

The Optimal Payment Reduction Ratios for a Catastrophe Bond

by

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B. Econ., Shanghai Jiaotong University, 2012

Project Submitted in Partial Fulfillment
of the Requirements for the Degree of

Master of Science

in the

Department of Statistics and Actuarial Science
Faculty of Science

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SIMON FRASER UNIVERSITY

Spring 2015

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Abstract

Catastrophe bonds, also known as CAT bonds, are insurance-linked securities that help to transfer catastrophe risks from insurance industry to bond holders. If there is a catastrophe, the CAT bond is triggered and the future bond payments are reduced. This project first presents a general pricing formula for a CAT bond with coupon payments, which can be adapted to various assumptions for a catastrophe loss process. Next, it gives formulas for the optimal payment reduction ratios which maximize two measurements of risk reduction, hedge effectiveness rate (HER) and hedge effectiveness (HE), respectively, and examines how the optimal payment reduction ratios help reinsurance or insurance companies to mitigate extreme catastrophe losses. Last, it shows how strike price, maturity, parameters of the catastrophe loss process and different interest rate assumptions affect the optimal payment reduction ratios. Numerical examples are also given for illustrations.

To my parents!

Acknowledgments

I would like to extend my sincere gratitude to my supervisor Dr. Cary Tsai. It has been such an privilege for me to be mentored by a man with great industry experience, solid technical background, especially in insurance-linked derivatives, and unlimited patience and kindness. This project would not be completed without his insightful guidance. Also, I would like to express my gratitude to the Department of Statistics and Actuarial Science at the Simon Fraser University for providing not only just knowledge, but also support and encouragement in the past two years.

I want to say thank you personally to Dr. Gary Parker and Dr. Liangliang Wang, too, for their prudent and thorough reviews on this project. I am also grateful to Dr. Yi Lu for providing me with the knowledge about loss models in her class, and for offering me industrial contacts that eventually lead to a full-time job.

Last, I would like to thank my friends and family for their unconditional love.

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Chapter 1

Introduction

1.1 Introduction to catastrophe bond

1.1.1 How CAT bond works

Catastrophe bonds, also known as CAT bonds, are insurance-linked securities that help to transfer catastrophe risks from insurance industry to bond holders. They were first issued in the mid-1990s by the American Insurance Group (Laster, 2001) and were used in the aftermath of Hurricane Andrew and the Northridge earthquake (as described in Wikipedia). In a typical case, an insurer or a reinsurer would set up a special purpose entity with investment banks. This entity would sell CAT bonds to investors like pension funds and set the proceeds aside. If there is a catastrophe, the bond is triggered. Future bond payment would be reduced and some of the proceeds would be used to pay for the damages; if no catastrophe occurs, the principle would go back to the investors just like a normal bond.

1.1.2 Overview of the CAT bond market

Once considered an exotic investment for specialists, CAT bonds are increasingly going mainstream these days (Chen, 2014). Strong demand from investors and growing supply from insurance and reinsurance companies are driving the growth of this market. According to data provider Artemis, the total outstanding CAT bond market is at around \$22 billion now, up from \$0 two decades ago. The market has been growing exponentially and picked up quickly after the global financial crisis in 2008 with an annual growth rate of outstanding

bonds at around 8%; in 2014 alone, around \$6.3 billion CAT bonds and other insurance-related securities were issued according to the statistics from Artemis.

On the supply side, increasing frequency and severity of catastrophes, especially in North America, is pushing risk takers from insurers to reinsurers to seek securitization of their risks in the financial market, pushing up supply of CAT bonds and other catastrophe related derivatives. A Munich Re study titled "Severe weather in North America", showed the number of weather-related loss events in North America almost quintupled in the past three decades. Also, as the CAT bond market matures and fees decrease, more and more companies are likely to follow this trend.

On the demand side, CAT bond is attractive to their investors in three ways. First, it serves as a great tool for alternative investments because it provides returns uncorrelated with the stock market and beyond control of market speculations. Second, CAT bonds generate market-beating yields, much higher than the record-low interest rate. The spread is now at about 4.7% according to John Seo, managing principal at Fermat Capital Management LLC. Third, more and more parties are turning to CAT bonds to hedge against extreme risks. Reinsurance companies are still the main players of the market with Swiss Re leading the underwriter list, while many insurance companies issue CAT bonds through special purpose vehicles like investment banks. In addition, some insurers are seeking insurance protection directly from the CAT bond market. Metropolitan Transportation Authority in New York issued a USD 200 million CAT bond in 2013 to fix its dark and flooded subway tunnels in case of another Hurricane Sandy (Catastrophe Bond: Perilous Paper, The Economist Oct 5th 2013). These new players would drive up demand not only in the short term as they poured into the market, but also in the long-term. More bond issuance tends to attract more competition to the underwriting market, thus tempering consulting fees in the long run.

This strong demand for CAT bonds is likely to grow further in the future. Institutions like pension funds, endowments and sovereignty wealth funds are allocating more and more assets to CAT bonds (Brendan Greeley, Pension Funds and Catastrophe Bonds: What Could Possibly Go Wrong? Bloomberg Business Week). According to Swiss Re, 14% of direct investment into CAT bonds comes from pension funds in 2012, rising from 0% a decade ago.

1.1.3 Trigger types of CAT bond

There are four types of CAT bond triggers, indemnity, modeled loss, industry loss, and parametric or parametric index. In case of indemnity, a bond is triggered by insurers' actual losses. This helps to mitigate company-specific risks, but would arouse the issue of moral hazard because companies have the very incentive to either overstate their losses to reduce bond payment or issue policies in high-risk areas since losses would ultimately be transferred to bond holders. In case of modeled loss, a bond is triggered by losses modeled by a third-party. When a catastrophe occurs, actual parameters of the catastrophe are plugged into the model to generate modeled loss. For industry loss trigger, the aggregate industry loss is used to determine the trigger of the bond. Data providers like Insurance Service Office of Verisk Inc. collect loss data from insurers to compile industry data. Industrial trigger helps to tackle moral hazards. Moreover, it helps insurers to avoid detailed loss information disclosure to their competitors (Ma and Ma (2013)). However, there is also a trade-off. As the trigger is not company specific, insurers are still exposed to basis risks when their losses differ significantly from industrial losses. The last type of trigger is parametric trigger. Specific parameters of a peril are chosen to serve as bond trigger. However, as the correlation between the chosen parameter and the actual loss varies case by case, many insurers are uncomfortable with using parametric trigger alone.

The model introduced in this project is based on industrial loss trigger. It can also be easily adapted to indemnity trigger by substituting loss parameters from industry parameters to company parameters and setting loss share to one. The loss triggered CAT bond is chosen in this project mainly due to its popularity; nowadays, company loss and industrial loss triggered bonds add up to around 75% of the market according to the statistics from Artemis.

1.2 Motivation

Measuring effectiveness of risk reduction is critical to both insurers and reinsurers. For insurers, effective measurements of risk reduction help them to make the best hedging decision. Underestimating risk reduction effects could lead to over-purchase of reinsurance or other risk-hedging products, adding unnecessary hedging costs to their products.

These extra costs would either be transferred to the policy owners eventually, which results in lost of market share to their competitors, or be borne by themselves, which leads to less profits. For reinsurers, measuring risk reduction of their clients is key to their own pricing excellence. With effective measurements, reinsurers would be able to quantify their contributions to the insurers, and charge a reasonable price for taking the risks.

Therefore, to keep up with the rising popularity of CAT bonds, new measurements of risk reduction tailored to the design of CAT bonds should be introduced. For insurers, these measurements would make it possible to compare CAT bonds with other traditional hedging solutions, and make the best decision. For reinsurers seeking securitization of their own risks, these measurements provide valuable information on how effective and efficient CAT bonds work. Moreover, informative measurements will introduce more transparency to the reinsurance market, drive down reinsurance price, lead to less premium and benefit policy holders in the long run.

Furthermore, effective quantification of risk reduction allows further study on pricing optimization. Though some efforts were devoted to the parameter optimization problems of life insurance-linked derivatives under some risk reduction measurements, there is no literature on parameter optimization problems related to catastrophe bonds yet. This project aims to find the optimal payment reduction ratios that maximize two key risk reduction measurements, the variance reduction amount per dollar spent on hedging, and the variance reduction ratio, respectively.

1.3 Outlines

This project is organized as follows. Chapter 2 reviews the previous work on models, approaches and limitations of pricing of CAT bonds. Chapter 3 first introduces notations and assumptions of the catastrophe loss process for the catastrophe bond in the project. Then, the expressions of the optimal payment reduction ratio of a CAT bond that maximize hedge effectiveness rate (HER) and hedge effectiveness (HE), respectively, are derived. In addition, we apply the results to a zero-coupon bond as an example. In Chapter 4, numerical experiments through Monte Carlo simulations are conducted to test the results obtained in Chapter 3 for illustrations. Also, effects of various parameters in the pricing model on the

optimal payment reduction ratios, such as strike price, maturity and parameters of the loss process, are examined.

Chapter 2

Literature review

Despite the rising popularity, limited research is devoted to CAT bond pricing. Among the current pricing literature, most are devoted to different approaches to modelling catastrophes or interest rates, and few has attempted to quantify risk reduction for CAT bonds. Furthermore, there is no literature on the parameter optimization based on risk reduction or any other measurement yet.

On the pricing side, different approaches were used to price CAT bonds. A few approaches assume stochastic processes with discrete time. Cox and Pederson (2000) proposed a CAT bond valuation model within the framework of representative agent equilibrium based on a model of the term structure of interest rates and a probability structure for catastrophe risk.

Under the assumption of continuous time, there are two main approaches. One approach is to follow the methodology of pricing credit derivatives in finance like defaultable bonds. These papers tend to model the probability of trigger directly by a stochastic process similar to modelling the probability of credit default. Baryshnikov, Mayo and Taylor (1998) presented an arbitrage-free solution to the pricing of zero-coupon and non-zero coupon CAT bonds under conditions of continuous trading through modelling the trigger probability directly by a compound doubly stochastic Poisson process. Burnecki and Kukla (2003) applied their results with PCS (Property Claim Service, ISO) data to calculate non-arbitrage prices of a CAT bond. Jarrow (2010) proposed a simple close form solution for valuing CAT bonds under LIBOR term structure of interest rates. The solution, applicable

under any arbitrage-free model, is based on a reduced model in the pricing literature of financial credit derivatives.

Another popular approach under continuous time is the actuarial approach. It uses an aggregate loss process to model the probability of trigger for industry loss triggered CAT bonds. Vaugirard (2003) developed a pricing model by an arbitrage approach through modelling the trigger process by an aggregate loss model based on a homogeneous compound Poisson process. Ma and Ma (2013) proposed a similar model based on a nonhomogeneous compound Poisson process. A mixed approximation method is also used to simplify the distribution of aggregate loss and estimate the trigger probability accordingly. Nowak and Romaniuk (2013) expanded Vaugirard's model and derived a general pricing formula for CAT bonds with a stepwise payoff and different interest rate dynamics. In addition to the two approaches above, Egami and Young (2008) proposed a method of pricing structured CAT bonds based on the assumption of utility indifference.

Although quite a few academic articles are devoted to quantifying risk reduction in life insurance-related securities, there is few such literature on CAT bonds. Lee and Yu (2007) proposed a contingent-claim framework for valuing reinsurance contract by modeling asset and liability, respectively, and examined how CAT bonds help to reduce the default risk of a reinsurance company. In this project, we introduce two risk reduction measurements, hedge effectiveness (HE) and hedge effectiveness rate (HER), used in mortality-linked securities, and study the parameter optimization problem based on these measurements. Hedge effectiveness (HE), for comparing the variance reduction ratios, was proposed by Li and Hardy (2011) and Li and Luo (2012) to measure effectiveness in hedging longevity risks. To further incorporate hedge cost into the measurement, Tsai and Yang (2015) proposed hedge effectiveness rate (HER) to quantify risk reduction amount per dollar spent in mortality and longevity hedging.

This project make a threefold contribution to the literature of catastrophe bond pricing. First, it gives a general price formula for non-zero coupon CAT bonds applicable under various assumptions of catastrophe loss processes. Though Vaugirard (2003) and Nowak and Romaniuk (2013) mentioned non-zero coupon CAT bonds in their literature, no specific pricing formula was given. This project completes the literature. Second, it defines two key

risk reduction measurements for catastrophe bonds. It introduces hedge effectiveness rate (HER), measuring the variance reduction amount per dollar spent on hedging, and hedge effectiveness (HE), measuring the variance reduction ratio for a catastrophe bond. These measurements not only quantify the effects of CAT bonds, but also make it possible to compare CAT bonds with other traditional hedging solutions. Third, it proves formulas for the optimal payment reduction ratios that maximize HER or HE, respectively. In the current literature, the payment reduction ratio is predetermined and treated as a constant. Rule of thumb knowledge indicates a higher risk should correspond to a higher payment reduction ratio when a CAT bond is triggered, but no precise guidance was ever given to help issuers of CAT bonds make the decision. This project completes the studies in CAT bond pricing and provides a general formula to calculate the optimal payment reduction ratio. It is applicable to both insurers and reinsurers, and can be applied under various catastrophe loss process.

Chapter 3

The model

This chapter first introduces notations and assumptions of the catastrophe loss process for the catastrophe bond defined in the project. It then derives the expressions for the optimal payment reduction ratios that maximizes hedge effectiveness rate (HER) and hedge effectiveness (HE), respectively. Last, it applies the results to a zero-coupon CAT bond and gives the explicit formulas for HER and HE as well.

3.1 Notations and assumptions

The following section gives notations and assumptions of the CAT bond defined in this project and the catastrophe process that triggers the bond.

3.1.1 Catastrophe loss process

Definition 1. We describe the catastrophe loss process with the following loss frequency and severity models.

1. **Frequency:**

- (a) **Number of catastrophe:** let $N(t)$ be a random variable representing the total number of catastrophe events before time t .
- (b) **Occurrence time:** define T_k as the occurrence time of the k^{th} catastrophe loss.

2. **Severity:** define X_k as the size of the k^{th} industry catastrophe loss occurred at time T_k . Industry catastrophe loss X_k refers to the sum of losses from all insurance companies caused by the k^{th} specific catastrophe.
3. **Aggregate loss:** define $L(t)$ as the aggregate industry catastrophe loss up to time t .
 - (a) $L(t) = \sum_{i=1}^{N(t)} X_i$ with $L(t) = 0$ when $N(t) = 0$.
 - (b) Let the cumulative distribution function of $L(t)$ be $F_{L(t)}$.
 - (c) Let $F_{L(t)|k}$ denote the conditional cumulative distribution function given k CAT losses occurred by time t .
4. **Independence:** we assume the CAT loss frequency $N(t)$ and the CAT loss severities $X_1, X_2 \dots$ are independent.
5. **Discounted aggregate loss:** let $Z(T)$ and $Z^*(T)$ be an insurance or a reinsurance company's total discounted aggregate losses before and after issuing a CAT bond maturing at time T , respectively.
 - (a) Assume the catastrophe loss process and the interest rate process are independent. This assumption was made by Vaugirard (2003) by assuming that investors recognize natural catastrophe risks are not correlated to financial risks and can be diversified away.
 - (b) Denote d as the insurance company's retention on the total loss on multiple policies in one catastrophe event (for example, earthquake) specified on a non-proportional reinsurance contract.
 - (c) Define m as the proportion of an industrial CAT loss an insurance company bears. We use the market share to estimate m in numerical experiments.
 - (d) Let δ denote the real constant force of interest rate;
 - (e) $Z(T) = \sum_{i=1}^{N(T)} e^{-\delta T_i} (mX_i - d)_+$ based on the assumptions above where $d > 0$ and $d = 0$ apply to the issuance of CAT bonds by reinsurance and insurance companies, respectively.
 - (f) The total discounted aggregate loss after issuing a CAT bond, $Z^*(T)$, is equal to the present value of the aggregate catastrophe loss paid to policy holders

subtracts the present value of the proceeds from selling the CAT bond plus the present value of the future payments of the CAT bond. Denote the bond price at time zero as P_0 . Then we will have $Z^*(T) = Z(T) - P_0 + A$.

3.1.2 Catastrophe bond

Definition 2. The CAT bond in this project satisfies the following assumptions:

1. **Maturity:** the CAT bond matures at time T .
2. **Coupon payments:** let C be the value of each coupon payment, and $n - 1$ be the total number of coupon payments before maturity.
3. **Face value:** let FV denote the face value of the CAT bond.
4. **Expense:** we assume the total expense issuing the CAT bond is proportional to its face value, that is, $\text{expense} = FV * \zeta$.
5. **Trigger condition:** the CAT bond is triggered when the aggregate industrial catastrophe loss, $L(t)$, goes over K before maturity, that is, $L(t) = \sum_{i=1}^{N(t)} X_i > K$ for some $t \leq T$. Here we use the total industrial loss to reduce moral hazards. It also helps to avoid detailed information disclosure of related insurance companies to their competitors (Ma and Ma (2013)).
6. **Trigger time:** let τ denote the time when $L(t)$ goes over K for the first time.
7. **Price:** Denote P_0 as the CAT bond price at time zero.
8. **Trigger payoff:** denote A as the present value of the total future payments of the CAT bond.
 - (a) If $\tau > T$, bond holders receive the original coupons and face value.
 - (b) If $\tau \leq T$, bond holders receive ω ($0 \leq \omega \leq 1$) times the original value of all the future payments due after τ , where $(1 - \omega)$ is the payment reduction ratio. Intuitively, it measures the leverage of a CAT bond hedging against catastrophe losses. A zero payment reduction refers to no leverage of hedging and a 100 % payment reduction refers to a full leverage of hedging.
 - (c) Based on (a) and (b), we have the present value of the payments of a CAT bond

as

$$A = \begin{cases} A_1 = \omega \sum_{i=1}^{n-1} C e^{-\frac{i\delta T}{n}} + \omega F V e^{-\delta T}, & 0 < \tau \leq \frac{T}{n}, \\ A_2 = C e^{-\frac{\delta T}{n}} + \omega \sum_{i=2}^{n-1} C e^{-\frac{i\delta T}{n}} + \omega F V e^{-\delta T}, & \frac{T}{n} < \tau \leq \frac{2T}{n}, \\ \vdots, \\ A_h = \sum_{i=1}^{h-1} C e^{-\frac{i\delta T}{n}} + \omega \sum_{i=h}^{n-1} C e^{-\frac{i\delta T}{n}} + \omega F V e^{-\delta T}, & \frac{(h-1)T}{n} < \tau \leq \frac{hT}{n}, \\ \vdots \\ A_n = \sum_{i=1}^{n-1} C e^{-\frac{i\delta T}{n}} + \omega F V e^{-\delta T}, & \frac{(n-1)T}{n} < \tau \leq T, \\ A_{n+1} = \sum_{i=1}^{n-1} C e^{-\frac{i\delta T}{n}} + F V e^{-\delta T}, & \tau > T, \end{cases}$$

where $h = 1, 2, \dots, n$ and $\sum_{i=n_1}^{n_2} C e^{-\frac{i\delta T}{n}} = 0$ for $n_1 > n_2$.

9. Assume investors are neutral towards catastrophe risks. As a result, the price of the CAT bond is the expectation of the future payments of the CAT bond.

3.2 HE and HER

The following section derives expressions for ω^* and ω^{**} that maximize HER and HE, respectively. It is organized as follows. First, we derive the expressions for trigger probabilities, $E(A)$, $[E(A)]^2$, $E(A^2)$ and $E[A \times Z(T)]$. Then we express HE and HER in terms of $E(A)$, $[E(A)]^2$, $E(A^2)$ and $E[A \times Z(T)]$. Last, we calculate ω^* and ω^{**} by taking the first derivative of HER and HE with respect to ω , respectively, and set them to zero.

Definition 1. HER and HE are defined by

$$HER = \frac{Var[Z(T)] - Var[Z^*(T)]}{(1 + \zeta)P_0}$$

and

$$HE = \frac{Var[Z(T)] - Var[Z^*(T)]}{Var[Z(T)]}.$$

We can see from the definition above that HE is the variance reduction ratio, and HER is the variance reduction amount per dollar spent on hedging. Next, we define $\rho(\frac{(h-1)T}{n}, \frac{hT}{n})$

as the probability that a CAT bond is triggered between the $(h - 1)^{th}$ and h^{th} coupon payments. For $h = 1$, $\rho(0, \frac{T}{n})$ stands for the probability that a CAT bond is triggered before the first coupon payment; for $h = n - 1$, $\rho(\frac{(n-1)T}{n}, T)$ denotes the probability that a CAT bond is triggered between the last coupon payment and maturity. We use $\rho(T, \infty)$ to denote the probability that a CAT bond is not triggered before maturity.

Lemma 1. An expression for $\rho\left(\frac{(h-1)T}{n}, \frac{hT}{n}\right)$ is given by

$$\rho\left(\frac{(h-1)T}{n}, \frac{hT}{n}\right) = \sum_{k=0}^{\infty} F_{L(\frac{(h-1)T}{n})|k}(K) \times \left\{ P\left[N\left(\frac{(h-1)T}{n}\right) = k\right] - P\left[N\left(\frac{hT}{n}\right) = k\right] \right\}.$$

Proof.

We first calculate $\rho\left(0, \frac{hT}{n}\right)$ by conditioning on $N\left(\frac{hT}{n}\right)$ as

$$\begin{aligned} \rho\left(0, \frac{hT}{n}\right) &= \Pr\left[L\left(\frac{hT}{n}\right) = \sum_{i=1}^{N\left(\frac{hT}{n}\right)} X_i > K\right] \\ &= \sum_{k=0}^{\infty} \left(1 - F_{L(\frac{hT}{n})|k}(K)\right) \times P\left[N\left(\frac{hT}{n}\right) = k\right] \\ &= \sum_{k=0}^{\infty} P\left[N\left(\frac{hT}{n}\right) = k\right] - \sum_{k=0}^{\infty} F_{L(\frac{hT}{n})|k}(K) \times P\left[N\left(\frac{hT}{n}\right) = k\right] \\ &= 1 - \sum_{k=0}^{\infty} F_{L(\frac{hT}{n})|k}(K) \times P\left[N\left(\frac{hT}{n}\right) = k\right]. \end{aligned}$$

Similarly,

$$\rho\left(0, \frac{(h-1)T}{n}\right) = 1 - \sum_{k=0}^{\infty} F_{L(\frac{(h-1)T}{n})|k}(K) \times P\left[N\left(\frac{(h-1)T}{n}\right) = k\right].$$

Therefore,

$$\begin{aligned} &\rho\left(\frac{(h-1)T}{n}, \frac{hT}{n}\right) \\ &= \rho\left(0, \frac{hT}{n}\right) - \rho\left(0, \frac{(h-1)T}{n}\right) \\ &= \sum_{k=0}^{\infty} F_{L(\frac{(h-1)T}{n})|k}(K) \times \left\{ P\left[N\left(\frac{(h-1)T}{n}\right) = k\right] - P\left[N\left(\frac{hT}{n}\right) = k\right] \right\}. \end{aligned}$$

Lemma 2. The CAT bond price at time zero is $P_0 = E(A) = C_{E(A), \omega} \times \omega + C_{E(A), \omega^0}$, where

$$C_{E(A), \omega} = \sum_{j=1}^n \left[\sum_{i=j}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right] \rho \left(\frac{(j-1)T}{n}, \frac{jT}{n} \right)$$

and

$$\begin{aligned} C_{E(A), \omega^0} &= \sum_{j=1}^n \left[\sum_{i=1}^{j-1} C \times e^{-\frac{i\delta T}{n}} \right] \rho \left(\frac{(j-1)T}{n}, \frac{jT}{n} \right) \\ &\quad + \left[\sum_{i=1}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right] \rho(T, \infty). \end{aligned}$$

Proof.

$$\begin{aligned} E(A) &= E[E(A|\tau)] \\ &= \left[\sum_{i=1}^{n-1} \omega C \times e^{-\frac{i\delta T}{n}} + \omega FV \times e^{-\delta T} \right] \rho(0, \frac{T}{n}) \\ &\quad + \left[C \times e^{-\frac{\delta T}{n}} + \sum_{i=2}^{n-1} \omega C \times e^{-\frac{i\delta T}{n}} + \omega FV \times e^{-\delta T} \right] \rho(\frac{T}{n}, \frac{2T}{n}) \\ &\quad + \dots + \left[\sum_{i=1}^{n-1} C \times e^{-\frac{i\delta T}{n}} + \omega FV \times e^{-\delta T} \right] \rho(\frac{(n-1)T}{n}, T) \\ &\quad + \left[\sum_{i=1}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right] \rho(T, \infty) \\ &= \sum_{j=1}^n \left[\sum_{i=1}^{j-1} C \times e^{-\frac{i\delta T}{n}} + \sum_{i=j}^{n-1} \omega C \times e^{-\frac{i\delta T}{n}} + \omega FV \times e^{-\delta T} \right] \\ &\quad \times \rho \left(\frac{(j-1)T}{n}, \frac{jT}{n} \right) + \left[\sum_{i=0}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right] \rho(T, \infty) \\ &= \sum_{j=1}^n A_j \times \rho \left(\frac{(j-1)T}{n}, \frac{jT}{n} \right) + A_{n+1} \times \rho(T, \infty) \end{aligned}$$

Let $E(A) = C_{E(A), \omega} \times \omega + C_{E(A), \omega^0}$, where we use $C_{E(A), \omega}$ to denote the coefficient of ω in the expression of $E(A)$. Then

$$C_{E(A), \omega} = \sum_{j=1}^n \left[\sum_{i=j}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right] \rho \left(\frac{(j-1)T}{n}, \frac{jT}{n} \right)$$

and

$$\begin{aligned} C_{E(A), \omega^0} &= \sum_{j=1}^n \left[\sum_{i=1}^{j-1} C \times e^{-\frac{i\delta T}{n}} \right] \rho \left(\frac{(j-1)T}{n}, \frac{jT}{n} \right) \\ &\quad + \left[\sum_{i=1}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right] \rho(T, \infty). \end{aligned}$$

Similarly, we can derive an expression for $[E(A)]^2$.

Lemma 3. $[E(A)]^2 = C_{[E(A)]^2, \omega^2} \times \omega^2 + C_{[E(A)]^2, \omega} \times \omega + C_{[E(A)]^2, \omega^0}$ where

$$\begin{aligned} C_{[E(A)]^2, \omega^2} &= (C_{E(A), \omega})^2, \\ C_{[E(A)]^2, \omega} &= 2C_{E(A), \omega} \times C_{E(A), \omega^0}, \end{aligned}$$

and

$$C_{[E(A)]^2, \omega^0} = (C_{E(A), \omega^0})^2.$$

Proof.

Let $[E(A)]^2 = C_{[E(A)]^2, \omega^2} \times \omega^2 + C_{[E(A)]^2, \omega} \times \omega + C_{[E(A)]^2, \omega^0}$. Since

$$\begin{aligned} [E(A)]^2 &= (C_{E(A), \omega} \times \omega + C_{E(A), \omega^0})^2 \\ &= (C_{E(A), \omega})^2 \times \omega^2 + 2C_{E(A), \omega} \times C_{E(A), \omega^0} \times \omega + (C_{E(A), \omega^0})^2, \end{aligned}$$

we have

$$\begin{aligned} C_{[E(A)]^2, \omega^2} &= (C_{E(A), \omega})^2, \\ C_{[E(A)]^2, \omega} &= 2C_{E(A), \omega} \times C_{E(A), \omega^0}, \end{aligned}$$

and

$$C_{[E(A)]^2, \omega^0} = (C_{E(A), \omega^0})^2.$$

Lemma 4. $E(AZ) = C_{E(AZ), \omega} \times \omega + C_{E(A^2), \omega^0}$, where $Z = Z(T)$,

$$C_{E(AZ), \omega} = \sum_{j=1}^n \left[\sum_{i=j}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right] E \left[Z \left| \frac{(j-1)T}{n} < \tau \leq \frac{jT}{n} \right. \right] \\ \times \rho \left(\frac{(j-1)T}{n}, \frac{jT}{n} \right)$$

and

$$C_{E(AZ), \omega^0} = \sum_{j=1}^n \left[\sum_{i=1}^{j-1} C \times e^{-\frac{i\delta T}{n}} \right] E \left[Z \left| \frac{(j-1)T}{n} < \tau \leq \frac{jT}{n} \right. \right] \rho \left(\frac{(j-1)T}{n}, \frac{jT}{n} \right) \\ + \left[\sum_{i=1}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right] E(Z|\tau > T) \rho(T, \infty).$$

Proof.

$$E(AZ) = E[E(AZ|\tau)] \\ = E[A \times E(Z|\tau)] \\ = A_1 E \left(Z \left| 0 < \tau \leq \frac{T}{n} \right. \right) \rho \left(0, \frac{T}{n} \right) + A_2 E \left(Z \left| \frac{T}{n} < \tau \leq \frac{2T}{n} \right. \right) \rho \left(\frac{T}{n}, \frac{2T}{n} \right) + \dots \\ + A_n E \left(Z \left| \frac{(n-1)T}{n} < \tau \leq T \right. \right) \rho \left(\frac{(n-1)T}{n}, T \right) + A_{n+1} E(Z|\tau > T) \rho(T, \infty) \\ = \sum_{j=1}^n A_j E \left(Z \left| \frac{(j-1)T}{n} < \tau \leq \frac{jT}{n} \right. \right) \rho \left(\frac{(j-1)T}{n}, \frac{jT}{n} \right) + A_{n+1} E(Z|\tau > T) \rho(T, \infty) \\ = \sum_{j=1}^n \left[\sum_{i=1}^{j-1} C \times e^{-\frac{i\delta T}{n}} + \sum_{i=j}^{n-1} \omega C \times e^{-\frac{i\delta T}{n}} + \omega FV \times e^{-\delta T} \right] \\ \times E \left(Z \left| \frac{(j-1)T}{n} < \tau \leq \frac{jT}{n} \right. \right) \rho \left(\frac{(j-1)T}{n}, \frac{jT}{n} \right) \\ + \left[\sum_{i=0}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right] E(Z|\tau > T) \rho(T, \infty).$$

Let $E(AZ) = C_{E(AZ), \omega} \times \omega + C_{E(A^2), \omega^0}$. Then

$$C_{E(AZ), \omega} = \sum_{j=1}^n \left[\sum_{i=j}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right] E \left(Z \left| \frac{(j-1)T}{n} < \tau \leq \frac{jT}{n} \right. \right) \\ \times \rho \left(\frac{(j-1)T}{n}, \frac{jT}{n} \right)$$

and

$$C_{E(AZ), \omega^0} = \sum_{j=1}^n \left[\sum_{i=1}^{j-1} C \times e^{-\frac{i\delta T}{n}} \right] E \left(Z \left| \frac{(j-1)T}{n} < \tau \leq \frac{jT}{n} \right. \right) \rho \left(\frac{(j-1)T}{n}, \frac{jT}{n} \right) \\ + \left[\sum_{i=1}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right] E(Z|\tau > T) \rho(T, \infty).$$

Based on the definition of A , we can derive an expression for $E(A^2)$.

Lemma 5. $E(A^2) = C_{E(A^2), \omega^2} \times \omega^2 + C_{E(A^2), \omega} \times \omega + C_{E(A^2), \omega^0}$ where

$$C_{E(A^2), \omega^2} = \sum_{j=1}^n \left[\sum_{i=j}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right]^2 \rho \left(\frac{(j-1)T}{n}, \frac{jT}{n} \right), \\ C_{E(A^2), \omega} = 2 \sum_{j=1}^n \left[\sum_{i=1}^{j-1} C \times e^{-\frac{i\delta T}{n}} \right] \left[\sum_{i=j}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right] \\ \times \rho \left(\frac{(j-1)T}{n}, \frac{jT}{n} \right),$$

and

$$C_{E(A^2), \omega^0} = \sum_{j=1}^n \left[\sum_{i=1}^{j-1} C \times e^{-\frac{i\delta T}{n}} \right]^2 \rho \left(\frac{(j-1)T}{n}, \frac{jT}{n} \right) \\ + \left[\sum_{i=1}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right]^2 \rho(T, \infty).$$

Proof.

$$\begin{aligned}
E(A^2) &= E[E(A^2|\tau)] \\
&= \sum_{j=1}^n (A_j)^2 \rho\left(\frac{(j-1)T}{n}, \frac{jT}{n}\right) + (A_{n+1})^2 \rho(T, \infty) \\
&= \sum_{j=1}^n \left[\sum_{i=1}^{j-1} C \times e^{-\frac{i\delta T}{n}} + \sum_{i=j}^{n-1} \omega C \times e^{-\frac{i\delta T}{n}} + \omega FV \times e^{-\delta T} \right]^2 \\
&\quad \times \rho\left(\frac{(j-1)T}{n}, \frac{jT}{n}\right) + \left[\sum_{i=1}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right]^2 \rho(T, \infty).
\end{aligned}$$

Let $E(A^2) = C_{E(A^2), \omega^2} \times \omega^2 + C_{E(A^2), \omega} \times \omega + C_{E(A^2), \omega^0}$. Then

$$\begin{aligned}
C_{E(A^2), \omega^2} &= \sum_{j=1}^n \left[\sum_{i=j}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right]^2 \rho\left(\frac{(j-1)T}{n}, \frac{jT}{n}\right), \\
C_{E(A^2), \omega} &= 2 \sum_{j=1}^n \left[\sum_{i=1}^{j-1} C \times e^{-\frac{i\delta T}{n}} \right] \left[\sum_{i=j}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right], \\
&\quad \times \rho\left(\frac{(j-1)T}{n}, \frac{jT}{n}\right),
\end{aligned}$$

and

$$\begin{aligned}
C_{E(A^2), \omega^0} &= \sum_{j=1}^n \left[\sum_{i=1}^{j-1} C \times e^{-\frac{i\delta T}{n}} \right]^2 \rho\left(\frac{(j-1)T}{n}, \frac{jT}{n}\right) \\
&\quad + \left[\sum_{i=1}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right]^2 \rho(T, \infty).
\end{aligned}$$

Theorem 1. Hedge effectiveness rate (HER) = $\frac{2E(A)E[Z(T)] + [E(A)]^2 - E(A^2) - 2E[A \times Z(T)]}{(1+\zeta)E(A)}$.

Proof.

Recall that

$$HER = \frac{\text{variance reduction}}{\text{hedge cost}} = \frac{\text{Var}[Z(T)] - \text{Var}[Z^*(T)]}{(1+\zeta)P_0}$$

where

$Z^*(T)$ = the present value of the aggregate catastrophe loss
 – the present value of the proceeds from selling the CAT bond
 + the present value of the future payments of the CAT bond
 $= Z(T) - P_0 + A$.

Since $P_0 = E(A)$, we have

$$\begin{aligned}
 E\{[Z^*(T)]^2\} &= E\{[Z(T) - E(A) + A]^2\} \\
 &= E\{[Z^*(T)]^2\} + [E(A)]^2 + E(A^2) - 2E(A) \times E[Z(T)] + 2E[A \times Z(T)] - 2[E(A)]^2 \\
 &= E\{[Z^*(T)]^2\} - [E(A)]^2 + E(A^2) - 2E(A) \times E[Z(T)] + 2E[A \times Z(T)].
 \end{aligned}$$

Under no arbitrage condition, we also have

$$E[Z^*(T)] = E[Z(T) - P_0 + A] = E[Z(T)] - P_0 + E(A) = E[Z(T)].$$

Therefore,

$$\begin{aligned}
 &Var[Z(T)] - Var[Z^*(T)] \\
 &= E[Z(T)^2] - \{E[Z(T)]\}^2 - E[Z^*(T)^2] - \{E[Z^*(T)]\}^2 \\
 &= 2E(A)E[Z(T)] + [E(A)]^2 - E(A^2) - 2E[A \times Z(T)].
 \end{aligned}$$

Corollary 1. Both HER and HE are zero at $\omega = 1$.

Proof.

Recall that

$$A = \begin{cases} A_1 = \omega \sum_{i=1}^{n-1} C e^{-\frac{i\delta T}{n}} + \omega F V e^{-\delta T}, & 0 < \tau \leq \frac{T}{n}, \\ A_2 = C e^{-\frac{\delta T}{n}} + \omega \sum_{i=2}^{n-1} C e^{-\frac{i\delta T}{n}} + \omega F V e^{-\delta T}, & \frac{T}{n} < \tau \leq \frac{2T}{n}, \\ \vdots, \\ A_h = \sum_{i=1}^{h-1} C e^{-\frac{i\delta T}{n}} + \omega \sum_{i=h}^{n-1} C e^{-\frac{i\delta T}{n}} + \omega F V e^{-\delta T}, & \frac{(h-1)T}{n} < \tau \leq \frac{hT}{n}, \\ \vdots \\ A_n = \sum_{i=1}^{n-1} C e^{-\frac{i\delta T}{n}} + \omega F V e^{-\delta T}, & \frac{(n-1)T}{n} < \tau \leq T, \\ A_{n+1} = \sum_{i=1}^{n-1} C e^{-\frac{i\delta T}{n}} + F V e^{-\delta T}, & \tau > T. \end{cases}$$

When $\omega = 1$, we have

$$A = \sum_{i=1}^{n-1} C e^{-\frac{i\delta T}{n}} + F V e^{-\delta T}$$

for all $\tau > 0$. Consequently, A becomes a constant. Since

$$\text{Var}[Z^*(T)] = \text{Var}[Z(T) - P_0 + A] = \text{Var}[Z(T) - E(A) + A],$$

we have $\text{Var}[Z^*(T)] = \text{Var}[Z(T)]$. As a result, by definition,

$$HER = \frac{\text{Var}[Z(T)] - \text{Var}[Z^*(T)]}{(1 + \zeta)E(A)} = 0$$

and

$$HE = \frac{\text{Var}[Z(T)] - \text{Var}[Z^*(T)]}{\text{Var}[Z(T)]} = 0$$

at $\omega = 1$.

Let $HER = \frac{a\omega^2 + b\omega + C}{(1 + \zeta)E(A)}$. Plug in the results from Lemmas 2-5 for $E(A)$, $[E(A)]^2$, $E[AZ(T)]$ and $E(A^2)$, respectively, into the ratio for HER in Theorem 1, we have the alternative expression for HER.

Corollary 2.

$$HER = \frac{a\omega^2 + b\omega + C}{(1 + \zeta)E(A)}$$

and

$$HE = \frac{a\omega^2 + b\omega + C}{\text{Var}[Z(T)]}$$

where

$$a = C_{[E(A)]^2, \omega^2} - C_{E(A^2), \omega^2},$$

$$b = 2C_{E(A), \omega} \times E[Z(T)] + C_{[E(A)]^2, \omega} - C_{E(A^2), \omega} - 2C_{E(AZ), \omega},$$

and

$$c = 2C_{E(A), \omega^0} \times E[Z(T)] + C_{[E(A)]^2, \omega^0} - C_{E(A^2), \omega^0} - 2C_{E(A^2), \omega^0}.$$

Theorem 2. ω^* maximizing HER is given by

$$\begin{aligned} \omega^* &= \max \left(\frac{-C_{E(A), \omega^0} + \sqrt{R}}{C_{E(A), \omega}}, 0 \right) \\ &= \begin{cases} \frac{-C_{E(A), \omega^0} + \sqrt{R}}{C_{E(A), \omega}}, & R \geq [C_{E(A), \omega^0}]^2, \\ 0, & R < [C_{E(A), \omega^0}]^2, \end{cases} \end{aligned}$$

where

$$R = [C_{E(A),c}]^2 - \frac{C_{E(A),\omega}(b \times C_{E(A),\omega^0} - c \times C_{E(A),\omega})}{a}.$$

As a result, the optimal payment reduction ratio maximizing HER is $(1 - \omega^*)$.

Proof.

To maximize HER, we take the first derivative of $(1 + \zeta)HER$ with respect to ω and set the result to zero. The first derivative of $(1 + \zeta)HER$ is given by

$$\begin{aligned} (1 + \zeta) \frac{\partial HER}{\partial \omega} &= \frac{2a \times \omega + b}{C_{E(A),\omega} \times \omega + C_{E(A),\omega^0}} - \frac{C_{E(A),\omega}(a \times \omega^2 + b \times \omega + c)}{(C_{E(A),\omega} \times \omega + C_{E(A),\omega^0})^2} \\ &= \frac{(2a \times \omega + b)(C_{E(A),\omega} \times \omega + C_{E(A),\omega^0}) - C_{E(A),\omega} \times a \times \omega^2}{(C_{E(A),\omega} \times \omega + C_{E(A),\omega^0})^2} \\ &\quad - \frac{C_{E(A),\omega} \times b \times \omega + C_{E(A),\omega} \times c}{(C_{E(A),\omega} \times \omega + C_{E(A),\omega^0})^2} \\ &= \frac{C_{E(A),\omega} \times a \times \omega^2 + 2a \times C_{E(A),\omega^0} \times \omega + (b \times C_{E(A),\omega^0} - c \times C_{E(A),\omega})}{(C_{E(A),\omega} \times \omega + C_{E(A),\omega^0})^2}. \end{aligned}$$

Since

$$\begin{aligned} C_{E(A),\omega} &= \sum_{j=1}^n \left[\sum_{i=j}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right] \rho \left(\frac{(j-1)T}{n}, \frac{jT}{n} \right) > 0, \\ C_{E(A),\omega^0} &= \sum_{j=1}^n \left[\sum_{i=1}^{j-1} C \times e^{-\frac{i\delta T}{n}} \right] \rho \left(\frac{(j-1)T}{n}, \frac{jT}{n} \right) \\ &\quad + \left[\sum_{i=1}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right] \rho(T, \infty) > 0, \end{aligned}$$

and $\omega > 0$, we have $\omega \neq -\frac{C_{E(A),\omega^0}}{C_{E(A),\omega}}$. Therefore, $\frac{\partial HER}{\partial \omega} = 0$ gives

$$a \times C_{E(A),\omega} \times \omega^2 + 2a \times C_{E(A),\omega^0} \times \omega + (b \times C_{E(A),\omega^0} - c \times C_{E(A),\omega}) = 0.$$

Let

$$\begin{aligned} \Delta &= 4a^2(C_{E(A),\omega^0})^2 - 4a \times C_{E(A),\omega}(b \times C_{E(A),\omega^0} - c \times C_{E(A),\omega}) \\ &= 4a^2 \left[(C_{E(A),\omega^0})^2 - \frac{C_{E(A),\omega}(b \times C_{E(A),\omega^0} - c \times C_{E(A),\omega})}{a} \right] \\ &= 4a^2 R. \end{aligned}$$

Case 1. If $R < 0$, then $\Delta = 4a^2R < 0$ and no real root exists. Since $C_{E(A), \omega} > 0$ and

$$\begin{aligned} a &= C_{[E(A)]^2, \omega^2} - C_{E(A)^2, \omega^2} \\ &= \left\{ \sum_{j=1}^n \left[\sum_{i=j}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right] \rho \left(\frac{(j-1)T}{n}, \frac{jT}{n} \right) \right\}^2 \\ &\quad - \sum_{j=1}^n \rho \left(\frac{(j-1)T}{n}, \frac{jT}{n} \right) \left[\sum_{i=j}^{n-1} C \times e^{-\frac{i\delta T}{n}} + FV \times e^{-\delta T} \right]^2 < 0 \end{aligned}$$

by Jensen's inequality. We conclude that $\frac{\partial HER}{\partial \omega}$ is a downward-open parabola with a negative vertex, which implies that $\frac{\partial HER}{\partial \omega} < 0$ holds for all ω . As a result, HER is steadily decreasing in $\omega \in [0, \infty)$. Therefore, HER is maximized at $\omega^* = 0$ for $R < 0$.

Case 2. If $R \geq 0$, we have $\Delta = 4a^2R \geq 0$ and there exist two real roots,

$$\omega_1 = \frac{-C_{E(A), \omega^0 + \sqrt{R}}}{C_{E(A), \omega}} \text{ and } \omega_2 = \frac{-C_{E(A), \omega^0 - \sqrt{R}}}{C_{E(A), \omega}} < \omega_1$$

where

$$R = C_{E(A), \omega^0}^2 - \frac{C_{E(A), \omega} (b \times C_{E(A), \omega^0} - c \times C_{E(A), \omega})}{a}.$$

Case 2.1. If $R < (C_{E(A), \omega^0})^2$, then $\omega_2 < \omega_1 < 0$.

Since

$$(1 + \zeta) \frac{\partial HER}{\partial \omega} = \frac{C_{E(A), \omega} \times a (\omega - \omega_1) (\omega - \omega_2)}{(C_{E(A), \omega} \times \omega + C_{E(A), \omega^0})^2}$$

and $a < 0$ as proved in Case 1, we have a downward-open parabola with two negative roots. As a result, $\frac{\partial HER}{\partial \omega} < 0$ holds for all $\omega \geq 0$, which implies HER is steadily decreasing in $\omega \in [0, \infty)$. Therefore, $\omega^* = 0$ maximizes HER.

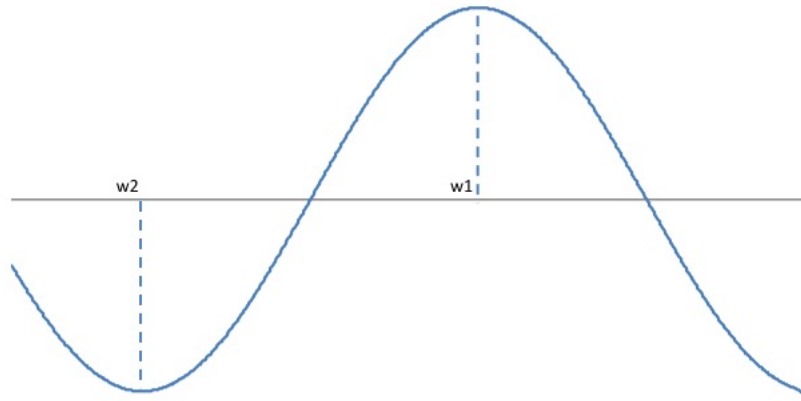
Case 2.2. If $R \geq (C_{E(A), \omega^0})^2$, then $\omega_2 < 0 \leq \omega_1$.

Since $C_{E(A), \omega} > 0$ and $a < 0$,

Case 2.2.1 $\omega > \omega_1$,

$\frac{\partial HER}{\partial \omega} < 0$ holds for all $\omega > \omega_1$, which implies HER is decreasing in $\omega > \omega_1$ as shown in Figure 3.1.

Figure 3.1: HER for Cases 2.2.1 to 2.2.3 (illustrative)



Case 2.2.2 $\omega_1 \leq \omega \leq \omega_2$,

$\frac{\partial HER}{\partial \omega} \geq 0$ holds for $\omega \in [\omega_1, \omega_2]$, which implies HER is increasing in $\omega \in [\omega_2, \omega_1]$ as shown in Figure 3.1.

Case 2.2.3 $\omega < \omega_2$,

$\frac{\partial HER}{\partial \omega} < 0$ holds for $\omega < \omega_2$, which implies HER is decreasing in $\omega < \omega_2$ as shown in Figure 3.1.

Based on the cases above, we conclude that

$$\omega^* = \omega_1 = \frac{-C_{E(A), \omega^0} + \sqrt{R}}{C_{E(A), \omega}}$$

maximizes HER.

Theorem 3. ω^{**} maximizing HE is given by

$$\omega^{**} = \max\left(-\frac{b}{2a}, 0\right) = \begin{cases} -\frac{b}{2a}, & b > 0, \\ 0, & b \leq 0. \end{cases}$$

As a result, the optimal payment reduction ratio maximizing HE is $(1 - \omega^{**})$.

Proof.

Recall that $HE \times Var[Z(T)] = a\omega^2 + b\omega + c = a\left(\omega + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}$ and $a < 0$.

Case 1. $-\frac{b}{2a} \geq 0$ for $b > 0$.

HE is a downward-open parabola with a positive axis of symmetry. Therefore, we conclude that $\omega^{**} = -\frac{b}{2a}$ maximizes HE.

Case 2. $-\frac{b}{2a} < 0$ for $b < 0$.

HE is a downward-open parabola with a negative axis of symmetry. As a result, HE is decreasing in $\omega \in [0, \infty)$. Therefore, we conclude that $\omega^{**} = 0$ maximizes HE.

3.3 Special case: zero-coupon bond

In this section, we apply the results obtained in the preceding section to a zero-coupon bond.

By setting n to one in Lemmas 2-5 and plugging in the results for $E(A)$, $[E(A)]^2$, $E[AZ(T)]$ and $E(A^2)$, respectively, into Corollary 1, we obtain the expressions of HER and HE for a zero-coupon CAT bond as follows:

Corollary 3. For a zero-coupon CAT bond,

$$\begin{aligned} HER &= \frac{-FV \times e^{-\delta T} \rho(0, T) [1 - \rho(0, T)]}{(1 + \zeta)[\rho(0, T) \times \omega + 1 - \rho(0, T)]} \omega^2 \\ &+ \frac{2 \{FV \times e^{-\delta T} [1 - \rho(0, T)] - \alpha\} \rho(0, T)}{(1 + \zeta)[\rho(0, T) \times \omega + 1 - \rho(0, T)]} \omega \\ &+ \frac{2\beta [1 - \rho(0, T)] - FV \times e^{-\delta T} \rho(0, T) [1 - \rho(0, T)]}{(1 + \zeta)[\rho(0, T) \times \omega + 1 - \rho(0, T)]} \end{aligned}$$

and

$$\begin{aligned}
HE &= \frac{-(FV)^2 e^{-2\delta T} \rho(0, T) [1 - \rho(0, T)]}{Var[Z(T)]} \omega^2 \\
&+ \frac{2FV \times e^{-\delta T} \{FV \times e^{-\delta T} [1 - \rho(0, T)] - \alpha\} \rho(0, T)}{Var[Z(T)]} \omega \\
&+ \frac{2\beta \times FV \times e^{-\delta T} (1 - \rho(0, T)) - (FV)^2 e^{-2\delta T} \rho(0, T) [1 - \rho(0, T)]}{Var[Z(T)]}
\end{aligned}$$

where

$$\alpha = E[Z(T)|0 < \tau \leq T] - E[Z(T)]$$

and

$$\beta = E[Z(T)] - E[Z(T)|\tau > T].$$

By differentiating HER and HE above with respect to ω , respectively, and setting the results to zero, we can calculate ω^* and ω^{**} for a zero-coupon CAT bond as follows:

Corollary 4. For a zero-coupon CAT bond,

$$\begin{aligned}
\omega^* &= \begin{cases} \frac{\rho(0, T) - 1 + \sqrt{R_0}}{\rho(0, T)}, & R_0 \geq [(1 - \rho(0, T))]^2, \\ 0, & R_0 < [(1 - \rho(0, T))]^2, \end{cases} \\
&= \max \left(\frac{\rho(0, T) - 1 + \sqrt{R_0}}{\rho(0, T)}, 0 \right)
\end{aligned}$$

and

$$\begin{aligned}
\omega^{**} &= \max \left(1 - \frac{\alpha}{FV \times e^{-\delta T} [1 - \rho(0, T)]}, 0 \right) \\
&= \begin{cases} 1 - \frac{\alpha}{FV \times e^{-\delta T} [1 - \rho(0, T)]}, & \alpha < FV \times e^{-\delta T} [1 - \rho(0, T)], \\ 0, & \alpha \geq FV \times e^{-\delta T} [1 - \rho(0, T)], \end{cases}
\end{aligned}$$

where

$$R_0 = 1 - 2\rho(0, T) - \frac{2(\alpha + \beta)\rho(0, T)}{FV \times e^{-\delta T}},$$

$$\alpha = E[Z(T)|0 < \tau \leq T] - E[Z(T)]$$

and

$$\beta = E[Z(T)] - E[Z(T)|\tau > T].$$

Chapter 4

Numerical Experiments

In this chapter, we conduct Monte Carlo simulations based on the formulas obtained in Chapter 3 to demonstrate how parameters, especially ω , affects HER and HE empirically. All results are based on 100,000 rounds of simulations unless specified. Models with different parameters are tested. We use the parameters of the catastrophe loss process given in Nowak and Romaniuk (2013).

This chapter is organized as follows. Section 1 gives assumptions we use in empirical studies. Section 2 gives simulation methodology. In Section 3, we first study the loss distributions of the reinsurer before and after issuing the CAT bond. Then we analyze the relationships between ω and HER and between ω and HE, and consider CAT bonds with different coupon payments. Next, we study the impacts of loss share (m), retention (d), strike price (K) and maturity (T) on ω^* and ω^{**} , respectively. Last, we analyze the sensitivities of ω^* and ω^{**} to the parameters of the catastrophe loss process, μ , λ and σ , respectively.

4.1 General assumptions

The following section outlines all the common assumptions used in the numerical experiments in addition to the ones stated in Chapter 3.

4.1.1 Catastrophe time and size

For each model, the catastrophe frequency $N(t)$ is modelled by a homogeneous counting process. We further assume the catastrophe severities, $X_1, X_2, X_3, \dots, X_{N(t)}$, are independent, and follow a lognormal distribution. These assumptions are consistent with the low frequency and high severity nature of catastrophe events.

We limit our empirical studies to a homogeneous Poisson process with intensity λ and independent and identical lognormal distributions with parameters μ_{LN} and σ_{LN} mainly for simplicity in calculations. We admit that other counting process and loss distributions might be appropriate as well. For the counting process, Lin et al. (2008) used a Markov-modulated Poisson process to model catastrophe arrivals. Ma and Ma (2013) used a non-homogeneous Poisson process and applied a mixed approximation method to the aggregate loss distribution. For the loss distribution, other heavy tailed distributions, like Weibull and Burr, are also widely used in actuarial literature (Ma and Ma (2013)).

4.1.2 Other assumptions and notations

Assumptions and notations in addition to the catastrophe loss process are listed below.

- **HER^{**} and HE^*** : define HE^{**} as the hedge effectiveness at ω^{**} and HER^* as the hedge effectiveness rate at ω^* .
- **Face value (FV)**: the face value FV is set at \$10,000,000,000 multiplied by the loss share (m) of any specific insurance company. Loss share is taken into consideration because the more loss a company is expected to have, the more catastrophe bonds should be issued to hedge that loss.
- **Coupon rate (C)**: for a non-zero coupon CAT bond, the coupon rate C is assumed to be fixed at 10% throughout the empirical studies.
- **Strike price (K)**: we assume K is a certain percentile of the aggregate industrial loss distribution; different percentiles are tested.
- **Loss share (m)**: the loss share m can be estimated by the market share of the insurer or reinsurer. For instance, the average retention level of the insurers in the industry is 60% and one reinsurer has around 30% market share in the reinsurance

market. Then m can be set at $(1 - 60\%) \times 30\% = 0.12$. Different assumptions for m are considered in numerical calculations. In particular, $m = 100\%$ represents the whole insurance industry.

- **Expense loading (ζ):** the expense loading is set at 1%.
- **Retention level of the insurance company (d):** two kinds of retention assumptions are tested in the following illustrations. First, we consider retention level $d = K \times m / \lambda$. Under this scenario, the insurance company takes a CAT loss up to the retention level, and the reinsurance company takes the rest of the loss and issues a CAT bond to securitize it. Since the strike price, K , is based on the aggregate industrial losses on a series of catastrophes, it is reasonable to assume that each company would retain loss at around $(K \times \text{loss share} / \text{average number of catastrophes})$ on each catastrophe. Second, we consider a zero retention. If an insurance company issues a CAT bond on its own, there will be no retention.

4.2 Simulation methodology

In each round of simulation, we take the following steps:

1. Generate the inter-occurrence time of the catastrophe events, S_1, S_2, \dots , from the exponential distribution with mean $1/(\lambda T)$.
2. Calculate the occurrence time of the i^{th} catastrophe event, $T_i = \sum_{j=1}^i S_j$, where $i = 1, 2, 3, \dots$.
3. Find the number of catastrophe events before time T , $N(T) = k^*$, where k^* is a positive integer such that $T_{k^*} \leq T$ and $T_{k^*+1} > T$.
4. Generate the size of the i^{th} catastrophe, X_i , from the lognormal distribution with parameters μ_{LN} and σ_{LN} .
5. If $L(T) = \sum_{i=1}^{N(T)} X_i \leq K$ then the trigger time $\tau > K$. Otherwise, find the smallest positive integer i^* such that $L(T) = \sum_{i=1}^{N(T_i^*)} X_i > K$; determine the trigger time $\tau = T_{i^*}$ and the interval $\left(\frac{(j-1)T}{n}, \frac{jT}{n} \right)$ containing τ .

6. Calculate the total discounted aggregate loss before issuing the CAT bond,

$$Z(T) = \sum_{i=1}^{N(T)} e^{-\delta T_i} (mX_i - d)_+.$$

After 100,000 simulations, we obtain the estimates of

1. the trigger probability $\rho(\frac{(j-1)T}{n}, \frac{jT}{n})$ which is estimated by the times that $(\frac{(j-1)T}{n}, \frac{jT}{n})$ containing τ over 100,000, $j = 1, 2, \dots, n$, and $\rho(T, \infty)$ which is equal to the counts of $\tau > T$ over 100,000;
2. the expected present value of $Z(T)$, $E[Z(T)]$, which is the average of the 100,000 simulated $Z(T)$'s; and
3. the expected $Z(T)$ conditioning on the trigger time, $E \left[Z(T) \left| \frac{(j-1)T}{n} < \tau \leq \frac{jT}{n} \right. \right]$: a simulated $Z(T)$ with a specific trigger time is obtained in each round of simulation. The 100,000 $Z(T)$'s are first categorized into different trigger time intervals $(\frac{(j-1)T}{n}, \frac{jT}{n})$, $j = 1, 2, \dots, n$ and (T, ∞) . Then $E \left[Z(T) \left| \frac{(j-1)T}{n} < \tau \leq \frac{jT}{n} \right. \right]$ is estimated by the average of the $Z(T)$'s in each trigger time interval.

With the results simulated above, we are able to calculate ω^* , ω^{**} , HER^* , HE^{**} and HER and HE at different ω based on the formulas obtained in Chapter 3.

4.3 Empirical results

In the following section, we first study the profit and loss distribution of the reinsurer or insurer before and after issuing a CAT bond. Then we analyze the relationship between ω and HER and between ω and HE . CAT bonds with different coupon payments are considered. Next, we study the impacts of loss share (m), retention (d), strike price (K) and maturity (T) on ω^* and ω^{**} , respectively. Last, we analyze the sensitivities of ω^* and ω^{**} to the parameters of the catastrophe loss process, μ , λ and σ , respectively.

4.3.1 Impacts of ω^* on the profit and loss distribution

In this section, we analyse how the optimal ω^* affects the profitability of a reinsurer. The values of parameters and the model can be found in Table 4.1. In addition, we further

Figure 4.1: Distributions of $Y(T)$ and $Y^*(T)$

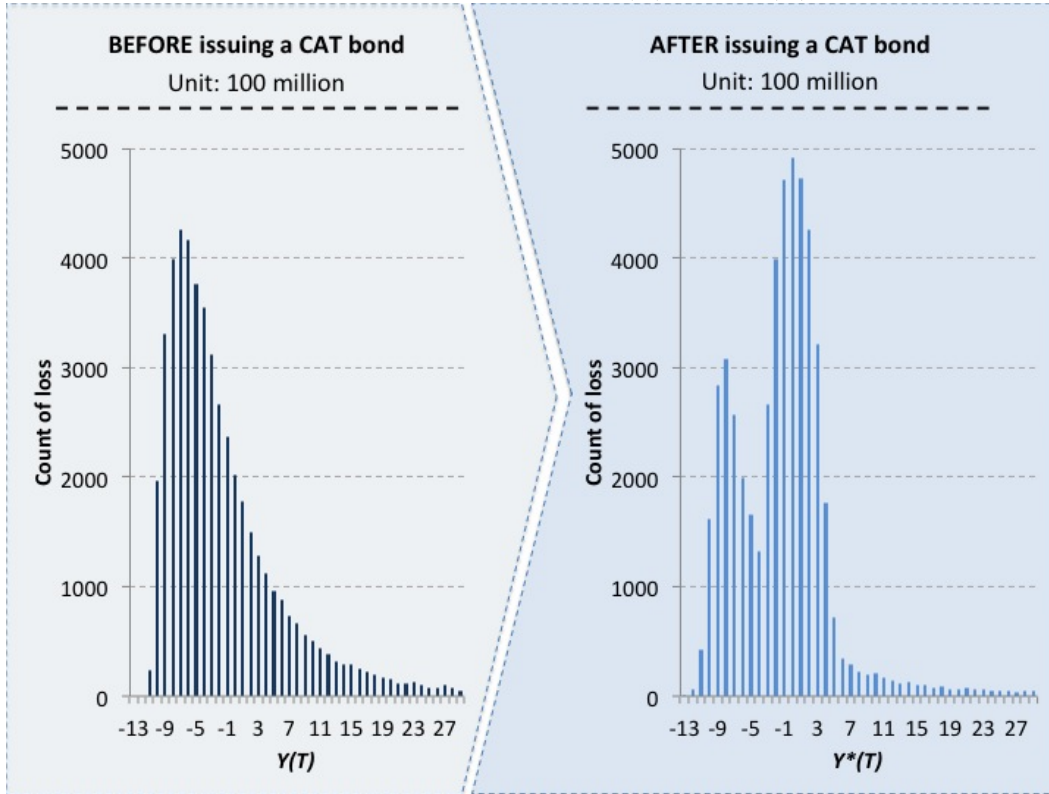


Figure 4.2: Distributions of $Y(T)$ and $Y^*(T)$

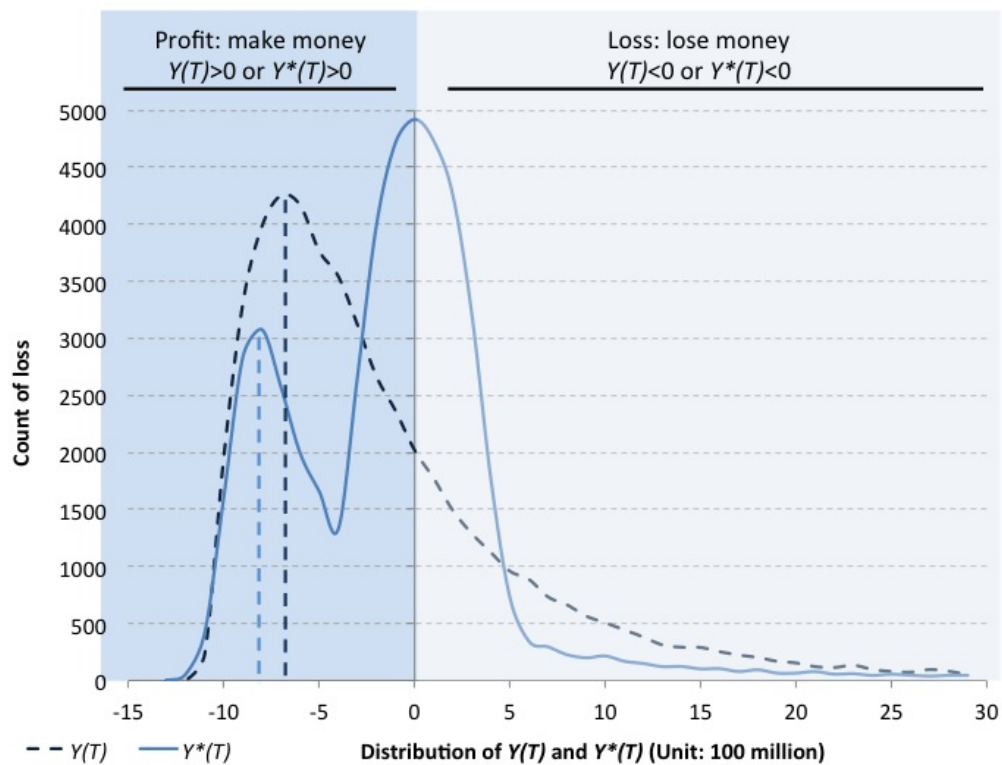


Table 4.1: Model and parameter assumptions for the impact of ω^* on distributions of $Y(T)$ and $Y^*(T)$

	Parameters
Force of interest (δ)	0.02
Homogeneous Poisson process	$\lambda = 31.7143$
Lognormal distribution	$\mu_{LN} = 17.357, \sigma_{LN} = 1.7643$
Face value (FV)	$10,000,000,000 \times m$
Coupon payments (C)	10% FV
Number of coupon payments ($n - 1$)	4
Expense loading (ζ)	1%
Loss share (m)	30%
ω^*	0.5482111
Maturity (T)	1
Retention (d)	$K \times m / (\lambda T)$
Strike price (K)	the median of $L(T)$
Number of simulations	50,000

assume the CAT bond issues 4 coupons annually, the loss share is 30%, and the strike price is the median of $L(T)$. We choose the above parameters for the base scenario and later we will study the impacts of these parameters, respectively.

The following steps are taken to obtain the empirical profit and loss distributions before and after issuing a CAT bond. We first follow the simulation methodology described in Section 4.2 and obtain the empirical distribution of $Z(T)$. To further study the profitability of the reinsurer, we deduct a premium from $Z(T)$ where the premium is the expectation of $Z(T)$ plus a loading γ ; that is, the premium equals $(1 + \gamma)E[Z(T)]$. Let $Y(T) = Z(T) - (1 + \gamma)E[Z(T)]$. By definition, the reinsurer makes money when $Y(T)$ is negative and loses money when $Y(T)$ is positive. To simulate the profit and loss distributions after issuing the CAT bond, we first obtain ω^* based on the results in Chapter 3. Then we price the CAT bond at ω^* and calculate $Z^*(T)$, the present value of total catastrophe loss after issuing the CAT bond at ω^* . By repeating the above process for 50,000 times, we obtain the empirical distribution of $Z^*(T)$. Next, we deduct the premium from $Z^*(T)$. Since $E[Z(T)] = E[Z^*(T)]$ as shown in Chapter 3, let $Y^*(T) = Z^*(T) - (1 + \gamma)E[Z(T)]$. Similar to $Y(T)$, the insurer or reinsurer makes money when $Y^*(T)$ is negative and loses money when $Y^*(T)$ is positive.

Table 4.2: Results of $Y(T)$ and $Y^*(T)$ based on 50,000 simulations

		0% loading		20% loading	
		$Y(T)$	$Y^*(T)$	$Y(T)$	$Y^*(T)$
Probability of making money	%	66%	52%	74%	71%
Expectation	\$	0	0	-200,928,637	-200,928,637
Minimum	\$	-1,004,643,187	-1,283,541,830	-1,205,571,824	-1,484,470,467
Maximum	\$	45,190,674,408	44,400,379,972	44,989,745,771	44,199,451,335
95% VaR	\$	1,028,259,460	473,116,931	827,330,822	272,188,294
95% TVaR	\$	2,430,680,145	1,611,449,759	2,229,751,508	1,410,521,122

Figures 4.1 and 4.2 illustrate the profit and loss distributions of an reinsurer before and after issuing a CAT bond with a zero loading. The second quadrant represents profit where $Y(T)$ and $Y^*(T)$ are negative, while the first quadrant represents loss where $Y(T)$ and $Y^*(T)$ are positive. As shown in Figures 4.1 and 4.2, issuing a CAT bond helps to reduce the variance of loss. With a flattened left tail, the CAT bond mitigates extreme losses. By issuing the CAT bond, the reinsurance company has significantly less chances of suffering losses over 500 million, while having more chances of losing 0 to 500 million as a trade-off. It is worth noting that the peak of the distribution of $Y(T)$ at around 650 million profit is only partially shifted to the new peak after hedging at around zero, the break-even point. A local maximum at around 900 million profit is kept to the left of the original peak. This is a desirable property since ω^* limits the extent of hedging trade-off and manages to capture the most part of profitability.

From Table 4.2, we observe that with a zero loading, the probability of making money after hedging, $P[Y^*(T) < 0]$, is 52 %, compared with 66 % ($=P[Y(T) < 0]$) before hedging. On the bright side, the 95% Value at Risk drops to 0.47 billion loss from 1.03 billion loss before hedging, the 95% Tail Value at Risk falls to 1.61 billion loss from 2.43 billion loss before hedging, and the maximum loss reduces from 45.19 billion loss to 44.40 billion loss after issuing the CAT bond. To further raise the probability of making money, the reinsurer can increase the loading γ . As shown in Table 4.2, the probabilities of making money before and after hedging increase to 74% and 71 %, respectively, under a 20 % loading.

4.3.2 Impacts of ω on HER and HE

Table 4.3: Model and parameter assumptions for the impacts of ω on HER and HE

	Parameters
Force of interest (δ)	0.02
Homogeneous Poisson process	$\lambda = 31.7143$
Lognormal distribution	$\mu_{LN} = 17.357, \sigma_{LN} = 1.7643$
Face value (FV)	$10,000,000,000 \times m$
Coupon payments (C)	10% FV
Number of coupon payments ($n - 1$)	0, 1, 2, 3 as shown on the figures
Expense loading (ζ)	1%
Loss share (m)	30%
Maturity (T)	1
Retention (d)	$K \times m / (\lambda T)$
Strike price (K)	the median of $L(T)$

We examine how HER and HE change with ω , respectively, for different values of n . The values of the parameters and the model can be found in Table 4.3.

As shown in Figure 4.3, for the number of coupon payments $n - 1 = 0, 1, 2, 3$, each HER follows a parabola-like curve; the HER for $n = 3$ rises from around -200 million at $\omega = 0$, reaches its maximum about 150 million at around $\omega = 0.53$ and then decreases to zero at $\omega = 1$. HER is always zero at $\omega = 1$ because $\omega = 1$ refers to no payment reduction when the CAT bond is triggered. Thus, for $\omega = 1$, the payments of a CAT bond are exactly the same no matter it is triggered or not. As a result, the variance before and after hedging would stay the same and HER becomes zero, which is also proved in Corollary 1 of Chapter 3. It is also worth noting that even though the HERs in Figure 4.3 look like parabolas, they are not parabolas exactly. Based on Corollary 2 in Chapter 3, the numerator of HER is a polynomial of degree two in ω and its denominator is linear in ω . Furthermore, a negative HER at the left corner is the results of the over-hedging of the reinsurance company and the retention of the insurance company, while the HERs close to zero on the right is the result of insufficient hedging. Here over-hedging means that too much leverage is applied for hedging. When ω is small and hence the leverage is high, significant deviations of the payoff of a CAT bond can raise the variance of loss of the reinsurer substantially, hence

Figure 4.3: Hedge effectiveness rate versus ω

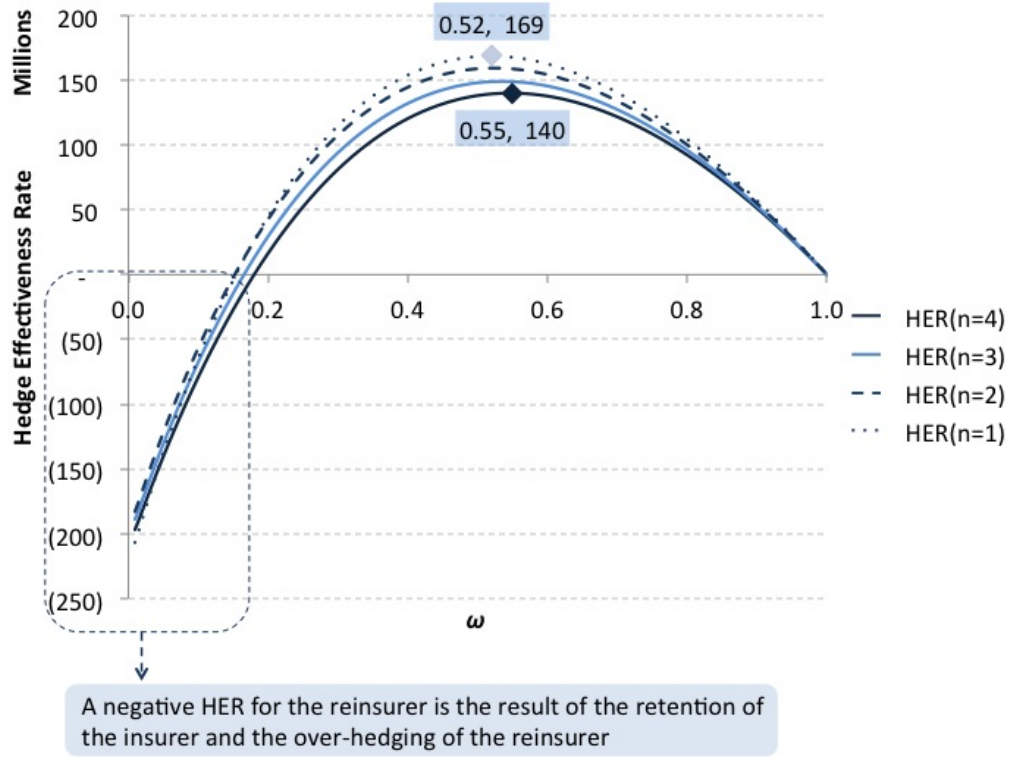
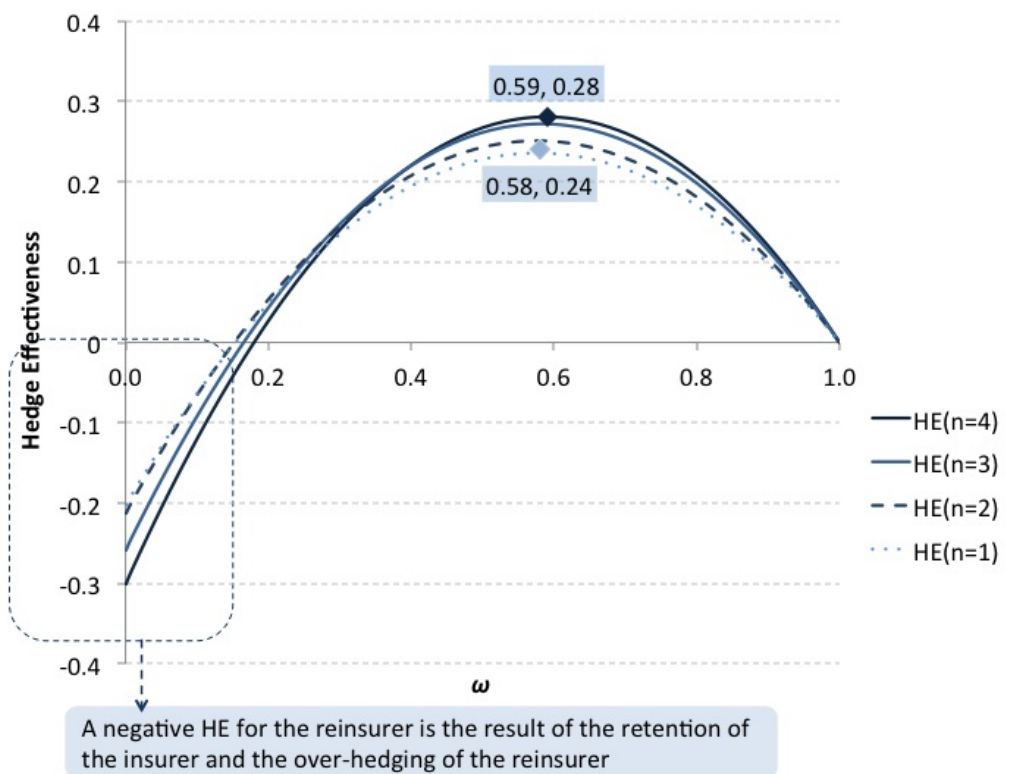


Figure 4.4: Hedge effectiveness versus ω



producing a negative HER, which results in over-hedging. On the contrary, if ω is too big, the payment reductions generated by a CAT bond are too small compared to the losses, which results in insufficient hedging. For the retention, we will further study its effects later.

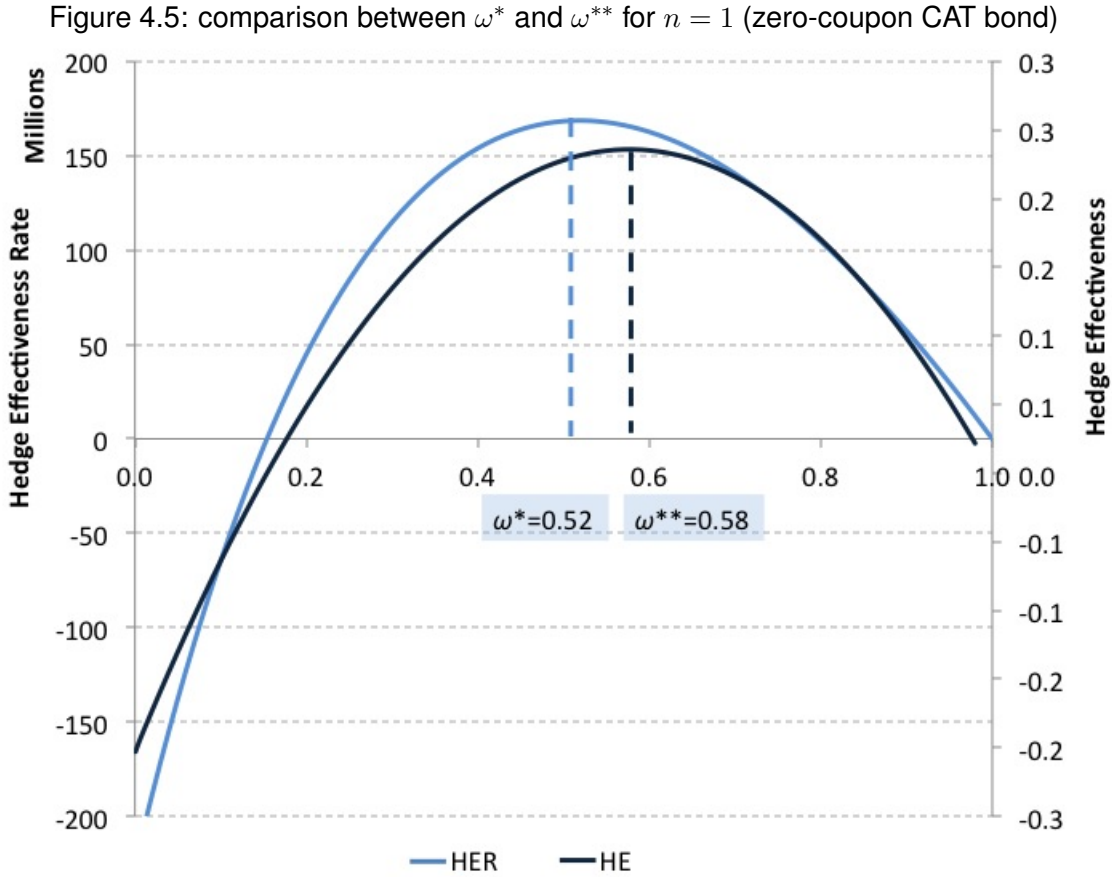
As shown in Figure 4.4, HE has the same parabola shape as HER. Different from HER, HE is exactly a parabola since it has a polynomial of degree two in ω in the numerator and a constant in the denominator. Like HER, as ω increases, HE starts from a negative zone at $\omega = 0$, gradually increases and reaches its maximum at ω^{**} , and decreases to zero at $\omega = 1$. Same as HER, HE is always zero at $\omega = 1$. The far left negative part of HE represents over-hedging where we take more leverage than needed, while the far right zero part represent insufficient hedging where the leverage of CAT bond is too small to mitigate the effects of CAT losses. Here we are able to achieve HE up to only 28% due to the limits of the one-step binomial tree structure of the CAT bond payoff. An effective solution to this problem is to adopt a step-wise payoff structure such as a set of $K_1, K_2, K_3 \dots$ instead of just one K . However, this is beyond the scope of this project.

The number of coupon payments has limited effects on the shape of both HER and HE. As shown in Figures 4.3 and 4.4, all lines follow a parabola shape and lay closely to each other. Moreover, as shown in Figure 4.3, ω^* increases with n . Given the same ω , the positive HER decreases with n because the CAT bond price increases faster than the amount of variance reduction does as n increases, which leads to a lower HER. For HE, the reverse patterns are observed since HE does not take price into consideration.

From Figure 4.5, we further compare ω^* and ω^{**} in the same plot of HER and HE for $n = 1$. As shown in the graph, $\omega^* = 0.52$ and $\omega^{**} = 0.58$ for $n = 1$, which are fairly close to each other. This implies the costs of risk reduction have limited negative impacts on the effectiveness of hedging with a CAT bond in general.

4.3.3 Impacts of the loss share and the retention level

In this section, we analyze how the loss share m and the retention d will affect ω^* , ω^{**} , HE^{**} and HER^* . The values of the parameters and the model can be found in Table 4.4.



From Figure 4.6, it is clear that ω^* increases while HER^* decreases as the retention level d of the insurer increases. The retention left truncates the aggregate loss distribution. Therefore, both the variances of the loss distributions before and after issuing a CAT bond decrease. For ω^* , since the variance of the loss distribution before issuing a CAT bond decreases, we need less leverage to hedge, and thus the optimal ω increases. The decrease in HER^* is the results of both a smaller amount of variance reduction and a higher price. The amounts of variance reduction before and after hedging are smaller when the retention is higher because a higher retention generates losses closer to each other. Meanwhile, since ω^* increases, the price of the CAT bond increases too. Same as ω^* and HER^* , ω^{**} increases in d , while HE^{**} decreases in d as shown on Figure 4.7.

Figure 4.6: HER, loss share and retention for the zero-coupon CAT bond

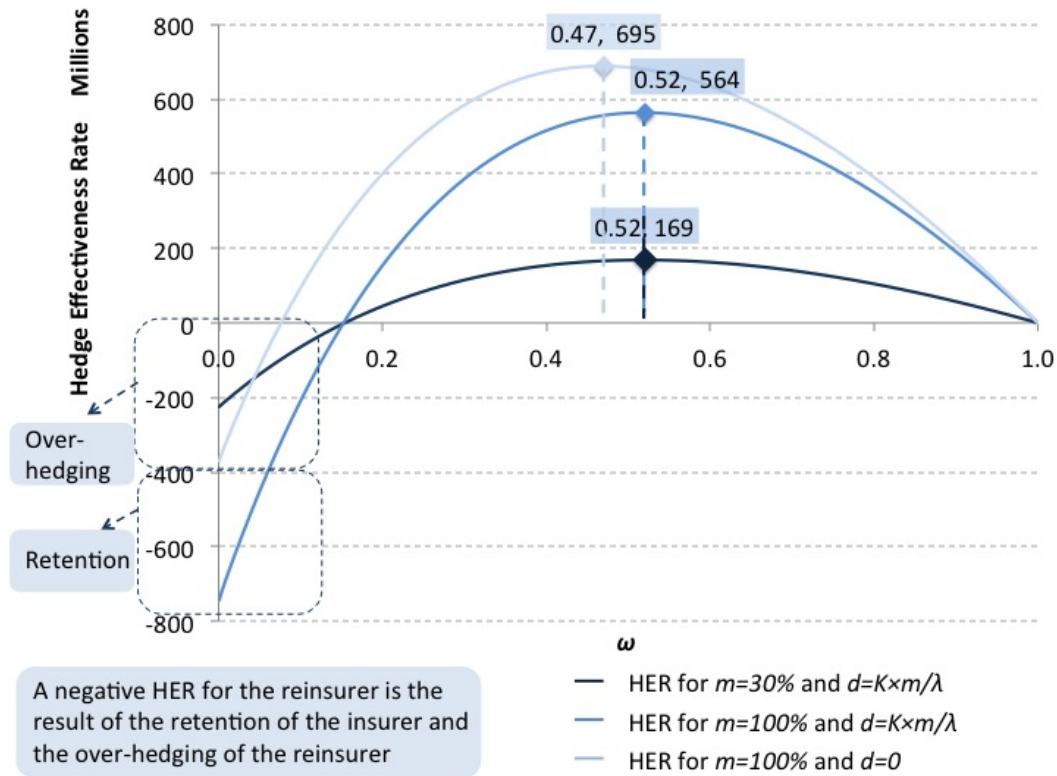


Figure 4.7: HE, loss and retention for the zero-coupon CAT bond

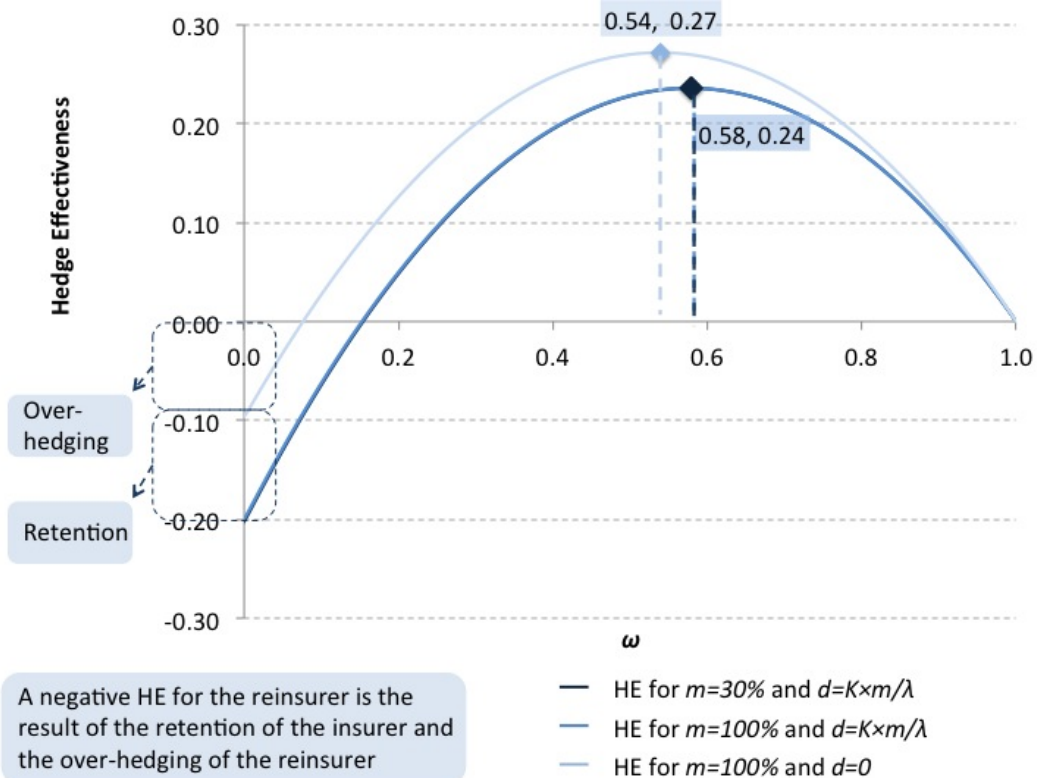


Table 4.4: Model and parameter assumptions for the impacts of the loss share m and the retention level d

	Parameters
Force of interest (δ)	0.02
Homogeneous Poisson process	$\lambda = 31.7143$
Lognormal distribution	$\mu_{LN} = 17.357, \sigma_{LN} = 1.7643$
Face value (FV)	$10,000,000,000 \times m$
Coupon payments (C)	zero coupon bond
Expense loading (ζ)	1%
Loss share (m)	30% or 100 % as specified in Figures 4.6 and 4.7
Maturity (T)	1
Retention (d)	$K \times m / (\lambda T)$ or 0 as specified in Figures 4.6 and 4.7
Strike price (K)	the median of $L(T)$

Another interesting conclusion from Figures 4.6 and 4.7 is that the loss share m will not affect ω^* and ω^{**} of a zero-coupon CAT bond provided that the face value FV is proportional to the loss share m and the retention d is proportional to the strike price k , which is proved theoretically in Appendices A and B. In addition to ω^* and ω^{**} , HE at any given ω is also independent of the loss share for a zero-coupon CAT bond as shown in Figure 4.7, which is proved theoretically in Appendix B.

4.3.4 Impacts of the strike price

In this section, we analyze ω^* and ω^{**} , HER^* and HE^{**} as a function of strike price K . The values of the parameters and the model can be found in Table 4.5.

As shown in Figure 4.8 (a), ω^* decreases as the strike price increases. Given ω , a higher strike price results in a smaller trigger probability, and hence a smaller variance reduction. Therefore, to maximize the variance reduction per dollar of hedging cost, we need a smaller ω to magnify the variance reduction and decrease the CAT bond price.

Also, we can observe that the HER^* first increases from the 50%tile to 90%tile of the aggregate industrial loss and then decreases as the strike price increases. To understand this pattern, we need to consider two extreme scenarios, 0%tile and 100%tile. The CAT bond payments are the same no matter it is triggered or not, which leads to a zero variance

Figure 4.8: HE^{**} , HER^* , ω^* and ω^{**} versus strike price

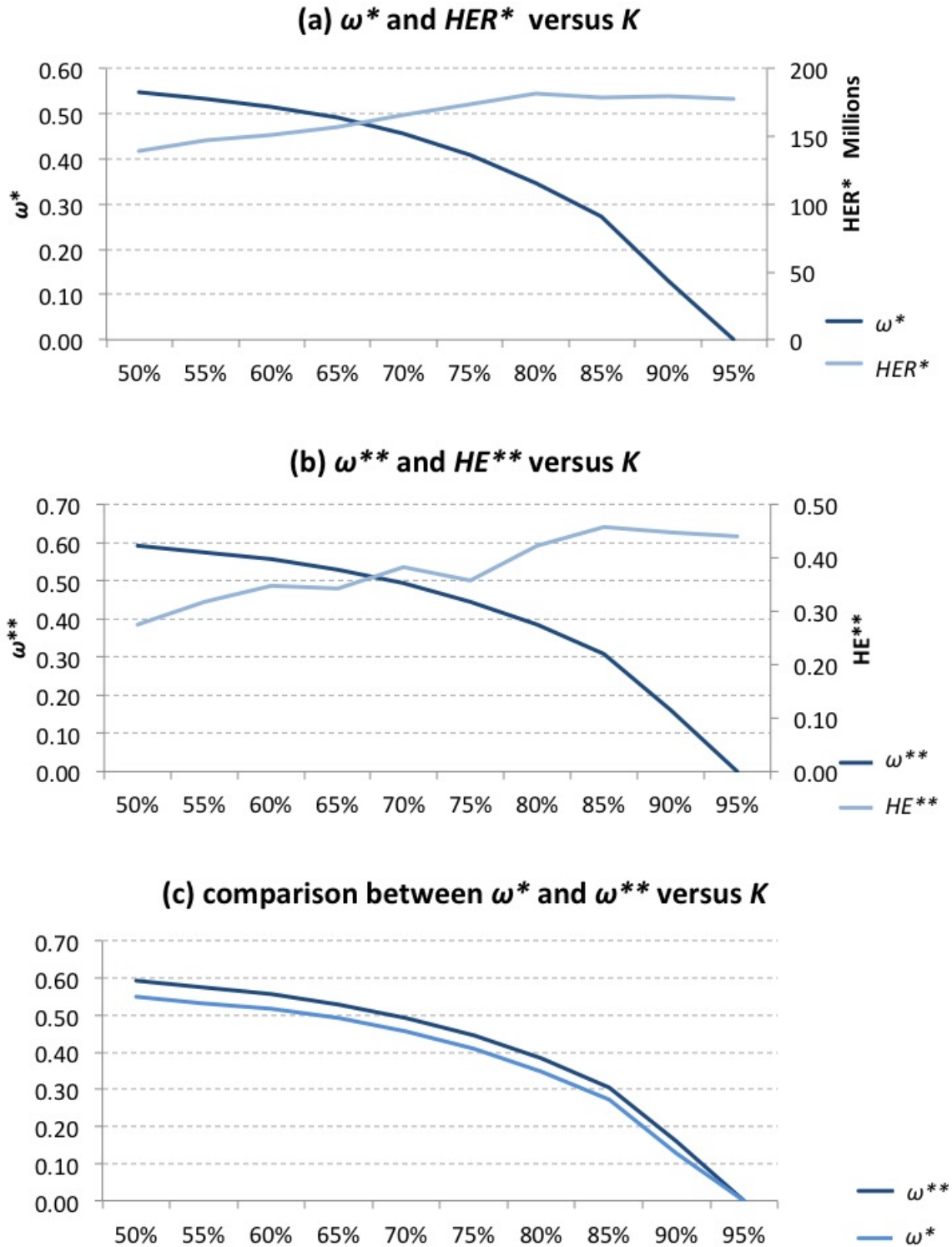


Table 4.5: Model and parameter assumptions for the impacts of the strike price K

	Parameters
Force of interest (δ)	0.02
Homogeneous Poisson process	$\lambda = 31.7143$
Lognormal distribution	$\mu_{LN} = 17.357, \sigma_{LN} = 1.7643$
Face value (FV)	$10,000,000,000 \times m$
Coupon payments (C)	10% FV
Number of coupon payments ($n - 1$)	3
Expense loading (ζ)	1%
Loss share (m)	30%
Maturity (T)	1
Retention (d)	$K \times m / (\lambda T)$
Strike price (K)	from 50%-tile to 95%-tile of $L(T)$

reduction and hence a zero HER for all ω . Therefore, with a zero on the far left and a zero on the far right, the HER^* first increases from zero then decreases to zero as the percentile of the aggregate industrial loss ranges from 0% to 100%.

As shown on Figure 4.8 (b), ω^{**} and HE^{**} follow similar paths as ω^* and HER^* . Furthermore, at any given strike price, ω^* is always not greater than ω^{**} as shown in Figure 4.8 (c). The difference between ω^* and ω^{**} represents the trade-off between the over-hedging of a higher risk and a lower hedging cost.

4.3.5 Impacts of the maturity

We examine ω^* and ω^{**} , HE^{**} and HER^{**} as a function of maturity T . The values of the parameters and the model can be found in Table 4.6.

As shown in Figures 4.9 and 4.10, both ω^* and ω^{**} decrease as the maturity T increases. Intuitively, given a fixed frequency of catastrophe, the longer the maturity, the more the uncertainty. Therefore, to mitigate the extra uncertainty, ω^* should be small enough to generate deviated bond payoffs that work in the opposite direction of the losses.

Also, from Figure 4.9, HER^* increases in the maturity T . This is because, by the definition of HER , the variance reduction amount increases much faster than the CAT bond price

Figure 4.9: HER^* and ω^* versus maturity

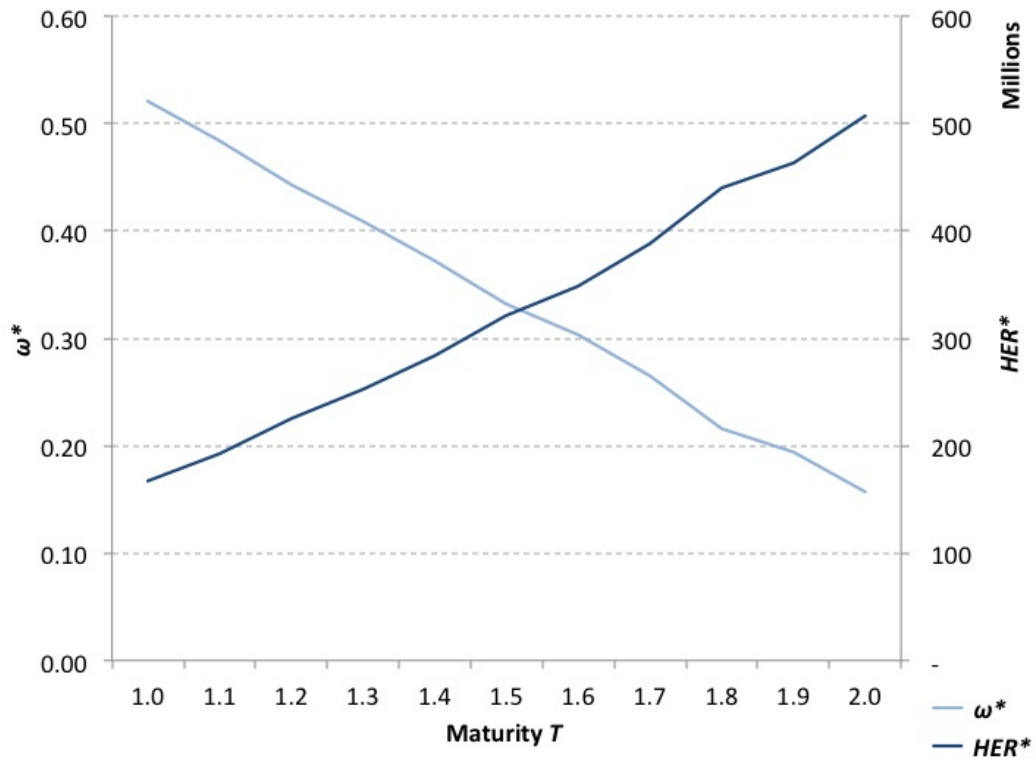


Figure 4.10: HE^{**} and ω^{**} versus maturity

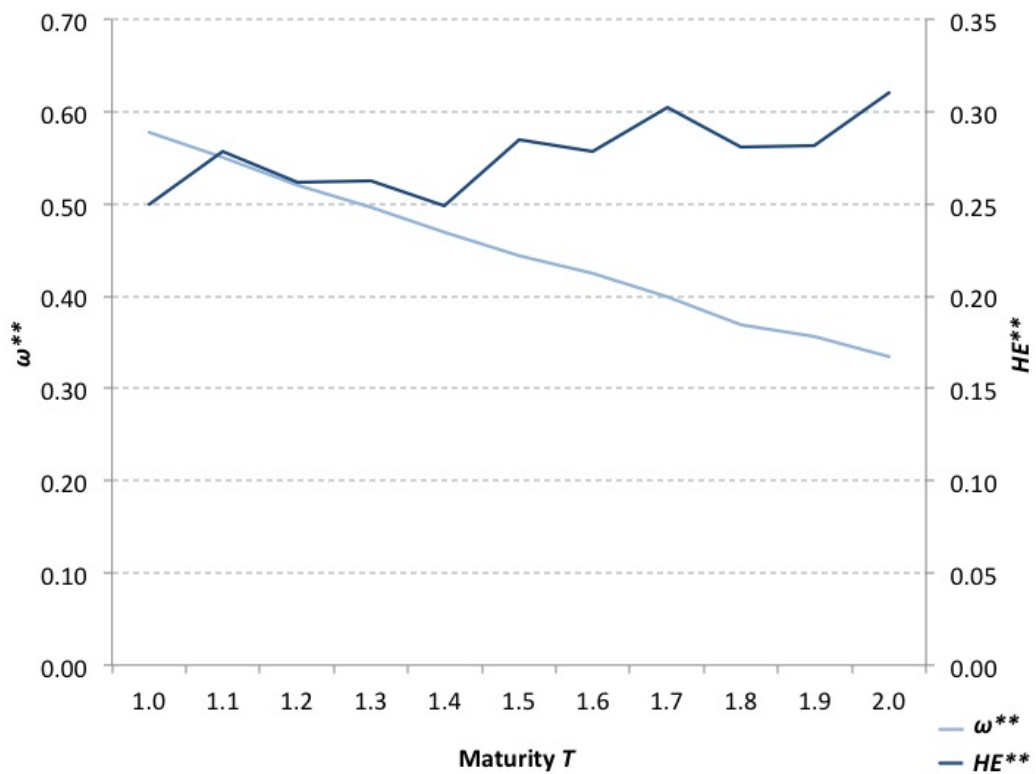


Table 4.6: Model and parameter assumptions for the impacts of the maturity T

	Parameters
Force of interest (δ)	0.02
Homogeneous Poisson process	$\lambda = 31.7143$
Lognormal distribution	$\mu_{LN} = 17.357, \sigma_{LN} = 1.7643$
Face value (FV)	$10,000,000,000 \times m$
Coupon payments	zero-coupon bond
Expense loading (ζ)	1%
Loss share (m)	30%
Maturity (T)	1.0, 1.1, 1.2, 1.3... ,2.0
Retention (d)	$K \times m / (\lambda T)$
Strike price (K)	the median of $L(T)$

in general. However, for HE^{**} , the similar pattern is not observed because HE does not take the price of a CAT bond into consideration.

4.3.6 Sensitivity test

We test the sensitivities of HER and HE to the catastrophe loss process. Under the assumption of a homogeneous Poisson process with lognormal CAT losses, the expected aggregate loss $E[Z(T)]$ is $\lambda T \times e^{\mu_{LN} + \frac{1}{2}\sigma_{LN}^2}$. We increase (decrease) the expectations of frequency and severity of the loss process by 10% through adjusting λ , μ_{LN} and σ_{LN} , respectively. The values of parameters and the models can be found in Tables 4.7, 4.8 and 4.9.

For the frequency, we adjust λ by 10% up (10% down) with other parameters unchanged so that the mean of the total aggregate loss increases (decreases) by 10%. As shown in Figures 4.11 and 4.12, both ω^* and ω^{**} decrease as the catastrophe loss frequency increases. The reason is that a larger aggregate CAT loss demands a higher leverage to hedge. As a result, ω^* and ω^{**} decrease so that the CAT bond can generate more deviated payoffs in the opposite direction of the CAT losses to mitigate its effect. Similarly, if the catastrophes are less frequent, we expect smaller losses. Consequently, we need a lower leverage, and hence higher ω^* and ω^{**} .

Table 4.7: Model and parameter assumptions for sensitivity test on λ

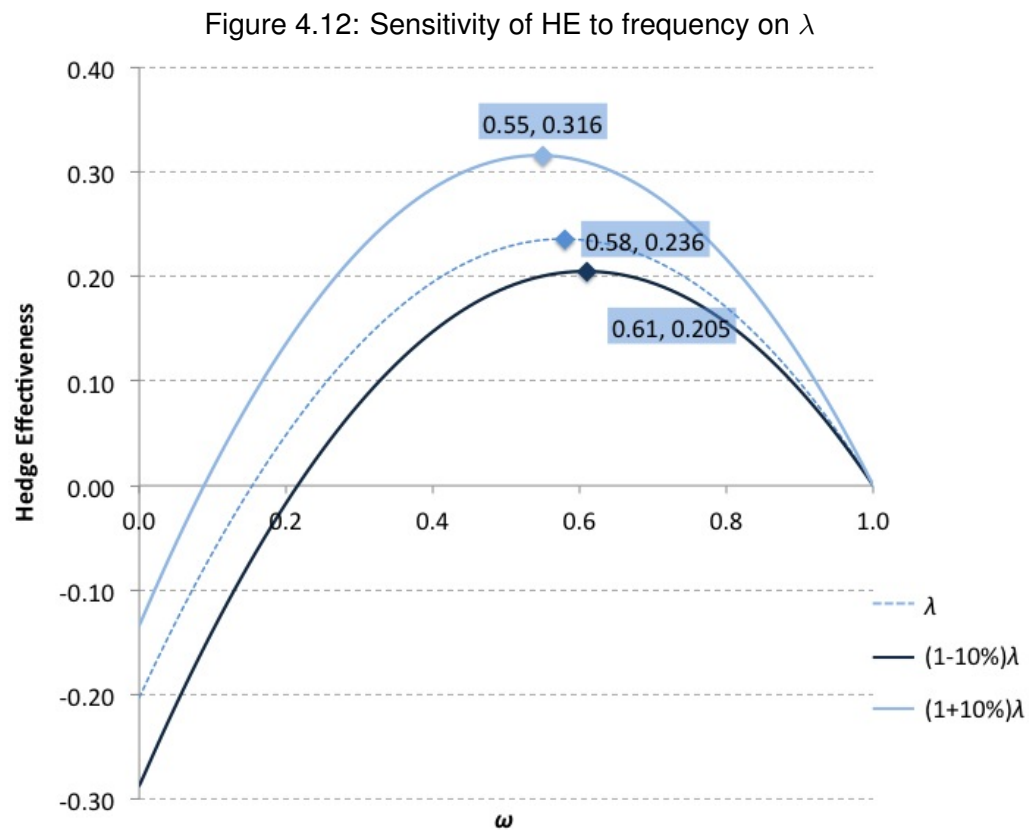
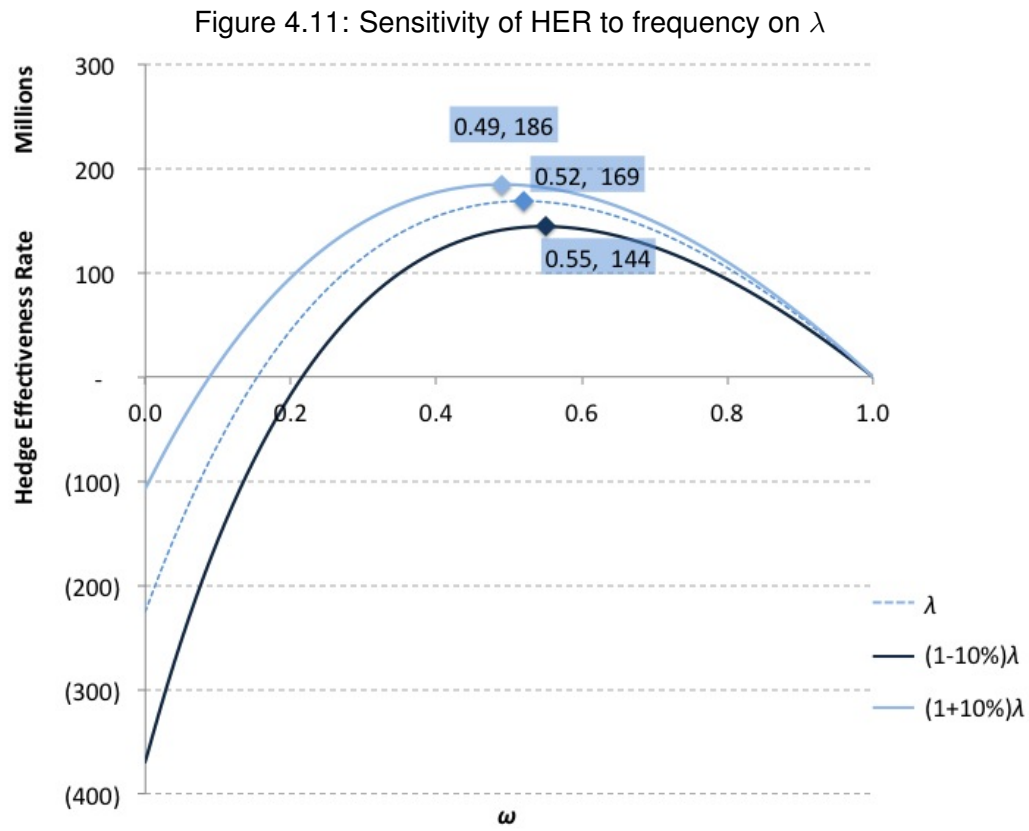
	Parameters
Force of interest (δ)	0.02
Homogeneous Poisson process	$\lambda = 31.7143 \times (1 - 10\%)$ or $\lambda = 31.7143 \times (1 + 10\%)$
Lognormal distribution	$\mu_{LN} = 17.357, \sigma_{LN} = 1.7643$
Face value (FV)	$10,000,000,000 \times m$
Coupon payments (C)	zero-coupon bond
Expense loading (ζ)	1%
Loss share (m)	30%
Maturity (T)	1
Retention (d)	$K \times m / (\lambda T)$
Strike price (K)	the median of $L(T)$

Table 4.8: Model and parameter assumptions for sensitivity test on μ_{LN}

	Parameters
Force of interest (δ)	0.02
Homogeneous Poisson process	$\lambda = 31.7143$
Lognormal distribution	$\mu_{LN} = 17.357 \times (1 + 0.549\%)$ or $\mu_{LN} = 17.357 \times (1 - 0.607\%), \sigma_{LN} = 1.7643$
Face value (FV)	$10,000,000,000 \times m$
Coupon payments (C)	zero-coupon bond
Expense loading (ζ)	1%
Loss share (m)	30%
Maturity (T)	1
Retention (d)	$K \times m / (\lambda T)$
Strike price (K)	the median of $L(T)$

Table 4.9: Model and parameter assumptions for sensitivity test on σ_{LN}

	Parameters
Force of interest (δ)	0.02
Homogeneous Poisson process	$\lambda = 31.7143$
Lognormal distribution	$\mu_{LN} = 17.357, \sigma_{LN} = 1.7643 \times (1 + 3.016\%)$ or $\sigma_{LN} = 1.7643 \times (1 - 3.444\%)$
Face value (FV)	$10,000,000,000 \times m$
Coupon payments (C)	zero-coupon bond
Expense loading (ζ)	1%
Loss share (m)	30%
Maturity (T)	1
Retention (d)	$K \times m / (\lambda T)$
Strike price (K)	the median of $L(T)$



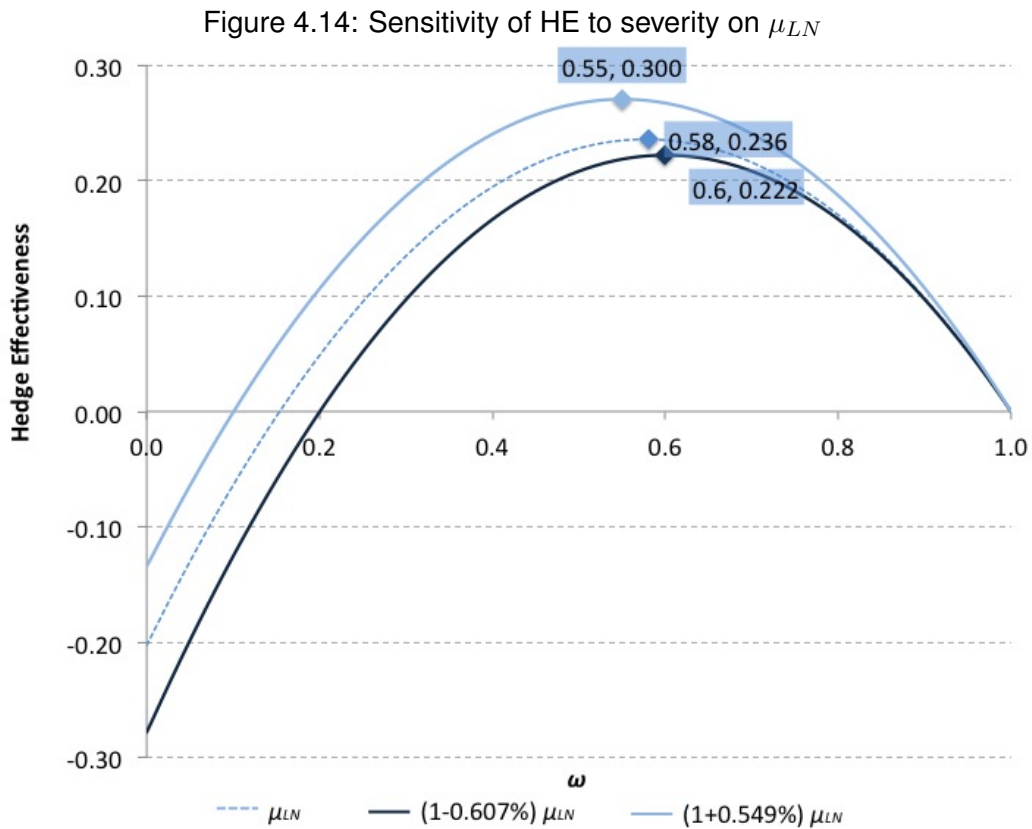
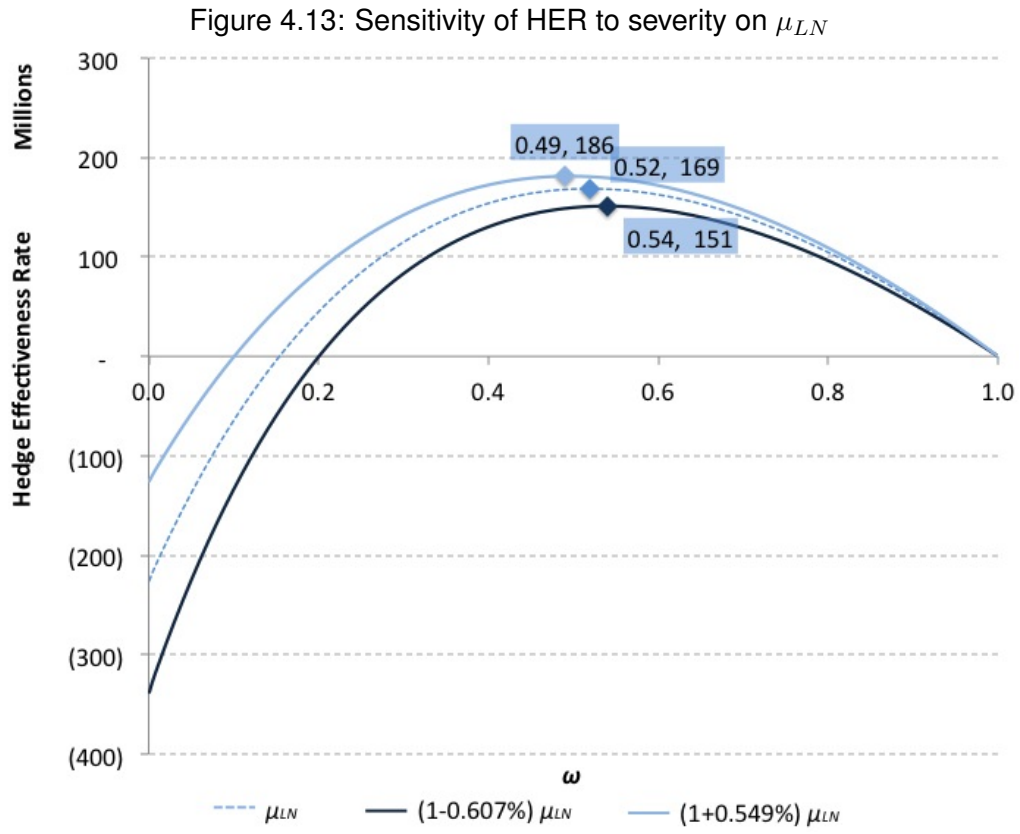


Figure 4.15: Sensitivity of HER to severity on σ_{LN}

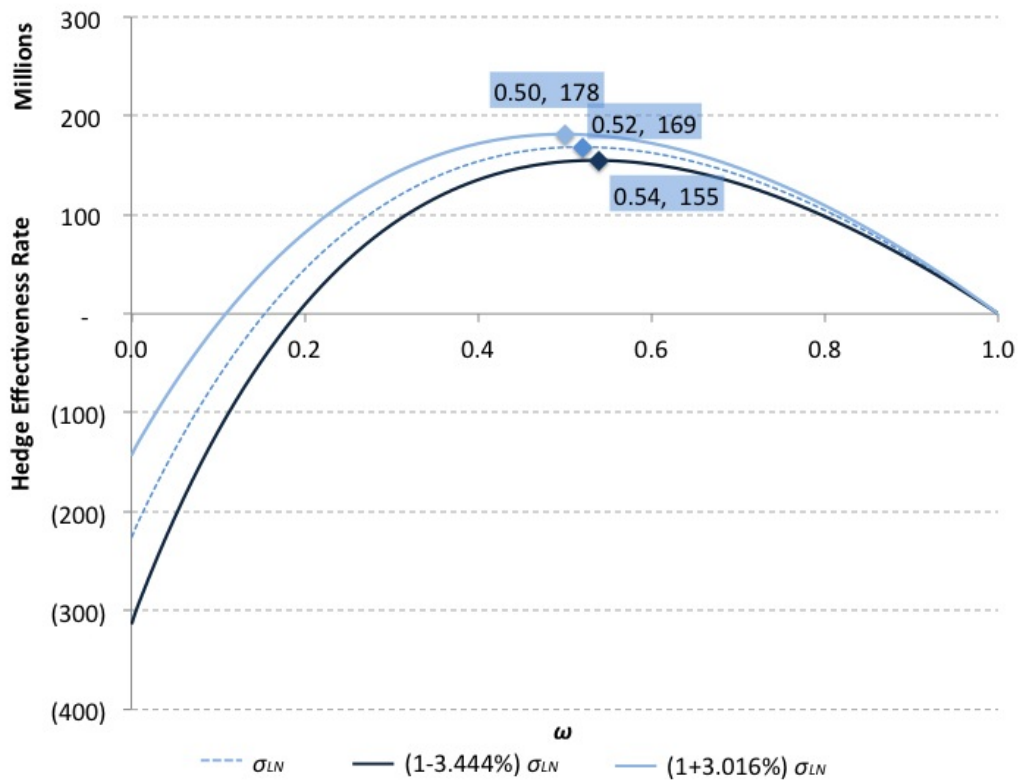
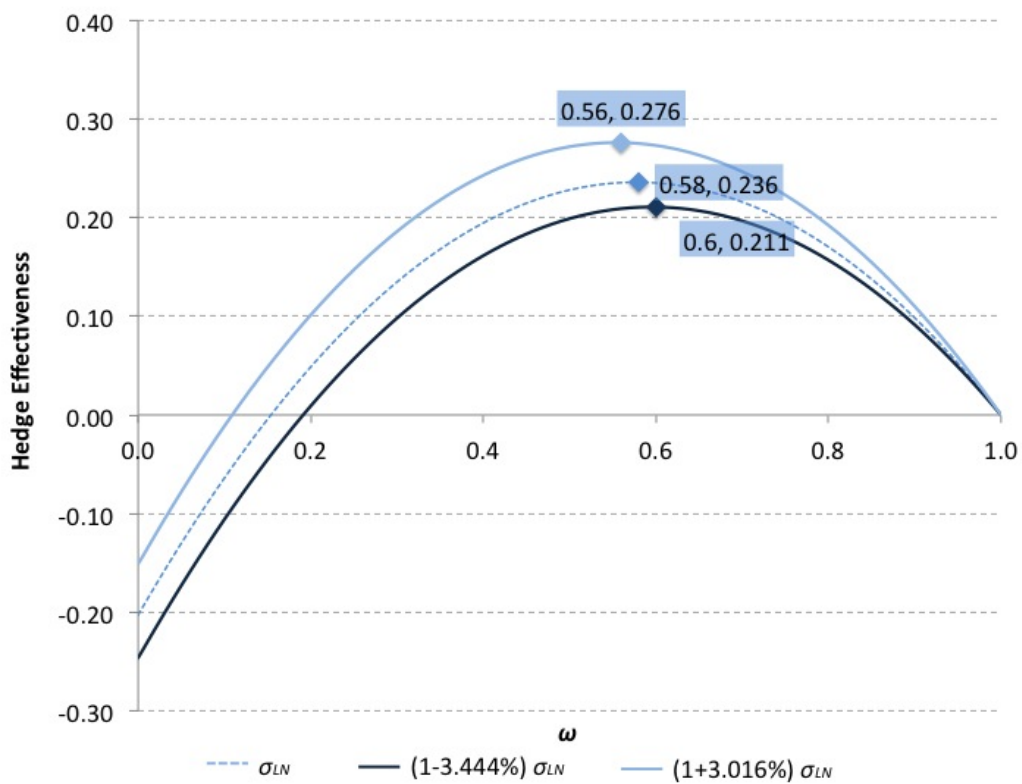


Figure 4.16: Sensitivity of HE to severity on σ_{LN}



We can also observe from Figures 4.11 and 4.12 that at any given ω less than 1, both HER and HE increase in the loss frequency. Given an ω , the price of a CAT bond increases when the loss frequency decreases (Vaugirard (2003)). Meanwhile, when the loss frequency is reduced, we have a smaller aggregate catastrophe loss. Therefore, failing to raise ω results in over-hedging, driving up the variance after hedging, and hence reducing the variance reduction amount. Similarly, an ω that significantly over-hedges in the base scenario will be partially justified when the loss frequency is higher, which results in a higher amount of variance reduction. At the same time, the price decreases as the frequency increases (Vaugirard (2013)). As a result, given the same ω , HER is higher (lower) if we slightly increase (decrease) the loss frequency. The same argument for the changes in the variance reduction amount applies to HE as well. As mentioned above, given an ω , a lower frequency drives up the variance after hedging due to over-hedging. Since HE is defined as one minus the variance after hedging over the variance before hedging, HE decreases when the loss frequency decreases.

Similar patterns can be observed when we adjust the severity up or down by 10% through μ_{LN} and σ_{LN} , respectively, as shown in Figures 4.13, 4.14, 4.15 and 4.16. For μ_{LN} , a 0.549% increase in μ_{LN} results in a 10% increase in the expected aggregate catastrophe loss, while a 0.604% decrease in μ_{LN} results in a 10% decrease in the expected aggregate catastrophe loss. For σ_{LN} , given the other parameters unchanged, the expected aggregate catastrophe loss increases by 10% if σ_{LN} increases by 3.016%, while it decreases by 10% if σ_{LN} decreases by 3.444%. Justifications for the patterns observed in Figures 4.13, 4.14, 4.15 and 4.16 are the same as these for Figures 4.11 and 4.12, which all show that ω^* and ω^{**} decrease (increase) and HER^* and HE^{**} increase (decrease) as the expected aggregate catastrophe loss is raised (reduced) by 10%.

We should note that the argument above for higher HER and HE at any given ω only applies to a slight increase in the expected aggregate loss. When the expected catastrophe loss increases so significantly that the catastrophe loss payments dominate the cash flows, the effects of insufficient hedging will dominate in most cases. Consequently, over-hedging is unlikely and the argument above does not hold. We can look at Figures 4.17 and 4.18 for illustrations. If we increase μ_{LN} by 10%, the expected aggregate loss will be 5.67 ($e^{0.1 \times \mu_{LN}}$) times larger. As a result, to optimize hedging, we need to fully leverage the CAT

Figure 4.17: HER under an extreme scenario: when μ_{LN} increases by 10%

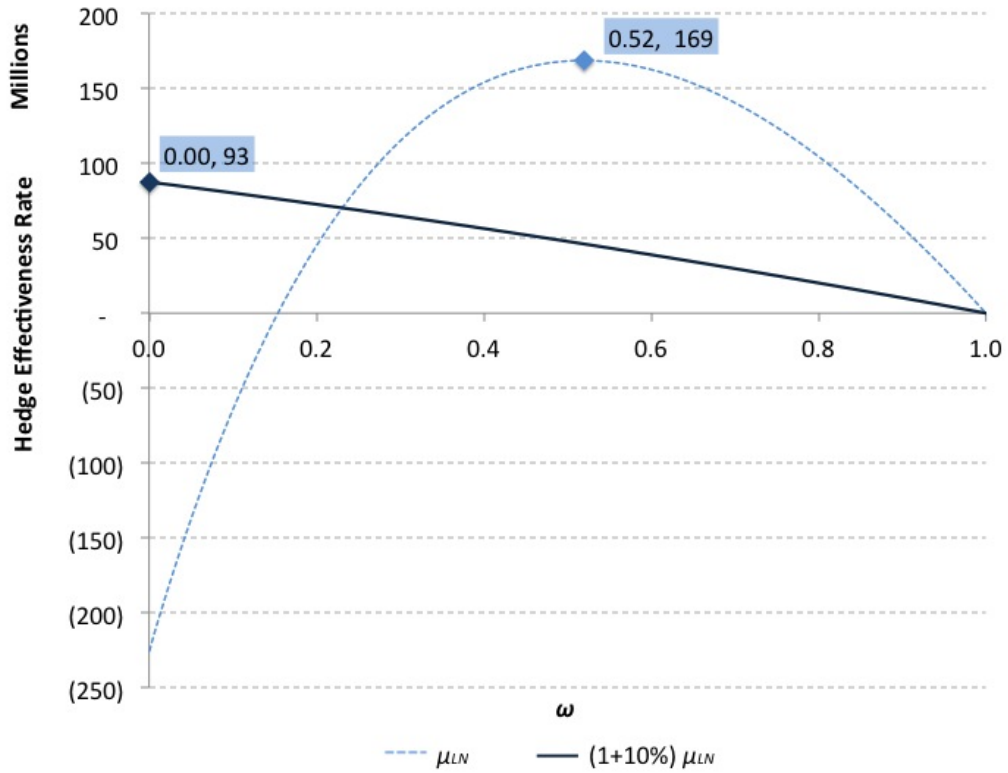
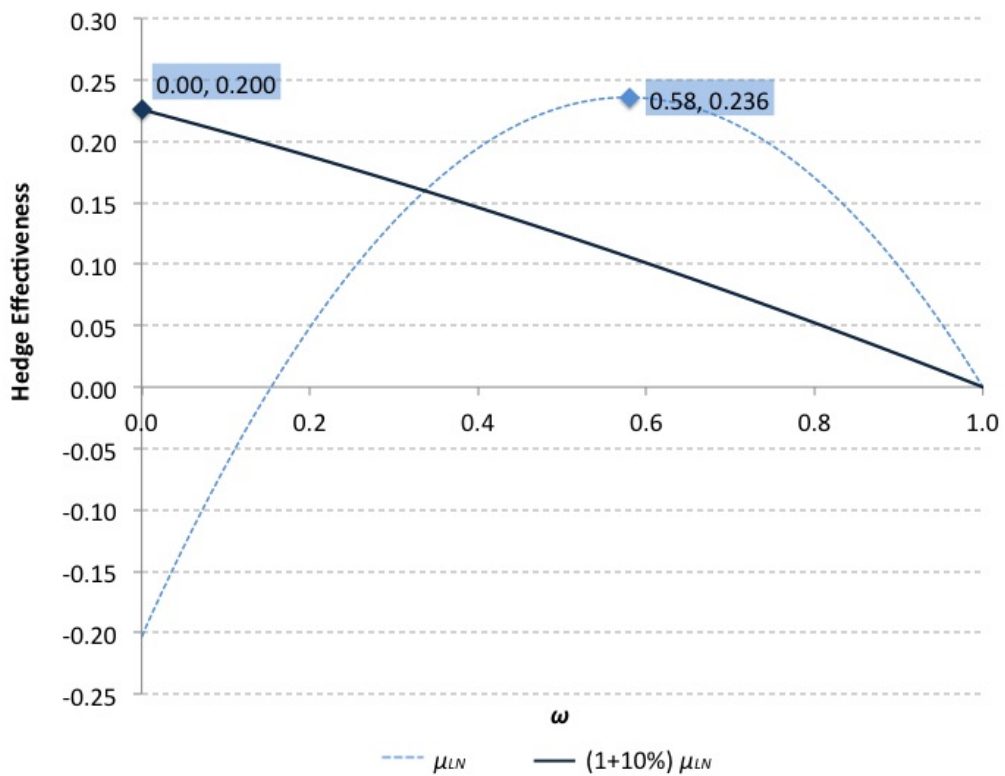


Figure 4.18: HE under an extreme scenario: when μ_{LN} increases by 10%



bond (that is, to use a zero ω) to mitigate the high CAT losses. However, since the payoffs of the CAT bond is too small compared to the CAT losses, the hedging effect of the CAT bond is limited. Consequently, both HE^{**} and HER^* are lower even if we fully hedge.

Chapter 5

Conclusion

This project makes a threefold contribution to the literature of catastrophe bond pricing. First, it gives a price formula for non-zero coupon CAT bonds applicable under various assumptions for a catastrophe loss process. Second, two key risk reduction measurements, HE and HER, defined for catastrophe bonds make it possible to compare CAT bonds with other traditional hedging solutions. Third, the proposed formulas for the optimal payment reduction ratios which maximize the above risk reduction measurements provide precise guidance to CAT bond issuers to make the best decision.

Furthermore, we arrive at the following conclusions through numerical experiments. First, we illustrate empirically that the optimal payment reduction ratios significantly reduce the variance of loss on catastrophes, limit the extent of hedging tradeoffs and manage to capture most part of profitability. Second, we demonstrate that HE and HER at any given ω decrease as the retention level increases. Third, we prove for zero-coupon CAT bonds, given the face value FV is proportional to the loss share m and the retention d is proportional to the strike price K , ω^* and ω^{**} are not affected by the loss share. Fourth, we illustrate that ω^* and ω^{**} decrease as the strike price or the maturity increases. Fifth, we conclude that minor changes in the parameters of the catastrophe loss process will not make significant impacts on ω^* , ω^{**} , HER^* and HE^{**} .

We can extend our work to the field of exotic option pricing in finance. By applying the same concept to a binary option, we can compute the optimal fixed return which is defined as the ratio between the higher and the lower payoffs minus one. Furthermore,

we can incorporate some path dependent features of the Asian option into a binary option. For instance, the "Asian binary option" is triggered when the average stock price before maturity goes over the strike price. In this case, the "Asian binary option" works similarly as a CAT bond.

Last, we acknowledge that hedging with a series of zero-coupon CAT bonds with different face values and different ω_s might be more effective than hedging with a non-zero coupon CAT bond. However, this is beyond the scope of this project.

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Appendix A

Proof.

Since

$$Z(T) = \sum_{i=1}^{N(T)} e^{-\delta T_i} (mX_i - d)_+$$

and

$$d = K * m/\lambda,$$

$Z(T)$ is proportional to m . Consequently, $E[Z(T)|0 < \tau \leq T] - E[Z(T)|\tau > T]$ is proportional to m . From Corollary 4 with $FV = m \times FV_0$ in Chapter 3, we have

$$\begin{aligned} R_0 &= 1 - 2\rho(0, T) - \frac{2(\alpha + \beta)\rho(0, T)}{mFV_0 \times E(e^{-\int_0^T \delta_s ds})} \\ &= 1 - 2\rho(0, T) - \frac{2\{E[Z(T)|0 < \tau \leq T] - E[Z(T)|\tau > T]\} \rho(0, T)}{mFV_0 \times e^{-\delta T}}. \end{aligned}$$

Therefore, the common factor m in both the numerator and the denominator can be cancelled out. As a result, ω^* for a zero-coupon bond is independent of the loss share m .

Appendix B

Proof.

We will first show P_0 , A and $Z^*(T)$ for a zero-coupon CAT bond with $FV = m \times FV_0$ are proportional to m at any given ω , respectively; then we prove HE at any given ω is independent of m .

For a zero-coupon CAT bond,

$$\begin{aligned} P_0 &= m\omega FV_0 \times e^{-\delta T} \rho(0, T) + mFV_0 \times e^{-\delta T} \rho(T, \infty) \\ &= mFV_0 \times e^{-\delta T} [\omega\rho(0, T) + \rho(T, \infty)] \end{aligned}$$

and

$$A = \begin{cases} mFV_0 \times e^{-\delta T}, & \tau > T, \\ \omega mFV_0 \times e^{-\delta T}, & \tau \leq T \end{cases}$$

are proportional to m . Furthermore, since $Z^*(T) = Z(T) - P_0 + A$ and $Z(T)$, P_0 and A are proportional to m as approved in Appendix A and above, respectively, $Z^*(T)$ at any given ω is proportional to m as well. Last, by definition,

$$HE = \frac{Var[Z(T)] - Var[Z^*(T)]}{Var[Z(T)]}.$$

Since $Z(T)$ and $Z^*(T)$ are proportional to m , both $Var[Z(T)]$ and $Var[Z^*(T)]$ are proportional to m^2 , and thus the common factor m^2 in the numerator and the denominator of HE is cancelled out. Therefore, HE is independent of m at any given ω . Consequently, HE** is independent of m .