta r}. \tag{6.34}$$

In order to show that the model, in respect this covariance structure, is identified, it must be shown that all unconstrained elements of matrices Λ_r and $\Theta_{\delta r}$ are identified (note that Φ_r is completely constrained, by constraints 4 and 5). Now, as noted in section 6.1.3, a necessary condition of identification is that the number of covariance equations is greater than or equal to the number of independent, unconstrained parameters. In total, given the $(p_j + p_v)$ elements in X, there are $(p_j + p_v)(p_j + p_v + 1) / 2$ independent covariance equations; and, in terms of independent, unconstrained parameters, $(k + 2)(p_j + p_v)$ in Λ_r, and $(p_j + p_v)$ in $\Theta_{\delta r}$. In total, then, there are $(k + 3)(p_j + p_v)$ unconstrained parameters.

For conditions in which $(p_j + p_v)$ is large, and k is relatively small, the necessary condition can be met. For example, if $p_j = p_v = 6$ and $k = 3$, then there are 78 covariance equations and 60 unconstrained parameters. We therefore move to a consideration as to whether Λ_r can be identified.

6.4.2.1 *Evaluation of Identification of Λ_r*

There are five general forms of the covariance equations: two for covariances of elements of X_j, two for covariances of elements of X_v, and one for covariances of elements, one from X_j, one from X_v.

For items *l* and *l+1* of X_j, it follows from equation 4.22 and the constraints above that:

$$\Sigma_r[l,l] = (\lambda_{jg}[l])^2 + (\lambda_{js_j}[l])^2 + (\lambda_{js_1}[l])^2 + (\lambda_{js_2}[l])^2 + \ldots + (\lambda_{js_k}[l])^2 + \Theta_{\delta C}[l,l] \; ; (6.35)$$

$$\Sigma_r[l, l+1] = (\lambda_{jg}[l])(\lambda_{jg}[l+1]) + (\lambda_{js_j}[l])(\lambda_{js_j}[l+1]) +$$

$$(\lambda_{js_1}[l])(\lambda_{js_1}[l+1]) + ... + (\lambda_{js_k}[l])(\lambda_{js_k}[l+1]) \,. \qquad (6.36)$$

For items u and $u+1$ of X_v, it follows from equation 4.22 and the constraints above that:

$$\Sigma_r[p_j + u, p_j + u] = (\lambda_{vg}[u])^2 + (\lambda_{vs_v}[u])^2 +$$

$$(\lambda_{vs_1}[u])^2 + ... + (\lambda_{vs_k}[u])^2 + \Theta_{\delta C}[p_j + u, p_j + u] \,; \qquad (6.37)$$

$$\Sigma_r[p_j + u, p_j + u+1] = (\lambda_{vg}[u])(\lambda_{vg}[u+1]) + (\lambda_{vs_v}[u])(\lambda_{vs_v}[u+1]) +$$

$$(\lambda_{vs_1}[u])(\lambda_{vs_1}[u+1]) + ... + (\lambda_{vs_k}[u])(\lambda_{vs_k}[u+1]) \,. \qquad (6.38)$$

For items l of X_j and u of X_v,

$$\Sigma_r[l, p_j + u] = (\lambda_{js_1}[l])(\lambda_{vs_1}[u]) + ... + (\lambda_{js_k}[l])(\lambda_{vs_k}[u]) \,. \qquad (6.39)$$

Now, consider the two sets of loadings $\lambda_{js_1}[l]...\lambda_{js_k}[l]$ and $\lambda_{js_1}[l+1]...\lambda_{js_k}[l+1]$. Equation 6.36 is the only equation in which these two sets of loadings co-occur. However, note that even if $\lambda_{js_j}[l+1]$ and $\lambda_{js_1}[l+1]...\lambda_{js_k}[l+1]$ were known, there remain $k+2$ unknowns in this equation, i.e., $\lambda_{jg}[l]$, $\lambda_{js_j}[l]$ and $\lambda_{js_1}[l]...\lambda_{js_k}[l]$. This equation would not, therefore, solve for $\lambda_{js_1}[l]...\lambda_{js_k}[l]$. This is true for any two elements of X_j. Furthermore, because of the structure of equations 6.35-6.39, there is no manner in which such equations for elements of Σ_r may be combined to isolate elements of $\lambda_{js_1}...\lambda_{js_k}$, regardless of the

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values of k and p_j (the reader is encouraged to verify this). It follows that $\lambda_{js_1}...\lambda_{js_k}$ are not

identified. By equation 6.38 and the same logic, $\lambda_{vs_1}...\lambda_{vs_k}$ are not identified. Therefore,

neither Λ_r nor the model as a whole are identified.

6.4.3 **Model Fit and Parameter Estimation**

A prerequisite for valid determination of model fit and estimation of parameters is

model identification. However, the model is not identified, as shown in 6.3.2.

Furthermore, there are no additional, reasonable constraints that could be placed on the

model that would cause the model to be identified. Therefore, under the structural

equation model implied by Kang and MacDonald, procedures for estimating, or

performing hypothesis testing in relation to, $\kappa_P[2] - \kappa_C[2]$, are not available.

6.5 **Summary**

In this chapter we specified four models, each a sub-model of the Multi-

Population Linear Factor Model, one for each distinct proto-framework extracted in

Chapters 4 and 5. Each model was an extension of the corresponding proto-framework

and/or was defined by additional constraints. We showed: *i)* a structural equation model

consistent with the proto-framework of Chapman and Chapman is identified, and allows

for estimation of the parameter of interest, $\kappa_P[2] - \kappa_C[2]$, constituting a solution to the

inferential problems housed under the term "Psychometric Confound", as articulated by

Chapman and Chapman (as well as Salthouse and Coon) ; *ii)* under structural equation

models consistent with the other proto-frameworks articulated in Chapter 5 (of Baron and Treiman, Knight and Silverstein, Kang and MacDonald), the model is not identified, and, hence, an estimate of the parameter of interest is not available.

Chapter 7.

Summary and Conclusions

We now summarize the key points of the current work and clarify what, exactly, has been accomplished. We began by describing a general inferential problem (section 2.2): an investigator is interested in identifying whether a Specific Ability Deficit, or S-Deficit, exists in population P, however, the ability-score distributions $s_j|P$ and $s_j|C$ are not available, but only inferential information about the distributions $c_{lj}|P$ and $c_{lj}|C$. Strategies must therefore be invented that take, as input $c_{lj}|P$ and $c_{lj}|C$, and yield, as output, decisions about S-Deficits (there also exists the related inferential problem of identifying Differential Ability Deficits, or D-Deficits, without access to ability-score distributions $s_j|P$, $s_j|C$, $s_v|P$, and $s_v|C$).

We went on to describe claims made by Chapman and Chapman and the authors of alternative accounts to the effect that strategies traditionally employed to overcome the general inferential problem are confounded (see sections 2.3 and 3.3). In essence, it was claimed that the strategies and conclusions of numerous researchers, in particular those who had drawn inferences regarding S- or D-deficits, were faulty / false, as past investigators had failed to consider either: *i*) that their test composites scale not only in respect to the specific ability of interest, but also in respect to a general ability, which we have named g; and/or *ii*) that their test composites scale not only in respect to the specific

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ability of interest but also in respect to addition abilities $s_1 \ldots s_k$; and/or *iii*) that their test composites have differential sensitivity to g and/or $s_1 \ldots s_k$.

We described the inferential solutions proposed by Chapman and Chapman and the authors of the alternative accounts meant to overcome the confounds described (see sections 2.4 and 3.3). However, we noted that the claims of existence of the confounds, as well as the inferential solutions, were proposed by the Chapmans and other authors without initial establishment of an adequate technical framework (see sections 2.6, 3.3.5). Only incomplete proto-frameworks were extractable from the work of these authors. Therefore, despite the influence of the work of Chapman and Chapman and their commentators apropos the issue of the Psychometric Confound (as described in sections 3.1 and 3.2), we had illustrated, by the conclusion of Chapter 3, a pressing need for a proper technical articulation of the issues surrounding the term "Psychometric Confound".

We then established a basis for ruling on the truth of the claims of the Chapman and Chapman and the authors of the alternative accounts by asserting that these investigators implicitly employed classical and modern test theory. By articulating this test theory (sections 4.1, 4.2), and translating the proto-frameworks extracted and claims made into the theory's technical and mathematical language (sections 4.3, 4.4, chapter 5), we were able to adjudicate the claims made. We found that, under the technical frameworks established, the claims of the existence of confounds in past strategies for overcoming the general inferential problem were true, however, all inferential methods

proposed by Chapman and Chapman and the authors of the alternative accounts were faulty.

Employing the tools of structural equation modeling, we then (Chapter 6) determined whether valid methodological solutions for the drawing of conclusions regarding S-Deficits are possible. We found that the structural equation model consistent with the work of Chapman and Chapman (and also, Salthouse and Coon), when constrained in reasonable ways, is capable of producing estimates of the parameter of interest. However, structural equation models consistent with the proto-frameworks of other authors, as extracted in Chapter 5, were not capable of producing estimates of the parameter of interest.

Overall, then, the current work shows that if a technical framework, based on classical and modern test theory, and consistent with the proto-framework implied by the work of Chapman and Chapman, is held to be the case, then Psychometric Confound, as composed of a general inferential problem and sub-confounds, exists, but can be overcome through the use of modern test theory methods. In contrast, each of the alternative accounts of the Psychometric Confound implies a technical framework under which are unsolvable, the essential inferential problems housed under the term "Psychometric Confound".

It should be noted, of course, that our treatment of the problem stands or falls as a contribution to the underpinnings of psychopathology research, as stands or falls classical and modern test theory, itself. But if this be a defect (and it may well be), then it is a

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defect inherited from the discipline, itself, wherein the perhaps insidious involvement of classical and modern test theory, in forms both tacitly and explicitly articulated, is everywhere to be found. It is simply the case that psychopathology research, at least within the sub-areas considered in the current work, is tied, both historically and spiritually, to concepts drawn from classical and modern test theory. These latter constitute the lenses through which problems are viewed. This is not to say that other quantitative approaches are not possible, but simply that, as it stands, this theory lies at the conceptual core of the research area. Accordingly, core concepts such as "ability", "difficulty", and "discriminating power" depend, for their coherence, upon the reasonableness of classical and modern test theory. In offering our analysis of past work and articulating a solution, then, we have employed classical and modern test theory as a foundational framework. However, we have not dealt with the question of the usefulness or accuracy of classical and modern test theory as a description of particular scientific contexts. Therefore, the researcher is not relieved of the responsibility to justify, for particular phenomena under study, the relevance of our analysis and solution, tied inextricably, as they are, to these very particular quantitative presuppositions.

Table 1.   Summary of Conclusions Regarding Claims

Authors	Claims	True/False
Chapman and Chapman	CC1	T
	CC2	T
	CC3	F
	CC4	F
	CC5	F
	CC6	F
Baron and Treiman	BTC1	T
	BTC2	T
	BTC3	F
Salthouse and Coon	SC1	F
	SC2	F
Knight and Silverstein	KSC1	T
	KSC2	F
	KSC3	F
	KSC4	F
	KSC5	F
	KSC6	F
	KSC7	F
	KSC8	F
	KSC9	T
	KSC10	T
Kang and MacDonald	KMC1	T
	KMC2	T

Figure 1.   **Covariance Structure for Chapman and Chapman Model**
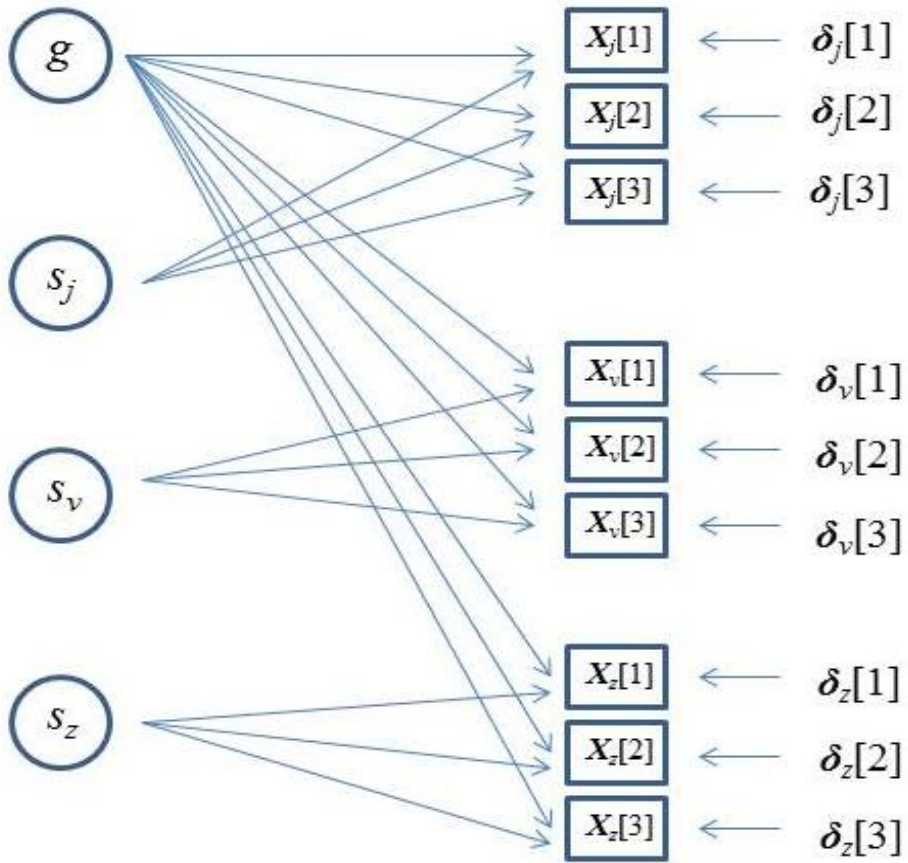
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