BROOKS-TYPE RESULTS FOR COLORING OF DIGRAPHS

by

Ararat Harutyunyan

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Name: Ararat Harutyunyan

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Examining Committee: Dr. Luis Goddyn
Chair

Dr. Bojan Mohar, Senior Supervisor

Dr. Matt DeVos, Supervisor

Dr. Pavol Hell, SFU Examiner

Dr. Bruce Reed, External Examiner,
Professor of Computer Science,
McGill University

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Abstract

In the thesis, the coloring of digraphs is studied. The chromatic number of a digraph D is the smallest integer k so that the vertices of D can be partitioned into at most k sets each of which induces an acyclic subdigraph.

A set of four topics on the chromatic number is presented. First, the dependence of the chromatic number of digraphs on the maximum degree is explored. An analog of Gallai's Theorem is proved and some algorithmic questions involving list colorings are studied. Secondly, an upper bound on the chromatic number of digraphs without directed cycles of length two is obtained, strengthening the upper bound of Brooks' Theorem by a multiplicative factor of $\alpha < 1$. Thirdly, evidence is provided for the global nature of the digraph chromatic number by proving that sparse digraphs with maximum degree Δ can have chromatic number as large as $\Omega(\Delta/\log\Delta)$, as well as showing the existence of digraphs with arbitrarily large chromatic number where every constant fraction of the vertices is 2-colorable. Finally, a generalization of digraph coloring to acyclic homomorphisms is considered, and a result linking D-colorability and girth is presented.

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Chapter 1

Preliminaries

Digraphs have not been as thoroughly studied in the literature as (undirected) graphs. In this thesis, we study the chromatic number of digraphs. One of the most common ways of defining the chromatic number of a directed graph D is to forget the orientation of the edges of D and define the chromatic number of D as the chromatic number of the underlying graph (we call this the *orientation-forgetful chromatic number* of D). This seems to be the most common coloring parameter for digraphs studied in the literature by researchers. The disadvantage of this definition is that digraphs with very different structures can have the same chromatic number if their underlying graphs are the same.

We study a particular coloring variant of digraphs called the *dichromatic number* in [49]. This is the smallest integer k such that the vertex set of the digraph D can be partitioned into k acyclic sets. We have decided to call this parameter the *chromatic number* of D for the reasons that will become apparent later. The problem does not seem to have received much attention in the literature until recently. We will show that the chromatic number is the natural coloring invariant for digraphs by presenting some old and new results that generalize analogous results from graph coloring.

In the next section, we present some common notation and definitions used in the thesis. More specific terms are defined throughout the thesis when they are needed.

1.1 Basic notation and definitions

In this section we give the main definitions and terminology that is used in the thesis. The basic definitions below can be found in [63].

A *graph* $G = (V, E)$ consists of a set V of *vertices* and a set E of *edges* and a relation that associates with each edge two vertices called its *endpoints*. The edge $uv \in E$ joins vertices u and v. A *loop* is an edge $vv \in E$ from a vertex v to itself. *Multiple edges* are edges having the same pair of endpoints. A graph is called *simple* if it does not have any loops or multiple edges. Every graph considered in this thesis is simple unless otherwise stated. The *order* of G is the number of vertices of G and the *size* of G is the number of edges. Vertices u and v are said to be *adjacent* or *neighbors* if they are the endpoints of the same edge. The *degree* of a vertex v, denoted by $deg(v)$, is the number of neighbors of v. The maximum degree of a graph G is denoted by $\Delta(G)$. G is a *d-regular graph* if every vertex of G has degree d. Given a vertex $v \in V$, the *open neighborhood* of v, denoted $N(v)$, is defined as $N(v) = \{u : uv \in E\}$. A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. The *length* of a cycle is the number of vertices in the cycle. The *girth* of a graph is the length of its shortest cycle. If the graph does not have any cycles, its girth is infinite. A *subgraph* of a graph G is a graph H such that $V(H) \subset V(G)$ and $E(H) \subset E(G)$. Let S be a subset of vertices. We denote by $G[S]$ the graph that has the vertex set S and edge set $E(S)$ where $uv \in E(S)$ if and only if $uv \in E(G)$. $G[S]$ is called the *induced* subgraph on S. G is *connected* if for all $u, v \in V$ there is a path in G containing u and v. A *cut-vertex* in a graph G is a vertex v whose removal increases the number of connected components of G. A maximal connected subgraph of G that has no cut-vertex is called a *2-connected component* or a *block* of G. A *cut-edge* (or *bridge*) is an edge whose removal increases the number of connected components of G. A graph is *planar* if it can be drawn in the plane so that no two edges cross. A set $S \subset V$ is *independent* if no two vertices in S are adjacent. We denote by $\alpha(G)$ to be the size of the largest independent set in G. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer k so that $V(G)$ can be partitioned into k independent sets.

Now, we introduce some definitions and notations for digraphs. The notation is standard and we refer the reader to [6] for an extensive treatment of digraphs. A *digraph* is obtained from a graph by giving each edge an orientation. We use xy to denote the arc joining vertices x and y, where x is called the *initial vertex* and y is called the *terminal vertex* of the arc xy. We denote by $A(D)$ the set of arcs of the digraph D. The vertex set of D will be denoted by $V(D)$. Digraphs discussed in this thesis will not have loops or parallel arcs. Such digraphs are called *simple*. We do allow, however, the existence of two arcs between two vertices going in opposite directions. For $v \in V(D)$ and $e \in A(D)$, we denote by $D - v$ and $D - e$ the subdigraph of D obtained by deleting v and the subdigraph obtained by removing e, respectively. We let $d_D^+(v)$ and $d_D^-(v)$ denote the *out-degree* (the number of arcs whose initial vertex is v) and the *in-degree* (the number of arcs whose terminal vertex is v) of v in D, respectively. The *total degree* of a vertex v is $d^+(v) + d^-(v)$. A vertex $v \in V(D)$ is said to be *Eulerian* if $d^+(v) = d^-(v)$. The digraph D is *Eulerian* if every $v \in V(D)$ is Eulerian. We say that u is an *out-neighbor (in-neighbor)* of v if vu (uv) is an arc. We denote by $N^+(v)$ and $N^-(v)$ the set of out-neighbors and in-neighbors of v, respectively. Every undirected graph G determines a *bidirected* digraph $D(G)$ that is obtained from G by replacing each edge with two oppositely directed edges joining the same pair of vertices. If D is a digraph, we let $G(D)$ be the *underlying undirected graph* obtained from D by "forgetting" all the orientations of the arcs. A digraph D is said to be *(weakly) connected* if $G(D)$ is connected. The *blocks* of a digraph D are the maximal subdigraphs D' of D whose underlying undirected graph $G(D')$ is 2-connected. We say that D is *strongly connected* if for every vertex u and v, there is directed path from u to v. A *cycle* in a digraph D is a cycle in $G(D)$ that does not use parallel edges. A *directed cycle* in D is a subdigraph forming a directed closed walk in D whose vertices are all distinct. A directed cycle consisting of exactly two vertices is called a *digon*. A vertex set $S \subset V(D)$ is called *acyclic* if the induced subdigraph $D[S]$ has no directed cycles. A *k-coloring* of D is a partition of $V(D)$ into k acyclic sets. The minimum integer k for which there exists a k-coloring of D is the *chromatic number* $\chi(D)$ of the digraph D.

1.2 Overview of the thesis

The rest of the thesis is organized as follows. In Chapter 2, we discuss the known results in the literature and motivate the problem. In Chapter 3, we prove Gallai's Theorem for list coloring of digraphs and study the algorithmic complexity of list coloring digraphs. In Chapter 4, we derive an upper bound on the chromatic number of digon-free digraphs in terms of maximum average degree of the digraph. In Chapter 5, we derive analogs of two well-known theorems in graph theory which show that the chromatic number, like the chromatic number, is not a local parameter. In Chapter 6, we look at the extension of the chromatic number to acyclic homomorphisms and derive a result about D-colorability and girth. In Chapter 7, we highlight related problems and future work.

Chapter 2

The Chromatic Number of a Digraph

In this chapter we introduce the problem studied in this thesis - the chromatic number of a digraph (see Chapter 1). We give a motivation for studying this digraph invariant by discussing some results for the chromatic number of (undirected) graphs and see how they generalize to digraphs. We will see that the digraph chromatic number introduced in Chapter 1 is the natural coloring invariant for digraphs. We also discuss some results in the literature and look at related problems.

Recall, that given a digraph D, the *chromatic number* $\chi(D)$ of D is the smallest integer k such the vertices of D can be colored with k colors so that no directed cycle is monochromatic. This coloring parameter was first introduced by Neumann-Lara [49] in 1982. There are a few papers that appeared in the literature on the topic in the following decade. However, recently there seems to be a newfound interest in the chromatic number due to some results that highlight its close relationship with the chromatic number of an (undirected) graph. In the rest of this chapter, we closely study this relationship.

2.1 Brooks theorem for graphs and digraphs

Recall that the *chromatic number* $\chi(G)$ of a graph G is the smallest integer k such that the vertices of G can be colored with k colors so that no two adjacent vertices receive the same color. Note that chromatic number of a graph G is equal to the chromatic number of its bidirected digraph $D(G)$. One of the earliest results in graph coloring is the following theorem of Brooks [13].

Theorem 2.1.1. *Let G be a connected graph of maximum degree Δ. Then $\chi(G) \leq \Delta + 1$ with equality only for odd cycles and complete graphs.*

For digraphs, it is not hard to see that the following tight upper bound holds, as proved by Neumann-Lara [49].

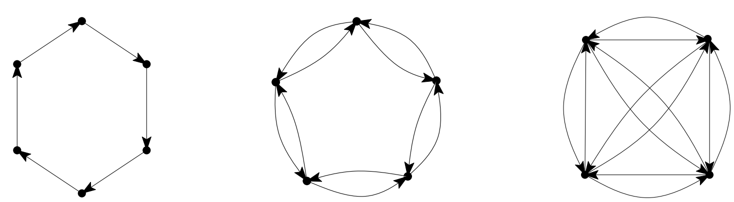
Theorem 2.1.2 ([49]). *Let D be a digraph and denote by Δ_o and Δ_i the maximum out-degree and in-degree of D, respectively. Then*

$$\chi(D) \leq \min\{\Delta_o, \Delta_i\} + 1.$$

It turns out that Brooks' Theorem has an analog for digraphs. We say that a digraph D is *k-critical* if $\chi(D) = k$ and for every vertex v, $\chi(D - v) < \chi(D)$. Mohar [46] proved the following theorem.

Theorem 2.1.3 ([46]). *Suppose that D is a k-critical digraph in which every vertex v satisfies $d^+(v) = d^-(v) = k - 1$. Then one of the following cases occurs:*

1. *$k = 2$ and D is a directed cycle of length $n \geq 2$.*
2. *$k = 3$ and D is a bidirected cycle of odd length $n \geq 3$.*
3. *D is bidirected complete graph of order $k \geq 4$.*



(a) (b) (c)

The above theorem shows that the only obstructions preventing a critical $k-1$-regular digraph from being $k-1$-colorable are the obvious ones. Note that odd cycles in Brooks' theorem are replaced by odd bidirected cycles and cliques are replaced by bidirected cliques. However, we also have an additional structure – the directed cycle. This new structure will also appear later when we study Gallai's Theorem for digraphs.

2.1.1 Spectral version of Brooks' Theorem

Brooks' Theorem also has its analog in spectral graph theory. Give a simple graph G of order n, one can define the *adjacency matrix* $A = A(G)$ for G. This is the $n \times n$ matrix $A = (a_{ij})$, where $a_{ij} = 1$ if vertex i is adjacent to vertex j, and 0 otherwise. The eigenvalues of $A(G)$ reveal a lot of information about G and are well treated in the literature, see for example [17, 31]. Let $\lambda_n(G)$ be the largest eigenvalue of $A(G)$. It is not hard to show that $\Delta(G)$ is always an upper bound on $\lambda_n(G)$. Wilf [64] proved the following Brooks-type result about the chromatic number. Note that the upper bound in the theorem is at most the upper bound of $\Delta + 1$ in Brooks' Theorem.

Theorem 2.1.4 ([64])**.** *Let G be a simple graph. Then $\chi(G) \leq \lambda_n + 1$ with equality if and only if G is an odd cycle or a complete graph.*

Surprisingly, this result generalizes to the digraph chromatic number. Given a simple digraph D of order n, the *adjacency matrix* $A(D)$ of D is the $n \times n$ 0-1 matrix where $a_{ij} = 1$ if ij is an arc and 0 otherwise. Note that $A(D)$ not necessarily a symmetric matrix. The *spectral radius* of D, denoted by $\rho(D)$, is the largest modulus of an eigenvalue of $A(D)$. It is known from the Perron-Frobenius theorem (see, for example [36]) that $\rho(D)$ is an eigenvalue of D with a corresponding non-negative eigenvector. More properties on the spectra of digraphs can be found in [14]. It turns out that the spectral radius gives a Brooks-type theorem on the digraph chromatic number, as shown by Mohar [46].

Theorem 2.1.5 ([46])**.** *Let D be a loopless digraph. Then*

$$\chi(D) \leq \rho(A(D)) + 1. \tag{2.1}$$

If D is strongly connected, then equality holds in 2.1 if and only if D is one of the digraphs listed in cases (1)-(3) in Theorem 2.1.3 for $k = \chi(D)$.

This theorem is further evidence that the digraph chromatic number is the natural coloring parameter for digraphs. Note that the assumption of strong connectivity in the above theorem is required – it is known that *transitive* tournaments (tournaments which are acyclic) have all eigenvalues equal to zero. Incidentally, this also shows that the upper bound in the above theorem will not hold for the "orientation-forgetful" chromatic number of digraphs.

Lin and Shu [43] recently obtained another spectral result for the chromatic number of digraphs. Let $D_{n,k}$ be the set of digraphs of order n and chromatic number k. They characterize the digraph which has the maximal spectral radius in $D_{n,k}$.

2.2 Further motivation - the circular chromatic number

In recent years, researchers have worked on different coloring invariants of graphs that refine the chromatic number. One of the well-studied problems in this area is the circular chromatic number. The circular chromatic number of a graph G is greater than $\chi(G) - 1$ and at most $\chi(G)$. Thus, it is a refinement of the chromatic number. Recently, the circular chromatic number was generalized to digraphs. In this section, we explore the relationship between the circular chromatic number and the chromatic number of a digraph. As we shall see, the circular chromatic number of a digraph refines the digraph chromatic number in much the same fashion as the circular chromatic number of a graph refines the chromatic number.

2.2.1 Circular chromatic number of graphs

Let G be a simple graph. For $q \in \mathbb{Q}$, we define a *circular q-coloring* of G to be a map $\phi : V(D) \to S_q$, where S_q is the circle of perimeter q, such that for all $xy \in E(G)$, $\phi(x) \neq \phi(y)$ and the shortest distance $d_S(\phi(x), \phi(y))$ from $\phi(x)$ to $\phi(y)$ on the circle is at least 1. We say that G is *q-circular colorable* if there exists a circular q-coloring ϕ for G. The *circular chromatic number* of G, denoted by $\chi_c(G)$, is defined as

$$\chi_c(G) = \inf\{q : G \text{ has a circular } q\text{-coloring}\}.$$

This notion is studied by many authors in the literature. We refer the reader to a survey by Zhu [67]. The circular chromatic number $\chi_c(G)$ refines the chromatic number $\chi(G)$ in the following sense.

Theorem 2.2.1. *For any graph G,*

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G).$$

It is known that if $\chi_c(G) = r$, then G is r-circular colorable, i.e. the infimum is attained.

The circular chromatic number was first introduced by Vince in 1988 [62] as 'the star-chromatic number'. The original definition of Vince, which is equivalent to the above definition, is as follows. For two integers $1 \leq d \leq k$, a *(k,d)-coloring* of a graph G is a coloring c of the vertices of G with colors $\{0, 1, ..., k-1\}$ such that for all $xy \in E(G)$, $d \leq |c(x) - c(y)| \leq k - d$. The circular chromatic number is defined as

$$\chi_c(G) = \inf\{k/d : \text{ there is a } (k,d)\text{-coloring of } G\}.$$

The definition of circular chromatic number has a natural extension to digraphs as treated by Mohar [45] and Bokal et. al [8]. A *circular q-coloring* of a digraph D is a function $\phi : V(D) \to S_q$ such that for all $xy \in E(D)$, $\phi(x) \neq \phi(y)$ and the distance $d_S(\phi(x), \phi(y))$ from $\phi(x)$ to $\phi(y)$ in the clockwise direction on the circle is at least 1. If D has at least one arc, we define the *circular chromatic number* $\chi_c(D)$ as

$$\chi_c(D) = \inf\{q : D \text{ has a circular q-coloring}\}.$$

If D has no arcs, then we define $\chi_c(D) = 1$. The above definition was introduced in [8]. As opposed to the circular chromatic number for graphs, it is possible that D does not admit a $\chi_c(D)$-coloring, i.e., the infimum is not necessarily attained. However, an alternative definition of a circular coloring overcomes this problem. Let $q \geq 1$. A map $\phi : V(D) \to S_q$ is called a *weak circular q-coloring* of D if, for every arc $uv \in A(D)$, either $\phi(u) = \phi(v)$ or the distance $d_S(\phi(u), \phi(v))$ from $\phi(u)$ to $\phi(y)$ in the clockwise direction on the circle is at least 1, and for every $x \in S_q$, the *color class* $\phi^{-1}(x)$ is an acyclic vertex set of D. It is easy to see that $\chi_c(D)$ is equal to the infimum of all real numbers $q \geq 1$ for which there exists a weak circular q-coloring of D. It turns out that results by Mohar [45] show that this infimum is always attained; i.e., every digraph D admits a weak circular $\chi_c(D)$-coloring. Moreover, it is also shown in [45] that $\chi_c(D)$ is a rational number for every D.

Interestingly, the circular chromatic number of a digraph is related to its analog for graphs. If G is a simple graph, then $\chi_c(G) = \chi_c(D(G))$, where $D(G)$ is the bidirected digraph obtained from G by replacing each edge with two oppositely oriented arcs. The following extension of Theorem 2.2.1 shows that the digraph circular chromatic number is related to the digraph chromatic number.

Theorem 2.2.2 ([8]). *For every digraph D, $\chi(D) - 1 < \chi_c(D) \leq \chi(D)$.*

Since the proof is short, we present it here.

Proof. Given a k-coloring of D, we can map each of the k color classes to a single point on S_k so that two points corresponding to two color classes are at least distance 1 apart. This shows that a k-coloring of D determines a weak circular k-coloring of D, proving the second inequality.

For the first inequality, let $p = \chi_c(D)$, $k = \lceil p \rceil$, and $\epsilon = p/2n$, where n is the order of D. Let c be a circular $(p+\epsilon)$-coloring. Then $S_{p+\epsilon}$ can be written as the disjoint union of $k+1$ arcs $A_0, A_1, ..., A_k$, each of length less than 1, and such that $c^{-1}(A_0) = \emptyset$. Let $V_i = c^{-1}(A_i)$, for $i = 1, ..., k$. Clearly, each V_i is acyclic. The partition of $V(D)$ into these acyclic sets gives a k-coloring of D. □

2.3 Some preliminary results and tournaments

Following the introduction of the chromatic number by Neumann-Lara several papers appeared on the subject. In particular, the study of tournaments has received some attention. Neumann-Lara and Urrutia [53] proved the existence of an infinite family of vertex-critical r-chromatic regular tournaments for every $r \geq 3$, $r \neq 4$. In particular, they proved the following theorems.

Theorem 2.3.1 ([53]). *For each pair of positive odd integers $r = 2l+1$, $i \geq 7$, there exists a vertex-critical r-chromatic regular tournament with $3^{l-1} \cdot i$ vertices.*

Theorem 2.3.2 ([53]). *For each even integer $r = 2l, l \geq 3$, and each odd $i \geq 7$, there exists a vertex critical $2l$-chromatic regular tournament with $3^{l-1} \cdot i$ vertices.*

The authors actually show a method of constructing such tournaments. They also conjecture that there exists an infinite family of vertex-critical 4-chromatic *circulant tournaments*. Circulant tournaments are defined as follows. Let Z_{2n+1} be the set of integers mod $2n+1$ and J an n-subset of $Z_{2n+1} - \{0\}$ such that for every $w \in Z_{2n+1}$, $w \in J$ if and only if $-w \notin J$. The circulant tournament $C_{2n+1}(J)$ is defined by $V(C_{2n+1}(J)) = Z_{2n+1}$, $A(C_{2n+1}(J)) = \{ij : i, j \in Z_{2n+1}, j - i \in J\}$.

In [51], Neumann-Lara solves the aforementioned conjecture in the affirmative. He also conjectures the following:

Conjecture 2.3.3 ([51]). *There is an infinite family of vertex-critical r-chromatic circulant tournaments for each $r \geq 3$.*

Neumann-Lara [52], settles the above conjecture for all $k \geq 3$, $k \neq 7$. In [5], the authors prove the conjecture for $k = 7$ and construct other infinite families of critical k-chromatic circulant tournaments for general k.

2.3.1 Extremal results on tournaments

The study of the chromatic number of tournaments has received some attention in the literature. Neumann-Lara [50] showed that the minimum order of a 3-chromatic tournament is seven, and the minimum order of a 4-chromatic tournament is eleven. In particular, he proved the following theorem.

Theorem 2.3.4 ([50])**.** *There are exactly four non-isomorphic 3-chromatic tournaments of order 7, and one 4-chromatic tournament of order 11.*

All the tournaments in the above theorem are also characterized.

Erdős, Gimbel and Kratsch [24] derived an extremal result for general digraphs. For a graph G, define $d(G) = \max\{\chi(D) \mid D \text{ is an orientation of } G\}$. For an integer k, let $d(k)$ be the minimum number of edges a graph G satisfying $d(G) = k$ can have. Then the following bounds on $d(k)$ hold.

Theorem 2.3.5 ([24])**.** *There exist positive constants c_1, c_2 such that*

$$c_1 k^2 \log^2 k \leq d(k) \leq c_2 k^2 \log^2 k.$$

The first inequality in Theorem 2.3.5 implies that any digraph D of order n has $\chi(D) = O(\frac{n}{\log n})$. The second inequality implies that there exists a digraph D of order k with $\chi(D) = \Omega(\frac{k}{\log k})$. The proof of the above theorem relies on previous results. Here, we give shorter proofs from first principles. We say that *almost all tournaments* have a property P if the probability that the random tournament T_n on n vertices obtained from K_n by randomly orienting the edges satisfies property P with probability tending to 1 as n approaches infinity. Given a digraph D, we let $\alpha(D)$ be the largest acyclic set of vertices in D.

Theorem 2.3.6. *Almost all tournaments of order n have chromatic number at least $\frac{1}{2}\left(\frac{n}{\log n+1}\right)$.*

Proof. Let T_n be the random tournament of order n. Let A be a fixed subset of vertices of T_n of size $2\log n + 2$. Note that if the subdigraph $T_n[A]$ induced by A is acyclic then there is an ordering of the vertices of A such that all the arcs of $T_n[A]$ go forward with respect to the ordering. Thus,

$$\begin{aligned}\mathbb{P}[\alpha(T_n) \geq 2\lceil\log n\rceil + 2] &\leq \binom{n}{2\lceil\log n\rceil + 2}\mathbb{P}[A \text{ is acyclic }] \\ &\leq \binom{n}{2\lceil\log n\rceil + 2}(2\lceil\log n\rceil + 2)!\left(\frac{1}{2}\right)^{\binom{2\lceil\log n\rceil+2}{2}} \\ &\leq n^{2\lceil\log n\rceil+2} \cdot \frac{1}{n^{2\lceil\log n\rceil+1}} \cdot \frac{1}{n^2} = o(1).\end{aligned}$$

Since $\chi(D) \geq \frac{|V(D)|}{\alpha(D)}$ for any digraph D, the theorem follows. □

In the proof of the above theorem we used the fact that with high probability a random tournament has no acyclic set of size greater than $2\log n + 2$. On the other hand, it is known (see, for example, [20, 59]) that every tournament has an acyclic set of size at least $c\log_2 n$, for some positive constant c. The following lemma can be readily derived.

Lemma 2.3.7. *Let D be a tournament of order n. Then D has an acyclic set of size at least* $\log n$*.*

Proof. Greedily pick vertices to be in the acyclic set in the following manner. In the first step pick any vertex v, and remove from the graph either the set of its in-neighbors or out-neighbors, whichever is smaller in size. Then pick one of the remaining neighbors and put it in the acyclic set. Repeat this process until there are no vertices remaining. Clearly, the resulting set is acyclic. Since in each iteration we remove at most $(n-1)/2$ vertices from the graph, we pick at least $\log n$ vertices. □

Theorem 2.3.8. *Let T be a tournament of order n. Then $\chi(T) \leq \frac{n}{\log n}(1 + o(1))$.*

Proof. The theorem is intuitively clear from lemma 2.3.7. In each iteration, using a single color we color and remove from the digraph a large acyclic set whose existence is guaranteed by Lemma 2.3.7. Of course, as we remove a color class from the digraph the size of the subsequent acyclic set decreases. However, it turns out that we still only need roughly $\frac{n}{\log n}$ colors. We now make the argument more precise.

We color the vertices of the digraph using the procedure described above until there are at most $\left\lfloor \frac{n}{\log^2 n} \right\rfloor$ uncolored vertices remaining. At this point we stop the procedure and greedily finish by assigning a new color to each uncolored vertex. This gives a valid coloring.

During each iteration of the first phase of the procedure we remove at least $\log\left(\frac{n}{\log^2 n}\right) \geq \log n - 2\log\log n$ vertices. Hence, there are at most $\frac{n}{\log n - 2\log\log n}$ such iterations. Hence the total number of colors needed is at most $\frac{n}{\log n - 2\log\log n} + \lfloor\frac{n}{\log^2 n}\rfloor \leq (1+o(1))\frac{n}{\log n}$.

□

Theorem 2.3.8 is thought to have extension to all digraphs, where n is replaced by the maximum total degree of a vertex. McDiarmid and Mohar [44] conjectured the following:

Conjecture 2.3.9 ([44]). *Every digraph D without digons and with maximum total degree Δ has $\chi(D) = O(\frac{\Delta}{\log \Delta})$.*

2.3.2 The digraph chromatic number and the Erdős - Hajnal Conjecture

The chromatic number of tournaments is also related to the well-known Erdős-Hajnal conjecture. The Erdős-Hajnal conjecture is one of the fundamental conjectures in Ramsey theory. Recall that for a graph G, $\alpha(G)$ is the size of the largest independent set in G and $\omega(G)$ is the order of the largest complete graph in G. It is known by Ramsey theory that if G is a graph of order n, then $\max\{\alpha(G), \omega(G)\} \geq \frac{1}{2}\log_2 n$ (see, [27]). Erdős [21] showed that this is essentially best possible by proving the existence of a graph G with $\max\{\alpha(G), \omega(G)\} < 2\log_2 n$. However, it may be true that forbidding subgraphs will yield a polynomial lower bound on $\max\{\alpha(G), \omega(G)\}$ rather than logarithmic.

If H is not an induced subgraph of G, then we say that G is an *H-free* graph. Erdős and Hajnal [25] conjectured the following.

Conjecture 2.3.10 (Erdős-Hajnal Conjecture). *For every graph H, there exists a positive $\epsilon = \epsilon(H)$ such that every H-free graph G with n vertices has* $\max\{\alpha(G), \omega(G)\} \geq n^\epsilon$.

The conjecture is known to hold for some classes of graphs. Alon, Pach and Solymosi [3] raised the following conjecture on tournaments that has a similar flavor. Recall that a tournament is called *transitive* if it is acyclic.

Conjecture 2.3.11 ([3]). *For every tournament T, there exists a positive constant $\epsilon = \epsilon(T)$ such that every T-free tournament with n vertices has transitive subtournament of size at least n^ϵ.*

We have shown above that every n-vertex tournament has transitive subtournament of order at least $\log_2 n$. Results of Erdős et al. [20] and Spencer [59] show that up to a multiplicative constant this is best possible. Alon, Pach and Solymosi [3] prove the following.

Theorem 2.3.12 ([3])**.** *Conjecture 2.3.10 and Conjecture 2.3.11 are equivalent.*

P. Seymour [58] raised the following question: is it true that for every tournament H, there exists a constant $c = c(H)$ such that every H-free tournament T has $\chi(T) \leq c$? It turns out that the answer to this question is negative as shown by Berger et al. [7]. One can pose the following weaker conjecture.

Conjecture 2.3.13 ([7])**.** *For every tournament H, there exist $c > 0$ and $\epsilon < 1$, such that if T is an H-free tournament, then $\chi(T) \leq c|V(T)|^{\epsilon}$.*

It is not hard to see that Conjecture 2.3.13 is equivalent to Conjecture 2.3.11, and thus to the Erdős-Hajnal Conjecture.

2.4 Planar digraphs and vertex-arboricity

For planar digraphs there is the following conjecture, raised by Neumann-Lara (1982) and Skrekovski (2001).

Conjecture 2.4.1. *If D is a planar digraph without directed cycles of length 2, then $\chi(D) \leq 2$.*

The conjecture is still very much open and seems quite difficult. Some results in this area follow from the theory of *vertex arboricity* of graphs. The *vertex arboricity* or *point-arboricity* of a graph G, denoted by $a(G)$, is the minimum number of sets in a partition of $V(G)$ into sets each of which induces a forest. The following observation is clear.

Observation 2.4.2. *If G is a graph and D is any orientation of the edges of G, then $\chi(D) \leq a(G)$.*

It is known that Conjecture 2.4.1 does not hold for vertex arboricity as proved by Chartrand, Kronk and Wall [15].

Theorem 2.4.3 ([15])**.** *$a(G) \leq 3$ when G is planar, and this bound is sharp.*

Note that Theorem 2.4.3 implies that $\chi(D) \leq 3$ for every digon-free planar digraph D.

For general graphs, the following upper bound on vertex-arboricity is given by Kronk and Mitchem [42].

Theorem 2.4.4 ([42]). *If G is a connected graph that is neither a complete graph of odd order nor a cycle, then $a(G) \leq \lceil \frac{\Delta(G)}{2} \rceil$.*

Planar graphs without short cycles have vertex arboricity two, as shown by Raspaud and Wang [55].

Theorem 2.4.5 ([55]). *If $k \in \{3, 4, 5, 6\}$, then $a(G) \leq 2$ for every planar graph G having no k-cycles.*

Raspaud and Wang [55] also show the following.

Theorem 2.4.6 ([55]). *If G is a planar graph with no two triangles having vertices that are shared or adjacent, then $a(G) \leq 2$.*

Theorem 2.4.5 with Observation 2.4.2 implies that $\chi(D) \leq 2$ for every planar digon-free digraph D having no k-cycles if $k \in \{3, 4, 5, 6\}$. Similarly, Theorem 2.4.6 implies that $\chi(D) \leq 2$ for every digon-free digraph D without triangles.

Albertson [1] proposed the following weaker version of Conjecture 2.4.1.

Conjecture 2.4.7 ([1]). *Every planar digraph of order n without digons has an acyclic set of vertices of size at least $n/2$.*

It may be true that $n/2$ in the above conjecture could be replaced by αn for some $\alpha > 1/2$. Below we show that α cannot be greater than $3/5$.

Proposition 2.4.8. *The largest acyclic set in the digraph D below is at most six.*

Proof. Suppose, for contradiction, that D has an acyclic set S of size seven. Then S contains exactly four vertices from one of the two directed pentagons. By symmetry, we may assume that the outer pentagon contains four vertices of S. Now, it is easy to see that the inner pentagon cannot contain more than two vertices of S, a contradiction. □

The above conjecture is a weakening of a much older conjecture due to Albertson and Berman [2].

Conjecture 2.4.9 ([2]). *Let G be a planar graph graph of order n and let k be the size of a largest set of vertices in G which induces a forest. Then $k \geq \frac{n}{2}$.*

By results of Borodin [11], it is known that $k \geq 2n/5$.

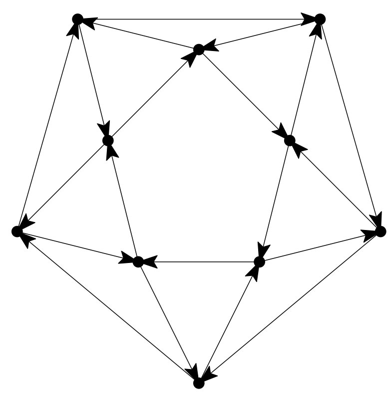


Figure 2.1: A digraph D on ten vertices with $\alpha(D) \leq 6$.

Chapter 3

Gallai's Theorem and List Coloring of Digraphs

3.1 Introduction

A theorem of Gallai [30] describes the structure of low degree vertices in graphs that are critical for the chromatic number. It states that the induced subgraph on the vertices of degree $k-1$ in a k-critical graph is composed of blocks that are either complete graphs or odd cycles. In this chapter, we consider the chromatic number of digraphs and show that Gallai's theorem can be extended to this setting. It is interesting to note that another structure appears in addition to cliques and odd cycles. These are directed cycles of any length. For a parallel, we observe that this kind of graphs also occur in the version of Brooks' Theorem for digraphs, see Theorem 3.1.3 below.

The Gallai theorem has a natural setting in terms of list colorings. For undirected graphs, it can be viewed as a list coloring problem where the list at each vertex has the same number of available colors as the degree of that vertex. The coloring problem for this type of lists is easily solvable for undirected graphs. However, as we show in Section 3.3, the list coloring problem of this type on digraphs is NP-hard.

List colorings and Gallai trees

A graph G is *k-color-critical* or *k-critical* if $\chi(G) = k$ and $\chi(H) < \chi(G)$, for every proper subgraph $H \subset G$. The minimum degree of a k-critical graph is at least $k-1$. A classical theorem of Gallai [30] states that in every k-critical graph, the vertices of degree $k-1$ induce a graph whose blocks are either odd cycles or complete graphs. Because of this result, a connected graph all of whose blocks are either odd cycles or complete graphs is called a *Gallai tree*.

A natural setting of applying Gallai's theorem is that of list colorings. Given a graph G and a list $L(v)$ of colors for each vertex v, we say G is *L-colorable* if there is a proper coloring of G (i.e. each color class is an independent set) such that each vertex v is assigned a color from $L(v)$. Having a k-critical graph G, one may assume that we have (somehow) colored vertices of degree larger than $k-1$ with $k-1$ colors and that only vertices whose degree in G is $k-1$ are left to be colored. Denote the subgraph induced by the vertices of degree $k-1$ by S. Now, each vertex $v \in V(S)$ has a list $L(v)$ of available colors, and $|L(v)| = \deg_S(v)$. This setting is used to formulate Gallai's theorem for list colorings. It was obtained independently by Borodin [12] and Erdős et al. [26]. Kostochka et al. [41] generalized it to hypergraphs.

Theorem 3.1.1 ([12],[26])**.** *Let G be a connected graph, and L a list-assignment for G. Suppose that $|L(x)| \geq \deg(x)$ for each $x \in V(G)$, and G is not L-colorable. Then G is a Gallai tree.*

The following strengthening of the previous theorem has been proved by Thomassen [61], while the generalization to hypergraphs can be found in [41].

Theorem 3.1.2. *Let L be an arbitrary list-assignment for a graph G. Let X be a subset of vertices such that $G[X]$ is connected and $|L(x)| \geq \deg_G(x)$ for each $x \in X$. Assume that $G-X$ is L-colorable. If G is not L-colorable, then $G[X]$ is a Gallai tree and $|L(x)| = \deg_G(x)$ for every $x \in X$.*

Digraph colorings and Brooks' Theorem

Note that the blocks in Gallai's theorem for undirected graphs are precisely complete graphs and odd cycles, which also appear in Brooks' theorem. For digraphs, a version of Brooks' theorem was proved in [46], as mentioned in a previous chapter.

Theorem 3.1.3 ([46]). *Suppose that D is a k-critical digraph in which for every vertex $v \in V(D)$, $d^+(v) = d^-(v) = k - 1$. Then one of the following cases occurs:*

1. *$k = 2$ and D is a directed cycle of length $n \geq 2$.*
2. *$k = 3$ and D is a bidirected cycle of odd length $n \geq 3$.*
3. *D is bidirected complete graph of order $k \geq 4$.*

Note that the last two cases of Theorem 3.1.3 are the analogues of odd cycles and complete graphs in the undirected version of Brooks' and Gallai's theorems. Thus, it is expected that the first case of Theorem 3.1.3 will appear in the Gallai's theorem for digraphs, which is proved in the sequel.

The rest of the chapter is organized as follows. In Section 3.2, we derive an analogue of Gallai's theorem for directed graphs. In Section 3.3, we consider algorithmic questions for list coloring a digraph.

3.2 List coloring and Gallai's Theorem

We define list colorings of digraphs in an analogous way as for undirected graphs. Let $\mathcal{C}$ be finite set of colors. Given a digraph D, let $L : v \mapsto L(v) \subseteq \mathcal{C}$ be a *list-assignment* for D, which assigns to each vertex $v \in V(D)$ a set of colors. The set $L(v)$ is called the *list* (or the set of *admissible colors*) for v. We say D is *L-colorable* if there is an *L-coloring* of D, i.e., each vertex v is assigned a color from $L(v)$ such that every color class induces an acyclic subdigraph in D. We say that D is *L-critical* if D is not L-colorable but every proper subdigraph of D is L-colorable. Clearly, by saying that a subdigraph H is L-colorable, we use the restriction of the list-assignment L to $V(H)$. The main result of this section is the following digraph analogue of Gallai Theorem.

Theorem 3.2.1. *Let D be a connected digraph, and L an assignment of colors to the vertices of D such that $|L(v)| \geq \max\{d^+(v), d^-(v)\}$. Suppose that D is not L-colorable. Then D is Eulerian and every block of D is one of the following:*

(a) *directed cycle,*

(b) *an odd bidirected cycle, or*

(c) *a bidirected complete digraph.*

Moreover, for each block B of D, there is a set C_B of colors so that for each vertex $v \in V(D)$, we have

$$L(v) = \{\cup C_B \mid B \text{ is a block of } D \text{ and } v \in V(B)\}.$$

Furthermore, $|L(v)| = d^+(v)$, implying that the blocks B containing v have pairwise disjoint color sets C_B.

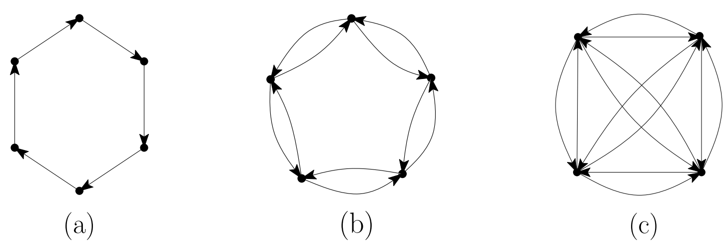


Figure 3.1: Possible blocks in Gallai trees: (a) a directed cycle, (b) a bidirected odd cycle, and (c) a bidirected complete graph.

The proof of Theorem 3.2.1 relies on several lemmas. The first of these gives information about the lists of L-critical Eulerian digraphs.

Lemma 3.2.2. *Let D be an Eulerian digraph, and let L be an assignment of colors to the vertices of D. Suppose that $|L(v)| = d^+(v)$ $(v \in V(D))$ and that D is L-critical. Given a vertex $v \in V(D)$, let f be an L-coloring of $D - v$. Then the following holds:*

1. *$L(v) = \{f(u) \mid u \in N^-(v)\} = \{f(w) \mid w \in N^+(v)\}$, and so each color in $L(v)$ appears exactly once in $N^-(v)$ and once in $N^+(v)$.*

2. *If u is a neighbor of v with $f(u) = c$, then uncoloring u and coloring v with c gives an L-coloring of $D - u$.*

Proof. If a color $c \in L(v)$ would not appear on the out-neighborhood of v, we could color v by c and obtain an L-coloring of D. Similarly, each color $c \in L(v)$ also appears on the in-neighborhood of v. This establishes the first claim.

To prove the second claim, remove color c from u and color v with c. Suppose, without loss of generality, that u is an out-neighbor of v. Since c appeared on the out-neighbors of v only once, we get an L-coloring of $D - u$. $\square$

Lemma 3.2.3. *Let D be a connected digraph. Let L be an assignment of colors to the vertices of D with $|L(v)| \geq \max\{d^+(v), d^-(v)\}$ for each $v \in V(D)$. Suppose that D is not L-colorable. Then*

1. *D is Eulerian and $|L(v)| = d^+(v) = d^-(v)$ for every $v \in V(D)$.*

2. *D is L-critical.*

Proof. To prove 1), we will use induction on $|V(D)|$. The claim is clear if $|V(D)| = 1$. If $|V(D)| = 2$, then D is either a directed edge (and hence L-colorable since $L(v) \neq \emptyset$ for $v \in V(D)$) or a digon, in which case 1) holds. So, assume now that $|V(D)| \geq 3$.

Suppose there exists a vertex $v \in V(D)$ such that $d^+_D(v) \neq d^-_D(v)$. Let $D' = D - v$. If D' was L-colorable so would be D, since one of the colors in $L(v)$ would not appear among the in-neighbors or out-neighbors of v. Thus, D' is not L-colorable. Then D' has a connected component D'' that is not L-colorable. Applying the induction hypothesis to D'', we conclude that D'' is Eulerian and $|L(u)| = d^+_{D''}(u) = d^-_{D''}(u)$ for every $u \in V(D'')$. Now, choosing a vertex $u \in V(D'')$ which is a neighbor of v we obtain that $d^+_{D''}(u) = |L(u)| \geq \max\{d^+_D(u), d^-_D(u)\} \geq d^+_{D''}(u)+1$, a contradiction. Therefore, $|L(v)| = \min\{d^+_D(v), d^-_D(v)\} = \max\{d^+_D(v), d^-_D(v)\}$, and the result follows.

To prove 2), we use induction on $|A(D)|$. The claim is clearly true when $|A(D)| \leq 2$. So, suppose $|A(D)| \geq 3$. Now, let $e = uv$ be any arc, and let $D' = D - e$. Let D'' be any component of D'. By part 1), D is Eulerian which implies that D'' is not Eulerian. Therefore, by the induction hypothesis, D'' is L-colorable. Similarly, if there exists a second component of D', it is also L-colorable. Therefore, D' is L-colorable, and thus D is L-critical. □

Let $C = v_1v_2...v_k$ be a cycle (not necessarily directed) in a digraph D. Let f be a coloring of $D - v_1$. A *shift of colors around* C is a color assignment g for $D - v_1$, where $g(v_2) = f(v_3)$, $g(v_3) = f(v_4)$, ..., $g(v_k) = f(v_2)$ and $g(v) = f(v)$ for $v \in V(D) \backslash V(C)$. Let us observe that in the case of Eulerian L-critical graphs, Lemma 3.2.2 guarantees that g is a (proper) L-coloring of $D - v_1$ since g can be obtained by repeatedly using part (2) of Lemma 3.2.2: first we uncolor v_2 and color v_1, then uncolor v_3 and color v_2, etc. until the last step when we uncolor v_1 and color v_k. This fact will be used throughout this section.

Lemma 3.2.4. *Let D be a connected digraph, and L an assignment of colors to the vertices of D such that $|L(v)| = \max\{d^+(v), d^-(v)\}$ for each $v \in V(D)$. Suppose that D is not L-colorable. Let C be a cycle of length 3 or 4 in the underlying graph $G(D)$. If the orientation of the edges of C in D is not cyclic (i.e., $E(C)$ does not induce a directed cycle in D), then $V(C)$ induces a complete bidirected graph in D.*

Proof. By Lemma 3.2.3, D is Eulerian and L-critical. First, assume that $C = v_1v_2v_3$ has length three. We may assume that the edges of C are directed as follows: v_3v_1, v_1v_2 and v_3v_2. We will show that the arcs v_1v_3, v_2v_3 and v_2v_1 are also present in D. Consider a coloring f of $D - v_1$. Let $f(v_2) = a$. If $f(v_3) = a$, then uncoloring v_3 and coloring v_1 with a would give an L-coloring of $D - v_3$ where v_3 has two out-neighbors colored a, a contradiction by Lemma 3.2.2. Therefore, $f(v_3) = b \neq a$. Now, the out-neighbor of v_1 that is colored b must be on the cycle C since otherwise doing a shift of colors around C we would get a new L-coloring of $D - v_1$ with v_1 having two out-neighbors colored b, so we could complete the coloring. The only way the out-neighbor of v_1 colored b is on C is when $v_1v_3 \in A(D)$. By a similar reasoning, $v_2v_1 \in A(D)$. To show the existence of the arc v_2v_3, consider an L-coloring of $D - v_3$ and the cycle C' consisting of the arcs v_1v_2, v_1v_3, and v_3v_2. The same proof as above shows that $v_2v_3 \in A(D)$. This settles the case when C is a cycle of length 3.

Suppose now that $C = v_1v_2v_3v_4v_1$ is a 4-cycle, and assume that the arcs of C are not cyclic. We may assume that the vertex v_1 has both vertices, v_2 and v_4, as its out-neighbors. Now, by criticality, $D - v_1$ is L-colorable. Moreover, every coloring f assigns different colors to v_2 and v_4 by Lemma 3.2.2. So suppose $f(v_2) = a$ and $f(v_4) = b$, $a \neq b$. Now, $f(v_3) \neq a$, since otherwise making the counter-clockwise shift of colors around C we would get two out-neighbors of v_1 colored a. Similarly, if we do a clockwise shift of colors around C we deduce that $f(v_3) \neq b$. Therefore, assume $f(v_3) = c$, $c \neq a, b$. Now, if we do a clockwise shift of colors around C we get that the color a disappears in the out-neighborhood of v, unless the vertex v_3 is an out-neighbor of v_1. Thus, by Lemma 3.2.2, $v_1v_3 \in A(D)$.

Now, regardless of the orientation of edges v_2v_3 and v_3v_4, the two triangles $v_1v_2v_3$ and $v_1v_3v_4$ have acyclic orientations and therefore by the first part of the proof, these sets induce bidirected cycles in D. Therefore, we have that C is a bidirected cycle that also has the chords v_1v_3 and v_3v_1. Now we apply the same proof to the cycle C' with arcs $v_2v_3, v_3v_4, v_4v_1, v_2v_1$ in which v_2 has two out-neighbors. We conclude that also the chords v_2v_4 and v_4v_2 are in D. This completes the proof of the lemma. □

Using Lemma 3.2.4, we now obtain the following.

Lemma 3.2.5. *Let D be a connected digraph, and L an assignment of colors to the vertices of D such that $|L(v)| = \max\{d^+(v), d^-(v)\}$ for each $v \in V(D)$. Suppose that D is not L-colorable. Let $C = v_1v_2...v_kv_1$, $k \geq 3$, be a cycle of length k in the underlying graph. Suppose that the orientation of the edges of C is not cyclic. Then the following holds:*

1. *If k is even, then $V(C)$ induces a complete bidirected subdigraph in D,*
2. *If k is odd, then $V(C)$ either induces a complete bidirected cycle or a complete bidirected subdigraph in D.*

Proof. By Lemma 3.2.3, D is Eulerian and L-critical. We proceed by induction on k. The cases $k = 3$ and $k = 4$ are established by Lemma 3.2.4. So we assume that $k \geq 5$. First, suppose that k is odd. We may assume that the two neighbors of v_1 on the cycle C, v_2 and v_k, are an out-neighbor and an in-neighbor, respectively. Such a vertex must exist by parity. We consider two cases. First, suppose there is a chord incident to v_1, say v_1v_i, $2 < i < k$. Then regardless of the orientation of the edge v_1v_i, one of the two cycles $v_1v_2...v_iv_1$ and $v_1v_iv_{i+1}...v_kv_1$ has acyclic orientation. By induction, we must have the arcs v_1v_i and v_iv_1 present in D. The arcs v_1v_i and v_iv_1 divide the cycle C into an odd cycle and an even cycle. Suppose $C_1 = v_1v_2...v_i$ is the even cycle. We can make sure that C_1 has its edges oriented acyclically by appropriately picking either the arc v_1v_i or v_iv_1. Thus, by induction, C_1 induces a complete bidirected digraph. Similarly, $C_2 = v_1v_iv_{i+1}...v_kv_1$ induces either a bidirected cycle or a bidirected clique. Now, consider the cycle $C_3 = v_2v_iv_{i+1}...v_kv_1v_2$. We can choose the appropriate bidirected arcs to ensure that C_3 has acyclic orientation. Since C_3 is an even cycle and it is shorter than C, it follows that C_3, and hence also C_2, induces a complete bidirected digraph. It remains to show that every vertex on C_1 has bidirected arcs to every vertex on C_2. But this is clear, since for any v_j on C_1, $v_1v_jv_iv_{i+1}...v_kv_1$ is an even cycle and thus induces a complete bidirected graph by the same argument as used above.

Now, suppose there is no chord incident to v_1. Let f be an L-coloring of $D - v_1$. First, we claim that $f(v_k) \neq f(v_2)$. Suppose, for a contradiction, that $f(v_k) = f(v_2) = a$. By making a shift of colors around C, we conclude that $f(v_3) = a$. Moreover, by repeatedly making a shift of colors around C, we conclude that all the original colors on C were equal to a. Let v_i be a vertex on C that has both of its neighbors on C as in-neighbors. Passing the color of v_2 to v_1 (by using Lemma 3.2.2(2)), the color of v_3 to $v_2, \cdots$, the color of v_i to

v_{i-1}, we get a proper L-coloring of $D - v_i$. But now v_i has two in-neighbors colored a, so we can complete the coloring to a coloring of D, a contradiction. So we may assume that $f(v_2) = a$ and $f(v_k) = b$, $a \neq b$. Now, the out-neighbor of v_1 that has color b must be v_k for otherwise doing a shift of colors we would get a coloring of $D - v_1$ with two out-neighbors colored b. So, $v_1 v_k \in A(D)$. By a similar argument, $v_2 v_1 \in A(D)$. Now, consider the vertex v_2 and a coloring of $D - v_2$. Since the edges $v_1v_2, v_2v_1, v_1v_k, v_kv_1$ exist, we can change C to a non-directed cycle C' in which v_2 has an in-neighbor and an out-neighbor. As above, we either get a bidirected clique or both arcs v_2v_3 and v_3v_2. Repeating this argument, we deduce that $V(C)$ induces a bidirected cycle or a bidirected clique.

Next, suppose k is even. We may assume that v_1's neighbors on C, v_2 and v_k, are both in-neighbors. We claim that there is a chord of C incident to v_1 and directed inwards (i.e., v_1 has another in-neighbor on C). Suppose not. Consider a coloring of $D - v_1$ and let $f(v_2) = a$ and $f(v_k) = b$. Now if we do a shift of colors around C we deduce that $f(v_3) = f(v_5) = f(v_7) = \cdots = f(v_{k-1}) = b$. But this is impossible since after performing a shift of colors in the opposite direction, we will obtain a valid coloring of $D - v_1$ with v_k and v_2 both colored b. Therefore, there is an arc $v_iv_1 \in A(D)$. If this arc divides C into two even cycles, then by an inductive argument similar to the case when k is odd we can deduce that C is a complete bidirected digraph. Therefore, assume that i is odd so that v_iv_1 splits the cycle C into two odd cycles $C_1 = v_1v_2...v_iv_1$ and $C_2 = v_1v_iv_{i+1}...v_kv_1$. By induction, we have that all the edges of C are actually bidirected arcs. Also, we know that $v_iv_1, v_1v_i \in A(D)$. Next, we show that there must be further chords incident to v_1 in addition to those coming from v_i. Suppose not. Consider a coloring g of $D - v_1$, and suppose $g(v_2) = a$, $g(v_k) = b$ and $g(v_i) = c$. Now, if we do shift of colors around C_1, we conclude that $g(v_2) = g(v_4) = \cdots = g(v_{i-1}) = a$ and $g(v_3) = g(v_5)... = g(v_i) = c$. Similarly, doing shift of colors around C_2 we conclude that $g(v_i) = g(v_{i+2}) = g(v_{k-1}) = c$ and $g(v_{i+1}) = g(v_{i+3}) = \cdots = g(v_k) = b$. Since $k \geq 6$, if we now do two shifts of colors around C, we will get a coloring of $D - v_1$ where there is the same color appearing twice in the neighborhood of v_1, contradicting Lemma 3.2.2. Therefore, there are other chords incident to v_1 beside the ones coming from v_i. This implies that one of the cycles C_1 or C_2 is divided into an even cycle and an odd cycle and we are done by a similar argument as in the case when k is odd. $\square$

Now, we can prove the main result of this section.

Proof of Theorem 3.2.1. By Lemma 3.2.3, D is Eulerian and L-critical. First, we prove the first claim of the theorem. Let H be a block of D, for which none of (a)-(c) applies. Note that H cannot be a single arc by L-criticality. The theorem is clear if $|V(H)| \leq 3$. Note that H cannot be a non-directed cycle or a cycle with some but not all edges bidirected, since every such cycle induces new arcs by Lemma 3.2.5. So we may assume that $|V(H)| \geq 4$ and that H (as an undirected graph) is not a cycle. Then there are two vertices in H with three internally vertex-disjoint paths between them, say P_1, P_2, P_3. Two of these paths, say P_1 and P_2, create a cycle C of even length. We claim that the cycle C induces a complete bidirected graph. Suppose not. Then C is a directed cycle by Lemma 3.2.5. This implies that at least one of the cycles $P_1 \cup P_3$ or $P_1 \cup P_2$ is not directed. By applying Lemma 3.2.5 again, this new cycle induces at least a bidirected cycle and therefore some of the arcs of C are bidirected. But this is a contradiction, which shows that C induces a complete bidirected digraph.

Let v be any vertex of H that is not on C. Since H is a block, there are two paths P and Q from v to C whose only common vertex is v. Now, simply take an even cycle C' that contains the path $P \cup Q$ and one or two additional arcs of C. We may choose the arcs of C' so that C' is a non-directed cycle. Now, Lemma 3.2.5 shows that C' induces a complete bidirected digraph. By using different vertices of C when making C' (by possibly including more than two arcs of C), we conclude that every vertex of $P \cup Q$ is adjacent to each other and to every vertex on C. Therefore, if we take any maximal bidirected clique K in H we conclude that all the vertices of H are on K. Hence, H is a complete bidirected digraph.

It remains to prove the last part of the theorem. Let us consider a block B of D. Note that B satisfies one of (a)–(c). If $B = D$, then it is easy to see that the only list assignment L, for which D is not L-colorable, has all lists $L(v)$, $v \in V(D)$, equal to each other. So, we may assume that $B \neq D$. Next, we L-color $D' = D - V(B)$. After this, each vertex $v \in V(B)$ is left with at least $d_B^+(v)$ colors that do not appear on $N(v)$. Let $L'(v) \subseteq L(v)$ denote these colors. Now, every L'-coloring of B gives rise to an L-coloring of D, so B is not L'-colorable. But since $|L'(v)| \geq d_B^+(v)$ for all $v \in V(B)$, we conclude, by the same arguments as above, that $|L'(v)| = d_B^+(v)$ for each $v \in V(B)$ and that all lists $L'(v)$ are the same. By denoting this common color set by C_B, we obtain the last part of the theorem. Since $|L(v)| = d^+(v)$, it is easy to see that the color sets C_B of all blocks B containing v are pairwise disjoint. $\square$

Note that the condition $|L(v)| \geq \max\{d^+(v), d^-(v)\}$ in Theorem 3.2.1 cannot be strengthened to, say, $|L(v)| \geq d^+(v)$, since we could take any digraph which has a vertex with no out-neighbors and an empty list of colors. However, this becomes possible if we know that the digraph is L-critical.

Corollary 3.2.6. *Let D be a connected digraph and L an assignment of colors to the vertices of D such that $|L(v)| \geq d^+(v)$, for every $v \in V(D)$. Suppose that D is L-critical. Then D is Eulerian, and hence the conclusions of Theorem 3.2.1 hold.*

Proof. If D is not Eulerian, then there exists a vertex $v \in V$ with $d^+(v) > d^-(v)$. Consider an L-coloring of $D - v$. Now, since $|L(v)| \geq d^+(v) > d^-(v)$, there is a color $c \in L(v)$ that does not appear on the in-neighborhood of v. Coloring v with color c gives an L-coloring of D, a contradiction. □

The next corollary obtains a similar result when the criticality condition is dropped, but we insist that vertices whose out-degree is larger than their in-degree have an extra admissible color.

Corollary 3.2.7. *Let D be a connected digraph, and L an assignment of colors to the vertices of D such that $|L(v)| \geq d^-(v)$ if $d^+(v) \leq d^-(v)$ and $|L(v)| \geq d^-(v) + 1$ otherwise. Suppose that D is not L-colorable. Then D is Eulerian, and hence the conclusions of Theorem 3.2.1 hold.*

Proof. We use induction on $|A(D)|$. If $|A(D)| \leq 3$ and D is not Eulerian, then D is L-colorable for any choice of L. So, we may assume from now on that $|A(D)| \geq 4$.

We first show that D is L-critical. Let $e = uv$ be an arc of D and suppose for a contradiction that $D - uv$ is not L-colorable. Consider a component C of $D - uv$ that is not L-colorable. By the induction hypothesis, we have that C is Eulerian and that conclusions of Theorem 3.2.1 hold. If $u \in V(C)$ (say), then u is not an Eulerian vertex in D, so $|L(u)| > d_C^+(u)$, which contradicts the conclusions of Theorem 3.2.1 for C.

Now, suppose that D is not Eulerian. Since $\sum_v d^+(v) = \sum_v d^-(v) = |A(D)|$, there exists a vertex v such that $d^+(v) > d^-(v)$. Then $|L(v)| \geq d^-(v) + 1$. Remove an arc e incident to v from D, and choose an L-coloring of $D - e$. Now, putting the edge e back, we see that we still have a color in $L(v)$ not appearing on the in-neighborhood of v, allowing us to complete the coloring to an L-coloring of D, a contradiction. □

The reader may wonder why require an additional color for non-Eulerian vertices. As we shall see in the next section, the situation changes drastically if this were not the case.

3.3 Complexity of list coloring of digraphs with Brooks' condition

It is natural to ask whether the condition of Corollary 3.2.7 can be relaxed to $|L(v)| \geq \min\{d^+(v), d^-(v)\}$. It turns out that the answer is negative even if the digraph is L-critical. There is an example on four vertices; see Figure 3.2, where the numbers at the vertices indicate the corresponding lists of colors. Further examples of digraphs that are L-critical with $|L(v)| \geq \min\{d^+(v), d^-(v)\}$ for every $v \in V(D)$, and yet do not admit a block decomposition described by Theorem 3.2.1, are not hard to construct, as shown by Figure 3.3. One can extend the construction in Figure 3.3 to get counterexamples of any order by subdividing any of the arcs.

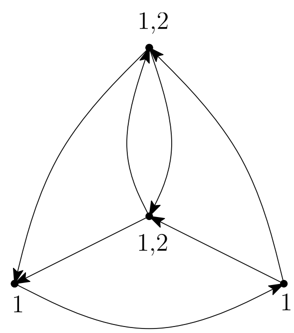


Figure 3.2: An L-critical digraph with $|L(v)| \geq \min\{d^+(v), d^-(v)\}$ that is not Eulerian

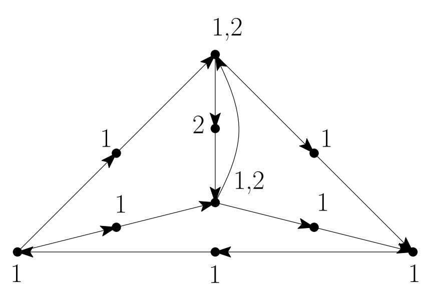


Figure 3.3: Constructing an L-critical digraph with $|L(v)| \geq \min\{d^+(v), d^-(v)\}$ of arbitrary order that is not Eulerian

Not only are there many such examples, it turns out that the list coloring problem restricted to such a class of instances is NP-hard. This (surprising) fact and its proof is the subject of the remainder of this section.

Computational complexity of digraph colorings has been studied by several authors. We have the following complexity theorem for digraphs proven in Bokal et al. [8].

Theorem 3.3.1 ([8])**.** *Let D be a digraph. It is NP-complete to decide whether $\chi(D) \leq 2$.*

Stronger results were obtained by Feder, Hell and Mohar [28]. These results will be discussed in a later chapter.

We study the following problem.

> PROBLEM: LIST COLORING WITH BROOKS' CONDITION
> INSTANCE: A digraph D, a list-assignment L such that for every vertex $v \in V(D)$, $|L(v)| = \min\{d^+(v), d^-(v)\}$.
> QUESTION: Is the digraph D L-colorable?

If we restrict the instances to planar graphs, we get the PLANAR LIST COLORING PROBLEM WITH BROOKS' CONDITION.

Theorem 3.3.2. *The* PLANAR LIST COLORING PROBLEM WITH BROOKS' CONDITION *is NP-complete.*

For a polynomial time reduction, we shall use the following problem, which was proved to be NP-complete in [29].

> PROBLEM: PLANAR $(\leq 3, 3)$-SATISFIABILITY
> INSTANCE: A formula Φ in conjunctive normal form with a set C of clauses over a set X of boolean variables such that
>
> (1) each clause involves at most three distinct variables,
>
> (2) every variable occurs in exactly three clauses, once positive and twice negative, and
>
> (3) the graph $G_\Phi = (X \cup C, \{xc \mid x \in X, x \in c \in C \text{ or } \neg x \in c \in C\})$ is planar.
> QUESTION: Is Φ satisfiable?

Proof. Clearly, every list coloring problem is in NP since after guessing an L-coloring, one can check in polynomial time whether each color class induces an acyclic subdigraph using Breadth-First-Search.

Let the formula Φ be an instance of PLANAR $(\leq 3, 3)$-SATISFIABILITY. Note that $G = G_\Phi$ is a bipartite graph with bipartition $\{X, C\}$. We create an instance of list coloring for digraphs as follows.

- Direct all the edges of G from X to C.
- For each $x \in X$, we create a new vertex x' and add the arcs $x'x$ and c_1x', c_2x', where c_1, c_2 are the two clauses that contain $\neg x$.
- Add the arc c_3x, where c_3 is the clause containing the literal x.
- For every variable $x \in X$, we define two colors, x and $\bar{x}$. For each $x \in X$, set $L(x) = \{x, \bar{x}\}$. For each $c \in C$, we set $L(c) = \{\bar{x} \mid x \in c\} \cup \{x \mid \neg x \in c\}$. Finally, let $L(x') = \{x\}$ for every x'.

Let D be the resulting digraph. We first claim that D is planar. Note that the graph G_Φ is assumed to be planar. Clearly, adding the arcs c_3x, where c_3 is the clause containing the literal x, preserve the planarity. All that remains to show is that the vertices x' and their incident arcs can be added in a way as to preserve the planarity. But this is clear because we can add the vertices x' one by one in the face defined by the vertices x, c_1 and c_2, where c_1 and c_2 are the clauses containing $\neg x$.

Next, we consider the sizes of the lists. Clearly, every $x \in X$ has out-degree 3 and in-degree 2 because x appears in three clauses, twice negative and once positive. Therefore, $|L(x)| = \min\{d^+(v), d^-(v)\}$. For a given clause $c \in C$, for every arc xc we have exactly one of the two arcs cx or cx'. Therefore, $d^+(c) = d^-(c) = |L(c)|$. Now, every x' has in-degree 2 and out-degree 1, which implies that $|L(x')| = \min\{d^+(x'), d^-(x')\}$. Therefore, all the list sizes match with minimum degree. Now, we claim that Φ is satisfiable if and only if D is L-colorable.

Suppose first that f is an L-coloring of D. Define a truth assignment ϕ as follows: $\phi(x) = \texttt{true}$ if $f(x) = x$ and $\phi(x) = \texttt{false}$ if $f(x) = \bar{x}$. We need to show that every clause c is satisfied. If $f(c) = x$ for some variable x, then $\neg x \in c$. Also, $f(x) \neq x$ for otherwise we would have a monochromatic triangle $cx'x$ of color x. Therefore, $f(x) = \bar{x}$, thus $\phi(x) = \texttt{false}$, and hence c is satisfied. Similarly, if $f(c) = \bar{x}$, then $x \in c$. Further, $f(x) = x$ for otherwise we would have a monochromatic digon. Therefore, $\phi(x) = \texttt{true}$ and c is satisfied.

Conversely, let ϕ be a satisfying truth assignment. Define the following L-coloring f: $f(x) = x$ if $\phi(x) = \texttt{true}$, and $f(x) = \bar{x}$ if $\phi(x) = \texttt{false}$. For each clause c, choose a variable x which satisfies c and set $f(c) = x$ if $\neg x \in c$, and $f(c) = \bar{x}$, if $x \in c$. Clearly, $f(x') = x$ for all x'. To see that f is a coloring, consider an arc xc. We claim that $f(x) \neq f(c)$. Suppose $f(x) = x$ (the other case is similar) and that $\neg x \in c$. Since $f(x) = x$, $\phi(x) = \texttt{true}$ which implies that $\neg x = \texttt{false}$. Therefore, $f(c) \neq x$. Thus, no arc from X to C is monochromatic, so f is a coloring. This completes the proof. $\square$

We note that the above proof implies the following immediate corollary.

Corollary 3.3.3. *List coloring of digraphs is NP-complete even if restricted to planar digraphs where each vertex v has $d_0(v) = \min\{d^+(v), d^-(v)\} \leq 3$ and the list size for v is equal to $d_0(v)$.*

Proof. Note that all the vertices v of the digraph D in the above proof satisfy the conditions $d_0(v) \leq 3$ and $d_0(v) = |L(v)|$. $\square$

Next, we consider the problem where the list sizes of vertices with $d^+(v) > d^-(v)$ have an additional color.

> PROBLEM: LIST COLORING WITH RELAXED BROOKS' CONDITION
> INSTANCE: A digraph D, a list-assignment L such that for every vertex $v \in V(D)$ with $d^+(v) \leq d^-(v)$, $|L(v)| \geq d^+(v)$, and for every vertex v with $d^+(v) > d^-(v)$, we have $|L(v)| \geq d^-(v) + 1$.
> QUESTION: Is the digraph D L-colorable?

Theorem 3.3.4. *The problem* LIST COLORING WITH RELAXED BROOKS' CONDITION *can be solved in linear time $O(|V(D)| + |A(D)|)$.*

Proof. Note that it is sufficient to provide an algorithm for connected digraphs because we can then apply it to all the components. We first give an algorithm for the Eulerian instances of D, and then show that the general case can be reduced to the Eulerian case.

So suppose D is Eulerian. We will apply Theorem 3.2.1. If there exists a vertex $v \in V(D)$ such that $|L(v)| > d^+(v)$, then D is L-colorable by Theorem 3.2.1. So we may assume that

$|L(v)| = d^+(v)$ for all $v \in V(D)$. We first find the blocks of D; this can be done in time $O(|V(D)| + |A(D)|)$ using Depth-First-Search, see for example [16]. By Theorem 3.2.1, if there exists a block of D that is not of type (a)–(c), then D is L-colorable. So we may assume that all blocks of D are of type (a), (b) or (c). Let B be a leaf block in the block-cutpoint tree of D. If $B = D$, then as mentioned in the proof of Theorem 3.2.1, D is not L-colorable if and only if all the lists of D are the same. This can be checked in linear time. Otherwise, let $v \in V(B)$ be the single cut-vertex in B. If there are two vertices in $u, w \in V(B)\backslash\{v\}$ with $L(u) \neq L(w)$ or there exists a vertex $x \in V(B)\backslash\{v\}$ such that $L(x) \nsubseteq L(v)$, then D is L-colorable by Theorem 3.2.1. Therefore, we may assume that for all $u, w \in V(B)\backslash\{v\}$, $L(u) = L(w)$ and $L(u) \subseteq L(v)$. In this case, it is easy to see that D is L-colorable if and only if $D - (V(B)\backslash\{v\})$ is L'-colorable, where $L'(v) = L(v)\backslash L(u)$, for some $u \in V(B)\backslash\{v\}$, and $L'(x) = L(x)$ for all $x \in V(D)\backslash V(B)$. Thus, we can reduce the problem by deleting a leaf block B at each step by using at most $O(|V(B)| + |A(B)|)$ time, which results in a $O(|V(D)| + |A(D)|)$ overall time.

Next, suppose that D is not Eulerian. We give a linear time reduction to the Eulerian case. Since $\sum_v d^+(v) = \sum_v d^-(v) = |A(D)|$, there exists a vertex u such that $d^+(u) > d^-(u)$. Consider $D - u$. We claim that D is L-colorable if and only if $D - u$ is L-colorable. Clearly, if D is L-colorable then $D - u$ is L-colorable. Now, suppose $D - u$ is L-colorable, and let f be such a coloring. Since $d^+(u) > d^-(u)$, we have that there is a color in $L(u)$ that does not appear in the in-neighborhood of u. By using such a color, we can complete the coloring of $D - u$ to an L-coloring of D.

Repeating this reduction we will obtain a (possibly empty) digraph D^* such that $d^+_{D^*}(v) = d^-_{D^*}(v)$ for every $v \in V(D^*)$. Since $d^+(v) \geq d^+_{D^*}(v)$, it follows that $|L(v)| \geq d^+_{D^*}(v) = d^-_{D^*}(v)$. Now, using the algorithm for the Eulerian case, we can decide whether each component of D^* is L-colorable. Then clearly D is L-colorable if and only if each component of D^* is L-colorable.

To keep the list of vertices v with $d^+(v) > d^-(v)$, and updating this list after every vertex-removal takes overall linear time. We only need to consider at most $\min\{d^+(v), d^-(v)\} + 1$ colors at v, so when comparing the lists in the blocks we only need $O(|V(D)| + |A(D)|)$ time. Thus, it takes $O(|V(D) + |A(D)|)$ time to reduce D to the Eulerian digraph D^*. Since we need linear time to decide whether an Eulerian digraph is L-colorable, we have an $O(|V(D)| + |A(D)|)$ algorithm. $\square$

Chapter 4

Brooks Theorem for Digraphs of Girth Three

4.1 Introduction

Brooks' Theorem states that if G is a connected graph with maximum degree Δ, then $\chi(G) \leq \Delta + 1$, where equality is attained only for odd cycles and complete graphs. The presence of triangles has significant influence on the chromatic number of a graph. A result of Johansson [38] states that if G is triangle-free, then $\chi(G) = O\left(\Delta / \log \Delta\right)$. In this chapter, we show that Brooks' Theorem for digraphs can also be improved when we forbid directed cycles of length 2.

Digraph colorings and the Brooks Theorem

Recall that for digraphs, a version of Brooks' theorem was proved in [46]. Here, a digraph D is *k-critical* if $\chi(D) = k$, and $\chi(H) < k$ for every proper subdigraph H of D.

Theorem 4.1.1 ([46]). *Suppose that D is a k-critical digraph in which for every vertex $v \in V(D)$, $d^+(v) = d^-(v) = k - 1$. Then one of the following cases occurs:*

1. *$k = 2$ and D is a directed cycle of length $n \geq 2$.*

2. *$k = 3$ and D is a bidirected cycle of odd length $n \geq 3$.*

3. *D is bidirected complete graph of order $k \geq 4$.*

A tight upper bound on the chromatic number of a digraph was first given by Neumann-Lara [49].

Theorem 4.1.2 ([49]). *Let D be a digraph and denote by Δ_o and Δ_i the maximum out-degree and in-degree of D, respectively. Then*

$$\chi(D) \leq \min\{\Delta_o, \Delta_i\} + 1.$$

In this chapter, we study improvements of this result using the following substitute for the maximum degree. If D is a digraph, we let

$$\tilde{\Delta} = \tilde{\Delta}(D) = \max\{\sqrt{d^+(v)d^-(v)} \mid v \in V(D)\}$$

be the maximum geometric mean of the in-degree and out-degree of the vertices. Observe that $\tilde{\Delta} \leq \frac{1}{2}(\Delta_o + \Delta_i)$, by the arithmetic-geometric mean inequality (where Δ_o and Δ_i are as in Theorem 4.1.2). We show that when $\tilde{\Delta}$ is large (roughly $\tilde{\Delta} \geq 10^{10}$), then every digraph D without digons has $\chi(D) \leq \alpha\tilde{\Delta}$, for some absolute constant $\alpha < 1$. We do not make an attempt to optimize α, but show that $\alpha = 1 - e^{-13}$ suffices. To improve the value of α significantly, a new approach may be required.

It may be true that the following analog of Johansson's result holds for digon-free digraphs, as conjectured by McDiarmid and Mohar [44].

Conjecture 4.1.3 ([44]). *Every digraph D without digons has $\chi(D) = O(\frac{\tilde{\Delta}}{\log \tilde{\Delta}})$.*

If true, this result would be asymptotically best possible in view of the chromatic number of random tournaments of order n, whose chromatic number is $\Omega(\frac{n}{\log n})$ and $\tilde{\Delta} > \left(\frac{1}{2} - o(1)\right) n$, as shown by Erdős et al. [24].

We also believe that the following conjecture of Reed generalizes to digraphs without digons.

Conjecture 4.1.4 ([56]). *Let Δ be the maximum degree of (an undirected) graph G, and let ω be the size of the largest clique. Then*

$$\chi(G) \leq \left\lceil \frac{\Delta + 1 + \omega}{2} \right\rceil.$$

If we define $\omega = 1$ for digraphs without digons, we can pose the following conjecture for digraphs. Note that a digraph is Δ-*regular* if $d^+(v) = d^-(v) = \Delta$ for every vertex v.

Conjecture 4.1.5. *Let D be a Δ-regular digraph without digons. Then*

$$\chi(D) \leq \left\lceil \frac{\Delta}{2} \right\rceil + 1.$$

Conjecture 4.1.5 is trivial for $\Delta = 1$, and follows from Lemma 4.3.2 for $\Delta = 2, 3$. We believe that the conjecture is also true for non-regular digraphs with Δ replaced by $\tilde{\Delta}$.

The rest of the chapter is organized as follows. In Section 4.2, we improve Brooks' bound for digraphs that have sufficiently large degrees. In Section 4.3, we consider the problem for arbitrary degrees.

4.2 Strengthening Brooks' Theorem for large $\tilde{\Delta}$

The main result in this section is the following theorem.

Theorem 4.2.1. *There is an absolute constant Δ_1 such that every digon-free digraph D with $\tilde{\Delta} = \tilde{\Delta}(D) \geq \Delta_1$ has $\chi(D) \leq \left(1 - e^{-13}\right) \tilde{\Delta}$.*

The rest of this section is the proof of Theorem 4.2.1. The proof is a modification of an argument found in Molloy and Reed [47] for usual coloring of undirected graphs. We first note the following simple lemma.

Lemma 4.2.2. *Let D be a digraph with maximum out-degree Δ_o, and suppose we have a partial proper coloring of D with at most $\Delta_o + 1 - r$ colors. Suppose that for every uncolored vertex v there are at least r colors that appear on vertices in $N^+(v)$ at least twice. Then D is $\Delta_o + 1 - r$-colorable.*

Proof. The proof is easy – since many colors are repeated on the out-neighborhood of v, there are many colors that are not used on $N^+(v)$. In particular, there are at most $\Delta - r$ distinct colors appearing on $N^+(v)$. Thus, one can "greedily" extend the partial coloring. □

Proof of Theorem 4.2.1. We may assume that $c_1\tilde{\Delta} < d^+(v) < c_2\tilde{\Delta}$ and $c_1\tilde{\Delta} < d^-(v) < c_2\tilde{\Delta}$ for each $v \in V(D)$, where $c_1 = 1 - \frac{1}{3}e^{-11}$ and $c_2 = 1 + \frac{1}{3}e^{-11}$. If not, we remove all the vertices v not satisfying the above inequality and obtain a coloring for the remaining graph with $\left(1 - e^{-13}\right) \tilde{\Delta}$ colors. Now, if a vertex does not satisfy the above condition either one of $d^+(v)$ or $d^-(v)$ is at most $c_1\tilde{\Delta}$ or one of $d^+(v)$ or $d^-(v)$ is at most $\frac{1}{c_2}\tilde{\Delta}$. Note that $1 - e^{-13} > \max\{c_1, 1/c_2\}$. This ensures that there is a color that either does not appear in the in-neighborhood or does not appear in the out-neighborhood of v, allowing us to complete the coloring.

The core of the proof is probabilistic, and we refer the reader to Appendix A for all the probabilistic tools used in the sequel. We color the vertices of D randomly with C colors, $C = \lfloor \tilde{\Delta}/2 \rfloor$. That is, for each vertex v we assign v a color from $\{1, 2, ..., C\}$ uniformly at random. After the random coloring, we uncolor all the vertices that are in a monochromatic directed path of length at least 2. Clearly, this results in a proper partial coloring of D since D has no digons. For each vertex v, we are interested in the number of colors which are assigned to at least two out-neighbors of v and are retained by at least two of these vertices. For analysis, it is better to define a slightly simpler random variable. Let $v \in V(D)$. For each color i, $1 \le i \le C$, let O_i be the set of out-neighbors of v that have color i assigned to them in the first phase. Let X_v be the number of colors i for which $|O_i| \ge 2$ and such that all vertices in O_i retain their color after the uncoloring process.

For every vertex v, we let A_v be the event that X_v is less than $\frac{1}{2}e^{-11}\tilde{\Delta} + 1$. We will show that with positive probability none of the events A_v occur. Then Lemma 4.2.2 will imply that $\chi(D) \le (c_2 - \frac{1}{2}e^{-11})\tilde{\Delta} \le (1 - e^{-13})\tilde{\Delta}$, finishing the proof. We will use the symmetric version of the Lovász Local Lemma (see Appendix, Theorem A.3.2). Note that the color assigned initially to a vertex u can affect X_v only if u and v are joined by a path of length at most 3. Thus, A_v is mutually independent of all except at most $(2c_2\tilde{\Delta}) + (2c_2\tilde{\Delta})^2 + (2c_2\tilde{\Delta})^3 + (2c_2\tilde{\Delta})^4 + (2c_2\tilde{\Delta})^5 + (2c_2\tilde{\Delta})^6 \le 100\tilde{\Delta}^6$ other events A_w. Therefore, by the symmetric version of the Local Lemma, it suffices to show that for each event A_v, $4 \cdot 100\tilde{\Delta}^6\mathbb{P}[A_v] < 1$. We will show that $\mathbb{P}[A_v] < \tilde{\Delta}^{-7}$. We do this by proving the following two lemmas.

Lemma 4.2.3. $\mathbb{E}[X_v] \ge e^{-11}\tilde{\Delta} - 1$.

Proof. Let X'_v be the random variable denoting the number of colors that are assigned to exactly two out-neighbors of v and are retained by both of these vertices. Clearly, $X_v \ge X'_v$ and therefore it suffices to consider $\mathbb{E}[X'_v]$.

Note that color i will be counted by X'_v if two vertices $u, w \in N^+(v)$ are colored i and no other vertex in $S = N(u) \cup N^+(v) \cup N(w)$ is assigned color i. This will give us a lower bound on $\mathbb{E}[X'_v]$. There are C choices for color i and at least $\binom{c_1\tilde{\Delta}}{2}$ choices for the set $\{u, w\}$. The probability that no vertex in S gets color i is at least $(1 - \frac{1}{C})^{|S|} \ge (1 - \frac{1}{C})^{5c_2\tilde{\Delta}}$. Therefore, by linearity of expectation, and using the inequality $(1-p)^n > e^{-pn-p}$, we can estimate:

$$\begin{aligned}\mathbb{E}[X'_v] &\geq C\binom{c_1\tilde{\Delta}}{2}\left(\frac{1}{C}\right)^2\left(1-\frac{1}{C}\right)^{5c_2\tilde{\Delta}} \\ &\geq c_1(c_1\tilde{\Delta}-1)\exp(-5c_2\tilde{\Delta}/C-1/C) \\ &\geq \frac{\tilde{\Delta}}{e^{11}}-1\end{aligned}$$

for $\tilde{\Delta}$ sufficiently large. ☐

Lemma 4.2.4. $\mathbb{P}\left[|X_v - \mathbb{E}[X_v]| > \log\tilde{\Delta}\sqrt{\mathbb{E}[X_v]}\right] < \tilde{\Delta}^{-7}$.

Proof. Let AT_v be the random variable counting the number of colors assigned to at least two out-neighbors of v, and Del_v the random variable that counts the number of colors assigned to at least two out-neighbors of v but removed from at least one of them. Clearly, $X_v = AT_v - Del_v$ and therefore it suffices to show that each of AT_v and Del_v are sufficiently concentrated around their means. We will show that for $t = \frac{1}{2}(\log\tilde{\Delta})\sqrt{\mathbb{E}[X_v]}$ the following estimates hold:

Claim 1: $\mathbb{P}\left[|AT_v - \mathbb{E}[AT_v]| > t\right] < 2e^{-t^2/(8\tilde{\Delta})}$.

Claim 2: $\mathbb{P}\left[|Del_v - \mathbb{E}[Del_v]| > t\right] < 4e^{-t^2/(100\tilde{\Delta})}$.

The two above inequalities yield that, for $\tilde{\Delta}$ sufficiently large,

$$\begin{aligned}\mathbb{P}[|X_v - \mathbb{E}[X_v]| > \log\tilde{\Delta}\sqrt{\mathbb{E}[X_v]}] &\leq 2e^{-\frac{t^2}{8\tilde{\Delta}}} + 4e^{-\frac{t^2}{100\tilde{\Delta}}} \\ &\leq \tilde{\Delta}^{-\log\tilde{\Delta}} \\ &< \tilde{\Delta}^{-7},\end{aligned}$$

as we require. So, it remains to establish both claims.

To prove Claim 1, we use a version of Azuma's inequality found in [47], called the Simple Concentration Bound (see Appendix A, Theorem A.4.2).

Theorem 4.2.5 (Simple Concentration Bound)**.** *Let X be a random variable determined by n independent trials $T_1, ..., T_n$, and satisfying the property that changing the outcome of any single trial can affect X by at most c. Then*

$$\mathbb{P}[|X - \mathbb{E}[X]| > t] \leq 2e^{-\frac{t^2}{2c^2 n}}.$$

Note that AT_v depends only on the colors assigned to the out-neighbors of v. Note that each random choice can affect AT_v by at most 1. Therefore, we can take $c = 1$ in the Simple Concentration Bound for $X = AT_v$. Since the choice of random color assignments are made independently over the vertices and since $d^+(v) \leq c_2\tilde{\Delta}$, we immediately have the Claim 1.

For Claim 2, we use the following variant of Talagrand's Inequality (see Appendix A, Theorem A.4.4).

Theorem 4.2.6 (Talagrand's Inequality)**.** *Let X be a nonnegative random variable, not equal to 0, which is determined by n independent trials, $T_1, \dots, T_n$ and satisfyies the following conditions for some $c, r > 0$:*

1. *Changing the outcome of any single trial can affect X by at most c.*

2. *For any s, if $X \geq s$, there are at most rs trials whose exposure certifies that $X \geq s$.*

Then for any $0 \leq \lambda \leq \mathbb{E}[X]$,

$$\mathbb{P}\left[|X - \mathbb{E}[X]| > \lambda + 60c\sqrt{r\mathbb{E}[X]}\right] \leq 4e^{-\frac{\lambda^2}{8c^2 r\mathbb{E}[X]}}.$$

We apply Talagrand's inequality to the random variable Del_v. Note that we can take $c = 1$ since any single random color assignment can affect Del_v by at most 1. Now, suppose that $Del_v \geq s$. One can certify that $Del_v \geq s$ by exposing, for each of the s colors i, two random color assignments in $N^+(v)$ that certify that at least two vertices got color i, and exposing at most two other color assignments which show that at least one vertex colored i lost its color. Therefore, $Del_v \geq s$ can be certified by exposing $4s$ random choices, and hence we may take $r = 4$ in Talagrand's inequality. Note that $t = \frac{1}{2}\log\tilde{\Delta}\sqrt{\mathbb{E}[X_v]} \gg 60c\sqrt{r\mathbb{E}[Del_v]}$ since $\mathbb{E}[X_v] \geq \tilde{\Delta}/e^{11} - 1$ and $\mathbb{E}[Del_v] \leq c_2\tilde{\Delta}$. Now, taking λ in Talagrand's inequality to be $\lambda = \frac{1}{2}t$, we obtain that $\mathbb{P}[|Del_v - \mathbb{E}[Del_v]| > t] \leq \mathbb{P}[|Del_v - \mathbb{E}[Del_v]| > \lambda + 60c\sqrt{r\mathbb{E}[X]}]$. Therefore, provided that $\lambda \leq \mathbb{E}[Del_v]$, we have the confirmed Claim 2.

It is sufficient to show that $\mathbb{E}[Del_v] = \Omega(\tilde{\Delta})$, since $\lambda = O(\log\tilde{\Delta}\sqrt{\tilde{\Delta}})$. The probability that *exactly* two vertices in $N^+(v)$ are assigned a particular color c is at least $\frac{c_1\tilde{\Delta}^2}{2}C^{-2}(1 - 1/C)^{c_2\tilde{\Delta}} \approx 2e^{-10}$, a constant. It remains to show that the probability that at least one of these vertices loses its color is also (at least) a constant. We use Janson's Inequality (see Appendix, Theorem A.2.1). Let u be one of the two vertices colored c. We only compute the probability that u gets uncolored. We may assume that the other vertex colored c is not a neighbor of u since this will only increase the probability. We show that with large probability there exists a monochromatic directed path of length at least 2 starting at u. Let $\Omega = N^+(u) \cup N^{++}(u)$, where $N^{++}(u)$ is the second out-neighborhood of u. Each vertex in Ω is colored c with probability $\frac{2}{\tilde{\Delta}}$. Enumerate all the directed paths of length 2 starting at u and let P_i be the i^{th} path. Clearly, there are at least $(c_1\tilde{\Delta})^2$ such paths P_i. Let A_i be the set of vertices of P_i, and denote by B_i the event that all vertices in A_i receive the same color. Then, clearly $\mathbb{P}[B_i] = \frac{1}{(\lfloor \tilde{\Delta}/2 \rfloor)^2} \geq \frac{4}{\tilde{\Delta}^2}$. Then, $\mu = \sum \mathbb{P}[B_i] \geq \frac{4}{\tilde{\Delta}^2} \cdot (c_1\tilde{\Delta})^2 = 4c_1^2$. Now, if $\delta = \sum_{i,j:A_i \cap A_j \neq \emptyset} \mathbb{P}[B_i \cap B_j]$ in Janson's Inequality satisfies $\delta < \mu$, then applying Janson's Inequality, with the sets A_i and events B_i, we obtain that the probability that none of the events B_i occur is at most e^{-1}, and hence the probability that u does not retain its color is at least $1 - e^{-1}$, as required. Now, assume that $\delta \geq \mu$. The following gives an upper bound on δ:

$$\begin{aligned} \delta &= \sum_{i,j:A_i \cap A_j \neq \emptyset} \mathbb{P}[B_i \cap B_j] = \sum_{i,j:A_i \cap A_j \neq \emptyset} \frac{1}{(\lfloor \tilde{\Delta}/2 \rfloor)^3} \\ &\leq (c_2\tilde{\Delta})^2 \cdot 2c_2\tilde{\Delta} \cdot \frac{8}{(\tilde{\Delta}-2)^3} < 32, \end{aligned}$$

for $\tilde{\Delta} \geq 100$. Now, we apply Extended Janson's Inequality (see Appendix, Theorem A.2.2). This inequality now implies that the probability that none of the events B_i occur is at most $e^{-c_1^2/4}$, a constant. Therefore, by linearity of expectation $\mathbb{E}[Del_v] = \Omega(\tilde{\Delta})$. □

Clearly, since $\mathbb{E}[X_v] \leq c_2\tilde{\Delta}$, Lemmas 4.2.3 and 4.2.4 imply that $\mathbb{P}[A_v] < \tilde{\Delta}^{-7}$. This completes the proof of Theorem 4.2.1. □

4.3 Brooks' Theorem for small $\tilde{\Delta}$

The bound in Theorem 4.2.1 is only useful for large $\tilde{\Delta}$. Rough estimates suggest that $\tilde{\Delta}$ needs to be at least in the order of 10^{10}. The above approach is unlikely to improve this bound significantly with a more detailed analysis. In this section, we improve Brooks' Theorem for all values of $\tilde{\Delta}$. We achieve this by using the result on list colorings found in Chapter 3.

Theorem 4.3.1 ([33]). *Let D be a connected digraph, and L an assignment of colors to the vertices of D such that $|L(v)| \geq d^+(v)$ if $d^+(v) = d^-(v)$ and $|L(v)| \geq \min\{d^+(v), d^-(v)\} + 1$ otherwise. Suppose that D is not L-colorable. Then D is Eulerian, $|L(v)| = d^+(v)$ for each $v \in V(D)$, and every block of D is one of the following:*

(a) *a directed cycle (possibly a digon),*

(b) *an odd bidirected cycle, or*

(c) *a bidirected complete digraph.*

D is said to be *k-choosable* if D is L-colorable for every list-assignment L with $|L(v)| \geq k$ for each $v \in V(D)$. We denote by $\chi_l(D)$ the smallest integer k for which D is k-choosable. Now, we can state the next result of this section.

Lemma 4.3.2. *Let D be a connected digraph without digons, and let $\tilde{\Delta} = \tilde{\Delta}(D)$. If $\tilde{\Delta} > 1$, then $\chi_l(D) \leq \lceil \tilde{\Delta} \rceil$.*

Proof. We apply Theorem 4.3.1 with all lists $L(v)$, $v \in V(D)$ having cardinality $\lceil \tilde{\Delta} \rceil$. It is clear that the conditions of Theorem 4.3.1 are satisfied for every Eulerian vertex v. It is easy to see that the conditions are also satisfied for non-Eulerian vertices. Now, if D is not L-colorable, then by Theorem 4.3.1, D is Eulerian and $d^+(v) = \lceil \tilde{\Delta} \rceil$ for every vertex v. This implies that D is $\lceil \tilde{\Delta} \rceil$-regular. Now, the conclusion of Theorem 4.3.1 implies that D consists of a single block of type (a), (b) or (c). This means that either D is a directed cycle (and hence $\tilde{\Delta} = 1$), or D contains a digon, a contradiction. This completes the proof. □

We can now prove the main result of this section, which improves Brooks' bound for all digraphs without digons.

Theorem 4.3.3. *Let D be a connected digraph without digons, and let $\tilde{\Delta} = \tilde{\Delta}(D)$. If $\tilde{\Delta} > 1$, then $\chi(D) \leq \alpha(\tilde{\Delta} + 1)$ for some absolute constant $\alpha < 1$.*

Proof. We define $\alpha = \max\left\{\frac{\Delta_1}{\Delta_1+1}, 1 - e^{-13}\right\}$, where Δ_1 is the constant in the statement of Theorem 4.2.1. Now, if $\tilde{\Delta} < \Delta_1$ then by Lemma 4.3.2, it follows that $\chi(D) \leq \lceil \tilde{\Delta} \rceil \leq \alpha(\tilde{\Delta}+1)$. If $\tilde{\Delta} \geq \Delta_1$, then by Theorem 4.2.1 we obtain that $\chi(D) \leq \left(1 - e^{-13}\right) \tilde{\Delta} \leq \alpha(\tilde{\Delta}+1)$, as required. □

An interesting question to consider is the tightness of the bound of Lemma 4.3.2. It is easy to see that the bound is tight for $\lceil \tilde{\Delta} \rceil = 2$ by considering, for example, a directed cycle with an additional chord or a digraph consisting of two directed triangles sharing a common vertex. The graph in Figure 4.2 shows that the bound is also tight for $\lceil \tilde{\Delta} \rceil = 3$. It is easy to verify that, up to symmetry, the coloring outlined in the figure is the unique 2-coloring. Now, adding an additional vertex, whose three out-neighbors are the vertices of the middle triangle and the three in-neighbors are the remaining vertices, we obtain a 3-regular digraph where three colors are required to complete the coloring.

Another example of a digon-free 3-regular digraph on 7 vertices requiring three colors is the following. Take the Fano Plane and label its points by 1,2,...,7. For every line of the Fano plane containing points a, b, c, take a directed cycle through a, b, c (with either orientation). There is a unique directed 3-cycle through any two vertices because every two points line in exactly one line. This shows that the Fano plane digraphs are not isomorphic to the digraph from the previous paragraph. Finally, it is easy to verify that the resulting digraph needs three colors for coloring.

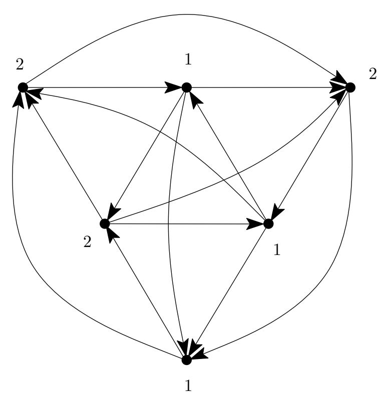


Figure 4.1: Constructing a 3-regular digraph D with $\chi(D) = 3$.

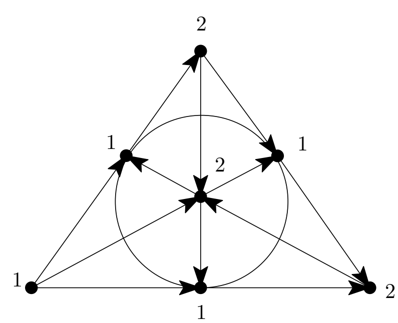


Figure 4.2: Constructing a 3-chromatic 3-regular digraph from the Fano plane.

Note that the digraphs in the above examples are 3-regular tournaments on 7 vertices. It is not hard to check that every tournament on 9 vertices has $\lceil \tilde{\Delta} \rceil = 4$, and yet is 3-colorable (simply choose three vertices that do not induce a directed triangle and color them with same color; the remaining 6 vertices can 2-colored). In general, we pose the following problem.

Question 4.3.4. *What is the smallest integer Δ_0 such that every digraph D without digons with $\lceil\tilde{\Delta}(D)\rceil = \Delta_0$ satisfies $\chi(D) \leq \Delta_0 - 1$?*

Note that this is a weak version of Conjecture 4.1.5. By Theorem 4.2.1, Δ_0 exists. However, we believe that Δ_0 is small, possibly equal to 4. The following proposition shows that the above holds for every $\lceil\tilde{\Delta}\rceil \geq \Delta_0$.

Proposition 4.3.5. *Let Δ_0 be defined as in Question 4.3.4. Then every digon-free digraph D with $\lceil\tilde{\Delta}(D)\rceil \geq \Delta_0$ satisfies $\chi(D) \leq \lceil\tilde{\Delta}(D)\rceil - 1$.*

Proof. The proof is by induction on $\lceil\tilde{\Delta}\rceil$. If $\lceil\tilde{\Delta}\rceil = \Delta_0$ this holds by the definition of Δ_0. Otherwise, let U be a maximal acyclic subset of D. Then $\lceil\tilde{\Delta}(D-U)\rceil \leq \lceil\tilde{\Delta}(D)\rceil - 1$ for otherwise U is not maximal. Since we can color U by a single color, we can apply the induction hypothesis to complete the proof. □

As a corollary we get:

Corollary 4.3.6. *There exists a positive constant $\alpha < 1$ such that for every digon-free digraph D with $\lceil\tilde{\Delta}(D)\rceil \geq \Delta_0$, $\chi(D) \leq \alpha\lceil\tilde{\Delta}\rceil$.*

Proof. Let $\alpha = \max\left\{\frac{\lceil\Delta_1\rceil}{\lceil\Delta_1\rceil+1}, 1 - e^{-13}\right\}$, where Δ_1 is the constant in the statement of Theorem 4.2.1. Now, applying Theorem 4.2.1 or Proposition 4.3.5 gives the result. □

Chapter 5

Non-locality of the digraph chromatic number

5.1 Introduction

In this chapter we prove, using standard probabilistic approach, that two further analogues of graph coloring results carry over to digraphs. The first result provides evidence that the digraph chromatic number, like the graph chromatic number, is a global parameter that cannot be deduced from local considerations. The second result, see Theorem 5.3.1, shows that there are digraphs with large chromatic number k in which every set of at most $c|V(D)|$ vertices is 2-colorable, where $c > 0$ is a constant that only depends on k. The analogous result for graphs was proved by Erdős [23] with the assumption being that all sets of at most cn vertices are 3-colorable. Both the 3-colorability in Erdős' result and 2-colorability in Theorem 5.3.1 are best possible.

Concerning the first result, it is well-known that there exist graphs with large girth and large chromatic number. Bollobás [9] and, independently, Kostochka and Mazurova [40] proved that there exist graphs of maximum degree at most Δ and of arbitrarily large girth whose chromatic number is $\Omega(\Delta/\log\Delta)$. We present a theorem (Theorem 5.2.1) that provides an extension to digraphs.

The bound of $\Omega(\Delta/\log\Delta)$ from [9, 40] is essentially best possible: a result of Johansson [38] shows that if G is triangle-free, then the chromatic number is $O(\Delta/\log\Delta)$. Similarly, Theorem 5.3.1 is also essentially best possible: we showed that every tournament on n

vertices has chromatic number at most $\frac{n}{\log n}(1 + o(1))$. In general, it may be true that the following analog of Johansson's result holds for digon-free digraphs, as conjectured by McDiarmid and Mohar [44].

Conjecture 5.1.1. *Every digraph D without digons and with maximum total degree Δ has $\chi(D) = O(\frac{\Delta}{\log \Delta})$.*

Theorem 5.2.1 shows that Conjecture 5.1.1, if true, is essentially best possible.

5.2 Chromatic number and girth

First, we need some basic definitions. The *total degree* of a vertex v is the number of arcs incident to v. The *maximum total degree* of D, denoted by $\Delta(D)$, is the maximum of all total degrees of vertices in D.

It is proved in [8] that there are digraphs of arbitrarily large digirth and chromatic number. Our result is an analogue of the aforementioned result of Bollobás [9] and Kostochka and Mazurova [40]. Note that the result involves the girth and not the digirth.

Theorem 5.2.1. *Let g and Δ be positive integers. There exists a digraph D of girth at least g, with $\Delta(D) \le \Delta$, and $\chi(D) \ge a\Delta/\log\Delta$ for some absolute constant $a > 0$. For Δ sufficiently large we may take $a = \frac{1}{5e}$.*

Proof. Our proof is in the spirit of Bollobás [9]. We may assume that Δ is sufficiently large.

Let $D = D(n,p)$ be a random digraph of order n defined as follows. For every $u, v \in V(D)$, we connect uv with probability $2p$, independently. Now we randomly (with probability 1/2) assign an orientation to every edge that is present. Observe that D has no digons. We will use the value $p = \frac{\Delta}{4en}$, where e is the base of the natural logarithm.

Claim 5.2.2. *D has no more than Δ^g cycles of length less than g with probability at least $1 - \frac{1}{\Delta}$.*

Proof. Let N_l be the number of cycles of length l in D. Then, by linearity of expectation

$$\mathbb{E}[N_l] \le \binom{n}{l} l!(2p)^l \le n^l(2p)^l \le (\tfrac{\Delta}{4})^l.$$

Therefore, the expected number of cycles of length less than g is at most Δ^{g-1}. So the probability that D has more than Δ^g cycles of length less than g is at most $1/\Delta$ by Markov's inequality (see Appendix A, Theorem A.1.2). $\square$

Claim 5.2.3. *There is a set A of at most $n/1000$ vertices of D such that $\Delta(D - A) \leq \Delta$ with probability at least $\frac{1}{2}$.*

Proof. As in [9], define the *excess degree of D* to be $ex(D) = \sum_{d_i > \Delta}(d_i - \Delta)$, where d_i is the total degree of the i^{th} vertex. Clearly, there is a set of at most $ex(D)$ arcs (or vertices) whose removal reduces the maximum total degree of D to Δ. Let X_d be the number of vertices of total degree d, $d = 0, 1, ..., n-1$. Then $ex(D) = \sum_{d=\Delta+1}^{n-1}(d - \Delta)X_d$.

Now, we estimate the expectation of X_d. By linearity of expectation, and using the bound $\binom{n}{k} \leq (\frac{en}{k})^k$, we have:

$$\begin{aligned}
\mathbb{E}[X_d] &\leq n\binom{n-1}{d}(2p)^d \\
&\leq n\left(\frac{e(n-1)}{d}\right)^d\left(\frac{\Delta}{2en}\right)^d \\
&\leq n\left(\frac{\Delta}{2d}\right)^d.
\end{aligned}$$

Therefore, by linearity of expectation we have that, for Δ sufficiently large,

$$\begin{aligned}
\mathbb{E}[ex(D)] &\leq \sum_{d=\Delta+1}^{n-1} nd\left(\frac{\Delta}{2d}\right)^d \\
&\leq \frac{n\Delta}{2}\sum_{d=\Delta+1}^{n-1}\left(\frac{\Delta}{2d}\right)^{d-1} \\
&\leq \frac{n\Delta}{2}\sum_{d=\Delta+1}^{n-1}\left(\frac{1}{2}\right)^{d-1} \\
&\leq \frac{n\Delta}{2}\cdot\frac{(\frac{1}{2})^\Delta}{1-\frac{1}{2}} \\
&= n\cdot\frac{\Delta}{2^\Delta} \\
&\leq \frac{n}{2000}.
\end{aligned}$$

Now, by Markov's inequality, $\mathbb{P}[ex(D) > n/1000] < 1/2$.

$\square$

Let $\alpha(D)$ be the size of a maximum acyclic set of vertices in D. The following result will be used in the proof of our next claim and also in Section 5.3.

Theorem 5.2.4 ([60]). *Let $D \in D(n, p)$ and $w = np$. There is an absolute constant W such that: If p satisfies $w \geq W$, then, asymptotically almost surely (on n),*

$$\alpha(D) \leq \left(\frac{2}{\log q}\right)(\log w + 3e),$$

where $q = (1-p)^{-1}$.

Claim 5.2.5. *Let $\alpha(D)$ be the size of a maximum acyclic set of vertices in D. Then $\alpha(D) \leq \frac{4en\log\Delta}{\Delta}$ with high probability, where the asymptotic is in terms of n .*

Proof. Since Δ is sufficiently large, Theorem 5.2.4 applies and using the fact $1 - p \leq e^{-p}$, the result follows. $\square$

Now, pick a digraph D that satisfies the three claims. After removing at most $n/1000 + \Delta^g \leq n/100$ vertices, the resulting digraph D^* has maximum degree at most Δ and girth at least g. Clearly, $\alpha(D^*) \leq \alpha(D)$. Therefore, $\chi(D^*) \geq \frac{n(1-1/100)}{\frac{4en\log\Delta}{\Delta}} \geq \frac{\Delta}{5e\log\Delta}$. $\square$

5.3 Local 2-colorings and the chromatic number

A result of Erdős [23] states that there exist graphs of large chromatic number where the induced subgraph on any constant fraction number of the vertices is 3-colorable. In particular, it is proved that for every k there exists $\epsilon > 0$ such that for all n sufficiently large there exists a graph G of order n with $\chi(G) > k$ and yet $\chi(G[S]) \leq 3$ for every $S \subset V(G)$ with $|S| \leq \epsilon n$.

The 3-colorability in the aforementioned theorem cannot be improved. A result of Kierstead, Szemeredi and Trotter [39] (with later improvements by Nelli [54] and Jiang [37]) shows that every 4-chromatic graph of order n contains an odd cycle of length at most $8\sqrt{n}$.

We prove the following analog for digraphs. Our proof follows the proof of the result of Erdős found in [4].

Theorem 5.3.1. *For every k, there exists $\epsilon > 0$ such that for every sufficiently large integer n there exists a digraph D of order n with $\chi(D) > k$ and yet $\chi(D[S]) \leq 2$ for every $S \subset V(D)$ with $|S| \leq \epsilon n$.*

Proof. Clearly, we may assume that $\log k \geq 3$ and $k \geq \sqrt{W}$, where W is the constant in Theorem 5.2.4. Let us consider the random digraph $D = D(n,p)$ with $p = \frac{k^2}{n}$ and let $0 < \epsilon < k^{-5}$.

We first show that $\chi(D) > k$ with high probability. Since k is sufficiently large, Theorem 5.2.4 implies that $\alpha(D) \leq 6n \log k/k^2$ with high probability. Therefore, almost surely $\chi(D) \geq \frac{1}{6}k^2/\log k > k$.

Now, we show that with high probability every set of at most ϵn vertices can be colored with at most two colors . Suppose there exists a set S with $|S| \leq \epsilon n$ such that $\chi(D[S]) \geq 3$. Let $T \subset S$ be a 3-critical subset, i.e. for every $v \in T$, $\chi(D[T] - v) \leq 2$. Let $t = |T|$. For every $v \in T$, $\min\{d^+_{D[T]}(v), d^-_{D[T]}(v)\} \geq 2$ for otherwise a 2-coloring of $D[T] - v$ could be extended to $D[T]$. Therefore, every vertex in T has total degree of at least 4 in $D[T]$ which implies that $D[T]$ has at least $2t$ arcs. The probability of this is at most

$$\begin{aligned}
\sum_{3 \leq t \leq \epsilon n} \binom{n}{t} \binom{2\binom{t}{2}}{2t} \left(\frac{k^2}{n}\right)^{2t} &\leq \sum_{3 \leq t \leq \epsilon n} \left(\frac{en}{t}\right)^t \left(\frac{et(t-1)}{2t}\right)^{2t} \left(\frac{k^2}{n}\right)^{2t} \\
&\leq \sum_{3 \leq t \leq \epsilon n} \left(\frac{e^3 t k^4}{4n}\right)^t \\
&\leq \epsilon n \max_{3 \leq t \leq \epsilon n} \left(\frac{7tk^4}{n}\right)^t \qquad (5.1)
\end{aligned}$$

If $3 \leq t \leq \log^2 n$, then $\left(\frac{7tk^4}{n}\right)^t \leq \left(\frac{7\log^2 n k^4}{n}\right)^t \leq \left(\frac{7\log^2 n k^4}{n}\right)^3 = o(\frac{1}{n})$.

Similarly, if $\log^2 n \leq t \leq \epsilon n$, then $\left(\frac{7tk^4}{n}\right)^t \leq (7\epsilon k^4)^t \leq (\frac{7}{k})^t \leq (\frac{7}{k})^{\log^2 n} = o(\frac{1}{n})$.

These estimates and (5.1) imply that the probability that $\chi(D[S]) \leq 2$ is $o(1)$. This completes the proof. $\square$

We show that 2-colorability in the previous theorem cannot be decreased to 1 due to the following theorem.

Theorem 5.3.2. *If D is a digraph with $\chi(D) \geq 3$ and of order n, then it contains a directed cycle of length $o(n)$.*

Proof. In the proof we shall use the following digraph analogue of Erdős-Posa Theorem. Reed et al. [57] proved that for every integer t, there exists an integer $f(t)$ so that every digraph either has t vertex-disjoint directed cycles or a set of at most $f(t)$ vertices whose removal makes the digraph acyclic.

Define $h(n) = \max\{t : tf(t) \leq n\}$. It is clear that $h(n) \to \infty$. Let c be the length of a shortest directed cycle in D.

If D has $h(n)$ vertex-disjoint directed cycles, then $ch(n) \leq n$ which implies that $c \leq \frac{n}{h(n)} = o(n)$. Otherwise, suppose that $h(n) = t$. There exists a set S of vertices with $|S| = f(t)$ such that $V(D) \backslash S$ is acyclic. Since $\chi(D) \geq 3$, we have that $\chi(D[S]) \geq 2$, which implies that S contains a directed cycle of length at most $|S| = f(t) \leq \frac{n}{t} = \frac{n}{h(n)} = o(n)$. $\square$

Chapter 6

Acyclic Homomorphisms

6.1 Introduction

In this chapter, we study a generalization of the digraph chromatic number. All the new results discussed here can be found in [32]. The main result of this chapter is Theorem 6.2.3, which can be found, along with an alternate proof, in [32]. For undirected graphs, a natural generalization of coloring is the homomorphism of graphs. Given graphs G and H, a *homomorphism* from G to H is a function $\phi : V(G) \to V(H)$ such that for every $uv \in E(G)$, $\phi(u)\phi(v) \in E(H)$. It is well-known (and easy to see) that a graph G is r-colorable if and only if there exists a homomorphism from G to the complete graph K_r. In general, we say that G is *H-colorable* if there is a homomorphism from G to H. Graph homomorphisms have been studied extensively in the literature and we refer the reader to [35].

One can generalize the notion of the digraph chromatic number. In a similar fashion, our digraphs are simple, i.e. loopless and without multiple arcs. However, we allow two vertices u, v to be joined by two oppositely directed arcs, uv and vu.

An *acyclic homomorphism* of a digraph D into a digraph C is a function $\phi\colon V(D) \to V(C)$ such that:

(i) for every vertex $v \in V(C)$, the subdigraph of D induced by $\phi^{-1}(v)$ is acyclic;

(ii) for every arc $uv \in E(D)$, either $\phi(u) = \phi(v)$, or $\phi(u)\phi(v)$ is an arc of C.

If digraphs C and D are obtained from undirected graphs G and H, respectively, by replacing every edge by two oppositely directed arcs, then acyclic homomorphisms between

C and D correspond to usual graph homomorphisms between G and H. In this sense, acyclic homomorphisms can be viewed as a generalization of the notion of homomorphisms of undirected graphs.

In the same way as usual graph homomorphisms generalize the notion of graph colorings, the acyclic homomorphisms generalize colorings of digraphs, where complete graphs are replaced by complete bidirected graphs. Motivated by this, we say that a digraph D is *C-colorable* if there is an acyclic homomorphism from D to C.

Acyclic homomorphisms were introduced in [28]. The authors studied the complexity of D-coloring. They proved the following theorems.

Theorem 6.1.1 ([28])**.** *Let D be a digraph that contains a directed cycle. Then the acyclic D-coloring problem is NP-complete.*

Let C_3 be the directed triangle. Then the above theorem can be strengthened.

Theorem 6.1.2 ([28])**.** *The acyclic C_3-coloring problem is NP-complete even when restricted to planar digraphs.*

Let C_2 be the directed two cycle, i.e., the digon. It is easy to see that for a digraph D, $\chi(D) \leq 2$ if and only if D is C_2-colorable. We mentioned previously that deciding 2-colorability for general digraphs is NP-complete. The next result strengthens this theorem.

Theorem 6.1.3 ([28])**.** *The acyclic C_2-coloring problem is NP-complete even when restricted to planar digraphs.*

6.2 D-colorable digraphs of large girth

A classical result of Erdős [22] asserts that for all integers k and g there exist graphs with chromatic number k and with girth at least g. Bollobás and Sauer [10] strengthened this result by showing that there are such graphs which are, moreover, uniquely k-colorable. Zhu [66] extended Bollobás and Sauer's result to homomorphisms into general graphs. Rather recently, the results of [66] have been extended by Nešetřil and Zhu [48] to give a simultaneous generalization of Zhu's two primary results. The results of this chapter extend these theorems to digraphs with acyclic homomorphisms.

Zhu [66] generalized Erdős' result as follows.

Theorem 6.2.1 ([66]). *If G and H are graphs such that G is not H-colorable, then for every positive integer g, there exists a graph G^* of girth at least g that is G-colorable but not H-colorable.*

To recover Erdős' result, we simply take $G = K_k$ and $H = K_{k-1}$.

For digraphs, the following analog of Erdős' theorem was proved by Bokal et. al. [8].

Theorem 6.2.2 ([8]). *For every $g \geq 3$ and $k \geq 1$, there exists a digraph D with digirth at least g and $\chi(D) \geq k$.*

The proof of the above theorem is in the same vein as that of Erdős. In fact, the method of proof yields a stronger result: the digirth in the statement of Theorem 6.2.2 can be replaced with girth. The purpose of this chapter is to extend Theorem 6.2.2 to acyclic homomorphisms. We will prove the following.

Theorem 6.2.3. *If D and C are digraphs such that D is not C-colorable, then for any positive integer g, there exists a digraph D^* of girth at least g that is D-colorable but not C-colorable.*

6.3 Proof of Theorem 6.2.3

This section is devoted to the proof of Theorem 6.2.3. Suppose that $V(D) = \{1, 2, \ldots, k\}$ and that $q = |E(D)|$. Let n be a (large) positive integer, and let D_n be the digraph obtained from D as follows: replace every vertex i with a stable set V_i of n ordered vertices $v_1, v_2, ..., v_n$, and replace each arc ij of D by the set of all possible n^2 arcs from V_i to V_j. Clearly, $|V(D_n)| = kn$ and $|E(D_n)| = qn^2$.

Now fix a positive $\varepsilon < 1/(4g)$. Our random digraph model $\mathcal{D} = \mathcal{D}(D_n, p)$ consists of those spanning subdigraphs of D_n in which the arcs of D_n are chosen randomly and independently with probability $p = n^{\varepsilon - 1}$.

As usual in nonconstructive probabilistic proofs of results of this nature, the idea is to show that most digraphs in $\mathcal{D}$ have only a few short cycles, and for most digraphs $H \in \mathcal{D}$, the subdigraph of H obtained by removing an arbitrary small set of arcs is not C-colorable. Choosing an $H \in \mathcal{D}$ with both these properties, we can force the girth to be large by deleting an arc from each short cycle. Since the set A_0 of deleted arcs is small, the resulting digraph $H - A_0$ satisfies the desired conclusion of Theorem 6.2.3.

To make this description more precise, let $\mathcal{D}_1$ denote the set of digraphs in $\mathcal{D}$ containing at most $\lceil n^{g\varepsilon} \rceil$ cycles of length less than g, and let $\mathcal{D}_2$ be the set of digraphs $H \in \mathcal{D}$ that have the property that $H - A_0$ is not C-colorable for any set A_0 of at most $\lceil n^{g\varepsilon} \rceil$ arcs. We will show that

$$|\mathcal{D}_1| > \left(1 - n^{-\varepsilon/2}\right) |\mathcal{D}| \tag{6.1}$$

and

$$|\mathcal{D}_2| > \left(1 - e^{-n}\right) |\mathcal{D}| . \tag{6.2}$$

Since (6.1) and (6.2) imply that $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \varnothing$ (for sufficiently large n), there exists a digraph $H \in \mathcal{D}_1 \cap \mathcal{D}_2$. Now $H \in \mathcal{D}_1$ implies that there is a set A_0 of at most $\lceil n^{g\varepsilon} \rceil$ arcs whose removal leaves a digraph $D^* := H - A_0$ of girth at least g, while $H \in \mathcal{D}_2$ means that D^* is not C-colorable. Thus, it remains to establish (6.1) and (6.2).

Proof of (6.1). The expected number N_ℓ of cycles of length ℓ in a digraph $H \in \mathcal{D}$ is at most

$$\binom{kn}{\ell} (\ell - 1)!\, p^\ell \tag{6.3}$$

since there are $\binom{kn}{\ell}(\ell - 1)!$ ways of choosing a cyclic sequence of ℓ vertices as a candidate for a cycle, and such an ℓ-cycle occurs in $\mathcal{D}$ with probability either 0 or p^ℓ. It is easy to see that the product of the first two factors in (6.3) is smaller than $(kn)^\ell/\ell$. Therefore, if n is large enough, then

$$\sum_{\ell=2}^{g-1} N_\ell \le \sum_{\ell=2}^{g-1} \frac{1}{\ell} (kn^\varepsilon)^\ell < k^{g-1} n^{(g-1)\varepsilon} < n^{-\varepsilon/2} n^{g\varepsilon}.$$

Now (6.1) follows easily from Markov's Inequality. □

Proof of (6.2). We shall argue that $|\mathcal{D} \setminus \mathcal{D}_2| < e^{-n}|\mathcal{D}|$. If $H \in \mathcal{D} \setminus \mathcal{D}_2$, then there is a set A_0 of at most $\lceil n^{g\varepsilon} \rceil$ arcs of H so that $H - A_0$ admits an acyclic homomorphism h to C. Let $k' = |V(C)|$. By the pigeonhole principle, for each $i \in V(D)$, there exists a vertex $x_i \in V(C)$ such that $|V_i \cap h^{-1}(x_i)| \ge n/k'$. Define $\phi\colon V(D) \to V(C)$ by setting $\phi(i) = x_i$. Since $n/k' \gg n^{g\varepsilon}$, the set $V_i \cap h^{-1}(x_i)$ contains a subset W_i of cardinality $w := \lceil n/(2k') \rceil$ such that no arc in A_0 has an end vertex in W_i.

Since D is not C-colorable, the function ϕ is not an acyclic homomorphism. Therefore, either there is an arc $ij \in E(D)$ such that $\phi(i) \neq \phi(j)$ and $\phi(i)\phi(j)$ is not an arc of C, or there is a $v \in V(C)$ such that the subdigraph of D induced on $\phi^{-1}(v)$ contains a cycle.

We first consider the case when ij is an arc of D such that $\phi(i) \neq \phi(j)$ and $\phi(i)\phi(j)$ is not an arc of C. Since h is an acyclic homomorphism, there are no arcs from W_i to W_j in $H - A_0$. By the definition of W_i and W_j, neither are there such arcs in H.

Let us now estimate the expected number M of pairs of sets $A \subseteq V_i$, $B \subseteq V_j$, with $|A| = |B| = w$, such that $ij \in E(D)$ and such that there is no arc from A to B in $H \in \mathcal{D}$ (we call such a pair A, B a *bad pair*). By the linearity of expectation, we have

$$M = q\binom{n}{w}^2 (1-p)^{w^2} < q\Big(\frac{n^w}{w!}\Big)^2 (1-p)^{w^2} = q\frac{(n^2(1-p)^w)^w}{(w!)^2}. \tag{6.4}$$

Since w grows linearly with n, for sufficiently large n we have

$$n^2(1-p)^w < e^{-2k'} \quad \text{and} \quad \frac{q}{(w!)^2} < \frac{1}{2}.$$

Therefore Markov's Inequality and (6.4) yield

$$\Pr(\exists \text{ a bad pair}) < \frac{e^{-n}}{2}. \tag{6.5}$$

Suppose now that there is a $v \in V(C)$ such that D contains a cycle Q whose vertices are all in $\phi^{-1}(v)$. Suppose that $Q = i_1 i_2 \cdots i_t$. Observe that $2 \leq t \leq k$. Since $\phi(Q) = \{v\}$, we conclude that $h(W_{i_1}) = h(W_{i_2}) = \cdots = h(W_{i_t}) = \{v\}$. Since h is an acyclic homomorphism, the subdigraph of H induced on $W_{i_1} \cup W_{i_2} \cup \cdots \cup W_{i_t}$ is acyclic.

Let us consider all sequences of sets $U_{j_1}, U_{j_2}, \ldots, U_{j_\ell}$ such that, for $r = 1, 2, \ldots, \ell$, we have $U_{j_r} \subseteq V_{j_r}$ and $|U_{j_r}| = w$, and the vertex sequence $j_1 j_2 \cdots j_\ell$ is a cycle in D. Let $U(\ell)$ the subdigraph of H induced on $U_{j_1} \cup U_{j_2} \cup \cdots \cup U_{j_\ell}$. Let $P_\ell := \Pr(U(\ell) \text{ is acyclic})$. We say that this sequence is *bad* if $U(\ell)$ is acyclic. Since the expected number N of bad sequences is the sum of the corresponding expectations over all possible cycle lengths, we have

$$N \leq \sum_{\ell=2}^{k} \binom{k}{\ell} (\ell-1)! \binom{n}{w}^{\ell} P_\ell. \tag{6.6}$$

In order to bound N, we first bound the probabilities P_ℓ.

Lemma 6.3.1. *There exists a constant $\gamma > 0$ (not depending on n) such that $P_\ell \leq e^{-\gamma n^{1+\varepsilon}}$ for every integer $\ell \in \{2, 3, \ldots, k\}$.*

The proof invokes the Janson Inequalities (see Appendix, Theorems A.2.1 and A.2.2).

Proof of Lemma 6.3.1. We use the Janson Inequalities. Here, Ω denotes the set of all potential arcs (in D_n, as defined at the start of Section 6.3) between the sets U_{j_i}, $i =$

$1, ..., \ell$, (introduced just prior to our statement of Lemma 6.3.1); each arc in Ω appears with probability p.

Let s be a (large) multiple of ℓ; the value of s will be independent of n and specified below. Now, let us enumerate those cycles of D_n that are of length s, and that cyclically traverse $U_{j_1}, U_{j_2}, ..., U_{j_\ell}$ s/ℓ times. For $j \geq 1$, denote by S_j the arc set of the jth such cycle and by $\mathcal{B}_j$ the event that the arcs in S_j all appear in H (i.e. the cycle determined by S_j is present in H). Let the random variable X count those $\mathcal{B}_j$ that occur. Since $\Pr(X = 0)$ (the probability that there is no such cycle of length s) is an upper bound for P_ℓ (the chance that $U(\ell)$ is acyclic), we can bound P_ℓ by bounding $\Pr(X = 0)$, and estimating the latter quantity is exactly the purpose of Janson's Inequalities. In the Janson paradigm, the value of Δ is defined by

$$\Delta = \sum_{S_i \sim S_j} \Pr(\mathcal{B}_i \cap \mathcal{B}_j), \tag{6.7}$$

where $S_i \sim S_j$ if the two cycles determined by S_i and S_j have at least one arc in common.

First, we find an upper bound for Δ. Letting i remain fixed, we (rather crudely) obtain

$$\Delta \leq n^s \sum_{j: S_i \sim S_j} \Pr(\mathcal{B}_i \cap \mathcal{B}_j), \tag{6.8}$$

since each $|U_r| \leq n$ and each $|S_i| = s$. The sum on the right side satisfies

$$\sum_{j: S_i \sim S_j} \Pr(\mathcal{B}_i \cap \mathcal{B}_j) \quad \leq \quad \sum_{r=1}^{s-1} \binom{s}{r} p^{2s-r} w^{s-(r+1)}. \tag{6.9}$$

The binomial coefficient in (6.9) accounts for the number of ways to choose the arcs of $S_i \cap S_j$, the power of p is $\Pr(\mathcal{B}_j | \mathcal{B}_i) \Pr(\mathcal{B}_i)$, the power of w reflects the facts that each U-set has cardinality w and, with i fixed, there are at most $s - (r+1)$ vertices in the S_j-cycle not already in the S_i-cycle. Recalling that $w = \lceil n/(2k') \rceil$ (so that $w < n$), using the gross bound $\binom{s}{r} < 2^s$, and replacing p with $n^{\varepsilon - 1}$, we find that

$$\sum_{j: S_i \sim S_j} \Pr(\mathcal{B}_i \cap \mathcal{B}_j) \ < \ 2^s \sum_{r=1}^{s-1} p^{2s-r} n^{s-(r+1)} \ = \ 2^s \sum_{r=1}^{s-1} n^{2\varepsilon s - s - r\varepsilon - 1} \ < \ 2^s s n^{2\varepsilon s - s - \varepsilon - 1}.$$

With (6.8), the last estimate yields

$$\Delta < 2^s s n^{2\varepsilon s - \varepsilon - 1}. \tag{6.10}$$

Next, we find a lower bound for $\mu := E[X]$. Since there are ℓ U-sets, each containing w vertices, and each ordered choice of s/ℓ vertices from each (up to the choice of the first vertex) contributes 1 to X with probability at least p^s, we have

$$\mu \;\geq\; \frac{1}{s}\binom{w}{s/\ell}^{\ell}\left[\left(\frac{s}{\ell}\right)!\right]^{\ell} p^s.$$

Therefore,

$$\mu \;\geq\; \frac{1}{s}\left(\frac{w!}{(w-s/\ell)!}\right)^{\ell} p^s \;\geq\; \frac{1}{s}\left(w-\frac{s}{\ell}\right)^{s} p^s \;\geq\; \frac{1}{s}\left(\frac{n}{4k'}\right)^{s} n^{\varepsilon s - s} \;=\; \frac{n^{\varepsilon s}}{s(4k')^s}. \tag{6.11}$$

We distinguish two cases.

Case 1: $\Delta \geq \mu$.

Here, we have the hypotheses of the Extended Janson Inequality, which, along with our bounds (6.10), (6.11) gives

$$\Pr(X=0) \;\leq\; e^{-\mu^2/(2\Delta)} \;<\; e^{-n^{1+\varepsilon}/(2s^3(32k'^2)^s)}.$$

Case 2: $\Delta < \mu$.

Now we have the hypotheses of the basic Janson Inequality, which together with (6.11) gives

$$\Pr(X=0) \;\leq\; e^{-\mu+\Delta/2} \;<\; e^{-\mu/2} \;\leq\; e^{-n^{\varepsilon s}/(2s(4k')^s)}.$$

Let $s > 1 + (1+\varepsilon)/\varepsilon$ be a multiple of ℓ. Then the last bound shows that

$$\Pr(X=0) \;\leq\; e^{-n^{1+\varepsilon}(n^{\varepsilon}/(2s(4k')^s))} \;\leq\; e^{-n^{1+\varepsilon}}.$$

Since s and k' are constants (not depending on n), in either case we see that

$$P_\ell \;\leq\; \Pr(X=0) \;\leq\; e^{-\gamma n^{1+\varepsilon}}$$

for some constant $\gamma > 0$. This gives us Lemma 6.3.1. $\square$

We return to our estimation of the expected number N of bad sequences in (6.6), repeated here for convenience:

$$N \;\leq\; \sum_{\ell=2}^{k}\binom{k}{\ell}(\ell-1)!\binom{n}{w}^{\ell} P_\ell.$$

Using Lemma 6.3.1 to bound the factors P_ℓ in this sum shows that for n large enough,

$$N \leq \sum_{\ell=2}^{k} \binom{k}{\ell} (\ell-1)! \binom{n}{w}^{\ell} e^{-\gamma n^{1+\varepsilon}} < \sum_{\ell=2}^{k} \frac{e^{-n}}{2k} < \frac{e^{-n}}{2}. \tag{6.12}$$

From (6.12) and Markov's Inequality, we conclude that

$$\Pr(\exists \text{ a bad sequence}) < \frac{e^{-n}}{2}. \tag{6.13}$$

Since ϕ fails to be an acyclic homomorphism exactly when there exists a bad pair or there exists a bad sequence, (6.5) and (6.13) now show that

$$|\mathcal{D} \setminus \mathcal{D}_2| \leq (\Pr(\exists \text{ bad pair}) + \Pr(\exists \text{ bad sequence}))\, |\mathcal{D}| < e^{-n}\, |\mathcal{D}|\,,$$

which yields (6.2). □

6.4 Uniquely D-colorable digraphs

The notion of colorability can be extended to unique colorability. A graph (digraph) G is *uniquely H-colorable* if it is surjectively H-colorable and for any two coloring ϕ, ψ of G, there is an automorphism π of H such that $\phi = \pi\psi$. A graph (digraph) G is a *core* if it is uniquely G-colorable.

Theorem 6.2.3 has the following similar result. The proof can be found in [32].

Theorem 6.4.1 ([32]). *For any core D and any positive integer g, there is a digraph D^* of girth at least g that is uniquely D-colorable.*

Theorem 6.4.1 is a generalization of its graph analog proved by Zhu [66].

Theorem 6.4.2 ([66]). *For any graph H that is a core and any positive integer g, there is a graph H^* of girth at least g that is uniquely H-colorable.*

Theorem 6.4.1 immediately applies to digraph circular colorings as discussed in the next section

6.5 Circular chromatic number of digraphs

Recall that there are digraphs with arbitrary large digirth and chromatic number. In [8], the authors proved the following generalization to the circular chromatic number.

Theorem 6.5.1 ([8])**.** *There exist digraphs with arbitrary large girth and arbitrary large circular chromatic number.*

Here we present a theorem that generalizes the above result.

Let $d \geq 1$ and $k \geq d$ be integers. Let $C(k, d)$ be the digraph with vertex set $\mathbb{Z}_k = \{0, 1, \dots, k-1\}$ and arcs

$$E(C(k,d)) = \{ij \mid j - i \in \{d, d+1, \dots, k-1\}\},$$

where the subtraction is considered in the cyclic group $\mathbb{Z}_k$ of integers modulo k.

Acyclic homomorphisms into $C(k, d)$ are an important concept because of their relation to the circular chromatic number of digraphs; cf. [8]. It is shown in [8] that D is $C(k, d)$-colorable if and only if $k/d \geq \chi_c(D)$.

In [32], the authors show that Theorem 6.4.1 implies the following generalization of Theorem 6.5.1.

Theorem 6.5.2 ([32])**.** *If k and d are relatively prime integers and $1 \leq d \leq k$, then for every integer g, there exists a uniquely $C(k, d)$-colorable digraph of girth at least g and with circular chromatic number equal to k/d.*

Chapter 7

3-colorings of planar graphs

7.1 Introduction

In Chapter 2, we mentioned that every graph can be 3-colored so that each color class induces a forest and that this bound is sharp (see Chartrand et al. [15]). In this chapter, we show that there are in fact exponentially many 3-colorings of this kind for any planar graph. The same result holds in the setting of 3-list-colorings.

Let us recall that a partition of vertices of a graph G into classes $V_1 \cup \cdots \cup V_k$ is an *arboreal partition* if each V_i $(1 \le i \le k)$ induces a forest in G. A function $f\colon V(G) \to \{1, \dots, k\}$ is called an *arboreal k-coloring* if $V_i = f^{-1}(i)$, $i = 1, \dots, k$, form an arboreal partition. The *vertex-arboricity* $a(G)$ of the graph G is the minimum k such that G admits an arboreal k-coloring.

It is an easy consequence of 5-degeneracy of planar graphs that every planar digraph D without cycles of length at most 2 and its associated underlying planar graph G satisfy

$$\chi(D) \le a(G) \le 3. \tag{7.1}$$

The main result of this chapter is a relaxation of Conjecture 2.4.1 and a strengthening of the above stated inequality (7.1). In particular, we prove the following.

Theorem 7.1.1. *Every planar graph of order n has at least $2^{n/9}$ arboreal 3-colorings.*

Corollary 7.1.2. *Every planar digraph of order n without cycles of length at most 2 has at least $2^{n/9}$ 3-colorings.*

Let us observe that Theorem 7.1.1 cannot be extended to graphs embedded in the torus since $a(K_7) = 4$ and K_7 admits an embedding in the torus. However, for every orientation D of K_7, we have $\chi(D) \leq 3$ (and in some cases $\chi(D) = 3$), so it is possible that Corollary 7.1.2 extends.

The proof of Theorem 7.1.1 is deferred until Section 7.4. Actually, we shall prove an extended version in the setting of list-colorings. Given a list-assignment L for the vertices of graph G, we say that L is a *k-list-assignment* if $|L(v)| = k$ for every $v \in V(G)$.

Theorem 7.1.3. *Let L be a 3-list-assignment for a planar graph G of order n. Then G has at least $2^{n/9}$ L-colorings.*

Corollary 7.1.2 then extends to the list coloring of digraphs.

7.2 Unavoidable configurations

We define a *configuration* as a plane graph C together with a function $\delta\colon V(C) \to \mathbb{N}$ such that $\delta(v) \geq deg_C(v)$ for every $v \in V(C)$. A plane graph G *contains* the configuration (C, δ) if there is an injective mapping $h\colon V(C) \to V(G)$ such that the following statements hold:

(i) For every edge $ab \in E(C)$, $h(a)h(b)$ is an edge of G.

(ii) For every facial walk $a_1 \dots a_k$ in C, except for the unbounded face, the image $h(a_1) \dots h(a_k)$ is a facial walk in G.

(iii) For every $a \in V(C)$, the degree of $h(a)$ in G is equal to $\delta(a)$.

If v is a vertex of degree k in G, then we call it a *k-vertex*, and a vertex of degree at least k (at most k) will also be referred to as a *k^+-vertex* (*k^--vertex*). A neighbor of v whose degree is k is a *k-neighbor* (similarly k^+- and *k^--neighbor*).

We will prove the following theorem.

Theorem 7.2.1. *Every planar triangulation contains one of the configurations listed in Fig. 7.1.*

Proof. The proof uses the discharging method. Assume, for a contradiction, that there is a planar triangulation G that contains none of the configurations shown in Fig. 7.1. We shall refer to these configurations as $Q_1, Q_2, \dots, Q_{23}$.

Let G be a counterexample of minimum order. To each vertex v of G, we assign a *charge* of $c(v) = \deg(v) - 6$. A well-known consequence of Euler's formula is that the total charge is always negative, $\sum_{v \in V(G)} c(v) = -12$. We are going to apply the following discharging rules:

R1: A 7-vertex sends charge of 1/3 to each adjacent 5-vertex.

R2: A 7-vertex sends charge of 1/2 to each adjacent 4-vertex.

R3: An 8^+-vertex sends charge of 1/2 to each adjacent 5-vertex.

R4: An 8^+-vertex sends charge of 3/2 to each adjacent 4-vertex whose neighbors have degrees $8^+, 7, 6, 6$.

R5: An 8^+-vertex sends charge of 3/4 to each adjacent 4-vertex whose neighbors have degrees 8^+, 8^+, 7^+, 6.

R6: An 8^+-vertex sends charge of 1/2 to each adjacent 4-vertex whose neighbors have degrees $8^+, 7^+, 7^+, 7^+$.

R7: An 8^+-vertex sends charge of 1 to each adjacent 4-vertex whose neighbors have degrees $8^+, 8^+, 6, 6$ or $8^+, 7, 7, 6$.

Let $c^*(v)$ be the *final charge* obtained by applying rules R1–R7 to all vertices in G. We will show that every vertex has non-negative final charge. This will yield a contradiction since the initial total charge of -12 must be preserved.

We say that a 4-vertex is *bad* if its neighbors have degrees $8^+, 7, 6, 6$, i.e., the rule R4 applies to it and its 8^+-neighbor. Let us observe that the clockwise order of the neighbors of a bad vertex is $8^+, 7, 6, 6$ (or $8^+, 6, 6, 7$) since Q_7 is excluded.

First, note that G has no 3^--vertices since the configuration Q_1 is excluded and since a triangulation cannot have 2^--vertices.

4-vertices: Let v be a 4-vertex. Note that v cannot have a 5^--neighbor since Q_2 is excluded. If all of v's neighbors have degree at most 7, then they all have degree exactly 7 since Q_6, Q_7 and Q_8 are excluded. Since the vertex v has initial charge of -2, and each 7-neighbor sends a charge of 1/2 to it, the final charge of v is 0.

Now, assume that v is adjacent to an 8^+-vertex. First, assume that the remaining three neighbors v_1, v_2, v_3 of v are all 7^--vertices. The vertices v_1, v_2, v_3 cannot have all degree 6 since Q_8 is excluded. If $\deg(v_1) = 7$ and $\deg(v_2) = \deg(v_3) = 6$, then the rules R2 and R4 imply that v receives a charge of 2, resulting in the final charge of 0. If $\deg(v_1) = \deg(v_2) = 7$ and $\deg(v_3) = 6$, then by rules R2 and R7, v again receives a charge of 2. The case where $\deg(v_1) = \deg(v_2) = \deg(v_3) = 7$ is similar through rules R2 and R6.

Next, assume that v has exactly two 8^+-neighbors v_1, v_2. If the remaining two vertices v_3, v_4 are both 7-vertices, then rules R2 and R7 imply that v receives a total charge of at least 3, giving it the final charge of 1. If the remaining two vertices are both 6-vertices, then rule R7 implies that v receives a total charge of 2, resulting in 0 final charge. Therefore, we may assume that $\deg(v_3) = 7$ and $\deg(v_4) = 6$. In this case, both v_1 and v_2 send a charge of $3/4$ to v by R5, and v_3 sends a charge of $1/2$, resulting in a final charge of 0 for v.

Finally, assume that v has at least three 8^+-neighbors. By rule R5 (if v has a 6-neighbor), or by rules R2 and R6 (if v has a 7-neighbor), or by rule R6 (otherwise), we see that v receives a total charge of at least 2, so $c^*(v) \geq 0$.

5-vertices: Let v be a 5-vertex. Note that v is not adjacent to a 4-vertex. If all neighbors of v are 7^--vertices, then exclusion of Q_8 and Q_{10} implies that v has at least three 7-neighbors. By R1, each such neighbor sends a charge of $1/3$ to v. Since v has initial charge of -1, its final charge is at least 0. Next, suppose that v has an 8^+-neighbor. If v has at least two 8^+-neighbors, then by rule R3, v receives a charge of $1/2$ from each of them, resulting in the final charge of at least 0 for v. Therefore, we may suppose that v has exactly one 8^+-neighbor. If v has at least two 7-neighbors, then by R1 and R3, v receives a total charge of at least $1/2 + 1/3 + 1/3 > 1$, resulting in a positive final charge for v. Finally, if v has at most one 7-neighbor, then we get the configuration Q_8 or Q_{10}.

6-vertices: They have initial charge of 0, and by the discharging rules, they do not give or receive any charge, which implies that they have a final charge of 0.

7-vertices: They have an initial charge of $+1$ and they send charge only to 4-vertices and 5-vertices. Let v be a 7-vertex. If v has no 4-neighbors then it has at most three 5-neighbors since Q_{11} is excluded. Therefore, it sends a charge of $1/3$ to each such vertex, resulting in the final charge of 0 for v. Next, suppose that v has at least one 4-neighbor. Since Q_{12} is excluded, v has at most one other 5^--neighbor. Therefore, v sends a charge of at most $1/2 + 1/2 = 1$, resulting in the final charge of at least 0 for v.

8-vertices: An 8-vertex v has initial charge of $+2$. Since Q_{13} is excluded, v has at most two 4-neighbors. Now, suppose that v has exactly two 4-neighbors, say v_1 and v_2. We consider two subcases. First, assume that v has a 5-neighbor. Excluding Q_2 and Q_{15}, no vertex in $N(v_1) \cap N(v)$ and $N(v_2) \cap N(v)$ has degree at most 6. If the two vertices in $N(v_1) \cap N(v)$ are both 7-vertices, then v_1 has no 6^--neighbor (Q_2 and Q_{16} being excluded). This implies that v sends charge of $1/2$ to v_1. Otherwise, the two vertices in $N(v_1) \cap N(v)$ are an 8^+ and a 7^+-vertex, respectively. This implies that by rules R5 and R6, v sends charge of $3/4$ or $1/2$ to v_1. Therefore, in all cases, v sends no more than $3/4$ charge to v_1. An identical argument shows that v sends a charge of at most $3/4$ to v_2. Since v sends a charge of $1/2$ to a 5-vertex, we have that v sends a total charge of at most $3/4 + 3/4 + 1/2 = 2$. Secondly, assume that v has no 5-neighbors. Consider v_1. Excluding Q_7 and Q_{17}, v_1 is not a bad 4-vertex. Therefore, v sends charge of at most 1 to v_1. An identical argument shows that v sends charge of at most 1 to v_2. Therefore, the final charge of v is non-negative.

Next, suppose that v has exactly one 4-neighbor, say v_1. First, suppose that v_1 is a bad 4-vertex. Excluding Q_7 and Q_{16}, v has at most one 5-neighbor. Since v sends a charge of at most $3/2$ to v_1 and charge $1/2$ to its 5-neighbor, its final charge is at least 0. Thus, we may assume that v_1 is not a bad 4-vertex. Then v sends at most charge of 1 to v_1. Because Q_{18} is excluded, v has at most two 5-neighbors, to each of which it sends a charge of $1/2$. Therefore, v sends a total charge of at most $1 + 1/2 + 1/2 = 2$, which implies that it has a non-negative final charge.

Finally, suppose that v has no 4-neighbors. Excluding Q_{19}, v has at most four 5-neighbors, to each of which it sends charge of $1/2$. Therefore, the final charge of v is again non-negative.

9-vertices: A 9-vertex v has a charge of $+3$. Since Q_{20} is excluded, v has at most three 4-neighbors. First, suppose that v has exactly three 4-neighbors. Since Q_{20} is excluded, v has no 5-neighbor and since Q_{21} is excluded, none of its 4-neighbors are bad. Therefore, in this case v sends charge of at most 1 to each 4-neighbor, resulting in a non-negative final charge. Secondly, suppose that v has exactly two 4-neighbors. We consider two subcases. For the first subcase, suppose that none of the 4-neighbors are bad. Now, v has at most two 5-neighbors since Q_{22} is excluded. This implies that v sends total charge of at most $1 + 1 + 1/2 + 1/2 = 3$ to its neighbors, resulting in a non-negative final charge for v. For the second subcase, assume that v has at least one bad 4-neighbor. Now, the exclusion of Q_{21} implies that v has no 5-neighbors. Thus, v sends total charge of at most $3/2 + 3/2 = 3$, and therefore $c^*(v) \geq 0$. Thirdly, suppose that v has exactly one 4-neighbor. The exclusion of Q_{22} implies that v has at most three 5-neighbors, and hence it sends out a total charge of at most $3/2 + 1/2 + 1/2 + 1/2 = 3$, resulting in $c^*(v) \geq 0$. Lastly, assume that v has no 4-neighbors. Excluding Q_4 we see that v has at most six 5-neighbors. This implies that v sends a total charge of at most $6 \times 1/2 = 3$ to its neighbors, thus $c^*(v) \geq 0$.

10-vertices: A 10-vertex v has a charge of $+4$. Let $v_1, \ldots, v_{10}$ be the neighbors of v in the cyclic order around v. If v_i is a bad 4-neighbor of v and $\deg(v_{i-1}) = 7$, $\deg(v_{i+1}) = 6$, then the absence of Q_3 and Q_9 implies that $\deg(v_{i+2}) \geq 6$ and $\deg(v_{i-2}) \geq 5$. The absence of Q_5 also implies that if v_{i+3} is another bad 4-neighbor, then $\deg(v_{i+2}) = 7$, thus $\deg(v_{i+4}) = 6$ and $\deg(v_{i+5}) \geq 6$ (all indices modulo 10). By excluding Q_{23} and Q_4, we conclude that if v has two bad 4-neighbors, then it has no other 4-neighbor and has at most two 5-neighbors. This implies that $c^*(v) \geq 0$. Suppose now that v has one bad 4-neighbor, say v_2. We may assume $\deg(v_1) = 7$, $\deg(v_3) = 6$ and by the arguments given above, $\deg(v_{10}) \geq 5$, $\deg(v_4) \geq 6$. Excluding Q_4, v can have at most four 5-neighbors. Thus, the only possibility that $c^*(v) < 0$ is that v has 3 more 4-neighbors (and the only way to have this is that the 4-neighbors are v_5, v_7, v_9) or that v has two more 4-neighbors and two 5-neighbors (in which case 4-neighbors are v_5, v_7 and 5-neighbors are v_9, v_{10}). In each of these cases, we see, by excluding Q_3 and Q_5, that $\deg(v_4) \geq 7$, $\deg(v_6) \geq 7$ and $\deg(v_8) \geq 7$. Thus, excluding Q_9, v sends charge of at most $3/4$ to each of v_5 and v_7 and at most 1 together to both v_9 and v_{10}. Thus, $c^*(v) \geq 4 - 3/2 - 2 \times 3/4 - 1 = 0$.

Suppose now that v has no bad 4-neighbors. If v has five 4-neighbors, then they are (without loss of generality) v_1, v_3, v_5, v_7, v_9 and excluding Q_3 and Q_4 we see that $\deg(v_j) \geq 7$ for $j = 2, 4, 6, 8, 10$. This implies (by the argument as used above) that v sends charge of at most $3/4$ to each 4-neighbor, thus $c^*(v) \geq 4 - 5 \times 3/4 > 0$. Similarly, if v has one 5-neighbor v_1 and four 4-neighbors v_3, v_5, v_7, v_9, then we see as above that v sends charge of at most $3/4$ to each 4-neighbor, and thus $c^*(v) \geq 4 - 4 \times 3/4 - 1/2 > 0$. If v has three 4-neighbors, then the exclusion of Q_2 and Q_4 implies that it has at most two 5-neighbors. Similarly, if v has two 4-neighbors, then it has at most four 5-neighbors. If v has one 4-neighbor, then it has at most five 5-neighbors. If v has no 4-neighbors, it has at most six 5-neighbors. In each case, $c^*(v) \geq 0$.

11^+-vertices: Let v be a d-vertex, with $d \geq 11$. Let $v_1, \ldots, v_d$ be the neighbors of v in cyclic clockwise order, indices modulo d. Suppose that v_i is a bad 4-vertex. Then we may assume that $\deg(v_{i-1}) = 7$ and $\deg(v_{i+1}) = 6$ (or vice versa), since Q_7 is excluded. By noting that the fourth neighbor of v_i has degree 6, we see that $\deg(v_{i+2}) \geq 6$ (since Q_3 is excluded) and $\deg(v_{i-2}) \geq 5$ (since Q_9 is excluded). If v_i is a good 4-vertex, then its neighbors are 6^+-vertices. Now, we redistribute the charge sent from v to its neighbors so that from each bad 4-vertex v_i we give $1/2$ to v_{i-1} and $1/2$ to v_{i+1}, and from each good 4-vertex v_i we give $1/4$ to v_{i-1} and $1/4$ to v_{i+1}. We claim that after the redistribution, each neighbor of v receives from v at most $1/2$ charge in total. This is clear for 4-neighbors of v. A 5-neighbor of v is not adjacent to a 4-vertex since Q_2 is excluded, so it gets charge of at most $1/2$ as well. The claim is clear for each 6-neighbor of v since it is adjacent to at most one 4-vertex (Q_3 is excluded). If a 7-neighbor v_j of v satisfies $\deg(v_{j+1}) = \deg(v_{j-1}) = 4$, the exclusion of Q_9 implies that both v_{j-1} and v_{j+1} are good 4-vertices. Thus, the claim holds for 7-neighbors of v. An 8^+-neighbor of v cannot be adjacent to a bad 4-neighbor of v, and therefore it receives charge of at most $1/2$ from v after the redistribution. This implies that if $d \geq 12$, then the final charge at v is $c^*(v) \geq c(v) - \frac{1}{2}d \geq 0$.

Thus, it remains to consider the case when $d = 11$. In this case the same conclusion as above can be made if we show that either the redistributed charge at one of the vertices v_i is 0, or that there are two vertices whose redistributed charge is at most $1/4$. If there exists a good 4-vertex, then there exists a good 4-vertex v_i, one of whose neighbors, say v_{i-1}, gets $1/4$ total redistributed charge. This is easy to see since $d = 11$ is odd and Q_9 is excluded. Let $t \geq 0$ be the largest integer such that $v_i, v_{i+2}, \ldots, v_{i+2t}$ are all good 4-neighbors of v. Then it is clear that v_{i+2t+1} has total redistributed charge $1/4$ and that $v_{i-1} \neq v_{i+2t+1}$ (by parity). This shows that the total charge sent from v is at most 5, thus the final charge $c^*(v)$ is non-negative. Thus, we may assume that v has no good 4-neighbors. If v has a bad 4-neighbor v_i, then we may assume that $\deg(v_{i-1}) = 7$ and $\deg(v_{i+1}) = 6$. As mentioned above, we conclude that $\deg(v_{i+2}) \geq 6$. We are done if this vertex has 0 redistributed charge. Otherwise, v_{i+2} is adjacent to another bad 4-neighbor v_{i+3} of v. Since $v_i, v_{i+1}, v_{i+2}, v_{i+3}$ do not correspond to the excluded configuration Q_5, we conclude that $\deg(v_{i+2}) = 7$. Now we can repeat the argument with v_{i+3} to conclude that v_{i+6}, v_{i+9} are also bad 4-vertices and $\deg(v_{i+8}) = 7$. However, since $\deg(v_{i-1}) = 7$, we conclude that v_{i+9} cannot be a bad 4-vertex and hence there is a neighbor of v with redistributed charge 0.

Thus, v has no 4-neighbors. Now the only way to send charge $1/2$ to each neighbor of v is that all neighbors of v are 5-vertices. However, in this case we have the configuration Q_4. $\square$

7.3 Reducibility

This section is devoted to the reducibility part of the proof of our main result, Theorem 7.1.3. Let G be a planar graph and L a 3-list-assignment. It is sufficient to prove the theorem when G is a triangulation. Otherwise, we triangulate G and any L-coloring of the triangulation is an L-coloring of G. Of course, we only consider arboreal L-colorings, and we omit the adverb “arboreal” in the sequel.

A configuration C contained in G is called *reducible* if $|C| \leq 9$ and any L-coloring of $G - C$ can be extended to an L-coloring of G in at least two ways. Showing that every triangulation G contains a reducible configuration will imply that G has at least $2^{|V(G)|/9}$ arboreal L-colorings.

Here we prove our main theorem by showing that each configuration from Section 5.2 is reducible. The following lemma will be used throughout this section to prove reducibility.

Lemma 7.3.1. *Let G be a planar graph, L a 3-list-assignment for G, and $v_1, \dots, v_k \in V(G)$. Let $G_i = G - \{v_{i+1}, \dots, v_k\}$ for $i = 0, \dots, k$ and suppose that:*

(1) *for every $i = 1, \dots, k$, $deg_{G_i}(v_i) \leq 5$, and*

(2) *there exists an i such that $deg_{G_i}(v_i) \leq 3$.*

Then every arboreal L-coloring of G_0 can be extended to G in at least two ways. If only (1) *holds, then every arboreal L-coloring of G_0 can be extended to G.*

Proof. Let f be an L-coloring of G_0. Since v_1 has degree at most 5 in G_1, there is a color $c \in L(v_1)$ such that c appears at most once on $N_{G_1}(v_1)$. Therefore, coloring v_1 with c gives an L-coloring of G_1. Repeating this argument, we see that the L-coloring of G_0 can be extended to an L-coloring of G by consecutively L-coloring $v_1, v_2, \dots, v_k$. If (2) holds for i, then there are actually two possible colors that can be used to color v_i. Therefore, every L-coloring of G_0 can be extended to G in at least two ways. □

Lemma 7.3.2. *Configurations $Q_1, \dots, Q_5$, $Q_8, \dots, Q_{12}$, $Q_{16}, \dots, Q_{19}$, Q_{21}, Q_{22} listed in Fig. 7.1 are reducible. The configuration Q'_{23} that is obtained from Q_{23} by deleting the pendant vertex with $\delta(v) = 4$ is also reducible.*

Proof. For these configurations Q_i, Q'_j we simply apply Lemma 7.3.1. The corresponding enumeration $v_1, \dots, v_k$ ($k = |V(Q_i)|$ or $k = |V(Q'_j)|$) is shown in Figure 7.2 and the vertex for which condition (2) of Lemma 7.3.1 applies is shown by a larger circle. □

Lemma 7.3.3. *Configuration Q_6 in Fig. 7.1 is reducible.*

Proof. Let u be the 4-vertex and let u_1, u_2, u_3, u_4 be its neighbors in cyclic order and let C be the cycle $u_1u_2u_3u_4$. Suppose that $\deg(u_1) = \deg(u_2) = 7$, $\deg(u_3) \leq 7$ and $\deg(u_4) = 6$. Let f be an L-coloring of $G - \{u, u_1, u_2, u_3, u_4\}$. Now, consider u_2. If there are at least two ways to extend the coloring f to u_2, then we can obtain at least two different colorings for G by sequentially coloring u_3, u_1, u_4, u using Lemma 7.3.1. Therefore, we may assume that $L(u_2) = \{1, 2, 3\}$ and that colors 1 and 2 each appear exactly twice on $N(u_2)$. Now, let us color u_2 with color 3. We now consider coloring u_1 and u_3. We claim that at least one of u_1 and u_2 must be forced to be colored 3. Otherwise, we color u_1 and u_2 without using color 3, then we color u_4 arbitrarily (this is possible since u is yet uncolored). Now, if $3 \in L(u)$, then we can color u with 3 since u_2 has no neighbor of color 3. Moreover, there is at most one color (other than color 3) that can appear on the neighborhood of u twice. Therefore, u has another available color in its list. Therefore, there are two ways to color u. Similarly, we get two different colorings of u when $3 \notin L(u)$. This proves the claim, and we may assume that $L(u_1) = \{a, b, 3\}$, u_1 is forced to be colored 3, and that the four colored neighbors of u_1 not on C have colors a, a, b, b. Now, we color u_3 arbitrarily with a color c. We may assume that $c \neq 3$, for otherwise we color u_4 arbitrarily and we will have two available colors for u. To complete the proof it is sufficient to show that u_4 can be colored with a color that is not c, for then we could color u with at least two different colors. If u_4 is forced to be colored c, then for every color $x \in L(u_4)$, $x \neq c$, the color x must appear at least twice on $N(u_4)$. This implies that the three colored neighbors of u_4 not on the cycle have colors $3, y, y$, for some color y and that $3, y \in L(u_4)$. But recall that u_1 and u_2 have no neighbors outside C having color 3. Therefore, coloring u_4 with color 3 gives a proper coloring of $G - u$. Now, u can be colored with at least two colors to obtain a coloring of G. □

Lemma 7.3.4. *Let u be a 4-vertex, and suppose u_1, u_2, u_3, u_4 are the neighbors of u in cyclic order. Suppose that* $\deg(u_1) \leq 6$, $\deg(u_2) \leq 7$ *and* $\deg(u_3) \leq 6$*. This configuration is reducible. In particular, the configuration Q_7 in Fig. 7.1 is reducible.*

Proof. Let f be an L-coloring of $G - \{u, u_1, u_2, u_3\}$. Suppose that $f(u_4) = 3$. Let C be the cycle $u_1u_2u_3u_4u_1$. Now, consider u_1. Since only four of u_1's neighbors are colored and $f(u_4) = 3$, we can color u_1 with a color other than 3, say 2. Now, consider coloring u_2. We have two cases. First suppose that it is possible to color u_2 with a color that is not 2. In this case, we color u_2 with a color $x \neq 2$, and then arbitrarily color u_3 with a color y (this is possible since u is not yet colored). Now, if u does not have two colors on $N(u)$, each appearing twice, we have two different available colors in $L(u)$. Therefore, we may assume that $x = 3$ and $y = 2$, and that $2, 3 \in L(u)$. Now, let $z \in L(u)\backslash\{2, 3\}$. Clearly, coloring u with color z gives a proper coloring of G. But by planarity of G, for one of the colors 2 and 3, coloring u with this color will not create a monochromatic cycle since a 2-colored path joining u_1 and u_3 and a 3-colored path from u_2 to u_4 would cross. Therefore, there are two colors available for u.

Next suppose that u_2 is forced to be colored with color 2. Let $L(u_2) = \{a, b, 2\}$. Since u_2 is forced to be colored 2, we have that the four neighbors of u_2 not on C have colors a, a, b, b. Now, consider u_3. We may assume that u_3 is forced to be colored with color 3 for otherwise we could color u with two different colors afterwards. This implies that $2, 3 \in L(u_3)$ and that the three neighbors of u_3 not on C have colors $2, d, d$, where $d \in L(u_3)\backslash\{2, 3\}$. Note that this way we get one coloring extension of f. We need to get another one. Now, since u_3 cannot be colored 2, and u_2 has no neighbor outside C of color 2, it follows that u_1 must have a neighbor of color 2 not on C. Now we color u_3 with color 3. Since u_1 has five colored neighbors and color 2 appears on $N(u)$ at least twice, we may change the color 2 of u_1 to another color in its list. Now, an extension of this coloring to u gives us the second L-coloring. □

Lemma 7.3.5. *A configuration consisting of an 8-vertex that is adjacent to at least three 4-vertices (configuration Q_{13}) is reducible.*

Proof. Let v_1 be an 8-vertex and suppose v_2, v_3, v_4 are 4-vertices adjacent to v_1. Let C be the cycle on the neighbors of v_1. Let $L(v_1) = \{1, 2, 3\}$. Consider a 3-coloring f of $G_1 = G - \{v_2, v_3, v_4\}$. We may assume that v_1 is colored 3. Since every L-coloring of G_1 extends to G, we may assume that v_1 cannot be recolored. Thus, (at least) two of its neighbors are colored 1 and two are colored 2. If no neighbor of v_1 is colored 3, then we can extend the coloring to v_2 in two ways since the vertex v_1 cannot be part of a monochromatic cycle. Thus, color 3 appears exactly once on $N(v_1)$ and colors 1 and 2 appear precisely twice. It is also clear that $3 \in L(v_2) \cap L(v_3) \cap L(v_4)$. Let v_5 be the neighbor of v_1 with $f(v_5) = 3$. Without loss of generality, v_5 is not the neighbor of v_2 on the cycle C. If v_2 has no neighbor colored 3 except v_1, then we may extend the coloring f to $G - \{v_3, v_4\}$ in at least two ways. We can then extend these colorings to G. Therefore, we may assume that v_2's neighbor distinct from its neighbors on the cycle C is colored 3. Now, v_2's neighbors on the cycle both have the same color for otherwise we can extend f to $G_2 = G - \{v_3, v_4\}$ in at least two ways. Therefore, we may assume that the neighbors of v_2 on the cycle are colored 1 and that $1 \in L(v_2)$. But now, by planarity, coloring v_2 by either 1 or 3 gives a proper L-coloring of G_2. Coloring v_2 with the other remaining color in its list gives a second coloring of G_2. Both of these colorings can be then extended to G. Therefore, there are at least two ways to extend a coloring of G_1 to G. $\square$

Lemma 7.3.6. *Let u be an 8-vertex and assume its neighbors (in the clockwise cyclic order) are $u_1, \ldots, u_8$ and let C be the 8-cycle $u_1u_2\ldots u_8u_1$. Suppose that u is adjacent to two 4-vertices, one 5-vertex, and a 6-vertex that is adjacent to either the 5-vertex or one of the two 4-vertices on C. Then this configuration (Q_{14} or Q_{15}) is reducible.*

Proof. Suppose that $\deg(u_i) = \deg(u_j) = 4$, $\deg(u_k) = 5$ and $\deg(u_l) = 6$, where $i, j, k, l \in \{1, \ldots, 8\}$ and $i \neq j$. First assume that u_k and u_l are adjacent on C. We may assume $l = k+1$. Let $L(u) = \{1, 2, 3\}$ and consider an L-coloring f of $G - \{u, u_i, u_j, u_k, u_l\}$. Without loss of generality, we may assume that colors 1 and 2 each appear exactly twice on $N(u)$ in the coloring f. Otherwise, there are two ways to extend the coloring f of $G - \{u, u_i, u_j, u_k, u_l\}$ to a coloring of $G - \{u_i, u_j, u_k, u_l\}$, and applying Lemma 7.3.1 we can extend each of these to a coloring of G. Therefore, color 3 does not appear in the neighborhood of u in the coloring f. We color u with color 3 to obtain a coloring g of $G - \{u_i, u_j, u_k, u_l\}$. Now, consider the 6-vertex u_{k+1}. Since u_{k+1} has only five colored neighbors so far, we have at least one available color for it from its list $L(u_{k+1})$. If $3 \notin L(u_{k+1})$ we color u_{k+1} arbitrarily with an available color. If $3 \in L(u_{k+1})$, we color u_{k+1} with 3 if color 3 does not appear on $N(u_{k+1})\backslash\{u\}$. If color 3 appears on $N(u_{k+1})\backslash\{u\}$, we color u_{k+1} with any other color in its list except 3 (this is possible since the remaining three colored neighbors of u_{k+1} can forbid only one additional color from $L(u_{k+1})$). Now, consider one of the 4-vertices, say u_i. We may assume that $u_i \neq u_{k+2}$, otherwise we consider u_j. First, assume that $3 \notin L(u_i)$. Since u_i has only three colored neighbors and u is colored 3, there are at least two available colors in $L(u_i)$ that can be used to color u_i. Each coloring then can be extended to a coloring of G by Lemma 7.3.1. Therefore, we may assume that $3 \in L(u_i)$. Recall that no neighbor of u, except possibly u_{k+1}, is colored 3. Therefore, u_i can be colored with color 3 without creating a monochromatic cycle of color 3, since any such cycle must use the vertex u_{k+1}, and by assumption if u_{k+1} is colored 3, it has no neighbor except u that is colored 3. Therefore, the four colored neighbors of u_i can forbid at most one color from $L(u_i)$, which implies that we can color u_i with two different colors. Now, applying Lemma 7.3.1 to $G - \{u_k, u_j\}$, we see that each of these two colorings can be extended to a coloring of G.

Next, assume that u_l and u_j are adjacent on C. We may assume that $u_l = u_{j+1}$. If $u_i \neq u_{j+2}$, then the above proof works also in this case. Thus, we have $u_i = u_{j+2}$. However, in this case we can use Lemma 7.3.1 (with $v_1 = u_i, v_2 = u, v_3 = u_{j+1}, v_4 = u_j, v_5 = u_k$), where property (2) applies for v_1. □

Lemma 7.3.7. *A configuration consisting of a* 9*-vertex adjacent to at least three* 4*-vertices and at least one other* 5^-*-vertex is reducible. In particular,* Q_{20} *is reducible.*

Proof. Let v_1 be a 9-vertex and suppose v_2, v_3, v_4 are 4-vertices and v_5 is a 5-vertex adjacent to v_1. Let $L(v_1) = \{1, 2, 3\}$. Consider an L-coloring f of $G_1 = G - \{v_1, \ldots, v_5\}$. This coloring can be extended to v_1 and henceforth to $v_2, \ldots, v_5$ by Lemma 7.3.1. We may assume that v_1 is forced to be colored 3; otherwise we are done. This implies that each of colors 1 and 2 appear on $N(v_1)$ at least twice. If color 3 does not occur on $N(v_1)$, then we can extend f to v_2 in at least two ways since color 3 does not give any restriction on the extension to v_2, and the remaining three neighbors of v_2 prevent at most one color to be used. Therefore, we may assume that colors 1 and 2 appear exactly twice and color 3 appears exactly once on $N(v_1)$ in the coloring f. Let v_6 be the neighbor of v_1 with $f(v_6) = 3$. Without loss of generality, v_2 is not contained in the triangular faces containing the edge v_1v_6. Let v_7, v_8 be the common neighbors of v_1 and v_2. If v_2 has no neighbor of color 3 except v_1, or if $3 \notin L(v_2)$, then we can extend the coloring f to v_2 in at least two ways. We can then extend these colorings to G. Therefore, we may assume that the neighbor of v_2 distinct from v_1, v_7, v_8 is colored 3. Now, v_7 and v_8 both have the same color, for otherwise we can extend f to v_2 in at least two ways. This implies that we may assume that v_7 and v_8 are colored 1 and that $1 \in L(v_2)$. But now, by planarity, coloring v_2 by 1 or 3 gives rise to a proper coloring since a path joining v_7 and v_8 colored 1 and a path colored 3 joining v_1 and the fourth neighbor of v_2 would cross. Coloring v_2 with the other remaining color in its list gives another extension of f. Both of these colorings can be then extended to G by Lemma 7.3.1. This shows that the considered configuration is reducible. □

7.4 Proof of the main theorem

It is easy to see that every plane graph is a spanning subgraph of a triangulation; we can always add edges joining distinct nonadjacent vertices until we obtain a triangulation.

Proof of Theorem 7.1.3. The proof is by induction on the number of vertices, $n = |G|$. We may assume that G is a triangulation. By Theorem 7.2.1 and Lemmas 7.3.2–7.3.7, G contains a reducible configuration C on at most $k \leq 9$ vertices. By the induction hypothesis, $G - V(C)$ has at least $2^{(n-k)/9}$ arboreal L-colorings. Since C is reducible, each of these colorings extends to G in at least two ways, giving at least $2 \times 2^{(n-k)/9} \geq 2^{n/9}$ arboreal L-colorings in total. □

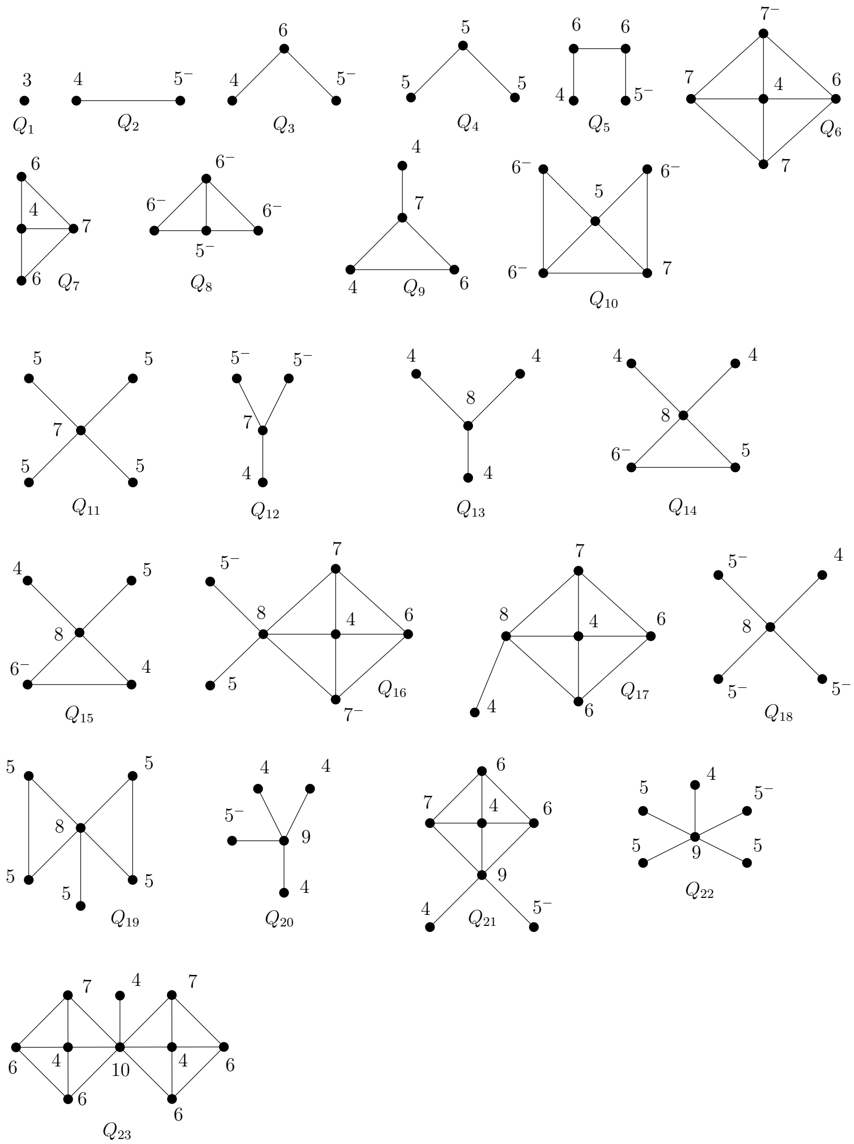


Figure 7.1: Unavoidable configurations. The listed numbers refer to the degree function δ, and the notation d^- at a vertex v means all such configurations where the value $\delta(v)$ is either d or $d-1$.

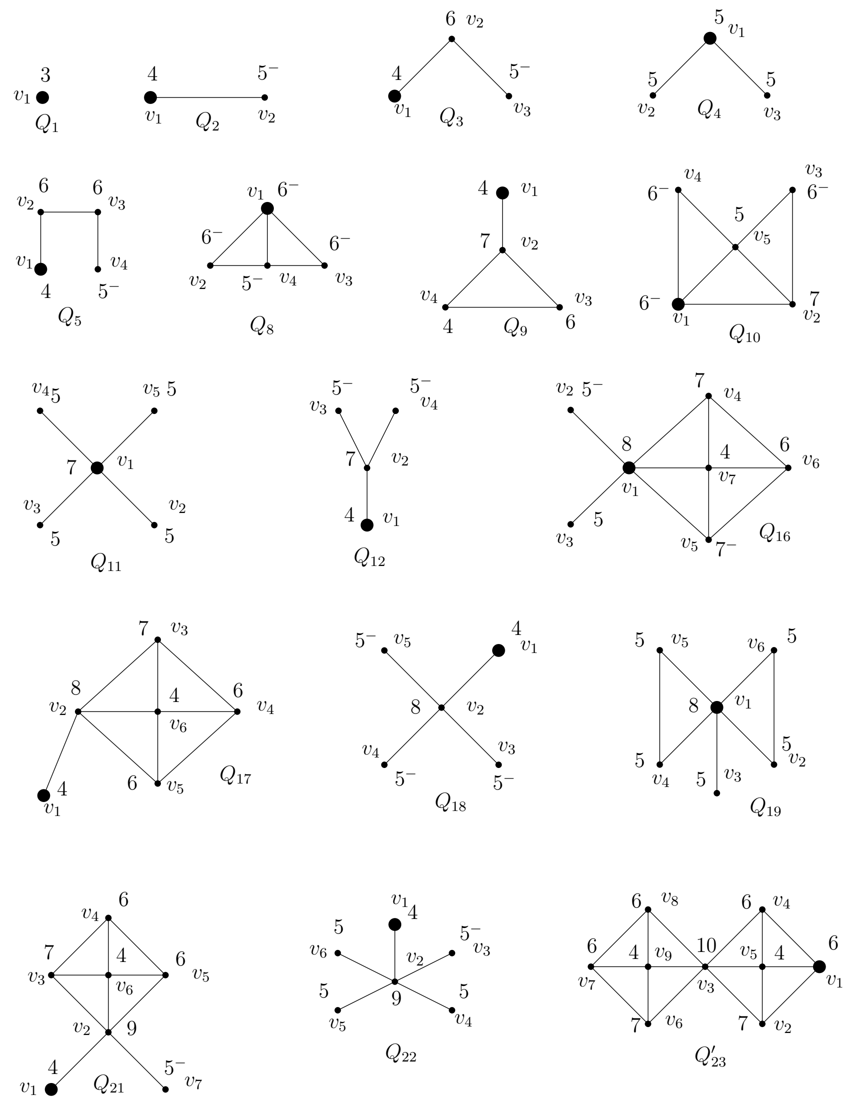


Figure 7.2: Lemma 7.3.1 applies to several configurations.

Chapter 8

Conclusion and Future Work

In this chapter, we outline possible directions for future research.

8.1 Upper bounds on $\chi(D)$ in terms of $\Delta(D)$

In the thesis, we proved that every digon-free digraph D has $\chi(D) \le (1-e^{-13})\tilde{\Delta}$. A natural question is if the proof of the theorem can be extended to list colorings, i.e. is it true that $\chi_l(D) \le (1-e^{-13})\tilde{\Delta}$? The main difficulty seems to be in showing that the concentration inequalities would still hold. We also mentioned that the constant $1-e^{-13}$ is probably not best possible and conjecture the following.

Conjecture 8.1.1. *Let D be a digon-free digraph. Then*

$$\chi(D) \le \left\lceil \frac{\tilde{\Delta}}{2} \right\rceil + 1.$$

We may also generalize the above mentioned conjecture to list colorings. In general, we believe that the asymptotic order of the chromatic number of a digon-free digraph should be $\tilde{\Delta}/\log\tilde{\Delta}$, as conjectured by Mohar and McDiarmid [44].

Conjecture 8.1.2. *Every digraph D without digons has $\chi(D) = O(\frac{\tilde{\Delta}}{\log\tilde{\Delta}})$.*

Again, we believe this should generalize to list colorings as well. As mentioned in Chapter 2, the conjecture is best possible due to results known about tournaments (see Theorem 2.3.6).

8.2 The planar digraph conjecture

The main conjecture concerning colorings of planar digraphs is that every digon-free planar digraph is 2-colorable, see Conjecture 2.4.1. Since this is likely to be difficult, we pose the following relaxations of this conjecture.

Conjecture 8.2.1. *Let D be a digon-free planar digraph such that all cycles of length 3 (directed or otherwise) are vertex-disjoint. Then $\chi(D) \leq 2$.*

The above conjecture is a weakening of a conjecture for vertex-arboricity of graphs posed by Raspaud and Wang [55] in 2008. The authors conjectured that every planar graph where all triangles are vertex-disjoint have vertex-arboricity at most 2. Clearly, this conjecture would imply Conjecture 8.2.1 since a 2-coloring of a graph G gives a 2-coloring of digraph D, where D is obtained by arbitrarily orienting the edges of G. We can also try weakening Conjecture 2.4.1 by forbidding directed cycles of certain lengths.

Conjecture 8.2.2. *Let D be a planar digraph with digirth at least 4. Then $\chi(D) \leq 2$.*

If we also forbid non-directed triangles of length three then the above conjecture follows trivially. This follows from the fact that every triangle-free planar graph has a vertex of degree at most three. This implies that the digraph has a vertex v with $\min\{d^+(v), d^-(v)\} \leq 1$. Now, applying induction on $D - v$ we get that D is 2-colorable.

One can also state the following weakening of Conjecture 8.2.2.

Conjecture 8.2.3. *There exists a k such that every planar digraph with digirth at least k is 2-colorable.*

8.3 The relationship between $\chi(G)$ and $\chi(D)$

Recall that every graph G has an orientation D of edges so that $\chi(D) = 1$. Neumann-Lara asked the question whether there is an orientation D of the edges of G so that $\chi(D)$ is large. Clearly, $\chi(D)$ is always bounded above by $\chi(G)$ for any orientation D. Neumann-Lara conjectured the following.

Conjecture 8.3.1. *For every $k \geq 1$ there exists an $r = r(k)$ so that every graph G with $\chi(G) \geq r$ has an orientation of edges D such that $\chi(D) \geq k$.*

The conjecture clearly holds for $k = 1$. For $k = 2$, it is easy to observe that we may take $r(k) = 3$. However, we do not even know if the conjecture holds for $k = 3$. Conjecture 8.3.1 also holds for complete graphs where $r = \theta(k \log k)$ as can be seen from the discussion of the chromatic number of tournaments in Chapter 2.

8.4 Chromatic polynomial for digraphs

An interesting notion is the *chromatic polynomial of a digraph*. For a (undirected) graph G and positive integer x, $P(G; x)$ is defined to be the number of colorings of G with x colors. Note that we distinguish between colorings with the same color class partitions where the colors of the classes are different. It can be shown that $P(G; x)$ is a polynomial in x for every graph G. Hence, $P(G; x)$ is called the *chromatic polynomial of the graph G*. One of the main motivations for studying $P(G; x)$ is that $\chi(G) = \min\{k : P(G; k) > 0\}$. For every graph G, the chromatic polynomial is known to admit the following nice recurrence.

Theorem 8.4.1. *Let G be a graph and $uv \in E(G)$. Then*

$$P(G; x) = P(G - uv; x) - P(G/uv; x), \tag{8.1}$$

where G/uv is the graph obtained from G by deleting the edge uv and identifying vertices u and v.

One can similarly define a chromatic polynomial for a digraph. Given a digraph D and a positive integer x, we can define $P(D; x)$ to be the number of colorings of D with x colors. One can derive that $P(D; x)$ is a polynomial in x from results on hypergraph polynomials. Note that a *hypergraph* $\mathcal{H} = (X, \mathcal{E})$ is a set X of elements called vertices and a set $\mathcal{E}$ of subsets of X called *hyperedges* or simply edges. Note that graphs are precisely those hypergraphs where each hyperedge contains two elements. The chromatic polynomials have been studied in the setting of hypergraphs. The following theorem is from [19].

Theorem 8.4.2. *Given a hypergraph $\mathcal{H} = (X, \mathcal{E})$, where $X = \{v_1, ..., v_n\}$ and $\mathcal{E} = \{e_1, ..., e_m\}$, let $f_{x_1,...,x_m}(\mathcal{H}, \lambda)$ denote the number of different λ-colorings of $\mathcal{H}$ satisfying the condition that in each edge e_i there appear at least x_i different colors. Then f is a polynomial in λ.*

The case where all $x_i = 2$ has been proved earlier in [18], along with an inclusion-exclusion recurrence. Note that digraph coloring can be formulated as a hypergraph coloring, where the vertices of the hypergraph are the vertices of the digraph and the hyperedges are all the sets of vertices which contain a directed cycle in the digraph. Then Theorem 8.4.2 with $x_i = 2$ for all i immediately implies the following.

Proposition 8.4.3. *$P(D; x)$ is a polynomial in x for every digraph D.*

It is easy to compute the chromatic polynomial of the following digraphs.

Proposition 8.4.4. *Let D be an acyclic digraph of order n. Then $P(D; x) = x^n$.*

Proof. Every assignment of colors to vertices of D is a proper coloring and there are x^n color assignments. □

Proposition 8.4.5. *Let $D = \vec{C_n}$, the directed cycle of length n. Then $P(D; x) = x^n - x$.*

Proof. Every assignment of colors to vertices of D is a proper coloring except those where all vertices are assigned the same color in $\{1, 2, ..., x\}$. There are x such assignments. Since the total number of color assignments is x^n the result follows. □

We can also express the chromatic polynomial of a digraph in terms of the strongly connected components.

Proposition 8.4.6. *Let D be a digraph and let $D_1, D_2, ..., D_k$ be the strongly connected components of D. Then*

$$P(D; x) = \prod_{i=1}^{k} P(D_i; x).$$

Proof. Note that two strongly connected components D_i and D_j, $i \neq j$, do not share a vertex in common for otherwise D_i and D_j would form a single component. Therefore, $P(D; x) \leq \prod_{i=1}^{k} P(D_i; x)$. Now, for $i = 1, ..., k$, let π_i be an x-coloring of D_i. We claim that $\pi = \cup_{i=1}^{k} \pi_i$ is an x-coloring of D. If not, then there is a monochromatic directed cycle that uses vertices of at least two components D_i and D_j. But this implies that the block digraph of strongly connected components of D is not acyclic, a contradiction. □

In general, it seems difficult to get a recursive formula for $P(D; x)$ similar to 8.1. However, we can show the following.

Proposition 8.4.7. *Let C be a directed cycle in a digraph D such that no edges of C appear in any other cycle of D. Then*

$$P(D; x) = P(D - E(C); x) - P(D/C; x),$$

where D/C is the digraph obtained from D by deleting $E(C)$ and identifying all vertices of C.

Proof. Note that, by assumption, C cannot have any chords. Since no arc of C appears in any other cycle, it follows that every x-coloring of $D - E(C)$ is an x-coloring of D except for those colorings which have vertices of C colored with the same color. Thus, it is easy to see that the number of x-colorings of $D - E(C)$ where the set $V(C)$ is monochromatic is $P(D/C; x)$. □

We can also reduce vertices of low degree.

Proposition 8.4.8. *Let v be a vertex with $\min\{d^+(v), d^-(v)\} = 0$ in a digraph D. Then $P(D; x) = xP(D - v; x)$.*

Proof. Note that $D_1 = v$ is a strongly connected component of D. The result now follows by Proposition 8.4.6. □

Proposition 8.4.9. *Let D be a digraph and v a vertex with $d^+(v) = d^-(v) = 1$. Let w be the in-neighbor of v and u be the out-neighbor of v, where $w \neq u$. Let D' be the digraph with $V(D') = V(D)\backslash\{v\}$ and $E(D') = E(D - v) \cup \{wu\}$. Then*

$$P(D; x) = P(D'; x) + (x - 1)P(D - v; x).$$

Proof. We first claim that the number of k-colorings of $D - v$ with no monochromatic uw-path (i.e. a path starting at u and ending at w) is $P(D'; x)$. Note that every coloring of $D - v$ with no monochromatic uw-path is a coloring of D' and no proper coloring of D' can have a monochromatic uw-path. This establishes the claim. Next, note that the number of colorings of $D - v$ with a monochromatic uw-path is $P(D - v; x) - P(D'; x)$. Now, a coloring of $D - v$ where there is no monochromatic path from u to w can be extended to a coloring of D in x ways by coloring v arbitrarily. On the the other hand, a coloring of $D - v$ with a monochromatic uw-path can be extended to D by coloring v with one of the $x - 1$ colors that do not appear on u and w. Thus, we have

$$\begin{aligned} P(D;x) &= xP(D';x) + (x-1)[P(D-v;x) - P(D';x)] \\ &= P(D';x) + (x-1)P(D-v;x). \end{aligned}$$

□

8.4.1 The chromatic polynomial and planar digraphs

We mentioned that for a digraph D, $P(D;x)$ is closely related to $\chi(D)$. Note that Conjecture 2.4.1 is equivalent to stating that for every digon-free planar digraph D, we have $P(D;2) > 0$. Since this conjecture seems difficult, we may try to find a lower bound on $P(D;3)$. Since every digon-free planar digraph is 3-colorable (see Chapter 2), we know that $P(D;3) > 0$. In Chapter 7, we proved that $P(D;3) \geq 2^{n/9}$, where n is the order of D. On the other hand, we believe that $P(D;2)$ should be finite for any digon-free digraph D.

Conjecture 8.4.10. *There exists a constant C such that $P(D;2) < C$ for any digon-free digraph D.*

Conjecture 8.4.10 seems to be difficult. In fact, it does not seem to be easy even if we replace C by a polynomial in $|V(D)|$.

8.5 Hedetniemi's Conjecture for digraphs

In this section, we propose an analog of Hedetniemi's Conjecture for digraphs. First, we need a definition. Given graph G and H, the *direct product* $G \times H$ of G and H is the graph with vertex set $V(G \times H) = V(G) \times V(H)$ where two vertices (u, u') and (v, v') are adjacent if and only if u is adjacent with v and u' is adjacent with v'. Hedetniemi's conjecture states:

Conjecture 8.5.1 (Hedetniemi's Conjecture)**.** *Let G and H be simple graphs. Then*

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

It is easy to verify that $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$ by simply considering the natural homomorphism projections $G \times H \to G$ and $G \times H \to H$. Aside from small values of $\chi(G)$ and $\chi(H)$, Hedetniemi's conjecture is largely open. Zhu [65] generalized the conjecture to circular colorings.

Conjecture 8.5.2 (Zhu's Conjecture)**.** *Let G and H be simple graphs. Then*

$$\chi_c(G \times H) = \min\{\chi_c(G), \chi_c(H)\}.$$

One can also ask if Hedetniemi's conjecture generalizes to digraphs. Given digraphs D and D', the *direct product* $D \times D'$ of D and D' is the digraph with vertex set $V(D \times D') = V(D) \times V(D')$ where there is an arc from vertex (u, u') to vertex (v, v') if and only if uv is an arc in D and $u'v'$ is an arc in D'.

We conjecture that the following analog of Hedetniemi's conjecture holds.

Conjecture 8.5.3. *Let D and D' be simple digraphs. Then*

$$\chi(D \times D') = \min\{\chi(D), \chi(D')\}.$$

It is easy to see that $\chi(D \times D') \leq \min\{\chi(D), \chi(D')\}$.

Proposition 8.5.4. *Conjecture 8.5.3 holds if* $\min\{\chi(D), \chi(D')\} \leq 2$.

Proof. Suppose that $\min\{\chi(D), \chi(D')\} = 1$. We may assume that $\chi(D) = 1$ and it follows that D is acyclic. Note that $D \times D'$ cannot contain a cycle for otherwise the projection of the cycle onto D would be a cycle in D. Therefore, $\chi(D \times D') = 1$.

Next, suppose that $\min\{\chi(D), \chi(D')\} = 2$. Let $C = v_1 v_2 ... v_k v_1$ be a directed cycle in D and $C' = u_1 u_2 ... u_l$ be a directed cycle in D'. Then clearly the walk $(v_1, u_1)(v_2, u_2)...$ will eventually (after $lcm(k, l)$ steps) reach (v_1, u_1). Therefore, $D \times D'$ contains a cycle and hence $\chi(D \times D') = 2$. □

Appendix A

Probabilistic Preliminaries

Here we present all the probabilistic tools used in the thesis. The results presented here can be found in [4] and [47]. The most fundamental property used in probabilistic analysis is the linearity of expectation.

Theorem A.0.5 (Linearity of Expectation)**.** *Let* $X_1, X_2, ..., X_l$ *be random variables. Then*

$$\mathbb{E}[\sum_{i=1}^{l} X_i] = \sum_{i=1}^{l} E[X_i].$$

A.1 The First Moment Method

The first moment method can essentially be summarized as follows.

Theorem A.1.1 (The First Moment Principle)**.** *Let* X *be a random variable. If* $\mathbb{E}[X] \leq t$ *then* $\mathbb{P}[X \leq t] > 0$.

The following inequality is frequently used in probabilistic analysis.

Theorem A.1.2 (Markov's Inequality)**.** *For any positive random variable* X*,*

$$\mathbb{P}[X \geq t] \leq \frac{\mathbb{E}[X]}{t}.$$

If X is positive and integer-valued, Markov's inequality implies the following:

Theorem A.1.3. $\mathbb{P}[X > 0] \leq \mathbb{E}[X]$.

The first moment method allows us to bound from above the probability that a random variable is large by computing its expected value. The power of the method relies in the fact that expected values are usually straightforward to compute due to linearity of expectation. In the next section we discuss bounding from above the probability that a random variable is small.

A.2 The Poisson Paradigm

When we have a random variable X that depends on many rare occurring and mostly independent indicator random variables we would like to say that X has distribution that is close to Poisson. In particular, we would like to say that $\mathbb{P}[X = 0] \approx e^{-\mu}$ where μ is the expectation of X. In this section, we present inequalities that achieve this.

A.2.1 The Janson Inequalities

Given a set of bad events B_i, $i \in I$, each of which has a small probability of occurring, we would like to say that $\mathbb{P}[\cap_{i\in I}\bar{B}_i]$ is small. If the events B_i are mutually independent of each other then this indeed is the case since $\mathbb{P}[\cap_{i\in I}\bar{B}_i] = \prod_{i\in I}\mathbb{P}[\bar{B}_i]$. Janson Inequalities (see Chapter 8, [4]) to make a similar claim if the B_i are "almost" independent.

Let Ω be a finite universal set and let R be a random subset of Ω constructed as follows. For each $r \in \Omega$, we put $r \in R$ with some probability p_r, independently. Let A_i, $i \in I$ be subsets of Ω, where I is a finite index set. Let B_i be the event that $A_i \subseteq R$. That is, B_i is the event that all the elements of A_i "won" their random coin flips and were put in R. Let X_i be the indicator random variable for B_i, i.e. $X_i = 1$ if the event B_i occurred and 0 otherwise. Set $X = \sum_{i\in I} B_i$. Note that X counts the number of events B_i that occur, and therefore, $\mathbb{P}[X = 0] = \mathbb{P}[\cap_{i\in I}\bar{B}_i]$. For $i, j \in I$, we write $i \sim j$ if $i \neq j$ and $A_i \cap A_j \neq \emptyset$. We define

$$\Delta = \sum_{i\sim j}\mathbb{P}[B_i \cap B_j].$$

The sum above is over all ordered pairs. Note that if $i \neq j$ and not $i \sim j$, then the events B_i and B_j are independent. This means that Δ is a kind of total measure of mutual dependence of the B_i. Finally, we set

$$\mu = \mathbb{E}[X] = \sum_{i\in I}\mathbb{P}[B_i].$$

Theorem A.2.1 (The Janson Inequality). *Let B_i, $i \in I$, Δ and μ be as above. Then*

$$\mathbb{P}[X = 0] = \mathbb{P}[\cap_{i \in I} \bar{B}_i] \leq e^{-\mu + \frac{\Delta}{2}}.$$

Note that when $\Delta \geq 2\mu$, the bound in Theorem A.2.1 is useless. Fortunately, the following extension can often be applied.

Theorem A.2.2 (The Extended Janson Inequality). *Under the assumptions of Theorem A.2.1 and the further assumption that $\Delta \geq \mu$,*

$$\mathbb{P}[X = 0] = \mathbb{P}[\cap_{i \in I} \bar{B}_i] \leq e^{-\frac{\mu^2}{2\Delta}}.$$

A.3 The Lovász Local Lemma

If we have n mutually independent events each of which holds with a positive probability p, we know that all events hold simultaneously with probability $p^n > 0$. The local lemma generalizes this statement to events which are only locally dependent.

Theorem A.3.1 (The Local Lemma). *Let $A_1, ..., A_n$ be events in an arbitrary probability space. A directed graph $D = (V, E)$ on the set of vertices $V = \{1, 2, ..., n\}$ is called a dependency digraph for the events $A_1, ..., A_n$, if for each i, $1 \leq i \leq n$, the event A_i is mutually independent of all the events $\{A_j : (i, j) \notin E\}$. Suppose that $D = (V, E)$ is a dependency digraph for the above events and suppose there are real numbers $x_1, ..., x_n$ such that $0 \leq x_i < 1$ and $\mathbb{P}[A_i] \leq x_i \prod_{(i,j) \in E}(1 - x_j)$ for all $1 \leq i \leq n$. Then*

$$\mathbb{P}[\cap_{i=1}^{n} \bar{A}_i] \geq \prod_{i=1}^{n}(1 - x_i).$$

In particular, with positive probability no event A_i holds.

In practice, the following version is usually the most useful.

Theorem A.3.2 (The Local Lemma; Symmetric Version). *Let $A_1, ..., A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all the other events A_j but at most d, and that $\mathbb{P}[A_i] \leq p$ for all $1 \leq i \leq n$. If $4pd \leq 1$, then $\mathbb{P}[\cap_{i=1}^{n} \bar{A}_i] > 0$.*

A.4 Concentration Inequalities

The first moment method states that a random variable X is at most $\mathbb{E}[X]$ with positive probability. Often we would like to show that X is very close to $\mathbb{E}[X]$ with very high probability. If this is the case, we say that X is *concentrated*. Concentration inequalities are widely used in probabilistic combinatorics. The most common example of a concentrated random variable is the binomial which is defined as follows. Suppose $X = \sum_{i=1}^{n} X_i$, where each X_i is a random variable that takes value 1 with probability p and 0 otherwise. If the X_i are all mutually independent, we say that X is a binomial random variable and write it as $X = BIN(n, p)$. The following theorem, due to Chernoff, shows that the binomial random variable $BIN(n, p)$ is strongly concentrated around its mean np.

Theorem A.4.1 (Chernoff Bound). *For any $0 \le t \le np$,*

$$\mathbb{P}[|BIN(n,p) - np| > t] < 2e^{-t^2/3np}.$$

Chernoff type bounds generalize to other random variables which are functions of independent trials. The following theorem (see [47]) is an example of one such generalization.

Theorem A.4.2 (Simple Concentration Bound). *Let X be a random variable determined by n independent trials $T_1, ..., T_n$, and satisfying the property that changing the outcome of any single trial can affect X by at most c. Then*

$$\mathbb{P}[|X - \mathbb{E}[X]| > t] \le 2e^{-\frac{t^2}{2c^2 n}}.$$

Typically, c in Theorem A.4.2 is a constant not depending on n and t is a constant fraction of $\mathbb{E}[X]$. Therefore, the above bound is generally good when $\mathbb{E}[X] = \Omega(n)$. Fortunately, under some additional conditions we can still get strong concentration of X even if $\mathbb{E}[X] = o(n)$. This can be achieved by Talagrand's Inequality. The original inequality yields a concentration around the median $Med(X)$ of a random variable X.

Theorem A.4.3 (Talagrand's Inequality (Median)). *Let X be a nonnegative random variable, not equal to 0, which is determined by n independent trials, $T_1, \ldots, T_n$ and satisfies the following conditions for some $c, r > 0$:*

1. *Changing the outcome of any single trial can affect X by at most c.*

2. *For any s, if $X \ge s$, there are at most rs trials whose outcomes certify that $X \ge s$.*

Then for any $0 \le t \le Med(X)$,

$$\mathbb{P}\left[|X - Med(X)| > t\right] \le 4e^{-\frac{t^2}{8c^2 r Med(X)}}.$$

To be clear, condition 2 states that for any s, there is a set of trials $T_{i_1}, ..., T_{i_{f(s)}}$ for some $f(s) \le rs$ so that changing the outcomes of all the other trials cannot cause X to be less than s. In other words, showing the outcome of the trials $T_{i_1}, ..., T_{i_{f(s)}}$ is sufficient to demonstrate that $X \ge s$.

A problem with Theorem A.4.3 is that medians are often difficult to compute and therefore the inequality may not be easy to apply. Fortunately, there exists the following version of the inequality that replaces $Med(X)$ with $\mathbb{E}[X]$.

Theorem A.4.4 (Talagrand's Inequality (Mean))**.** *Let X be a nonnegative random variable, not equal to 0, which is determined by n independent trials, $T_1, \ldots, T_n$ and satisfies the following conditions for some $c, r > 0$:*

1. *Changing the outcome of any single trial can affect X by at most c.*

2. *For any s, if $X \ge s$, there are at most rs trials whose outcomes certify that $X \ge s$.*

Then for any $0 \le t \le \mathbb{E}[X]$,

$$\mathbb{P}\left[|X - \mathbb{E}[X]| > t + 60c\sqrt{r\mathbb{E}[X]}\right] \le 4e^{-\frac{t^2}{8c^2 r \mathbb{E}[X]}}.$$

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