Teaching Through Problem Solving: Bridging the Gap Between Vision and Practice

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B.Sc., Simon Fraser University, 2005

Thesis Submitted in Partial Fulfillment
of the Requirements for the Degree of
Master of Arts

In The
Individual Program
Faculty of Education

SIMON FRASER UNIVERSITY

Summer 2012

Approval

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Degree: **Master of Arts (Education)**

Title of Thesis: ***Teaching Through Problem Solving: Bridging the Gap Between Vision and Practice***

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Date Defended/Approved: August 10, 2012

Abstract

This thesis examines a mathematics classroom where students are taught mathematics through problem solving. Weekly observations of two different classrooms were done. One was an Honors of Math 10 class and the other was a Math 11 class. In addition to the observations, student interviews were conducted throughout the semester with the intent of speaking to a number of students with a broad range of ability levels and diverse attitudes about mathematics. The aim was to explore whether or not students will buy in. Is it possible to create this dynamic, problem solving environment consistently within a classroom? Can it be done within the context of the current curriculum? How does teaching through problem solving affect low achieving students? And how does this pedagogy influence students' relationships with mathematics? The findings show that creating this environment is possible even within the current provincial curriculum and moreover, that this pedagogy generally led students to experience a deeper engagement with mathematical ideas.

Keywords: Problem Solving; Mathematics; Education; Engagement; Relationship with Mathematics; Affect

To my family,

for their unconditional support and encouragement during this time.

Acknowledgements

I would like to thank Dr. Peter Liljedahl for his contributions as senior supervisor during the development of this thesis. Without his contagious passion for mathematics education and problem solving this thesis would never have come to be.

I would also like to thank Dr. Stephen Campbell for his contribution to shaping the final document.

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Chapter 1.

Introduction

The first time I experienced strong feelings about mathematics education was in Grade Five. My teacher was trying to teach the long-division algorithm and I was staring, completely perplexed, at the board as we multiplied and subtracted and numbers moved up and down. It felt random – seemingly with no rhyme or reason. I could not figure out why the numbers were moving as they were or what we were trying to accomplish. I certainly could not reproduce these mysterious manipulations. I remember needing extra help and my teacher sat with me and did several more examples of long-division to no avail. They all seemed equally mysterious to me. Eventually she gave up and declared that I would never be any good at "Math". This statement appeared on my first fifth-grade report card.

The next time that I had a formative experience in mathematics education was in junior high. I wrote the Fermat mathematics contest. I was completely enthralled with the problems that I found there. It was like nothing I had ever experienced in a mathematics classroom. We had to think and trouble-shoot – and we had not been "taught" how to do any of the problems. These contests broke away from the traditional mathematical experience in which students are only given problems on a test that they have been previously shown how to do. Instead, they were specifically designed with the intent that students would not have any prior knowledge of how to solve the problems. Students were given marks for innovative mathematical thinking. The problems fascinated me and I continued to write every mathematics contest that I could get my hands on.

These contests were considered optional and auxiliary to the classroom experience. I never had a teacher address the problem solving content of these contests within the scope of the traditional environment. We were told about the existence of the contests and students who wished to write it were sent to the library to do so. However, there was no notion that students could prepare for their experiences in writing the contests. Moreover, there was no discussion or reflection on the contest experience within the classroom environment.

Outside of the mathematics contests, however, my classroom experience as a student was very traditional. We worked primarily from textbooks. Almost every lesson began with notes and was followed by an assignment from the text. We were assigned only the odd numbered questions because these were the ones with the answers in the back. Occasionally, we would have a quiz and each unit was concluded with a test that was mostly multiple-choice with a few written response questions at the end.

These experiences illustrate one of the common fundamental problems in mathematics education today. Mathematics is often taught as a series of steps to memorize and reproduce. These steps are dictated by the teacher who is the source of mathematical knowledge in the classroom. This is not the type of mathematical experience that contemporary mathematics curriculum calls for. The social conventions of algebra and the various mathematical notations are simply a language to communicate much grander ideas. If teachers focus only on teaching conventions and 'rules', students are missing out on the essence and beauty of the subject. It is no wonder that so many students leave school thinking that mathematics is boring and useless. In my experiences as a student within the British Columbia educational system, I felt that many of my peers left school with this significant hole in their mathematical educational experience.

As a student teacher, I was introduced to the idea of teaching mathematics through problem solving and I felt like I had found my passion. To me, this was how mathematics should be experienced. I started developing a vision of what my classroom would look like. Students would be engaged and working together to solve problems. It would be as exciting to them as the mathematics contests were to me. There would be an atmosphere of enthusiasm as students shared ideas and helped each other. I would be there to encourage and guide the class in their experience of mathematics. Somehow, through this we would cover the entire curriculum and my students would still do well on the exams that are so important to administrators and educators in this province. I believed in this vision. It spoke to me and always resonated as the way that mathematics should be experienced by students.

And yet, my souvenirs from my first teaching contract are a collection of 4-inch binders sitting on my bookshelf containing very precise fill-in-the-blank notes on every topic that my students need to learn about along with worksheets, pre-tests, and (nearly identical) tests, dozens of practice government exams, and the occasional project (to be done if time permits). Why is this? How can it be that my ideals and my practice are so far apart? If I believe that teaching through problem solving is the best way for students to experience and learn mathematics, why is problem solving only an afterthought in my practice? Am I alone in my experience?

I believe that my answer to the last question is “no”. I am not alone. I suggest that the reason for this is rooted in fear. When teachers enter the teaching profession there are a lot of external pressures. Pressure to fit in with the other teachers, to use the same textbooks, and give standard tests. There is also the pressure of the government exams. No new teacher wants their students to do poorly at the end of the year. I feared failure. How could I radically depart from the traditional teaching environment if I was not sure that my less traditional methods would really work in practice? There are many stakeholders in the education system; the most important of which are the students. How can I take such a large risk in my classroom when the price of my mistakes could be paid by my students?

Within my professional practice, one of my largest goals is to create a dynamic learning environment for my students. This will not be possible until these fears are addressed. The first hurdle to overcome is to answer some questions surrounding feasibility that have prevented me from attempting to implement teaching through problem solving in my classroom. These questions are:

1. Will students buy in?
2. Is it possible to create this dynamic problem solving environment consistently within a classroom? Can it be done within the context of the current curriculum?
3. How does teaching through problem solving affect low-achieving students?

4. How does a problem solving environment impact students' relationship with mathematics?

In writing this thesis, I have done an overview of literature that has shaped my beliefs on teaching mathematics through problem solving. This literature review is presented in Chapter 2. Through my own personal experience and the research that was conducted in the literature review some questions emerged and these questions form the specific goals of this thesis which are discussed in detail in Chapter 3. Research was then conducted in a mathematics classroom that is being taught largely with the philosophy of teaching through problem solving. The methodology of this research is presented in Chapter 4. Chapter 5 is a brief excerpt dedicated to painting a picture of the research classroom. The results of the research and some qualitative data are presented in Chapter 6. Finally, Chapter 7 and 8 provide some conclusions and discussion to bring the research together with the literature review in order to answer the research questions.

Chapter 2.

Literature Review

“The man who does not read books has no advantage over the man who cannot read them.”

- Mark Twain

I am an educator. I am also a life-long learner. Literature has been fundamental to my own professional development and has played a formative role in both my vision for a classroom and my pedagogy. It has served both to inspire and to inform. The following is a summary of the literature that has proven most influential to me in my work on this thesis. When I was first introduced to the idea of teaching mathematics through problem solving, in my teacher training program, I was inspired to do further reading. This naturally led to the work of the National Council of Teachers of Mathematics. The NCTM served as a jumping off point and led me to delve deeper into the topic of teaching through problem solving. The literature summarized in this review has been collected over time and has fundamentally shaped my perspectives on mathematics education. The culmination of all of these readings resulted in the development of the four research questions that form the basis of this thesis.

The literature that shaped my motivation, research questions, and methodology throughout this thesis fell into two categories. The first category was literature directly related to problem solving and teaching mathematics through problem solving. Within this problem solving literature, I present the work of Polya to provide some historical context, followed by *Principles and Standards for School Mathematics,* the work of the National Council of Teachers of Mathematics, which provides a vision of teaching through problem solving. The work of the NCTM provided inspiration for me to look further into the ideas of teaching through problem solving and this lead me to read works by Leone Burton, John Mason, Kaye Stacey, and Robert Fischer. The literature on problem solving that is presented in this review shaped my motivation and research questions, as well as my choice of classroom. It also influenced the lens through which I completed the observations and interviews and analyzed the data.

Problem solving has a significant affective component. It was clear, upon doing these readings on problem solving and attempting to teach through problem solving in my own classroom, that any quest to effectively implement teaching through problem solving would require significant research on affect. As such, the second category of literature included in this review consists of some works that contributed useful theories of affect. This literature provided frameworks on which to analyze the data that was collected through the observations and the interviews. Through the process of examining student beliefs I initially struggled to communicate a sense that these beliefs about mathematics were being socially developed and were changing over time in the classroom. The work of Yackel and Cobb in the area of sociomathematical norms provided a theoretical foundation and vocabulary on which to clearly express these ideas and analyze belief-related data.

A. Problem Solving Literature

Historical Context

Polya's work *How to Solve It* was written in 1945. It served, and continues to serve, as a prevailing text on how to teach problem solving in mathematics. This work has had a significant impact on how problem solving is presented in mathematics classrooms to this day and thus was formative in creating the current educational environment that inspired the work within this thesis. Polya addresses many of the predominant issues in teaching problem solving including:

- The teacher's role in teaching problem solving
- A process for problem solving
- Issues around being stuck
- The importance of reflection
- The value of problem posing.

In this work, Polya states that students learn to solve problems through imitation and practice (Polya, 2004). He writes; "When a teacher solves a problem before the class, he should dramatize his ideas a little and he should put to himself the same questions which he uses when helping the students" (Polya, 2004, p. 5). He paints a picture of the teacher asking a series of questions to guide the student and argues that whenever possible, the questions that are asked should be as general as possible so that the same questions can be used in almost any problem. As questions are repeatedly useful, the students should eventually internalize them and be able to ask them to themselves when they get stuck. He emphasizes questions like "Did you use all the data?", "What is the unknown?", "Do you know a related problem?" and "What are the data?" He writes "Begin with a general question or suggestion of our list, and if necessary, come down gradually to more specific and concrete questions or suggestions till you reach one which elicits a response in the student's mind" (Polya, 2004, p. 20).

While Polya argues that students must do as much of the problem solving on their own as is possible, he portrays an image of problem solving that shows the teacher quite involved in the student's work. The teacher is ready and able to ask guiding questions should the student get stuck.

In *How to Solve It,* Polya presents four phases of mathematical problem solving: understand the problem, make a plan, carry out the plan, and review and discuss (Polya, 2004). He argues that for the first phase, the student should be able to state the problem fluently and point out the unknown, the data, and the condition in the problem. Thus, "the teacher can seldom afford to miss the questions: *What is the unknown? What are the data? What is the condition?"* (Polya, 2004, p. 7). If applicable, the student should then draw a figure and point out the unknown, the data, and the condition. For the second phase, the student must make a plan. Polya argues that this will require a bright idea. In this phase, "the best that the teacher can do for the student is to procure for him, by unobtrusive help, a bright idea" (Polya, 2004, p. 9). The teacher should then ask leading questions to help the student figure out a bright idea. The teacher should be prepared if even fairly explicit hints are "insufficient to shake the tupor of the students; and so he should be prepared to use a wide gamut of more and more explicit hints" (Polya, 2004, p. 12). In the third phase of carrying out the plan, the teacher's job is mainly to ensure that the student checks each step. In the final phase the teacher should encourage to student to check their result, see if they can derive it differently, and examine whether their solution could connect to any other problems (Polya, 2004).

In discussing being stuck, Polya writes "No idea is really bad unless we are uncritical. What is really bad is to have no idea at all" (Polya, 2004, p. 99). Polya believes that being stuck on a problem is not a good state. When a student becomes stuck Polya suggests specializing and perhaps trying to solve some related subsidiary problem. He writes "human superiority consists in going around an obstacle that cannot be overcome directly, in devising a suitable auxiliary problem when the original problem appears insoluble" (Polya, 2004, p. 51). He also recommends that taking a break from the problem may be useful. "The fact is that a problem, after prolonged absence, may return into consciousness essentially clarified, much nearer to its solution that it was when it dropped out of consciousness" and "there are certain moments in which it is better to leave the problem alone for a while" (Polya, 2004, p. 198).

Previously, it was mentioned that Polya writes about students learning to solve problems through imitation and practice. It is important here to emphasize that the value of reflection is not absent from Polya's writing. Throughout his book Polya consistently mentions the importance of the teacher making the student check each step for accuracy and correctness. Moreover, his fourth phase of problem solving is to review and discuss the completed solution. He writes:

> *By looking back at the completed solution, by reconsidering and re-examining the result and the path that led to it, they could consolidate their knowledge and develop their ability to solve problems. A good teacher should understand and impress on his students the view that no problem whatever is completely exhausted. Their remains always something to do; with sufficient study and penetration, we could improve any*

solution and, in any case, we can always improve our understanding of the solution (Polya, 2004, p. 15).

For Polya, problem posing becomes an extremely important part of the problem solving process. He writes “Good problems and mushrooms of certain kinds have something in common; they grow in clusters. Having found one, you should look around; there is a good chance that there are some more quite near” (Polya, 2004, p. 65). He argues that any result that we obtain by solving one problem can open up new possibilities for solutions to new problems. The challenge, in his mind, is to find problems that are both interesting and accessible. Polya also recommends problem posing as a way to regain interest in a problem when our thoughts begin to wander and we are “in danger of losing the problem all together” (Polya, 2004, p. 210).

Presenting a Vision

In *Principles and Standards for School Mathematics (2000)*, the National Council of Teachers of Mathematics, NCTM, presents their vision of what mathematics education should be. They write:

> *Imagine a classroom, a school, or a school district where all students have access to high-quality, engaging mathematics instruction. There are ambitious expectations for all, with accommodation for those who need it. Knowledgeable teachers have adequate resources to support their work and are continually growing as professionals. The curriculum is mathematically rich, offering students opportunities to learn important mathematical concepts and procedures with*

> *understanding. Technology is an essential component of the environment. Students confidently engage in complex mathematical tasks chosen carefully by teachers. They draw on knowledge from a wide variety of mathematical topics, sometimes approaching the same problem from different mathematical perspectives or representing the mathematics in different ways until they find methods that enable them to make progress. Teachers help students make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or disprove those conjectures. Students are flexible and resourceful problem solvers. Alone or in groups and with access to technology, they work productively and reflectively, with the skilled guidance of their teachers. Orally and in writing, students communicate their ideas and results effectively. They value mathematics and engage actively in learning it (NCTM, 2000, A Vision for School Mathematics, para. 1).*

The NCTM admits that this is an idealized vision. However, this is a vision that I believe we should be working towards. In the Standards documents, the NCTM offers ways to help educators move in this direction. Their document is divided into two categories. The first is “Content Standards”, which describes the specific content that should be learned. The second is “Process Standards”, which highlight the ways that students should acquire and use their mathematical knowledge (NCTM, 2000). It is the Process Standards that have greatly influenced this research.

The first Process Standard is Problem Solving. The NCTM argues that Problem Solving should not be an isolated part of the mathematics curriculum, but rather should be integrated throughout each of the content standards. In fact, it should be one of the primary means of acquiring new mathematical understanding (NCTM, 2000). They write “students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking” (NCTM, 2000, Problem Solving Standard, para. 1).

The second Process Standard is Reasoning and Proof. The NCTM would like to see students recognize reasoning and proof and fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and investigate mathematical arguments and proofs, and select and use various types of reasoning and methods of proof. “Students should see and expect that mathematics makes sense” (NCTM, 2000, Reasoning and Proof Standard, para. 2). While this declaration seems trivial, it is not actually always assumed by students that mathematics should make sense. Many students that I have taught expressed surprise when I insisted that they should not write things down that do not make sense to them. In fact, they had no expectation that mathematics would ever make sense. I wanted students to ask questions and engage with the ideas being presented until the concepts made sense. I did not want them to simply believe me and copy down my words because I was in a position of authority. Students can become accustomed to the idea that mathematics is a series of steps, dictated by the teacher, and that repeating the steps correctly will return the same answer as the back of the book. However, there may be no expectation that these steps are logically derived, or that they could solve problems themselves without being given an algorithm to follow. The NCTM (2000) argues that this standard cannot be met simply by incorporating a unit on geometry proofs, for example, into the curriculum. The pedagogical goal of valuing proof must be integrated throughout students' classroom experiences.

The third Process Standard is the “Communication Standard”. The NCTM (2000) directs that students should be able to organize and consolidate their mathematical thinking through communication, communicate their mathematical thinking clearly, analyze and evaluate the mathematical thinking and strategies of others, and use the language of mathematics to express mathematical ideas precisely. They argue that conversations exploring mathematical ideas from multiple perspectives help students sharpen their thinking and make more connections. Moreover, deeper understanding can be gained through mathematical disagreements and debate (NCTM, 2000). This level of debate and discussion cannot be accomplished if a classroom environment is established where the teacher is the authority and the students' role is to listen. In particular, procedure-based activities do not usually provide opportunities for worthwhile discussion. On the other hand, interesting problems that create opportunities for puzzlement tend to provide opportunities for rich discussion.

The fourth Standard is the “Connections Standard”. Within it the NCTM (2000) argues that students should be able to recognize and use connections among mathematical ideas, understand how mathematical ideas interconnect, and recognize and apply mathematics in contexts outside of mathematics. Mathematics should not be presented as a series of disconnected sub-topics of study.

All of these standards are supported by a classroom environment that is rooted in problem solving. However, there is a tension that exists between the vision that is presented in these standards, a vision that most educators would support, and the reality of what is being implemented in classrooms. The challenge for many educators, including myself, is to bridge the gap between our current educational practice and this vision.

Getting Practical

The *Teaching Mathematics Through Problem Solving* books were put out by the NCTM. In these books, the NCTM reiterates their vision for mathematics education. The classroom vision that is described within these books provided the inspiration for this thesis. The NCTM then extends this vision and offers practical advice for teachers on how to teach mathematics through problem solving. They recommend the following process for teaching mathematics through problem solving:

> *1. Identify the important mathematics that students are to learn.*
> *2. Choose a problem that is likely to connect with this mathematics.*
> *3. Pose the problem, and make sure that students have sufficient understanding of it without solving it for them.*
> *4. Allow them to explore the problem and come up with one or more solutions.*
> 5. *Guide the students in reflecting on the problem, their work, and the mathematical ideas that were uncovered* (NCTM, 2003, p. 15).

The primary argument that the NCTM makes is that teaching mathematics through problem solving leads to greater levels of understanding on the part of the students. They argue that “knowing how to execute procedures does not ensure that students understand what they are doing. To understand, students must get inside these topics; become curious about how everything works” (NCTM, 2003, p. 3).

One of the most striking ideas in these the book for secondary teachers is that teachers need to allow mathematics to be problematic for students. As a mathematics teacher, it is easy to fall into the role of trying to remove the struggle for our students. We often want to make mathematics as clear, understandable, and easy for our students as possible. This instinct comes from a place of great compassion. We do not want to see others struggling and so we react by providing solutions – a clear path for passing the course and the final exam. It is this compassion and desire to relieve struggle that has resulted in a collection of binders, each filled with step-by-step notes and examples, that is currently sitting in boxes in my basement. However, *Teaching Mathematics Through Problem Solving* argues that it is precisely the mathematical struggle that builds strong mathematicians. We want our students to “emerge as young mathematicians, not mathematical libraries” (NCTM, 2003, p. 21).

This philosophy of allowing mathematics to be problematic for our students changes the role of the teacher from that of an informational front-end loader. We would need to know how to give our students the right information at the right times – and not before. The NCTM gives three rules of thumb for this. First, students should not be expected to derive or discover social conventions on their own. Things like notation and symbols should be presented by the teacher. However, the right time to do this is *when students need them* (NCTM, 2003, p. 11). Secondly, teachers can present alternative solutions that the students did not come up with on their own. Thirdly, teachers can highlight ideas that are imbedded in the students' methods. Often students will come up with a method without realizing the full implications of their idea. Teachers can often provide deeper insight in this regard.

The NCTM gives three reasons that we should be teaching mathematics through problem solving (NCTM, 2003, p. 20):

> 1. It helps students understand that mathematics develops
> through a sense-making process.
> 2. It deepens students' understanding of underlying mathematical
> ideas and methods.
> 3. It engages students' interest.

This book dedicates a chapter to research that has been done in the area of problem solving. The authors summarize that U.S. students who have worked with problem solving approaches tend to perform at least as well, or better, than students who have worked traditionally. However, they emphasize that this kind of large-scale research does not address the pedagogical differences between individual teachers. They argue that further research into teaching mathematics through problem solving "should focus on the details of classroom implementation in addition to studying student outcomes associated with the adoption of particular curricula" (NCTM, 2003, p. 255).

Overall, these books provide a very strong philosophical and practical foundation for teachers interested in teaching mathematics through problem solving. They are designed to be a strong resource for teachers interested in moving to a more progressive pedagogy and they serve to offer solutions to many practical questions that educators making this transition would have.

The Level of the Learner

Thinking Things Through, by Leone Burton, is a resource on how to incorporate problem solving successfully in to a mathematics classroom. In it, Burton addresses the following questions:

1. What makes problem solving different?
2. Why is problem solving important?
3. How to get going
4. The problem solving process
5. What about individual differences?
6. Have the students gained?

In response to the first question, “What makes problem solving different?”, Burton argues that students' attitudes will be different if taught through problem solving than if they are taught in a traditional manner. She argues that when students are taught through problem solving more emotional energy is expended in the process and a deeper commitment is made to the conclusions (Burton, 1984).

In response to the second question about the importance of problem solving Burton writes:

> *The greatest value of this approach is in the effect it has on the classroom. Hesitancy and dependency in pupils are replaced by confidence and autonomy.*

> *Dislike turns to enjoyment, indeed enthusiasm. Low self-images are replaced by expressions of authority... The overwhelming importance of problem solving, therefore, is in the opportunity it provides for teachers and pupils to enter into a spirit of enquiry and through that spirit to establish different styles of teaching and learning* (Burton, 1984, p. 10).

When discussing the issue of how to get going, Burton argues that providing pupils with a choice of problems is important and will help students feel that the problem 'belongs' to them. This idea of the problem belonging to the students is key. She argues that this choice will affect students' attitudes positively (Burton, 1984). Another significant point that Burton makes is that the teacher's role needs to shift from providing information to asking questions. It is important that at all times it is the student doing the thinking. Otherwise, the result is that the teacher takes ownership of the problem away from the student. In a successful problem solving environment the responsibility for thinking rests with the student (Burton, 1984).

On the issue of whether or not the students have gained, Burton writes "Frequently pupils perceive what is being taught quite differently from the teacher. In looking for pupil gains in these circumstances, the teacher might decide that the lesson was not successful whereas the pupils might have learnt something quite powerful that was unintended" (Burton, 1984, p .50). She suggests some specific signs that the students have benefited from their problem solving experience. One sign is a sense of commitment to what is being done. Another is a sense of enjoyment. She points out that these signs are qualitative rather than quantitative in nature. Evidence of these benefits needs to be gathered through listening and observation which requires an educator to be paying attention to more than just the academic goals within a classroom.

Introducing the Extra-Logical

Thinking Mathematically is intended as an interactive book requiring the reader to engage in mathematical problem solving. In it, Mason, Burton, and Stacey (1985), break problem solving into three phases: entry, attack, and review. Their goal is for the reader to engage in their own process of mathematical thinking and then transfer their skills and help develop mathematical thinking in others (Burton, Mason, & Stacey, 1985). Chapter 9 is dedicated to developing mathematical thinking. In it, the authors argue that focusing on the emotional and psychological states of problem solvers seems to be a necessary part of helping people develop creative mathematical thinking. The approach that they recommend involves practice and reflection. However, they make the important point that this practice and reflection take time:

> *The rapid question/answer format of many mathematics classrooms is the antithesis of the time and space upon which developing mathematical thinking depends; so is the notion that mathematical thinking is the product of practising on repetitive mathematical examples, each done as quickly as possible* (Mason et al., p. 149).

They argue that time needs to be taken to allow for real engagement with the problem and for quality reflection.

The authors point out that a teacher needs to build a supportive environment that will help build mathematical confidence in their students. This confidence can only be built when students achieve and reflect on personal successes. They define this supportive environment as being questioning, challenging, and reflective (Mason et al., 1985). “There is a significant difference between an atmosphere which expects children to supply correct answers, and one where conjectures are made, challenged and modified, where the demand is Convince!” (Mason et al., 1985, p. 154).

Beyond Facts

In *Teaching Children to Think*, Robert Fisher addresses the issue of “if education is supposed to be about teaching young people to think, why does the education system produce so many unthinking people?” (Fischer, 2005, p. 128). He claims that thinking can be taught (or at least facilitated) by mentors in the child’s life. He begins with what most of us already believe: reproductive learning continues to be necessary but it is no longer sufficient. What our children need for their future survival is innovative learning. He argues that creative thinking and critical thinking are not separate entities but are interrelated. Moreover, he believes that the boundaries between the arts and sciences need to be broken down and replaced with problem solving as the heart of education (Fischer, 2005). In mathematics, we should not just be teaching critical thinking; we should be teaching creative thinking as well. This can partially be done through problem solving which applies both creative and critical thinking (Fischer, 2005).

Fischer lists some strategies for supporting children in problem solving (Fischer, 2005, p. 91):

1. Describe with interest what the child is doing. “It looks to me as if you are....”
2. Ask the child what she is doing.
3. Support the process when needed.

The third point here is tricky. Children sometimes need clues or to be shown a tool to help them solve the problem. However, it is important that ownership of the problem remains with the child.

Fischer also argues that good problem solvers have confidence in their ability. He recommends that teachers start with easy tasks to build confidence and encourage children to take their time (Fischer, 2005). Not many students enter a high school mathematics class confident in their ability to solve problems. Fischer argues that “a sense of incompetence is one of the most difficult deficiencies to reverse. It can be repeatedly confirmed by a child’s low expectations and become a self-fulfilling prophecy [...] Positive strategies are needed that will mediate in the child a confidence in his own reasoning powers. This self-reliance can be built up only by a pattern of successful responses to challenge” (Fischer, 2005, p. 112).

However, providing these positive experiences is just the first step. Children must also be able to effectively reflect on these experiences. How do you get children to be reflective? Fischer argues that you do it by encouraging discussion and getting students to talk things through. He writes “there is a common misapprehension that it is thinking or reflection that generates the need to talk. More often it is dialogue that generates reflection and thoughtful response” (Fischer, 2005, p. 130).

Moving education beyond a focus on facts and into a focus on thinking, reasoning, and problem solving is essential – and it is a common refrain. Again, educators need to search for ways to align these values with practice.

B. Affect and Beliefs Literature

Problem solving is an unavoidably emotional phenomenon. These emotions have the potential to be either a help or a hindrance depending on students' beliefs about themselves and about mathematics. By their very nature, problems generally evoke strong affective responses and these responses, in turn, affect how students approach and engage a problem. Students' beliefs, attitudes, emotions, and values will be a dominant factor in whether or not they are able to genuinely engage in a problem. As such, affective issues are likely to be a determining factor in whether or not teaching through problem solving can be effective and were central to this research.

Attempting a Definition

One of the pinnacle academic works in research on affect in mathematics education is *Affect and Mathematical Problem Solving: A New Perspective* edited by D.B. McLeod and V.B. Adams. Prior to the publication of this book in 1989, there was little agreement among researchers on a working definition of the term “affect”. In the final chapter, ‘Beliefs, Attitudes, and Emotions: New views of Affect in Mathematics Education’, McLeod argues that describing affect as comprised of beliefs, attitudes, and emotions offers a useful framework for analyzing affective factors in problem solving (McLeod & Adams, 1989). These three factors vary in their intensity and their stability as well as their cognitive involvement (McLeod & Adams, 1989).

Teaching through problem solving is likely to produce strong affective responses from students and the traditional practice of teachers to try and minimize or eliminate emotional responses is not sufficient to make affect a constructive component of the math classroom. McLeod and Adams' work is based on the work of George Mandler who argues that most errors in problem solving will lead to negative affect. This may result in abandonment of the task, panicky quasi-random attempts at a solution, and general disorganization. In contrast “smooth progression of some planned course of problem solving will produce little in the way of arousal” (McLeod & Adams, 1989, p. 13). However, even though stress and strong emotions tend to inhibit cognition, “a well-informed individual can use stress constructively” (McLeod & Adams, 1989, p. 10). Moreover, under this view of emotions, if students expect errors to occur when problem solving then the errors will not be as surprising and thus will generate less extreme emotional responses. Along this line of reasoning, Mandler does not believe that students should be protected from errors in problem solving. “It is advantageous - at least in the problem solving situation – for the individual to have been exposed to the school of hard knocks” (McLeod & Adams, 1989, p. 15). He argues that students need to be prepared for errors in problem solving. That being said, he does also recommend developing less demanding acquisition paths for the more emotionally reactive learners (McLeod & Adams, 1989).

A Tetrahedral Model

Previously, I discussed McLeod and Adams' definition of affect as being comprised of emotions, attitudes, and beliefs. In a later work by Valerie A. DeBellis and Gerald A. Goldin entitled *Affect and Meta-Affect in Mathematical Problem-Solving: A*

Representational Perspective this view of affect is extended to include a fourth sub-domain of 'values/morals/ethics' (DeBellis & Goldin, 2006). They present a tetrahedral model of the subdomains of affect where each vertex can be understood to be interacting dynamically with the others within each individual problem-solver.

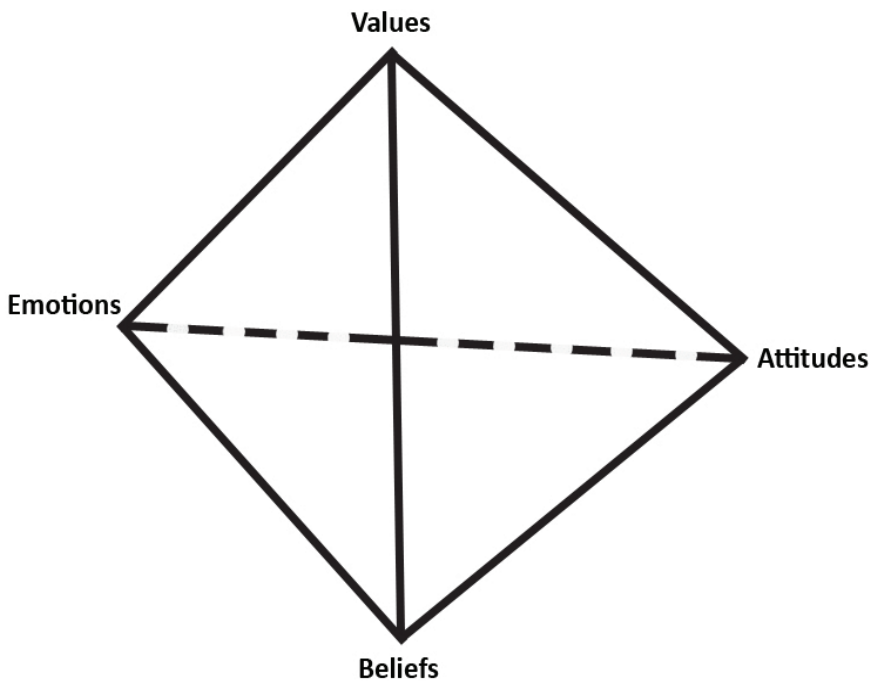

Figure 1.

Moreover, each vertex also interacts with the corresponding vertex in other individuals; furthermore each individual's affect also is affected by the social and cultural conditions surrounding the individual within the learning environment (DeBellis & Goldin, 2006). They argue that "affect may empower or disempower students in relation to mathematics" (DeBellis & Goldin, 2006, p. 134). DeBellis and Goldin's (2006) definition of affect is the definition of affect that I will be using throughout this thesis.

DeBellis and Goldin also address the ideas of mathematical intimacy and mathematical integrity. Mathematical intimacy “involves deeply-rooted emotional engagement, vulnerability, and the building of mathematical meaning and purpose for the learner” (DeBellis & Goldin, 2006, p. 137). Mathematical intimacy connects the mathematics with one’s sense of self-worth. However much teachers of problem solving may wish for their students to experience this kind of mathematical intimacy, DeBellis and Goldin also point out that part of the problem solving experience is ‘intimate betrayal’ caused by unexpected outcomes, failures, or the negative reactions of others. They argue that learning to survive this betrayal is a meta-affective capability (DeBellis & Goldin, 2006). Mathematical integrity “describes an individual’s affective psychological posture in relation to when mathematics is ‘right’, when a problem solution is satisfactory, when the learner’s understanding suffices, or when mathematical achievement deserves respect or commendation” (DeBellis & Goldin, 2006, p. 138).

An Alternate Framework

There is research that offers contrary, or perhaps alternative, theoretical frameworks to those of McLeod, Goldin, and DeBellis. In a paper entitled “Embodied Cognition and the Mathematical Emotional Orientation” by Edward N. Drodge and David A. Reid, the authors reject the dichotomization of emotion and reason that had been made by previous authors like McLeod, Goldin, and DeBellis. It is impossible to do justice to the complexities of their theory in these few short pages but a brief description of their key ideas will provide an alternate framework to that of McLeod, DeBellis and Goldin which also frequently proves useful for analyzing affective responses of students when they are engaged in problem solving. Drodge and Reid argue that cognition “is a unified activity incorporating perceiving, emotioning, reasoning, acting, and being" (Drodge & Reid, 2000, p. 251). They reference Damasio (1994) in arguing that emotioning and reasoning are not two separate activities that influence each other but are, rather, two ways of looking at one activity and also that emotioning is a fundamental part of decision making (Drodge & Reid, 2000).

In this work, the authors present the idea of a mathematical 'emotional orientation' and offer it as a useful theoretical framework for analyzing mathematical activity. Briefly, they describe emotional orientation as the shared beliefs, values, ways of communicating, and forms of acting that members of an academic discipline share (Drodge & Reid, 2000). To lend some clarity to this concept, Drodge and Reid provide some examples of emotioning that they consider to be part of the mathematical orientation. The list includes (but is not limited to): feeling that isomorphic structures are significant, requiring more than examples for a convincing argument, valuing patterns, and wanting to make connections (Drodge & Reid, 2000). Some elements of the mathematical emotional orientation include:

> 1. *Behaving as if mathematical objects are real.*
> 2. *Behaving and claiming that deductive logic is a natural, inevitable, and specific way of reasoning.*
> 3. *Being able to recognize and produce proofs.*
> 4. *Preferring proofs and theorems that are "elegant"*
>
> (Drodge & Reid, 2000, p. 256).

Drodge and Reid also list three qualities that they consider to be integral to the mathematical emotional orientation that came from Polya's "moral qualities of the scientist":

> 1. *Intellectual courage: The ability to make conjectures that require revision of one's beliefs.*
> 2. *Intellectual honesty: The willingness to revise a conjecture in the face of a contradiction.*
> 3. *Wise restraint: The recognition that some forms of resistance to a conjecture are not appropriate mathematical behaviors and should not undermine the conjecture* (Drodge & Reid, 2000, pp. 256-257).

Within the scope of this theoretical framework, students who are operating from an emotional orientation which is not mathematical would be less likely to take mathematical risks (like making a conjecture). This is precisely because they do not place value on the qualities listed above (Drodge & Reid, 2000).

Overall, Drodge and Reid offer an alternative theoretical framework with which to interpret problem solving behavior. They hope to convince their readers that emotion can be a positive force in the doing of mathematics.

Relating Pedagogy and Student Beliefs

In her paper, *Personal Epistemology and Mathematics: A Critical Review and Synthesis of Research,* Krista Muis does a critical examination of the academic literature relating to epistemology and mathematics. Within her paper, she avoids classifying beliefs as appropriate/inappropriate or naive/sophisticated as these labels place implied judgement on beliefs. Instead, she distinguishes beliefs as availing or nonavailing. Availing beliefs are those which advantage learning whereas nonavailing beliefs are those that either have no influence on learning outcomes or negatively influence learning (Muis, 2004).

Muis examined research on students' beliefs about mathematics and she found that students at all levels held nonavailing beliefs about mathematics. Firstly, they believe that mathematics is unchanging and that the goal of doing mathematics is to find *the* right answer. They also believe that knowledge about mathematics is handed to them by an authority figure such as a teacher or a textbook and that they do not believe that they are capable of constructing mathematical knowledge or solving mathematical problems on their own. They also hold a belief that mathematics ability is innate (a belief in a math-gene). Students also commonly perceive mathematics as isolated bits of information and they do not see the relationships among the different concepts. Finally, they believe that mathematics should be learned quickly and that it is important to solve problems quickly (Muis, 2004).

Next, Muis examines the possible nature of the development of beliefs. After a thorough examination of the relevant research in the field, she comes to the conclusion that students' beliefs generally become more availing over time. She also concludes that the classroom environment has significant impact on students' beliefs about mathematics. "Classrooms that rely on more teacher-centred forms of instruction – wherein the teacher tells the students how to solve problems, demonstrates how to use formulas, and engages students in activities that are expected to be completed in a short period of time with high rates of success – are coupled with students' beliefs that the teacher is the source of knowledge and that learning should be quick" (Muis, 2004, p. 338).

The third issue that Muis addresses in her paper is the effect of epistemological beliefs on behavior. Her conclusion is that the preceding research in the field had revealed significant relationships between students' beliefs and their academic achievement. However, she is careful to say that researchers cannot currently make a clear cause-and effect claim about traditional instruction and these non-availing beliefs (Muis, 2004).

Finally, Muis discusses the idea of whether making a pedagogical change in classroom instruction can have resulting changes in students' beliefs. Her research found positive results. The studies that she examined "provide evidence that students' beliefs are malleable and that change can occur by changing the methods by which students engage in learning" (Muis, 2004, p. 362). Moreover, she argues that these changes can occur within a relatively short period of time (some interventions were between 2 months and 1 year). However, there is the deeper question of what will happen when a student goes from a progressive classroom back into a traditional classroom. Will their new beliefs be sustained even when the students are placed back in a traditional environment? Muis' argues that if students do not have specific awareness of their beliefs then their beliefs will be less resilient. That is, new beliefs that have come from being in a more constructivist mathematics classroom may revert to less availing beliefs if the student is placed back into a traditional classroom. She argues that giving students an explicit awareness of their beliefs through discussion may be necessary to help ensure resilience of the availing beliefs (Muis, 2004).

In her conclusion, Muis writes:

> *It appears that there is a need for a major shift away from the traditional instructional style at the elementary and secondary levels. Specifically, the instructional implications of this research suggest that teachers should engage students in actively learning mathematics using a constructivist-oriented approach to teaching mathematics* (Muis, 2004, p. 367).

Sociomathematical Norms

In their paper, *Sociomathematical Norms, Argumentation, and Autonomy in Mathematics*, Erna Yackel and Paul Cobb develop the idea of sociomathematical norms and connect this concept to students' developments of mathematical beliefs. Sociomathematical norms are a way of defining what constitutes acceptable mathematical activity in a classroom. They argue that this set of norms is both influenced by the beliefs of the members of the classroom and, reflexively, influences the beliefs of the members of the classroom community. “These normative understandings are continually regenerated and modified by the students and the teacher through their ongoing interactions” (Yackel & Cobb, 1996, pp. 458-477).

The role of the teacher is central to the development of sociomathematical norms. Yackel and Cobb (1996) reference Voigt (1995) in arguing that the teacher necessarily represents the discipline of mathematics in the classroom. As such, a teacher's responses to students’ explanations legitimize some aspects of mathematical activity while sanctioning others. The classroom that Yackel and Cobb are using to conduct their research is founded on an inquiry tradition of mathematics instruction. This is a more progressive classroom environment that moves away from the idea of the teacher as the sole provider of mathematical knowledge. If students' prior experience in mathematics was that teachers were typically the only members of the classroom community to provide mathematical explanations, as was the case for the students in the classroom being studied, then students were likely to rely on authority and status to develop rationales rather than relying on mathematical validity.

Within this paper Yackel and Cobb argue that it is important that students develop a belief that explanations to solutions should be mathematical rather than 'status-based'. The authors state that “what constitutes acceptable mathematical reasons is interactively constituted by the students and the teacher in the course of classroom activity” (Yackel & Cobb, 1996). As students develop an understanding of what constitutes an acceptable explanation, they may assume the role of making judgements about other students’ solutions. Yackel and Cobb (1996) argue that this is evidence of a deeper understanding of what constitutes explanation.

Moreover, Yackel and Cobb (1996) connect the development of sociomathematical norms to the desired goal of intellectual and social autonomy. They argue that “it is the analysis of sociomathematical norms implicit in the inquiry mathematics tradition that clarifies the process by which teachers foster the development of intellectual autonomy” (Yackel & Cobb, 1996). In order for students to achieve intellectual autonomy, they must first develop a clear set of beliefs about what constitutes different, insightful, efficient, or acceptable solutions. In the process of negotiating these sociomathematical norms, students construct specific mathematical beliefs and values that help form their judgements (Yackel & Cobb, 1996).

In conclusion, the construct of sociomathematical norms provides a useful tool for analyzing the development of students' mathematical beliefs as a social process. It also underscores the importance of the role of the teacher in operating as a representative of the mathematical community and establishing effective sociomathematical norms within the classroom.

Walking Away

The preceding literature review has been included to give the reader context for the motivation behind the research questions, methodology, analysis, and conclusions of this thesis. Polya (2004) presents some historical context for the evolution of problem solving in the mathematics classroom. This evolution has lead to the current controversial dichotomy between traditional and progressive mathematics education. Throughout this thesis, I will use the term “progressive” to describe a pedagogy that reflects the ideals presented by the NCTM (2000). In contrast, the term “traditional” will be used to describe a classroom environment that embodies the teacher as the 'giver' of knowledge. In this environment the teachers' primary role is to dictate a lesson and the student’s primary role is to listen and then imitate the procedures that have been taught. The remaining problem solving literature serves to present a vision for the future of teaching mathematics through problem solving and provides some philosophical and practical resources for teachers who are trying to work within this pedagogy. In *Thinking Things Through,* Burton (1984) provides practical advice on how to incorporate problem solving into a classroom. In *Thinking Mathematically,* Mason, Burton and Stacey (1985) challenge their reader to engage in problem solving and use the experience to help develop mathematical thinking in others. Robert Fisher (2005) emphasizes the importance of innovative learning as well as both creative and critical thinking.

Affect was an issue that permeated the research in this thesis. McLeod, DeBellis, and Goldin (1992) offer a tetrahedral definition of affect that was adopted as a working definition in this research. The research of Drodge and Reid (2000) served to provide an alternate framework for analyzing affective issues. Muis (2004) provided some literature that specifically addressed the impact of students' beliefs. Within this thesis, her work became particularly applicable in analyzing interviews in which students shared their beliefs or when analyzing behaviour presented in class that provided evidence of a particular belief. Her designation between availing and nonavailing beliefs in mathematics was particularly useful in analyzing many of the classroom observations that were conducted. The work of Yackel and Cobb (1996) on developing the construct of sociomathematical norms was useful in analyzing the development of beliefs among students.

Chapter 3.

Research Questions

"You can tell whether a man is clever by his answers. You can tell whether a man is wise by his questions."

- Naguib Mahfouz

In Chapter 1, I discussed the discrepancy between my educational beliefs and my pedagogy throughout the beginning of my teaching career. My hope is to complete this thesis with a personal sense of how I can build a classroom that is grounded in problem solving and whether I can do this without unduly risking my students' success. I believe that there are other teachers working within a predominantly traditional educational environment who are having a similar struggle and I hope that my work will be of some practical benefit to them and answer some of their questions as well.

The four previously unaddressed questions noted in the introduction reflect barriers that have prevented me from making a serious move towards aligning my values with my practice and implementing the methods encouraged by the NCTM in *Teaching Mathematics Through Problem Solving*. These questions are presented in more detail in this section.

Question #1

Will students ‘buy in’?

In my introduction I mentioned the idea that the students are the most important stakeholders in the educational system. When I envision a classroom where mathematics is taught entirely through problem solving, the students' interests are the primary consideration. I want them to have a dynamic classroom where they can experience the beauty and interest of mathematics and I want them to walk out of the classroom with skills that will serve them well in their future endeavours. However, this vision requires students to buy in. Experience shows that simply because something is good for our students does not mean they will embrace it wholeheartedly. I would like to explore whether students will fight Mr. H on his efforts to reform the social norms in his class or whether they will embrace this new pedagogy.

In my field study for this Masters program I taught a series of problem solving workshops within another teacher's classroom. I was somewhat discouraged by the lack of enthusiasm some students had for the problems. I thought that the diversion from the regular curriculum would be met with enthusiasm and relief and, for some students, this was true. However, there were many students whose attitudes and demeanour remained much more jaded. I postulate that the negative student attitudes I observed were, at least in part, a result of the dramatic change in expectations and the perception that these new problems were simply an addition to the regular workload. It is possible that if an environment of problem solving were to be established from the beginning of the course that it would be a more fruitful experience.

Throughout my time teaching in secondary schools I have had many conversations with teachers about teaching through problem solving. There is a persistent belief among teachers that their students would never buy into problem solving and, because they do not believe that their students would be willing to accept this method, they are unwilling to attempt it. The NCTM has expressed in their writings that students do, indeed, buy in. However, despite this claim by the NCTM that students will accept a problem solving method of teaching mathematics, local teachers do not believe that it can work in their classrooms, with their students and the current curriculum.

We are at a point in time in British Columbia right now where a new mathematics curriculum has just been phased in. A large part of the intent behind the new curriculum is for teachers to teach differently and for problem solving to be a large part of every mathematics classroom. However, in talking to my colleagues it has become clear to me that for most teachers, the only change in their daily practice has been to use a new textbook. Some of them went so far as to photocopy new worksheets. Not one of the colleagues I have talked with has used this new curriculum as an opportunity to rethink their pedagogy and make changes to their classroom environment or routines. In order to convince educators, including myself, to make changes in practice it is extremely important that the question of students' engagement is addressed within this research.

Question #2

(a) Is it possible to create this dynamic, problem solving environment consistently within a classroom? (b) Can it be done within the context of the current curriculum?

I am fairly confident that I could create a problem solving environment that is interesting, fun, and successful in promoting mathematical thinking. Indeed, there are many fun mathematical problems that students will immediately enthusiastically engage with. Showing mathematical card tricks (tricks that work because of an underlying mathematical rule that can be expressed algebraically) and asking students to figure out how the tricks work is an example of this. I have a book on my shelf dedicated entirely to math-magic. This could be the foundation of a really fun mathematics course. However, this alone will not meet the long term needs of my students. At the end of the day, educators need to prepare their students both for final exams and their post-secondary mathematics experiences. Many of my students are going to leave my class and walk into a first year calculus or other mathematics course at university. Their instructors will be assuming that they have covered and understood the entire Mathematics 12 curriculum. Time is an inescapable reality in education. Can this desired environment be created without sacrificing students' preparedness for final exams and their future mathematics classes?

Question #3

How does teaching through problem solving affect low-achieving students?

In speaking with mathematics teachers about teaching through problem solving, I found that one of the most common things preventing people from implementing this practice is that they do not believe that their students have the ability to come up with the mathematical ideas and discover the concepts for themselves. This suggests a widely-held belief that teaching through problem solving caters to the high-achieving students and will potentially leave some kids behind. The NCTM argues that low-achieving students do not need the teacher to solve or trivialize a problem for them (NCTM, 2003). However, many teachers that I have conversed with still express concern about the abilities of their lowest-achieving students to engage with the level of mathematical problem that would be required in a senior mathematics classroom.

Question #4

How does a problem solving environment impact students' relationships with mathematics?

This is a very large question and probably deserves its own quantitative study. What students believe and how they feel about mathematics is an extremely important outcome of the classroom environment that teachers create. However, gathering large amounts of quantitative data is beyond the scope of this study. My goal in this thesis is to gather some qualitative data from individual students about their thoughts on this more progressive pedagogy. I would like to discover whether they feel that the problem solving has been beneficial to them. Have their beliefs about mathematics and about themselves in relation to mathematics changed? Do they enjoy it more or less? Do they find it more or less difficult? Is it more or less stressful?

I intend to address these questions from the perspective of an observer of a mathematics classroom that is taught through problem solving and to gather observations that, hopefully, disclose the perspectives of the students participating in that class.

Chapter 4.

Methodology

My research questions have their roots in the larger question of possibility. As mentioned in the introduction, I have a vision for the classroom environment that I would like to create. I would like to know if it is possible. This research represents my initial attempt at answering these questions by taking a 'proof-by-existence' approach to a solution. As such, I conducted a series of observations in a mathematics classroom that is being taught primarily through problem solving and collected qualitative data in the form of observation notes and interviews. There were four main phases of the research process:

1. Choosing a classroom
2. Observations
3. Interviews
4. Data Analysis

The intent and the philosophy behind choosing a classroom, the framework and focus for the observations, the choice of student interviews, and the methods used for analyzing the data all require some justification here.

Choosing a classroom

This thesis examines whether it is possible for a teacher with the requisite skills and dedication to effectively implement teaching through problem solving. Thus, the choice of the classroom and teacher was essential to this research. It was necessary to choose a teacher who had both the skill set and the motivation that would make it possible for him or her to have success with this pedagogy.

The classroom that was chosen belongs to a teacher, hereafter referred to as Mr. H, currently exploring and implementing the philosophy of teaching through problem solving. Mr. H recently completed his Master's degree in Mathematics Education at Simon Fraser University. As such, he has received formal education on the methods of teaching through problem solving. He is a young, dynamic teacher and with his advanced education, including in-depth exposure to problem solving methods, behind him I felt that he was a candidate likely to be successful in creating a classroom with a foundation in problem solving. As the results of this research will show, I was correct in my judgements.

Observations

The research for this thesis included of a series of weekly observations in Mr H's classroom over the course of one semester. It began at the start of the school year in September of 2010 and continued through to December 2010. Mr. H was working to create a classroom environment that was rooted in problem solving rather than the more traditional lecture-notes format. In order to collect data that reflected students' diverse ability levels, two classes were observed and students from both classes were interviewed. One was a linear Honors Mathematics 10 class (C-Block) and the other was a semestered Principles of Mathematics 11 class (D-Block). This means that the Honors 10 students had class every second day while the Principles 11 students came to class every day. In the Honors 10 class I completed ten observations. In the Principles 11 class I completed nine observations. The difference was due to the fact that on one particular week there was only one day that would work for the observations. On this particular day there was no class for the Principles 11 students. The observations were done once a week with a short break in the research when Mr. H took some time off for the birth of his son.

During my observations, I collected field notes on incidents that I witnessed happening around me. When observing an entire class it is more difficult to assess whether students are actively engaged. Therefore, in order to answer the question of *will students buy in,* it was necessary to focus on an individual or small group level so that I could look for specific evidence of genuine engagement and mathematical thinking within their conversations. This focus on the individual or small group was also essential in answering questions about the low achieving students and about students' relationships with mathematics. Throughout the classroom observations that are part of this research, I kept a particular eye out for the low-achieving students and watched for evidence of engagement and learning in these students.

It became clear fairly quickly in the research that trying to gather general data on every aspect of the classroom would be too broad. However, there were some elements of the progressive classroom environment that were ubiquitous and distinguished this environment from a traditional environment. Group work was a fundamental component of every class, as was a lack of teacher directed steps or notes. There were also affective components of the classroom that were unique to a problem solving environment and it became clear very early on in the observations that I wanted to know more about the impact of emotions and beliefs on the success of teaching through problem solving. It also became clear very early on in the observations that students' motivations were frequently linked to grades and I wanted to further explore how this would impact the environment as the semester progressed. Therefore, my observations naturally focused on central factors that made themselves repeatedly apparent or that consistently differed from a more traditional environment. These five central factors were:

- Emotions and observable beliefs
- Group work
- A lack of dictated steps and solutions that would normally be provided by the teacher
- Grades
- Notes

Throughout the research, I attempted to remain an impartial observer and refrained from getting involved in the events that were happening around me. In the beginning, while the students were getting used to my role in the classroom, they would occasionally ask me mathematical questions in their problem solving to which I simply replied that I could not help them.

Interviews

In order to get a deeper understanding of the impact that this environment had on individual students I conducted a series of individual interviews. In all, twelve interviews were conducted. Students were chosen for these interviews for several reasons. It was important to interview a diverse range of students. I wanted to speak with both high and low-achieving students, males and females, introverts and extroverts, students who entered the class loving mathematics and students who entered the class dreading it. Each interviewee was chosen because that student stood out to me for some reason on that particular day. However, this is not to say that the students chosen were always the student dominating the class. Sometimes they stood out because of their silence. The majority of the time, I chose to interview a student because I saw evidence of strong thoughts or opinions about the central factors listed above. Because it was important to me that I interview both high and low-achieving students, twice it was based on their grades.

Interviews generally began with a question related to the five central factors. I would ask how a student felt about the group work or the lack of predetermined steps to solve a problem. However, after this initial question I allowed the student to lead the conversation and the interview questions were generally guided by their comments. I considered these interviews to be an opportunity to listen more deeply to students. The interviews turned out to be a very rich source of data. Moreover, they frequently served to provide a more accurate affective perspective on the observational data. Throughout the analysis component of the research there were several times when the analysis of the classroom observations was significantly altered by the direct conversations that happened with students outside of the classroom. The interviews allowed me to have a window beyond what was directly observable in the classroom and to gain some knowledge of students' affective responses to this more progressive pedagogy.

Data Analysis

Once all of the observations and interviews were completed I examined the data using a constant comparative methodology. This is a methodology that comes from grounded theory (Charmaz, 2000). Throughout the research, the observations focused on five central factors that were either ubiquitous or that differentiated Mr. H's class from a more traditional environement. After the research was completed, I went through the data that was organized around each of these central factors and searched for emerging themes that were common to each of these factors. As each theme emerged, I went recursively back through the data looking for evidence of that theme. The evidence collected and the themes that emerged were then analyzed within the context of my research questions.

Chapter 5.

Painting a Picture – Mr. H's Class September 29th

This anecdote was carefully chosen from my field notes as a representation of Mr. H's general practice. The intent is to give the reader a sense of what teaching through problem solving looked like for the purpose of this research. The lesson took place approximately one month into the course on September 29th in the grade 10 class. This gave Mr. H enough time to establish classroom routines and for students to adjust to new, dramatically different expectations. It also gave time for some sociomathematical norms to develop within the class (Yackel & Cobb, 1996).

Students are smiling and chatting as they file into Mr. H's mathematics class after lunch. As they enter the room each student searches for their number in a grid on the board. When they find it they call out for the other numbers in today's randomly assigned group. The classroom is loud as students yell out their numbers searching for their group members. There are no desks in the classroom, only large tables, and when students find their group members they stake claim to their table and sit down.

At 12:30, the bell rings. Mr. H walks to the front of the room and writes a linear system on the board for students to solve by graphing and substitution:

$$5x + 6y = -11$$

$$-7x + 9y = 14$$

He doesn't say anything but the class gets right to work in their groups. The room buzzes with chatter. Students check their answers with the members of their group and with neighbouring groups.

Fifteen minutes later, most groups are still working. Mr. H speaks up. “Is this really giving you guys that much trouble?” He had expected them to be done already. He decides to break out some hello kitty dolls. “When you are done the problem, stand up the hello kitty doll so I can see who's done.” A couple groups stand up their dolls but the others continue working. Mr. H puts an extension on the board for the groups that finished early.

Ten minutes later Mr. H pulls the class together. “This is exactly what I wanted to see. It takes a month before this happens.” He is very happy with the way that the class has been working together and are helping each other. He calls on Sabrina[1] to do the first problem on the board. She walks to the board but is clearly nervous about starting. She stands facing the board and, mute, she starts writing. Mr. H tells her to talk.

“Oh, I need to talk?” says Sabrina quietly. The class rustles with friendly laughter. She says “So... do you guys see where I got it?” The class laughs again. Sabrina blushes but explains her solution in more detail. The class applauds when she finishes. Some students listened attentively and others chatted in their groups. Mr. H thanks her and compliments her for being so articulate.

[1] All students’ names are pseudonyms.

Mr. H begins a lesson on elimination. However, his example does not have any variables in it. He introduces the idea that you can add or subtract equations from each other with an example that only involves numbers. As the class realizes that they can add the equations together they respond with a mix of “WHAT!?!” and “Ahhhh...”. Aaron grabs his head like his brain is exploding. He is smiling.

Mr. H asks them to make the first column of

$$12 - 7 = 5$$

$$5 \quad + 3 = 8$$

disappear. He is asking the student to eliminate the 12 and the 5 using elimination.

There are mutters from the class that this is too easy as they quickly work in their groups to do it. Alex blurts out the answer (to multiply the first equation by 5 and the second equation by 12 before subtracting them). Mr. H draws their attention back to the problem that is still on the board from the beginning of the period. Originally, they had solved it using graphing and substitution. Now he asks them to find a way to solve it somehow using what they have just learned about adding and subtracting equations.

All of the groups engage quickly and work together to figure out the elimination method. There are a lot of smiles as students work and the room is filled with a very positive energy. When Mr. H brings the class back together to formalize the solution all of the groups appear to have figured out how to apply the elimination method on their own. He hands out one sheet of notes anyway but does not discuss the notes with the class. It serves only a reference tool if they choose to use it. At 1:40 the bell rings and students head off to their next class.

This example has been presented to briefly illustrate the environment that Mr. H created in his classroom. The amount of direct instruction that he gave was limited to essential information or to formalizing the mathematics that students had themselves uncovered. Moreover, this direct instruction was done only after engaging students in actively thinking about problems. Students did not enter his classroom and get out their notebooks expecting a lesson to be dictated to them. Instead, they entered the classroom expecting to be given a problem to engage with. It should be noted that a system of two linear equations is an exercise that frequently appears in traditional textbooks. However, if students have been told how to do the question then it does not qualify as a ‘problem’. What defines this question as a problem is the context in which it was presented. Students were not told how to do it. Instead, they were helped to understand what the problem was asking and then they were expected to figure out how to solve it for themselves. Direct instruction was not completely absent from Mr. H’s classroom. However, notes were given after the students had engaged with mathematical thought and these notes were an auxiliary rather than a central feature of the classroom.

Chapter 6.

Evidence and Results

The data that were collected throughout the weekly observations focused on five recurring central factors that were very prevalent within the classroom environment. These were:

- Emotions and observable beliefs
- Group work
- A lack of dictated steps and solutions that would normally be provided by the teacher
- Grades
- Notes

This chapter presents some evidence from the observations and interviews that address each of these five central factors. The evidence presented here is not exhaustive. I have chosen these exemplars because I believe them to be representative. A discussion of the evidence and an initial analysis of each incident are presented to illuminate some of the relevant issues that emerged throughout the research.

Emotions and Observable Beliefs

This central factor of emotions and observable beliefs deserves some clarification here. I am using the tetrahedral model of affect presented by presented by DeBellis and Goldin. In this model, affect is comprised of values, attitudes, beliefs, and emotions all of which influence and affect each other. As researchers before me have addressed, some components of affect are easier to quantify than others. In a large-scale, quantitative study beliefs are easier to measure than emotions because they can be examined using questionnaires and self-report data. McLeod argues that in the past research has focused more on the effect of the affective domain on cognition. In particular, emotions have received less attention than beliefs from the researchers "probably due to the fact that research on affective issues has mostly looked for factors that are stable and can be measured by questionnaires" (McLeod, 1992, p. 582). He argues that detailed studies of a small number of subjects would allow the researcher to focus more on the impact that emotions have on students when they are problem solving (McLeod, 1992). This was particularly true in the scope of this thesis where emotions, though transient and less quantifiable, were an observable, palpable component of the mathematics classroom. Thus, this central factor of 'Emotions and Observable Beliefs' addresses only the components of affect for which there were readily observable evidence in the classroom.

The First Day

The very first day in the Math 10 class provided some interesting insights into students' initial beliefs about mathematics. Mr. H presented an article "*A Mathematicians'*

Lament"[2] and provoked discussion about it. The article presents a vision of what an art class would be like if it were taught with the same methods that are traditionally used to teach mathematics. The implication in the article is that mathematics is boring, not because of the subject, but because of how it is taught. The resulting discussion provided some clear evidence about the students' beliefs in mathematics and their relationship with the subject. The comments that emerged from the students included:

- *"Is it really supposed to be about fun though?" (referring to mathematics)*
- *"It's too late... since elementary school it's always been about memorization... I don't think I'd be willing to change that now."*
- *Michael: "I think math is pretty boring"*
- *Saba: "Math can be art and enjoyable but depending on the teacher it can be all about the facts."*
- *Jason: "I think people have valid points about math being useless... unapplicable [sic]."*
- *At one point in the discussion, Mr. H makes a reference to the Gauss mathematics contests and several students react by yelling "YES!" These students have presumably enjoyed these problem solving contests in the past.*

This discussion served to reinforce the need for change in mathematics education. Upon entering this class, the relationships that these students have with mathematics are rooted in the beliefs that mathematics is all about memorizing facts and procedures, that it is useless, and that it is predominantly not enjoyable.

[2] A brief handout based on the full publication of *A Mathematicians' Lament* (Lockhart, 2009).

The students in Mr. H's class are not unique in the beliefs that they are expressing in this discussion. As mentioned, Muis (2004) summarized a list of non-availing beliefs about mathematics that are commonly held by students. The beliefs that the students in Mr.H's class shared and the common beliefs that were found in this study overlap. Muis (2004) also gives evidence of studies that have shown that classrooms that rely on teacher-centred instruction are coupled with several of these non-availing beliefs.

The NCTM also addresses the problem of student beliefs in their preface to *Teaching Math Through Problem Solving.* They argue that there are serious long-term consequences to students maintaining these beliefs about mathematics. The NCTM is concerned that students are leaving school with a superficial and disconnected understanding of mathematics and that the mathematical knowledge is not transferable when it may be needed at a later date (NCTM, 2003). Certainly, the beliefs that the students in Mr. H’s class have expressed are cause for concern if they are left unaltered.

In *Teaching Children to Think*, Robert Fischer also stresses the importance of a move away from the traditional 'transmission' method of teaching. He suggests that the traditional method of teaching assumes that the ideas are in the teacher's head and that the children need to be told these ideas. He calls this method of teaching the “transmission model” (Fischer, 2005, p. 152) and he argues that it is primarily suited for low-level tasks. He also insists that students must experience mathematics as more than simply rules and facts learned in isolation. It is evident from the students’ discussion around *‘A Mathematicians Lament’* that the students in Mr. H’s class have been profoundly impacted by primarily experiencing mathematics as sets of rules and facts learned in isolation. Fischer argues that students need to experience the structure and interconnectedness of mathematics:

> *The trouble with rules is that they are easily forgotten. Relational understanding implies knowing the reasoning behind the rules and this understanding can be gained if the child has thought through and can reconstruct the rules for himself. This learning tends to be deeper, more lasting and more easily recalled to memory (Fischer, 2005, p. 170).*

Weeks Two and Three

The second and third weeks of observations also provided some insight into students' beliefs and their relationships with mathematics at the beginning of the semester. In particular, there was frequent evidence of an initial belief that finishing a mathematics problem quickly is good. On September 16th Joe, a very high-achieving student, gets stuck on a problem and immediately asks Mr. H how long it took him to solve it. I asked him if he felt like he needed to beat Mr. H's time, he responded by saying “No. I'm just thinking. It would be pretty hard to beat Mr. H.” This illustrates a belief that an expert problem-solver, in this case his teacher, would finish a problem quickly and thus be difficult to “beat”. In the Math 11 class on this same day Andre, a struggling student, gave up on a problem after working on it for less than one minute. In both of these students we see evidence of a belief that finishing a mathematics problem quickly is good. Andre's behaviour illustrates that he believes that if he does not know the answer to a problem immediately that he will be unable to do it. These observations support the work of Muis (2004) who summarized much of the current research on educational beliefs about mathematics including the belief that mathematics problems should be solved quickly.

The Semester Progresses

In contrast to the beliefs that emerged from the discussion on *A Mathematicians' Lament*, evidence of new beliefs emerged towards the end of the semester. November 24th C Block, I did an interview with Sabrina. I asked her whether she was enjoying mathematics this year.

> *Sabrina: "Yeah. <hesitant>. It's a little bit more interesting than math usually is. I mean, most people, including myself, go into math thinking 'oh no. More math. More difficult things that I have to remember and everything'. But it's pretty easy-going. Like, the mood in math makes it a little bit better. It's not always like work, work and it's not as stressed out."*

Sabrina was not the only student to comment on the change in the mood in mathematics class. Every student that I interviewed expressed that mathematics class was more enjoyable now and that their relationship with mathematics had changed. Many students had gone from dreading mathematics, to enjoying it. Several students remained concerned about the lack of notes, and this concern will be discussed in more detail later, but, without exception, the students that I spoke with all found this class to be more enjoyable than their previous traditional classes.

Abi's Story

Throughout the semester, I noticed that many students' beliefs about themselves and their abilities in mathematics made a slight shift. They started to believe that mathematics was something that they could derive for themselves. About one month into the research I started to get comments from students that Mr. H could give them the minimum amount of information and that they could then figure it out on their own. At first, I did not register the significance of these comments. They were made in passing during the interviews. For example, at the end of the Math 11 class on October 5th, as students were filing out of class, I walked out with Abi and Jetta. They had been working on a problem about boats that the class had only managed to partially solve and Mr. H would not show them how to complete. However, he did tell them it would be on the test. I asked “Are you guys okay with the fact that he didn't teach you explicitly how to do it?” Abi replied “Now that he said it's on the test I kind of wish... but, I mean, he kind of did tell us how to do it. He gave us the equations. We can probably figure it out from there.”

If I compare this conversation to my experience teaching and observing in more traditional mathematics classes, students acclimatized to a traditional environment would complain viciously if a question appeared on a test that was even a slight variation on what they had been “taught” how to do. Students' expectations are shaped by the classroom norms that develop. In a traditional classroom environment the established sociomathematical norm is that mathematical knowledge comes from the teacher (Yackel & Cobb, 1996). As a result, suddenly changing this expectation on a test and requiring students think and develop their own mathematical ideas would break this established norm. However, in a more progressive classroom where a sociomathematical norm is established that students will be expected to figure out problems for themselves and be able to articulate solutions to these problems, students are very willing to accept this demand. This shift in students' attitudes and relationship with mathematics became even more noticeable as the semester progressed.

On September 29th, in D-Block, Abi was trying to solve a system of three equations and made a mistake but could not find it. Mr. H would not tell her where it was but encouraged her to keep trying. She wanted to give up: “I was only here for the last 10 minutes yesterday. I don't know if I have the rules right.” Mr. H sighed and looked at her work. He pointed to her mistake (a negative sign). After seeing her mistake Abi said “I hate math. I hate math. I hate math. I am so angry right now!” Shortly after this, Mr. H spoke with the class about matrices and why they work. Abi spent the rest of this discussion staring quietly at the board and looking disengaged and very disheartened.

This strong negative emotional response to being stuck and the resulting impact that it had on her ability to engage for the remainder of the period prompted me to pull Abi aside for an interview. I was concerned that if Abi's emotional responses to being stuck prevented her from engaging with the lesson then perhaps a classroom environment taught entirely through problem solving would be detrimental to her.

During the interview I learned that Abi suffers from an anxiety disorder. She is in Mr. H's class this year because she had been in a more traditional class the year before and had done poorly. She felt that in the more traditional environment she was unable to understand the material and this was so disheartening to her that she disengaged from the class and did not do the assigned work for the entire second half of the semester.

When I asked her if she was understanding more in Mr. H's class she responded “Yes. The other teachers make it so complicated to understand. They try to shove like so much information and then they're like 'okay. Apply it.' But he gives us probably the minimum information that we need and the rest will work itself out.” This comment indicates that Abi believes that much of the volume of information that she was getting from previous teachers actually made it more difficult for her to understand. For her, the “minimum information” that Mr. H gives is less overwhelming and serves her better.

I asked Abi about her emotional response in class. She brushed it off saying “It's just a thing that everyone does” and she clarified that her stress was a result of having missed the previous class and feeling behind. While she was in class, the emotions were very strong and prevented her from finishing the problem or re-engaging. However, a few minutes later when we are sitting together talking about it, she is completely calm and does not feel that her emotions were a significant reflection on the success of Mr. H's teaching methods. She said that this environment allowed her to stay engaged in the course because, as a whole, she was understanding more. This temporary nature of emotions is supported by McLeod's theory of affect. He argues that emotions are transitory by their nature and while they are powerful, they are limited in duration (McLeod, 1992). In contrast, he argues that attitudes and beliefs are more permanent in their nature. This conversation, combined with Abi's comments on October 5th that were discussed at the beginning of this section show evidence of an increase in self-efficacy and a shift in her attitudes and beliefs about mathematics.

The Semester Concludes

November 24th, D Block: This was my final observation. I returned at the end of the period after doing an interview with a student. Students had been working in groups at the boards to try and figure out three problems on parabolas that Mr. H had refused to tell the students how to do. When I left, Simone was lost and trying to figure out an algebraic way to solve the problem. Upon my return I found Simone explaining the second problem to Irena and three members from another group.

> *Irena: "I get it!"*
>
> *Irena and Simone are smiling and excited. They tell Mr. H that they got it.*
>
> *Mr. H: "You got it?"*
>
> *Simone: "Yeah, It was the same way that they did it (gesturing to the boys beside them) but WE did it."*
>
> *Eric: "Did you get it?"*
>
> *Simone: "Yeah, we explained it to ourselves!"*
>
> *Simone and Irena go to help Moby at the board in applying their method for problem #2 to problem #1. There is now a large group of students gathered around working on problem #1 on the board. They continue working together for the last few minutes of class.*

This was a really beautiful moment to witness during my final observation. The problems that the students were working on were not contrived to be fun or exciting. They were simply three problems to write the equation of a parabola with different given information. These problems were no different than the problems that would be assigned from the textbook in a more traditional classroom.

However, in contrast to a traditional classroom, the enthusiasm in the room was palpable. Students were excited and working in teams to solve these problems. Moreover, they took incredible pride in figuring it out for themselves. We see evidence of this when Simone says “but WE did it” with a strong emphasis on the word 'WE'. There is also clear evidence of genuine understanding by the students. Students had figured out the solution to the problems on their own and they demonstrated the ability to teach it to each other. One effective way to tell if a student understands a concept is to see if they can teach it to someone else. Burton writes “convincing a group member of an idea is a good test of how well the pupil understands it herself” (Burton, 1984, p. 52). As such, I take the students’ abilities to discover and teach each other their methods of writing equations for these parabolas as evidence of deep mathematical understanding of these problems.

The contrast between this final interaction among the students while they were working on a series of problems on parabolas and the discussion of *'A Mathematicians’ Lament’* that was presented at the beginning of this section shows a significant affective shift among these students. Their beliefs about mathematics, and their relationships with mathematics have changed. I argue that these students could not engage in these mathematical problems in the manner that they have if they still believed that mathematics was all about memorization, that it could not be enjoyable, and that it was something that they needed to be taught explicitly how to do by an authority figure. This incident, not atypical for the observations later in the semester, also illustrates how genuine engagement in a mathematical problem can lead to a deeper level of understanding. As such, it provides evidence in support of the potential success of teaching through problem solving.

Group Work

Group work is a fundamental part of teaching through problem solving. In fact, it is one of the features of teaching through problem solving that is most readily observable. You could use a camera to take a snapshot of the entire class and see a significant difference from a traditional environment. Naturally, one of the predominant central factors that emerged throughout my observations in Mr. H's class was how the group work changed the learning experience for the students. Almost everything is done in groups. Students are expected to rely on each other and work together to solve problems. Within this section I look deeper into the interactions within the individual groups to give some insight into the dynamic interplay between students and how that interplay impacts the students.

The Benefits of a Team

In my observations there were many examples of group work leading to genuine engagement with mathematics. When I look through the observational notes that I have collected, there are clear moments in every class where groups are working together and engaging in mathematical ideas. Here is one such example from C Block on October 5th, approximately one month into the course:

> *Mr. H: "Let's have some fun" - he puts a system of three equations on the board. The students have only ever seen systems of two equations.*

Sabrina, Michael, and Wilhelm are in one group.

Michael: "if you isolate one variable on each side..."

Sabrina is trying to graph it.

Michael: "if you isolate one on each then that could be on the y-side."

Wilhelm uses elimination to get x=7. He tells his group "x=7".

Michael gets frustrated with Wilhelm. "Well don't just say that! You have to show us."

Wilhelm explains.

Sabrina: "I'm confused. You got rid of the z's. What about the y's?"

Michael: "Now you can plot it."

Sabrina: "Wait, so you just got rid of the x's?"

Michael and Sabrina watch as Wilhelm gets a system of two equations and isolates y so that they can graph it.

Sabrina gets confused because her calculator won't graph it with the z's.

Michael easily substitutes x's for the z's to enter it into the calculator.

Sabrina: "It makes a triangle. There are 3 intersections."

Wilhelm: "I thought we couldn't graph it because it's 3D."

Michael: "But we eliminated one variable."

Michael: "Wait! We have this <points to his equations> eliminate one more variable."

Mr. H interrupts. "We only have 5 minutes left. I want to make sure we clean this up."

Now, in a more traditional classroom environment this lesson would have been experienced very differently. All of the teachers that I have worked with plan one entire period devoted to solving a system of three equations. The lesson begins with the steps to solve a system of three equations. Here are the steps as recovered from my own collection of binders from my first year of teaching:

> *Step 1. Eliminate the same variable from the first 2 equations and the second 2 equations. This creates a system of two equations.*
>
> *Step 2. Solve the system of two equations.*
>
> *Step 3. Plug this result into one of the original equations to solve for the third variable.*
>
> *Step 4. Write the answer as a point.*

Customarily, students are given these notes and a couple of examples and then are left to do several more practice systems. They may be told that exactly one system of three equations will appear in the written portion of the test.

In contrast, the students in Mr. H's class were left to figure out how to solve this system with their group members. We can see that individually they may not have been able to do it but as a team they manage to figure it out. Wilhelm figured out how to get it down to a system of two equations and Michael figured out how to graph and solve that system. While Wilhelm had strong algebraic skills, if he were not working in a group he may not have realized that he could graph the resulting system of two equations. Michael helped him with this. Sabrina had weaker mathematical skills but she was also getting a lot out of the group work. While she did not contribute any new ideas to the group, she was actively involved in following what Michael and Wilhelm were doing and in the end she also managed to solve the system. By working in a group, each of these students was more deeply engaged in the mathematical ideas behind this problem.

Moby

November 17th, D Block: I pulled Moby aside for an interview. Moby had been pretty vocal in his dislike of previous mathematics classes and his enjoyment of this one. He had failed his last class but appeared to be a strong leader in Mr. H's class. At the time of the interview, he was getting an A. He told me that this was the first time he had ever got an A in mathematics. I wanted to talk with him about group work. I asked if he found the group work helpful.

> *Moby: "Yeah. That's it. I think there are two things that are really positive about that. One, when you are with your peers you are at the same level as them so you understand exactly what they understand. When the teacher tries to explain it to you, he obviously knows everything about the lesson. If he tries to explain it to you he might not know the right way and what you're comprehending. But if you're interacting with people you won't fall asleep and if they understand something and you understand the other thing..."*
>
> *[...]*

Me: “I notice the stuff you do in class is so group-work oriented. What happens when you're in a test by yourself?”

Moby: “Well, it's like the group-work prepares you for the test. When you're in a group and you don't understand something, you ask. Or, when you're working with other people they depend on you. And then you understand. So, when you're alone on a test I feel like I really know how to do the question.”

I chose Moby's response to these questions because he was able to articulate his thoughts with clear and concise language. However, this was a typical response that I got from many students during formal and informal conversations, particularly from students who had previously been, or still were, low achieving. Many of them professed that the group-work and the reliance on their peers had increased their level of mathematical understanding far beyond that of a traditional environment where information comes primarily from the teacher.

Colton

On Novermber 17th I did an interview with Colton. I decided to pull him aside to speak with him because on this particular day he was not contributing anything to his group and, while he was pretending to work on the problems and turn pages in his notebook, he was not actually engaging in the problems. Here is a section of that interview:

> *Me: Do you find that the group work is beneficial? I noticed that you are kinda quiet in your group.*
>
> *Colton: Um. I don't know. It can be, at times for me but I'm not really into it.*
>
> *Me: How come?*
>
> *Colton: I don't know. It's just like, I'm not on par with the rest of the class. Everyone else is way ahead of me.*
>
> *Me: So you don't feel like you have something to offer to your group.*
>
> *Colton: Yeah.*
>
> *Me: That sounds difficult.*
>
> *[…]*
>
> *Me: How do you feel when you walk into the classroom? Do you feel positive about what's going to happen or is it intimidating or...?*
>
> *Colton: Positive.*
>
> *Me: If you had a choice, would you choose to be in this class or would you choose to be in a class with more one-on-one?*
>
> *Colton: This class.*
>
> *Me: Yeah?*

Colton: Because I'm pretty sure that I can ask for help from my other group members.

Colton articulated to me that struggled in mathematics because he lacked a lot of the prior knowledge that other students may take for granted. He expressed mixed feelings on the group work in class. On one hand, he felt that it was helpful to be able to get assistance from his peers, but on the other he struggled with the group work because he felt that he did not have anything to contribute. Colton continued to struggle with this throughout the semester. In terms of his relationship with mathematics, Colton said that he was having more fun and that he was "more keen on learning". Overall, even though he felt badly about the fact that he did not get to contribute as much to his group as he would have liked, he preferred to be in this class than a traditional one.

The example of group work with Michael, Wilhelm, and Sabrina illustrates a high level of engagement in mathematics among these students. The interview with Moby presents evidence that Moby believes that the group work has also helped to develop an increase in understanding. Colton expresses that the group work has helped him become 'more keen on learning' and that he is enjoying mathematics more although he still does not feel that his skills are as strong as the other students in his class. Many of these students have benefited from the group work in Mr. H's class. The literature that was presented in the literature review also corroborates the benefits of group work. For example, the NCTM advocates for small group work in mathematics because participating in small discussions may be less intimidating for some students than whole class discussions. They also argue that it presents opportunities for more students to verbalize their questions and thinking (NCTM, 2003). As Vygotsky has shown, when children are in collaborative or cooperative situations they are able to function at an intellectually higher level (Fischer, 2005). Moreover, Fischer writes “There is a common misapprehension that it is thinking or reflection that generates the need to talk. More often it is dialogue that generates reflection and thoughtful response” (Fischer, 2005, p. 130).

If we combine the arguments of Fischer and the NCTM we see that if we create a classroom norm that students will work in small groups, our students will have more opportunities to engage in dialogue than if the norms of our class are founded exclusively on whole class discussion. Moreover, the more often that children engage in dialogue, the more they will engage in reflection and thoughtful response. The combination of current educational literature and the evidence gathered through the observations and interviews in this research provides a strong argument for using a pedagogy that relies heavily on group work.

Lack of Pre-determined Steps/ Solutions

One significant pedagogical difference between a traditional mathematics classroom and a classroom that is taught through problem solving is that steps and solutions are not provided by the teacher. This is extremely important when creating an atmosphere of teaching through problem solving. It is essential that ownership of the problem remains with the student. If the teacher supplies the answers or interjects too much, then the teacher owns the problem (Burton,1984).

However, there is the potential for tension between the teacher and the student because within our school system a classroom norm has frequently been established that the teacher's role is to provide the answers. The student's role is to learn them. Therefore, in our observed class, it may have been perceived by the students that Mr. H was being unfair, or not sufficiently doing his job when he refused to supply answers and procedures for solving problems. This section examines this tension by analyzing the data collected throughout the observations and the interviews.

David

September 22nd, C-Block: Mr. H wrote an equation on the board: 2x+4y=22.

He asked the students to think of a point that satisfies the equation. One student raised his hand and answered. Peter then put a second equation on the board:

$$5x-2y=7$$

He said: “Okay. There's a point that satisfies both. In your groups figure it out. I don't care how you do it.” The students proceed to work in their groups on solving the problem.

When Mr. H pulled the class' attention back to the front of the room and asked them to share their thoughts, David shared his own theory for solving a system of linear equations. He explained to the class that he could guess the solution by looking at odd versus even numbers. “In 5x-2y=7 the 2y is even and the 7 is odd, therefore 5x must be odd, therefore x must be odd.” He found x=7. While this is not an efficient method for solving any linear system, it shows evidence that David has genuinely engaged with the problem and thought about what solutions might be reasonable. This engagement would not have occurred if Mr. H had explicitly told the students how to solve a system of equations. Moreover, it shows that David has a deep understanding of what it means for a particular point to be a solution to a given system of equations.

In a later interview, November 24th, I specifically asked David about a problem that Mr. H had put on the board. Mr. H had told the students that it would be on the test, but he then refused to tell them how to do it. David responded to my enquiry by saying “Because we already had the prior knowledge. We just had to think more in order to know how to do it.” From this interview, it is clear that David has a willingness to engage more deeply with the problem.

Don't Tell Me!

On September 22nd in C-Block Michael, Jessica, and William were working together on a problem. They were stuck. Marley came over from another group and offered to tell Jessica the answer. In response, Jessica plugged her ears and yelled “la la la la la”. She did not want to hear the answer. This incident illustrates a shift in this student's relationship with mathematics. Jessica progressed from working on mathematics solely to get the right answer, to valuing the process of figuring things out for herself.

In not providing students with a predetermined set of steps to follow, Mr. H allowed Jessica to place importance on figuring out the process for herself. It is important to note that this was not an isolated incident. This happened multiple times on that day and it also occurred in D-Block when Michael came over and told Jonathan the answer to a problem. Jonathan responded: “Show me how you got it. Don't just tell me the answer.” The process for solving a problem became as important as the answer to many of these students. This was only 3 weeks into the semester.

Muis argues that students who are in a teacher-centred environment hold beliefs that the goal of mathematics is to get the right answer as quickly as possible (Muis, 2004, p. 338). In contrast to this, when placed in a problem solving environment Michael and Jessica showed clear evidence that their goal was to understand how to solve the problem. They valued the process of solving the problem, not just the result.

The evidence shows that a positive sociomathematical norm was established in Mr. H's class wherein students came to believe that they needed to understand the method for solving a problem. Simply providing an answer was no longer sufficient for them. It is important to note that this incident has been chosen as an exemplar, but it is not the only time that students demanded that other students supply reasoning rather than answers. This expectation became routine as the semester progressed. Thus, it became a true sociomathematical norm in Mr. H's class.

A High-achieving Student

November 10th, C Block: I did an interview with Joe. He was the top-achieving student in the honors class. In the interview, I asked him what he thought about the fact that Mr. H did not give him any predetermined steps or method for solving the problems. He said “Well, I guess it helps a lot if you, like, think about how to do it yourself because then you actually remember how to do it as opposed to writing it on the board and everyone copies it down mechanically.”

Joe made it clear throughout the interview that he felt that he would have been successful in any classroom environment and he was always successful in the past in more traditional environments. His definition of success relied on achieving high grades. However, he also commented that the lack of predetermined steps led him to achieve a greater understanding of the mathematics and he believed that he would retain what he had learned.

This classroom environment provides the potential for other less academic, although no less important, advantages for a student like Joe. Burton speaks specifically to social changes and, in particular, to the expectations that future employers will have for students to have collaborative problem solving skills. She argues that there are invaluable social opportunities that arise naturally when students are forced to work in groups. These include learning how to evaluate the skills of peers, collaborate, or handle dominant personalities, ensuring that others make contributions to the work, and accepting responsibility (Burton, 1984). While Joe expresses that his academic achievement would be high in any classroom environment, this opportunity for social development is unique to an environment that relies on collaboration and group work.

Evidence has been provided to show that in a classroom environment rooted in problem solving the lack of predetermined steps presented by the teacher has led to an increase in a willingness to engage in mathematical thinking, as in the case of David, as well as a shift in the relationship that students have with mathematics, as shown by Jessica and Jonathan placing significant value on the process rather than the solution to a problem. Moreover, even the highest-achieving student in the class, Joe, commented that while he would achieve high-grades in any environment, the lack of predetermined steps provided by Mr. H has led him to have an increased level of understanding.

Grades

Assessment is an unavoidably complex topic in education. Many teachers struggle to find assessment methods that go beyond traditional summative assessments and to use assessment methods that provide students with opportunities for deeper learning. Moreover, grades can both effectively motivate or demotivate students. In the past I have felt that my students would not engage in mathematical activity without the knowledge that they were going to be assessed on their efforts. However, I have also struggled with students wanting to do only the minimum amount of work required to achieve their desired grade. The focus came away from understanding the mathematics and shifted towards getting correct answers.

At the time of this research, Mr. H had just begun his quest to implement teaching through problem solving in his classroom. Hence, much of his assessment methods remained rooted in a traditional pedagogy. This section does not aim to evaluate the benefits of differing assessment methods but rather to continue looking at the relationship between problem solving and grades and assessment from the perspective of the students within this class.

Grades as Motivation

Joe is a very high-achieving student. On October 27th, Mr. H handed back some tests to C-Block.

> *Joe: “I feel scared. I feel scared...”*
>
> *Joe: “Oh, wait. I got 100%. Wait, how'd I lose half a mark?”*
> *Students look over their tests and compare marks.*
>
> *Almost all of the students look to their friends to correct their mistakes.*
>
> *Mr. H: “The test was out of 41. The highest mark was 40.5 so that's what it's out of.*
>
> *Joe: “What was the average of the other class? Did anyone get 41?”*
>
> *Mr. H: “Does it matter?”*
>
> *Joe: “Yes, it does!”*
>
> *Mr. H doesn't answer. He takes questions about the test.*

In this incident, Joe demonstrated that he was exceedingly motivated by grades to be a high-achieving student. This desire is likely to push him to solve every problem and study hard for his exams. However, his motivation to do mathematics was still largely external. He was very concerned about the half mark that he lost, not because he perceived that he did not understand a key mathematical concept, but because there may have been a student in the other class that got a better mark than him. At the time of this incident, Joe's relationship with mathematics was rooted in competition and ego rather than curiosity. He felt a need to have done better than everyone else in mathematics and he believed that this would be accomplished if he got the highest grade. Darien Shannon (2009) argues that studying behaviour in response to failure can reveal the beliefs that students have about intelligence. Joe's response to his temporary perception of failure reveals that he likely has an entity view of intelligence (Shannon, 2009). That is, he likely believes that intelligence is fixed and that he can do little to change it. Thus, he likely believes that needing to put significant effort into his studies would imply that he has lower intelligence. This theory is also supported by Joe's boasts to me that he generally puts very little effort into math class and finds it easy. There is danger in this belief system and this danger is particularly prevalent in a problem solving environment. If Joe is outperformed by his peers he is likely to lose self-esteem. Also, "one who holds a fixed view of intelligence is very susceptible to feelings of inadequacy when they are faced with negative feedback. It is common to see students give up on a question when they have been unsuccessfully trying to solve it for only a few minutes" (Shannon, 2009, p. 21).

Grades as Demotivation

Jack is a student who has struggled in mathematics classes in the past. On September 16th , D Block, Jack expressed that he was not going to work on a problem at home because his goal in the class was to get fifty percent and then he would be done. His primary motivation to do mathematics was external – the pressure of a grade, rather than curiosity or a desire to understand a mathematical problem. This attitude has the potential to be problematic for Jack. By setting the bar at fifty percent he granted himself permission to abandon problems that he sees as difficult. It is evident that the presence of traditional assessment is doing nothing to improve Jack's relationship with mathematics. Jack does not love mathematics or appreciate the beauty and interconnectedness of the subject. Rather, his relationship is rooted in avoidance and, at the time of this study, the grades served as a self-imposed barrier between him and the mathematical problem.

Moby was a student who had struggled in mathematics classes in the past. On September 29th, in D Block, Mr. H put a system of three equations on the board and the groups got to work trying to figure out how to solve it. Moby, Michael, and Mohammed were working together. Mohammed was confused. Moby empathized and said that he also got confused with so many steps. However, it was clear, by his ability to answer his groups' questions, that he understood what he was doing. He feigned a struggle to empathize with his group members but he was helping them work their way through the problem. Moby said “Once you get 'c' everything else is easy...” During the work on this problem Moby was engaged and confident. He was supportive and encouraging of his group members as they went through the process of trying to solve the problem. He alleviated their frustration by pretending to share in it and by providing some scaffolding to help them as they solve the problem.

However, when the next problem was put on the board Moby and Mohammed got different answers and began debating who was right. At one point in the discussion, Moby shut it down by saying “*Screw it, there's only one of those on the test*.” He then abandoned the problem. When he encountered difficulty, he reverted to using grades as an excuse to abandon a problem. He was no longer motivated by the mathematics or by a desire to solve the problem and he decided that the external motivator (grades) was insufficient to keep him engaged with the problem. In fact, this external motivational factor was counter-productive in that it served as an excuse for disengaging from the problem.

Overall, grades can serve to motivate students to work hard and persist in mathematics but they can also serve as an excuse to disengage or abandon a mathematical problem. However, even when grades serve to motivate students to work hard, there is a danger that the focus of this hard work will be on satisfying the grading scheme rather than on understanding the mathematics. This may reenforce some non-availing beliefs about mathematics. In particular, it may reenforce the belief that the primary goal of mathematics instruction is to have students do well on exams (Muis, 2004). It is important to consider the impact of assessment methods very carefully.

Notes

Notes are a challenge for teachers who are trying to teach mathematics through problem solving. Giving notes somewhat undermines the philosophy and environment that they are trying to establish. When teachers stand at the front of the room and dictate notes, they are solidifying their role as the primary provider of knowledge in the classroom. However, in teaching through problem solving teachers are trying to encourage students to look to themselves as sources of knowledge. It is certainly possible, with some creativity, to incorporate notes in a way that the notes have been generated from the students rather than from the teacher. However, it is clear that the role of notes in the progressive classroom will be significantly diminished from the more traditional environment. Some students will let the notes go without a second thought and will even be relieved not to have to spend any more time sitting and copying what a teacher writes on the board. Other students rely heavily on the notes as a resource in studying for exams and as a way to recall for themselves what happened in class. As teachers, we do not want these students feeling like they have to spend the period frantically copying down what each group does for fear of not otherwise having something to refer to at a later date. Mr. H dealt with the problem by occasionally handing out a single photo-copied sheet of notes at the end of the period that summarized that day's key concept with an example or two. Students were welcome to have the notes to place in their binders or to ignore them altogether.

Marley

On November 4th I interviewed Marley. She had a history of being a very high-achieving mathematics student. She told me that she enjoyed mathematics because she could study and get it by doing a bunch of practice questions. I asked her about her previous teachers and she laughed: "*I don't remember them being like Mr H before. My last teachers, they usually give me notes and I worked. They would give me notes and tell me to do questions and I would work on it at home*." Marley enjoyed this system. She made a comment "*I'm better with equations than imagination*." Marley expressed that she would prefer it if Mr. H would give more notes. She said "*the main reason we are here is to study so I think he should give a portion of the notes and then he can go on with the stuff for fun*."

This discussion reveals Marley's belief that mathematics is the stuff that was contained in her previous teachers' notes. The things that Mr. H is doing are 'fun' and therefore should come after the real mathematics. Previously, I had asked if she would be happy if the course had continued similarly to the way that it had been run for the first month (through problem solving). She had responded "*I would not be happy. I think it's a waste of time*."

This interview reveals a lot about Marley's relationship with mathematics and the beliefs that she holds. Marley's previous enjoyment in mathematics stemmed largely from the fact that she could feel successful at it simply through hard work and practice. She is a very diligent student and is more than willing to take notes and imitate the procedures on a slew of practice questions until she has mastered a particular skill. This process of learning has shaped Marley's beliefs about the nature of mathematics. She believes that the real mathematics is what is contained in those notes. She therefore would like Mr. H to give the notes first and not waste time on the "stuff for fun". Marley is in danger of leaving school with this distorted understanding of the nature of mathematics. As discussed earlier in this paper in the section on affect, such beliefs can carry with them consequences. These consequences include students leaving school with a superficial and disconnected understanding of mathematics and the mathematical knowledge that they do acquire may not be transferable when it is needed at a later date (NCTM, 2003).

Marley stated both that she is good at mathematics and that she is not good with imagination. This illustrates her belief that mathematics does not require imagination. Based on her previous mathematics experience, she is right. Fischer writes "a reproduced or stereotyped product does not count as creative, no matter how fine it is in terms of craftsmanship” (Fischer, 2005, p. 24). Indeed, she has not needed to be creative to be successful in her previous mathematics classes. She simply had to take notes and diligently imitate what was in those notes. Her previous traditional classroom experience has shaped her relationship with mathematics and after two months in Mr. H's class this relationship has not changed. Instead, she created a distinction between the real mathematics (notes) and the fun stuff that Mr. H does.

Sabrina

November 24th, C Block – I did an interview with Sabrina. She was a diligent student and had always worked a year ahead on her mathematics. Now that she was in the honors class she was feeling somewhat overwhelmed because “*Now it's kinda like I have to work to get it. But it's always been kinda easy*.” I asked her to compare this year to previous years.

> *Sabrina: “It's different <laughs>. We don't take as many notes so I can't really refer back to anything. I find this year I have to look through the textbook to figure out a concept. So that's a bit of a challenge. I guess it's prep for University and all that but I liked the notes.”*

This is a consistent theme with several of the high-achieving students. They feel that they need to be able to refer to notes in order to learn and understand some of the mathematical concepts. Earlier in the interview she also commented that she felt that she just needed more practice to understand. Both Sabrina and Marley believed that understanding came through practice and that the notes were an essential component to their learning.

Abi

On September 29th, about one month into the course I did an interview with Abi. As mentioned, she suffered from anxiety and had previously really struggled in mathematics. We discussed the idea of notes. Here is a section of that interview:

> *Abi: The other teachers make it so complicated to understand. They try to shove like so much information and then they're like 'okay. Apply it'. But he gives us probably the minimum information we need and then the rest will work itself out.*
>
> *Me: Do you find that it does?*
>
> *Abi: Yeah, I do. From him giving us one thing and then he will be like 'try this with this question' instead of giving us all the information at once and then... you know what I mean?*
>
> *Me: So instead of giving you lots of examples and then asking you to try it he's giving just the main concept.*
>
> *Abi: I was this far in last year and I had no idea what I was doing and now I don't have a doubt at all.*
>
> *[...]*
>
> *Me: So, are you enjoying math this year?*
>
> *Abi: I wouldn't say I'm enjoying it yet but it's a more positive attitude instead of negative like 'oh, I have to go to math'.*

From this interview we can see that the lack of notes in Mr. H's class was slowly having a positive effect on Abi's relationship with mathematics. She was not yet planning to become a mathematician but her anxiety was less and she no longer dreaded and feared coming to class. She found it very overwhelming to have so much information pushed at her in mathematics class and she appreciated being given only the essentials and then left to play with it and digest it. Her interview and class participation demonstrate that she was slowly starting to feel like mathematics was something that she could accomplish.

The Uprising

November 10th, D Block was a particularly interesting and dramatic class to observe. Mr. H wanted to show his students how to derive the quadratic formula. However, this topic had him conflicted in his pedagogical philosophy. On one hand, he believed in teaching through problem solving and had managed to get through the curriculum up to this point without relying heavily on a lecture/notes format. He believed that students should not be blindly using the quadratic formula without understanding the proof behind it. However, he did not believe that his students were capable of deriving the proof on their own. With this in mind, he attempted to present the proof in a lecture to the class simply so that they had seen the proof before they started using the formula. The following is a segment from the field notes that I took during that lesson:

Mr. H starts deriving the Quadratic Formula. Anders gets frustrated when fractions come up.

Jonathan: "This is retarded."

Abi: "Are you following this?"

Jordan: "I'm lost."

Abi: "Me too."

Abi (yells): "No-one is following you!"

The whole class agrees / laughs.

Mohammed: "What the hell Mr. H!"

Mr. H laughs at himself. "Who is getting this?"

3 students raise their hands.

Mr. H laughs at himself again but keeps going.

The class starts yelling sarcastically "I get it!" and begging him to stop.

Mr. H: "Okay guys, you don't want to be people who use this thing and don't understand how it works."

Abi: "But you didn't teach us! You just showed it on the board. If you taught it we'd probably get it.

The class continues protesting the lecture.

Mr. H gets them to settle down and finishes. Abi/ Jetta/ Jordan/ Jonathan/ Mohammed/ Anders are not getting it. They are now just sitting quietly.

Mr. H gives them another question to do.

Jetta: "I'm not getting this."

Mohammed: "Me either"

Anders is working on his own.

Mr. H walks by.

Anders: "Wait. Wait. Wait... Am I doing this right?"

Mr. H: "You're doing it right. But you can keep going."

Abi: "So, are we supposed to use the formula?"

Mr. H does not answer.

Abi: "Just a simple yes/no. Do I use the formula?"

Anders: "Mr. H, is this right?"

Mr. H: "You can go further... 4/10?"

Anders: "2/5. But I didn't change the square root."

Jonathan: "Mr. H!"

Abi: "Mr H. Is this right?"

Mr. H walks away. He goes to the front and talks about simplifying expressions and using the quadratic formula. He does the question on the board.

There are many interesting layers to this interaction between Mr. H and his class. I would propose that if this lesson had been presented at the beginning of the semester, very few, if any, students would have protested. This is a lesson that is part of almost every traditional mathematics classroom and students generally sit in their seats and quietly copy down their notes. However, when Mr. H tried to give these same notes after two months of teaching through problem solving, the proof was met with strong protests from the students. They got frustrated and yelled. At first they spoke out, but when Mr. H made them settle down and finished anyway they eventually sat quietly and listened. After this exchange, we see that the students became a lot more teacher-dependent. They began continuously asking Mr. H if they are right whereas previously they had been able to work within their groups and look to their peers for their primary support. In this incident, Mr. H provided the students with notes and examples but this has led to a decrease in thinking and engagement for the remainder of the period.

This incident, rather than being presented as a criticism of Mr. H's pedagogy on this day, serves as strong evidence of the effectiveness of teaching mathematics through problem solving in changing students' relationships with mathematics and mathematics education. As Goldin writes:

> *Mathematics educators who set out to modify existing, strongly-held belief structures of their students are not likely to be successful addressing only the content of their students' beliefs, or only the warrants for their beliefs. It will be important to provide experiences that are sufficiently rich, varied, and powerful in their emotional content to foster the students' construction of new meta-affect (Goldin, 2002, p. 71).*

Clearly, Mr. H has been successful in doing this. In particular we see a shift in Abi's beliefs about what it means to teach mathematics when she said "*but you didn't teach us. You just showed it on the board. If you taught it we'd probably get it*". The other students' protests are strong evidence that they agreed with her. Abi no longer believed that writing notes on the board qualifies as teaching mathematics. This is despite the fact that all of her experience prior to this course involved teachers giving notes at the board.

This change in student beliefs corresponds with Muis' findings. She writes: “These studies provide some evidence that students' beliefs are malleable and that change can occur by changing the methods by which students engage in learning” (Muis, 2004, p. 362). Mr. H essentially removed all but the most basic notes from his class. This is one of the many aspects of this pedagogy that has helped change the relationship that these students have with mathematics. In particular, they no longer believe that mathematics is learned exclusively from a teacher giving notes at the board. The lack of notes has reenforced the belief that mathematics is something that they can and should figure out for themselves.

It is informative to examine this incident in relation to the idea of sociomathematical norms. Drodge and Reid argue:

> *Students’ obligations to the social norms established within a mathematics context were important in determining the emotional acts that arose within that classroom. The patterns of social interactions that exist in all classrooms are, therefore, important in understanding why particular emotional responses are evident* (Drodge & Reid, 2000, p. 264).

By taking notes out of his pedagogy and refusing to implement an external source of all mathematical knowledge, Mr. H allowed students to see that mathematics is something that they can figure out for themselves. As a result, the pattern of social interactions in the classroom had changed. The classroom norm that Mr. H developed became that students were to rely on themselves and their peers for mathematical knowledge. In this incident, however, Mr. H tried to break away from the new expectations that he himself had created. The response of the students was to protest. However, after they realized that their protests were not going to change the course of the lesson, they opted to fall back into the old social norm of listening passively to the lesson. This old set of expectations continued to dominate the environment for the remainder of the period and the students functioned as if the teacher were once again the source of all mathematical knowledge.

Chapter 7.

Emerging Themes

During the course of my observations, I noticed that many of the interesting phenomena in the classroom were centred around group work, a lack of predetermined steps and solutions given by the teacher, grades, notes, as well as emotions and observable beliefs. Direct evidence relating to each of these sociomathematical norms was observed and recorded and these then became the “central factors” that I used to focus my observations and analysis.

Upon examination and analysis of the central factors in this thesis, several themes that cut across all of these categories began to surface. This section reconsiders the data that has already been analyzed with the intent of re-conceptualizing it as a unified body of work. I will bring to the forefront some themes that have been emerging throughout discussion of the central factors. In distinguishing these three main themes I am not intending to draw the conclusion that each of theme is completely independent from the others. While I present these themes separately, it is important to recognize that they do interact with and influence each other. I also recognize that there are instances where a single incident in the classroom could be analyzed from the perspective of more than one theme. The goal in this thesis is not to provide an exhaustive discussion that analyzes every theme that could emerge out of each incident. Rather, I intend to use each incident in the data to support a particular theme and conversely, for each theme to have sufficient data to support it.

Relationships with Mathematics

The first theme that emerged through a re-examination of the central factors is a more subtle component of affect. The data and analysis show significant evidence of a change in students' *relationships with mathematics*. This emerging theme also happens to answer the research question about how students' relationships with mathematics will change. According to the tetrahedral definition of affect that I am using that includes emotions, attitudes, beliefs, and values/morals/ethics, students' relationships with mathematics are part of the affective domain but they are not directly observable. In this small, qualitative study inferences needed to be made from the evidence that I gathered in order to draw conclusions about potential changes in students' relationships with mathematics.

Data showed explicit evidence of students' initial beliefs about mathematics in the class discussion of *A Mathematicians' Lament* on the first day of class. It was clear that many students held non-availing beliefs about mathematics. This is common among students who come from a traditional mathematics environment that is rooted in notes and homework assignments (Muis, 2004).

About one month into the semester students started to get frustrated with each other for attempting to share answers to a problem without sharing a method. This happened over multiple weeks in both the honors and the regular mathematics classes. This behavior showed evidence of a significant change in these students' relationships with mathematics. Whereas they initially believed that the purpose of mathematics is to get the correct answer by following predetermined procedures, their behavior illustrated that they were learning to value coming up with the process for themselves. This behavior also illustrated an increase in self-efficacy among the students. Most students from a traditional mathematics classroom believe that mathematics is something that only particularly gifted people create (Muis, 2004). In contrast, these students, after being immersed in an environment rooted in problem solving, came to believe that mathematics was something that they could figure out for themselves.

As mentioned in Chapter 6, Mr. H is still in the beginning stages of implementing teaching mathematics through problem solving in his classroom and he is still using primarily traditional grading methods in his class. Overall, there was evidence that these grades could serve both to motivate and demotivate students in problem solving. High and low-achieving students had different responses to grades. However, the presence of assessment definitely impacted students' relationships with mathematics. The evidence from these observations showed that the traditional assessment methods that were used generally reenforced nonavailing beliefs both among high and low achieving students.

As was seen, low-achieving students frequently used grades as an excuse to abandon a task that was frustrating them. For example, Jack was a student who had not been very successful in his previous mathematics classes. There was evidence of Jack putting a barrier between himself and mathematics at the beginning of the semester when he refused to work on a problem at home because his goal was to do just enough to get fifty percent. It is not the grades themselves that created a negative relationship with mathematics for Jack; this relationship was likely built up from many previous experiences over the course of many years. Rather, grades served as a crutch in supporting the negative relationship. Jack used grades as an escape route to avoid difficult problems. Rather than seeing being 'stuck' as an honorable state from which much can be learned as Mason, Burton, and Stacey (1985) advocate, Jack still had a relationship with mathematics that was rooted in avoidance rather than curiosity. We see similar behavior in Moby who had previously been a low-achieving student in his traditional mathematics classes but was doing quite well in this more progressive environment. He also used grades as an excuse to abandon a problem when it got difficult.

In high achieving students, grades generally had a different impact. I examined Joe's story. He was highly motivated by grades. However, his efforts were focused primarily on the grades and on getting the highest grade rather than on understanding the mathematics or appreciating the beauty and elegance of different solutions. His concern over his missed half-mark on a test was not rooted in concern that he may lack understanding of an important concept. He was much more concerned about how he stacked up against others. In this case, the traditional grading scheme was supporting an ego-based relationship with mathematics.

It is evident that implementing assessment methods that are more consistent with the new classroom norms that Mr. H has established could be beneficial in building a more positive relationship with mathematics for both the high and low-achieving students in these classes. However, it is also possible that the nature of assessment being a source of external motivation is, by its very nature, problematic for helping students develop positive relationships with mathematics. Examining the effects of different grading schemes on students' relationships with mathematics is an opportunity for future research but goes beyond the research questions presented in this paper.

In Marley's interview she expresses that she would like Mr. H to give the notes first and do the “fun stuff” later. A heavy reliance on notes in Marley's traditional classes has dramatically shaped her relationship with mathematics. Marley believed that the notes *were* the real mathematics and that they were the most important part of mathematics class. Marley perceived that she had a positive relationship with mathematics because she enjoyed the fact that she could master it through practice and repetition. Many students abandoned this belief fairly early in the semester but Marley's previous success under the traditional paradigm had her quite attached to this view of mathematics. She viewed mathematics as a serious endeavour rather than as a subject that might be played with and explored for fun. Sabrina expressed similar sentiments in her interview.

In contrast to these high-achieving students' perspectives on notes, I presented Abi's story. She found that the notes in her previous classes caused her a lot of anxiety because of the volume of information that was being front-end loaded. By reducing the notes to the bare minimum and only giving the students essential information, such as notation customs and basic concepts, Mr. H allowed Abi to work at overcoming her anxiety and building a more positive relationship with mathematics. There were also high-achieving students who appreciated the lack of notes. Abi's story was closer to the feelings of the majority of the students. Complaints about a lack of notes were rare and reserved for only a couple of the highest-achieving students.

I found that the students who desired the presence of traditional notes did so because they had a belief that doing mathematics necessarily involved copying notes and then doing practice questions. In this way, even though more notes were desired by these students, giving more notes would actually be supporting a nonavailing belief system, the consequences of which have already been discussed in this paper.

Overall, cross the grain analysis of each of the five central factors revealed an emerging theme of students' *relationships with mathematics*. There was direct evidence that Mr. H's attempts to break away from a traditional pedagogy had the consequence of producing positive changes in his students' relationships with mathematics and helped to develop more availing belief systems.

Thinking and Engagement

Another theme that emerged across this research was a theme of *thinking and engagement*. Many aspects of Mr. H's classroom afforded opportunities for students to become deeply engaged in mathematical thinking. There were also many challenges that provided the potential for disengagement. As I was sifting through the data, it became clear that there was much to be learned by examining the moments when students engaged or disengaged with a problem.

Problem solving is an unavoidably emotional experience. Throughout my observations in Mr. H's class, emotions were the most frequent reason for abandonment of a problem. We saw an example of this when Abi got stuck and had such a strong emotional response that she not only abandoned the problem but was unable to engage for the remainder of the period. In contrast, by the end of the semester, students were more able to work through difficult emotions and continue to engage in a problem. I presented an example of this from my observations on the last day where Simone, Irena, Eric, and Moby were working in a group to solve problems involving equations of parabolas. Initially, they did not understand how to do it. Irena made comments like “I'm going to cry if we get a bad mark on this" and Simone made a comment "I'm scared." However, these feelings did not cause them to disengage from the problem. By the time I came back from an interview, the students were excitedly working at the board and had figured it out. They took a lot of joy in knowing that they figured it out for themselves.

Overall, in this classroom environment where students are required to work in groups, students were often involved in thinking very deeply about mathematics with a level of engagement that would be very rare or non-existent in a traditional classroom where students are left to work in isolation. I provided an example of this when Michael, Wilhelm and Sabrina are trying to solve a system of three equations. They are working together discussing mathematics in a way that illustrates a deep level of thinking and engagement.

The lack of predetermined steps and the lack of provided solutions to problems undeniably lead to deeper thinking and engagement on behalf of the students. An example of this was presented when David uses his own reasoning to solve a linear system using properties of odd and even numbers. He had been deeply involved in figuring out the problem for himself and ended up coming up with a unique and creative solution. While the goal of the lesson was for students to uncover a method that would work for all linear systems and David's method does not meet this criterion, his solution does show that he engaged deeply in the problem.

There was equally as much to be learned by examining the moments in the class where students refused to engage. An example of this was presented when Mr. H attempted to give notes on the proof of the quadratic formula using direct instruction. The students launched a protest. Abi's comment "*But you didn't teach us! You just* *showed it on the board. If you taught it we'd probably get it*", shows a significant change in her relationship with mathematics and pedagogy. For her, giving notes no longer qualified as teaching. The lack of formal notes in this class has clearly left its mark on her beliefs about mathematics education. Moreover, we see that even though this lesson confused her, it did not have her doubting her own abilities. She believed, even though Mr. H did not, that the students would be able to figure it out for themselves.

Overall, a cross the grain analysis of the data revealed a theme of *thinking and engagement* that permeated each of the central factors. Certainly, in this classroom, teaching through problem solving has lead to a deeper level of thinking and engagement than might have been present in a traditional classroom.

Confusion and Understanding

The third theme that emerged throughout this research was a theme of *confusion and understanding*. Problem solving, by its nature, will create confusion. If students are not at least initially confused when working on a problem, then it was not a real problem to begin with. The five central factors presented in this paper all represent things in Mr. H's classroom that contribute to the creation of confusion and the development of understanding. Confusion is part of the foundation of the classroom environment. Mr. H's pedagogy is founded on the idea of giving students a problem that they will initially not know how to solve. As such, students will inevitably experience initial feelings of confusion. Whether or not this confusion can transform into understanding is the key to the success of teaching through problem solving.

One place where this was clearly evident was group work. Evidence of this was provided in Moby's interview. Moby spoke eloquently about the impact of group work on his level of understanding. He felt that working in groups enabled him to ask questions when he was stuck. He also commented that when he was working with peers he was “less likely to fall asleep” and that explanations provided by peers were frequently more understandable than explanations provided by a teacher. We saw evidence of Moby working well in a group by pretending to be at the same level as his peers and joining in their confusion in order to help them on a problem. His ability to have his own solution to the problem and yet still join in with his peers and work on the problem from their perspective showed a deep level of mathematical understanding that was being supported and fostered by working in a group.

Moby was not the only student whose level of understanding benefited from group work. There were many students who had clearly benefited from peer interactions and there were many times when students were clearly feeding off of each other's enthusiasm and joy as they figured out problems for themselves. A beautiful example of this was provided on the last day when Simone was working with a group to figure out the equations of some parabolas with different given information. The fact that they figured out the methods for themselves and were able to teach these methods to each other shows clear evidence of understanding. Burton also supports the idea that having students convince each other of their arguments is a good way to gather evidence of understanding (Burton, 1984).

The data also spoke to the impact on this pedagogy on the high-achieving students' levels of understanding. Examples were presented of Joe and Sabrina. Their interviews lead to some interesting insights on their levels of understanding in this class.

Joe felt that while he would be successful in a more traditional classroom environment, the lack of predetermined steps and solutions in this environment pushed him to have a deeper level of mathematical understanding because he needed to figure it out for himself. Sabrina, on the other hand, is also a high-achieving student but still felt a need for notes and practice in order to feel that she truly understood the mathematics.

Overall, a cross analysis of each of the central factors brought to the forefront a theme of *confusion and understanding* that was related to, although distinct from, the theme of *thinking and engagement*. Generally speaking, the data showed that teaching through problem solving lead to deeper levels of understanding on behalf of the students. The difference was generally most dramatic among mid to low achieving students.

The Results in a Framework

I aim to bring together all of the central factors that focused the research and observations and the resulting themes that emerged from a cross-grain analysis into one cohesive framework. There were five central factors that were used to focus the research: emotions and observable beliefs, the lack of answers and dictated steps that would normally be provided by the teacher, group work, notes, and grades. As I closely examined and analyzed each of these central factors three themes emerged. These were: *students' relationships with mathematics, thinking and engagement*, and *confusion and understanding*.

Taken together, the presence of these themes across each of the five aspects of a problem solving classroom that were studied here can be visualized as fitting into a frame:

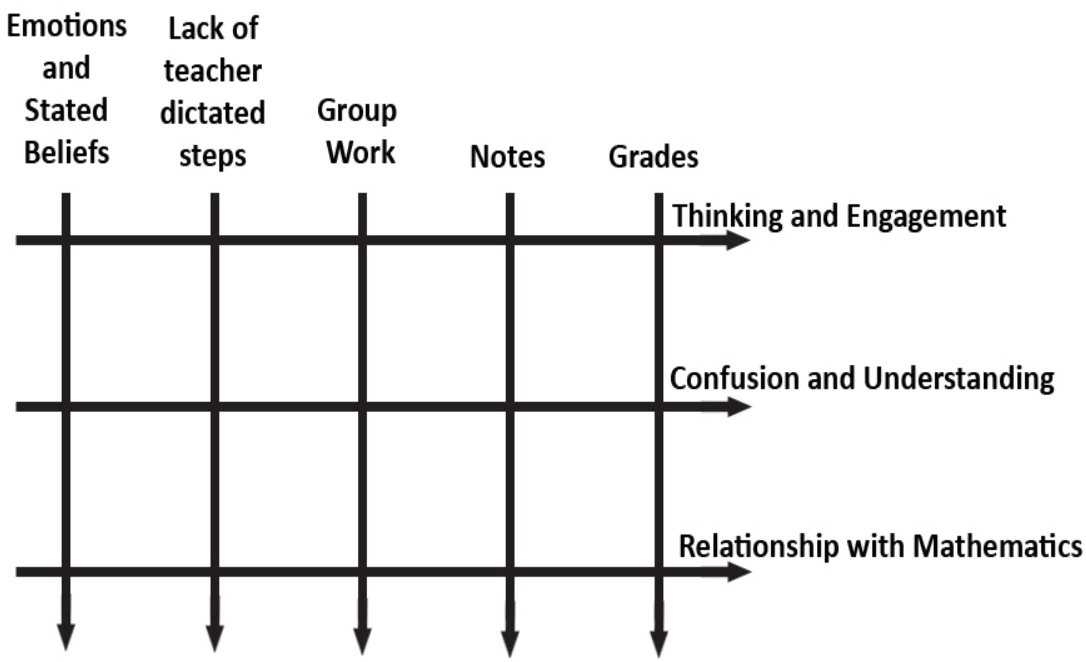

Figure 2

The three themes run across each of the five central factors that were the focus of the data collection phase of this research. Each of the anecdotes that I have shared in Chapter 6, Evidence and Results, correspond to one of these intersection points and serve to illustrate how a particular theme emerges from the central factors. For example, I looked at incidents where the lack of teacher-dictated steps or solutions influenced each of the themes of *thinking and engagement, confusion and understanding*, and *students' relationships with mathematics*.

As is the nature of education, many of these incidents were, in fact, very complex and could have been viewed from the perspective of several intersection points. However, I chose to analyze and present each event based on what I felt was the predominant central factor and theme surfacing in the moment. The intent was not to force an event to conform to one intersection point but rather, to illustrate how each theme emerged from the five central factors.

It should also be noted that in choosing to focus my observations on five aspects of a problem solving classroom that I felt were the most ubiquitous and prevalent, it is probable that other potential central factors were not addressed. I speculate that it is likely that these three emergent themes would be visible even in any additional aspects of a problem solving classroom.

Chapter 8.

Conclusion: Back to the Beginning

In Chapter 3, I presented four research questions that developed through my own personal experience and through the research presented in the literature review. Each of these unanswered questions presented a barrier in my own personal growth as an educator. They were questions that prevented me from taking the next step in my professional development and from aligning my beliefs with my practice. The questions were:

1. Will students buy in?
2. (a) Is it possible to create this dynamic, problem solving environment consistently within a classroom?

   (b) Can it be done within the context of the current curriculum?
3. How does teaching through problem solving affect low-achieving students?
4. How does a problem solving environment impact students' relationships with mathematics?

My hope was that answering these questions would allow me to take the next steps in moving towards a more progressive pedagogy that is rooted in teaching through problem solving. It is time to return to these initial questions and seek to provide some answers.

Will Students 'Buy In'?

In Chapter 7, evidence was presented from my final observation that described students actively and excitedly engaged in working on problems to write the equations of parabolas given different starting information. After observing this lesson I can say with confidence that it is certainly possible to get students to embrace to classroom environment that is created by teaching through problem solving. I have never before witnessed such palpable enthusiasm about parabolas as I did during my final observation. Moreover, these were not only the highest achieving students. It was a diverse group of both low and high-achieving students working together in teams.

This buy-in happened in phases. At first, students were actively engaged in working in the problems mostly because the problems presented were inherently interesting and easily accessible by students of all ability levels. Even though the lack of predetermined steps provided challenges for the students, when I conducted the individual interviews they all expressed that they understood why he did not give them predetermined steps or answers to the problems and they were glad he did not. Students would occasionally ask Mr. H if they were doing things right, or get visibly frustrated when they were stuck. It is important to learn that what students will request or tell you that they want when they are stuck on a problem may be very different than what they want upon reflection. The example of Abi clearly illustrated this. Complaints during the process of problem solving are not, in fact, evidence that students are not buying in.

Some students, in particular a couple of high-achieving students, remained sceptical about the lack of notes and traditional lessons. They felt a need to get the 'work' done. This scepticism was rooted in their nonavailing beliefs about the nature of mathematics. However, despite this scepticism, they were still willing to engage and try their best on the problems that were presented in class because they were highly motivated to be successful at everything that a teacher asked them to do. As several of them directly expressed, they would be successful in any environment.

Overall, it was hard not to be impressed with how quickly and wholeheartedly students bought into this new classroom environment. It was important to learn that while negative emotions were a natural response to students being stuck and did lead to negative verbal comments, students were still able to recognize the value in the process. Upon reflection, students were able to brush off the negative feelings and appreciate the gains in understanding that came from figuring things out for themselves and in hindsight students would embrace the confusion. This finding is in line with the research done by Krista Muis which found that students' belief systems could change within a relatively short period of time (Muis, 2004). The themes that emerged throughout this research provide some clarity around students' reasons for buying into this process. In general students began to see mathematics as a means of attacking and solving problems for themselves and to recognize that it does not always need to be handed to them by an authority figure. This change in their beliefs about mathematics and themselves in relation to mathematics enabled them to persevere through the times when they were confused and struggling. Moreover, they began to believe that genuine engagement in a problem would lead to a greater level of understanding.

Is it possible to create this dynamic problem solving environment consistently within a classroom? Can it be done within the context of the current curriculum?

The example of Mr. H's class serves as a proof-by-existence that it is possible to create this classroom environment and still students interested and engaged in problems that come from the curriculum. However, this is not to argue that it can be done perfectly or easily. Mr. H is in the beginning stages of developing his classroom environment. Overall, he has been remarkably successful. However, regardless of one's personal pedagogy, the quest to improve one's teaching is never ending. We saw evidence of Mr. H struggling with the conflict between his beliefs and the pressure to teach students the proof of the quadratic formula. While he believes in teaching through problem solving, he was unable to find a way to cover the proof of the quadratic formula using these methods. As a result, Mr. H tried to revert to a more traditional transmission model of teaching and attempted to dictate the algebraic proof of the quadratic formula. The resulting protests from the students revealed how completely they had bought into the problem solving environment. Upon reflection, I believe that it is not impossible to cover this proof within the pedagogy of teaching through problem solving. While it may be beyond the ability of most students to derive this proof on their own using algebraic methods, this does not prohibit them from engaging with the proof through problem solving. For example, perhaps they could be asked to compare and contrast the proof with the method of solving a quadratic equation by completing the square.

In conclusion, it is possible to create a dynamic classroom environment rooted in problem solving. Moreover, even concepts that seem boring at a surface level or that are boring when presented in a traditional manner can be more interesting when taught through problem solving. Students believed that their engagement in serious thinking about mathematics often led to an increase in their understanding of that mathematics. A particular example that was presented in this paper involved students writing the equations of parabolas. The proof of the quadratic formula was a notable exception. We have seen evidence that Mr. H is on a journey rather than having reached a destination. Teaching through problem solving will be a long term goal that will require flexibility, reflection, and constant personal and professional development by the educator. What is most important to note is that the students have not suffered for being on this journey with Mr. H. Despite the fact that his pedagogy is not perfectly aligned with his ideals, students are experiencing a dynamic mathematical environment and they are learning the curriculum. Students have learned that there are alternatives to the social and sociomathematical norms that are established in a more traditional environment. In turn, this change in norms has inevitably influenced their relationships with mathematics. This shift to more availing beliefs about mathematics is likely to make their knowledge more transferable when it is needed at a later date (NCTM, 2003).

How does teaching through problem solving affect low-achieving students?

The answer to this question is complex and very much depends on the individual student. Students struggle for a myriad of reasons. Several examples emerged throughout this research. Abi struggles with anxiety that is so severe that she was unable to participate in a traditional mathematics class. Moby's main problem was boredom in a traditional class. He continuously fell asleep in class or skipped going all together. Both of these students benefited greatly from this more progressive pedagogy. I also did an interview with Colton who has struggled academically with mathematics in the past. Colton found the group work to be a very socially challenging experience because he felt that his mathematical skills were so weak that he did not have anything to contribute to the group. However, he did say that he believed that he was enjoying mathematics more this year and that he appreciated his ability to get help from his peers.

Moby's story provides a dramatic example of the power of teaching mathematics through problem solving. In a traditional environment he was bored and, by his own admission, either slept through or skipped most classes. However, in an environment where he was pushed to figure things out for himself and work with others he emerged as a leader in the classroom. His level of thinking and engagement in the class dramatically increased. Academically, he went from failing to getting an 'A' demonstrating an increase in understanding. Moreover, Moby showed evidence of dramatic changes in his relationship to mathematics. He was excited about the subject and found it interesting. He stated that he enjoyed coming to class and working on the problems.

Abi still struggled with anxiety. This was a particular challenge for her when she got stuck working on a problem. However, while her frustration may have overwhelmed her in the moment and caused her to disengage, she expressed unconditionally that her level of understanding was much higher and that she had a more positive relationship with mathematics. Moreover, when talking about thinking and engagement she mentioned that her anxiety was even more overwhelming in a more traditional class because she never felt that she understood the curriculum. In this more progressive class, the emotional payoff that occurred when she finally understood helped her to maintain some perspective on her moments of anxiety.

Overall, the results for each of the individual low-achieving students that I examined were diverse. The degree of the positive effects of these methods varied. However, there was no evidence of a single student who was done a disservice by participating in this class rather than a more traditional environment.

How does a problem solving environment impact students' relationships with mathematics?

The answer to this question emerges naturally out of the answers to the previous three questions and out of the theme of *relationship with mathematics*. In general, students unanimously and unequivocally state that a classroom rooted in problem solving is more enjoyable. Many students have shown evidence of a change in their beliefs about mathematics and have started to hold more availing beliefs. I have already discussed several examples of both low-achieving and high-achieving students and how their relationships with mathematics have changed for the better. The exception to this is

Marley, a very high-achieving student, who has created a distinction between the 'real math' and the 'fun stuff' that Mr. H does rather than change her beliefs about the nature of mathematics.

It is unclear whether this more positive relationship with mathematics will persist beyond the setting of this classroom. As Muis (2004) argues, it is likely that unless students are made explicitly aware of their beliefs and are encouraged to reflect on them, that their beliefs could revert back if they are again placed in a traditional classroom environment.

What have I learned as a researcher and an educator?

This has been an invaluable learning opportunity in developing my skills as an educational researcher. I have never done a research project of this scale before. Hence, the entire process has been an entirely new experience. From trying to concisely articulate my questions, to planning the particular methodology, to completing observations while trying to maintain an impartial position in the classroom, to finally sitting down with the massive amounts of data that I had collected, sifting through it applying the constant comparative methodology to look for emerging themes, to eventually staring at a blank word document thinking “okay now, just write a thesis”, and finally editing and re-editing the document to clarify thoughts and arguments - it has been quite illuminating.

When I embarked on this study it was not with the intent to uncover a universal or inalienable theory about mathematics education. I approached this thesis with a predetermined vision for the kind of teacher that I want to be and the type of classroom environment that I would like to create. As such, I was not an impartial observer within the classroom. I was biased in my desire to see it work. Moreover, I did not choose to explore other pedagogical frameworks for instruction. I also recognize that there are limitations to a study of this scale. I am one researcher, and I observed only one teacher and two classes in one school. Throughout the process I inevitably needed to make choices that would significantly impact the outcome of this study. Choosing to limit the study to a single classroom narrowed the scope of the study and limited my ability to generalize my conclusions beyond a proof-by-existence. I was able to show that it is possible for this pedagogy to be effective. However, I was unable to make the claim that it would always be effective. Another limiting factor is that I chose the five central factors around which to focus my observations and the emerging themes came from my own interpretation of these observations. Focusing my observations on these five central factors will have narrowed my attention and may have limited my awareness of other occurrences within the classroom. It has been my intent throughout this thesis to be upfront about the aspects of this research that relied upon my own judgement and interpretations. However, for myself, I do not feel that these limitations have diminished the lessons that came out of this research or the impact that these lessons will have on my future educational practice.

The parts of the process that had the most lasting impact on me are those that were pivotal in creating personal growth for me as an educator. At the end of all of this, I will not be attempting to embark on a career in educational research. I will be intending to continue shaping lives as an educator. This has been my passion and the root of my motivation in writing this thesis. I am walking away from this research with a solid belief that aligning my ideals and my practice is attainable. It is possible to break out of the traditional paradigm in mathematics education and in doing so provide a much richer experience for students. Moreover, I no longer worry that taking this risk may have possible negative consequences for these students. I am confident that moving towards a more progressive pedagogy will not only make their mathematics experience much more enjoyable, but also will lead to deeper levels of understanding of the curriculum. It will undoubtedly take courage to move in a new direction and there will be many challenges along the way. However, with the research questions answered, there is nothing holding me back from taking the first steps towards implementing teaching mathematics through problem solving in my classroom.

"Whatever you can do, or dream you can, begin it.
Boldness has genius, power, and magic in it"

-- Goethe

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