ON THE PSYCHOLOGICAL AND PHYSIOLOGICAL FOUNDATIONS OF STRUCTURE IN GEOMETRY: A STUDY IN EDUCATIONAL NEUROSCIENCE

by

Kerry Handscomb
B.Sc. (First) University of London, 1982
B.Ed. University of British Columbia, 2001
M.Sc. Simon Fraser University, 2005

DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

In the
Faculty of Education

© Kerry Handscomb 2010

SIMON FRASER UNIVERSITY

Summer 2010

All rights reserved. However, in accordance with the *Copyright Act of Canada*, this work may be reproduced, without authorization, under the conditions for *Fair Dealing*. Therefore, limited reproduction of this work for the purposes of private study, research, criticism, review and news reporting is likely to be in accordance with the law, particularly if cited appropriately.

APPROVAL

Name:	**Kerry Handscomb**
Degree:	**Doctor of Philosophy**
Title of Thesis:	**On the Psychological and Physiological Foundations of Structure in Geometry: A Study in Educational Neuroscience**

Examining Committee:

Chair: Peter Liljedahl, Assistant Professor

Stephen Campbell, Associate Professor
Senior Supervisor

Nathalie Sinclair, Assistant Professor
Committee Member

Nicholas Jackiw, Adjunct Professor
Committee Member

Teleconference from Malaysia

Allan MacKinnon, Associate Professor
Internal/External Examiner

Carol MacLeod
Anthropology Department, Langara College
External Examiner

Date Defended/Approved: April 27, 2010

SFU SIMON FRASER UNIVERSITY
LIBRARY

Declaration of Partial Copyright Licence

The author, whose copyright is declared on the title page of this work, has granted to Simon Fraser University the right to lend this thesis, project or extended essay to users of the Simon Fraser University Library, and to make partial or single copies only for such users or in response to a request from the library of any other university, or other educational institution, on its own behalf or for one of its users.

The author has further granted permission to Simon Fraser University to keep or make a digital copy for use in its circulating collection (currently available to the public at the “Institutional Repository” link of the SFU Library website <www.lib.sfu.ca> at: <http://ir.lib.sfu.ca/handle/1892/112>) and, without changing the content, to translate the thesis/project or extended essays, if technically possible, to any medium or format for the purpose of preservation of the digital work.

The author has further agreed that permission for multiple copying of this work for scholarly purposes may be granted by either the author or the Dean of Graduate Studies.

It is understood that copying or publication of this work for financial gain shall not be allowed without the author’s written permission.

Permission for public performance, or limited permission for private scholarly use, of any multimedia materials forming part of this work, may have been granted by the author. This information may be found on the separately catalogued multimedia material and in the signed Partial Copyright Licence.

While licensing SFU to permit the above uses, the author retains copyright in the thesis, project or extended essays, including the right to change the work for subsequent purposes, including editing and publishing the work in whole or in part, and licensing other parties, as the author may desire.

The original Partial Copyright Licence attesting to these terms, and signed by this author, may be found in the original bound copy of this work, retained in the Simon Fraser University Archive.

Simon Fraser University Library
Burnaby, BC, Canada

Last revision: Spring 09

ABSTRACT

Perception has structure. Aspects of this structure are relevant for image-based geometrical objects and relations between them, referred to as schematic perception and inferencing, respectively. Perception of geometrical structure, is a specific cognitive function. Without direct perception of structure mathematical reasoning may be inefficient and inaccurate. It is important for mathematics educators to understand the nature of schematic perception and to identify ways in which it can be nurtured in students. The main focus of this thesis is a specific aspect of image-based geometrical reasoning, the schematic nature of geometrical diagrams. The research framework is educational neuroscience. Selected results from mathematics education research pertaining to geometrical reasoning are constrained and informed by selected results from the neurosciences pertaining to the cerebral cortex and cerebellum, and vice versa. These two epistemological domains are integrated coherently with a theoretical framework that draws on embodied cognition and the neutral monism of Spinoza. A cognitive network model of the cerebral cortex enables concepts to be understood in an extensional (i.e., generalized) sense. It may also permit an explication of the distinction between procedural reasoning and conceptual reasoning and a re-evaluation of mathematics education theories of concept formation. However, the extensional concepts of the cerebral cortex are too inexact for mathematical application. I argue that a functional role of the cerebellum is to schematize these extensional concepts of the cerebral cortex, and then these schematic concepts may be understood in an intensional (i.e., abstracted) sense. I suggest there are implications for mathematics education theories of abstraction and generalization. I present the hypothesis that decontextualization in the presentation of mathematical concepts may be a significant factor in the development of students' ability for schematic perception and inferencing from geometrical diagrams.

Keywords: Mathematics education, embodied cognition, geometry, educational neuroscience, cerebellum, cognitive network, generalization, abstraction, decontextualization.

DEDICATION

To my dear wife, Connie

ACKNOWLEDGEMENTS

I offer my sincere thanks to staff and students at the Engrammetron Educational Neuroscience Laboratory at Simon Fraser University, who provided a stimulating and supportive intellectual environment.

Special thanks are due to my senior supervisor, Dr. S. R. Campbell, Director of the Engrammetron. He inspired my research, gave me room to grow intellectually, and encouraged my explorations.

My parents were unwavering in their belief in me. They have been, and still are, a rock on which I can lean, and I owe them heartfelt thanks.

Lastly, my eternal love and gratitude goes to my wife, Connie. Her support was unbroken and multi-facetted. I owe her more than I can ever repay.

TABLE OF CONTENTS

Approval ii
Abstract iii
Dedication iv
Acknowledgements v
Table of Contents vi
List of Figures ix
Prologue: The Angles in a Circle Theorem 1
Chapter 1: General Introduction 11
- 1.1 Introduction 11
- 1.2 Geometrical Reasoning: A Clarification of Scope 23
- 1.3 Property Analysis 30
- 1.4 Educational Neuroscience 34

Chapter 2: Literature Review 42
- 2.1 Philosophy of Mind 42
- 2.2 The Cerebral Cortex and Cognitive Neuroscience 47
- 2.3 The Cerebellum and Cognitive Neuroscience 55
  - 2.3.1 Physiology and Phylogenesis of Cerebellum 56
  - 2.3.2 Neural Connections between Cerebral Cortex and Cerebellum 58
  - 2.3.3 Uniformity of Cerebellar Histology 62
  - 2.3.4 Theories of Cerebellar Function 63
  - 2.3.5 Dysmetria of Thought 67
  - 2.3.6 Selective Attention 70
  - 2.3.7 The Cerebellum and Geometrical Reasoning 73
- 2.4 Mathematics Education Research 74
  - 2.4.1 The Cerebral Cortex and Mathematics Education 75
  - 2.4.2 The Cerebellum and Mathematics Education 76
  - 2.4.3 Decontextualization 78

Chapter 3: Psychology and Neurophysiology 82
- 3.1 Neutral Monism 83
- 3.2 Neural Correlates 88
- 3.3 Neural Correlates for Image-based Geometrical Reasoning 96
  - 3.3.1 Spatial Aspects of Arithmetical Reasoning 96
  - 3.3.2 Spatial Aspects of Geometrical Reasoning 99

Chapter 4: A Theory of the Cerebral Cortex 111
4.1 Cognitive Network Theory 112
4.2 Two Models of Attention 128
  4.2.1 Five-level Clinical Model of Attention 129
  4.2.2 Four-component Neurophysiological Model of Attention................... 131
4.3 Duration.. 137
4.4 Essential Properties and Incidental Properties 148
4.5 Schematic Concepts and Schematic Percepts 155
Chapter 5: The Cerebellar Schematization Hypothesis 169
5.1 Large-scale Anatomy of the Cerebellum 173
5.2 Phylogenesis of the Cerebellum 175
5.3 Cerebrocerebellar Connectivity 181
  5.3.1 Methods for Determining Cerebrocerebellar Connections 182
  5.3.2 The Broad Picture of Cerebrocerebellar Connectivity 184
  5.3.3 Neuronal Tracing Studies.. 185
  5.3.4 FCMRI Studies.. 188
  5.3.5 The Classical Understanding of Cerebrocerebellar Connectivity 192
  5.3.6 The Closed-loop Understanding of Cerebrocerebellar Connectivity 194
  5.3.7 Localization in the Cerebellum 199
5.4 Uniformity of Cerebellar Histology 206
5.5 The Motor Cerebellum and the Non-motor Cerebellum........................... 208
5.6 The "Mental Motor" Cerebellum 220
  5.6.1 Skilful Manipulation of Ideas 220
  5.6.2 Internal Models of Mental Models 222
5.7 Several Other Theories of the Functional Role of the Cerebellum 223
  5.7.1 Timing 225
  5.7.2 Sequencing 227
  5.7.3 Working Memory.. 228
  5.7.4 Sensorimotor Imaging 230
  5.7.5 Procedural Learning... 231
  5.7.6 Linguistic Processing 232
  5.7.7 Visuospatial Ability 234
5.8 Dysmetria of Thought 235
5.9 Selective Attention As a Functional Role of the Cerebellum 241
5.10 Summary of the Arguments... 251
  5.10.1 Structural Arguments 253
  5.10.2 Substantiating Arguments.. 255
  5.10.3 "Seeing the General in the Particular"... 256
5.11 Conclusion.. 256
Chapter 6: Mathematics Education and the Cerebellum 258
6.1 "Seeing the General in the Particular" 260
6.2 Cerebral Learning and Cerebellar Learning 264
6.3 Decontextualization 266
6.4 Decontextualization and Mathematics Education Theory........................ 271

6.4.1 Decontextualization, Situated Learning Theory, and Constructivism 272
6.4.2 Decontextualization in Other Areas of Mathematics Education Theory 274
6.5 Conclusion 278
Chapter 7: Potential Implications and Future Directions 282
7.1 Mathematics Education and the Cerebral Cortex 283
7.1.1 Procedural Reasoning versus Conceptual Reasoning 284
7.1.2 Concept Formation 291
7.2 Theories of Generalization and Abstraction in Mathematics Education 295
Chapter 8: Reflection and Critique 309
8.1 Contributions 309
8.2 Limitations 312
8.3 Conclusion 313
Reference List 315

LIST OF FIGURES

Figure P.1. The Angles in a Circle Theorem.. 6
Figure P.2. Different versions of the Angles in a Circle Theorem 7
Figure 1.1. Cerebellum, cerebral cortex, and world. 18
Figure 1.2. Pasch's Axiom. 26
Figure 1.3. Proposition I1 from the *Elements*... 26
Figure 3.1. Dorsal stream and ventral stream.. 101
Figure 4.1. Anterior cortex (red) and posterior cortex (blue). 118
Figure 4.2. Bergson's cone diagram.. 145
Figure 4.3. Relationship of concepts, percepts, and properties........................ 156
Figure 4.4. A property that is attended to. 157
Figure 4.5. Happy face. 162
Figure 5.1. Large-scale structure of the cerebellum. 174

PROLOGUE: THE ANGLES IN A CIRCLE THEOREM

The investigation in the coming pages closes with a distinction between “cerebral learning” and “cerebellar learning,” and a discussion of the implications of this distinction for mathematics education with respect to geometry. In these first few pages I would like to present a brief psychological analysis, focusing on the example of the Angles in a Circle Theorem, in order to illustrate the distinction between the two types of learning. When the argument appears to digress later on, the Angles in a Circle Theorem should be kept in mind—a classroom context for geometrical image-based thinking and learning. Readers may refer back to this prologue to help ground their understanding of my efforts in the more familiar educational environment of high school geometry.

There are (at least) two distinct aspects to the learning of geometrical concepts. Firstly, students must gain knowledge of the concepts themselves: the ability to recognise a triangle, for example, including its various properties. I refer to development of this ability as *cerebral learning*, for reasons that shall become apparent later. Cerebral learning may be identified in subject areas other than geometry. Geometry is unusual because of the extent to which it emphasizes a second aspect of learning: development of the ability to attend only to those properties of a given situation that are essential to the geometrical concept under consideration. In other words, only the *triangularity* of the figure is perceived, not its background, not its colour, not its size and orientation, and not even its particular shape. I refer to acquisition of the capability to perceive only the essential properties of a geometrical situation as *cerebellar learning*, again for reasons that become apparent later. With cerebellar learning attention is directed toward the geometrical essence of a situation. Properties of the situation that are incidental to its geometrical essence are excluded from attention. This study primarily concerns cerebellar learning, which is of particular importance to geometry.

As just indicated, properties are either *essential* or *incidental*. The essential properties of a given concept are those that are constant over all instantiations of a given concept; the incidental properties, on the other hand, vary over the instantiations of a given concept. For example, all triangles have the property that they consist of three line segments connected at three vertices—an essential property. On the other hand, triangles may exist in a variety of colours, which implies that colour is an incidental property. Note that even the particular shape of a triangle, when considering the triangularity in itself, is not an essential property, because triangles may appear in a variety of particular shapes. The shapes of circles or squares, on the other hand, do not vary across particular instantiations, meaning that shape, in these cases, *is* an essential property, according to my criterion.

Note that properties are only essential or incidental with respect to a given concept. For the concept “isosceles triangle,” for example, the property of having two sides of equal length is essential. However, with respect to the concept “triangle,” the property of a triangle figure that it has two sides of equal length is incidental.

Cerebral learning results in students acquiring *extensional concepts*, the ability to recognise instantiations of a concept without necessarily focusing attention on the mathematical essence of the concept. Cerebellar learning, on the other hand, results in students acquiring *intensional concepts*, the ability to recognise instantiations of a concept while *in addition* focusing attention on the essence of the image-based geometrical concept. I refer to the process of focusing attention on the essence of a geometrical concept as *schematization*. Note that schematization may never be achieved perfectly, and incidental properties may continue to hover at the edge of attention. In this case, the more accurate term is that concepts are *relatively intensional* or *relatively schematized*. The qualifier “relatively” is implicit whenever intensional concepts and schematization are mentioned throughout this dissertation. Note in addition that schematization can refer to relationships between objects in a geometrical diagram—these, too, may be essential properties of the diagram. The notion of schematization includes both *schematic perception* and *schematic inferencing*, the latter referring to these relationships.

Decontextualization refers to presentation of a geometrical situation in such a way as to minimize the number of apparently incidental properties. If the triangle figure is presented as starkly as possible, as a black-line figure on a white background, then the triangle has been decontextualized as far as possible. Decontextualization can never be achieved perfectly. For example, every particular instantiation must be drawn with a certain size or a certain shape, and these are incidental properties. Strictly speaking, the more accurate term is *relative decontextualization*. A geometrical figure can be decontextualized relatively by removing obvious non-geometrical distractors and by isolating the figure. The qualifier “relative” is implicit whenever decontextualization is mentioned throughout this dissertation.

One of my hypotheses is that cerebellar learning, acquisition of the ability to schematize, to perceive the geometrical essence, is facilitated if geometrical situations are decontextualized—presented abstractly, purely, and symbolically. In other words, students may find it easier to learn to perceive the essential triangularity of a triangle if the triangle is presented in a manner that involves the fewest number of incidental properties. After all, geometry *in itself* is abstract, pure, and symbolic. If geometrical structure can be isolated from incidental distractors as far as possible, then geometrical structure can more easily become the focus of attention. In consequence, geometrical reasoning may be clearer and possibly even more accurate.

The question now is, how can cerebellar learning be encouraged? Dynamic geometry software such as Geometer’s Sketchpad (Jackiw, 1991) enables students to manipulate geometrical situations so as to generate large numbers of examples that instantiate a given set of geometrical constraints. Thereby, students investigate for themselves, empirically, those characteristics of a geometrical situation that remain invariant under the given constraints. These invariant features are the essence of the geometrical situation. Dynamic geometry software is an ideal way to demonstrate geometrical essences, within an environment in which figures are already decontextualized to a large degree. Thereby, dynamic geometry software may promote cerebellar learning. Dynamic geometry software, as a particular pedagogical tool, is not discussed at any length in this dissertation. However, the power of Geometer’s Sketchpad in example generation is examined, for example, in Marrades and Gutiérrez (2000); and further research on Geometer’s Sketchpad and pedagogy is contained, for example, in Battista and Borrow (1997), Sinclair (2000), and Christou, Mousoulides, Pittalis, and Pitta-Pantazi (2005).

It is necessary to be very careful in interpreting the type of learning—cerebral or cerebellar—that is occurring as students vary a triangle figure in Geometer’s Sketchpad. Focus on the essential triangularity is a cerebellar aspect of learning. However, learning that accompanies the variation of size, orientation, and shape may well be *cerebral* to the extent that the student recognizes that a number of particular figures belong to a common category. Appreciation of the essential triangularity, the one thing that is common to all instantiations of the triangle, as it varies in Geometer’s Sketchpad, is the *cerebellar* aspect of the exercise.

For the triangle concept, cerebral learning and cerebellar learning are inextricably mixed. Nevertheless, I argue in this dissertation that they correspond to quite distinct brain physiological processes. A more complex example may be able to tease out the psychological distinction between the two kinds of learning. (I clarify the physiological distinction in Chapter 6.) The Angles in a Circle

Theorem from high-school geometry is a good illustration of the difference between cerebral and cerebellar processes.

The so-called Angles in a Circle Theorem has been part of the high-school geometry curriculum from time immemorial. It is Proposition 20 from Book 3 of Euclid's *Elements* (Heath, 1908/1956): *the central angle opposite the arc of a circle is twice the angle at the circumference opposite the same arc*. As I have discussed in Handscomb (2005), it is a characteristic of high-school geometry that geometrical concepts are represented diagrammatically and that diagrammatic information is utilized in geometrical arguments. *Image-based reasoning in geometry* is distinguished in this way from formal geometrical reasoning, in which it is not permitted to use diagrammatic information for geometrical arguments. Consequently, visual representations of the Angles in a Circle Theorem are important at the high-school level.

The Angles in a Circle Theorem is most frequently represented by means of a diagram like that in Figure P.1.

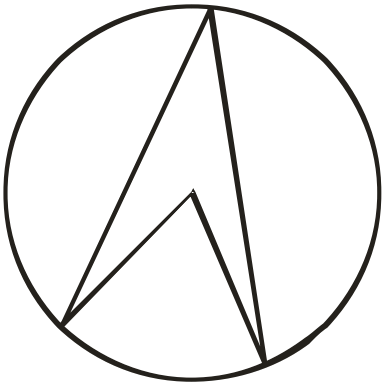

Figure P.1. The Angles in a Circle Theorem.

In my own teaching practise I refer to it as the “Star Trek Theorem,” because of its inadvertent similarity to the pop-culture icon. However, the Angles in a Circle Theorem has a number of representations that are qualitatively quite different. These are shown in Figure P.2.

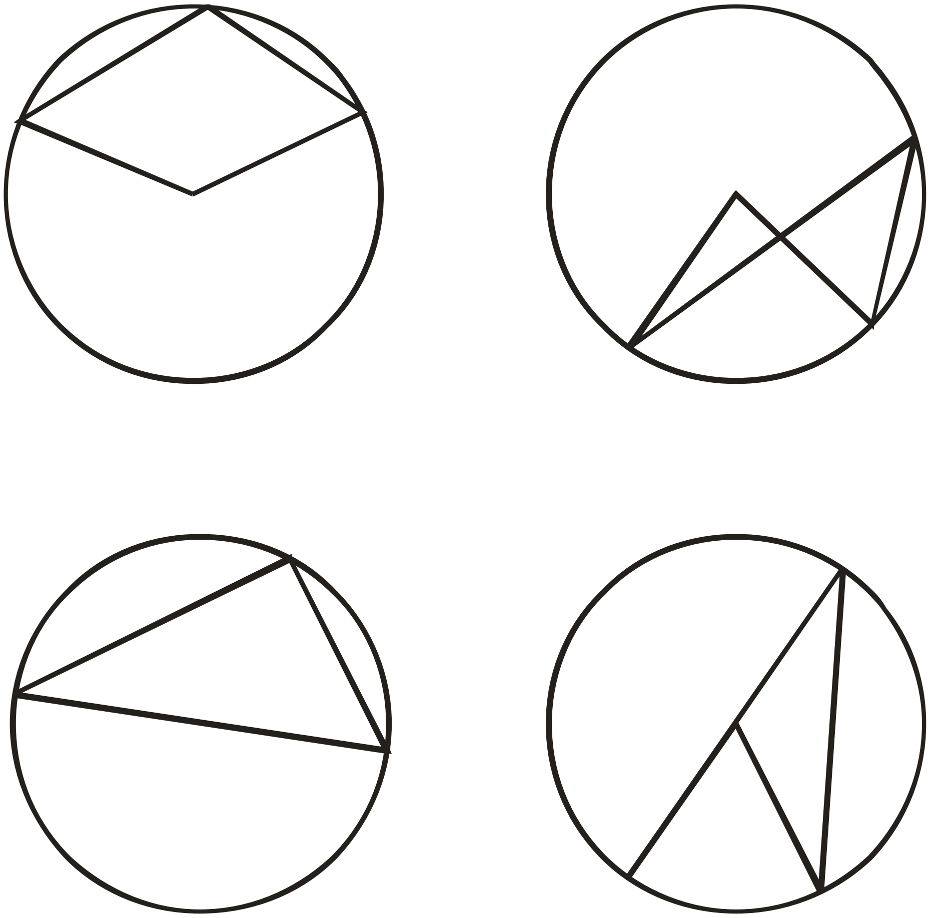

Figure P.2. Different versions of the Angles in a Circle Theorem.

In addition to its Star Trek representation, the Angles in a Circle Theorem may appear as a convex quadrilateral, or as a “bow tie,” or as various triangular formations. In my experience students find it very difficult to recognise the Angles in a Circle Theorem as a single theorem in all of its manifestations. In consequence, there is a tendency for teachers not to present it as a single theorem, but as *a series of five different theorems*, or at least as five versions of a single theorem, which students have to learn separately. (Admittedly, the proofs can differ between versions.)

It is inefficient for students to learn the Angles in a Circle Theorem in five separate forms. But there are difficulties in teaching it as one, single theorem. Unavoidably, the Star Trek icon, or the convex quadrilateral, or the bow-tie, or the triangles are associated with the theorem as faux essential properties. Students are unable to “see” the essence of the theorem in itself detached from the specific properties of its various manifestations. In other words, they are unable to schematize the theorem in itself. The Angles in a Circle Theorem contains an angle at the centre that measures twice the angle at the circumference; but *in addition* it looks like the Star Trek icon, or *in addition* it looks like a bow tie, and so on. In consequence, students’ understanding of the Angles in a Circle Theorem is inextricably bound to various incidental properties of the geometrical diagram. Cerebral learning is expansive and connective, resulting in an extensional concept. It is unavoidable that peripheral properties are associated with the concept, and its boundaries are indistinct.

In order to learn the Angles in a Circle Theorem as a single theorem it is necessary for students to acquire the ability to focus attention only on the essential aspects of the theorem in all of its various manifestations. They must clearly perceive the angle at the circumference opposite a circular arc and a central angle opposite the same arc—and then the conclusion of the theorem follows—no matter the specific appearance of the diagram. In other words, *cerebellar learning* is required. Students must learn to suppress from attention incidental aspects of the diagram; in other words, they must schematize. Cerebellar learning is restrictive and focused, resulting in an intensional concept. The concept is perceived sharply and purely.

Dynamic geometry software such as Geometer's Sketchpad may enable students to accomplish the cerebellar learning necessary for the five different versions of the Angles in a Circle Theorem to collapse into one. Geometer's Sketchpad allows students to vary the positions of the vertices on the circumference. As the central angle increases to 180^o, the diagram becomes a right-angled triangle inscribed in a circle; and as the central angle increases over 180^o, the circle contains a convex quadrilateral. On the other hand, with an acute or obtuse central angle, variation of the position of the angle at the circumference gives the other three diagrams. In all of these manipulations students are able to verify mechanically, internal to the software, that the central angle remains double the angle at the circumference. It is possible to achieve the same result with students drawing numerous diagrams by hand, with ruler and compasses, and measuring the angles with protractors, but the traditional technique cannot match the continuous example generation and immediate feedback of Geometer's Sketchpad.

The Angles in a Circle Theorem is an important example because it dissociates cerebral learning and cerebellar learning fairly clearly. Often the two are difficult to separate, as in the triangle example. With regard to decontextualization, it is impossible to decontextualize the Angles in a Circle Theorem further than the diagrams in Figure P.2. Unavoidably, there are distracting incidental properties, even though *the incidental properties in themselves are geometrical*.

For other types of geometrical situation incidental properties are non-geometrical and can be eliminated readily. “Word problems,” for example, may be taken out of context and represented symbolically. I hypothesize later that when this is done, cerebellar learning is enhanced.

The purpose of this prologue is to demonstrate the psychological distinction between cerebral and cerebellar learning and thereby to ground this dissertation in K-12 mathematics education. I hope that readers recall the example of the Angles in a Circle Theorem, in the pages to come, when the discussion appears to diverge from classroom application.

CHAPTER 1: GENERAL INTRODUCTION

Direct perception of structure is a cognitive function that is indispensable for geometrical reasoning. My research is a working through of this idea and its implications for mathematics education with regard to geometry. This first chapter introduces the main ideas in the dissertation, clarifies my scope, explains the method of property analysis, and situates the work of this dissertation within the emerging research framework of educational neuroscience.

1.1 Introduction

My investigation of geometrical reasoning is concerned with certain structural aspects of *visual* perception. There may be other structural aspects of visual perception, of more relevance to the artist, for example, than the mathematician, so that geometry does not exhaust the structure of visual perception. Moreover, geometry is not delimited by the structural aspects of visual perception, but entails also mathematical procedures, such as logical reasoning, which engage cognitive functions other than perception. Nevertheless, it is at the intersection of geometry and visual perception that this dissertation is primarily concerned.

Let me be more specific. Euclidean objects such as points, straight lines, and circles still form the backbone of the school geometry curriculum (Handscomb, 2005). These objects and their combination and interaction, as appropriate to the K-12 curriculum, are the focus of the discussion herein. Image-based reasoning in geometry is the process of forming geometrical inferences from geometrical diagrams. My whole dissertation is concerned with a particular type of image-based inference, which may be referred to as *schematic inference.* Schematic inferences depend on *schematic perception*, perception only of the geometrical essence of the situation. I explain this restriction of scope more fully in the next section.

As the Prologue explains, a triangle in perception has various incidental properties, such as specific colour, size, orientation, and shape. These properties are irrelevant to its essential triangularity. As far as the triangle is concerned, these aspects of perception have nothing to do with the geometry of the triangle in itself. If incidental properties of the triangle remain at the forefront of perception, then geometrical reasoning risks confusion and inaccuracy. A goal of geometry education should be to enable students to attend to the geometrical structure of their perception.

To clarify: perception of structure is not the same as definition of structure. It is one thing to attend to the essential triangularity of a geometrical figure, and ignore incidental properties; it is another thing entirely to define a triangle by means of propositions. Perception of structure, rather than definition of structure, is my main concern.

I refer to perception of structure, in the sense above, as *schematic perception*. Schematic perception, as we see later, is a reinterpretation of the phrase “seeing the general in the particular,” (Mason & Pimm, 1984).

In one sense, the development of the argument in this dissertation is purely *psychological*. Indeed, it contains a psychological framework into which the cognitive function to “see the general in the particular” fits like a key into a lock. Along the way, some ideas in mathematics education theory are interpreted anew, albeit tentatively. In another sense, the development of the argument is *physiological*, or rather *neurophysiological*, in that the various psychological aspects of reasoning related to geometry have neural correlates in the activity of neural assemblages in the brain. The neurophysiological dimension is important for two reasons: firstly, research results in cognitive neuroscience substantiate the psychological argument; secondly, the neurophysiology actually constrains and informs the psychological theory, perhaps thereby underlining the significance of certain pedagogical approaches.

My engagement with selected literature from neurophysiology and cognitive neuroscience concerning the cerebral cortex and cerebellum leads to a hypothesis concerning image-based reasoning in geometry, specifically concerning schematic perception and schematic inferences from geometrical diagrams. Indeed, this doctoral thesis is, in essence, an attempt to account for certain aspects of the theory of geometrical image-based reasoning that I presented in my master’s thesis (Handscomb, 2005).

The theoretical framework must be able to accommodate psychological and physiological data coherently. The theory of embodied cognition (Varela, Thompson, & Rosch, 1991), as modified by Campbell (2001, 2003), and discussed in Chapter 3, is able to accomplish this goal. It results in a neutral monism, much like Spinoza's classical formulation, with a single ontological substrate and two epistemological categories, psychology and physiology.

There are other approaches to embodied cognition, such as the cognitive metaphors of Lakoff and Núñez (2000), in which straight lines or circles are "embodied" in actions such as walking in a straight line or turning in a circle. The level of analysis herein, however, concerns embodiment at the level of dendrites and synapses: conception of a straight line or circle, a higher cognitive function, is correlative to a specific pattern of neural activity. Dehaene (e.g., 1997) has already investigated the neural activity correlative to aspects of arithmetical reasoning, and my goal herein is to do something similar with respect to aspects of image-based reasoning in geometry.

The research method often associated with embodied cognition is neurophenomenology (Varela, 1996, 1999), according to which neurophysiological hypotheses constrain and inform hypotheses developed through phenomenological investigation, and vice versa. However, I have utilized the approach of cognitive psychology rather than formal phenomenology, while maintaining the fundamental orientation of reciprocal constraints between cognitive psychology and neurophysiology. Moreover, the cognitive psychology herein is grounded in mathematics education. In consequence, the research framework of this dissertation may be identified with the emerging research paradigm of educational neuroscience, specifically *mathematics* educational neuroscience.

There are two ways in which humans interact with the world: action and perception. Sensation that is afferent from the world contributes to perception, while action discharges in movement in the world.

It has long been recognized that there is a perception-action loop: action leads to movement, which alters sensation, and then perception, and finally subsequent action (Uexküll, 1926, cited in Fuster, 2003; Merleau-Ponty, 1942/2006). Human interaction with the world operates on a continuous cycle.

Concepts, as understood herein, are associations of percepts, such that each particular percept instantiates the concept. Geometrical reasoning is the main focus of the dissertation, and therefore *visual* concepts and percepts are pertinent to the discussion. Visual percepts are specific and particular in spatial terms, whereas visual concepts are non-specific and general in spatial terms. The neurophysiological interpretation of visual concepts, visual percepts, and visual properties is discussed in Chapter 3.

Similarly, procedures are associations of acts. In this case, however, individual acts are concatenated in temporal sequence. Procedures extend over time and are therefore non-specific in temporal terms, just as concepts are non-specific in spatial terms. The main concern in this dissertation is with a specific aspect of geometrical reasoning, and I propose that perception is more pertinent in this respect than action. Although I frequently make reference to action, my main area of focus is perception, specifically visual perception.

In the same way that acts and percepts connect via a perception-action loop that passes through the world in movement and sensation, concepts and procedures connect via a higher-level perception-action loop, although in this case the loop does not pass through the world exterior to the individual's body (Fuster, 2003).

At any moment in cognition, concepts are resolving to percepts and procedures are resolving to acts. Equivalently, percepts are associating in concepts and acts are associating in procedures. There is a kind of "standing wave" of cognition that is simultaneously top-down and bottom-up, leading from the general to the particular in one direction and from the particular to the general in the other direction. An adequate explanation of these notions presupposes the cognitive network structure of the cerebral cortex (Fuster, 2003) and the philosophy of duration (Bergson, 1896/2005), which are the main thrust of Chapter 4.

Concepts, as associations of percepts, are fuzzy, indistinct, and inadequate for geometrical reasoning purposes. A further level of processing is required in order to produce the crisp, pure concepts that are the raw material for geometrical reasoning. Indeed, those aspects of the concepts, and the percepts to which they resolve, that are *essential* for geometrical purposes must be attended to, and those aspects of concepts and percepts that are *incidental* to the geometry must be suppressed from attention. In Chapter 4 I explain that this process must happen simultaneously in the concept and the percept to which it resolves. Moreover, the phrase "seeing the general in the particular," coined by Mason and Pimm (1984), may be reinterpreted to mean attention to the essential aspects of a geometrical situation at the expense of its incidental aspects. The resulting pure concepts and pure percepts are *schematic*[1]. A similar process must occur with respect to procedures and the acts to which they resolve.

The discussion of the previous few paragraphs results in the diagram below. Figure 1.1 contains references to the anterior cortex and the posterior cortex, a division of the cerebral cortex that is shown more realistically in Figure 4.1, with the same colour scheme. With reference now to neurophysiology rather than psychology, neural activity in the anterior cortex corresponds to procedures and acts, whereas neural activity in the posterior cortex corresponds to concepts and percepts. Betz (1874, cited in Fuster, 2003) first noticed this division of the brain and nervous system, and it is a key component of Fuster's cognitive network theory.

Perhaps even more important for this dissertation is the notion that a functional role of the cerebellum in cognition is to schematize concepts and percepts. Indeed, providing grounds for substantiating this hypothesis is the goal of Chapter 5. The cerebellum is a neural structure that, traditionally, is not noted for its contribution to higher cognitive function. In fact, historically, theories of cerebellar function have tended to emphasize the contribution of the cerebellum to motor behaviour. However, in the latter part of the twentieth century research evidence, and the accompanying explanatory theories, began to accumulate that the cerebellum was indeed involved in areas of cognition other than motor behaviour. While no cerebellar theorist would deny the motor role of the

---

[1] The terms *schematic*, *schematize*, and *schematization*, as used herein, should not be confused with *schema* in the way that it is used in APOS theory (see Section 7.1.2). My use of these terms is analytic and implies focusing on essential aspects of a situation, whereas the schema of APOS theory is synthetic in that it establishes connections between disparate elements.

cerebellum, a substantial minority of cerebellar researchers are investigating the involvement of the cerebellum in higher cognitive functions such as reasoning and attention. MacLeod (2000) writes, “To ignore the function of the cerebellum in thought is to do a disservice to the exquisite interplay of neurological patterning in the whole brain” (p. 181).

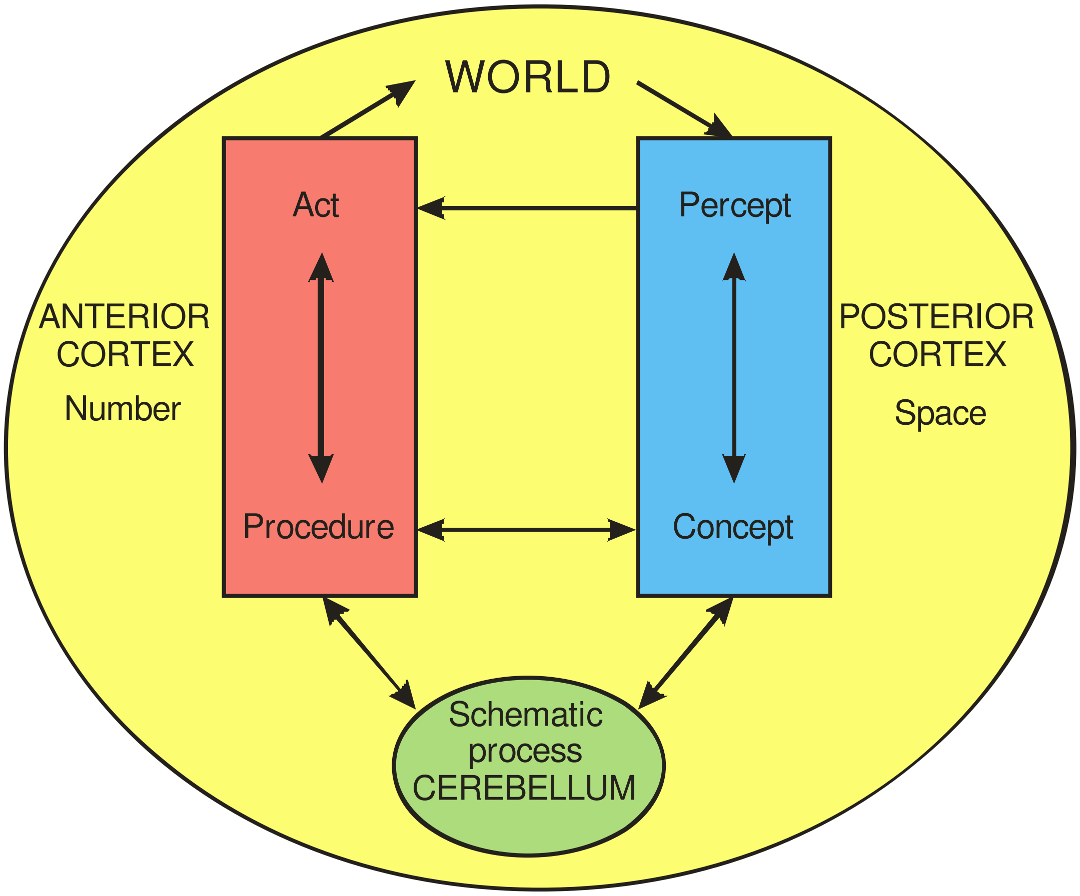

Figure 1.1. Cerebellum, cerebral cortex, and world.

A hypothesis of this dissertation, that the cerebellum is involved in schematizing geometrical concepts, was inspired by research on the role of the cerebellum in higher cognitive functions. However, the focus of my argument herein is unusual because of its emphasis on the posterior cortex and perception and its de-emphasis of the anterior cortex and action. I utilize well established research and theory on the cerebellum, but my perspective, coming from a background in mathematics education, with the goal of understanding geometrical reasoning and learning, is quite different from that of the cognitive neuroscientists. My interpretations of the cognitive neuroscience are taken from the perspective of a mathematics educator.

Even disregarding the cerebellum, however, there may be important consequences for mathematics education arising from the theory of the cerebral cortex developed in Chapters 3 and 4. These tentative consequences are discussed in the first part of Chapter 7. Very broadly speaking, the anterior cortex is involved with arithmetical and algebraic reasoning, because these areas of mathematics are concerned primarily with procedures rather than static percepts. Conversely, the posterior cortex is associated with geometrical reasoning, because geometry is primarily concerned with static percepts. Mathematical reasoning fuses procedural reasoning and conceptual reasoning, perhaps giving primacy to one or the other. The procedural reasoning-conceptual reasoning distinction has been important for mathematics education theory since the 1970s, under various guises. At the risk of oversimplification, it is tempting to postulate a deep underlying divide that has a neurophysiological basis in the brain.

Moreover, with regard to the theories of concept formation that have been influential in mathematics education, concepts typically emerge from procedures (with varying terminology between the different theories). However, it seems clear from the brief discussion so far that the converse possibility, whereby procedures emerge from concepts, should also be considered.

Thus, it appears that some mathematics education theory may be interpreted anew simply by reference to the neurophysiological substrates for different types of reasoning in the anterior cortex and posterior cortex. These ideas with respect to mathematics education and the role of the cerebral cortex are preliminary, and further research is required.

The hypothesized role of the cerebellum in schematization also has important implications for mathematics education, which are developed in Chapters 6 and 7. In particular, the schematization of concepts and percepts is closely related to the psychological process of *abstraction*, which in turn has been the focus of intense discussion by mathematics educators. Abstraction, as understood herein, is a process that is different from concept formation. Concept formation refers to the *extensional* concepts of the cerebral cortex prior to involvement of the cerebellum. In fact, as explained in Chapter 7, the extensional concepts of the cerebral cortex may be regarded as resulting from a process of *generalization* rather than abstraction. The effect of the cerebellum is to schematize these concepts, resulting in *intensional* concepts.[2] Abstraction, I propose in Chapter 7, may be identified with schematization. Again at the risk of

---

[2] Note that *extensional* and *intensional* both have technical meanings in logic (e.g., Fitting, 2007). Extension refers to the reference of a term, whereas intension refers to its meaning. My use of extensional and intensional is psychological (and neurophysiological) rather than logical. The use of extensional herein does recall its technical meaning in logic, in that an extensional concept is the collection of those percepts that may be regarded as the “reference” of the concept. However, my use of intensional is quite different in that it refers to a concept apprehended in its essence. I have decided to assign the additional technical meaning to intensional rather than coin a new term because of the connotations of “intension” with respect to “intensity,” “concentration,” and “attention,” and because of the opposition between “extension” and “intension.”

oversimplification, it is tempting to postulate a deep underlying divide that has a neurophysiological basis. Once again, however, this particular topic requires additional work, and is therefore placed in Chapter 7 as a potential implication of the research herein.

A second consequence of the cerebellar dimension to the theory, which belongs to the main line of reasoning, is discussed in Chapter 6. It involves the notion of decontextualization. As the Prologue explains, mathematical concepts are decontextualized if they are presented abstractly and symbolically, with few incidental properties. My hypothesis is that the role of the cerebellum in schematization is facilitated if mathematical situations are presented in a manner that is decontextualized. Decontextualization as a pedagogical strategy runs counter to some aspects of constructivism and situated learning, which are established theories of mathematics education. I do not claim that these valuable and interesting theories are wrong, but simply that they may not have all the answers. They may not even apply to cerebellar processes.

Although this dissertation is primarily theoretical, empirical research is frequently cited. In a sense, therefore, there is an indirect empirical component to the dissertation. Moreover, it prepares the ground for focused empirical research in the future. For instance, work remaining to be done as a natural extension of this doctoral thesis is to articulate the ideas contained herein with the theory of image-based reasoning of my master’s thesis. Such an articulation would constitute a clear program of research for educational neuroscience.

In summary, I develop a general framework that offers a new theoretical perspective for mathematics education based on research in the neurosciences. A hypothesis is that a functional role of cerebellum, with respect to its connections to the cerebral cortex, is to facilitate the cognitive ability to “see the general in the particular.” The cerebellum, in other words, may be responsible for pure, uncluttered thought of the type that is necessary for geometrical reasoning. Accordingly, there may be implications for mathematics education.

Before moving forward to a clarification of my scope, the property analysis, and the research method of educational neuroscience, I would like to establish clearly what I view as the contribution of this dissertation. From the perspective of mathematics education, my hypotheses concerning the functional role of the cerebellum and the pedagogical value of decontextualization may seem to be outside the mainstream of mathematics education research. Moreover, my background research in mathematics education may not achieve the depth and detail that would be required for a work solely concerned with the psychological aspects of mathematics education.

On the other hand, from the perspective of cognitive neuroscience my framework may seem overly simplistic and not fully nuanced—my engagement of the vast literature in cognitive neuroscience is necessarily partial. However, I am a mathematics educator primarily rather than a cognitive neuroscientist. My utilization of cognitive neuroscience research is restricted to areas that seem to be seminal or important for my work in mathematics education.

Therein lies the value and significance of my dissertation. It is a study in *educational neuroscience*, in which the data, results, and conclusions of mathematics education constrain and inform the data, results, and conclusions of cognitive neuroscience, and vice versa. My research occupies a middle ground between these two academic disciplines. If my work is viewed as research in mathematics education, it may be subject to criticism, and justifiably so from that perspective. Likewise, if my work is viewed as research in cognitive neuroscience, it might justifiably be a target for criticism.

However, I would ask my readers from backgrounds in mathematics education and cognitive neuroscience to situate themselves in a region that lies between these two academic disciplines in order to appreciate my work. My research belongs to educational neuroscience. It attempts to integrate and synthesize two separate and divergent areas of research. And this is the strength and contribution of my dissertation. This region between the two disciplines is, to my knowledge, virtually empty, and my aim is to break new ground in this area.

1.2 Geometrical Reasoning: A Clarification of Scope

In this section, I wish to further clarify the scope of my investigation with respect to geometrical reasoning and learning. As I discussed in the previous section, image-based reasoning in geometry, as I understand it, is concerned with certain structural aspects of visual perception involving Euclidean objects such as points, lines, and circles. There is a great deal more to geometrical reasoning than the image-based reasoning which concerns these structural aspects of visual perception. For example, deduction and geometrical construction are two other aspects of geometrical reasoning and learning. Moreover, there is a great deal more to visual perception than the structural aspects related to Euclidean objects. My research primarily lies at the intersection of visual perception and geometrical reasoning.

I have limited my scope to an image-based reasoning in geometry, in which students are permitted to make geometrical inferences based on their spatial, visual intuition of the geometrical diagram. More specifically, my investigation herein is concerned with schematic perception of geometrical diagrams. That is to say, when the observer sees only the essential aspects of a geometrical concept, as manifested in a geometrical diagram, the observer is able to ignore aspects of the diagram that do not involve the target concept. This is what I mean by schematic perception. Hence, schematic perception is an aspect of image-based geometrical reasoning. Schematic perception allows the observer to make schematic inferences from the geometrical diagram. Schematic inferences concern the essence of the geometrical concept, as manifested in a geometrical diagram. It is only this very specific and limited form of geometrical reasoning with which I am concerned in this dissertation.

I deal now with my justification for restricting the scope of my investigation in this way. The objection can be raised as to whether making schematic inferences from geometrical diagrams can rightfully be considered as a genuine aspect of geometrical reasoning. For instance, mathematicians and philosophers of mathematics may take issue with the notion that geometry necessarily requires inferences from diagrams. Indeed, Hilbert (1899/1971) initiated the program to fully axiomatize geometry, eliminating entirely the necessity for inferences from diagrams. Hilbert was successful in this endeavour—in formal, axiomatic geometry, diagrammatic inferences are prohibited. It is not my place in this dissertation to argue for, or against, any particular definition of geometry in a formal sense. My approach is more pragmatic—my concern is with the image-based reasoning processes required of students of geometry at the K-12 level.

Nevertheless, in order to clarify my position with respect to geometrical diagrams and image-based reasoning, and thereby to justify the limitation of the scope of my thesis, I undertake here a short digression in the history of geometry. My arguments in the next few paragraphs are largely adapted from Handscomb (2005).

Firstly, the axiomatic system of the Greek geometers was imperfect. It was pointed out by Pasch in 1882 in his *Vorlesungen über Neuere Geometrie* (Greaves, 2002, p. 66; Netz, 1999, p. 27) that the system of postulates and common notions in the *Elements* is inadequate for a logical deduction of all intuitively obvious features of geometrical images. Pasch's Axiom, for example, which cannot be deduced from Euclid's axioms, states, "If a line intersects one side of a triangle and misses the three vertices, then it must intersect one of the other two sides" (Weisstein, 2005a).

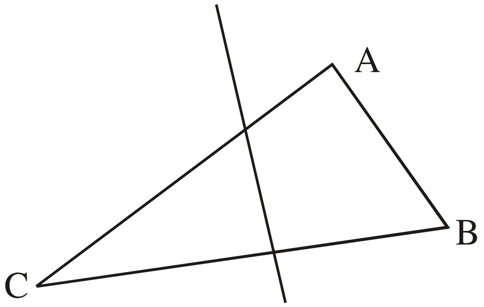

Figure 1.2: Pasch's Axiom.[3]

In addition, the proof of Euclid's Proposition 1 from Book I, the construction of an equilateral triangle on a given line segment, requires an inference from the diagram, because otherwise there is no way to confirm that the two circles of the construction actually intersect (Heath, 1908/1956, Vol. 1, pp. 241-243).

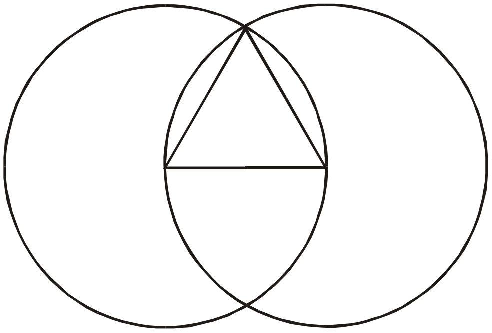

Figure 1.3: Proposition I1 from the *Elements*.[4]

It seems obvious from the diagram that the circles do intersect, but formally the geometer needs the Continuity Axiom (Weisstein, 2005b). I could cite several other examples from Euclid where diagrammatic inferences are necessary (Handscomb, 2005). If inferences from diagrams are not only permitted, but actually *necessary* for establishing results in geometry, then geometrical

---

[3] Reprinted from Handscomb (2005, p. 43).

[4] Reprinted from Handscomb (2005, p. 44).

diagrams in the period from Euclid to Hilbert were essential for the geometrical reasoning process rather than incidental adjuncts.

Secondly, Friedman (2000, pp. 186-187) argues that all the objects of Euclid's reasoning are iteratively constructed, with straight edge and compass, by means of Euclid's first three postulates, and are therefore finite systems of points, lines, and circular arcs—the starting point for the geometrical argument is the geometrical diagram so constructed. Moreover, according to Netz (1999) a geometrical argument of Euclidean times *necessarily* consisted of a lettered diagram, together with statement of the proposition and its proof. These points alone do not demonstrate the primacy of the geometrical diagram over the propositional argument, but, together with the first point, they imply the indispensability of the geometrical diagram in Euclidean times.

Indeed, the main thrust of Netz (1999) is an attempt to provide a justification for the generality of Greek geometrical arguments despite their use, indispensably, of particular diagrams and inferences from these particular diagrams. I discuss this point at some length in Handscomb (2005), with reference to high-school geometry and the necessity for students today likewise to use particular diagrams and to make inferences from particular diagrams.

According to Mueller (1981, p. 13), the Greeks themselves never provided an answer to the question of generality. Does it matter if the Greek style of geometrical reasoning lacked rigor according to Hilbertian standards? According to Netz (2004), "Archimedes' goal is not axiomatic perfection (where every axiom, and every application of an axiom, must be made explicit), but *truth*" (p.

42, author's italics)—and *truth*, according to Archimedes and his contemporaries, must have resided, at least partially, in the eye of the beholder, that is, in spatial, visual perception. Insofar as separating incidental from essential properties in geometrical objects and diagrams, and thereby drawing inferences, is part of *geometrical* reasoning, this captures what I mean by *image-based* reasoning.

Thirdly, geometry did not spring into life with the investigations of the Greeks. Image-based geometrical methods were utilized in Babylon and Egypt long before the Greeks formulated their axioms (e.g., Fowler & Robson, 1998; Silvester, 2001). Thales, according to legend, brought the Egyptian geometry to Greece, where it underwent considerable development before culminating in Euclid's *Elements* (Silvester, 2001). The Greeks were the first to (partially) formalize geometry with axioms. The Babylonian and Egyptian geometry that preceded Greek geometry must therefore have been a spatial, visual science.

Fourthly, the formal approach dominates mathematics at the university level. The mathematician's attitude, as Freudenthal (1973) puts it, is that the "quicksand of reality is no basis to build a mathematical system; mathematics should be protected against any contamination with non-deductive germs" (p. 403).

Should the perspective of higher mathematics affect curriculum content in secondary schools? If the van Hiele levels (van Hiele, 1986) have any validity, then it is clear that axiomatic geometry should not and cannot be taught to students unless they first have a strong foundation in spatial, visual geometry. Freudenthal (1973) concurs with van Hiele's analysis that geometrical ideas should be introduced by spatial and visual means. He argues that the real objective of geometry is grasping the space in which we live and breathe and move (p. 403). According to Freudenthal,

> Geometry can only be meaningful if it exploits the relation of geometry to the experienced space. If the educator shirks this duty, he throws away an irretrievable chance. Geometry is one of the best opportunities that exists to learn how to mathematize reality. (pp. 406-407)

Fifthly, given that axiomatic geometry should *not* be taught to students at the high school level, the question remains, *is* axiomatic geometry taught to students at the high-school level? The answer is, no (Handscomb, 2005). In a sense, the geometry taught at the high-school level is pre-Euclidean. Of course, students must learn some of the Euclidean theorems, and, at least at the senior high-school level, geometrical reasoning is nominally deductive. However, axioms are not taught. Inferences that might be justified by reference to an axiom are instead justified by spatial, intuitive understanding of the geometrical diagram. For example, the only way that a student can know that a line intersects the circle at zero, one, or two points is by experimenting with a diagram. This kind of image-based reasoning in geometry is a fact of life in the mathematics classroom at the high-school level and below.

Lastly, non-Euclidean geometries have been developed, for example, by varying the parallel-lines postulate. These geometries may defy visual, spatial intuition. Perhaps the existence of these non-Euclidean geometries can be taken as an argument against the significance of visual, spatial intuition in Euclidean geometry. To address this question, I use an analogy from abstract algebra. Ring theory was inspired by the arithmetic of the natural numbers. The more obscure objects of ring theory may seem distant from the simple numerical intuition of arithmetic. However, it would be peculiar to claim that developments in abstract algebra imply that arithmetic is not founded on simple numerical intuition. It would seem equally peculiar to claim that developments in non-Euclidean geometry imply that Euclidean geometry is not after all founded on visual, spatial intuition.

In sum, to insist that geometry is essentially an axiomatic science—and thereby to undermine the relevance, value and necessity of image-based reasoning in geometry—would be to view the thousands of years of history of geometrical reasoning through the limited lens ground out by Hilbert. It is, moreover, to ignore the way that geometry should be taught, and actually is taught, in high schools today. This is my justification for the focus on schematic perception in this dissertation. My approach is to view objects and inferences pertaining to schematic perception as aspects of image-based reasoning in geometry.

1.3 Property Analysis

The relationship between image-based concepts and percepts pertaining to K-12 geometry, and their analogues in action, depends on a *property analysis*. In my approach to property analysis, concepts are analyzed as associations of percepts, and percepts are analyzed as associations of incidental and essential properties. I present and discuss research in cognitive neuroscience that provides substantial support for my approach to property analysis, particularly with regard to the visual percepts and their associated properties that are of major concern for image-based geometrical reasoning. I utilize property analysis to elucidate the relationship between concepts and percepts, to clarify the difference between incidental and essential properties, and to establish the notion of equality of range.

With respect to equality of range, the property analysis enables development of the notion that concepts have potential properties, just as percepts have actual properties. For example, the concept of triangle has the potential property of “colour,” just as a triangle percept may have the actual property “green.” The *geometrical* triangle, purified and schematized, has no such property, in concept or percept. The movement from concept to percept represents a reduction in potential, whereas the movement in the opposite direction, from percept to concept, represents an increase in potential. With respect to this movement, within the cerebral cortex and without involvement of the cerebellum, it is not the case that properties are removed from the percept to create the concept. Likewise, it is not the case that properties are added to the concept to create the percept. Indeed, to each actual property of the percept there corresponds a potential property of the concept, and vice versa. In consequence, *concepts and percepts have the same range*. These points are discussed in detail in Chapter 4, based on the cognitive network approach to neural structures in the cerebral cortex (Fuster, 2003), as informed by Bergson’s (1896/2005) philosophy of duration.

Based on the property analysis, *essential properties* of an image-based concept are those properties that take the same value in every percept that is associated in the concept. Every triangle percept, for example, has the property of being three line segments connected at three vertices. It follows that this characteristic, as the Prologue illustrates, is an essential property. Consequently, with respect to the essential properties, the movement from percept to concept really does not correspond to an increase in potential. The full potential of the essential property is already actualised in the percept. Colour, on the other hand, is an *incidental property* of the triangle concept, because it can actualise in many ways in the percept.

With regard to geometry, some properties of the image-based concepts and percepts are essential and some are incidental. However, even ostensibly geometrical properties may be incidental with respect to the specific concept being examined, as I illustrate with reference to the Angles in a Circle Theorem in the Prologue—in other words, *image-based properties are essential or incidental with respect to a specific image-based concept*. If attention is focused only on those potential properties of a concept that are essential for the concept, then the concept has been schematized and purified with respect to the concept. Because of equality of range, the percept that resolves from the concept is constrained to those actual properties that are essential. In this case, with attention focused only on essential properties in concept and percept, the concept has no more potential than the percept; the concept really is, in a sense equivalent to the percept; the concept, in other words, is perceived in full generality in the percept. This is my reinterpretation of the phrase “seeing the general in the particular.”

To “see the general in the particular,” to perceive schematically, is the human cognitive function that enables image-based geometrical reasoning, specifically schematic perception—to gaze upon an object with a simple basic shape and allow the characteristics of this object except its basic shape to recede into the perceptual background. This exercise may be attempted with the Angles in a Circle Theorem of the Prologue. The goal is to see *only* a central angle opposite a circular arc and an angle at the circumference opposite the same arc—independently of its specific manifestations in the diagrams of Figures P.1 and P.2.

Just as the psychological process involving concepts and percepts has a physiological correlate, which consists of neural activity in the posterior cortex, the process of schematizing, constraining a concept or percept to its essential properties, also has its physiological correlate. Activity of the cerebellum with respect to its connections with the posterior cortex, I propose in Chapter 5, is just this correlate. Thus, the cerebellum reduces the fuzziness inherent in the associatively formed concepts in the cerebral cortex, resulting in crisp, pure concepts suitable for geometrical reasoning: *the cerebellum is a lens that sharpens cognition*. (The cerebellum may have other functional roles, even with respect to the posterior cortex, and any claims herein should not be regarded as exclusive.)

The property analysis technique is an important aspect of the theory developed herein. Property analysis relies on the cognitive network interpretation of concepts and percepts, but the specific implementation is an original contribution of this dissertation.

1.4 Educational Neuroscience

This dissertation constitutes a study in educational neuroscience. Educational neuroscience has been broadly defined by Campbell (2006) according to two main criteria:

> First, educational neuroscience is characterized by soundly reasoned and evidence-based research into ways in which the neurosciences can inform educational practice, and vice versa. Secondly, educational research in cognitive psychology informed by, and informing, cognitive neuroscience constitutes the core of educational neuroscience. (p. 442)

I hope the reader finds that the research in this dissertation is soundly reasoned and (at least indirectly) evidence-based. Moreover, ideas in cognitive psychology relevant to geometrical reasoning constrain and inform our neuroscientific knowledge, and vice versa, and therefore Campbell's second criterion is satisfied.

It is important to distinguish educational neuroscience, of the type pursued in this dissertation from the so-called *brain-based education* movement. A strong critique of brain-based education has been led by Bruer (1997, 1999, 2006). He argues that the results of cognitive neuroscience have been misconstrued and misapplied by educators, because they have over-interpreted them or taken them out of context. In particular, Bruer (1997) critiques the distinction between right-brain and left-brain learning and the notion that there are critical periods, during which certain types of learning are facilitated. He suggests that direct application of neuroscientific research to education is always a "bridge too far." The correct approach, according to Bruer, is to utilize cognitive psychology as an intermediate discipline between neuroscience and education. Consequently, the “bridge too far” is replaced by two smaller bridges that can be crossed more readily. The bridges from education to cognitive psychology and from cognitive psychology to neuroscience are already well established.

It is apparent, however, that these two smaller bridges, rather than the “bridge too far,” is basically the approach of this dissertation. A psychological analysis suggests the idea of schematization, the focusing of attention on essential properties, so that mathematical structure is perceived directly. There are powerful connections between the psychological idea of schematization and mathematics education, starting with the reinterpretation herein of the notion of “seeing the general in the particular” of Mason and Pimm (1984). On the other hand, the overall framework introduced below and articulated further in Chapter 3 permits the search for a neural process that corresponds to the psychological process of schematization. Chapter 5 develops the idea that this correspondence may involve the cerebellum. Cerebellar research has implications for the cognitive process of schematization, which in turn has implications for mathematics education.

The approach in which physiological ideas constrain and inform theories in cognitive psychology (and vice versa) is none other than Campbell’s (2006) second criterion defining educational neuroscience. In this respect, the field of mathematics education is often regarded as belonging to cognitive psychology, although with its own goals and terminology. The two bridges are already in place, and there is no question of “a bridge too far.” Educational neuroscience should not be confused with brain-based education. The approach of this dissertation is resolutely aligned with the former.

A foundational assumption of educational neuroscience is that cognitive experience and functioning (e.g., mathematical reasoning) is *embodied* in the neural activity of the brain (Campbell, 2001, 2003, 2006). In other words, there is a relationship between subjective cognitive functioning and objective neural activity, such that knowledge of one entails knowledge of the other. This notion is one form of the theory of *embodied cognition* (Varela, et al., 1991). There are other forms of the theory, and I mention here two other approaches that help situate educational neuroscience within the broader community of mathematics education.

Firstly, the cognitive metaphors of Lakoff and Núñez (2000) have attracted considerable attention among mathematics educators. Campbell (2010) comments on various interpretations of the notion of embodiment:

> [T]he notion of embodiment can be viewed in a variety of different ways. Embodiment can be considered in terms of concrete particulars. For instance, a chalk stroke on a blackboard can be considered as a concrete embodiment of the concept of a line, or a marble can be considered the embodiment of a sphere. This view of embodiment is very much akin to the Platonic view, whereby concrete particulars are mere shadows of ideas, which have transcendent existence of their own. Embodiment can also be viewed as being akin to the Aristotelian view, where ideas are somehow embodied, *qua* immanent, *within* concrete particulars. In this view, mathematical manipulatives, popular in mathematics education, embody mathematical ideas. (p. 317, author’s italics)

In contrast, Lakoff and Núñez explicitly claim, “Human mathematics is not a reflection of a mathematics existing external to human beings; it is neither transcendent nor part of the physical universe" (p. 349). According to Campbell, therefore, the view of embodiment propounded by Lakoff and Núñez differs from both the Platonic and Aristotelian approaches. The primacy in Lakoff and Núñez belongs not to the idea but to the physical situation that embodies the "idea-to-be." In other words, "Mathematics is a mental creation that evolved to study objects in the world" (p. 350).

The construct by which mathematics is lifted from the world, or rather created by human agency through experience with the world, is the *conceptual metaphor*. Lakoff and Núñez (2000) write, "Mathematical objects are embodied concepts—that is, ideas that are ultimately grounded in human experience and put together via normal human conceptual mechanisms, . . . [such as] conceptual metaphors" (p. 366). For example, a collection of physical objects, metaphorically, may be regarded as a number.

Almost all mathematics, according to Lakoff and Núñez (2000), has its ultimate origins in conceptual metaphors that depend on ordinary human activities such as walking, turning, gathering, and so on. The authors demonstrate that this is indeed a possibility by analyzing some advanced mathematical ideas in terms of primitive conceptual metaphors. (An exception to this rule, according to the authors, is the ability to perform arithmetical operations on small quantities, which is an innate ability in humans—see Section 3.3—and which does not therefore require conceptual metaphors.)

The embodiment of Lakoff and Núñez (2000) is to be taken literally on the scale of arms and legs, fingers and toes, and the way in which the body interacts with the world, macroscopically, as it were, in the performance of everyday human activities. These activities form the basis of conceptual metaphors. The meaning of “embodiment” in educational neuroscience is somewhat different.

In the first place, educational neuroscience, according to Campbell (2006), requires a basic metaphysical assumption, such as neutral monism (Handscomb, 2007), in order to connect subjective experience to objective neural activity. In this dissertation I develop the implications of a strong version of this assumption, that the structure of psychological functioning (specifically geometrical reasoning) reflects the structure of neural activity, and vice versa (see Section 3.1). There is no explicit metaphysical assumption of like kind in Lakoff and Núñez (2000). One has the sense of a Cartesian dualism, in which mind observes action in the external world and uses these observations, by means of metaphor, to compose mathematical ideas.

To be sure, Lakoff and Núñez (2000) are clear that neural activity underlies psychological functions such as mathematical reasoning. However, they are not explicit about the relationship between these two domains on the microscopic level of dendrites and synapses. Their argument remains at the macroscopic level of arms and legs, fingers and toes.

From the perspective of educational neuroscience, human beings are themselves part of the physical universe. The structure of our mathematics, therefore, reflects the structure that inheres in the physical universe. In other words, cognitive functioning is embodied in our interaction with the universe, whether microscopically or macroscopically. Specifically, the higher cognitive functions such as mathematical reasoning are embodied in brain structure and associated brain activity. This view has some advantages. Campbell (2010) explains,

> To help illustrate the unifying power of this embodied view with regard to *mathematics* educational neuroscience, consider Eugene Wigner's renowned reflections on the "unreasonable effectiveness of mathematics in the natural sciences" (1960). If mind (*res cogitans*) is fundamentally (i.e., ontologically) distinct from the material world *(res extensa*), it remains a great grand mystery as to why mathematics can be *applied* to the world so effectively. If mind is embedded within the material world, as the embodied view entails, mystery dissolves into expectation (Campbell, 2001). (p. 317, author's italics)

Is the "unreasonable effectiveness of mathematics" readily explicable within the framework of Lakoff and Núñez (2000)?

Lakoff and Núñez (2000) have offered one approach to embodied cognition that has been influential for mathematics educators. It begins from a different set of assumptions and proceeds in a different direction to the research herein. The authors' ideas are fascinating and far-reaching, but they appear not to tell the whole story. My approach in this dissertation is not the whole story either. I have disregarded considerations of language, culture, and communication, which must all belong to a complete theory of mathematics education. For readers concerned with such matters, I conclude this section on educational neuroscience with brief mention of another approach to embodied cognition, due to Radford, Bardini, Sabena, Diallo, and Simbagoye (2005) that *does* attempt to take account of language, culture, and communication.

Radford is known for his semiotic-cultural approach to mathematics education (e.g., 2008). In Radford et al. (2005), the theory of embodied cognition is augmented by the semiotic-cultural approach. After all, the objects that students relate with in an embodied sense—graphs, calculators, and so on—are shared cultural artefacts. According to the authors,

> [T]he world that the body encounters is a cultural world populated by other bodies, objects, signs, and meanings, a world already endowed with ethical, aesthetical, scientific and other values. These values provide the world with specific configurations that, instead of being neutral, qualify the body with the historicity of events and concepts deposited in language, artifacts, and institutions (Foucault, 2001, p. 1011). (p. 114)

It follows that

> an account of the embodied nature of thinking must come to terms with the problem of the relationship between the body as a locus for the constitution of an individual's subjective meanings and the historically constituted cultural system of meanings and concepts that exists prior to that particular individual's actions. (pp. 114-115)

Despite the relevance to mathematics education of culture, language, and communication, these considerations are beyond the scope of this dissertation. The cultural content of objects refers to shared meaning, and presumably to a shared structure of thought. A shared structure of thought should have implications for social neuroscience. Theoretically, the cultural content of objects should also fall within the range of educational neuroscience. However, that level of analysis is beyond what I attempt herein.

The embodied cognition of educational neuroscience, and of this dissertation in particular, is concerned primarily with embodiment in neural structure and activity. It is embodiment at a more fundamental level than that of Lakoff and Núñez (2000). On the other hand, the consolidation of embodiment at the level of synapses and neurons may make the journey back to communication and culture more arduous. I hope to demonstrate to the reader throughout the course of this dissertation that embodiment of thought in neural structure and neural activity is a promising level of analysis in mathematics education research.

▲►▼◄▲►▼◄▲►▼◄

This chapter has developed a broad outline of the dissertation. A specific cognitive function, the ability to “see the general in the particular” is identified and investigated. The theoretical foundation of embodied cognition and the research framework of educational neuroscience permit coherent integration of psychological and neuroscientific ideas. It is an attempt to formulate a view of image-based geometrical reasoning mutually constrained by due consideration given to both educational and neuroscientific research. More specifically, I propose a particular understanding of the functional role of the cerebellum in facilitating schematic inferences from geometrical diagrams, and a further hypothesis that decontextualization of geometrical concepts may be a valuable pedagogical strategy. An important component of this general introduction is a preview of the property analysis technique. Chapter 2 now investigates the relationship of this dissertation to prior research.

CHAPTER 2: LITERATURE REVIEW

The literature review in this chapter highlights the academic context of this dissertation, thereby developing the ideas introduced in the previous chapter. A variety of topics is discussed, including (1) philosophy of mind, (2) cognitive neuroscience of the cerebral cortex, (3) cognitive neuroscience of the cerebellum, and (4) mathematics education. Each of these areas is covered separately below. The multidisciplinary nature of the research herein prohibits exhaustive engagement of the literature of any of these areas. Many of the references cited are seminal or otherwise central to their fields.

2.1 Philosophy of Mind

The first requirement of the theoretical framework is to establish the psychological domain, the physiological domain, and the relationship between the two. The theory of embodied cognition is a natural choice, as it proposes a dynamic unity of mind and brain that gives priority to neither. The seminal reference for embodied cognition is Varela et al. (1991). The ontology and epistemology of embodied cognition are clarified by Campbell (2001, 2003). Handscomb (2007) points out the remarkable similarities between Campbell's formulation of the theory of embodied cognition and the neutral monism of Spinoza (1667/1996). My ontology and epistemology, therefore, is a classical Spinozistic neutral monism. Chapter 1 has already contrasted embodied cognition as understood herein with the cognitive metaphors of Lakoff and Núñez (2000) and the semiotic-cultural approach of Radford et al. (2005).

Given the foundational significance of the embodied cognition of Varela et al. (1991), it would have been natural to turn to neurophenomenology because neurophenomenology is the research method primarily associated with embodied cognition. In neurophenomenology, phenomenological hypotheses constrain and inform neurophysiological hypotheses, and vice versa. However, I use the approach of cognitive psychology for investigating subjective first-person experience, rather than the formal techniques of phenomenology. The research framework whereby the ideas and understandings of cognitive psychology (with special reference to mathematical reasoning and education) inform and constrain those of neurophysiology is that of educational neuroscience (Campbell, 2006), which is consciously utilized herein.

My thinking evolved in the following temporal order: (1) I proposed a psychological model of the process of image-based reasoning in geometry (Handscomb, 2005, 2006); (2) I began investigating the cerebellum for its potential contribution to mathematical reasoning, and specifically geometrical reasoning (inspired by Vandervert, 1997, 1999, 2003); (3) I began to consider that a functional role of the cerebellum is that it contributes to the human cognitive faculty to “see the general in the particular” (Mason & Pimm, 1984), which is important for geometrical thinking and learning; (4) I investigated cognitive neuroscience of cerebral cortex and cerebellum that had the potential to explicate this specific cerebellar function; (5) finally, I began to consider that the resultant theory could provide a novel framework for understanding research in mathematics education, potentially in a more unified and coherent way. The reader may appreciate the game of tennis, in which ideas bounce back and forth between psychological and neurophysiological accounts of cognition. This, indeed, is the research method of educational neuroscience.

With regard to neurophenomenology, which inspired educational neuroscience (S. R. Campbell, personal communication, September 2005), the two seminal papers are Varela (1996) and Varela (1999). Varela's ideas are developed further in Lutz and Thompson (2003). Varela (1995), Lachaux, Rodriguez, Martinerie, and Varela (1999), Varela, Lachaux, Rodriguez, and Martinerie (2001), and Lutz, Lachaux, Martinerie, and Varela (2002) focus on the neurophysiological aspect of neurophenomenology, whereas Varela and Shear (1999) and Depraz, Varela, and Vermesch (2000) concentrate on the phenomenological aspect. Thompson and Varela (2001) and Varela and Thompson (2003) discuss the notion of reciprocal causation between consciousness and neural activity. Neurophenomenology illustrates the idea of reciprocal restraints, whereby a theory of the subjective domain influences a theory of the objective domain, and vice versa. However, educational neuroscience is the research framework for this dissertation because it utilizes an approach based on cognitive psychology for the subjective domain, rather than phenomenology, as discussed in Campbell (2006).

Both Varela (1995) and Fuster (2003) refer to what Fuster calls the *cognitive code*, the relationship between specific cognitive functions and specific neural activities. The cognitive code recalls Spinoza's (1667/1996) claim that the "*order and connection of ideas is the same as the order and connection of things*" (E2 P7, author's italics). In the literature of cognitive neuroscience, the cognitive code is usually spoken of in terms of neural correlates. The search for neural correlates stems from the ideas of Gall (see Hollander, 1920, Part II, Section II), who proposed that cognitive functions were localized in brain regions. Although Gall's pseudo-science of phrenology failed to gain widespread acceptance, the crucial idea of localization stuck and was substantiated by the identification of Broca's and Wernicke's areas for language.

Lashley (1950/1988) subsequently demonstrated that not all cognitive function is localized in the brain. Specifically, memory is a distributed function. The stage was set for a connectionist reinterpretation of neuroscience, whose theoretical underpinnings were established in McClelland and Rumelhart (1987) and Rumelhart, McClelland, and The PDP Research Group (1987). Connectionist research is often driven by the desire to recreate brain-like structures rather than the desire to investigate the human brain per se. However, the cognitive network theory of Fuster (2003) is a practical, rather than theoretical, connectionism grounded in brain research and the theory of Hayek (1952/1976). It permits a balanced view of localization versus distribution of the brain activity correlative to cognitive functions.

Special care is needed when time is a factor in cognition. This issue is discussed by Bergson (1889/1960, 1896/2005, 1907/1944) and his followers Whitehead (1919/2007, 1925/1967) and Brown (2000, 2002). Simply put, objective time and subjective time are incommensurable. The subjective experience of James' (1890/1950) sensible present, Bergson's (1896/2005) duration, or Brown's (2002) virtual time corresponds to a certain "temporal thickness" that is present in an instant of objective time. The objective now, constituted perhaps of a single pulse of neural activity, may cover a substantial span of subjective time. There is potential for confusion with respect to the neural correlates of cognitive functions that have a temporal component. The procedures of the anterior cortex *do* extend into time. I do discuss the procedures of the anterior cortex in Chapter 3 and Chapter 4, but my approach to the procedures is tentative, and I have to admit that I do not have all the answers.

Bergson's (e.g., 1895/2005) philosophy of duration affords a view of cognition as a process that is simultaneously bottom-up and top-down. This idea is discussed with respect to the visual system in Gilbert and Sigman (2001) and is developed on a broader basis by Fuster (2003) and Roth and Hwang (2006).

The property analysis, discussed in Chapter 1, leads to the notion of schematization, or "seeing the general in the particular" (Mason & Pimm, 1984), as reinterpreted herein, in which attention in the concept-percept structure is constrained to those properties that are essential to the concept. Bergson's (1896/2005) metaphorical cone of duration becomes a cylinder, shedding light on the meaning of schematization and equality of range.

2.2 The Cerebral Cortex and Cognitive Neuroscience

A necessary stage in the research process was to identify physiological models of the cerebral cortex and the cerebellum, with the goal of finding the neural correlate of schematic perception. This section discusses the model chosen for the cerebral cortex and summarizes the related research literature in cognitive neuroscience.

Fuster's (2003; summarized, 2006) cognitive network theory was selected as a theoretical model of neural activity in the cerebral cortex. It facilitates an understanding of the relationship between concepts and percepts (and their analogues in action) and strongly suggests the property analysis, which is foundational for my arguments. It appears that Hayek's (1952/1976) theoretical psychology was the inspiration for cognitive network theory. The cognitive network theory is connectionist in orientation, although its practical approach aims to describe actual brain mechanisms rather than to theorize about brain-like structures.

An alternative to cognitive network theory would have been the Brainweb of Varela et al. (2001), because the Brainweb is explicitly the neurophysiological model for neurophenomenology. There were two reasons for choosing cognitive network theory over the Brainweb. Firstly, the research framework for this dissertation is educational neuroscience rather than neurophenomenology. Secondly, Friston (1994) distinguishes between neural connections that are functional (i.e., inferred statistically), and neural connections that are effective (i.e., structural): connections in the Brainweb are functional, whereas in cognitive network theory they are effective. A model based on effective connections has more conceptual clarity, and it facilitates the property analysis argument of this dissertation. It should be noted that effective neural connections have directionality, and that they are *efferent* from A to B or equivalently *afferent* to B from A.

The fundamental units of analysis in the brain are the neurons, which are linked via effective connections to form network structures. Actually, Fuster (2003) explains that groups of neurons in minicolumns are the smallest unit of analysis. A minicolumn, as a single unit, is linked to other minicolumns. Small, relatively localized networks join to form larger networks, which in turn connect to form ever larger, more widely distributed assemblies of neurons. These structures are *cognitive* networks to the extent that their activity is correlative to cognitive functions. Importantly, the cognitive networks are hierarchically structured.

According to cognitive network theory, there is a fundamental division of brain and nervous system into action and perception moieties. It was Betz (1874, cited in Fuster, 2003), who first identified this partition. Cognitive networks of the anterior cortex, broadly speaking in front of the Rolandic fissure, are correlative to action, whereas cognitive networks of the posterior cortex, broadly speaking behind the Rolandic fissure, are responsible for perception. According to Fuster, and substantiated by Koechlin, Ody, and Kouneiher (2003), there is a twin hierarchy of cognitive networks, one for the anterior cortex and one for the posterior cortex.

The action-perception division, both psychologically and physiologically, is absolute. Research into the neural correlates of higher cognitive functions is muddy water, through which many have waded. The sharp distinction between action and perception promotes clarity.

Action relates to time, whereas perception relates to space. Rising through the cognitive network hierarchy in the anterior cortex, cognitive networks correlate with concatenated acts (i.e., procedures) that are increasingly less specific in time. Rising through the cognitive network hierarchy in the posterior cortex, on the other hand, cognitive networks correlate with generalized perceptions (i.e., concepts) that are increasingly less specific in space.

There is a bidirectional flow of neural activity, bottom-up and top-down, in both halves of the cerebral cortex. In the posterior cortex, percepts associate into concepts and conversely concepts resolve to percepts; in the anterior cortex, acts associate into procedures and conversely procedures resolve to acts. This bidirectional flow is given a cognitive neuroscience interpretation by Fuster (2003), a philosophical interpretation by Bergson (1896/2005), an educational interpretation by Roth and Hwang (2006), and a visual-system interpretation by Gilbert and Sigman (2001).

Action, or rather the movement into which it discharges, is perceived, and subsequently action is guided by perception. The reciprocal relationship between perception and action, the perception-action loop, was first identified by Uexküll (1926, cited in Fuster, 2003), but it is central to the phenomenology of Merleau-Ponty (1942/2006) and the embodied cognition of Varela et al. (1991). Fuster argues that there are reciprocal connections between perception and action at all levels of their respective hierarchies. At the lowest level acts discharge as movement in the world, leading to sensation, and then to percepts. At higher levels, the perception-action loop is confined to the cerebral cortex, and does not pass through the world.

With regard to the ontogenesis of the cognitive network structure, Fuster (2003) makes reference firstly to Hebb's (1949/1964) laws and secondly to the experience-expectant/experience-dependent distinction of Black and Greenough (1986), Greenough, Black, and Wallace (1987), and Greenough and Black (1992). A hierarchical network structure is exactly what one would expect to result from the application of Hebb's laws, as explained in Chapter 4.

One way of thinking about the top-down flow in the posterior cortex, according to Fuster (2003), is that it is an attentional process, which inhibits the explosion of associative activity that could begin with afferent sensation. In addition, the reciprocal connections of the perception-action loop are attentional in that they mutually constrain cognitive networks in the respective hierarchies. I propose a third type of attentional process, involving the cerebellum, which is discussed below.

The focus in this dissertation is the posterior cortex. Firstly, as established earlier, image-based reasoning in geometry is an important part of geometrical reasoning, both historically and today in the mathematics classroom. The main topic of my investigation is schematic perception and inferencing from geometry diagrams, which is an aspect of image-based reasoning in geometry. In my opinion, geometry is the mathematical discipline of perception par excellence, which deeply implicates neural structures of the posterior cortex. Secondly, I would like to promote the view that higher cognitive function is shared between anterior cortex and posterior cortex rather than being centred in the prefrontal anterior cortex. Thirdly, analysis involving the temporal generality of the anterior cortex is more difficult than analysis involving the spatial generality of the posterior cortex. A more thorough investigation of procedures and the anterior cortex has to wait for now.

As mentioned above, the goal of understanding geometrical reasoning is foremost. There has been little work on the neurophysiological activity that corresponds to geometrical reasoning per se. However, there has been considerable research on the neurophysiology of vision.

Ungerleider and Mishkin (1982) and Mishkin, Ungerleider, and Macko (1983) developed the idea of the dorsal stream and ventral stream for visual processing. Accordingly, the dorsal stream is concerned with spatial features of objects in the peripheral visual field, whereas the ventral stream is concerned with identification of visual characteristics of objects in the central visual field. Other researchers have studied the two streams. There is agreement that the role of the ventral stream is visual perception. However, the functional significance of the dorsal stream has been disputed. According to Milner and Goodale (1995), the role of the dorsal stream is visual guidance of action. DeYoe and Van Essen (1988) argue that visual elements are processed along both dorsal stream and ventral stream in a complex, interacting manner, and therefore perhaps it is inappropriate to make an absolute distinction between the roles of the two streams for cognition. In any case, much of the research on the two streams has been conducted on primates. The human inferior parietal cortex has no primate homologue, although the inferior parietal cortex is usually associated with the dorsal stream. Husain and Nachev (2006) argue that a different role entirely must be sought for this brain structure. One of the proposals of this dissertation is that the inferior parietal cortex is responsible for visual concepts rather than visual percepts.

According to Kosslyn (1988), Roland and Gulyas (1994, 1995), and Kosslyn, Ganis, and Thompson (2001) visual imagery uses the whole visual system of the brain, except for the primary visual cortex. In other words, imaginary objects have substantially the same neural correlates as perceived objects where the perception is initiated by sensation. According to Kilner, Paulignan, and Broussaoud (2004), the same is true with respect to motor imagery and motor activity: the identical neural structures are involved in both, except that motor imagery does not utilize the primary motor cortex.

According to Grill-Spector and Malach (2004) the visual processing system has a specialization aspect as well as a hierarchical aspect. In other words, visual sensation is analyzed into elements (i.e., “properties”) as well as synthesized into unified percepts. Gilbert and Sigman (2001) proposed that there is a top-down flow of neural activity in the visual system as well as a bottom-up flow, as mentioned above. Roland and Gulyas (1995) identify brain activity in the parietal cortex accompanying perception of static geometrical figures, and

Pasupathy (2006) and Vinberg and Grill-Spector (2008) investigate shape recognition in the occipital cortex.

Shape recognition, at least, is approaching image-based geometrical reasoning. In comparison, Dehaene and colleagues have done considerable work on the specific neural correlates of arithmetical reasoning. The triple-code model for arithmetical reasoning was proposed in a seminal paper by Dehaene (1992). It was investigated further in Dehaene and Cohen (1995, 1997), Cohen and Dehaene (1995, 1996), Dehaene (1996), Dehaene, Piazza, Pinel, and Cohen (2004), and Dehaene, Molko, Cohen, and Wilson (2004). The spatial aspects of arithmetical reasoning are discussed in Dehaene (1997) and Hubbard, Piazza, Pinel, and Dehaene (2005). Dehaene (1997), in particular, is important for its localization of the innate “number sense” in the parietal cortex. Butterworth (1999) argues a similar point with respect to his “number module.” Dehaene, Izard, Pica, and Spelke (2006) investigate image-based reasoning in geometry rather than arithmetical, reasoning, but from the perspective of cultural rather than physiological universals. There has been no investigation of image-based geometrical reasoning comparable to that of arithmetical reasoning. I hope to redress the balance in a small way with this dissertation. Of course, my investigation only concerns schematic perception and inferences from geometrical diagrams, both aspects of image-based reasoning in geometry.

An important component of my thesis is interpreting the role of the cerebellum in attention with respect to patterns of neural activation in the posterior cortex. It is therefore relevant to engage neuroscientific literature concerning attention. I draw upon two main sources. Sohlberg and Mateer (1989) proposed a five-level hierarchical model of attention in neuropsychology, based on the temporal progress of recovery of attention following brain lesions. The most primitive form of attention is focused attention, followed by sustained attention, selective attention, alternating attention, and finally divided attention. My main concern herein is selective attention. It is important to note that the ability for alternating attention, for example, implies the ability for selective attention because of the hierarchical nature of the model. A complementary approach is Knudson's (2007) four-component neurophysiological model of attention. The four components are (1) working memory, (2) top-down sensitivity control, (3) bottom-up salience filters, and (4) competitive selection. Knudson makes substantial reference to hierarchies of neural networks, which recalls the cognitive network theory. Knudson's neurophysiological model of attention is consistent with the theoretical framework of this dissertation. I discuss these models and related matters in more detail in Chapter 4.

In summary, the goal of my investigation of the cerebral cortex is not to identify hard, specific neural correlates for image-based reasoning in K-12 geometry. It is quite probable that all kinds of cognitive networks at the upper ends of the hierarchies in posterior and anterior cortex contribute to geometrical reasoning. A study of the visual system points in the right direction for image-based geometrical reasoning, and moreover it justifies the division of the cognitive networks for vision into three categories, corresponding to visual concepts, visual percepts, and visual properties. These neural structures are important for image-based reasoning in geometry, but, like other types of cognitive network, they may be implicated in various forms of reasoning, geometrical or not. Most important is the cognitive network structure itself, which permits the property analysis.

2.3 The Cerebellum and Cognitive Neuroscience

A hypothesis that I develop in this dissertation concerns a functional role of the cerebellum. The cerebellum may be crucially important for image-based geometrical reasoning in particular, and perhaps mathematical reasoning more generally. This may come as a surprise. After all, classical theories of the cerebellum connect it with smooth, efficient motor behaviour. However, I suggest herein that the cerebellum is also necessary for the focused, crisp thinking that is necessary for mathematics. Indeed, the cerebellar schematization hypothesis (CSH) proposes that the cerebellum contributes to the schematization of the concepts and percepts of the posterior cortex.

There are two lines of reasoning in support of CSH. The first set of arguments concerns the phylogenesis of the cerebellum, connections between the cerebellum and the cerebral cortex, and the histology of the cerebellum; the second set of arguments interprets existing research and theory on the cerebellum to substantiate the hypothesis. Chapter 5 contains the various lines of reasoning in support of the CSH.

There is a vast literature on the cerebellum. I discuss herein only a small but central portion of research on the cerebellum, which I have chosen in an attempt to reflect the most important or seminal works in this area, as well as those that are most specifically related to my research. My thesis is primarily intended as educational neuroscience. Its main contribution lies in a mutually constraining dialogue between matters of concern to both education and the neurosciences. It does not, nor could it, offer a fully nuanced account of all the relevant associated literature from either of these vast fields. It constitutes a beginning, not a definitive end.

Sections 2.3.1 to 2.3.3 below concern the first line of reasoning concerning the structural argument in support of the CSH; Sections 2.3.4 to 2.3.6 mainly concern the second line of reasoning, providing interpretations of existing literature in support of the CSH; finally, Section 2.3.7 discusses the CSH in relation to mathematics education.

2.3.1 Physiology and Phylogenesis of Cerebellum

There are many sources for cerebellar physiology. Those cited herein are Altman and Bayer (1997), Ito (2006), Apps and Garwicz (2005), Ramnani (2006), and Sultan and Glickstein (2007). The cerebellum consists of a cerebellar cortex, which is a thin sheet of neural tissue, overlaying the cerebellar white matter. The cerebellar white matter contains the deep cerebellar nuclei, including the dentate nucleus. According to Voogd, Feirabend, and Schoen (1990), the cerebellar cortex expanded laterally in phylogenesis from vermis, through intermediate zone, to lateral zone (see Figure 5.1).

The phylogenesis of the cerebellum and related structures is important, not least because those structures that are of very recent origin phylogenetically may correspond to cognitive functions that are also of very recent origin phylogenetically. Passingham (1975) emphasizes the coordinated growth in phylogenesis of cerebellum and cerebral association cortex. The human cerebellum is compared with the primate cerebellum in Whiting and Barton (2003). Sultan and Glickstein (2007) make a comparative study of the cerebellum across many animal species. Paulin (2005) identifies the evolutionary origins of the cerebellum in the emergence of predator-prey behaviour in the early vertebrates.

The phylogenesis of the dentate nucleus and related cerebellar and cerebral areas is the foundation of the work of Dow and colleagues: Dow (1974, 1988), Leiner, Leiner, and Dow (1986, 1987, 1989, 1991, 1993), and Leiner and Leiner (1997). The dentate nucleus of the cerebellum is associated with non-motor functions. Leiner and Leiner emphasize (1) the information processing capabilities of cerebrocerebellar fibres, and (2) the inadequacy of primate study alone for investigation of the human cerebellum.

The work of MacLeod (2000) and MacLeod, Zilles, Schleicher, Rilling, and Gibson (2003) challenges the work of Dow and colleagues on the coordinated expansion of the cerebellum and dentate nucleus. Macleod, using a larger sample of primate brains, demonstrated that the inferior olive, rather than the dentate nucleus, expanded in phylogenesis when compared with the lateral zones of the cerebellar cortex. The inferior olive and red nucleus are closely associated with the cerebrocerebellar system (Leiner et al., 1987), as discussed below.

2.3.2 Neural Connections between Cerebral Cortex and Cerebellum

Schmahmann (1996) suggests that if there is a cerebellar contribution to higher cognitive function, then there must be a neurological substrate that supports this contribution. The cerebrocerebellar connections between phylogenetically recent cerebellar structures and association areas of the cerebral cortex constitute this neurological substrate.

There are two ways of determining the connections between the cerebral cortex and the cerebellum: neuronal tracing studies and functional connectivity magnetic resonance imaging (FCMRI). In neuronal tracing studies, a foreign element is introduced into the brain, and its spread along neural pathways is measured (Oztas, 2003). FCMRI correlates changes in blood oxygen levels of different regions of the brain, with the assumption that regions of strong correlation are structurally connected (Cordes et al., 2000). There are advantages and disadvantages to both methods. Neuronal tracing studies determine structural connections, but are restricted to monkeys; FCMRI studies may only indirectly identify structural connections, but are applicable to humans.

The results of neuronal tracing studies are discussed in Schmahmann and Pandya (1989, 1990, 1995, 1997), Middleton and Strick (1994, 1997), Clower, West, Lynch, and Strick (2001), and Strick, Dum, and Fiez (2009). Overall, it is established, at least with monkeys, that there are neural projections from the cerebral cortex to the cerebellum via the pons, and back to the cerebral cortex from the cerebellum via the thalamus. As well as reciprocal connections between cerebellum and anterior cortex—both motor and association areas—there are reciprocal connections between cerebellum and posterior cortex, specifically the posterior parietal cortex. Strick et al. discuss projections from phylogenetically recent regions of the dentate nucleus to posterior parietal cortex.

Leiner and Leiner (1997), and more recently Krienen and Buckner (2009), point out that primate studies may well be inadequate for studying the contribution of the cerebellum to higher cognitive function—there may well be substantial cerebrocerebellar connections in humans that have no homologue in monkeys.

Allen et al. (2005), using FCMRI methods, substantiated the claim that the cerebellum connects to higher association areas in the *human* parietal cortex. Krienen and Buckner (2009), in another FCMRI study, established that there are at least four distinct circuits between frontal cortex and cerebellum in humans, including connections to prefrontal cortex. Presumably, relationships between cerebellum and association areas in cerebral cortex support higher cognitive functions.

There are (at least) two perspectives on the architecture of cerebrocerebellar connectivity—the classical understanding and the closed-loop understanding. In the classical understanding, explained in Strick et al. (2009), the cerebellum gathers information from widespread regions of the cerebral cortex, which it processes and distributes to motor regions of the anterior cortex. For example, perceptual data, by this means, facilitate smooth, efficient motor behaviour. Updated variations of the classical idea take account of cerebellar connections to the prefrontal cortex, which permit the cerebellum to contribute also to higher cognitive functions. Stein, Miall, and Weir (1987) offer some empirical support for the classical notion. MacLeod (2005) cites evidence that the number of fibres from posterior cortex to cerebellum and from cerebellum to anterior cortex is greater than the number of fibres in the reverse directions, a circumstance which supports the classical view.

Against the classical understanding, neuronal tracing studies sustain the idea that the cerebellum is reciprocally connected to the cerebral cortex in distinct, segregated, closed-loop circuits. Middleton and Strick (1998) explain this understanding through their three principles of cerebrocerebellar connectivity. Kelly and Strick (2003), Krienen and Buckner (2009), and Strick et al. (2009) all argue for this point of view. The closed-loop architecture is also supported on general principles of neural connectivity (Varela et al., 1991).

The brain structures of the cerebrocerebellar loops, discussed above, include cerebral cortex, pons, cerebellum, and thalamus. The red nucleus and inferior olive are closely related to this system, with projections from cerebral cortex to red nucleus, from red nucleus to inferior olive, and from inferior olive to cerebellar cortex (Leiner et al., 1987; Leiner et al, 1993). Connections from the parietal cortex to the red nucleus are demonstrated by Dum and Strick (2003) and Habas and Cabanis (2007), which means that connections from the cerebral cortex to red olive and thence inferior olive may be implicated in the cerbrocerebellar loops between cerebellum and parietal cortex.

Brodal and Bjaalie (1997) review research on connections from cerebral cortex to cerebellum via the pons. They identified convergent and divergent neural connectivity between cerebral cortex and cerebellum, via the pons. In other words, a small region of the cerebral cortex may project to multiple, distributed regions of the cerebellar cortex; on the other hand, a small region of the cerebellar cortex receives projections from multiple, distributed regions of the cerebral cortex. This more complex connectivity does not preclude the possibility of closed, distinct cerebrocerebellar loops. For my thesis, I only need closed, distinct cerebrocerebellar loops at a fairly coarse-grained level of analysis. The convergence and divergence of connections may occur as a substructure of a single cerebrocerebellar loop. In fact, research cited by the authors, which I address in Chapter 5, substantiates closed, distinct cerebrocerebellar loops, provided these loops are taken to exist at a coarse-grained level of analysis.

Bjaalie and Brodal (1997) provide a more detailed report specifically on projections from the cat pons to the cerebellar paraflocculus. This research supports the results discussed above in Brodal and Bjaalie (1997). However, the paraflocculus is a very small part of the cerebellar cortex, and it is concerned with motor behaviour. It is certainly plausible that the paraflocculus as a whole belongs to a *single* cerebrocerebellar loop, within which there is convergence and divergence. Moreover, the authors suggest that their research is not necessarily generalizable to other areas of the cerebellar cortex.

Welker (1986) also appears initially to contradict the notion of closed, distinct cerebrocerebellar loops. The author mapped projections from somatosensory cortex to cerebellar cortex in several animal species. He discovered a fractured somatotopy in the cerebellum, so that one particular sensory region may project to widespread locations in the cerebellar cortex, supporting the work of Brodal and Bjaalie (1997). Once again, however, the results may refer to a single cerebrocerebellar loop at a different level of analysis. In any case, there is no necessity for the cerebellar component of a cerebrocerebellar loop to be highly localized in the cerebellum.

Lastly, the more recent research of Manni and Petrosini (2004) is a historical review of localization in the cerebellar cortex. These authors do give qualified support for the notion of cerebellar localization: research overall demonstrates that there is a somatotopic mapping of the human body within the cerebellar cortex, in contrast to the implication of Welker (1986) and Brodal and Bjaalie (1997). If this is the case for somatosensory cortex, there are grounds to suppose such mappings hold for other regions of the cerebral cortex as well.

2.3.3 Uniformity of Cerebellar Histology

According to Sultan and Glickstein (2007) the small-scale structure of the cerebellum is uniform throughout the cerebellum and the same for all mammals. Wolpert, Miall, and Kawato (1998), Paulin, Hoffman, and Assad (2001), Apps and Garwicz (2005), Ito (2006), and Ramnani (2006) all suggest that uniformity of cerebellar structure implies the uniformity of cerebellar function. The uniformity of function must apply, in some sense, no matter where the cerebellum connects to the cerebral cortex. Schmahmann (2004) refers to the uniform function of the cerebellum as the universal cerebellar transform. One of the arguments I make in favour of the CSH is that it may be regarded as an expression of the universal cerebellar transform with respect to cerebrocerebellar connections involving associative areas of the parietal cortex.

2.3.4 Theories of Cerebellar Function

Because of the evident motor dysfunction that results in many cases from cerebellar lesions, the dominant scientific paradigm is the “motor cerebellum.” Schmahmann (1997) and Glickstein, Strata, and Voogd (2009) explain the historical circumstances that resulted in the traditional view that the cerebellum facilitates smooth, efficient motor behaviour, a view that is encapsulated in Holmes (1939).

However, aspects of motor behaviour, such as the planning of motor activity, are higher cognitive functions. The discussions of Stein et al. (1987), Kim, Ugurbil, and Strick (1994), and Thach (1996) all indicate that the lateral cerebellar cortex and dentate nucleus are implicated in more sophisticated aspects of motor cognition than straightforward motor execution.

With regard to cognitive functions that are not directly implicated with motor behaviour, some older theories of the cerebellum, from the early nineteenth century, proposed alternatives to the dominant paradigm. Leiner et al. (1986), in their seminal paper, cited above, first proposed specifically that the cerebellum contributed to higher cognitive functions. Subsequently, Schmahmann (1991), Paulin (1993), Bloedel and Bracha (1997), Apps and Garwicz (2005), Ito (2006), and Ramnani (2006) are among those who argue that the motor-cerebellum paradigm needs to be re-examined. Desmond and Fiez (1998), Paquier and Mariën (2005), Timmann and Daum (2007), and Strick et al. (2009) all review theories of non-motor alternatives for a functional role of the cerebellum.

Nevertheless, Glickstein (2007) and Glickstein and Doron (2008) argue that the non-motor cerebellum is a misunderstanding. They suggest that the lesions in lesion studies are insufficiently localized for conclusions to be drawn about the cerebellar contribution to higher cognitive function. Moreover, sophisticated behaviour shared by humans and monkeys involves visual acuity and fine motor skills, as well as higher cognitive functions. These abilities are also of recent phylogenesis, and one need not search for more exalted contributions of the cerebellum. Working within the classical understanding of cerebrocerebellar connectivity, Glickstein supposed that connections to prefrontal areas from the cerebellum are responsible for eye movements. In reply, Strick et al. (2009) maintain that the prefrontal areas they investigated with respect to cerebrocerebellar connections are *not* associated with eye movement.

In any case, Bloedel and Bracha (1997) suggested that the motor/non-motor distinction may be misleading—all “non-motor” behaviours have a “motor” component, and vice versa. According to Gao et al. (1996), the motor dysfunctions following cerebellar lesions may reflect disruption of the *sensory* data needed for motor behaviour. Paulin (1993) has a similar idea, but with a vivid metaphor: a broken windscreen may result in “motor dysfunction” of the vehicle, despite the fact that the windscreen has no direct influence on the vehicle’s speed and trajectory. Bower (2007) questions whether the cerebellum is primarily sensory, which contributes to motor behaviour; or whether the cerebellum is primarily motor, which contributes to sensory integration.

The idea that the cerebellum is responsible for “skilful manipulation of ideas” developed from the cluster of work that emerged from Dow and colleagues on the dentate nucleus, mentioned above. The seminal article for the skilful manipulation of ideas theory is Leiner et al. (1986). It refers to the notion that manipulation of physical objects is analogous to manipulation of mental objects: if the cerebellum facilitates one, it may also facilitate the other. In Ito’s (1993, 2006, 2008) “internal models of mental models” idea, a “model” formed in the temperoparietal cortex is transferred to the prefrontal cortex, where it is influenced by means of a cerebrocerebellar loop. Skilful manipulation of ideas and internal models of mental models are both related to the motor paradigm of the cerebellum in that the cerebellum contributes to the manipulation of ideas rather than objects.

Likewise, the cerebellum may be associated with the performance of visuospatial tasks, as in Molinari, Petrosini, Misciagna, and Leggio (2004) and Molinari and Leggio (2007). However, objects in these visuospatial tasks are manipulated mentally, and motor imagery is necessary, if not overt motor behaviour.

Other theories of the functional role of the cerebellum are variously distant or close to the motor-cerebellum paradigm. Abbie (1934) envisioned that the cerebellum, with respect to its connections with the posterior cortex, was responsible for combining sensory data into a coherent whole; Prescott (1971) proposed that the cerebellum participates in emotional processes; Heath (1977) showed that the negative emotions of psychiatric patients can be controlled by electrical stimulation of the cerebellum; in the state-estimation theory of Paulin (1993) and Paulin et al. (2001), the cerebellum is used for tracking movement of other objects rather than regulating movement of the organism itself.

Keele and Ivry (1990) and Bengtsson, Ehrsson, Forssberg, and Ullén (2005) suggested that the cerebellum plays a crucial role in the timing of motor activities. Closely related to the "timing cerebellum" is the "sequencing cerebellum," suggested by Inhoff, Diener, Rafal, and Ivry (1989), Leggio et al. (2008), and Molinari et al. (2008), where sequencing is interpreted as a predictive function, verbally, spatially, and logically. Prediction requires knowledge of what has preceded, and Fiez, Raife, Balota, Schwarz, and Raichle (1996), Ben-Yehudah, Guediche, and Fiez (2007), and Molinari et al. (2008) suggest that the cerebellum has a role to play in working memory. The work of Paulin (1993) and Gao et al. (1996) has already been mentioned with respect to the cerebellum and sensation. Ackermann, Mathiak, and Riecker (2007) and Hanakawa, Dimyan, and Hallett (2008) further propose that the cerebellum contributes to sensorimotor imagery, including imagined speech. Petersen, Fox, Posner, Mintun, and Raichle (1989), Fiez, Petersen, Cheney, and Raichle (1992), and Raichle et al. (1994) suggest that the cerebellum is involved in linguistic processing. According to Doyson (1997), the cerebellum facilitates procedural learning; procedural learning coupled with spatial processing is suggested by the research of Molinari, Petrosini, and Grammaldo (1997).

The idea of the non-motor cerebellum remains controversial. Frank et al. (2007), Richter et al. (2007), and Timman and Daum (2007) all suggest that a cerebellar contribution to non-motor cognition is still in question. On the other hand, the cerebellar involvement in motor behaviour is undisputed. In this dissertation, I assume that there *is* a non-motor cerebellum, mediated by cerebrocerebellar loops involving association areas of parietal cortex. The dysmetria of thought hypothesis and the role of the cerebellum in attention, two ways of approaching the non-motor cerebellum, are discussed now.

2.3.5 Dysmetria of Thought

Different parts of the cerebellum connect to different parts of the cerebral cortex. As mentioned above, many researchers have suggested that uniformity of cerebellar structure implies the uniformity of cerebellar function, and this uniformity of function must apply, in some sense, no matter where the cerebellum connects to the cerebral cortex. Schmahmann (2004) refers to the uniform function of the cerebellum as the universal cerebellar transform.

The work of Schmahmann and colleagues represents another cluster of research centred on an important idea with respect to the functional role of the cerebellum. It is referred to either as the cerebellar cognitive affective syndrome, with respect to the cognitive dysfunction that results from cerebellar lesions, or dysmetria of thought, with respect to a conceptual explanation of that cognitive dysfunction. Schmahmann and Pandya (1997) coined the phrase “dysmetria of thought,” whereas the term “cerebellar cognitive affective syndrome” first appears in Schmahmann and Sherman (1998). Schmahmann (1998) reviews both notions. The cerebellar cognitive affective syndrome/dysmetria of thought hypothesis is developed further in Schmahmann (2004). The basic idea is that just as cerebellar dysfunction leads to “motor dysmetria,” inefficient, jerky motor behaviour, cerebellar dysfunction can also lead to “dysmetria” of the higher cognitive processes, “inefficient, jerky thought,” and that these are different expressions of a universal cerebellar transform. The research of Schmahmann and colleagues is largely based on lesion studies of patients with acquired cerebellar dysfunction. Tavano et al. (2007) confirmed that a cognitive affective syndrome is also present in patients with congenital cerebellar dysfunction.

In a sense, dysmetria of thought and dysmetria of movement imply cognitive stability, with respect to perception and action, respectively. Schmahman and Pandya (1997) refer to cognitive stability as “maintaining function steadily around a homeostatic baseline” (p. 55). I suggest that this “homeostatic baseline,” for perception, is equivalent to schematic conception and perception.

It should be noted that the understanding of dysmetria of thought, as developed by Schmahmann and colleagues, depends on their interpretation of cerebrocerebellar connections with prefrontal regions of the cerebral cortex (C. E. MacLeod, personal communication, May, 2010). In this dissertation, however, my framework is different: action depends on the anterior cortex, whereas perception depends on the posterior cortex—and according to Fuster (2003) there is a vast quantity of neuroscience research to support this view. The interpretation of dysmetria of thought developed by Schmahmann and colleagues does not necessarily accord with this primordial divide in cognition. Herein, I interpret the dysmetria of thought hypothesis with respect to posterior-cortex cerebrocerebellar connections. However, I do not see any major difficulties with this step because of the uniformity of cerebrocerebellar structure and the existence of closed-loop connections from cerebellum to parietal cortex. Moreover, the dysmetria of thought hypothesis was developed on the basis of clinical data. There is no reason to suppose that it represents a dysfunction of posterior cerebrocerbellar loops rather than a dysfunction of anterior cerebrocerebellar loops.

In addition, however, the universal cerebellar transform is in accord with a property analysis approach. The sequential resolution of acts from procedures involves the sequential selection of groups of action properties, which are the anterior cortex analogue of the properties of the posterior cortex. Presumably, the cerebellum facilitates this selection so that the procedure is performed smoothly and efficiently. If the cerebellum facilitates selection of action properties in the anterior cortex, then the universal cerebellar transform would imply selection of groups of perception properties in the posterior cortex. I propose that these perception properties are the essential properties of concepts. Although the action of the cerebellum with respect to the anterior cortex and posterior cortex is the same, the effect on behaviour is completely different because of the distinction between action and perception.

2.3.6 Selective Attention

I have already mentioned Sohlberg and Mateer’s (1989) hierarchical model of attention, one level of which is selective attention. Schematization of concepts and percepts of the posterior cortex is a form of selective attention. The CSH, therefore, is about the role of the cerebellum in selective attention with respect to the posterior cortex. If I can show that it is reasonable to suppose that the cerebellum does indeed have a functional role in selective attention, then the plausibility of the CSH, as a hypothesis pertaining first and foremost to mathematics educational neuroscience, will be enhanced.

The theory that the cerebellum plays a role in selective attention is truly distant from the paradigm of the motor cerebellum. Akshoomoff and Courchesne (1992) conducted experiments with patients with cerebellar lesions in switching attention between modalities, finding that cerebellar patients accomplished this task less efficiently. Strictly speaking, this research is relevant to alternating attention rather than selective attention, although, as mentioned above, selective attention is prior to alternating attention in the hierarchy of Sohlberg and Mateer (1989). Nevertheless, the cerebellum may be responsible for the jump from selective attention to alternating attention, rather than selective attention per se, so no firm conclusions can be drawn, even though the research of Akshoomoff and Courchesne is suggestive.

Courchesne et al. (1994) come close to a cognitive network interpretation of attention in their conceptual understanding of the role of the cerebellum in attention. Townsend et al. (1999) conducted experiments in switching attention, but this time to different foci within the *same* modality, again finding that cerebellar patients performed the task less efficiently. Once again, however, the research of Courchesne et al. and Townsend et al. concerns alternating attention rather than selective attention per se.

Allen, Buxton, Wong, and Courchesne (1997), using brain-imaging techniques on healthy participants, determined that the cerebellum is involved in the ability to focus visual attention on different aspects of a single object within a single modality. The research of Allen et al. appears to refer specifically to the role of the cerebellum in selective attention.

The experimental paradigm of Allen et al. (1997) may have involved working memory or procedural learning, and the activity of the cerebellum may have concerned these two processes rather than selective attention per se (C E. MacLeod, personal communication, May, 2010). With respect to working memory, the neurophysiological model of Knudson (2007) identifies working memory with focused attention. Nevertheless, further research is necessary to eliminate these possible confounding factors and truly to isolate the functional role of the cerebellum in selective attention.

The research of Schweizer, Alexander, Cusimano, and Stuss (2007) appears to isolate attention from all other cognitive processes, although the type of attention that they identify, once again, is not selective attention. The attentional blink is a (neuro)psychological mechanism whereby the observer cannot recognise the second occurrence of a stimulus presented rapidly after the first occurrence, because attention has been captured by the first occurrence and cannot be switched quickly enough to register the second occurrence.

Schweizer et al. (2007) noted that patients with cerebellar lesions had an amplified attentional blink, although they displayed no deficit in recognizing the stimulus the first time it occurred. They cited this as evidence that the cerebellum does not affect selective attention. The same point was made by Gottwald, Mihajlovic, Wilde, and Mehdorn (2003) in their research on the attentional abilities of cerebellar patients. In both cases, however, it is *focused* attention rather than *selective* attention in which the cerebellar patients suffer no deficit. Selective attention involves the ability to concentrate attention on an object despite the presence of distractors, and this ability is not investigated by Schweizer et al. nor Gottwald et al.

As I discuss in Chapter 3 and Chapter 4, selective attention, the ability to recognize an object despite the presence of distracting, incidental information, involves differential enhancement and suppression of the neural activity of the components of a cognitive network. In my view, distant though it seems from motor behaviour, selective attention is yet another version of the universal cerebellar transform. Attention in the posterior cortex, selective excitement of certain features of the cognitive networks of concepts, is analogous to the sequential selection of groups of action properties in the anterior cortex. The selective attention theory is important in that it is a purely posterior-cortex approach to cerebellar function, even though, according to the property analysis, it is analogous to the anterior-cortex role of the cerebellum.

An experimental paradigm has yet to be designed that truly isolates selective attention and the activity of the cerebellum with respect to the posterior cortex. Allen et al. (1997) and Schwiezer et al. (2007) both come close, but neither exactly hits the nail on the head, as it were. In Chapter 5 I make some preliminary suggestions for future empirical research that would go further towards identifying the role of the cerebellum in selective attention.

2.3.7 The Cerebellum and Geometrical Reasoning

There are several arguments for the CSH embedded in the discussion above, and in more detail in Chapter 5. The arguments in favour of the CSH are gathered together in summary form at the end of Chapter 5. If the CSH is assumed, then there may well be a role for the cerebellum in schematic perception and inferencing, which I have previously identified as specific aspects of image-based reasoning in geometry.

Schematization of concepts and percepts are important aspects of geometrical reasoning, equivalent to “seeing the general in the particular,” in a reinterpretation of the phrase coined by Mason and Pimm (1984). The suggestion that the cerebellum has a role in mathematical reasoning is not original to this dissertation. Vandervert’s (1997, 1999, 2003) ideas in this respect were my initial inspiration. However, Vandervert’s proposal is closer to theories such as Leiner et al.’s (1986) skilful manipulation of ideas than the theory of this dissertation. Vandervert relies on Mandler’s (1988, 1992) notion of image-schemas, and his focus is the prefrontal cortex-cerebellum relationship. The arguments and conclusions of this dissertation are quite different from Vandervert’s.

The conclusion that cerebellar activity facilitates certain aspects of image-based reasoning in geometry depends on the notions of schematic perception and inferences; the importance of schematic perception and inferences for geometrical reasoning; and a particular understanding of the cerebral cortex, the cerebellum, and the connections between the two. Accordingly, the theory has implications for mathematics education in K-12 geometry. Implications following from the theory do not constitute assertions of fact, but rather hypotheses that remain subject to empirical substantiation or refutation.

One implication I develop herein is the decontextualization hypothesis: if geometrical concepts are presented with few incidental distractors, then the work of the cerebellum in schematizing these concepts is facilitated. This hypothesis is intuitively appealing, given my work on the CSH developed herein.

2.4 Mathematics Education Research

The first phase in formulating my thesis was to identify the kinds of cognitive network in the cerebral cortex that correlate with the concepts, percepts, and properties of significance for geometrical reasoning. The second phase was to identify a cerebellar mechanism that can turn the fuzzy, indistinct concepts and percepts of the cerebral cortex into the crisp, clear schematized concepts and percepts that are adequate for K-12 image-based geometrical reasoning. Each phase has implications for geometry education and potentially more generally mathematics education, although some of these implications are tentative and require further work.

2.4.1 The Cerebral Cortex and Mathematics Education

Cognitive network theory can shed light on the various approaches to procedural learning versus conceptual learning. These approaches include Skemp (1976/2006), Hiebert and Lefevre (1986), and Rittle-Johnson, Siegler, and Alibali (2001). In very broad terms, and at the risk of oversimplification, there does indeed appear to be a common thread. Perhaps this common thread is pointing to a deep connection between the theories, and perhaps this deep connection is actually based upon functional differences between anterior cortex and posterior cortex, between action and perception. I propose that procedural reasoning activates primarily neural structures of the anterior cortex, whereas conceptual reasoning activates primarily neural structures of the posterior cortex. Presmeg's (1986, 1992) view of the distinction between visualizers and non-visualizers may be interpreted to mean that some students naturally prefer a conceptual, posterior-cortex approach to learning, whereas others naturally prefer a procedural, anterior-cortex approach.

Related to the distinction between procedural learning and conceptual learning are various theories of mathematical concept formation, including the reification theory of Sfard (1991) and the APOS theory of Czarnocha, Dubinsky, Prabhu, and Vidakovic (1999). The theories of Czarnocha et al. (1999) and Sfard (1991) can be understood to mean that concepts develop from procedures. Indeed, this has a natural interpretation in terms of the perception-action loop, from procedures to concepts, in the cognitive network theory.

In a cluster of papers, however, Tall and colleagues claim that the approach that always leads from procedure to concept is overly prescriptive, especially for geometry. According to Gray and Tall (1991, 1994), Tall (1995, 1999), and Tall, Thomas, Davis, Gray, and Simpson. (2000) the procept idea, which combines Piaget's pseudo-empirical abstraction and reflective abstraction, is applicable primarily to arithmetic and algebra. Piaget's empirical abstraction, on the other hand, is suitable for geometry. Mitchelmore (2002), also, argues that geometrical concepts result from empirical abstraction. The work of Tall and colleagues may be interpreted as an attempt to redress the imbalance in APOS, so that concepts may develop before procedures in geometry. It is reasonable to suppose that, just as concepts develop from procedures in the perception-action loop, procedures develop from concepts in the reverse order in the perception-action loop.

These implications for mathematics education of the theory of the cerebral cortex are preliminary and tentative. They should be regarded as topics for further research rather than as established positions.

2.4.2 The Cerebellum and Mathematics Education

A cerebellar contribution to cognition also has implications for mathematics education theory. It is important firstly to understand the various approaches to abstraction in the mathematics education literature. Dubinsky (1991) is a good source for a summary of Piaget's three forms of abstraction: empirical abstraction, pseudo-empirical abstraction, and reflective abstraction. Empirical abstraction, as I understand it, corresponds to concept formation in the posterior cortex, the association of percepts together to make concepts, resulting in concepts that are, in a sense, extensional (see p. 20, n. 2), and which I have interpreted to be a form of generalization. Pseudo-empirical abstraction may be understood as perception of acts, or in other words the operation of a lower-level perception-action loop from action to perception. Reflective abstraction, lastly, corresponds to the encapsulation of APOS, operation of the higher-level perception-action loop from procedures to concepts.

The cerebellar approach to abstraction is quite different. As explained previously, my hypothesis is that a functional role of the cerebellum is to focus attention on those aspects of a concept and percept that are essential. I approach Aristotle's ideas on abstraction through Lear (1982) and Mendell (2004). It seems that schematization may indeed be regarded as abstraction in the Aristotelian sense, resulting in intensional concepts rather than extensional concepts (see p. 20, n. 2). The cerebellum does not *create* concepts in this sense but acts to *purify* concepts of the cerebral cortex.

Some mathematics education research seems to refer to something similar to the cerebellar intensional concept. Mitchelmore and White (1995, 2004), for example, distinguish between abstract-general and abstract-apart concepts, the latter being mathematical concepts produced by formal definition. There is a similar distinction between concept image and concept definition in Tall and Vinner (1981).

On the other hand, *mathematical definition* is not the same as *mathematical perception*, in which the mathematical object is "seen" as being constituted only of those aspects that belong to its essence as a mathematical object. Godfrey (1910) called this the "geometrical 'eye.'" Harel and Tall (1991), Fischbein (1993), and Sierpinska (Boero et al., 2002) all describe this cognitive function in similar terms. This type of mathematical perception is referred to as “seeing the general in the particular” in the seminal paper by Mason and Pimm (1984). The authors used the phrase primarily for algebra, and I have reinterpreted it herein for geometry.

Once again, the ideas on abstraction and the cerebellum included in this dissertation, particularly my interpretation of Piaget, should be regarded as potential implications rather than established positions. Further research is necessary. On the other hand, a role for the cerebellum in “seeing the general in the particular”—or schematic perception—and the further hypothesis that decontextualization may facilitate certain aspects of mathematical learning, is my main point in this dissertation. The present chapter finishes with a brief examination of the issue of decontextualization.

2.4.3 Decontextualization

Decontextualization refers to the representation of image-based geometrical objects and relations (associated with schematic perception and inferencing respectively), without distracting, incidental information. Intuitively, it seems clear that decontextualization facilitates the cerebellum in schematizing concepts and percepts—if there are fewer incidental properties to begin with, then there is less work for the cerebellum to do in this respect. Following the CSH, this is my secondary hypothesis. The decontextualization hypothesis needs further investigation and substantiation, but at least herein I can indicate how it may be related to other research in mathematics education.

The hypothesis that decontextualization facilitates schematization is complemented by other research in mathematics education. Kaminski, Sloutsky, and Heckler (2008) and Koedinger, Alibali, and Nathan (2008) offer empirical substantiation of the pedagogical benefit of decontextualized, symbolic representations of mathematical situations (although the support is qualified in the latter). Approaching the same idea from a different direction, the deleterious effect of irrelevant information on mathematical reasoning is investigated in Bana and Nelson (1978), who cite a number of other studies. The idea of distractors is formalized in the framework of Hegarty and Kozhevnikov (1999), in which visual, as opposed to schematic, presentations of mathematical situations are seen to be potentially confusing.

On the other hand, decontextualization appears to contradict the principles of some influential theories in mathematics education, including constructivism (e.g., Confrey & Kazak, 2006) and situated learning (Lave & Wenger, 1991). Two points can be made. Firstly, bear in mind that the decontextualization proposal refers to *cerebellar* rather than *cerebral* processes. I believe the distinction I make between cerebellar and cerebral processes is a valid one. It follows from the theoretical framework and argument of this dissertation. Cerebral processes represent generalization, whereas cerebellar processes represent abstraction, at least with respect to image-based reasoning. This distinction exists precisely in the middle ground between mathematics education and cognitive neuroscience, and hence belongs to educational neuroscience. Within this middle ground, I am able to argue points that depend on the distinction between cerebellar and cerebral learning. It is not an arbitrary distinction, but rests on and is mutually constrained by considerations of concern to both education and the neurosciences. It is possible, despite my concentration on the CSH and related decontextualization hypothesis, that the cerebellar-cerebral distinction may be the most lasting contribution of my thesis. It provides a neurophysiological explanation of a conundrum that has occupied philosophers for thousands of years. Be that as it may, it is unclear whether the various theories of mathematics education are cerebellar or cerebral in scope, although it appears that they refer primarily to cerebral processes. If this is the case, then there is no opposition, and decontextualization for cerebellar purposes simply extends the discussion into another dimension.

My second point is that I question whether constructivism and situated learning are universally valid, even for cerebral processes. Anderson, Reder, and Simon (2000) critique constructivism and situated learning theory in this respect. Neither, of course, is it true that the decontextualization hypothesis would apply universally as a pedagogical strategy. There may well be affective or social factors that overwhelm the benefit of decontextualization for cerebellar learning. A specific pedagogical technique, such as decontextualization, may have its place, but only as part of an overall strategy that may well include other techniques, perhaps inspired by constructivism or situated learning.

▲▶▼◀▲▶▼◀▲▶ ▼◀

The theory of this dissertation was outlined in Chapter 1. Chapter 2 sketched the background from which the theory emerges and further delineated the theory itself. It is time now to begin the detailed exposition of the arguments. Chapter 3 firstly develops an ontology and epistemology that can support an educational neuroscience research framework. I discuss neural activity in the cerebral cortex that might correspond to image-based geometrical reasoning.

CHAPTER 3: PSYCHOLOGY AND NEUROPHYSIOLOGY

Psychology is not neurophysiology,[5] and knowledge of neither domain in itself can be extended to encompass the other. Given the research framework of this dissertation, which is educational neuroscience, utilizing insights from both, it is essential to provide for a theoretical environment in which both domains can be integrated coherently. The solution is to situate both psychology and physiology, specifically neurophysiology, within a broader framework.

The broader framework is a neutral monism, in which there is a single ontological substance and two epistemic categories, corresponding to psychological knowledge and neurophysiological knowledge. It is based on the embodied cognition of Varela et al. (1991), as modified by Campbell (2001, 2003), and informed by Spinoza's (1667/1996) classical formulation of neutral monism.

Spinoza (1667/1996) clearly indicates that there are precise correspondences between the two epistemic categories. These correspondences, between cognitive functions of the psychological domain and neural activity of the physiological domain, are referred to as neural correlates. The cognitive network theory of Fuster (2003) provides a model of the neurophysiological domain that is very useful for my purposes in this dissertation. The neural correlates of concepts, percepts, and properties are structurally

---

[5] Note that in this chapter the term neurophysiology implicitly includes neural activity as well as neural structure.

related according to cognitive network theory in a way that permits the property analysis in Chapter 4.

With regard to specific neural correlates for specific cognitive functions, there has been little work on determining the neural correlates for image-based geometrical reasoning per se. However, there has been substantial research on the neural correlates for arithmetical reasoning, spearheaded by Dehaene and colleagues. It seems clear from a review of this research that certain aspects of arithmetical reasoning are spatial, and therefore they have implications for image-based geometrical reasoning.

A second approach in the search for the neural correlates for image-based geometrical reasoning is to study the cognitive neuroscience of the visual system. Indeed, I argue in Chapter 1 that geometry is a spatial, visual subject, both in its origin and the way that it is, and should be, taught in the high schools. Chapter 2 has already reviewed the research literature in this area. The purpose herein is not to determine the exact nature of the neural correlates for image-based geometrical reasoning, but rather to exploit this body of research material to bolster the plausibility of my overall argument.

3.1 Neutral Monism

The research herein depends on psychology and physiology (specifically, *neuro*physiology). It is necessary to adopt a theoretical framework in which both domains of knowledge are integrated coherently. Accordingly, I have embraced the theory of embodied cognition, as developed in Varela et al. (1991) and extended by Campbell (2001, 2003). Two alternative approaches to embodied cognition, the cognitive metaphors of Lakoff and Núñez (2000) and the semiotic-cultural approach of Radford et al. (2005) were briefly reviewed already in Section 1.4. With respect to the method adopted herein, the research framework of educational neuroscience, Campbell's (2001, 2003) ideas are more appropriate.

A principal idea behind embodied cognition is *double embodiment*: we are beings who exist in the world; but also we are beings who perceive the world. According to Merleau-Ponty (1945/2005), whose phenomenology was a major inspiration for embodied cognition, "The world is inseparable from the subject, but from a subject which is nothing but a project of the world, and the subject is inseparable from the world, but from a world which the subject itself projects" (pp. 499-500). Campbell (2001) adds, "We are both embodied within the world and the world is embodied within us: *we are the world within itself*" (p. 6, author's italics).

Note that the "world that the subject projects" and the "world that projects the subject" should both be regarded as epistemic categories, mind and body, respectively. Idealism and realism are matters of perspective, corresponding to Husserl's phenomenological and natural attitudes, respectively (Campbell, 1998; Campbell & Handscomb, 2007). The *subjective* mental world of cognitive function flows from the idealist stance, and the *objective* world of physical activity that can be observed and measured flows from the realist stance.

The embodied point of view takes the body as the locus of experience. The very fact of being embodied and thereby embedded in the world, taking the natural attitude, means that the organism receives external stimuli that change the internal milieu. A change in the internal milieu in turn changes the way the organism acts, altering the subsequent stimulus it receives. Varela et al. (1991) describe this in poetic terms as “organism and environment enfold into each other and unfold from one another in the fundamental circularity that is life itself” (p. 217). According to Merleau-Ponty (1942/2006), “When the eye and the ear follow an animal in flight, it is impossible to say ‘which started first’ in the exchange of stimuli and responses” (p. 13). Organism and world in which it is embodied are a single, interactive structure.

It would be easy to generate confusion at this point. Consciousness, according to Thompson and Varela (2001), is an emergent property of neural states, and there is *reciprocal causation* between consciousness and neural states; Varela and Thompson (2003) consider “the issue of the causal efficacy of consciousness” (p. 266). However, “the fundamental circularity” and “exchange of stimuli and responses” should be regarded as entirely within the objective domain. Physical activity does not *cause* subjective cognition, or vice versa. Fuster (2003) writes, “A cognitive order, no matter how it is construed, cannot be causally related to a brain order” (p. viii), and I agree. This important point is worth emphasizing. The naïve belief is that the physical event of stubbing one’s toe “causes” the psychological event of pain. It does not—stubbing one’s toe *does* cause objective nociceptive activity in the nervous system, but that activity is not in itself pain. Pain is a subjective experience belonging to the epistemic category of mind rather than body.

Cognition, in the natural attitude, is defined by Varela et al. (1991) as "Enaction: A history of structural coupling that brings forth a world" (p. 206). The *history of structural coupling* is the dance between organism and the world. They move together in perfect synchrony, neither taking the lead, but both moving to the same melody. The boundaries of the organism do not stop at the physical shell of the body, but include organs, blood, and nerves—the body is itself part of the world that it enacts—and therefore cognition arises also in the *body's interaction with itself*. In this regard, the most characteristically human aspects of cognition, it may be assumed, are manifested in the neural activity of the brain.

Stepping back from physiology, and into psychology, this interaction between organism and world, most specifically neural activity, "corresponds to" subjective cognition. The term *cognitive function* is reserved for the subjective epistemic category, whereas *neural activity* refers to the objective epistemic category. Neural activity is *correlative* with cognitive function, and this relationship is investigated in the next section.

The view of embodied cognition, in which the subjective and objective are epistemic categories, is not, according to Campbell (1993), the original understanding of Varela et al. (1991). They appear, in fact, to endorse, albeit implicitly, a Cartesian dualism. Campbell (2001) explains this point and suggests modifying embodied cognition with a monist ontology:

> *The standard . . . enactivist notion of ‘double-embodiment’ is to adopt a ‘realist’ ontology with regard to being in the world, and an ‘idealist’ ontology with regard to the world in us. Ironically this pragmatic approach to enactivism appears to embrace the very Cartesian problematic it set out to reject. In contrast, the enactivist approach proposed [by the author] rejects both realism and idealism: i.e., both ontological poles of Cartesian dualism. Drawing instead on Merleau-Ponty’s metaphysical notion of ‘flesh’ as an ontological primitive, it is a view rooted in a definitely non-Cartesian ontological monism that takes the traditionally conceived objective ‘real’ world we are in and the subjective ‘ideal’ world within us to be manifestations of the same world*. (p. 3, author’s italics)

Accordingly, the “fundamental ontology of *flesh* . . . recognizes that organisms are comprised of *living* materials that can both touch and be touched” (Campbell & Handscomb, 2007, p, 4, authors’ italics). Campbell (2003) develops further the notion of embodied cognition with this alternative ontology. He writes, “Flesh is not reducible to mind or matter. . . . [S]omething conscious, aware, feeling, something yet more primitive manifesting as these, is innate in this foundational substance of the world” (p. 248).

Embodied cognition, in this sense, a monist ontology and dual epistemology, is a *neutral monism*. Handscomb (2007) reviews the remarkable similarities between Campbell’s (2001, 2003) formulation of embodied cognition and the classical neutral monism expounded by Spinoza (1667/1996). According to Spinoza, the world consists of a single substance, which can be known in two ways, thought and extension. These two attributes correspond, respectively, to the subjective and objective epistemic categories.

Spinoza’s neutral monism, or alternatively Campbell’s (2001, 2003) formulation of embodied cognition, defines the fundamental ontology and epistemology of this dissertation: a neutral monism with two epistemic categories, cognitive function and neural activity. The relationship between these two ways of knowing is discussed in the next section.

3.2 Neural Correlates

According to Spinoza (1667/1996) there is a precise correlation between the domains of thought and extension: the "*order and connection of ideas is the same as the order and connection of things*" (E2 P7, author's italics). Spinoza, in other words, argues that the structures of the two epistemic categories are identical.

The cognitive network theory of Fuster (2003) is an important tool in this dissertation. Fuster has a practical approach to the connection between the two domains:

> For clarity and simplicity, here I shall define my agenda as the search for a spatial and temporal order in the cerebral cortex that matches the cognitive order in every respect. By match I mean that the spatial or temporal constituents of the cortical order occupy the same relative space with respect to one another as the corresponding constituents of the cognitive order. Thus, a change or difference in the cortical order corresponds to a change or difference in the mental order. . . . [N]eural measures would correlate with behavioral or psychophysical measures of cognitive variables or changing states. From the aggregate of such relations, if demonstrable, we could legitimately conclude that the two orders are identical, and so are the structures, events, and processes in them. (p. 4)

His goal, in so many words, is an empirical substantiation of Spinoza (1667/1996). Fuster continues,

> Because neither the cortical nor the cognitive order is well known, however, we need working models of both to conduct our quest. The cortical model should accommodate as much scientific evidence as we now have on the cerebral cortex. The cognitive model should do the same with psychological evidence. (p. 4)

As far as this dissertation is concerned, Fuster's cognitive network theory is taken as the model for the "cortical order" (i.e., the objective domain), whereas the psychology of mathematics education models the "cognitive order" (i.e., the subjective domain). However, Fuster's arguments are largely concerned with the cerebral cortex, and it is necessary to extend the cortical order to include a "cerebellar order."

Nevertheless, I do not argue in favour of the universal application of the strong formulation of the relationship between the subjective and objective domains. For example, objective time is isomorphic to the number line: the interval between any two instants is indefinitely divisible. The subjective experience of time, however, is not like a single point on a number line, not "like a glow-worm spark illuminating the point it immediately covered, but leaving all behind in total darkness," in the words of James (1890/1950, p. 606). Subjective time proceeds from duration to duration, James' *sensible present*, in which each duration has a temporal thickness, and one duration shades into the next. Time in subjective experience can dilate or contract, according to mood—who has not felt "time drags" or "time flies"? Subjective time and objective time are not commensurable. Bergson (1889/1960) felt that "time, conceived under the form of an unbounded and homogeneous medium [i.e., the number line], is nothing but the ghost of space haunting the reflective consciousness" (p. 99). Great care must be taken with regard to the neural correlates of subjective experience when temporality is involved.

With respect to the concepts, percepts, and properties of mathematics, and specifically the visual, spatial aspects geometry, however, and more tentatively their analogues in action (all to be explained later); it is assumed that the structures of the two epistemic domains are indeed the same. It follows that conclusions with respect to the structure of neural activity may have implications for the structure of cognitive function, and vice versa. By restricting the scope of the investigation in this way, the arguments, at least with respect to the cerebral cortex, fall well within the span of cognitive network theory, or indeed of Spinoza's epistemology.

For a given cognitive function, the concurrent neural activity is a *neural correlate* for that cognitive function. The notion of neural correlate is crucial for this dissertation. Some of the ideas with respect to neural correlates are reviewed in this section and the next. Campbell and Handscomb (2007) is a review that complements the one herein, again from an embodied perspective.

The first strategy in the search for neural correlates is to look for areas of the brain in which cognitive function is localized. At the beginning of the nineteenth century, Gall (see Hollander, 1920, Part II, Section II) made a concerted effort to localize cognitive functions in the brain. In Gall's theory of phrenology 27 "brain organs" are responsible for various psychological proclivities. The relative development of these brain organs can be determined by examining bumps on the skull.

Gall's ideas failed to persuade the scientific community. However, the idea that activity of specific parts of the brain correlates with higher cognitive functions was given a massive boost later in the nineteenth century by the discovery of regions of the brain responsible for production of language (Broca’s area) and understanding of language (Wernicke’s area).

In recent times, the hypothesis that neural activity in particular regions of the brain correlates with specific psychological functions has been substantiated by lesion studies. Thus, if a cognitive function is impaired in an individual whose brain has been damaged in a particular region, then it is inferred that the impaired cognitive function is correlated with activity from the damaged area of the brain.

While neural correlates may be studied by “negative” methods such as brain lesions, they may also be investigated by “positive,” brain-imaging techniques such as functional magnetic resonance imaging (FMRI), positron emission tomography (PET), and electroencephalography (EEG). If brain-imaging data indicate a particular configuration of neural activity at a given time, and behavioural or psychological data indicate concurrent engagement in a particular cognitive function, then it may be concluded that the pattern of neural activity is correlative to the cognitive function.

Following the early successes of Broca and Wernicke, the main thrust of cognitive neuroscience has been to localize cognitive function in the brain, largely by means of brain-imaging studies. It is a vast area of research, which produces thousands of academic papers annually.

It appears incontrovertible that certain cognitive functions do correlate with neural activity that is relatively localized in the brain. For example, vision disappears with destruction of the occipital cortex, implying that brain activity in the occipital cortex is a necessary neural correlate of vision. According to Dehaene (1997),

> The extreme modularity of the human brain stands out as the main lesson to be gathered from studies of cerebral pathology. Each small region of the cortex appears to be dedicated to a specific function and may thus be viewed as a mental ‘module’ specialized in processing data from a distinct source. (p. 177)

Nevertheless, there are problems with respect to localizing some higher cognitive functions. Lashley (1950/1988), for example, demonstrated the difficulty of identifying a specific part of the brain responsible for a given memory. It appears that memory is distributed over a large part of the brain.

The truth lies somewhere on a continuum between the two extremes. Neural correlates of cognitive functions, generally, are neither fully localized in the brain nor fully distributed. Connectionist models of neural activity can explain the partially distributed nature of the neural correlates. A connectionist model is used in this dissertation, the cognitive network theory of Fuster (2003), which is discussed in Chapter 4.

A connectionist understanding of neural activity begins with the small-scale structure of the brain: neurons connected through synapses. The synapses are physiological links between neurons that connect neurons into neural arrays (also known as cell assemblies or cognitive networks). The brain is constantly active electrically and chemically. Neurons discharge and propagate electrical signals, or “fire.” The firing of neurons, usually the aggregate firing of large groups of neurons, is what is meant by *neural activity*. When one neuron fires, it may cause other neurons to fire that are connected to it via synapses. Electrical activity propagates throughout neural arrays. Neural connections that are repeatedly activated are strengthened (see Chapter 4), meaning that future activation is facilitated. In this way, the vast neural array that is the brain itself changes and “learns.”

Connectionist research was inspired by brain structure, but has proceeded largely independently of brain research per se. Connectionism may be thought of as a program to instantiate brain-like structures, with the neural connectivity of the brain as its inspiration. The connectionist program was initiated largely through the ideas of McClelland and Rumelhart on *parallel distributed processing* (PDP) (McClelland & Rumelhart, 1987; Rumelhart et al., 1987).

Connectionist models contain a large number of cells and connections between the cells, which are analogous to neurons and synapses, respectively. The rules for cell activation and learning mimic analogous processes in the brain. Remarkably, these PDP systems are able to reproduce sophisticated aspects of cognition, such as pattern recognition (Rumelhart et al., 1987).

The Brainweb is a theory of large-scale, integrated neural activity (Lachaux et al., 1999; Varela et al., 2001; Lutz et al., 2002), which is associated with the research program of neurophenomenology (Varela, 1996, 1999; Lutz & Thompson, 2003). Fuster’s (2003) cognitive network theory, rather than the Brainweb, is the neurophysiological model that is used herein. However, it is worth devoting a little space to the Brainweb and neurophenomenology, because the latter was the inspiration for educational neuroscience (S. R. Campbell, personal communication, September 2005), which is the research framework of this dissertation.

The interconnected cell assemblies that constitute the Brainweb are understood largely in terms of the *statistical* relationship of phase locking in gamma-frequency electroencephalographic data. A connectionist model would typically involve *structural* (i.e., electrochemical) connections between neurons. Nevertheless, given that a structural substrate is implied in the statistical analysis of phase-locking relationships, the Brainweb itself has an underlying connectionist orientation.

Varela (1995) proposes, "*A singular, specific cell assembly underlies the emergence and operation of every cognitive act*" (p. 82, author's italics). In other words, there is a well-defined mapping from the class of cognitive functions to the class of neural activities, provided the later is defined in terms of cell assemblies. This robust proposal is almost as strong as the Spinoza-Fuster hypothesis, discussed above.

The distinguishing characteristics of neurophenomenology are its method for collecting data on cognitive function and its approach to the relationship between models of cognitive function and neural activity. The original formulation of embodied cognition used Buddhist mindfulness/awareness techniques for investigation of the subjective domain (Varela et al., 1991). The method proposed for neurophenomenology, on the other hand, is Husserl's formal technique of phenomenological reduction, as discussed in Varela (1996), Varela and Shear (1999), and Depraz et al. (2000). Varela's (1996) key insight was that subjective psychological data can *constrain and inform* hypotheses that are developed by means of objective neurophysiological investigation, and vice versa. According to Varela, “Phenomenological accounts of the structure of experience and their counterparts in cognitive science relate to each other through reciprocal restraints” (p. 343). In other words, subjective cognitive experience can inform the analysis of objective physiological data, and vice versa. Despite the quotation from Varela (1995), above, it appears that the goal of neurophenomenology is not to produce exact neural correlates, but rather to offer successive refinements of the theoretical models of cognitive function and neural activity.

The research framework of this dissertation—at least with respect to concepts, percepts, and properties—is educational neuroscience, rather than neurophenomenology. Educational neuroscience is distinguished from neurophenomenology in two respects. Firstly, the approach to the subjective domain relies on cognitive psychology rather than phenomenology; secondly, the focus is more narrowly that of education—specifically geometry education in this dissertation. My intent is not to produce exact neural correlates, but rather to establish structural connections within the neural substrate of cognition that have implications for geometry education. In addition, the theoretical construct that is the Brainweb does not model neural activity with sufficient granularity for the purposes herein, and Fuster’s (2003) cognitive network theory is used instead.

This section has established the notion of neural correlate, implying a fundamental parallelism between the two epistemic categories of neutral monism. The next section investigates further the neural correlates of some specific cognitive functions that are relevant for arithmetical reasoning and image-based geometrical reasoning.

3.3 Neural Correlates for Image-based Geometrical Reasoning

There has been significant investigation of the neural correlates of arithmetical reasoning by Dehaene and colleagues, although there has been no research to date specifically on the neural correlates of imae-based geometrical reasoning. However, certain aspects of arithmetical reasoning are spatial (Dehaene, 1997; Hubbard et al., 2005), and their neural correlates are likely also to be neural correlates for image-based geometrical reasoning. In addition, the fundamental levels of image-based geometrical reasoning depend on visual perception and imagery, and there has been a great deal of research on both, some of which is discussed below.

3.3.1 Spatial Aspects of Arithmetical Reasoning

The triple-code model was proposed in Dehaene (1992) and investigated further in Dehaene and Cohen (1995, 1997), Cohen and Dehaene (1995, 1996), Dehaene (1996), Dehaene, Piazza, et al. (2004), and Dehaene, Molko, et al. (2004). In the triple-code model, numbers can be understood in three different ways, each of which is related to specific arithmetical reasoning processes, and each of which has specific neural correlates. The *auditory verbal word frame* is language based: numbers coded in this format would be related, for example, to rote-memorized multiplication tables stored in long-term memory. The *visual*

Arabic number form is symbol based: it is connected with determination of parity from the last digit and is necessary for multi-digit calculations. In the *analogue magnitude code* numbers are represented on an analogue oriented number line: the number line representation is used for number comparison and approximate calculation—in other words numbers are understood spatially.

With regard to neural correlates, the verbal code is associated with activity in the left hemispheric perisylvian areas, the visual code with activity in the left and right inferior ventral occipitotemporal areas, and the analogue code with activity in the left and right inferior parietal areas (Dehaene, Piazza, et al., 2004). Dehaene, Piazza et al. note that the perisylvian language network extends into the inferior parietal cortex and that the posterior superior parietal cortices are engaged in visual attention processes. Therefore, all three codes, to an extent, are associated with activity in the parietal cortex. Dehaene, Molko et al. (2004) write,

> Recent studies in human neuroimaging, primate neurophysiology, and developmental neuropsychology indicate that the human ability for arithmetic has a tangible cerebral substrate. The human intraparietal sulcus [BA 7/BA 39, see below] is systematically activated in all number tasks and could host a central amodal representation of quantity. (p. 218)

Dehaene (1997) contains a more extensive discussion of the neural substrate of arithmetical reasoning from the perspective of its being an innate ability shared by a number of species. This “number sense” involves small numerical quantities handled approximately. The neural activity for the number sense is bilaterally located in the inferior parietal cortex.

Dehaene (1997) argues that extension of the innate number sense to human arithmetic requires reasoning with symbols and number words as well as the spatial ability to visualize the number line, the three components of the triple code. Reasoning with symbols and number words, requiring language, must be located in the left hemisphere, including Broca's area and Wernicke's area. Dehaene claims also that the ability to visualize the number line is located in the left inferior parietal cortex, specifically left Brodmann Area (BA) 39.[6]

Dehaene's (1997) evidence for the lateralization of the number line ability is not extensive. In any case, the left-hemisphere/right-hemisphere divide is not a dimension that is investigated in this dissertation. For the purposes herein, the main division is between front and back of the cerebral cortex, anterior cortex and posterior cortex, respectively.

The number-line ability, nevertheless, is a spatial ability that is localized to the parietal cortex. According to Hubbard et al. (2005), "[T]he neural circuitry that is crucial for abstract representations of quantity is housed in the parietal lobe, in regions that overlap with the neural circuitry involved in spatial representations" (p. 440). They continue, "In the more distant future, it might become possible to study whether more advanced mathematical concepts that also relate numbers and space, such as Cartesian coordinates or the complex plane, rely on similar parietal brain circuitry" (p. 446). And then there is image-based geometry itself, which emphasizes spatial cognition. The implication is that certain aspects of

---

[6] The Brodmann Areas are a convenient way of referring to various parts of the cerebral cortex. Those Brodmann Areas of significance for this dissertation are shown in Figure 4.1.

higher cognitive function related to image-based reasoning in geometry are localized in the parietal cortex.

3.3.2 Spatial Aspects of Geometrical Reasoning

An investigation of the neural correlates of arithmetical reasoning turns up some spatial neural correlates that may have application to image-based reasoning in geometry. A broader investigation of the neuroscience of the visual system may be able to identify the neural correlates of visual concepts, visual percepts, and visual properties. A *visual concept* would correspond to the general idea of "triangle"; a *visual percept* would correspond to perception of a specific triangle; a *visual property* would correspond to the notion "three sides." These concept-percept-property distinctions are clarified in Chapter 4 from a cognitive-network perspective. I suggest that a major part of image-based geometrical reasoning is captured through an analysis of visual concepts, visual percepts, and visual properties, which belong to perception.[7]

Typically, afferent visual sensation from a geometrical diagram activates the cerebral visual system. There may even be visual imagery in the absence of sensation, such as when a geometrical diagram is imagined. In the latter case, the visual system, except for the primary visual cortex (V1, or BA 17), is activated still (Kosslyn, 1988; Roland & Gulyas, 1994, 1995; Kosslyn et al., 2001). The

---

[7] It should be noted that even from its origins in Ancient Greece geometry involved construction of diagrams as well as reasoning about static diagrams (Netz, 1999)—geometrical construction most assuredly requires action (as well as perception). However, my focus herein is image-based geometrical reasoning utilizing *preconstructed, static diagrams*. Even with respect to my references to dynamic geometry software (see Prologue), I view the dynamic geometry software as a tool to generate an unlimited number of static diagrams. Dynamic geometry is a different topic entirely and beyond the scope this dissertation.

visual system of the cerebral cortex must be a part of any discussion of the neural correlates of image-based geometrical reasoning.

As shown in Figure 3.1, afferent visual sensation enters the retina (at X-cells and Y-cells). Thereafter, neural excitation propagates through the lateral geniculate nucleus (LGN), the primary visual cortex (V1), and onwards through the various cortical areas associated with vision (Ozdemir & Black, 2005).

Ungerleider and Mishkin (1982) identified two subsystems of the visual system, the dorsal stream and the ventral stream. According to their theory, the dorsal stream is concerned with spatial features of objects in the peripheral visual field, whereas the ventral stream is concerned with identification of visual characteristics of objects in the central visual field. These are referred to as the “where” path and the “what” path, respectively. Mishkin et al. (1983) write,

> [T]he inferior temporal cortex participates mainly in the acts of noticing and remembering an object’s qualities, not its position in space. Conversely, the posterior parietal cortex seems to be concerned with the perception of the spatial relations among objects, and not their intrinsic qualities. (p. 415, cited from Mishkin, 1972).

Figure 3.1 is a schematic representation of cortical areas involved in the two streams, showing also that colour analysis in the visual system is part of the ventral stream.

As shown in Figure 3.1, the primary visual cortex, V1, contains three kinds of neurons. The neurons of the “blob” zones are responsible for colour processing, whereas neurons in the “interblob” areas are responsive to bars or lines (Ozdemir & Black, 2005). The neurons in Layer IVB are sensitive to movement and orientation of objects in visual space (ibid.). In approximate terms, the dorsal stream consists of the sequence of Brodmann Areas BA17 → BA18→ dorsal BA19 → BA39/40, whereas the ventral stream consists of the sequence BA17 → BA18 → ventral BA19 → BA37 → BA20.

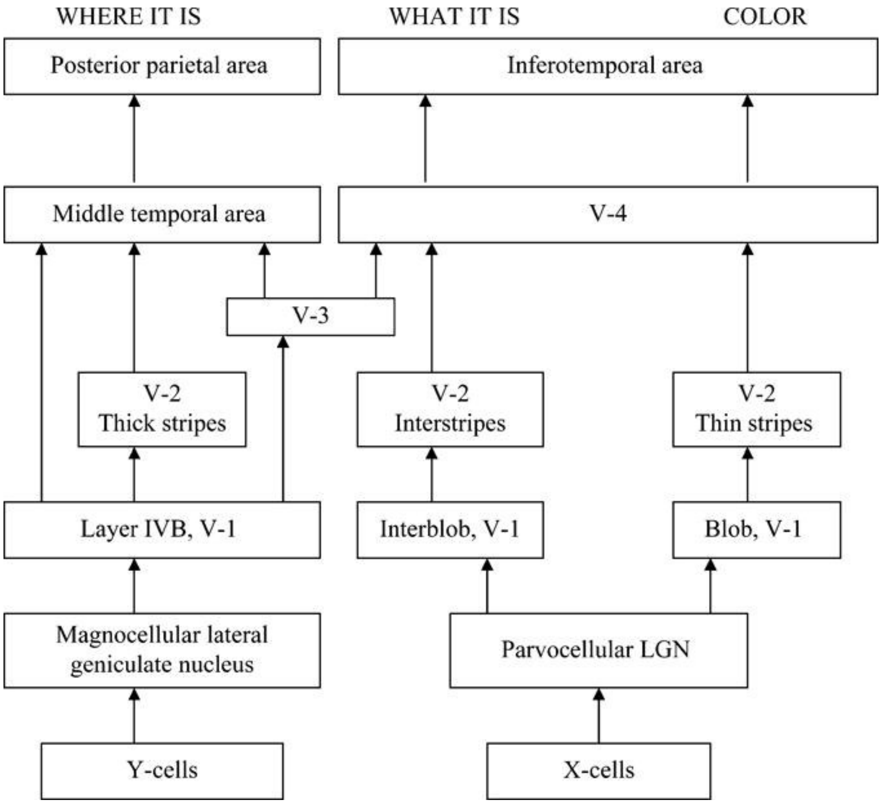

Figure 3.1. Dorsal stream and ventral stream.[8]

While the literature is universal in assigning the role of visual perception to the ventral stream, there is disagreement with respect to the dorsal stream. For example, Ungerleider and Mishkin (1982) claim that the role of the dorsal stream is spatial localization, whereas Milner and Goodale (2006) argue that the dorsal stream transmits information to the motor system for immediate use in reaching and grasping.

---

[8] Reprinted from Ozdemir and Black (2005, p. 66), with permission from Wolters Kluwer Health.

According to Husain and Nachev (2006), neither Ungeleider and Mishkin (1982) nor Milner and Goodale (2006) deal adequately with the role of the inferior parietal cortex. The theory of the two streams initially emerged from research with primates. Husain and Nachev point out that the inferior parietal cortex expanded greatly in humans compared with primates, particularly in its posterior aspects, the angular gyrus, and temperoparietal junction. The authors suggest that the functional role of the inferior parietal cortex is “detection of salient new items embedded in a rapid sequence of events and in maintaining or controlling attention over time” (p. 35).

On the other hand, it is discussed in Chapter 4 that activity in the inferior parietal cortex (BA 39) is correlative to certain aspects of mathematical reasoning. Indeed, Roland and Gulyas (1995) identified brain activity in the parietal cortex accompanying perception of static geometrical figures. It is possible that this brain activity is the neural correlate of the percept itself, but it is more likely that it is brain activity accompanying the visual concept of the geometrical figure, a notion that is discussed in more detail below. In Chapter 4, it is suggested more generally that associative areas of the parietal cortex, specifically BA 39, are the seat of conceptual reasoning. In any case, the point to take from Husain and Nachev (2006) is that the inferior parietal cortex must be regarded as apart from the dorsal stream model, or at least given a special role within the dorsal stream. The “spatial relationships” aspect of the dorsal stream may be understood in the sense of generalizing the particularities of perception that are the role of the ventral stream. In other words, the upper reaches of the dorsal stream, if BA 39 is included in this territory, may be associated with visual concepts.

Turning to the ventral stream, there has been a huge amount of research on visual perception and object recognition—see Wallis and Bülthoff (1998), Riesenhuber and Poggio (2000, 2002), and Kourtzi and DiCarlo (2006) for reviews. According to Grill-Spector and Malach (2004), there are two ways to understand the processing of visual information: hierarchical, in which localized, simple information is successively associated into more complex forms; and functional specialization, in which different aspects of the visual scene are processed along different paths. With regard to the hierarchical view,

> The underlying idea implies that object recognition is implemented in the brain through a series of processing stages, in which more global and invariant representations emerge up the hierarchy of the processing stream. The hierarchy is implemented through a gradual transition, from local representations that are closely tied to the retinal image to abstract representations that are closely linked to perception [and conception]. (p. 661)

With regard to functional specialization, Riesenhuber and Poggio (2002) write,

> Simple cells in primary visual cortex (V1) have small receptive fields and respond preferentially to oriented bars. Progressing along the ventral stream—thought to play a central role in object recognition in cortex [1,2]—neurons show an increase in receptive field size and in the complexity of their preferred stimuli [3]. (p. 162)

In the ventral stream, in particular, the neural responses are typically highly individuated: they identify colours, contrasts, orientations, and so on—see also Pasupathy (2006) and Vinberg and Grill-Spector (2008), particularly for investigations of shape recognition. According to Fuster (2003), “Cell assemblies, representing *partial features of sensory experience*, would constitute the ‘nodes’ of a memory network" (p. 10, italics added); these are the "sensory qualia of each modality" (p. 60). It is these partial features of sensory experience that are the *visual properties*.[9]

The notion of property here does not refer necessarily to a geometrical property such as "three sides" or "right angle." Properties, in the sense used here, are more fundamental, whereas geometrical "properties" are already aggregates of elementary properties. Note also that it is possible that the ventral stream is not the sole source of the neural correlates of properties, because specific spatial relationships in the dorsal stream may qualify, too. De Yoe and Van Essen (1988) argue that visual elements are processed along the dorsal stream and ventral stream in a complex, interacting manner.

With respect to Grill-Spector and Malach's (2004) hierarchical view, in the upper reaches of the ventral stream, in the inferotemporal cortex, individual aspects of the visual experience are associated together as *visual percepts*. According to Fuster (2003), the inferotemporal cortex (and indeed any visual areas in the ventral stream beyond V1) are visual unimodal association areas. He writes,

> Because the cells of unimodal association respond to stimuli that are more complex and spread over wider sectors of cortex than those of primary cortex, it is reasonable to conclude that they are part of networks representing those complex stimuli. Those networks, we can further infer, bind together the features that constitute the percepts they represent. Thus those networks would be formed by integration of the stimulus features that have been individually analyzed in sensory cortex. (pp. 71-72)

---

[9] Strictly speaking the brain activity referred to is the "neural correlate" of a visual property. The term "neural correlate" often will be omitted where it is obvious that the objective domain is being discussed.

Beyond the inferotemporal cortex, the seat of visual perception, the processing stream enters association areas of the temporal and parietal lobes. At various stages of the unimodal visual path, there are reciprocal connections sent to "temporal and parietal areas of multimodal convergence, presumably serving intermodal association, [that] have been termed *transmodal areas*" (Fuster, 2003, p. 68, author's italics). The highest level of visual perception, according to Fuster is at the junction of occipital, temporal, and parietal lobes, in other words, around BA 39. Fuster continues,

> [T]he streams of connectivity from primary sensory to transmodal areas mark not only trails of sensory processing but also an ascending ladder in a hierarchy of representations of perceptual knowledge. A massive literature of cognitive neuroscience substantiates this hierarchy. (p. 69).

This hierarchy, according to Fuster (2003), makes available "progressively higher categories of knowledge. By *higher categories of knowledge* [he] mean[s] knowledge of greater abstraction or generality" (p. 71, author's italics). The higher categories of knowledge in the visual modality may be termed *visual concepts*.

To summarize, Fuster (2003) clearly identifies three levels: primary association cortex, unimodal association cortex, and transmodal association cortex, which correspond, roughly, to visual properties, visual percepts, and visual concepts. These distinctions are supported by the other research cited in this section.[10] As discussed in Chapter 4, throughout the hierarchy, there are horizontal and vertical reciprocal connections. The formation of properties, percepts, and concepts is a process that is by no means simple or linear.

---

[10] The so-called *trans*modal association cortex may involve modalities other than the visual, bearing in mind that *visual* aspects only of activity in this region correspond to visual concepts.

Varela et al. (1991) make a special study of colour properties from the perspective of embodied cognition. Colour perception, in their interpretation, does not necessarily conform to independently existing features of a pre-given world, but emerges through the complex interaction of stimuli cascading through the visual system. Fuster (2003) also writes of emergent aspects of cognition, but his interpretation refers to higher cognitive functions in which “emergent” means that there is top-down processing from higher levels of cognition to lower levels. According to Fuster, emergence is less significant at lower levels of cognition, as bottom-up processing is more important at lower levels. However, top-down processing and bottom-up processing are features of all stages of cognition. The “emergence” of Varela et al. may be interpreted to be the effect of top-down processes at low levels of cognition within the visual system. According to Gilbert and Sigman (2001),

> Evidence that some [aspects of perceptual learning] are under top-down control suggests that perceptual learning arises from a combination of changes in local circuits at early cortical stages in sensory processing and feedback influences coming from higher order cortical areas. (p. 693)

This dual process is discussed in more detail in Chapter 4. It is interesting to speculate that other examples of “emergence” share similar characteristics.

Emergence, as such, applies not only to properties, but even more so to percepts. It is claimed, in Chapter 4, that percepts are associations of properties and concepts are associations of percepts. It is clear that a percept has component properties. However, it is not the case that a given collection of properties automatically coagulates to a specific percept. Quite simply, “causation,” as such, is bidirectional, and includes top-down processing from concepts as well as bottom-up processing from properties. In a given cognitive moment a percept resolves from a concept, just as properties are associated to form a percept. Percepts emerge at the confluence of two streams, bottom up from the properties, which have their origins in sensation, and top down from the concepts, which have their origins elsewhere—in “the deepest source of our knowing.”

In this section I wished at least to suggest candidates for localizing the neural correlates of visual concepts, visual percepts, and visual properties in the cerebral cortex. The objective, as always, is not to make detailed claims about the neural basis of image-based geometrical reasoning, but rather to suggest ways of understanding image-based geometrical reasoning that are compatible with the findings of cognitive neuroscience. In this way, cognitive neuroscience “constrains” psychological theories of mathematics education.

Now, the main thrust of the dissertation is a certain aspect of image-based geometrical reasoning, schematic inferences, which involves perception. However, the discussion also makes reference to action, and the notions of procedure, act, and action property, which are the analogues in action of concept, percept, and property, respectively. Just as visual perception interfaces with the world in sensation at the primary visual cortex (and its precursors in the retina and LGN), action interfaces with the world in the primary motor cortex (M1, or BA 4). Just as visual properties are the elementary components of visual percepts, *action properties* are the elementary components of acts. The action properties represent “specific movements defined by groups of muscles working in synergy and genetically preassigned to the cell populations of motor modules of area M1" (Fuster, 2003, p. 76). According to Fuster (2003) the grouping of muscles in M1 is by desired effect rather than somatotopically.

Next up from primary motor cortex is premotor cortex (BA 6), "where actions are represented in more concrete form, defined by trajectory and goal" (Fuster, 2003, p. 76). These are the *acts*, the analogues of percepts. A given act associates a group of action properties.

Finally, at the top level, in the prefrontal cortex (BA 8, 9, 10, 46), are "neuronal networks that represent programs or plans of action" (Fuster, 2003, p. 76). These are the *procedures*, the analogues of concepts. A given procedure associates a group of acts.

According to Koechlin et al. (2003), "[I]nformation processing underlying cognitive control [in anterior cortex] and perception [in posterior cortex] may obey common basic principles of neuronal computations" (p. 1184), which implies that procedures, acts, and action properties are likely to be organized in a way that is analogous to the organization of concepts, percepts, and properties. Just as in perception there is top-down and bottom-up processing between levels of the action hierarchy. Moreover, equivalent levels in action and perception, according to Fuster (2003), are reciprocally connected. The relationship between action and perception is described more fully in Chapter 4.

Just as concepts lack spatial specificity, procedures lack temporal specificity.[11] The supplementary motor area (SMA) is the upper, more medial part of BA 6. These cells are activated during the execution of specific sequences of movements rather than individual, component movements of a sequence (Fuster, 2003). This, according to Fuster is "a higher level of abstraction, executive *abstraction in the time domain*. The representations are no longer defined solely by spatial coordinates but also by the temporal coordinate" (p. 77, author's italics). Moreover, "The temporal dimension of action is paramount in the *lateral prefrontal cortex*" (p. 78, author's italics), at the top of the action hierarchy. According to Koechlin et al. (2003), "[T]he engagement of prefrontal regions along the posteroanterior axis is not primarily based on the relational complexity or memory load but on the *temporal structure* of representations underlying executive control" (p. 1184, italics added).

This section has suggested neural correlates for concepts, percepts, and properties in the visual modality, and their analogues in action. The broad division of cognition into action and perception, instantiated objectively in the anterior cortex and posterior cortex, respectively, is primordial. It must take priority over the left-right division that has received attention from educators concerned with brain-based learning (e.g., Bruer, 1997). In Chapter 7 I discuss

---

[11] Of course, procedures that are extended in time may become associated so that cognitive networks of larger size may correspond to more "general" procedures, whereas cognitive networks of more limited size may correspond to more "particular" procedures—no matter that "general" and "particular" procedures have the same temporal extent. I have not analyzed the cognitive networks of the anterior cortex in this respect, wishing to emphasize the temporality of the anterior cortex. The reader should bear in mind that alternative analyses are possible.

potential implications of this primordial division with respect to procedural reasoning and conceptual reasoning.

The goal of this section was to summarize some of the existing research in cognitive neuroscience. The evidence demonstrates the possibility of identifying neural correlates for aspects of image-based geometrical reasoning. However, as mentioned above, I am more concerned with overall structure rather than specific details.

Chapter 4 contains a more thorough investigation of concepts, percepts, and properties within the model of cognitive network theory. The discussion therein is non-specific with respect to modality, but the reader should bear in mind always that the goal is understanding image-based geometrical reasoning, specifically schematic inferences, and therefore vision should be foremost.

▲►▼◄▲►▼◄▲►▼◄

This chapter established neutral monism as a metaphysical foundation for the dissertation and discussed the notion of neural correlate. Specifically, I briefly summarized relevant research in cognitive neuroscience, with the goal of investigating the neural correlates for arithmetical reasoning and image-based geometrical reasoning. The next chapter develops a general model of neural activity in the cerebral cortex, in which the visual system is a special case. It completes the development of the theoretical framework.

CHAPTER 4: A THEORY OF THE CEREBRAL CORTEX

The previous chapter finished with a discussion of possible neural correlates for visual concepts, visual percepts, and visual properties. The cognitive network theory investigated below is a model that applies throughout the cerebral cortex, beyond vision, to all modalities of perceptual knowledge, as well as to action. It suggests an integration of procedural knowledge and conceptual knowledge, which may have consequences for theories of mathematics education (see Chapter 7).

According to the cognitive network theory, the visual system is hierarchical. A visual percept is an association of visual properties; a visual concept is an association of visual percepts. With regard to the hierarchy of knowledge, at least for the posterior cortex, Bergson's philosophy of duration provides an interesting and illuminating perspective. The discussion of duration leads into the property analysis and formalization of the notions of schematic concept and schematic percept. I propose that schematization of concepts and percepts is the essence of image-based geometrical reasoning..

4.1 Cognitive Network Theory

Fuster's (2003) cognitive network theory, which is concisely summarized in Fuster (2006), is a practical connectionism, in contrast to the theoretical connectionism of McClelland and Rumelhart (1987) and Rumelhart et al. (1987). It has similarities with the neural constructivism of Quartz and Sejnowski (1997), but sufficient granularity for identification of the neural correlates of concepts, percepts, and properties and their analogues in action. Before proceeding with the theory per se, it is necessary to review the neurophysiology of the cerebral cortex and how the cerebral cortex can "learn."

Throughout this dissertation there are many references to brain connectivity. This connectivity is to be understood in an effective (i.e., structural) sense, with actual physical connections, rather than in a functional sense, with connections that are inferred statistically (Friston, 1994). Neurons are connected structurally to other neurons, in other words, to form cell assemblies (or neural assemblies). In this dissertation, cell assemblies are referred to as *cognitive networks*, in accord with Fuster's (2003) terminology. Fuster uses the term *cognit* for the cognitive function that has a cognitive network as its neural correlate, as well as the cognitive network itself. According to Fuster,

> A cognit is an item of knowledge about the world, the self, or the relations between them. Its network structure is made up of elementary representations of perception or action that have been associated with one another by learning or past experience. (p. 14)

I do not use the term "cognit." Instead, the full phrase "cognitive network" is used for the objective correlate; the subjective equivalent is "cognitive function."

Axons extend from neurons to the dendrites of other neurons. Between the axon and dendrite is a synapse. When an electrical signal, an axon potential, passes down the axon, the resulting chemical or electrical activity across the synapse may cause the receiving neuron to fire electrically, resulting in further axon potentials. Single neurons connect in this fashion, and large populations of neurons connect on aggregate to other large populations of neurons in a similar way. Connectivity is directional, efferent *from* a particular region and afferent *to* a particular region.

The smallest structural unit of neural activity in the cerebral cortex, according to Fuster (2003), is the *minicolumn*, which contains around 100 neurons (see also Mountcastle, 1997). The electrons of one minicolumn connect, on aggregate, to the electrons of other minicolumns, and many minicolumns are linked together in this manner to form cognitive networks. Networks themselves may be linked to form higher-level network structures. On the large scale, whole regions of the cerebral cortex may be connected with other regions, linking vast populations of neurons through fibre pathways.

Structural neural connectivity itself is delivered through phylogenesis and ontogenesis. Hebb (1949/1964) proposed ways in which existing synaptic connections are modified, or "learn." His main postulate is,

> *When an axon of cell A is near enough to excite a cell B and repeatedly or persistently takes part in firing it, some growth process or metabolic change takes place in one or both cells such that A's efficiency, as one of the cells firing B, is increased*. (p. 62, author's italics)

Varela et al. (1991) suggest something similar: "[I]f two neurons tend to be active together, their connection is strengthened; otherwise it is diminished. Therefore, the system's connectivity becomes inseparable from its history of transformation" (p. 87). This rule implies that nascent cognitive networks become more sensitive and more distinct with use.

Hebb's (1949/1964) second rule for synaptic modification is that "any two cells or systems of cells that are repeatedly active at the same time tend to become 'associated,' so that activity in one facilitates activity in the other" (p. 70). In other words, when two neurons that are connected to the same neuron, or to different neurons within the same network, fire together repeatedly, then firing of one neuron alone precipitates firing of the second. Activation of part of a network, in other words, tends to activate the whole network. Or, if two networks overlap, activation of one facilitates activation of the other. These mechanisms, in themselves, tend toward the creation of a hierarchical network configuration.

Phylogenesis supplies the proclivity for the overall configuration of a cognitive network system. In ontogenesis, there is a process of synaptic proliferation followed by synaptic pruning (Black & Greenough, 1986; Greenough et al., 1987; Greenough & Black, 1992). According to Black and Greenough, in their seminal article,

> Mammalian development apparently relies on two basic categories of neural plasticity. One type, termed *experience-expectant* here, has evolved to utilize reliable experience common to nearly all members of a species in order to refine coarse neural systems. Other plasticity mechanisms, termed *experience-dependent* here, have evolved to maximize fitness of each individual animal by incorporating idiosyncratic information needed to find food, avoid predators, and so forth. (p. 2, authors' italics)

Experience-expectant neural plasticity, in other words, is determined phylogenetically, whereas experience-dependent plasticity is ontogenetic.

According to Black and Greenough, experience-dependent development continues into adulthood. Curiously, they do not cite the earlier research of Hebb (1949/1964). For the purposes of this dissertation, Hebb's ideas appear to offer greater conceptual clarity. It is assumed, therefore, that learning starts from a given neural connective structure, which is modified according to Hebbian principles. The reader should bear in mind, however, that there are, in all probability, several mechanisms of neural plasticity, which operate even in adulthood.

Fuster's (2003) theory is that the neural correlate of a cognitive function is the activity of a particular cognitive network. Fuster refers to this relationship as the *cognitive code*. According to Fuster, "A cognit is an item of knowledge about the world, the self, or the relations between them" (p. 14) and "[c]ognits are the structural substrate of all cognitive operations" (p. 14). As discussed earlier, Fuster assumes a strong version of the cognitive code, that the domains of cognitive function and neural activity have the same structure.

Below, it becomes clear that particular types of cognitive network are the neural correlates of concepts, percepts, and properties, respectively. I endorse Fuster's (2003) strong version of the cognitive code, at least with respect to the concepts, percepts, and properties relevant for geometrical reasoning.

Cognitive networks, according to Fuster (2003), are structured hierarchically. At the lowest level are networks whose activity correlates with elementary properties or action properties. Cognitive network theory is an empirical approach to the theoretical psychology of Hayek (1952/1963). Fuster agrees with Hayek that the overall configuration of the network system, in terms of possibilities for network formation, is phylogenetically determined, and that aspects of the network configuration that are constant across a species represent a kind of phylogenetic memory, the “memory of the species.”

Connections between network levels may be *convergent* or *divergent*. In other words, several lower-level networks may converge on a single higher-level network, or a lower-level network may connect with several higher-level networks. Connections between network levels are always reciprocal. Generally, a single network structure is *heterarchical*, consisting of substructures from several different hierarchical levels. At the lowest level networks are localized, but at higher levels they are increasingly diverse and interconnected, “gaining in width of distribution with every step as they fan out into more and higher areas, where they intersect other networks of different origin” (Fuster, 2003, p. 50). Individual neurons can belong to many different cognitive networks, and a lower-level network can belong to many different higher-level networks.

The Rolandic fissure separates the primary motor cortex from the primary sensory cortex. The Sylvan fissure separates the inferior frontal gyrus from the temporal cortex. The Rolandic fissure and the Sylvan fissure together divide the cerebral cortex into two hemispheres, the *anterior cortex*, consisting of the frontal cortex, and the *posterior cortex*, consisting of the parietal, temporal, and occipital cortices. According to Fuster (2003), cognitive networks of the anterior cortex correlate with action, interpreted in its broadest sense, whereas cognitive networks of the posterior cortex correlate with perception, again interpreted in its broadest sense. It was Betz (1874, cited in Fuster, 2003), according to Fuster, who first noticed that action is anterior and perception is posterior, and that this distinction runs from the spinal chord up through the cerebral cortex.

Note that my usage of the terms “action” and “perception” is meant to contrast with the terms “act” and “percept,” respectively. The latter pair of terms are used when their correlates are low-level networks in their respective hierarchies, whereas the former pair applies broadly, when their correlative cognitive networks are of the anterior cortex or posterior cortex, respectively.

In the literature of cognitive neuroscience, the primary division of the cerebral cortex is usually between left and right hemispheres. However, the primordial division between anterior and posterior cortices, between action and perception, is crucially important. The discussion throughout this dissertation refers primarily to the posterior cortex, although there needs to be reference to the anterior cortex.

According to Fuster (2003), the formation of cognitive networks follows largely the order in which axons are myelinated in ontogenesis, from primary sensory cortices, to unimodal association cortices, to transmodal association cortex, and also analogously in the anterior cortex. Those regions of the brain whose neurons are myelinated late in ontogenesis are associated with the most extensive cognitive networks. In Figure 4.1, below, the numbers correspond to Brodmann areas. The lighter coloured areas correspond to higher-level networks in the anterior and posterior hierarchies. This diagram agrees closely with the order that axons are myelinated in the respective regions.

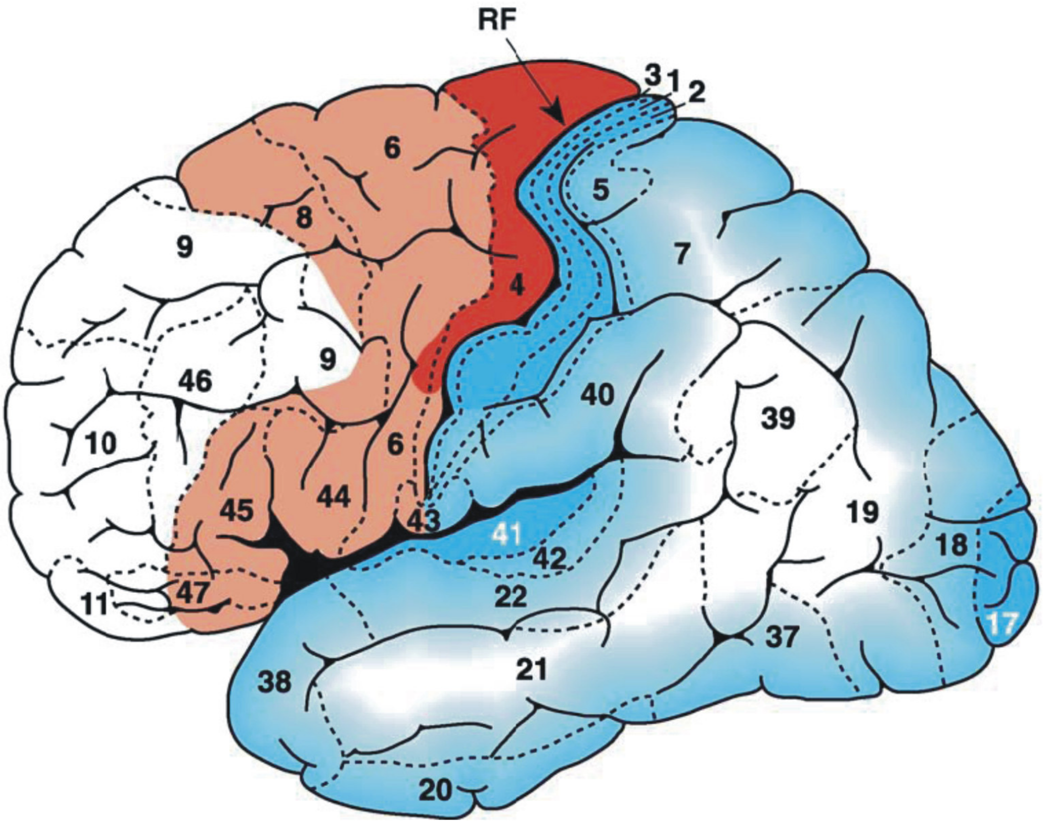

Figure 4.1. Anterior cortex (red) and posterior cortex (blue). Lighter shading shows areas corresponding to high-level cognitive networks; darker shading shows areas corresponding to low-level cognitive networks. Numbers refer to Brodmann areas.[12]

Note that BA 39 is central for networks of the highest level in the posterior cortex. A major focus of this dissertation is the functional significance of activity in BA 39.

It would be inaccurate to envisage distinct, discrete network levels in Fuster's (2003) theory. Nevertheless, several levels of network structure may be identified in broad outline, and a discussion of these levels is useful. In this respect, three levels of cognitive network in the posterior cortex are important, corresponding to networks belonging to primary sensory cortices, unimodal

---

[12] Adapted from Fuster (2006, p. 128), with permission from Elsevier.

association cortices, and transmodal association cortex, and then also their analogues in action.

Identification of specific cortical areas seems to imply that the cognitive networks of the respective levels are localized to these regions. This is not the case. Instead, these are the nodal regions, where lower-level networks are connected together, associated in network structures that extend well beyond these regions. Fuster (2003) writes, “[T]he function of every area or subarea of the cortex is defined by its afferent and efferent connections with other structures, as well as by its intrinsic processes” (p. 62).

According to Fuster (2003, pp. 71-72), quoted earlier, the cells in a unimodal association cortex belong to wider networks than those of a primary sensory cortex. These wider networks, he argues, integrate complex stimuli. A unified percept, correlative to a cognitive network of the unimodal association cortex, binds together elementary components of the stimulus. These elementary components depend on analysis of the stimulus in primary cortex. It follows that correlates of the various components of visual experience in the primary visual cortex, visual properties, in other words, are integrated as networks in the visual association cortex. As discussed in Section 3.3.2, the network structure of a visual percept is integrated at the upper reaches of the ventral stream, and may also extend to the dorsal stream.

A similar argument can be made for the integrative role of the transmodal association cortex. Networks of the highest level are distributed over the whole of the posterior cortex, and the role of the transmodal association cortex is integrative.

Now, to what cognitive function does activity involving the transmodal association cortex correspond? Surely, the transmodal association cortex would unify the correlates of the various perceptual modalities into a single percept constituted of vision, hearing, touch, and so on. Activity in the transmodal association cortex would still correlate with a *percept* rather than a still higher cognitive function. At the highest network levels, however, this is not the case. Fuster (2003) remarks,

> [A] high level cognit (e.g., an *abstract concept*) would be represented in a wide network of association cortex that has contacts with multiple lower-category networks with which it is associated—and which have contributed to its formation. (p. 82, italics added)

And elsewhere, he writes,

> At the highest level, in the upper reaches of the posterior association cortex, that is, in the broad confluence of the occipital, parietal, and temporal regions, lie distributed the most general and abstract cognits, the semantic memories and knowledge of facts and concepts that derive from sensory experience. Because such memories and knowledge derive from multiple experiences, and are largely generalizations of those experiences, their networks are the most widely distributed, with multiple associative anchors in cognits below. In global functional terms, therefore, cognits of ever-higher rank and generality develop from the bottom up mainly in divergent though also to some extent convergent-fashion. (p. 129)

According to Fuster (2003), “Upward convergence assists the binding of properties into higher categories, while upward divergence assists the distribution of common properties to different categories” (p. 96).

It seems that cognitive networks associated with “the upper reaches of the posterior association cortex” correlate with concepts. Moreover, “the broad confluence of the occipital, parietal, and temporal regions" appears to correspond to the inferior parietal cortex, BA 39. Indeed, I assume throughout this dissertation that concepts are correlative to activity in BA 39. Since an attempt to understand geometrical reasoning is a focus for this dissertation, my concern is solely the visual modality.

To clarify, a sensory stimulus from a triangle object activates neurons in the primary visual cortex, BA 17. Properties emerge as this activity cascades through the visual system, and then the triangle percept itself links these properties within the upper reaches of the ventral stream, the inferotemporal cortex, BA 20. When the cognitive networks for many different triangle percepts are linked together as a higher-level cognitive network in BA 39, the cognitive network of the triangle concept is formed.

Note that the linking together of percepts implies that the cognitive networks exist concurrently for multiple triangle percepts. And they do, of course, in memory, as neural structures that have been created and modified according to Hebbian processes. Every new triangle percept modifies the cognitive network structure of the posterior cortex, and the aggregate of these modifications is the triangle concept.

Fuster (2003) clarifies that association of low-level cognitive networks into higher-level cognitive networks does not lead to a proliferation of characteristics in the higher-level networks. In other words, activation of the high-level triangle network does not correlate somehow with a percept consisting of all specific triangles simultaneously. He writes, "Each class is thus defined by its members and the relationships between them, and not by their sum. . . . [T]he categorization is often accompanied by a degree of generalization, abstraction, and symbolization" (p. 61).

The cognitive network understanding of concepts is therefore extensional in the sense that the particular percepts that are associated in a concept may be regarded each as a "reference" of the concept (see p. 20, n. 2). According to Fuster (2003), however, a cognitive network

> cannot be a discrete and isolated network, uninfluenced by changes in other networks. A more appropriate view is that of a network with relatively firm connections at the core, made of repeatedly enhanced synaptic contacts, as well as weakly enhanced and *noncommitted* contacts 'around the edges.' It is difficult to determine with present methods what are the boundaries of the core and of the more plastic or labile periphery. (p. 82, author's italics)

In other words, the extensional concepts of the cerebral cortex lack clear boundaries and are therefore fuzzy and vague. These extensional concepts may be inadequate for the clear, accurate reasoning required for mathematics. One of the main goals of this dissertation is to uncover the neural mechanism whereby concepts are purified and abstracted, made intensional, in the way that this term is used herein (see p. 20, n. 2). Intensional concepts are abstracted concepts in an Aristotelian sense, as discussed in Chapter 7.

Learning corresponds in the cognitive network model to the formation and strengthening of cognitive networks. Repeated presentation of a stimulus leads to formation of low-level cognitive networks, according to Hebb's (1949/1964) first principle. Hebb's second principle naturally leads to the growth of higher level cognitive networks, those correlative to percepts, and then concepts. The cognitive networks of two similar percepts overlap, and then activation of either one activates both; the resulting cascade of activations corresponds to the cognitive network of the concept.

The formation of an entirely novel low-level cognitive network only incrementally changes the high-level network into which it is integrated. Conceptual learning is therefore slow. At the lower levels, on the other hand, “perceptual learning” is rapid. If learning, in this limited sense, is the working through of Hebbian principles, then learning is happening continuously. However, the term “learning” is usually reserved for more substantial alterations in the cognitive network structure.

A wave of neural activity begins with the sensory impact on a primary sensory cortex and flows upwards through the hierarchy of the posterior cortex. The wave begins with small, localized cognitive networks and spreads into larger, more diverse, more diffuse networks.

Interestingly, in addition to this centripetal wave, there is a complementary centrifugal wave that originates in high-level cognitive networks and cascades down through the posterior hierarchy. According to Fuster (2003),

> [E]very percept has two components intertwined, the sensory induced *re-cognition* of a category of cognitive information in memory [i.e., centripetal flow] and the categorization of new sensory impressions in the light of that retrieved memory [i.e., centrifugal flow]. Perception can thus be viewed as the interpenetration of new experiences based on assumptions from prior experience. (p. 84, author’s italics)

These two components of perception, centripetal and centrifugal, were discussed in the previous chapter and are examined further in the context of duration later in this chapter. It is apparent that the centrifugal flow is a natural concomitant of the centripetal flow in the process that is cognition.

With regard to memory, Fuster (2003) writes, “We perceive what we remember as well as remember what we perceive. Every percept is a historical event, a categorization of current sensory impressions that is entirely determined by previously established memory” (p. 84). Fuster is arguing that perception and memory are different terms for the same idea.

Equating perception and memory goes against the grain of cognitive psychology. Cognitive scientists have classified memory into various types, including short-term memory, long-term memory, episodic memory, and so on (e.g., Byrnes, 2001). However, perception as understood herein has a broader meaning than perception usually has in cognitive psychology. It seems clear that all memory with sensory qualities belongs to perception. For example, the image of a remembered face or a name is *perceived*, although it may not have the clarity and immediacy of a percept. The converse that all perception is memory may be more difficult to accept. However, activation in perception of a cognitive network is the *reactivation*, largely, of a cognitive network that already existed. Certainly the latter point is true with respect to concepts, which alter only incrementally with new percepts. However, even percepts themselves are memories, memories created at the moment, with new sensation, although still determined by “older memories,” the concepts, in the top-down flow. Existing memory surges to meet new sensation, to paraphrase Bergson (1896/2005).

Fuster (2003) defines the neural correlate of *attention*, with respect to cognitive networks, as the joint action of excitation and inhibition—activation of the "attended to" network is enhanced, while the activation of networks competing for attention is suppressed. According to Fuster (2003), the centrifugal flow from concept to percept networks in the perception hierarchy is a necessary attentional mechanism. Concepts suppress alternative perceptual interpretations of sensory data. He argues,

> If that were not the case, a discrete sensory stimulus would lead to a cascade of activation through innumerable associative links of context and past experience. In other words, the stimulus would lead to an excitatory explosion and to the submersion of the cognitive gestalt that it evokes in a morass of associative noise. (p. 98)

Elsewhere, he writes, "[D]iscrete aspects of perception are selectively modulated from experience, that is, enhanced or gated to maximize the yield from the processing of sensory information that experience tells us is most relevant at a given place and time" (Fuster, 2003, pp. 85-86). There are other attentional mechanisms. A second is discussed below, and a third in Chapter 5. Two theoretical models of attention are discussed in the next section.

In the theory so far, only cognitive networks in the cerebral cortex have been discussed. Fuster (2003) admits that precortical structures such as the amygdala and hippocampus have a role in memory. The precise nature of the precortical contribution to memory, I believe, has yet to be fully elucidated as far as cognitive network theory is concerned.

Much of what has been written with respect to the posterior cortex translates directly to the anterior cortex and action. Corresponding to the three levels of cognitive network in the posterior cortex are cognitive networks correlative to procedures, acts, and action properties, which were introduced in Section 3.3.2. According to Fuster (2003),

> Paralleling a hierarchy of perceptual knowledge, there is a *hierarchy of action knowledge*. . . . As is the case with percepts, all the actions of the organism can be categorized, from the bottom up, and stacked in a hierarchy of motor cognits. At the bottom reside the elements of action that are defined by discrete movements and muscle groups. Above them are the categories of action defined by goal and trajectory; and higher yet are the programs and plans. (pp. 61-62, author's italics)

As discussed earlier, the action hierarchy is temporal, whereas the perception hierarchy is spatial.

Cognitive networks need not be restricted to a single hemisphere. They may integrate both hemispheres, anterior and posterior. Merleau-Ponty's (1942/2006) "exchange of stimuli and response" (p. 13) is at the basis of Varela et al.'s (1991) embodied cognition, which was discussed in Section 3.1. Uexküll (1926, cited in Fuster, 2003) first identified what Fuster calls the *perception-action loop*. In the formulation of Uexküll, and also of Varela et al. (1991), the perception-action loop passes through the external world: the cognitive network correlative to an act discharges in movement; movement causes sensation; sensation activates the cognitive network correlative to the percept; only then, within the cerebral cortex, does the percept network close the loop by modifying the network of the succeeding act. This perception-action loop participates in the hierarchies of perception and action only at a low level. According to Fuster, however,

> At all levels of the central nervous system, the translation of perception to action is mediated through connections between sensory and motor structures. Both sensory and motor structures are hierarchically organized along the entire length of the nerve axis, the two tiers interconnected by reciprocal polysynaptic pathways that form a sort of ladder of connections between the sensory moiety and the motor moiety of the nervous system. (p. 107)

In other words, the perception-action loop at the level of percepts and acts is complemented by an analogous loop at the level of concepts and procedures. This higher-level loop does not pass through the external world, but remains confined to the cerebral cortex.

The perception-action loop may be regarded as a second attentional mechanism, operating from action to perception and from perception to action. Fuster (2003) writes, “[V]isuospatial attention appears to be a clear example of the cortical operations of the perception-action cycle, the continuous circular interaction of the organism with its environment” (p. 172). He continues,

> At all levels of the perception-action cycle, the feedback from sensory systems on motor systems is reciprocated by inhibitory feedback in the opposite direction. This motor-sensory feedback is an essential support of the *inhibitory control* over sensory systems that constitutes the exclusionary component of attention. (pp. 174-175, author’s italics)

The “inhibitory feedback” from anterior cortex to posterior cortex clearly can be interpreted as an attentional mechanism. Fuster does not specifically refer to the flow from posterior cortex to anterior cortex in the same way, but it seems apparent that there is mutual specification between the two halves of the cerebral cortex. Certainly perception guides action, and perhaps this too should be regarded as an attentional mechanism.

This section has discussed the cognitive network interpretation of concepts, percepts, and properties, and their analogues in action. It has identified reciprocal flows of neural excitation/inhibition in perception-action loops and also between levels in the anterior and posterior hierarchies. The bidirectional flow between levels is discussed in Section 4.3, in the context of Bergson's philosophy of duration, but firstly I would like to investigate two influential models of attention that I need to cite in Chapter 5.

4.2 Two Models of Attention

One of my positions in this dissertation is that a functional role of the cerebellum is a specific kind of attention. It is important, therefore, for me to engage neuroscientific theories of attention. The literature in this area is vast and beyond the scope of this thesis to review comprehensively. It is adequate for my purpose here to restrict consideration to two recent works in this area—the first from neuropsychology and a second from neurophysiology. Regarding the former, I briefly describe a five-level clinical model of attention developed by Sohlberg and Mateer (1989); with respect to the latter, I briefly describe a four-component model of attention developed by Knudson (2007). The former I interpret with respect to image-based geometrical reasoning; the latter I interpret with respect to cognitive network theory.

4.2.1 Five-level Clinical Model of Attention

Sohlberg and Mateer (1989) proposed a neuropsychological model of attention. It is a hierarchical model, which describes levels of attention recovery following brain lesions. Patients recover first the more primitive forms of attention. As recovery processes, patients gain access to increasingly sophisticated attentional processes. The five levels are focused attention, sustained attention, selective attention, alternating attention, and divided attention. I briefly describe each in turn.

At the lowest level is *focused attention*: “This is the ability to respond discretely to specific visual, auditory, or tactile stimuli” (Sohlberg & Mateer, 1989, p. 120). A patient with a severe deficit in focused attention will not be able to respond to external stimuli. With respect to image-based geometrical reasoning, focused attention would correspond, for example, to basic recognition of a triangle, when that triangle is presented simply, without distracting information. Focused attention itself does not imply any cognitive function, sustained or otherwise, beyond simple recognition.

When attentional focus can be maintained, the level of *sustained attention* has been reached: “This refers to the ability to maintain a consistent behavioral response during continuous and repetitive activity. It incorporates the notion of vigilance” (Sohlberg & Mateer, 1989, p. 121). A patient with a deficit in sustained attention can maintain responses to stimuli only for brief periods. The observer would be able to direct and maintain attentional focus on the triangle, for example, provided the triangle is presented simply, without distractors.

Once the ability has been reached to achieve an attentional focus in the face of distractors, the level of *selective attention* has been reached:

> This level of attention refers to the ability to maintain a behavioral or cognitive set in the face of distracting or competing stimuli. It thus incorporates the notion of 'freedom from distractibility.' Individuals with deficits at this level are easily drawn off task by *extraneous, irrelevant stimuli.*" (Sohlberg & Mateer, 1989, p. 121, italics added)

In other words, the observer can establish an attentional focus on the triangle even when it is embedded in a non-geometrical environment, or even embedded as a single aspect of a complex geometrical environment. Selective attention is an important concept for my thesis regarding schematic perception. I argue in Chapter 5 for the hypothesis that a functional role of the cerebellum is facilitation of selective attention.

When attentional focus can be shifted fluently between stimuli, the level of *alternating attention* has been reached: "This level of attention refers to the capacity for mental flexibility that allows individuals to shift their focus of attention and move between tasks having different cognitive requirements, thus controlling which information will be selectively attended to" (Sohlberg & Mateer, 1989, p. 121). A patient with a deficit at this level has difficulty changing tasks once attention has been directed to a given task. For example, the observer may be able to switch attention smoothly between two angles and a circle, when both are embedded in a complex geometrical situation, as illustrated in the Prologue. This mode of attention can be understood to help facilitate schematic inferencing over objects of schematic perception.

Lastly, with *divided attention*, the observer is able to juggle multiple tasks simultaneously: “This level involves the ability to respond simultaneously to multiple tasks or multiple task demands. Two or more behavioral responses may be required, or two or more kinds of stimuli may need to be monitored” (Sohlberg & Mateer, 1989, pp. 121-122). The authors explain that divided attention may involve either rapid and continuous alternating attention or unconscious processing of one of the tasks rather than real simultaneous attentional focus on multiple targets. A patient with a deficit in divided attention may find it difficult, for example, to hold a conversation while preparing a meal. In geometrical terms, the observer may generate a cognitive response to different aspects of schematic perception, without having to alternate attention between the two. In practical terms, conscious attention may be focused directly on one component of a diagram, while others are processed below the conscious level.

It is important to note that this clinical model of attention is hierarchical. In other words, attentional capacity at a given level in the hierarchy implicates attentional capacity at all lower levels. For example, the capacity for alternating attention implicates the capacity for selective attention. This is important to bear in mind when I discuss attention and the cerebellum in Chapter 5.

4.2.2 Four-component Neurophysiological Model of Attention

The second model of attention I consider is Knudson’s (2007) four-component neurophysiological model. The four components are (1) working memory, (2) top-down sensitivity control, (3) bottom-up salience filters, and (4) competitive selection. I deal with each of these in turn.

The previous section discussed memory from the cognitive-network perspective, and the next section discusses memory from a Bergsonian perspective. In both locations, I consider memory and perception as essentially the same cognitive function. Nevertheless, cognitive scientists have identified various forms of memory, one of them being working memory. Working memory according to the framework of this dissertation is a form of perception.

According to Knudson (2007), “Working memory holds a limited amount of information for periods of seconds while the information is evaluated and manipulated in a uniquely powerful and flexible fashion on the basis of the animal’s internal state and stored memories” (pp. 59-60). Working memory, therefore, lies close to the percept end of the percept-concept scale. Percepts change relatively rapidly, whereas concepts change slowly with the gradual accumulation of associated percepts, as discussed more fully in the next section. Working memory, essentially, consists of percepts, according to the framework established in this chapter and in Chapter 3.

According to Knudson (2007), “[W]orking memory represents the objects of attention” (p. 60). In other words, current, conscious percepts are those that are being attended to. Knudson also states that the capacity of working memory at any given time is limited to a single domain, such as verbal or visuospatial; moreover, the portions of the brain that participate in working memory depend on the particular cognitive domain that is being invoked by attention.

The visuospatial aspect of attention/perception is my main concern herein. Knudson cites evidence that areas of the cerebral cortex that participate in visuospatial working memory are the dorsolateral prefrontal cortex, the right inferior parietal cortex (BA 39, 40), and high-order visual areas in the occipital cortex. Executive control, the planning of behaviour, would involve the dorsolateral prefrontal cortex (Fuster, 2003). Knudson cites other research that indicates that the spatial working memory activates the posterior parietal cortex (BA 5, 7) as well as the inferior parietal cortex. The posterior parietal cortex belongs to the dorsal visual stream, whereas the inferior parietal cortex is identified as a central location for the cognitive networks of visual concepts (see Section 3.3.2). Knudson suggests that the posterior parietal cortex may be concerned primarily with the relative importance of stimuli (see below). However, Knudson also notes that there are substantial reciprocal connections between prefrontal cortex and posterior parietal cortex. Therefore, the posterior parietal cortex may be concerned with the attentional mechanism that is dependent on the perception-action loop, as identified by Fuster, and discussed in the previous section.

Knudson (2007) describes two ways in which attentional mechanisms improve information quality. Firstly, body and eyes may be orientated toward the target. Then, he writes, “A second strategy for improving information quality is to modulate the sensitivity of neural circuits that represent the information. . . . This top-down mechanism can improve the signal-to-noise ratio in all domains of information processing” (p. 62).

Knudson (2007) indicates that vast areas of the brain are associated with “attention-related bias signals” (p. 62)—he cites sub-cortical areas, as well as prefrontal cortex, including the cerebellum. He states that the “[t]op-down modulations of neural responsiveness are precise for the features upon which judgments will be made” (p. 63), and so, “[o]nly neurons with receptive fields that contain the stimulus . . . exhibit an increase in sensitivity” (p. 63). On the other hand, “The inhibition of neurons that are not tuned for the stimulus suggests that top-down bias signals activate local inhibitory circuitry, as well as excitatory circuitry” (p. 63).

Knudson (2007) cites evidence that visual area V4 (BA 17) is the target for top-down bias signals for line orientation, and the medial temporal area (BA 35, 36) is the target for top-down bias signals for the direction of stimulus motion.

Identification of top-down sensitivity control with a combination of excitation and inhibition of neural circuitry, together with its targeting of higher visual areas of the occipital cortex, is consistent with the definition of attention in the previous section and the research cited in Section 3.3.2. It seems plausible to consider that the centrifugal flow of neural activity from higher to lower levels of the cognitive network hierarchy of the posterior cortex, identified by Fuster (2003) as an important attentional mechanism, equates to top-down sensitivity control.

On the other hand, according to Knudson (2007),

> Information does not need to be modulated by top-down bias signals to gain access to working memory (Egeth & Yantis 1997, James 1890). Certain properties of the world can evoke exceptionally strong neural responses that may win access to working memory. (p. 64)

Accordingly, evocation of attentional processes by features inherent in the external world is referred to as bottom-up attention by Knudson.

Knudson (2007) suggests further that mechanisms of neural plasticity can create bottom-up salience filters. Temporally, responses to stimuli occuring frequently in *time* may diminish over time. Spatially, “Network connections can create filters for stimuli that occur infrequently in *space*” (p. 64, italics added), and “the unusually strong neural activation that results from these filters gives the representations of salient stimuli an advantage in the competition for access to working memory” (p. 64).

This dichotomy between time and space is a very interesting aspect of bottom-up attention. Attention may diminish with regard to occurrences that are frequent, but unusual aspects of the given environment, at a given time, “are perceived as ‘popping out’ from the scene” (p. 64).

Bottom-up attention is consistent with the centripetal flow of neural activity in the cognitive network hierarchy of the posterior cortex, as discussed in this chapter and in Chapter 3. Percepts, it was argued in Chapter 3, emerge at the confluence of the two flows, centrifugal and centripetal. And the emergence of percepts means that data coming from the external world has gained access to working memory.

The final component of attention in Knudson’s (2007) model is competitive selection. He writes,

> The selection of information for entry into working memory is a highly competitive process (Desimone & Duncan 1995). Information about the external world, from memory stores, and about the animal’s internal state is processed extensively and automatically in parallel hierarchies of networks in the central nervous system. (p. 69)

Which of these parallel hierarchies is chosen for focused conscious attention, according to Knudson, is the result of competitive selection. He suggests that cortical regions dominate stimulus selection, although subcortical areas have an important influence, and he concludes, “[I]nformation from the superior colliculus influences the selection process in parallel with information from the visual cortex” (p. 72).

Knudson (2007) makes substantial reference in his discussion of competitive selection to hierarchies of neural networks, which certainly recalls the cognitive network theory. In overall terms, Knudson’s model is consistent with the view of attention presented in the previous section. Working memory would seem to be equivalent to conscious perception, whereas top-down sensitivity control and bottom-up salience filters seem to correspond, respectively, to centrifugal and centripetal flows of neural activation.

In summary, the various neuropsychological forms of attention identified by Sohlberg and Mateer (1989) seem to correspond to some of the cognitive processes required for image-based geometrical reasoning. On the other hand, Knudson’s (2007) neurophysiological model appears to accord with the cognitive network theory of the previous section. I am particularly interested in selective attention. Schematization, the focusing of attention on those aspects of a geometrical situation that are most relevant for a given cognitive task, is a form of selective attention. The cerebellar schematization hypothesis, therefore, should be closely related to selective attention.

4.3 Duration

One of the most interesting features of the cognitive network model is the bidirectional flow of neural activity, up and down the cognitive network hierarchy, centripetal and centrifugal, from percepts to concepts and from concepts to percepts. This process is discussed now in the light of Bergson’s philosophy of duration.

A few remarks are necessary in order to account for the substantial disparities between orthodox Bergsonism and the theory outlined herein. Firstly, Bergson’s ideas, as represented primarily in Bergson (1896/2005), clearly identify him as a dualist rather than a monist, but a dualist of an unusual kind. There are two domains, matter and memory. All subjective cognitive experience is “memory.” These two domains connect at the single point where the organism acts in the world, where memory congeals into matter, as it were, and where action deposits new memory. The relationship between action and perception that I espouse herein is substantially different.

Secondly, Bergson (1896/2005) has a different view of the relationship between concepts and percepts. According to Bergson, in the surge from memory to action in the world, and the deposition of matter, a variety of memories, potentially, can congeal in matter as a single, specific act. Acts, therefore, are the principle that unifies percepts into concepts, rather than the association of percepts at higher hierarchical levels.

In view of these major deviations from orthodox Bergsonism, the content of this section should be regarded as *inspired by* Bergson rather than as a summary or recasting his ideas. At the least, I utilize Bergson's notions of memory and duration and his metaphor of the cone of duration.

Duration, the most important of Bergson's (e.g., 1907/1944) ideas, may be thought of as the totality of subjective lived experience, the survival of the past in the present. This section focuses on experience as manifested in perception, although the reader should bear in mind that there is an analogue in action.

Subjective experience of time is not instantaneous—the present retains a nimbus of the recent past. The indistinct experience of time that we actually live through was identified by James (1890/1950) as the *sensible present*. The sensible present may be thought of as the conscious content of subjective experience. According to Bergson (1903/1999),

> [T]here is no state of mind, however simple, which does not change every moment, since there is no consciousness without memory, and no continuation of a state without the addition, to the present feeling, of the memory of past moments. It is this which constitutes duration. Inner duration is the continuous life of a memory which prolongs the past into the present . . . . Without this survival of the past into the present there would be no duration, but only instantaneity. (p. 40)

Stretching behind the instantaneous present, as it were, is the sensible present. The knife-edge of the instantaneous present is continually moving forward, depositing, as it changes, new memories of the recent past and thereby transforming the sensible present. *Memory-images* are memories that retain perceptual characteristics. They belong to the sensible present and may be regarded as percepts. (The equivalence of memory and perception was briefly discussed in the previous section.) Further from the instantaneous present are *pure memories*, which I would like to interpret as concepts. (For Bergson, pure memories are still perceptual, although distant from actualization in the present instant as action.) Beyond concepts we need not go, although pure memory extends into deep duration, the ocean of the unconscious, and “the deepest source of our knowing.” Duration, therefore, contains not only the sensible present, consisting of percepts and concepts, but the entirety of a person’s lived experience, conscious and unconscious.

Concepts, as well as percepts, are a type of memory. Remember that concepts and percepts, or rather their neural correlates, belong to the perception hierarchy of cognitive networks. *Concepts and percepts are cognitive/neural objects of the same type*. There is no sharp distinction between the two, but rather a difference of degree, in their physical extent and complexity. Conceptual cognition is correlative to the activation of cognitive networks that, to a large extent, were already existent. In fact, since the cognitive networks of concepts are more stable than those of the fleeting percepts, conceptual cognition has more right to be regarded as memory, as the word is usually understood, than perceptual cognition. Conceptual cognition is deep memory created through layer after layer of perceptual cognition. It would be a mistake to think of memory of a time long ago, perhaps evoked with eidetic clarity, as “deep memory.” These memories, ancient in objective terms, are simply a variety of percept, and the more so as their sensory clarity is heightened. “Deep memories” are concepts.

Bergson (1896/2005) referred to experience of the instantaneous present as *pure perception*. According to Bergson, pure perception is action in the world. In the terms developed in this dissertation, it is possible to regard pure perception and action as connected through the lowest level of the perception-action loop, in which stimulus follows response without reflection. As mentioned above, this view is not orthodox Bergsonism.

The perception of objects presumed to exist in the external world consists of memories that have slipped back slightly from the present instant. Conscious perception of the external world, according to Bergson (1896/2005), is not perception at the present instant, but perception of the “slippage of the present into the past.” Bergson writes,

> Your perception, however instantaneous, consists then in an incalculable multitude of remembered elements; in truth, every perception is already memory. *Practically we perceive only the past*, the pure present being the invisible progress of the past gnawing into the future. (p. 150, author’s italics)

The present instant, then, marches forward depositing memory in its wake. This idea needs to be clarified. It is an error to suppose that an object that is a memory is receding in time. It would lead to two kinds of problem. In the first place, the idea of watching an object recede into memory, thereby fading and losing focus, animates the homunculus fallacy, because it supposes that one part of the mind is an observer of another part of the mind, and the well known paradox of infinite regress results—“Must we then say that for the hero’s reflections how to act to be intelligent he must first reflect how best to reflect how to act? The endlessness of this implied regress . . . . ” (Ryle, 1949/1963, p. 31). Secondly, the idea of watching the object recede into memory implies that it is possible to mark time independently of duration, and that duration is somehow imbedded in time that is separate from duration. Clearly this is not the case, because duration itself manifests subjective time. According to Bergson (1889/1960),

> The principle of identity is the absolute law of our consciousness: it asserts that what is thought is thought at the moment when we think it: and what gives this principle its absolute necessity is that it does not bind the future to the present, but only the present to the present: it expresses the unshakeable confidence that consciousness feels in itself, so long as, faithful to its duty, it confines itself to declaring the apparent present state of the mind. (pp. 207-208)

According to Brown (2002), *the sensible present is virtual time. A single instant of objective time may contain an age of subjective temporal experience*. It brings to mind the image of the drowning man, whose whole life flashes before his eyes—the whole of his duration becomes conscious in an instant of objective time.

Bergson (1889/1960) and his successor Whitehead (e.g., 1919/2007, 1925/1967) argued that subjective time was not commensurable by objective means, and clearly this is the case, given the discussion so far. We all understand the distinction between subjective and objective time from personal experience—one day can fly past, while another seems to drag on interminably, despite the fact that both days are 24 hours long. As mentioned earlier, it is necessary to take great care in identifying neural correlates when cognitive experience has a temporal component.

On the other hand, neural activity takes place in the objective world. According to Varela (1999), large-scale cell assemblies (the equivalent of cognitive networks) arise and subside on a timescale of around two to three seconds each. This time period is a cognitive moment. The actual *experience* of duration must be continuous, whether or not cognitive-network activation is continuous or pulses discretely. The space between cognitive moments is impenetrable, subjectively.

Now, imagine a row of lights, each of which can be on or off. In the following image, each line represents a cognitive moment. The numbers to the left of each line represent surface memory in the sensible present, or alternatively, percepts (or rather their neural correlates); numbers to the right represent deep memories, or alternatively, concepts (or rather their neural correlates). Illumination is propagated along the line; a light switches off and its neighbour to the right comes on in the next cognitive moment. The 1 appears to move to the right, but this is an illusion.

10000
01000
00100
00010
00001

A more accurate metaphor would utilize a different colour of light in each position, resulting in the following image.

10000
02000
00300
00040
00005

In other words, there is an illusion of movement accompanied by qualitative change. In the objective domain, activation cascades up the cognitive network hierarchy.

There is a second way to think about the behaviour of the row of lights. Instead of imagining the 1 to be receding to the right in the first image, imagine the row of 0s to be shunting to the left, displacing the 1. This second way of looking at the flashing lights corresponds to the idea that surface memories (percepts) resolve from deep memories (concepts). The two ways of understanding the behaviour of the row of lights are equivalent, and they are both illusions; in actuality there is no *movement* of memories, in either direction.

Suppose now that all lights in a row may be illuminated simultaneously, and that the brightness and colour of each light at one cognitive moment is able to influence the brightness and colour of the adjacent lights in the following cognitive moment. This is as close to the actual process of cognitive network activation, up and down the posterior hierarchy, as the metaphor can take us. And this cognitive network interpretation is none other than the neural correlate of subjective duration.

The very fact that memories are not objects moving from level to level is the reason that both movements, centripetal and centrifugal, can be entertained simultaneously. In fact, with duration as a process as I have described it, it does not make sense to have a flow in one direction without the other. According to Bergson (1896/2005),

> Does this not amount to saying that distinct perception is brought about by two opposite currents, of which the one, centripetal, comes from the external object, and the other, centrifugal, has for its point of departure that which we term ‘pure memory’? The first current, alone, would only give a passive perception with the mechanical reactions which accompany it. The second, left to itself, tends to give a recollection that is actualized—more and more actual as the current becomes more marked. Together, these two currents make up, at their point of confluence, the perception that is distinct and recognized. (pp. 127-128)

Roth and Hwang (2006) express the same idea, but from an educational perspective:

> [L]earning and development appear to simultaneously constitute a movement from concrete to abstract *and* a movement from abstract to concrete. In other words, it is not two concurrent movements one going from abstract to concrete and the other from concrete to abstract. Rather, it is the same movement (development, learning) that simultaneously goes from abstract to concrete and 'from concrete to abstract. (p. 335, authors’ italics)

And both of these ideas recall Fuster (2003, p. 84), quoted earlier.

A second illuminating metaphor is Bergson’s (1896/2005) cone diagram, shown in Figure 4.2. Concepts of increasing generality, higher up the cognitive network hierarchy, have more *potential*, because they have increasingly wide cognitive network structures that link networks correlative to percepts that are increasingly disparate. In contrast, as cognitive network structures decrease in physical extent, becoming narrower, as it were, they are correlative to cognition that increasingly has the clarity and immediacy of a percept. In other words, they are more *actual*. (Note that Bergson’s own interpretation of the cone is different.)

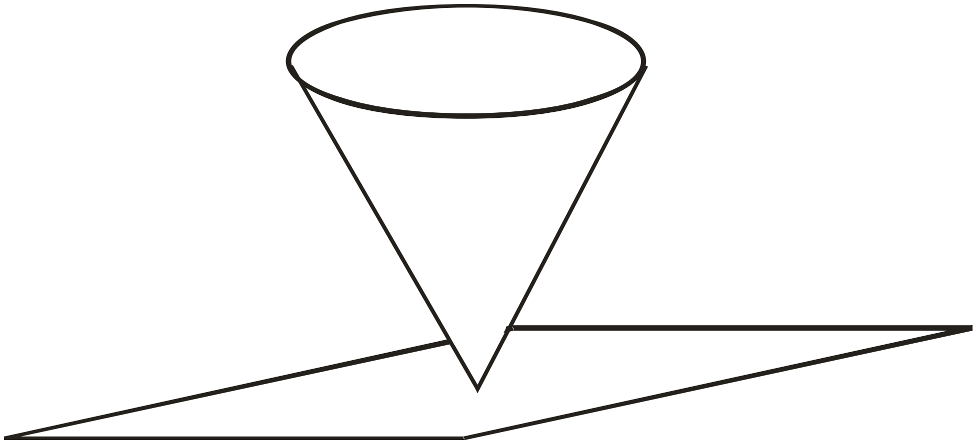

Figure 4.2. Bergson's cone diagram.[13]

The vertex of the cone represents the actuality of an exterior world in perception. The plane is the external world, indicating that the vertex of the cone is just one aspect of the external world, one perspective, at one location, at one time. It is important to recognise, for geometrical reasoning, that the vertex of the cone is the actual percept and not a geometrical diagram. In this sense, the plane may be regarded as the whole diagram and the vertex that part of the diagram that is perceived at a given moment. The gaze may move across the diagram, allowing the percept to vary. On the other hand, the gaze may remain fixed, while different aspects of the diagram in a specific location are perceived. In this sense, too, perception "ranges over" the external world. It is mistaken to imagine that the plane is simply a two-dimensional analogue of three-dimensional space.

The cone stretching above the plane represents memories that are channelled toward actualization as percepts. Close to the vertex of the cone is the instantaneous present. The widening of the cone represents increasing

---

[13] Adapted from Bergson (1896/2005, p. 152), with permission from Zone Books.

potential; the narrowing of the cone toward the vertex represents increasing actuality.

The notion that a concept represents potential implies that there may be a variety of ways in which the potential could become actual. In other words, “concept” should be understood in an extensional sense rather than in an intensional sense (see p. 20, n. 2). It is necessary to emphasize, however, that the concept is not a collection of individual percepts, but is something qualitatively different in the sense that it is the aggregate of relationships between percepts. This point has already been addressed in the discussion on cognitive networks. It should also be emphasized that the concept, as described, is not yet an abstraction, or intensional concept. As explained in Chapter 7, it may be regarded instead as a form of generalization. The properties of the percept are not selectively disregarded in the movement that is conceptualization.

In the next section I argue that the concept and percept have the same range. In other words, actual properties of the percept correspond one-to-one with potential properties of the concept. For example, the actual colour green of the percept corresponds to the potential of possible colours in the concept. Some concepts are wider in range than others. The percepts that actualize from these concepts also have wide range.

Now, an *essential* property of a percept takes the same value in all percepts that are associated in a concept. For example, all triangles have three sides, so three sides is an essential property of triangles. The essential properties, therefore, do *not* have more potential in the concept than in the percept. Consequently, Bergson's cone becomes a cylinder when the percept only has essential properties. In this case, the concept is referred to as a *schematic concept* and the percept that resolves from it is a *schematic percept*. Experiencing a schematic percept, in this sense, is my reinterpretation of "seeing the general in the particular" of Mason and Pimm (1984).

The schematic concept arises from focusing attention on essential properties. It is a different variety of attention than attentional feedback between hierarchical levels or the perception-action loop. Identifying the form of the neural correlate of this type of attention, especially with regard to image-based geometrical concepts, is a goal of this dissertation. All of these ideas are discussed more fully in the next section.

Note that the percept is inherently unstable, and frequently changes from moment to moment. On the other hand, the concept is more stable. Brown (2000) writes,

> Deep levels undergo slow transformation—gradual movement from one conceptual frame to another—a change obscured by the evanescent shifts at the surface. Attention is like a moving stream, the unseen depths of which run slowly. Waking objects are brief snapshots that dance over the glacial drifts of the core. (p. 60)

But if the concept has the same range as the percept, then the concept must also frequently change from moment to moment. Therefore, the stability of the concept, such as it is, cannot refer to *temporal stability*, but rather *stability of content*.

One more interesting idea emerges from contemplation of the cone (or cylinder) of duration: *depth of consciousness*. Consciousness, at least in perception, is made up of concepts and percepts, the contents of the sensible present. Depth of consciousness, which is my term not Bergson's, may be viewed as the penetration of the totality of duration by the sensible present. Non-human animals, we may suppose, have a narrow sensible present, in that their subjective experience is close to the instantaneous present: their depth of consciousness is small. According to Bergson (1907/1944), "The more duration marks the living being with its imprint, the more obviously the organism differs from a mere mechanism, over which duration glides without penetrating" (p. 42). It is depth of consciousness that allows humans to experience concepts, by penetrating further into duration.

The remainder of this chapter continues the discussion of schematic concepts and schematic percepts by means of a property analysis. The aim is to elaborate some of the ideas developed in this section in order that they can be applied in the chapters to come.

4.4 Essential Properties and Incidental Properties

There are three categories of cognitive/neural objects of interest for this dissertation: concepts, percepts, and properties (and their analogues in action). Concepts and percepts are not sharply distinguished, but rather one category shades gradually into the other. There is a dual flow of neural activity between the sensory interface of organism and world and the highest levels of the perception hierarchy. Multiple properties are associated in a percept, and multiple percepts are associated in a concept.

Individual properties can only be apprehended in the unity that is a perceptual moment. Perceptual experience is of necessity a synthesis of many properties, just as concepts are of necessity a synthesis of many percepts. However, some properties of a percept may be attended to more than others. It is the goal of this section and the next to develop the consequences of this idea.

The objective correlate of attention is the dual action of inhibition and excitation of cognitive networks, discussed earlier. One possible attentional mechanism is the feedback between hierarchical levels. Another is the feedback that occurs through the perception-action loop. There is at least one other attentional mechanism, which acts to narrow the cone (or cylinder) of duration.

It is possible that attention given to a property may decrease to the extent that it is almost fully eclipsed from perception. However, it is doubtful whether a given property ever can be entirely eliminated from a percept—it remains, highly attenuated, on the periphery of perception. In any case, inhibition of the property’s cognitive network, in itself, is a type of objective correlate of the property, a “shadow correlate,” and the property remains implicit in the percept.

In the Prologue I defined the *essential properties* to be those properties that are constant among the percepts that may resolve as instantiations of a given concept. Equivalently, the properties that must be present in order for a percept to be recognized as an instantiation of a given concept are the essential properties. Likewise, in the prologue I defined the *incidental properties* to be those properties that vary among the percepts that may resolve as instantiations of a given concept. Equivalently, also, properties that may or may not be present in order for a percept to be recognized as an instantiation of a given concept are the incidental properties. Thus, to be recognized as a triangle, a figure must have three sides; whether it has a right angle or not, it may still be recognised as a triangle—"having three sides" is essential to the image-based concept of triangle, in other words, whereas "having a right angle" is incidental to the image-based concept of triangle.

Note that the definition of essential property refers to "recognition." The percept is recognized provided it is assimilated to the cognitive network of a concept. Alternatively, and equivalently, the percept is recognized if it resolves from a concept. The recognition itself implies the existence of a larger cognitive structure within which the percept is accommodated. I deliberately chose not to use a term other than recognition that would imply language. It is not the case that a percept requires language for it to be recognized. This is true even for relatively simple geometric shapes. I can show you two rectangles, one embedded inside another. You could recognize that same shape when presented with it again. If one of the two rectangles were removed you would not recognize it as the same shape, and therefore the property of having two rectangles would be essential. Given a minute you might decide to give it the symbolic designation "frame," but the name would come later, and its main purpose would be communication.

I am not alone in considering that language does not belong to the essence of mathematical reasoning. In the work of Dehaene (1997), for example, some animals are shown to possess a primitive arithmetical facility called the “number sense,” and these same animal species obviously do not have language in the same way that humans have language. I would speculate that some animals are likewise possessed of an analogous primitive “geometry sense.” Hadamard (1945/1996), moreover, cites mathematicians for whom language is not a necessary component of their mathematical reasoning. He quotes Galton, for example, that verbal expression for communication is a poor substitute for the clarity of mathematical perception achieved prior to verbalization.

Nevertheless, the motivation behind this work is to understand image-based geometrical reasoning, specifically schematic perception and inferencing, and the objects of interest in geometry *do* have symbolic designations. Their essential properties are well known and well documented in mathematical practice. These symbolic designations and practices are *extrinsic* to the neural correlate of a geometrical percept. The *intrinsic* characteristics of the cognitive network of the concept determines whether properties are essential or incidental.

I have emphasized throughout that essential properties are constant among the percepts that resolve from a given concept, whereas incidental properties vary among the percepts that resolve from a given concept. Some questions naturally arise from this distinction, which should be dealt with before moving on.

When children are learning about geometry, the geometrical figures to which they are exposed are usually contained within the pages of a book. Every image-based geometry percept encountered by a child, therefore, satisfies the property “fits on a page.” Why is not “fits on a page” an essential property? If it were the case that every percept resolving from the triangle concept satisfied “fits on a page,” then children would not be able to recognize larger figures as triangles. Obviously this does not happen, and there are several reasons why.

Firstly, I challenge the idea that children only see geometrical figures as illustrations in books. Triangles, for example, are all around, in road signs, in architectural structures, in physical gestures, and so on. Perception of a triangle is a common feature of everyday experience, whether or not this perception is consciously designated “triangle.”

On the other hand, unlike triangles, perhaps more obscure geometrical figures really are only ever seen in geometry textbooks. In these cases, “fits on a page” must be an essential property. However, *what matters is not the absolute size of the figure, but rather the angle of vision that it subtends*. A circle on a page, for example, may well subtend a larger visual angle than the moon, no matter that the moon in absolute size is several orders of magnitude greater than the circle on the page. Moreover, angle of vision, in common experience, must be regarded as an incidental property, because the angle of vision subtended by any given geometrical figure can be changed by moving the figure closer to or further from the observer. There is no property of a figure, respecting its size, which can be regarded as essential just because the figure “fits on a page.”

Nevertheless, with obscure geometrical figures that are only usually encountered within the context of a geometry lesson, it has to be admitted that “seen within the context of a geometry lesson” is attached to them as an essential property. I admit that children may not be able to recognise such figures as examples of the concepts they have learned on the rare occasions that they encounter these figures outside of the context of a geometry lesson. However, this circumstance is a pedagogical failure rather than a flaw in my distinction between essential and incidental properties.

A similar question arises from the fact that geometrical figures in mathematics classes are usually presented as thin black lines on a white background. Why is not “drawn with thin black lines on a white background” an essential property? As above, the point can be made that the everyday experience of common geometrical figures such as triangles does not involve “drawn with thin black lines on a white background” as a property. Again, obscure geometrical figures that are usually only seen in a geometry class may well have “drawn with thin black lines on a white background” as an essential property, but the presence of this faux essential property is an educational shortcoming rather than a flaw in the essential-incidental distinction.

Lastly, is it not the case that “composed of lines with non-zero thickness” is an essential property of *all* geometrical figures (putting aside the possibility that the “lines” composing a figure are the boundaries between different media, and which therefore *are* of zero thickness)? Now, “composed of lines with non-zero thickness” is not generally a property that is useful or significant mathematically.

How can it be that this non-geometrical property is essential? Two points can be made in this regard. Firstly, although “composed of lines with non-zero thickness” may well be an essential property, its perception does not have to interfere with valid processes of image-based geometrical reasoning. It simply may not matter. If a teacher discovered the strange case that a child were mistakenly treating the non-zero thickness of lines as geometrically relevant, then the teacher would have to correct the child’s reasoning, and the child would learn some permissible limits for image-based geometrical reasoning. The same point can be made with to respect to other faux essential properties such as “fits on a page” or “drawn with thin black lines on a white background.”[14]

A second way of looking at the issue of lines with non-zero thickness is that the lines can be *arbitrarily thin*, without actually reaching zero thickness. One is reminded of Cauchy limits. In the limit, the lines really do have zero thickness, even though the limit is never perceived.

The various discussions of the difference between essential and incidental properties show that it is by no means a straightforward distinction, despite its apparent simplicity. There may be other complications which have yet to come to light. However, the basic idea is intuitively appealing and powerful: essential properties are those which are constant across multiple instantiations of a concept, whereas incidental properties are those which vary across multiple instantiations of a concept.

---

[14] I would like to suggest that all geometrical reasoning requires schematic concepts. The property “lines with non-zero thickness” is a counter example for the converse: there *are* schematic concepts that are irrelevant for image-based geometry, though not necessarily for geometry education.

4.5 Schematic Concepts and Schematic Percepts

Now, let us be clear about the nature of concepts. Is it the case that the concept is constituted of abstracted essential properties? If this were the case, then the percept would be formed by concatenating the essential properties with a suitably large collection of incidental properties. Thus, the percept of a triangle would be formed from the concept by the addition of colour, size, thickness of lines, and so on.

This does not happen. The percept *as a whole* resolves from the concept in a movement from potential to actual. The concept network, as an association of percepts, contains, implicit within itself, all possible values for every property of the percept. The properties of the percept all exist in potential in the concept.

As emphasized in the previous section, if a property is essential then it takes the same value in all percepts that are associated in the concept. A triangle percept, for example, however it varies in other ways, always has three sides. Incidental properties, on the other hand, are unstable, so that the triangle may have different orientations or, for that matter, different colours. If a property is essential, then it has no potential beyond that of its actualization in the percept.

In fact, it is this aspect, the measure of conceptual potential compared with perceptual actuality that determines whether a property is essential or incidental. There is no yardstick outside of the concept-percept structure that can decide on the degree to which a given property is essential or not. “Essential” and “incidental” are imminent within the concept-percept structure. Firstly, there is no need, therefore, for an additional neural mechanism, outside of the concept-percept structure itself, to determine whether or not properties are essential—the homunculus fallacy (see Section 4.3) is avoided. Secondly, with respect to a given essential property, the “fibre” of that property in Bergson’s cone of duration is cylindrical rather than conical. On the aggregate, therefore, if all properties are essential, Bergson’s cone is really a cylinder.

Figure 4.3 demonstrates a cognitive network approach to the relationship between the neural correlates of concepts, percepts, and properties.

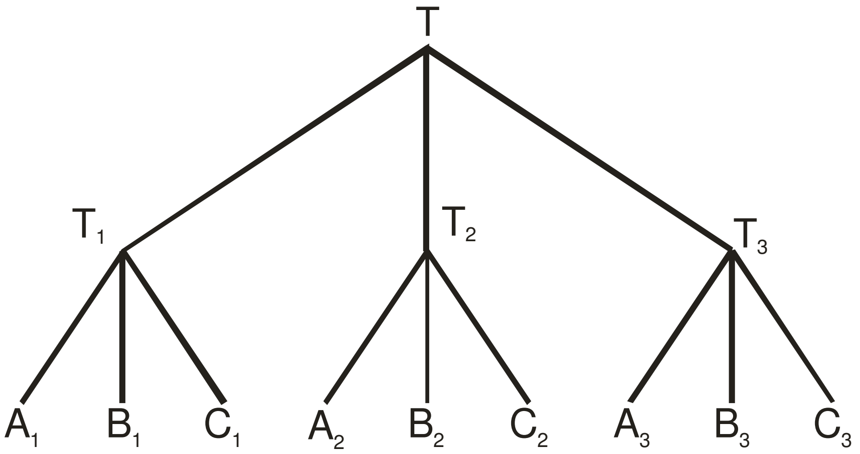

Figure 4.3. Relationship of concepts, percepts, and properties.

T is the neural correlate of a triangle concept, consisting of a cognitive network associating the cognitive networks of the three triangle percepts, T_1, T_2, and T_3. Likewise, T_1, for example, is the neural correlate of a triangle percept, consisting of a cognitive network associating the cognitive networks of the three properties, A_1, B_1, and C_1. It is important to understand that the nodes T, T_1, T_2, and T_3 do not have an independent existence as neural objects aside from their role in linking together the cognitive networks under them.

In centrifugal flow, T_1, for example, would resolve from T. Equivalently, the cognitive network of percept T_1, becomes excited while the cognitive networks of the others are suppressed. This is the attentional mechanism that operates between hierarchical levels. The remainder of the cognitive network of the concept is residually active and is the neural correlate of the concept that remains concurrently with that of the triangle percept.

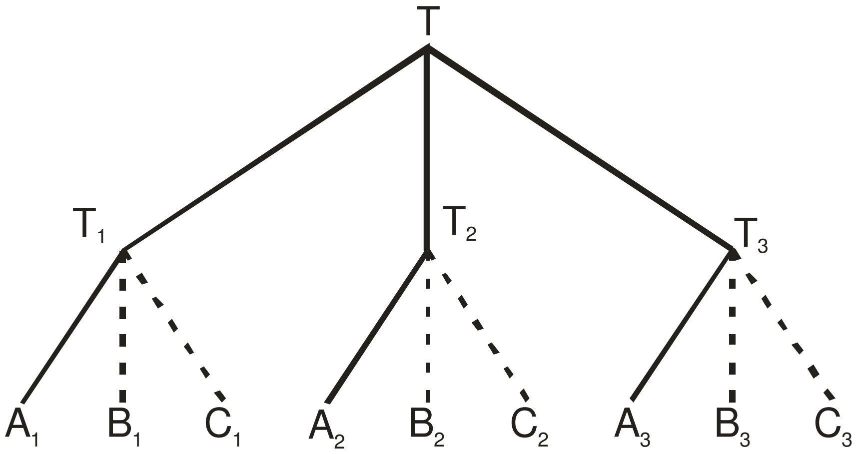

Figure 4.4. A property that is attended to.

Now, A_1, A_2, and A_3 represent three ways in which a given property can become actual, three different shapes, for example. If that property becomes attended to, then Figure 4.4 represents the neural correlate of this attention. The cerebellum may support an attentional mechanism for properties, as discussed in Chapter 5. Moreover, this attentional mechanism is *selective* in that the essential properties are emphasized while the incidental properties are inhibited.

Suppose now that triangle T_1 resolves from T, with its property A_1 now attended to. Any other properties of the triangle may yet be present in consciousness, but they remain peripheral. If A_1 is essential, then it is equal to A_2

and A_3. It follows that the perceptual experience is the same whichever triangle resolves from T. If the property A_1 is incidental, on the other hand, it may well differ from A_2 and A_3, and then the perceptual experiences of the other two triangles that resolve from T with this property attended to are different. Incidental properties vary, whereas the essential properties are constant. When the triangle is experienced as a schematic percept, with only essential properties, then it cannot be experienced otherwise than what it is, in all the purity of its triangularity.

Now, the raw concept is not an abstraction of essential properties. As explained above, it is extensional. However, the selective attention given to properties *is* a form of abstraction. It is abstraction in which both concept and percept are constrained, schematized in other words, simultaneously. At the percept end of the cone the percept becomes restricted to essential properties only, as the incidental properties are systematically ignored. At the concept end of the cone, if A_1, A_2, and A_3.are all equal because they are manifestations of an essential property, then the concept network of the concept, represented by all bold lines in Figure 4.4, is no more extensive than that of the concept network of any of the triangles that resolves from it. Bergson’s cone, in other words, is a cylinder.

The concept represented by the bold lines in Figure 4.4, and any of the triangles that resolve from it, have only one property. Usually, the concept has a number of properties associated with it, rather than just one, each of which corresponds to a cognitive network like the one in Figure 4.4. Each of these cognitive networks, corresponding to a single property each, may be regarded as the neural correlate of a *potential property*. The term *actual property* refers to those properties belonging to the percept, to distinguish them from potential properties. The neural correlates of the potential properties of a concept resolve to the neural correlates of the actual properties, just as the concept resolves to the percept. In fact, a potential property may be defined as a concept that would resolve to a percept with a single actual property.

All actual properties exist in potential in the concept; moreover, every potential property resolves to an actual property in the percept. Likewise, attention given to potential properties corresponds precisely to attention given to actual properties.

The *range* of a percept is determined by the variety of actual properties that belong to the percept. Percepts can have greater or lesser range. Likewise the range of a concept is determined by the variety of its potential properties. Given that a percept/concept typically has a large number of actual properties/potential properties, its range is really a measure of how many of these are the focus of attention.

It follows that *concepts and percepts have the same range*. Every actual property in the percept corresponds to a potential property in the concept from which it has resolved. Likewise, every potential property in the concept resolves to an actual property in the percept.

The notion of equality of range is crucial for the argument in this dissertation. I would like to test this idea with reference to the Angles in a Circle

Theorem, discussed in the Prologue. The Angles in a Circle Theorem, after all, is one of the most complex cases to analyze from high-school geometry.

Firstly, remember that the Angles in a Circle Theorem appears in five disparate versions. Its essence, however, is the same in each, and consists of a central angle opposite an arc and an angle at the circumference opposite the same arc, which is a potential property in the concept and an actual property in each precept that instantiates the Angles in a Circle Theorem.

However, the quadrilateral version of the Angles in a Circle Theorem, for example, contains a particular quadrilateral in any given percept of that version, whereas the “bow tie” version contains two particular triangles joined at a vertex (see Figure P.2). Now, each triangle percept contains a particular triangle, and the triangle concept is the general triangle. By analogy, it would seem that the Angles in a Circle Theorem concept should contain a general quadrilateral and a general pair of triangles connected at a vertex. But this would contradict equality of range, because the concept contains more potential properties than each of the individual percepts.

All is not lost. The Angles in a Circle Theorem is more complex that the simple triangle concept. The solution is to view it, if necessary, as inducing five separate, intermediate concepts, corresponding perhaps to five separate theorems, as suggested in the Prologue. The point is that the quadrilateral version, for example, contains a general quadrilateral as a potential property in the concept of the quadrilateral version. Moreover, since each percept contains a quadrilateral, the quadrilateral property is indeed essential to the concept-percept structure of the quadrilateral version. Equality of range is maintained. Similar arguments can be made for the other versions of the Angles in a Circle Theorem.

On the other hand, the Angles in a Circle Theorem in itself is a single concept, and the quadrilaterals, triangle pairs connected at a vertex, and so on, are really incidental properties of any percept instantiating the Angles in a Circle Theorem. However, the complete cognitive network of the Angles in a Circle Theorem unifies the cognitive networks of the intermediate concepts. In this overarching cognitive network, the general quadrilateral, the general triangle pair connected at a vertex, and so on are “actual properties” in the intermediate concepts. These “actual properties” have resolved from a potential property in the overarching concept. What is this potential property? Well, there is no name for it, but we can refer to it as “additional geometrical stuff.” The “additional geometrical stuff” then is definitely an incidental potential property of the Angles in a Circle Theorem because it resolves to a huge variety of actual properties in particular percepts. Moreover, equality of range is maintained.

The shear diversity of actual properties, or equivalently the nebulosity of the potential property “additional geometrical stuff” is the very reason why the Angles in a Circle Theorem is so difficult to learn as a single theorem. It is a tall order to ask high-school students always to focus attention on the essential property that two angles are opposite the same arc.

Two more examples may further clarify my meaning with respect to equality of range. Firstly, suppose a circle is presented containing some additional squiggle marks in its interior. In this case it is easy for students to see the essential property and to recognize it as a circle. However, the circle concept does not contain a potential property for the squiggle marks. How is equality of range maintained? My answer is that there are two possibilities. In the first case, the students are seeing only the circle, and are blocking the squiggle marks from their attention. Then, indeed, there is equality of range. In the second case, the students are still perceiving the squiggle marks, but—and this is important—the concept is not now circle, but “circle with squiggle marks.” In this case, too, equality of range is maintained because the “circle with squiggle marks” concept contains the potential property “squiggle marks.” Indeed, depending on the specific configuration of the squiggle marks, the circle itself may not even be at the forefront of attention (see Figure 4.5).

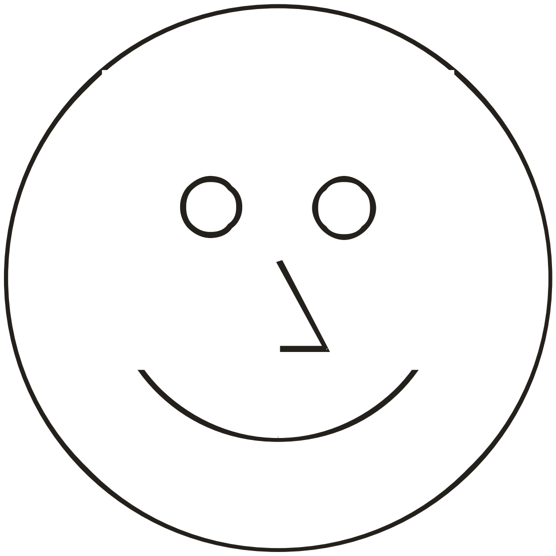

Figure 4.5. Happy face.

Lastly, a triangle recognized as a “blue triangle” is not an instantiation of the concept “triangle,” but rather the concept “triangle with colour”—maintaining equality of range. It is interesting to note, in this case, that the concept “triangle with colour” can never be schematized, except in the unlikely case that every triangle percept is blue, say. Schematizing the blue triangle percept for mathematical purposes would involve (1) ignoring the fact that the triangle has any colour at all (resulting in the concept “triangle” and the percept that resolves from it), and (2) ignoring all other properties that vary among those percepts that resolve from the triangle concept, such as orientation, specific shape, and so on.

The point of the last two examples is that I am dealing in this dissertation primarily with the psychological (and neurophysiological) reality of geometrical reasoning rather than formal geometry and its definitions. The concept-percept structure *always* complies with equality of range—the naming of the concepts and percepts is something introduced from outside this structure, and it may be necessary to adjust the concept labels according to the reality of perception and conception.

Now that I have clarified my meaning with some examples, it is time to return to generalities. If attention is directed to the essential potential properties, and therefore attention is also directed to the essential actual properties, as with the circle or the triangle in the preceding examples, then the concept is a *schematic concept*. A schematic concept resolves to a *schematic percept*. Note that schematization is seldom, if ever, perfect. While certain properties may be heavily emphasized in attention, other properties remain still, on the periphery of perception, present still, though attenuated.

According to Dehaene (1997), “Number emerges naturally as the most abstract representation of the permanence of objects in space—in fact, we can almost define number as the only parameter that remains constant when one removes object identity and trajectory” (p. 190), which certainly sounds like schematization. However, I would include shape and pattern, in addition to number, as aspects of objects that can be schematized for mathematics. Mathematical concepts, in other words, involve number, shape, and pattern. Specifically, the geometrical concepts investigated herein concern combinations of Euclidean objects such as points, straight lines, and circles. For efficient, accurate image-based geometrical reasoning, these concepts and percepts must be schematized.

Mason and Pimm’s (1984) notion of *generic example* is similar to that of schematic percept. The generic example is an object that, even in its particularity, is representative of the concept. According to Mason and Pimm, a generic example entails “seeing the general in the particular.” In this case, the triangle, for example, is perceived in the full generality of its essential triangularity. Its colour, size, orientation, and even its particular shape fade from the focus of attention. In this chapter I give a brief cognitive-network analysis of the notion of generic example. There is further discussion of “seeing the general in the particular” in Section 6.1.

Let us compare the two ideas—generic example and schematic percept. Firstly, let us suppose that a generic example is an object in subjective cognition. Otherwise, it exists objectively in the world, and there is no way to compare it with the schematic percept, which certainly is an object in subjective cognition. And, after all, it is the psychological reality of geometrical reasoning that I am investigating herein.

The generic example is a percept rather than a concept because of its particularity. The generic example is constituted of a number of actual properties, and since the concept has the same range as the percept that resolves from it, the concept is constituted of this same set of properties, except in potential. For the generic example to be a particular that is also general, each of its actual properties should be, in a sense, also potential properties. This is the case when the properties are essential to the geometrical concept under consideration. If all of its properties are essential, then the generic example is indeed a schematic percept. To “see the general in the particular,” to perceive generic examples, corresponds to schematizing concepts and percepts. Of course, this is my interpretation, which may differ from the original intent of Mason and Pimm (1984), as discussed in Section 6.1.

Is it not the case that the general is always seen in the particular? In a sense, it is. The non-schematic concept of tree resolves to the non-schematic percept of tree, and “tree” is a general concept. However, the backdrop to the non-schematic concept is a sea of potential properties, all of which, to varying degrees, are essential or incidental. And the composition of this sea of properties varies widely between individuals. Perhaps a process of schematization is necessary with respect to the tree, too, for a botanist, say, who wants to derive some conclusions about all trees. The botanist must focus on the essential properties of “tree.”

I discussed schematization of concepts with respect to image-based reasoning in geometry in Handscomb (2005), although with different emphasis. I observed that, at least in the high school, students are expected to use information derived from the diagram in a geometrical proof. In order to guarantee the generality of the proof with respect to the geometrical idea being investigated, the student must use only essential properties. The efficiency and accuracy of reasoning from the diagram is improved if the student’s perception of the diagram is schematic. It follows that the notion of schematic concept and schematic percept is important for understanding the reasoning processes required in high-school geometry. One of the ideas investigated in Handscomb (2005) was the mathematical justification of image-based reasoning in geometry—contrasting it with formal, axiomatic reasoning. In the present study, I am concerned with a psychological and physiological investigation of image-based reasoning in geometry—the neural structures that are the objective correlates for reasoning with schematic concepts and schematic percepts.

The main focus of this dissertation is the system of concepts, percepts, and properties whose neural correlates are cognitive networks in the posterior cortex. Image-based geometrical reasoning is largely a perceptual process and involves action only to a limited degree. However, in order to investigate mathematical reasoning more generally, it is necessary to consider also cognitive networks of the anterior cortex. It was discussed earlier that the analogues in the anterior cortex of concepts, percepts, and properties in the posterior cortex are procedures, acts, and action properties. Just as concepts are less specific in spatial terms, procedures are less specific in temporal terms—in other words, they are action plans, as it were. An act discharges in movement involving a certain goal or trajectory. The acts associated in a procedure fit together to give temporally coherent motor behaviour. Acts are composed of action properties, cognitive networks corresponding to the movement of small groups of muscles.

Now schematization of concepts and percepts implies reducing the number of properties to which attention is directed. If we ask what "schematization" can mean with respect to the anterior cortex, we can see that it must refer to something different. The procedure itself consists of a great many action properties, which have to resolve in groups, in a temporal order, in such a way that each succeeding act follows smoothly from the preceding act, and the procedure overall is coherent. The resolution of an act from a procedure represents selection of a certain group of action properties. This process can be regarded, in and of itself, as a constraint on the available action properties of the procedure, and this constraint is the anterior-cortex analogue of the schematization of the posterior cortex. Smooth, efficient motor behaviour results from a temporally coherent set of such constraints.

▲►▼◄▲►▼◄▲►▼◄

The cognitive network theory (Fuster, 2003) was selected as the theoretical model of the cerebral cortex. Extension of the cognitive network theory by means of the property analysis enables development of the notion of schematization, which is a focus on those properties of concepts and percepts that are essential. Bergson’s cone of duration, and its metamorphosis into a cylinder, is an illuminating metaphor for schematization.

While the first dimension of the theory is the cerebral cortex, which has been discussed so far, the second dimension is the cerebellum. A hypothesis of this dissertation is that it is a functional role of the cerebellum to schematize concepts and percepts of the posterior cortex. There are some arguments to support this hypothesis in Chapter 5.

CHAPTER 5: THE CEREBELLAR SCHEMATIZATION HYPOTHESIS

Chapter 4 completed the discussion of the cerebral cortex begun in Chapter 3. It is time now to turn to the cerebellum, the second dimension of my thesis.

The distinction between extensional concepts and intensional concepts was clarified in Chapter 4. Extensional concepts, or rather their neural correlates, emerge naturally from a cognitive network understanding of the cerebral cortex and the operation of Hebbian principles. However, extensional concepts are indistinct and inadequate for clear, accurate image-based geometrical reasoning. Intensional concepts, in which attention is directed to essential properties, are sharp and clear and suitable for image-based geometrical reasoning. Chapter 4 discussed the notion of intensional properties from a theoretical perspective, within the context of Bergson’s cone of duration. I concluded that experience of an intensional concept is equivalent to “seeing the general in the particular”—or schematic perception—a cognitive function that is necessary for image-based geometrical reasoning, specifically the schematic perception and inferencing from geometry diagrams that are the topic of my thesis.

So far I have not suggested a neural mechanism for schematizing concepts, for generating sharp intensional concepts from fuzzy extensional concepts. However, given the neutral monist framework of the dissertation and the notion of neural correlate, as discussed in Chapter 3, the cognitive function "seeing the general in the particular" must have a neural correlate. My hypothesis is that a functional role of the cerebellum, with respect to its connections with the associative areas of the posterior cortex, is precisely that of schematizing extensional concepts, resulting in intensional concepts:

The Cerebellar Schematization Hypothesis

A functional role of the cerebellum with respect to the posterior cortex is to schematize concepts and percepts.

Metaphorically, *the cerebellum is a lens that sharpens conceptual and perceptual cognition*. Arguments supporting the hypothesis are given in this chapter. The implications for mathematics education are discussed in Chapter 6.

The traditional interpretation of the functional role of the cerebellum is that it facilitates smooth, efficient motor behaviour—and motor behaviour belongs to the anterior cortex rather than the posterior cortex. In establishing the significance of the cerebellum for mathematics education it might seem natural to focus on the motor-cerebellum paradigm, particularly in view of the relationship between cognition and movement that is presupposed by embodied cognition.

However, my research is driven by the goal of understanding image-based geometrical reasoning. The visual concepts, percepts, and properties that must be considered in a theory of image-based geometrical reasoning reside in the posterior cortex. *Image-based geometrical reasoning is embodied in neural activity in the posterior cortex*.

Nevertheless, because of the bias of cerebellar research toward various forms of motor behaviour, I have to make considerable reference to the anterior cortex in my review of research on the cerebellum in this chapter. And it *is* necessary to review the broad sweep of cerebellar research rather than focusing only on those areas that directly support the argument for the CSH. Otherwise, the reader unfamiliar with the cognitive neuroscience of the cerebellum would receive an unbalanced presentation, which would weaken the credibility of the CSH in the final analysis.

Sections 5.1, 5.2, and 5.3 discuss, respectively, the gross anatomy of the cerebellum, the phylogenesis of the cerebellum and related structures, and the reciprocal neural connections between the cerebellum and the cerebral cortex. In the phylogenesis of the cerebellum and its connections to the cerebral cortex, there are two arguments that support the CSH. Section 5.4 discusses the uniformity of cerebellar histology, which supplies another argument in support of the CSH.

Section 5.5 discusses the controversy of non-motor cerebellum versus motor cerebellum. This section is important for context, in that I interpret the motor cerebellum with respect to the relationship of cerebellum and anterior cortex, whereas the non-motor cerebellum concerns the relationship of cerebellum and posterior cortex. Section 5.6 considers two theories of the “mental motor” cerebellum, which are variations on the motor cerebellum paradigm.

Section 5.7 discusses briefly, in turn, a number of other theories of the functional role of the cerebellum. The presentation in Section 5.7 is, strictly speaking, peripheral to the main flow of argument in Chapter 5. However, a review of alternative theories is necessary for completeness. In addition, I hope it enables the reader to gain a sense of the plausibility of the CSH and a sense also that the CSH rests comfortably alongside established theories of the functional role of the cerebellum.

Section 5.8 turns to an influential, overarching theory of the functional role of the cerebellum, the universal cerebellar transform—the breakdown of which is referred to as “dysmetria.” The CSH, I propose, is an application of the universal cerebellar transform to the posterior cortex. Thus, the evidence and argument in favour of the universal cerebellar transform may be conscripted to support the CSH. In Section 5.9, the evidence that the cerebellum is responsible for selective attention is a direct substantiation of the CSH. Section 5.10 brings together the various arguments for the CSH, and the chapter concludes in Section 5.11.

It is not my intention to suggest that the CSH in any way supersedes other ways of understanding the functional role of the cerebellum. Many of the theories discussed herein are significant and interesting, and the CSH should be regarded as complementary to them rather than conflicting. My interpretation of the evidence has been motivated by and is a reflection of my focus on and interests in geometrical image-based reasoning in mathematics education. It is this background and context that gives the CSH its peculiar characteristics when contrasted with other theories of cerebellar function.

5.1 Large-scale Anatomy of the Cerebellum

The cerebellum is located at the rear of the brain, above the pons and below the occipital cortex. It accounts for only ten percent of the total brain by weight but contains more neurons than the rest of the brain together (Glickstein, 2007). Like the cerebrum, the cerebellum consists of a thin cortical sheet covering a white matter core. The cerebellar cortex is highly convoluted, and only one tenth of it is visible on the surface; in comparison, one third of the cerebral cortex is visible on the surface (Sultan & Glickstein, 2007).

The large-scale anatomy of the cerebellum may be analyzed "horizontally" or "vertically." Vertically, the cerebellum contains two hemispheres and three divisions within each hemisphere. From the midline, these divisions are the *vermis*, the *intermediate zone*, and the *lateral zone*.

There are three pairs of *deep cerebellar nuclei* embedded within the cerebellar white matter. Each division of the cerebellar cortex projects to a different nucleus: the vermis projects to the *fastigial nucleus*, the intermediate zone to the *interpositus nucleus*, and the lateral zone to the *dentate nucleus*.

Horizontally, the cerebellum divides into three lobes. The most dorsal is the *anterior lobe*, the most ventral is the *flocculonodular lobe*, and between them lies the *posterior lobe*. The anterior lobe and posterior lobe account for most of the cerebellum. The flocculonodular lobe lies outside the vertical divisions of vermis/intermediate zone/lateral zone.

The flocculonodular lobe and adjacent parts of the vermis constitute the *archicerebellum*, the oldest part of the cerebellum phylogenetically. The *paleocerebellum*, of intermediate age, consists of the vermis and intermediate zones. The *neocerebellum* consists of the lateral zones. It is significant for the argument later that the lateral zones, phylogenetically the most recent part of the cerebellar cortex, project to the dentate nucleus (Leiner et al., 1987).

Figure 5.1 is a conceptual representation of the large-scale structure of the cerebellar cortex, as it might look if it were unfolded.[15]

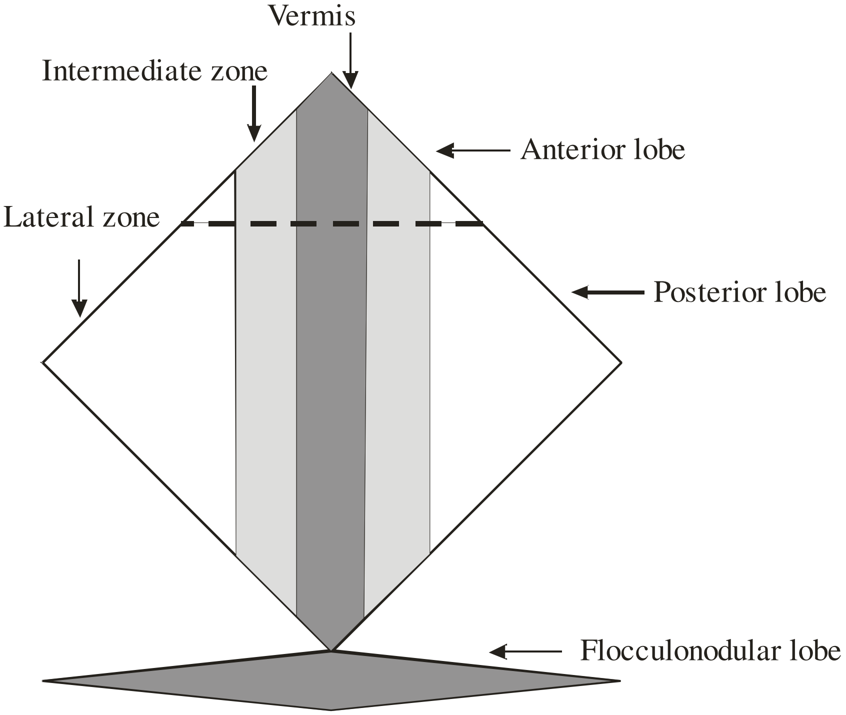

Figure 5.1. Large-scale structure of cerebellum.

---

[15] Altman and Bayer (1997) is the source for basic information on the large-scale structure of the cerebellum.

5.2 Phylogenesis of the Cerebellum

It is important to review the phylogenesis of the cerebellum and related cortical and subcortical structures. After all, phylogenetically recent physiological structures ought to provide neural correlates for phylogenetically recent cognitive functions.

The cerebellum evolved with the vertebrates; all vertebrates have a cerebellum except for primitive chordates such as amphioxus and the lampreys (Sultan & Glickstein, 2007). According to Paulin (2005),

> The circumstantial evidence indicates that at some point in the lower Cambrian, a metazoan developed the ability to prey on other metazoans. Metazoa responded in a diverse fashion, developing a variety of defense mechanisms. Among other things, predation created selection pressure for larger size and faster speeds . . . . One group, ancestors of the vertebrates, developed agility—the ability to attack and evade other metazoans. (p. 62)

Subsequently, the vertebrates were the only group of large, fast, agile animals to emerge from the Cambrian explosion (ibid.). The speed and agility of the vertebrates is generally attributed to the cerebellum (ibid.). No other animal has an analogous brain structure except the cephalopod molluscs, which are also fast-moving, biomechanically complex creatures (ibid.). Paulin writes, “The cerebellar cortex may have been the key innovation that allowed ancestors of the vertebrates to . . . [give] rise to the biggest, fastest, and fiercest creatures on the planet” (p. 63).

In all vertebrates there are connections from the spinal cord to the cerebellar vermis and its immediately adjacent cortex; these connections bring information to the cerebellum with respect to body position and movement (Sultan & Glickstein, 2007). According to Sultan and Glickstein,

> The histological structure of the cerebellum is virtually identical in all mammals and birds. In reptiles, amphibians, and fish the organization differs somewhat from that of mammals and birds, but in those species as well, the cerebellum consists of a characteristically large group of cells that receive sensory input and connect to a group of target cells that provide the output from the cerebellum. (p. 168)

According to Voogd et al. (1990), both ontogenetic and phylogenetic growth of the cerebellum occurs in the direction from medial to lateral. Only in the mammalian evolutionary line (and to a minor extent in birds) did the cerebellar hemispheres develop, with their characteristic intermediate and lateral zones (Sultan & Glickstein, 2007). Only in mammals are there extensive connections between the cerebral cortex and the cerebellum (ibid.). For many mammals, the major input to the cerebellum is not from the spinal cord itself but from the cerebral cortex via the pons (ibid.). The cerebral cortex projects to the intermediate and lateral zones, but not to the flocculonodular lobe or vermis (Ito, 2006).

As the mammalian line developed in the direction of larger, more sophisticated brains, cerebrocerebellar structural connections burgeoned. Leiner and Leiner (1997) write,

> In the mammals that evolved initially, which had small brains with little cerebral neocortex, the cerebellar connections were linked to sensorimotor areas of this cortex, which enabled cerebellar modules to participate in the processing of sensorimotor information; in the primates that evolved later, which had brains with larger cerebral ‘association’ areas, the links to some newly evolved cerebral areas could enable the cerebellum to participate as well in the processing of some cognitive information, particularly in the cerebral prefrontal cortex. (pp. 537-538)

According to Ito (2006), moreover,

> The intermediate zones of the cerebellar hemisphere developed in relation to voluntary movements in mammals. The lateral zones are related to high-order functions of the cerebral association cortex. The most lateral region of the cerebellar hemisphere in humans is likely related to cognitive functions. (p. 294)

It is these cognitive functions, dependent on the lateral zone of the cerebellum and the cerebral association cortex, that are of interest for this dissertation.

As discussed later, there is evidence that there are closed loops from the cerebral cortex to the cerebellar cortex via the pons, returning to the cerebral cortex via the deep cerebellar nuclei and the thalamus. In particular, these closed loops connect lateral zones of cerebellar cortex with association areas of posterior cortex. According to Whiting and Barton (2003, p. 4), investigations into the evolution of the primate brain should look at interconnected structures rather than focusing solely on individual brain regions. They write,

> Previous work on the evolution of the primate brain has generally focused on changes in individual structures. Brain structures, however, do not function in isolation, but rather contribute to distributed functional systems. The present analyses demonstrate correlated evolution among neocortex, cerebellum, vestibular complex and relay stations (pons and thalamus). (p. 10)

According to Dow (1974), in primates, the system including cerebral association cortex, lateral zones of the cerebellum, dentate nucleus, and thalamus has grown rapidly compared with the rest of the brain. Dow, noting that lesions of the dentate nucleus do not impact motor behaviour, muses, “One must ask why this system reached such extraordinary size in man if it is not concerned with the traditional control of motor activities. It completely overshadows the older parts of the cerebellum present in all subhuman forms” (p. 110). According to

Passingham (1975), the cerebellum has increased in concert with the cerebral association cortex; it is likely that the whole system has evolved as a single integrated structure. Whiting and Barton (2003) suggest that a focus on the cerebral association cortex as the principle area of change in primate evolution may be misplaced. Instead, “some attention should be paid to cortico-cerebellar circuits and the behavioral functions they may mediate” (p. 10).

According to Leiner et al. (1987), “In anthropoid apes a unique part evolved in the dentate nucleus of the cerebellum, and in the human brain this unique part enlarged enormously [4]” (p. 426). According to Ramnani (2006), the dorsal dentate, the phylogenetically older part of the dentate nucleus, connects primarily with motor areas of the cerebral cortex, whereas the phylogenetically newer ventral dentate, or neodentate, is connected with the prefrontal cortex—and also, as we see later, to association areas of the parietal cortex. Strick et al. (2009) write,

> Comparative studies suggest that the dentate has expanded in great apes and humans relative to the other cerebellar nuclei (Matano et al. 1985). Furthermore, most of this increase appears to be due to an expansion in the relative size of the ventral half of the dentate (Matano 2001). This observation implies that the nonmotor functions of the dentate grow in importance in great apes and humans. (p. 420)

The contribution of the cerebellum to the phylogenesis of cognition in the higher primates is discussed in MacLeod (2000) and MacLeod et al. (2003). MacLeod (2000) is interesting and significant because of its use of a larger sample group of primate brains than the research cited in the previous paragraphs. MacLeod found that the lateral cerebellum in apes and humans is almost three times larger than would be expected in a monkey with the same size of vermis. In addition, the dentate nucleus is actually smallest in comparison to the lateral zones in humans when compared with other primates. In other words, phylogenetic growth of the dentate nucleus did not keep up with the expansion of the lateral zones of the cerebellar cortex.

There are projections from cerebral cortex to red nucleus, from red nucleus to inferior olive, from inferior olive to cerebellar cortex, and back from cerebellum to red nucleus (Leiner et al., 1987). The red nucleus and inferior olive, therefore, belong with cerebral cortex, cerebellar cortex, pons, and thalamus in an expanded cerebrocerebellar system. According to MacLeod (2000), the *inferior olive*, rather than the dentate nucleus, expanded in phylogenesis with the expansion of the lateral zones. These results contrast with the research cited above, which emphasizes coordinated expansion of lateral zones and *dentate nucleus* in phylogenesis. Because of the larger sample size, I give greater weight to MacLeod's conclusions.

In the higher primates, the cellular composition and input-output connections of the red nucleus differ from those of other mammals (Nathan & Smith, 1982). In other mammals, much of the red nucleus consists of large cells associated with motor activity; in the higher primates, on the other hand, there are smaller cells, and the region of the red nucleus containing these smaller cells has enlarged.

The picture that emerges from this discussion of the phylogenesis is a three-stage development of the cerebellum and related structures. The cerebellum evolved with the first vertebrates. It consisted of the flocculonodular lobe and an undeveloped vermis. This primitive cerebellum connected with the spinal cord and vestibular system. Its function was body position and movement, saccadic eye movement, and balance. The second major stage was that of the mammals. The intermediate zone of the cerebellum developed, connected with motor areas of the cerebral cortex. The function of this system was smooth, efficient voluntary movement. In the third stage, in primates and most noticeably in humans, the lateral zones of the cerebellum developed in concert with the cerebral association cortex and related structures, including the inferior olive.

MacLeod et al. (2003) write,

> [The] . . . marked increase in the size of the cerebellar hemispheres for hominoids over monkeys. . . . has implications in hominoids for a common set of cognitive abilities that may be partly dependent on an increased participation of the neocerebellum. The superior abilities of the great apes to acquire artificial languages under human tutelage, and their more complex ordering of actions in food processing and tool use, could be explained in part by the increase in cerebellar processing abilities, especially in the lateral cerebellum. . . . [T]he differential expansion of the cerebellar hemispheres are indicative of a new balance in neurological functioning with the apes and humans, a qualitative change in primate brain dynamics that likely took place in the common ancestor to the Hominoidea. Although humans are unique in our enormous encephalisation, we nonetheless share this differential expansion of the lateral cerebellum with the apes. (p. 427)

To the list of abilities proposed by the authors I may add also schematic perception and inferencing—implying the ability to focus attention on the essential elements and interrelations thereof in a geometrical diagram.

According to the theoretical framework developed in Chapter 3, brain function follows brain activity, which, to an extent, depends on brain structure. It follows that activity of the recently evolved cerebrocerebellar system is the objective correlate of phylogenetically recent cognitive functions. Is it the case that these newly evolved functions are related to motor control? I think it more likely that they are related to higher cognitive functions that are specifically human, such as abstract reasoning. Phylogenesis of the cerebellum and related neural structures is the first argument to establish the plausibility of the CSH. Phylogenesis is not direct evidence, but it does provide circumstantial evidence. Of course, this would not preclude the possibility that the phylogenetic expansion of these structures is related to higher cognitive functions completely unrelated to image-based reasoning in geometry.

5.3 Cerebrocerebellar Connectivity

The second argument for the CSH concerns the association area of the parietal cortex and the connections between this area and phylogenetically recent cerebellar and related structures. According to Schmahmann (1996), "Consistent with the notion that in the nervous system, function is dependent on structure, if there is a cerebellar contribution to cognitive function then there must be a corresponding anatomic substrate that supports it" (p. 177).

In Section 5.3.1 I review two methods for establishing cerebrocerebellar connections, and the advantages and disadvantages of both. Section 5.3.2 discusses the broad picture of cerebrocerebellar connectivity in order to orient the reader for the presentation to follow. Sections 5.3.3 and 5.3.4 review the results of neuronal tracing studies and neural imaging methods, respectively. Sections 5.3.5 and 5.3.6 discuss the classical understanding and closed-loop understanding of cerebrocerebellar connectivity, respectively, and I give my reasons for emphasizing the latter. Finally, in Section 5.3.7 the discussion on cerebrocerebellar connectivity has implications for localization in the cerebellum, which supports the notion of diversity of cerebellar function, depending on which particular region of the cerebellum connects to which particular region of the cerebral cortex.

5.3.1 Methods for Determining Cerebrocerebellar Connections

Two methods have been used for investigating connections between the cerebellum and the cerebral cortex. Firstly, neural pathways can be identified structurally by means of axonal tracing. Substances are introduced to specific locations in the brain. These substances follow neural pathways by moving along the axons between neurons. Detection of the substance in another location in the brain establishes that there is a neural pathway between the two locations. Transmission of the substances may be anterograde or retrograde. In the former, movement is from the cell body to the axon terminal; whereas in the latter, movement is from the axon terminal to the cell body. In the literature on cerebrocerebellar connections retrograde tracing methods are most commonly used, with horseradish peroxidase or a virus such as herpes or rabies. When discussing this literature, I simply refer to it as “neuronal tracing” without specifying the detailed methodology in particular studies. A review of neuronal tracing methods is given in Otzas (2003).

The second method used to determine cerebrocerebellar connections is functional connectivity magnetic resonance imaging (FCMRI), which developed from the method of functional magnetic resonance imaging (FMRI). Cordes et al. (2000) is a seminal paper on FCMRI, in which the authors contrast FMRI and

FCMRI. In both methods the change in blood oxygen level is measured in various regions of the brain. The basic assumption is that uptake of blood oxygen in a specific region of the brain correlates with neural activity in that region. In FMRI participants are given a specific task to perform and then localized changes in blood oxygen levels are correlated with the cognitive function associated with that task. In FCMRI, on the other hand,

> Synchronous fluctuations in signal intensity in the brain are used to examine the strength of neural connections between different brain regions while they are not activated by a specific task. For the fcMRI study, the subject refrains from any cognitive activity as much as possible and receives no experimental stimuli from the investigator. (p. 1637)

Regions of the brain with correlated levels of activity in FCMRI are presumed to be functionally related.

There are advantages and disadvantages to neuronal tracing and FCMRI. Neuronal tracing detects structural connections in the brain, but it is limited to animal studies. FCMRI, on the other hand, can be used with human participants, but the connections are functional correlations rather than structural connections. We may assume, and researchers usually do, that functional correlations between brain regions are manifestations of a structural substrate. However, FCMRI is unable directly to determine the directionality of the presumed connections between brain regions—although directionality may be inferred from homologous connections in primate brains determined by neuronal tracing methods.

5.3.2 The Broad Picture of Cerebrocerebellar Connectivity

The broad picture of cerebrocerebellar connectivity is as follows: the cerebellar cortex projects to the dentate nucleus, which projects to the thalamus, which projects to the cerebral cortex, which projects to the pons, which lastly projects back to the cerebellar cortex (Leiner & Leiner, 1997). I argue below for the closed-loop understanding of cerebrocerebellar connections, in which distinct regions of the cerebellar cortex are reciprocally connected with distinct regions of the cerebral cortex in segregated loops. For example, association areas of the anterior cortex are reciprocally connected with regions of the lateral cerebellar cortex, passing through the neodentate nucleus (Leiner et al., 1987, Leiner & Leiner, 1997). Likewise, association areas of the posterior cortex are also reciprocally connected with (distinct) regions of the lateral cerebellar cortex, again passing through the neodentate (see below). The cerebrocerebellar loops connected to association areas of the cerebral cortex may subserve higher cognitive functions. For the thesis herein, I am particularly concerned to establish the existence of closed-loop circuits from cerebellum to parietal association cortex.

Most of my discussion of cerebrocerebellar closed-loop circuits in this chapter concerns cerebral cortex, cerebellum, pons, and thalamus. In the previous section, however, I mentioned that the red nucleus and inferior olive are two other subcortical structures belonging to the cerebrocerebellar system. There are projections from the cerebral cortex to the red nucleus, projections from the red nucleus to the inferior olive, and then projections from the inferior olive to the cerebellar cortex (Leiner et al., 1987). Dum and Strick (2003), using neuronal tracing techniques on primates, demonstrated that the parietal cortex projects to the red nucleus. Habas and Cabanis (2007), investigating red-nucleus connections in humans using diffusion tensor imaging axonal tracking, showed that the main cortical afferents to the red nucleus originate in prefrontal, temporal, occipital, and parietal cortices, including parietal association areas. In other words, it seems plausible that red nucleus and inferior olive are integrated components of the cerebrocerebellar closed-loop circuits, specifically those loops connecting lateral zones of the cerebellar cortex with association areas of the parietal cortex. Although the discussion below concerns cerebral cortex, cerebellum, pons, and thalamus rather than red nucleus and inferior olive, additional subcortical structures, particularly red nucleus and inferior olive, belong to this system.

5.3.3 Neuronal Tracing Studies

One of the earlier neuronal tracing studies is that of Schmahmann and Pandya (1989). The authors established that there were substantial projections from the posterior parietal cortex to cerebellum via the pons in rhesus monkeys. Schmahmann and Pandya (1990), again using neuronal tracing methods on rhesus monkeys, established that there are returning connections from the cerebellar cortex to temporal cortex and posterior parietal cortex via the thalamus. Schmahmann and Pandya (1989, 1990) also identified reciprocal connections from cerebellum to prefrontal cortex. The results of these two authors are also discussed in Schahmann and Pandya (1997). Returning connections from cerebellum to parietal cortex were confirmed by Clower et al. (2001) in a neuronal tracing study on cebus monkeys. Clower et al. established that there are projections from the dentate nucleus of the cerebellum to the inferior parietal cortex, probably via the thalamus.

Schmahmann and Pandya (1989) write,

> The heavy projections to the pontine nuclei from the posterior parietal cortex, and particularly from those caudal parietal regions that have prominent associative and limbic connections, seem to suggest that the corticopontocerebellar pathways permit a cerebellar contribution not only to the coordination of movement, but also to the modulation and integration of higher function. (p. 53)

Schmahmann and Pandya (1995) reiterate the significance for higher cognitive function of these structural connections.

In a more recent study on cebus monkeys, Strick et al. (2009) confirmed that there are structural connections from regions of the dentate nucleus to areas of the posterior parietal cortex. They write, “Currently, the information about cerebellar projections to areas in posterior parietal cortex is complex and incomplete. It is clear, however, that several areas in this cortical region are the target of output channels from the ventral dentate” (p. 418). The research of Strick et al. is interesting for its analysis of the dentate nucleus:

> A sizeable portion of the nucleus projects to parts of the prefrontal and posterior parietal cortex. The output channels to prefrontal and posterior parietal areas are clustered in a ventral and caudal region of the nucleus. Consequently, these output channels are spatially segregated from those in the dorsal dentate that target motor areas of the cortex. Thus, the dentate appears to be spatially subdivided into separate motor and nonmotor domains that focus on functionally distinct cortical systems. (p. 419)

According to Strick et al. (2009), comparative studies on apes and humans indicate that the ventral dentate—the “non-motor dentate,” or “neodentate”— has expanded considerably compared to other cerebellar nuclei. This phylogenetically recent non-motor dentate may account for as much as 50% of the dentate nucleus in the cebus monkey, according to the authors.

Earlier research by Middleton and Strick (1997), once again utilizing neuronal tracing methods on cebus monkeys, focused mainly on connections between cerebellum and anterior cortex. But even this study concluded, “[C]erebellar output has the potential to influence not only the primary motor cortex, but also widespread regions of the cerebral cortex in the frontal and *parietal lobes*” (p. 553, italics added).

Projections from the parietal cortex to the cerebellum are in accord with the classical view, discussed below, whereby the cerebellum collects neural information from the posterior cortex, which it distributes to the anterior cortex. However, the functional role of the returning connections from cerebellum to parietal cortex does not fit this model and needs to be explicated.

Neuronal tracing studies are very suggestive of the structural connections that may exist in humans between cerebellum and cerebral cortex. However, extrapolation of results for non-human primates to humans does have its limitations. According to Leiner and Leiner (1997),

> Although experimental data on nonhuman primates are important, they do not suffice, because the additional structures that evolved in the human brain, including the connections of these structures to each other, cannot be studied in experiments on the brains of nonhuman primates. Yet, precisely these additional connections in the human brain, particularly those linking the cerebellum to the language areas of the cerebral cortex, are of paramount importance to theorists who seek to explain the neural basis of human cognitive and linguistic capabilities. (pp. 540-541)

In particular, humans are capable of conceptual thought to a far higher degree (presumably) than monkeys. According to Krienen and Buckner (2009), “Our map of cortical correlations with posterior cerebellar hemispheres . . . suggests the possibility that there exist cerebro-cerebellar circuits in human prefrontal cortex that may not find a homologue in monkeys” (p. 2493). Research on monkeys, therefore, cannot produce neural correlates for conscious conceptual thought.

5.3.4 FCMRI Studies

Allen et al. (2005) is an important investigation in that it uses FCMRI to evaluate integration of the dentate nucleus with various cerebral-cortical regions in *humans*. This study was unable to establish the afference or efference of connections directly. However, since functional coherence was observed between the dentate nucleus and the thalamus, but not between the dentate nucleus and the pons (and it is well known, according to the authors, that the direction of flow is from the cerebellum to the thalamus and from the pons to the cerebellum), the authors supposed that they were finding efferent connections to the cerebral cortex from the cerebellum.

Allen et al. (2005) found a high degree of functional coherence within the cerebellum and also between the dentate nucleus and the thalamus. However, the main interest herein is in their findings with respect to the dentate nucleus and the cerebral cortex. They distinguished between connections of the right dentate and the left dentate. Generally speaking, dentate-cerebral connections are contralateral (ibid.). The authors' results are worth quoting at length:

Left dentate

> In parietal cortex, a single cluster of connectivity was observed in the right hemisphere, with peak activity in the inferior parietal lobule (BA 40). This cluster also extended into the supramarginal and postcentral gyri as well as into the angular gyrus (BA 39). Small areas of functional coherence were also found in the posterior cingulate and in medial aspects of the occipital lobe.
>
> Several clusters of signal coherence with the left dentate were observed in the frontal lobes, with three such areas in the right hemisphere. One was a small cluster centered in the superior frontal gyrus (BA 9) that extended into the middle and medial frontal gyri (BA 8 and BA 9). Another more medial frontal cluster with maximum intensity in the medial frontal gyrus (BA 8) also extended into the cingulate gyrus (BA 24 and BA 32). A third right frontal cluster, which was the largest area of functional coherence with the left dentate within the cerebral cortex, had its peak in the middle frontal gyrus (BA 46). This large dorsolateral prefrontal cluster also extended into Brodmann areas 9 and 10. In addition to functional coherence in the right frontal lobe, we observed two frontal clusters in the left hemisphere. One of these had its maximum intensity in the precentral gyrus (BA 6) and extended into the middle frontal gyrus (BA 8). The other was centered in BA 9 but extended into BA 10 and had its peak in BA 46. (p. 42)

Right dentate

> In cerebral cortex, three separate clusters were observed. One of these was centered in the middle occipital gyrus (BA 19), extending into the precuneus (BA 31), cuneus (BA 18), and middle temporal gyrus (BA 39). The other two cortical clusters were in the frontal lobes. One of these had its peak intensity in the right anterior cingulate gyrus (BA 24), but extended into the left cingulate (BA 24) and medial frontal (BA 6) gyri as well. The other frontal cluster was located in the superior and middle frontal gyri, with peak intensity in BA 9 and extension into BA 10 and BA 46. (pp. 42-43)

In conclusion,

> *The dentate nucleus showed robust functional coherence with the inferior parietal lobule (BA 39 and 40*) and with multiple loci in the dorsolateral prefrontal cortex (BA 9, 10, and 46). In certain cases such functional coherence was bilateral (e.g., between the left dentate and both the right and left prefrontal cortices). (p. 43, italics added)

It follows that dentate-nucleus connection to the highest association areas both in the anterior cortex and the posterior cortex is substantiated. In particular, the posterior association cortex, BA 39, is strongly linked to the dentate nucleus. As discussed above, on the other hand, the motor regions in the anterior cortex are connected to the intermediate zone and the interpositus nucleus, whereas the vermis and fastigial nucleus connect to the spinal chord and vestibular system. The fact that the cerebellum connects with a region of the cerebral cortex that is associated with conceptual thought is a second argument, following the phylogenetic conclusions, for supposing that the cerebellum is deeply implicated in higher cognitive functions.

As mentioned above, Allen et al. (2005) stated that the directionality of connections cannot be inferred directly from FCMRI data. Krienen and Buckner (2009) make the same point, but they also suggest further limitations of the method. In particular, FCMRI results depend on statistical correlations rather than structural connections. For this reason, FCMRI data can reflect indirect connections and its results can be ambiguous. For example, if three areas are simultaneously correlated, which are the primary connections, or are all three areas mutually interconnected? Taking account of these limitations, however, Krienen and Buckner, using FCMRI methods, identified at least four distinct "fronto-cerebellar circuits" (p. 2488).

Unlike Allen et al. (2005), Krienen and Buckner (2009) did not identify cerebellar connections with the posterior cortex. In fact, they confirmed that the primary occipital cortex does *not* project to the cerebellum. However, it may be fair to suggest that the authors did not find cerebrocerebellar connections to the parietal cortex because they were not looking for them, and they admit, "[A] great deal of cerebrocerebellar connectivity remains to be explored" (p. 2495).

The point of looking for cerebrocerebellar connections to areas of the cerebral cortex known to be involved in higher cognitive functions—such as the prefrontal cortex or the parietal cortex—is to suggest that the cerebellum itself is involved in higher cognitive functions.

It seems to be clear from neuronal tracing studies and FCMRI studies that these connections do indeed exist. Nevertheless, there are various interpretations of the architecture of these connections. I discuss the classical interpretation and its variations and also the closed-loop interpretation.

5.3.5 The Classical Understanding of Cerebrocerebellar Connectivity

According to Strick et al. (2009), the classical view of cerebrocerebellar connectivity is as follows:

> The cerebellum is massively interconnected with the cerebral cortex. The classical view of these interconnections is that the cerebellum receives information from widespread cortical areas, including portions of the frontal, parietal, temporal, and occipital lobes . . . (Glickstein et al. 1985, Schmahmann 1996). This information was then thought to be funneled through cerebellar circuits where it ultimately converged on the ventrolateral nucleus of the thalamus (e.g., Allen & Tsukahara 1974, Brooks & Thach 1981). The ventrolateral nucleus was believed to project to a single cortical area, the primary motor cortex (M1). Thus, cerebellar connections with the cerebral cortex were viewed as a means of collecting information from widespread regions of the cerebral cortex. (p. 414)

Stein et al. (1987) discuss some empirical support for the classical view. Firstly, the number of cerebrocerebellar fibres (anterior and posterior) can be compared with the number of corticocortical fibres. According to the authors, who presumably are referring to monkeys, five million fibres project from each side (left and right) of occipital cortex and parietal cortex to the cerebellum via the pons. For comparison, the corticocortical pathway linking parietal cortex and occipital cortex with prefrontal areas consists only of a few hundred thousand fibres.

Secondly, when the corticocortical pathway is severed in monkeys, they are still able to perform dextrous tasks (Stein et al., 1987). On the other hand, inactivation of the lateral cerebellar hemispheres in monkeys, the region of the cerebellum that takes projections from occipital and parietal areas, *does* affect the accuracy and efficiency of movement, as if the integration of visual data with action were impaired (ibid.). The authors propose that corticocortical fibres do indeed regulate movement, but that these paths are less efficient in accomplishing this task than the cerebrocerebellar route.

MacLeod (2005) gives a third argument in favour of the classical understanding. She cites evidence that fibre pathways from posterior cortex to anterior cortex via cerebellum are more abundant than the reverse, from anterior cortex to cerebellum or from cerebellum to posterior cortex. In other words, afferents and efferents from the cerebellum are not proportionally reciprocal.

A variation of the classical understanding, which takes note of the fact that the cerebellum projects to association areas of the prefrontal cortex as well as to motor areas, is that the cerebellum participates also in higher cognitive functions. This is MacLeod's (2005) view. She writes, basing her ideas on Piagetian principles of genetic epistemology,

> [T]he most concrete aspects of the cerebellum, its gross anatomy, cytoarchitecture, and connectivity with the rest of the brain, [are] related to its function to argue that the cerebellum acts as a mediator between peripheral and central nervous systems and hence ultimately affects the child's abstraction of logical principles from concrete experience. (p. 145)

As indicated above, however, there is evidence for projections from cerebellum to parietal cortex and also there is confirmation that the primary occipital cortex does *not* project to the cerebellum (Krienen & Buckner, 2009). The classical understanding appears to be less strongly supported in these respects. The closed-loop understanding, discussed below, rather than the classical understanding, appears to have additional empirical evidence in its favour, notwithstanding qualifications arising from the results and arguments discussed above.

It is probable that both views are true to an extent, the classical understanding and the closed-loop understanding, but it is beyond the scope of this dissertation to suggest ways in which they may be integrated coherently. For my purposes, I assume the primacy of closed-loop circuits between cerebral cortex and cerebellum. There is empirical support for this view based on research on human subjects, which I turn to now.

5.3.6 The Closed-loop Understanding of Cerebrocerebellar Connectivity

Against the classical understanding of cerebrocerebellar connectivity, and any variations that take into account cerebellar involvement in higher cognitive functions, there is the understanding that the cerebellum connects with various regions of the cerebral cortex in closed, distinct loops. In other words, the cerebellum receives projections from the inferior parietal cortex, for example, and projects back to the inferior parietal cortex; likewise, the cerebellum receives projections from the prefrontal cortex and projects back to the prefrontal cortex. Moreover, these loops operate from distinct and separate regions of the cerebellum, just as they operate from distinct and separate regions of the cerebral cortex.

The existence of these reciprocal connections between cerebellum and cerebral cortex may be established on general principles of brain connectivity. Varela et al. (1991) write, “A rule for the constitution of the brain is that if a region . . . A connects to B, then B connects reciprocally back to A. This law of reciprocity has only two or three minor exceptions” (p. 94).

With regard to cerebrocerebellar connections, in particular, however, there is evidence for the closed-loop understanding both from neuronal tracing studies on monkeys and from FCMRI studies on humans. Some of the evidence from neuronal tracing studies has already been discussed in Section 5.3.3, above, and is summarized in Schmahmann and Pandya (1997).

Middleton and Strick (1998), in another neuronal tracing study on monkeys, argue that cerebrocerebellar connections are always distinct, and reciprocal, and they formulated some general principles of cerebrocerebellar connectivity. Although their research is limited to monkeys, the results and arguments are structural and general rather than specific, and it is likely they are applicable to all primates, or possibly even to all mammals. From their results and the results of previous studies, the authors identified three general principles of cerebrocerebellar connectivity:

> *(1) Cerebral cortical areas that project to the cerebellum are the target of cerebellar output. . . .*
> *(2) Cerebral cortical areas that do not project to the cerebellum are themselves not the target of cerebellar output. . . .*
> *(3) The cerebellar output channels to different cortical areas are distinct.*
> (p. 349, authors' italics)

To clarify the third of these principles, "each population of neurons in the deep cerebellar nuclei that projects to a specific cortical area is organized into a distinct output channel" (p. 350). The fact that authors are referring explicitly to cerebrocerebellar closed-loop circuits is made clear in the quotation in the next paragraph.

Middleton and Strick (1998) also claim that cerebrocerebellar loops involving non-motor areas of the cerebral cortex are *not* concerned with motor behaviour. According to the authors,

> The cerebellum receives its inputs *and* it directs its outputs to multiple cortical areas. Thus, the cerebellum participates in multiple closed-loop circuits with the cerebral cortex. These loops do not appear to enable widespread cortical areas to gain access to the motor system, as previously thought. Rather, they represent a framework for massive parallel processing of motor and non-motor information. . . . [I]t is clear that the output of the cerebellum operates in both motor and cognitive domains. (p. 353, authors' italics)

In other words, the authors are arguing specifically against the classical understanding of cerebrocerebellar connectivity. Cerebellar output, for example, to the parietal cortex is correlated directly with local parietal functions rather than indirectly with motor functions. Nevertheless, there are corticocortical connections between anterior and posterior regions of the cerebral cortex (Stein et al., 1987; Fuster, 2003), and it seems reasonable to suppose that the posterior cerebrocerebellar loops are certainly involved *indirectly* with action (notwithstanding research with respect to severing these corticocortical connections, mentioned above). However, it is reasonable to assume that activity in posterior cerebrocerebellar loops *primarily* correlates with perception, just as activity in anterior cerebrocerebellar loops *primarily* correlates with action. Requiring that *all* output from the cerebellum involve action is hammering a square peg into a round hole.

The conclusions of Middleton and Strick (1998) are reinforced in Kelly and Strick (2003), where the authors write, "Importantly, cortical regions correlated with [specific] cerebellar sites were nonoverlapping, supporting the characterization of certain cerebral-cerebellar circuits as closed, segregated loops" (p. 2488). Moreover, according to Strick et al. (2009), "The cortical areas that are the target of cerebellar output also project via the pons to the cerebellar cortex (Glickstein et al. 1985, Schmahmann 1996). This observation suggests that cerebro-cerebellar connections may form a closed-loop circuit" (p. 420). Even more clearly, "[T]he fundamental unit of cerebro-cerebellar operations is a closed-loop circuit" (ibid., p. 414), and "[M]ultiple closed loop circuits represent a fundamental architectural feature of cerebro-cerebellar interactions" (ibid., p. 420).

The evidence and conclusions cited above for cerebrocerebellar closed-loop circuits are based, moreover, on neuronal tracing studies. There is no question with respect to the directionality of the connections, as there is with FCMRI studies. The FCMRI research of Allen et al. (2005), discussed above, establishes connections between cerebellum and extensive regions of cerebral cortex, including parietal cortex. The authors only dealt with the directionality of connections in passing. Krienen and Buckner (2009) also emphasize the difficulty of establishing directionality using FCMRI methods, although their results are suggestive of closed-loop circuits.

Krienen and Buckner (2009) established the existence of three segregated circuits connecting the cerebellum to three distinct regions of the prefrontal cortex. According to the authors, "Taken as a group, the regions of the cerebellum linked to prefrontal cortex occupy a significant portion of the posterior [cerebellar] hemisphere suggesting that, in humans, a large portion of the cerebellum may be dedicated to supporting cognitive functions" (p. 2491), and "[T]he cerebellum participates in multiple different networks subserving cognition" (p. 2491). The results of Krienen and Buckner do not indicate connections between cerebellum and parietal cortex. On the other hand, the authors are looking only for connections to the anterior cortex, and it is not surprising, therefore, that parietal connections do not appear in their results.

With respect to directionality, Krienen and Buckner (2009) make the interesting point that FCMRI probably picks up correlations from both afferent and efferent connections. Therefore, if there are only two regions that are simultaneously correlated, then these two regions must represent each other's afferent and efferent connections. In other words, the two regions belong to a closed-loop circuit. And this, indeed, is the kind of correlation that they found. They substantiated the idea of closed-loop architecture as follows:

> We . . . investigated whether the connectivity between a given frontal site and a cerebellar region is reciprocal and selective, that is whether maps produced by cerebellar seed regions exhibit 'closed-loop circuitry' by showing preferential connections with those frontal sites that originally produced the cerebellar correlations. (p. 2487)

The results were positive, and the authors state, "Projections from the cerebellum form closed-loop circuits" (p. 2489, Fig. 2).

Strick et al. (2009) argue if closed-loop circuits are the rule, then all areas of the cerebral cortex that project to the cerebellum also receive input from the cerebellum. These areas include, among others, the inferior parietal cortex and secondary visual cortex.

As discussed above, there is evidence also for the classical view and its variations. However, I have selectively chosen to emphasize the closed-loop understanding rather than the classical understanding in my research.

5.3.7 Localization in the Cerebellum

According to Strick et al. (2009) the older view of projections from the cerebellum is that they pass through the ventrolateral thalamus to the primary motor cortex. However, the ventrolateral thalamus has been found to contain multiple subdivisions, which in turn project to multiple regions of the cerebral cortex, including frontal, prefrontal, and posterior parietal cortex. Moreover, the pathways leading from cerebellar cortex back to cerebral cortex appear to be segregated at stages prior to the thalamus, in cerebellar cortex and dentate nucleus.

Within the context of the closed-loop perspective, the analysis of Strick et al. (2009) implies that the loops are distinct and segregated from cerebellar cortex, through dentate nucleus and thalamus, to cerebral cortex. The analysis of Strick et al. further implies that the functional role of a region in a lateral cerebellar hemisphere depends on the loop to which it belongs and where this loop connects to the cerebral cortex. Kelly and Strick (2003) argue that a functional map of the cerebellar cortex is likely to be as rich and complex as that of the cerebral cortex.

Brodal and Bjaalie (1997) review research on the connections from cerebral cortex to cerebellum via the pons. I discuss this article because it could be taken as evidence against localization in the cerebellum and closed, distinct cerebrocerebellar loops. The overall conclusion of the authors is that input from multiple, widespread cortical regions is integrated in localized regions of the cerebellar cortex; likewise, a single region in the cerebral cortex projects to multiple regions in the cerebellar cortex.

More specifically, Brodal and Bjaalie (1997) discuss both the afferent connections to the pons from the cerebral cortex and efferent connections from the pons to the cerebellar cortex. With respect to the first stage, from cerebral cortex to pons, the authors cite evidence of considerable divergence of neural connections: “even a very small volume of cortex projects to widespread parts of the pontine nuclei” (p. 231). However, these “widespread parts” of the pons are themselves clearly delineated. Thus, (for monkeys) localized regions of the cortex project to “lamella-like subspaces . . . . resembling the skins of an onion” (p. 231). The originating region of the cerebral cortex may be small, but the terminating region in the cerebellum, though relatively large, has very specific shape and boundary. Although these lamellae appear sharply delineated, the authors state, “[T]his evidence of segregation does not preclude the existence of considerable integration of inputs from various regions” (pp. 232-233).

Furthermore, Brodal and Bjaalie (1997) point out that different cortical areas may project to the same lamella. There is evidence for this kind of overlapping, according to the authors, for projections from the somatosensory cortex and visual cortex to the pons. Accordingly, “[M]ost pontine neurons integrate inputs from at least two—probably often more—different cortical sites.

When moving perpendicularly through the lamellae, one would presumably encounter pontine neurons with changing combinations of cortical inputs” (p. 233). In spite of this conclusion, convergence of inputs from the somatosensory cortex and visual cortex is the only specific example the authors give. Looking at Figure 5 (p. 236), which indicates the overlap in the cat pons of projections from somatosensory cortex and visual cortex, the area of coincidence appears to be quite small compared with the overall extent of the targeted areas in the cerebellum from the two regions of the cerebral cortex.

The second major aspect of Brodal and Bjaalie’s (1997) study concerns the pontocerebellar connections. The authors identified convergence as a property of these connections: “[A]xons from neurons in many parts of the pontine nuclei converge in a small volume of cerebellar cortex” (p. 236). A region of the pons that converges on a single area of the cerebellar cortex also has a lamella-like structure. In this way, the pons is segregated into lamellae also in terms of its connections to the cerebellar cortex.

Brodal and Bjaalie (1997) note further that the lamellae for efferents from the pons are less distinct than those for afferents to the pons. However, both sets of lamellae have basically the same shape and orientation. They write, “One notable difference between corticopontine and pontocerebellar lamellae is that the latter appear to be thicker and more fuzzy than the former” (p. 238). In consequence, “This would imply that *a pontocerebellar population projecting to even a fraction of a folium would be contacted by several corticopontine terminal fields—that is, influenced from several cortical sites*” (p. 238, original italics).

In summary,

> These overall patterns of connectivity must be viewed in conjunction with the marked divergence and convergence within both links of the cortico-ponto-cerebellar pathway, reviewed above. Thus, we concluded that one particular cortical region has access to widely separated parts of the cerebellum, and that one particular part of the cerebellar cortex would receive convergent inputs from different parts of the cortex. (p. 240)

More succinctly put, “each small part of the cerebellar cortex would receive a specific combination of messages from many different sites in the cerebral cortex” (p. 244). As this conclusion appears at odds to my thesis, I am behoved to engage this matter in some detail in what follows.

Firstly, a different interpretation of the double lamella-like structure in the pons strikes me as plausible. A small region of the cerebral cortex corresponds to a lamella in the pons; this lamella is clearly related to the efferent lamellae, which have “basically the same orientation and shape.” It seems plausible that an afferent lamella may be at least partially contained within an efferent lamella. Hence, it is possible that the corresponding projection to the cerebellar cortex is also contained within a small region. This is my preferred interpretation of the double lamella-like structure of the pons, which may actually *support* localization and the closed loops.

Secondly, Brodal and Bjaalie (1997) suggest a role for the cerebellum in integrating inputs from various regions of the cerebral cortex. However, they also admit to localization of cerebrocerebellar connections, though on a larger scale. They state:

> [S]tudies comparing projections from various cortical areas (Figs. 1, 5) give nevertheless *evidence of considerable segregation*. . . . Thus, it is obvious that the sensorimotor cortex has terminal regions different from those of the parietal cortex, that the premotor cortex projects differently from the motor cortex, and so forth. (pp. 231-232, italics added)

In more detail,

> [P]referred cerebellar target regions of each main cortical region can be identified. Thus, in the cat the sensorimotor region is most strongly connected with the anterior lobe, whereas the posterior parietal cortex directs its output especially to crus I, crus II and the dorsal paraflocculus (Brodal, 1987). The visual cortex seems to project primarily to the dorsal paraflocculus, (Robinson et al., 1984; Burne and Woodward, 1989; Broch-Smith and Brodal, 1990) whereas the cingulate gyrus focusses on the ventral paraflocculus (Aas and Brodal, 1989). The same *general* pattern appears to hold for the monkey (Brodal, 1979, 1982b; Glickstein et al., 1994) although there are some notable differences. (p. 240, original italics)

Hence, the authors themselves, as indicated in the preceding quotations, clearly advocate localization in the cerebellum, at least at a fairly large scale. In utilizing the existence of closed, distinct cerebrocerebellar loops in my thesis, I only *need* these loops to exist on the large scale. It is quite clear from Brodal and Bjaalie (1997) that cerebrocerebellar connections manifest complex convergent and divergent behaviour on passing through the pons. However, the level of analysis at which this complex behaviour is indicated is finer than I require for my thesis.

Thirdly, the integration of somatosensory and visual data in the cerebellum does not contradict my thesis. It is reasonable that somatosensory properties, should be associated with the visual percept, given that one's interaction with the object is tactile as well as visual. I do no analysis in this regard herein, but in my reading, Fuster's (2003) cognitive network framework does not preclude the possibility of integrating the two types of data.

A fourth point is suggested by the researchers themselves in another paper. Bjaalie and Brodal (1997) contains detailed research on the pontocerebellar connections summarized in Brodal and Bjaalie (1997). Bjaalie and Brodal investigate connections from the cat pons to the paraflocculus by means of neuronal tracing. However, the paraflocculus is just a small part of the cat cerebellum, concerned with motor behaviour. The authors give several reasons why the paraflocculus represents a special case in the cerebellar cortex; they write, “These considerations warn against generalizations based on our results on the paraflocculus alone” (p. 208). Therefore, one should be careful not to assume that this highly localized form of convergence-divergence behaviour necessarily extends to other regions of the cerebellum.

Welker (1986) in an earlier paper also appears at first to contradict the notion of closed, distinct cerebrocerebellar loops. The authors conducted research on projections from the somatosensory cortex to cerebellar cortex in several species of animals. They discovered evidence of a fractured somatotopy, in which projections to the cerebellum from a given sensory region exhibit multiple, widespread target zones in the cerebellar cortex. However, I may reuse the argument above, that these multiple, fractured locations in the cerebellar cortex may well belong to a single cerebrocerebellar loop. After all, there is no necessity for the cerebellar component of a cerebrocerebellar loop to be connected and spatially localized in the cerebellum.

Lastly, I would like to cite some additional, more recent research in favour of cerebellar localization. Manni and Petrosini (2004) provide a historical review of the notion of localization in the cerebellar cortex. Their investigation is concerned with motor behaviour and which regions of the cerebellar cortex are responsible for facilitating movement of which parts of the body. In other words, they investigate the notion of a somatotopic mapping from cerebellar cortex to body. They indicate that the debate is still going. However, with certain reservations, the authors conclude that there is indeed such a somatotopic mapping, involving the anterior lobe of the cerebellum and parts of the posterior lobe, with no contribution from the lateral cerebellar cortex, the most recent region of the cerebellar cortex, phylogenetically.

I propose that the lateral cerebellar cortex is responsible for certain aspects of higher cognitive functions with respect to cerebellar closed-loop circuits between association areas of the anterior cortex and the posterior cortex. Although Manni and Petrosini (2004) are concerned with motor behaviour, it would appear that their review is fully consistent with the notion of localization of higher cognitive function in the cerebellar cortex, as would be manifested by closed-loop connections to association areas of the cerebral cortex.

To summarize, within the context of the closed-loop assumption, certain regions of the lateral cerebellar cortex project to association areas of the parietal cortex, which are responsible, among other things, for visual, spatial conceptual reasoning. It is the goal of this chapter to establish one interpretation of the functional role of the cerebrocerebellar loops connecting cerebellum with association areas of parietal cortex. The very existence of these cerebrocerebellar loops is the second argument in support of the plausibility of the CSH.

5.4 Uniformity of Cerebellar Histology

In Section 5.3.6 I argued that the cerebrocerebellar connections take the form of distinct, segregated, closed-loop circuits. The functional role of the lateral cerebellar cortex depends upon where within the cerebral cortex any particular loop projects. The cerebellar cortex, therefore, may undertake a variety of tasks according to the regions of the cerebral cortex to which it projects.

However, the cerebellar cortex throughout has a highly uniform histology. According to Ramnani (2006), “[T]he beautifully regular and simple cellular organization in the cerebellar cortex is repeated in a crystalline manner across the entire cortex” (p. 511); Ito (2006) writes, “The outstanding feature of the cerebellum’s circuitry is the precision and geometric beauty of its arrangement” (p. 273). Ito argues, moreover, that the uniform structure “can be regarded as the neuronal machinery designed to process input information in some unique and essential manner” (p. 274).

In some sense, therefore, an identical cerebellar microstructure undertakes an identical processing load no matter where in the cerebral cortex it projects. This processing load manifests in different ways according to the variety of functional roles of the cerebral cortex.

Leiner et al. (1986) address the issue of a single cerebellar function from the perspective of phylogenesis. With the early vertebrates, the cerebellum evolved to regulate motor reflexes through connections with the brain stem. Then, these same cerebellar structures assumed alternative functions as new areas of the cerebral cortex evolved and the cerebellum became connected to these new areas. The cerebellum assumed the role of facilitating smooth, efficient motor behaviour as it was connected with the sensorimotor cortex. Eventually, we suppose, the cerebellum undertook some processing of higher cognitive functions through its connections with association areas of anterior and posterior cortex.

In humans the cerebellum is still connected to the brain stem, the sensorimotor cortex, and various association areas of the cerebral cortex. I am not overly concerned with the brain stem connections, which are beyond the scope of my investigation. With regard to the other connections, however, it is interesting to suppose that in some sense the cerebellum plays the same role in facilitating smooth, efficient motor behaviour as it does in modulating higher cognitive functions, in which one of these higher cognitive functions is the schematic nature of image-based reasoning in geometry. According to Schmahmann (2004), "[B]ecause cerebellar anatomy is essentially uniform throughout the structure, the basic work that cerebellum does in the nervous system should be constant as well. This we have referred to as the *universal cerebellar transform*" (p. 374, italics added). The universal cerebellar transform is discussed further in Section 5.8.

Many other researchers have suggested something like the universal cerebellar transform. Wolpert et al. (1998) write, "[M]ost people believe that there is a common computational operation performed by all cerebellar areas, although processing specific inputs and sending outputs to different extracerebellar targets" (p. 338). Paulin et al. (2001) propose that the cerebellum performs a particular computation that contributes to a range of tasks since it has a relatively simple, uniform circuit structure that is similar in all vertebrates. According to Apps and Garwicz (2005), "Given the uniform structure of the cerebellar cortex, the basic neural computation performed is assumed to be similar throughout, whether used for the control of autonomic functions, limb movements or higher functions such as language" (p. 299). Ramnani (2006) suggests that repetition of the specific organization and types of connections across the entire cerebellar cortex is evidence for the same type of information processing across the entire cerebellar cortex. All these ideas are variations of the universal cerebellar transform.

The notion of the universal cerebellar transform is another piece of the puzzle, a third argument in favour of the plausibility of the CSH. I argue below that cerebellar schematization with respect to the concepts and percepts of the posterior cortex is analogous to the role of the cerebellum in facilitating smooth, efficient motor behaviour through its connections with the primary motor cortex.

5.5 The Motor Cerebellum and Non-motor Cerebellum

The traditional view of the functional role of the cerebellum, discussed below, is that the cerebellum contributes to smooth, efficient motor behaviour. On the other hand, the cerebellum has been implicated in a number of cognitive functions not directly related to motor behaviour, such as planning and reasoning. There is some controversy in the research literature whether the cerebellum is solely concerned with motor behaviour, or whether there is also a "non-motor cerebellum." And then there is the question of dissociating the motor and non-motor components of the cerebellar contribution to cognition, given that the two appear to be inextricably bound. My view is that there certainly is a non-motor cerebellum, dependent on the *posterior* cerebrocerebellar loops—the motor cerebellum, in its various forms, is manifested in activity of *anterior* cerebrocerebellar loops. The distinction concerns the cerebellar contribution to perception and action, respectively. The motor cerebellum may be confounded with the non-motor cerebellum because any aspect of cognition has both an action component and a perception component. The CSH, in this interpretation, is a hypothesis about the non-motor cerebellum.[16] The purpose of this section is to discuss these issues in order to position the CSH better in relation to mainstream cerebellar research.

By the end of the eighteenth century, the gross anatomy of the cerebellum had been mapped, and speculation began in earnest as to its function (Glickstein et al., 2009). Luigi Rolando (1773-1831), by means of lesion studies, was the first to identify the role of the cerebellum in facilitating motor behaviour. Again through lesion studies, Pierre Flourens (1794-1867) made the distinction between motor execution and motor coordination, and proposed that the cerebellum was responsible for the latter. Luigi Luciani (1840-1919) identified the motor tremor

---

[16] It is worth emphasizing that the "non-motor cerebellum," in no way implies that the cerebellum is *exclusively* concerned with non-motor cognitive function. Rather, my use of the phrase "non-motor cerebellum" should be taken to imply that *the cerebellum has a functional role also in aspects of higher cognitive function that are* not *directly related to motor behaviour*—specifically in perception.

associated with cerebellar dysfunction. Holmes (1939) expresses the traditional view of the cerebellum as a facilitator of motor behaviour as follows:

> [I]n addition to regulating postural tone, the cerebellum reinforces or tunes up the cerebral motor apparatus, including subcortical structures with motor functions, so that they respond promptly to volitional stimuli and the impulses from them which excite muscular contractions are properly graded. (p. 29)

The traditional view of the cerebellum, therefore, is that its functional role is regulating motor reflexes and facilitating smooth, efficient motor behaviour; indeed, "[t]he study of the cerebellum has been dominated by interest in its role in movement and motor control" (Schmahmann, 1997, p. 4).

The particular type of motor control is thought to depend on the specific regions of cerebellum and cerebral cortex that are connected. For example, Stein et al. (1987), in their research on monkeys, indicate that the parietal cortex is connected to the lateral cerebellar hemispheres; inactivation of the lateral hemispheres, according to the authors, did not effect execution of movements, but rather their accuracy. These locations in the parietal cortex of the monkey, after all, belong to the dorsal stream, which is sensitive to movement.[17] On the other hand, the intermediate zone of the cerebellum, according to Stein et al., is probably concerned with controlling the actual execution of limb movements, through connections with the primary motor cortex.

Thach (1996) also differentiates between lateral and intermediate cerebellum in planning versus execution of movement. The dentate nucleus and

---

[17] See the discussion of dorsal stream versus ventral stream in Section 3.3.2—the human inferior parietal cortex, which connects to the lateral cerebellum, does not have a monkey homologue, and its functional interpretation with respect to the cerebellum cannot rest on monkey research.

lateral zone are important for the planning of movement, whereas the interpositus nucleus and intermediate zone are responsible for executing movement. According to the author,

> The prefrontal and premotor areas could still plan without the help of the cerebellum, but not so automatically, rapidly, stereotypically, so precisely linked to context, or so free of error. Nor would their activities improve so optimally with mental practice. (p. 411)

In an FMRI study by Kim et al. (1994) activation of dentate nucleus was indicated when participants were solving a pegboard puzzle. Much less activation of the dentate was detected when participants were simply moving pegs. Once again, the dentate and the lateral zone of the cerebellum, by implication, are associated with a higher cognitive function, the planning of movement.

The cerebellum, then, is crucial for various aspects of motor behaviour. Nobody doubts this. However, since the nineteenth century there have been streams of research investigating other possibilities (Schmahmann, 1997). After all, according to the discussion in this chapter so far, the phylogenesis of the cerebellum and its reciprocal connections with association areas of the posterior cortex imply that the cerebellum has a role also in non-motor cognition, including perhaps the higher cognitive functions such as the schematic perception and inferences in image-based geometrical reasoning. Before proceeding further, however, I should address the ongoing debate in the research literature with respect to whether or not the cerebellum is really implicated in non-motor cognition as well as motor cognition.

Leiner et al. (1986), in their seminal paper on this topic, were the first to propose a role for the cerebellum in higher cognitive functions. Their argument was based on the coordinated phylogenesis of cerebellar and related structures and the prefrontal cortex.

Schmahmann and Pandya (1989, 1990), discussed above, and other researchers such as Middleton and Strick (1994), demonstrated that the cerebellum is connected with parts of the prefrontal cortex and parietal cortex in monkeys, and presumably therefore also in humans. According to the latter, "[T]here is now convincing evidence for a cerebellar involvement in some aspects of cognitive processing, most notably verbal working memory which in turn may influence performance in a number of other cognitive domains" (p. 161); however, "[T]he empirical evidence available so far does not yet allow a firm conclusion or well-founded theoretical view concerning the mechanisms of a cerebellar involvement in cognitive function" (p. 161).

According to Middleton and Strick (1994), there are many studies reporting cerebellar activation during non-motor cognitive tasks. The difficulty, however, is to dissociate the cognitive element from motor aspects such as eye movement. Moreover, lesion studies are inconclusive for three reasons: firstly, it is difficult to find patients with lesions that are sufficiently localized; secondly, most lesion studies involve patients with chronic lesions, in which there is probably compensatory activity from other areas of the brain; thirdly, cerebral lesions can produce the same cognitive deficits as cerebellar lesions, but to an even greater extent (ibid.). The last of these difficulties may seem to challenge the view that the cerebellum is involved in higher cognitive functions. However, it is reasonable to suppose that if one end of a cerebrocerebellar loop—in the cerebral cortex—is disabled, then the whole loop is disabled. In addition the role of the cerebellum is that of *facilitator* for the function that is localized in the cerebral cortex, and so a cerebellar lesion would not disable the function but rather cause it to be exercised less efficiently.

The dissociation of motor and non-motor cognitive tasks is an important issue. With regard to the non-motor cerebellum, Bloedel and Bracha (1997) pose the following question:

> To many [the non-motor cerebellum] appeared to be a radical departure that was inconsistent with the traditional views of cerebellar function. Consequently, a question must be raised: is this view such a departure that it must be considered incorrect from the outset based on first principles; or is it a view that is not only tenable, but also instructive, suggesting the need to broaden our concepts of cerebellar function, revisit what has become a dichotomy between motor and cognitive functions, and consider the heterogeneous contributions the cerebellum can make to nervous system function? (p. 617)

The authors argue the interesting point that research generally, and in particular research on the cerebellum, tends to support the scientific paradigm within which the research is conducted. Since the motor cerebellum has been a dominant paradigm, the non-motor cerebellum may be “considered incorrect from the outset based on first principles.” On the other hand, if the underlying assumption is that the cerebellum *is* responsible for non-motor cognitive functions, then research results tend to support this alternative view.

Bloedel and Bracha (1997) also suggest that the distinction between motor and non-motor cognitive tasks is artificial. According to the authors, all paradigms that demonstrate motor dysfunction may also be regarded as displaying non-motor cognitive dysfunction, and vice versa—all “motor” tasks have non-motor cognitive components and all “non-motor” cognitive tasks have motor components—a consequence of the perception-action loop. Accordingly,

> [T]he ultimate interpretation depends on the assignment of one of the prevailing schemes of behavioral organization and nervous system function on a selective if not exclusive basis. To us this approach is at best confusing and more importantly fails to interpret the observations in the most meaningful context.
>
> It must be asked whether the traditional emphasis on concepts pertaining to motor function results in an excessively restrictive attempt to categorize deficits produced by cerebellar lesions solely as dysfunctions in the motor domain. (p. 619)

Paulin (1993) makes a similar point. Because cerebellar dysfunction causes deficits in motor control, researchers have claimed that the function of the cerebellum is motor control. Paulin points out that this argument is fallacious. His metaphor is that a stone shattering the windscreen of a car may well lead to some wobbling of the car’s trajectory. However, no one suggests that the function of the windscreen is motor control. It is part of a larger system, one of whose functions is motor control. In fact, one of the effects of the broken windscreen is to impact the driver’s *perception* of the road ahead. According to Paulin, “Current widespread acceptance of the motor control theory of cerebellar function is based on an impressive accumulation of supporting evidence, rather than a critical evaluation of that theory in the light of all of the evidence” (p. 40).

In the state-estimation theory of Paulin (1993) and Paulin et al. (2001), the cerebellum facilitates the tracking of movement of other objects rather than regulating movement of the organism itself. In the state-estimation, theory, the motor cerebellum is a secondary consideration. Perception may lead to movement, but it is the cerebellum's role in perception per se that is primary.

Specifically referring to motor dysfunction caused by perceptual deficit, Gao et al. (1996) write,

> Cerebellar deficits in voluntary movement, such as incoordination and ataxia, may reflect disruption of the sensory data (from the medial cerebellum-controlled muscle spindle system) on which the motor system depends, rather than disruption of cerebellar computations of smooth motor performance per se (*11*). . . . *[T]he sensory and motor components of task performance have not been well dissociated*. (p. 547, italics added)

I agree with Paulin (1993), Middleton and Strick (1994), Gao et al. (1996), and Bloedel and Bracha (1997) with respect to the potential for confounding the motor cerebellum with the non-motor cerebellum. As we see later in this chapter, however, there are certain cognitive functions, such as selective attention, which appear to be largely dissociated from motor behaviour. I propose that the cerebellar role with respect to motor behaviour, including even higher cognitive functions that are in a sense analogous to motor behaviour, is mediated by cerebrocerebellar loops of the anterior cortex. On the other hand, I propose that the non-motor cerebellum depends on cerebrocerebellar loops of the posterior cortex. The dichotomy, therefore, is between action and perception, and depends on whether the cerebellum connects with the anterior cortex or posterior cortex.

Although I have emphasized the closed-loop understanding of cerebrocerebellar connectivity, this notion is not universally accepted, as discussed above. Much of the literature on the role of the cerebellum with respect to higher cognitive function assumes that higher cognitive function must of necessity involve the prefrontal cortex. Indeed, in the classical understanding and its variations, the cerebellum gathers data from diverse regions of the cerebral cortex, including posterior cortex, and funnels this data to anterior cortex—to the prefrontal cortex, in particular, for higher cognitive functions.

Glickstein and Doron (2008), operating within the classical understanding of cerebrocerebellar connectivity, and admitting connections to the prefrontal cortex, deny that the cerebellum plays a role in higher cognitive functions. According to the authors,

> The original syllogism [of Leiner et al., 1986] seems to be that the cerebellum, and particularly the cerebellar hemispheres, is particularly large in humans, so the cerebellum is a likely brain structure for cleverness. (An alternative syllogism might be that humans, apes, and monkeys are most skillful in the use of their fingers…). Three sorts of evidence have been put forward to support the idea of a role for the cerebellum in cognitive functions; neuropsychological deficits in patients with cerebellar lesions, activation of the cerebellum in normal subjects as they perform a cognitive task, and anatomical connections showing links to and from the cerebellum of structures in the cerebral cortex that are known or thought to be involved in cognition. (p. 590)

Glickstein (2007) writes that humans and monkeys have other things in common besides cleverness—they are good with their fingers and eyes, for example—which undermines the phylogenetic argument. With regard to the lesion evidence, Glickstein and Doron (2008) make the same point as Middleton and Strick (1994), mentioned above, that it is very difficult to find patients with cerebellar lesions that are localized enough for strong conclusions to be drawn about cerebellar function, and "[m]any of the papers demonstrating cognitive deficits following cerebellar damage may be due to concomitant damage to other brain structures" (p. 593).

According to Glickstein (2007), imaging evidence and connectivity evidence for the cerebellar contribution to higher cognitive function are both unreliable:

> Activation of the human cerebellum in fMRI scans may be related to actual or planned movement of the eyes, vocal apparatus or fingers. Anatomical evidence is interpreted on the basis of cognitive functions of the structures that receive an input from the cerebellum, but these same structures may be more related to the control of eye movements than to cognition. (p. 825)

The same points are reiterated in Glickstein and Doron (2008):

> Anatomical evidence at best reveals only a weak connection between the cerebellum and cerebral cortical areas involved in cognition. . . . . Neural activity in the cerebellum during cognitive tasks may be associated with actual or planned eye movements. (p. 593)

Glickstein and Doron (2008) admit, however,

> If an area of the cerebral cortex that is known to function in cognitive tasks projects to the pons or receives an input from the cerebellar nuclei, that would be evidence for a role for the cerebellum in such tasks.(p. 590)

With regard to closed-loop connections between cerebellum and association areas of the inferior parietal cortex, which is outside the scope of Glickstein and Doron’s (2008) investigation, I believe that the criterion mentioned in the previous quotation is indeed present.

Glickstein (2007) and Glickstein and Doron (2008) agree that the massive expansion of the cerebellum in recent phylogenesis needs some explanation. A functional role must be found for it. Their answer is that the higher primates, including humans, developed great manual dexterity and visual acuity, and the facilitation of these functions is indeed sufficient reason for the growth of cerebellar and related structures. Of course, the cerebellum is massively connected to the primary motor cortex, which accounts for manual dexterity. With respect to visual acuity, the authors argue that the connections of the cerebellum to areas of the prefrontal cortex thought to be responsible for higher cognitive functions are really connections to areas responsible for eye movements. It follows that the apparent cerebellar activity during higher cognitive processing corresponds to actual or planned eye movements rather than the higher cognitive functions themselves.

In reply to Glickstein and Doron (2008), Strick et al. (2009) point out that in their experiments on cebus monkeys they were very careful to avoid prefrontal areas associated with eye movement. In addition, the regions of the dentate that project to the prefrontal cortex are quite distinct from those dentate output channels known to be associated with eye movement: “Thus, there is considerable evidence that the cerebellar output to prefrontal cortex is distinct from the cerebellar control of eye movements” (p. 422).

Nevertheless, the functional role of the cerebellum with respect to higher cognition is still in doubt. According to Richter et al. (2007), for example, patients with chronic cerebellar abnormalities showed no cognitive dysfunctions in a bedside test—although the authors admitted that their screening process may not have been sensitive enough to pick up cognitive deficits. Frank et al. (2007), reporting on lesions studies in children and adolescents, failed to replicate the results of earlier studies that the cerebellum affects cognitive function. According to the authors,

> Although a role of the cerebellum in specific aspects of non-motor functions seems obvious it is still an open question which cognitive functions are involved, why and to what extent. It is a matter of ongoing discussion whether or not cognitive dysfunction belongs to the symptoms of cerebellar disease. Overall, disorders appear to be mild and far less frequent than disorders observed following lesions of cerebral areas. . . . There is ongoing need of well-controlled lesion studies, which show that disorders are due to cerebellar lesions independent of motor dysfunction and other confounding factors. (p. 242)

As mentioned earlier, however, the cerebellum is a *facilitator* of cerebral functions rather than their initiator. It is not surprising that cerebellar lesions should prove less damaging to cognition than cerebral lesions.

Timman and Daum (2007), in their introduction to a series of articles on the cerebellum and cognition in a special issue of *Cerebellum*, claim that, after two decades of research, there is no convincing theory of a cerebellar contribution to cognition. They do not dismiss the possibility of cerebellar involvement in higher cognitive functions, but argue the understanding of how this works is still deficient.

As with the issue discussed earlier with respect to the classical understanding versus the closed-loop understanding of cerebrocerebellar connectivity, the motor cerebellum versus the non-motor cerebellum is still a controversial topic. Just as earlier I chose to emphasize the closed-loop understanding to carry my argument forward, at this point I choose to emphasize the non-motor cerebellum. Thus, closed-loop connections to the anterior cortex and posterior cortex mediate, respectively, cognitive functions involving action and cognitive functions involving perception. The former concern the “motor cerebellum” (and its variations with respect to higher cognitive function), whereas the latter concern the “non-motor cerebellum.” The CSH refers to perception and the posterior cortex and the associated higher cognitive functions, such as schematic inferences in image-based geometrical reasoning— the CSH does not undermine the cerebellar contribution to motor behaviour, in all its variations.

5.6 The “Mental Motor” Cerebellum

In the various theories of the “mental motor” cerebellum, higher cognitive functions are understood with respect to operations of the prefrontal cortex. By analogy, ideas are manipulated as representations in the prefrontal cortex in the same way that objects are manipulated by the hands via the primary motor cortex. The cerebellum facilitates skilful operations in either case.

The mental-motor cerebellum is a variation of the motor cerebellum, discussed in Section 5.5, above, and as such it still lies within the motor-cerebellum paradigm. It is interesting to compare the mental-motor cerebellum with the CSH. The former does not yet take account of the posterior cerebrocerebellar loops, and neither, to my mind, does it dissociate with clarity the action and perception moieties of cognition.

5.6.1 Skilful Manipulation of Ideas

The notion of skilful manipulation of ideas originated with Dow’s (1974, 1988) early investigations of the dentate nucleus. The lateral dentate, or neodentate, is phylogenetically more recent than the medial dentate nucleus. Dow (1988) points out that the neodentate and the expanded lateral zone of the cerebellar cortex appeared at the same time as the rapid expansion of the anterior and posterior association areas of the cerebral cortex, and that the two developments are anatomically and functionally related. He notes that lesions of the lateral zone of the cerebellar cortex and neodentate alone do *not* produce the classical signs of motor dysfunction. Consequently, it is reasonable to suppose that these regions of the brain are responsible for cognitive functions other than those of motor control. Dow’s hypothesis was developed in Leiner et al. (1986, 1987, 1989, 1991, 1993) and Leiner and Leiner (1997).

The seminal article is Leiner et al. (1986), in which the authors write, “The hypothesis states that in the human brain the newest cerebrocerebellar loops could contribute to skilled mental performance in much the same way that the older loops contribute to skilled motor performance” (p. 443). They continue,

> [A] detailed examination of cerebellar circuitry suggests that its phylogenetically newest parts may serve as a fast information-processing adjunct of the association cortex and could assist this cortex in the performance of a variety of manipulative skills, including the skill that is characteristic of anthropoid apes and humans: *the skillful manipulation of ideas.* (p. 444, italics added)

By “association cortex” in this context, it is clear that the authors are referring to the anterior association cortex. In other words,

> [T]he cerebellum functions, in effect, as a general-purpose computer whose special-purpose applications can differ in each species, depending on the input-output connections that evolved between the cerebellum and the other parts of the brain. Connections with the brain stem could enable the cerebellum to function as an adaptive control device, regulating motor reflexes such as the vestibulo-ocular reflex. Connections with the sensorimotor neocortex could allow the cerebellum to extend its adaptive capabilities to programming the manipulations of hand or speech muscles. Finally, *the connections with the prefrontal cortex could allow the cerebellum to extend such programming to the ideational manipulations that precede planned behavior.* (p. 450, italics added)

Leiner et al. (1987) clarify,

> Through its communications with the [pre-]frontal cortex, then, the human cerebellum may be contributing to the dexterity with which the cerebral cortex executes procedural skills, both motor and mental. Such a dual cerebellar capability is consistent with information-processing theory, which holds that motor capabilities and mental capabilities both can be produced by the same information-processing mechanisms. (p. 428)

These researchers are still, as it were, viewing the cerebellum purely as a motor control device, except that the manipulation of ideas is “motor control of ideas.” This is a big step forward, but it does not address the existence and function of posterior cerebrocerebellar loops. One of the ideas in this dissertation is that higher cognitive functions, correlative to activity in the posterior cortex, are not even faux motor functions: perception of a triangle is just that, with no motor component.

5.6.2 Internal Models of Mental Models

Ito (1993, 2006, 2008) considerably advanced the theory of the role of the cerebellum in non-motor behaviour. Ito (2006) makes special note of the anterior cerebrocerebellar loop from BA 46 in the prefrontal cortex. He argues that this loop supports the hypothesis that the cerebellum is a location for forward models from regions of the anterior cortex other than those directly implicated with motor behaviour. Accordingly,

> The internal model hypothesis has been expanded conceptually to apply to cognitive functions such as thought . . . . This capacity can be considered to be a manipulation of mental models that are small-scale models of reality. These may be used by the mind to anticipate events requiring reason and an explanation. (p. 295)

Ito (2006) proposes that “a mental model of an image, idea, or concept is formed in the temporoparietal association cortex” (p.295). Subsequently, this mental model is manipulated as an object in the prefrontal cortex “just as the motor cortex moves body parts during voluntary movements” (p. 295). Ito, therefore, at least suggests the importance of the posterior cortex for higher cognitive functions. However, he does not directly address the posterior cerebrocerebellar loops, and he is still working within the paradigm of the cerebellum as a motor control device. Therefore, the mental models in the posterior cortex have to be transposed to the anterior cortex first before the cerebellum becomes involved.

5.7 Several Other Theories of the Functional Role of the Cerebellum

This section discusses several possibilities that, to a greater or lesser extent, are disjoint from the motor cerebellum paradigm. The content of Section 5.7 is, strictly speaking, peripheral to the main line of argument. However, it is necessary, for the sake of completeness, to review various alternative theories of the functional role of the cerebellum. I hope the reader will agree that the CSH fits comfortably with the alternative theories discussed in this section and throughout Chapter 5 as a whole.

Despite the traditional view of the cerebellum as facilitating motor behaviour, whether “mental motor” behaviour or not, many researchers have suggested alternatives. Abbie (1934) investigated the connection from temporal cortex to the cerebellum via the pons. He proposed that the cerebellum combined sensory data into a coherent whole. According to Schmahmann (1997),

> It seems unlikely that Abbie had considered a direct role for the cerebellum in [non-motor] cognitive tasks, but he drew attention to at least the possible contribution of the cerebellum to motor performance that incorporates or reflects a creative [i.e., perceptual] purpose. (p. 9)

On the basis of primate studies, Prescott (1971) concluded that the cerebellum participates in emotional processes. Minimal sensory stimulus, he hypothesized, prompts the cerebellum to generate unusual movement patterns, such as psychopathological rocking and head banging. Heath (1977) showed that electrical stimulation of the cerebellum can moderate the negative emotions of psychiatric patients. By the 1970s, it had become apparent that a re-evaluation of the orthodox view of the cerebellum as being limited to motor function was overdue (Schmahmann, 1997). Schmahmann (1991), Desmond and Fiez (1998), Paquier and Mariën (2005), Timmann and Daum (2007), and Strick et al. (2009) all review various non-motor theories of the cerebellum. I discuss several of these theories below.

In overview, Section 5.7.1 reviews the theory that the cerebellum is involved in the timing of motor activities, which is in accord with my view of the temporal aspect of anterior-cortex activity and the known connections between cerebellum and anterior cortex. The temporal patterns of Section 5.7.1 may be regarded as special cases of a more general sequencing function of the cerebellum, which is reviewed in Section 5.7.2. Sequencing also includes aspects of verbal processing, which is the topic of Section 5.7.6. Sequencing, moreover, seems to have a perceptual component and to be related directly to the CSH. The cerebellar contributions to working memory and procedural learning are discussed in Sections 5.7.3 and 5.7.5, respectively. Sections 5.7.4 and 5.7.7 discuss the role of the cerebellum in sensorimotor integration and visuospatial ability, respectively. Both of these proposed functional roles of the cerebellum seem to be related to the classical understanding of the cerebrocerebellar connectivity, discussed in Section 5.3.5, above. The more one studies the incredibly diverse literature of cerebellar research, the more one acquires a sense of its basic unity.

5.7.1 Timing

According to Keele and Ivry (1990), timing is a general computational process that needs to be applied in a variety of tasks. These tasks include rhythmic movements of the body, judgment of duration, and comparisons of the velocities of moving objects. The authors proposed that these tasks utilize a common neural system for temporal computation. They proposed, on the basis of lesion studies, that the neural system responsible for timing was the cerebellum or the cerebellum and related structures.

Cerebellar patients were tested on their accuracy to perceive time intervals or to produce time intervals by tapping. Keele and Ivry (1990) established that these cerebellar patients demonstrated deficits in both types of task. They concluded,

> The general thesis . . . is that it may be fruitful to approach the brain in terms of computations that are demanded by different tasks. We have suggested that one such basic computation is timing, and we think evidence implicates the integrity of the cerebellum as critical for accurate timing. (p. 203)

Bengtsson et al. (2005) studied the production of temporal patterns by healthy subjects, whether by tapping with the left hand or right hand or by vocalization. For a given subject, the authors observed that the same pattern of deviation from the ideal was present independently of how the temporal pattern was produced. They proposed therefore that subjects use a single representation of the temporal pattern, no matter how the pattern is manifested. By means of an FMRI study, the authors were able to localize neural activity during the production of rhythmic sequences to the cerebellum, as well as to other cortical and subcortical brain regions. The authors concluded, “[These brain regions] constitute a neural substrate for the everyday experience that a well-learned rhythm can as easily be produced with the hand, with the foot or by singing” (p. 3264).

Cognitive networks of the anterior cortex have a temporal component, as is clear from the discussion in Section 4.1. The temporal aspect of timing would therefore imply involvement of the anterior cortex. Indeed, the timing hypothesis may be a particular way of looking at the motor cerebellum, rather than a non-motor theory of the cerebellum, per se. According to Strick et al. (2009),

> With the loss of precise timing information and control, motor commands and internal cognitive states may no longer be appropriately selected and sequenced at a fine level. Thus, motorically, an individual may become less coordinated, and, cognitively, they may exhibit problems with task-shifting and other forms of executive control. (p. 425)

It is clear that timing and motor behaviour are intimately related, and it may be one of those cases, discussed above, where it is difficult to separate motor and non-motor aspects of cerebellar function.

5.7.2 Sequencing

Sequencing is the capacity to arrange items in order, whether verbally, spatially, or behaviourally (Leggio et al., 2008). Inhoff et al. (1989) demonstrated a slowed reaction time in generating sequences in patients with cerebellar dysfunction. Leggio et al. also found that cerebellar patients were deficient in sequencing ability. Specifically, patients with left cerebellar lesions were affected more spatially, whereas patients with right cerebellar lesions were affected more verbally.

Leggio et al. (2008) suggested that sequence processing was the cerebellar contribution to higher cognition. Moreover, the authors proposed that their results with respect to localized differences in sequencing ability were evidence for the closed-loop interpretation of cerebrocerebellar connectivity, discussed above:

> [T]he presence of right/left and pictorial/verbal differences is in agreement with the idea that cerebro-cerebellar interactions are organized in segregated cortico-cerebellar loops in which specificity is not related to the mode of functioning, but to the characteristics of the information processed. (p. 1332)

Molinari et al. (2008) reiterate the hypothesis that sequence detection might represent the main contribution of the cerebellum to brain functioning, where sequencing is understood as “ identifying patterns that allow a response to be anticipated” (p. 611). According to the authors, this general mechanism would apply to simple motor responses such as eye blinking as well as to complex social behaviour. They write,

> [T]he cerebellum compares sequences of incoming stimuli to detect similarities and differences in order to predict what is coming next or to alert if a prediction is not fulfilled. If this is the basic operational model of the cerebellum, then its fingerprints should be recognizable in all domains of cerebellar functions. (p. 613)

Molinari et al. (2008) specifically discuss connections between cerebellum and parietal cortex for sequencing. With respect to the posterior cerebrocerebellar loops, outside the temporal domain, “detection of similarities and differences” sounds quite similar to the notion that the cerebellum identifies those properties of a concept that are constant across multiple presentations versus those properties that vary across multiple presentations. In other words, the cerebellum would distinguish between essential properties and incidental properties.

5.7.3 Working Memory

According to Molinari et al. (2008), for the cerebellum to be engaged in sequencing tasks, “detecting similarities and differences” among incoming stimuli, there should be a cerebellar role with respect to working memory. The authors write, “[I]t is tempting to hypothesize a specific importance of cerebellar processing in maintaining short-term memory buffers” (p. 614).

According to Ben-Yehudah et al. (2007), evidence across multiple studies suggests that the cerebellum is indeed involved with verbal working memory, and that the cerebellum is part of an articulatory rehearsal system. For example, in a neuroimaging study by Fiez et al. (1996), participants were asked to keep five words in memory silently. Greater activity was found in the prefrontal cortex and cerebellum when compared with a control task.

To account for the cerebellar role in verbal working memory Ben-Yehudah et al. (2007) consider “two general theories of cerebellar function, error-driven adjustment and internal timing” (p. 199). By “error-driven adjustment” the authors are referring to “detection of a discrepancy between predicted and actual sensory consequences” (p. 199). It appears that these two theories refer to the topics of the previous two sections, sequencing and timing, respectively. The authors propose that both theories of the cerebellum can be understood as special cases of a theory of “the cerebellum as a predictive device in various cognitive processes” (p. 199).

Working memory, sequencing, and timing, as theories of the functional role of the cerebellum, are mutually implicated. It is difficult to separate the timing aspect of working memory from its sequencing aspect. And sequencing and timing both, apparently, need working memory. When reviewing the various theories of the functional role of the cerebellum, one often has the sense that these theories are not disjoint, but different perspectives on a single function.

5.7.4 Sensorimotor Imagery

The role of the cerebellum in sensorimotor integration has already been discussed with respect to the classical understanding of cerebrocerebellar connectivity. The notion of Gao et al. (1996) is that it is very difficult to dissociate the motor cerebellum from its role in sensation. Indeed, Bower (1997) asks whether the cerebellum is primarily motor, and then the motor cerebellum supports sensory discrimination; or whether it is primarily sensory, and then the sensory cerebellum supports motor behaviour.

This section discusses now the idea that the cerebellum facilitates sensorimotor imagery. According to Kilner et al. (2004) sensorimotor imagery, such as imagined speech, utilizes many of the same neural processes as actual motor behaviour or actual speech. The cerebellum is implicated in motor behaviour or speech that is acted out, or expressed, and so the cerebellum should also be involved in imagined movement or imagined speech.

In an FMRI study, Hanakawa et al. (2008) showed that the cerebellum is active during imagined finger tapping. Similarly Ackermann et al. (2007) demonstrated involvement of the cerebellum in imagined speech. They write,

> [A] prearticulatory verbal code bound to reciprocal right cerebellar/left frontal interactions might represent a common platform for a variety of cerebellar engagements in cognitive functions. The distinct computational operation provided by cerebellar structures within this framework appears to be the concatenation of syllable strings into coarticulated sequences. (p. 202)

It is significant that Ackermann et al. (2007) refer to reciprocal, closed-loop circuits in their paper. Moreover, the sequencing cerebellum of Section 5.7.2 is present as an aspect of the "prearticulatory verbal code."

With respect to the variety of cognitive tasks engaged by the cerebellum due to its involvement in imagined speech, Strick et al. (2009) explain that verbal encoding by humans underpins a variety of cognitive tasks, not just speech for communication, per se: "[I]ntuitively, [one uses] one's internal (imagined) voice to represent, maintain, and organize task-relevant information and conscious thoughts" (p. 426). The language capacity of humans, according to the authors, is pre-eminent for a variety of higher cognitive functions.

5.7.5 Procedural[18] Learning

The sequencing role of the cerebellum and the cerebellar contribution to working memory tend towards the idea that the cerebellum is involved in learning. According to Doyson (1997), "Skill learning (also called 'procedural memory') refers to the capacity to acquire an ability through practice" (p. 274). The difference now, in comparison with the earlier discussions, is that procedural learning applies to long-term, rather than short-term, acquisition of a skill. Note also that acquisition of the skill in procedural learning may be implicit, in that it does not require conscious recall for its utilization.

Doyson (1997) reviews a number of results obtained by neuroimaging, concluding that the cerebellum "is critical for the acquisition of skills in motor and

---

[18] Note that "procedure" here is not identical with the technical use of "procedure" in Chapter 4, although there is substantial overlap.

visuomotor modalities" (p. 287). Doyson remarks also on the phenomenon that cerebellar activation decreases upon acquisition of a skill, which is discussed in the context of linguistic processing in the previous section. He remarks that the decrease in activity may be due to methodological disparities, but

> There is however, an alternative interpretation to be found within the framework of the phases of skill learning. . . . The profile that emerges from [the studies cited] is that the cerebellum is active in the early 'composition' and 'proceduralization' phases of learning, but that its level of activation drops significantly when subjects have achieved an asymptotic level of performance or when subjects are performing a task that is overlearned. At the same time this reduction in cerebellar activity appears to be coupled to an increase in [activity] in specific cortical and/or subcortical regions. (p. 289)

According to Molinari et al. (1997) a review of the literature indicates that the cerebellum contributes to spatial information processing. The authors conducted experiments on rats with partial cerebellectomies. The rats were slow to acquire the ability to navigate a maze, a task which combines spatial information processing with procedural memory. The authors concluded that there is a cerebellar role particularly in the procedural aspects of spatial functions.

5.7.6 Linguistic Processing

Sections 5.7.2 and 5.7.4 indicated a role for the cerebellum in language. Now, language has both anterior cortex and posterior cortex components, in execution and understanding of language, respectively. The papers by Petersen et al. (1989), Fiez et al. (1992), and Raichle et al. (1994) demonstrate that the right lateral cerebellum is involved in linguistic processing, the higher language functions.

By means of a PET study, Petersen et al. (1989) found activity in the right lateral cerebellum when participants were asked to generate verbs from nouns, presented auditorily or visually. Lateral hemisphere activity was not present when participants were reading aloud, nor when they were repeating nouns that they had heard. According to the authors, these results indicate a “cognitive” rather than sensory or motor role for the cerebellum in the generation of verbs from nouns.

The results of Petersen et al. (1989) were substantiated by Fiez et al. (1992) in a study of a patient with a lesion to the right lateral zone of the cerebellum. The patient was unable to generate verbs from nouns efficiently.

Raichle et al. (1994) took the results of Petersen et al. (1989) and Fiez et al. (1992) a step further by investigating the effects of learning. In other words, participants were allowed to practice generating verbs from a given list of nouns. After ten minutes of practice, activity decreased in the right lateral cerebellar hemisphere, as determined by PET methods. In addition, however, activity in the left prefrontal cortex, thought by the authors to be reciprocally connected to the right lateral cerebellum, also decreased. This result is consistent with the notion that incidental properties are suppressed and essential properties, differentially, are enhanced.

5.7.7 Visuospatial Ability

Sequencing, timing, working memory, sensorimotor imagery, and procedural learning are all involved in motor behaviour at one level or another, which implicates the anterior cortex. In the previous section on linguistic processing, both action and perception are implicated, anterior cortex and posterior cortex. All of the ostensibly non-motor theories of the functional role of the cerebellum investigated so far seem to involve a motor component, even if that motor component is motor imagery.

On the other hand, “visuospatial ability” seems to be a candidate for a direct role for the cerebellum in the posterior cortex. After all, the visual system lies entirely within the posterior cortex.

Molinari et al. (2004, reviewed in Molinari and Leggio, 2007) is a typical study, in which patients with cerebellar lesions have to complete various visuospatial tasks. These tasks include the Benton Line Orientation Test, in which a line has to be matched to one of a set of lines that has the same orientation; the Minnesota Paper Form Board Test Revised, in which two halves of a partitioned object are given and they have to be matched to one of five presented whole objects; and a differential spatial aptitude test, in which an unfolded cube must be matched to one of four three-dimensionally represented cubes. The authors also gave a fourth test of more general ability, the Wechsler Adult Intelligence Test Revised, but their analysis focused on visuospatial aspects of this test, such as the picture arrangement task. Cerebellar patients, it was found, were deficient in their ability to complete the various visuospatial tasks.

A careful review of the experimental paradigm indicates that all of the tests of visuospatial aptitude require motor imagery. As such, they could have been discussed in Section 5.7.4. The authors themselves conclude, “[L]esions of the cerebellar circuits affect visuospatial ability. The *ability to rotate objects mentally* is a possible functional substrate of the observed deficits” (Molinari et al., 2004, p. 235, italics added). “Visuospatial ability,” in this sense, is a form of sensorimotor imagery, and is implicated in motor behaviour.

5.8 Dysmetria of Thought

The discussion of the various motor, “mental motor,” and other theories of the functional role of the cerebellum has not yet uncovered a direct contribution to cognition for the posterior cerebrocerebellar loops alone. Action has been overt or implicit in every case. There *is* a theory of the functional role of the cerebellum that appears to isolate perception and the posterior cortex. Before discussing that theory, however, I would like to address now the ideas of Schmahmann and colleagues on dysmetria of movement and dysmetria of thought.

As I mentioned earlier, one has a sense, in studying the various theories of the functional role of the cerebellum, that a common thread runs through them all. They are different ways of understanding a single, overarching cerebellar function, the universal cerebellar transform. And this is exactly what one would expect, of course, given the uniformity of cerebellar histology and the idea that function follows structure.

The cognitive affective syndrome (Schmahmann & Sherman, 1998) and dysmetria of thought (Schmahmann & Pandya, 1997) may be regarded as a failure of the universal cerebellar transform with respect to non-motor cognition. I propose that dysmetria of thought interpreted for posterior cerebrocerebellar loops is none other than a failure to schematize concepts and percepts. This correspondence between the two theories of the functional role of the cerebellum is another argument in favour of the plausibility of the CSH.

Schmahmann and Sherman (1998) investigated patients who had sustained cerebellar damage. Their methods consisted of neurological examinations, bedside clinical tests, and neuropsychological tests. The authors identified a pattern of behavioural abnormalities that they dubbed the *cerebellar cognitive affective syndrome* (see also Schmahmann, 1998). The cerebellar cognitive affective syndrome includes deficiencies in executive function (e.g., planning, abstract reasoning), spatial cognition, personality (e.g., blunting of affect or disinhibited and inappropriate behaviour), and language. According to the authors, “The net effect of these disturbances in cognitive abilities appears to be a general lowering of intellectual function” (p. 562).

Note that Schmahmann and Sherman’s (1998) cerebellar cognitive affective syndrome concerns patients with *acquired* cerebellar lesions. However, Tavano et al. (2007) confirm a remarkably similar syndrome in patients with *congenital* cerebellar malformations.

The conclusions of Schmahmann and Sherman (1998), on the significance of the cerebellum for non-motor cognition, are reinforced by Schmahmann (2004) in a summary of research which utilizes functional imaging studies on healthy participants:

> [N]onmotor domains that invoke a cerebellar contribution include sensory processing and discrimination, mental imagery, motor learning, classical conditioning, nonmotor learning and memory, linguistic processing, attentional modulation, timing estimation, emotion perception and experience, visual spatial memory, executive function (including verbal working memory, strategy, reasoning, and verbal fluency), and autonomic functions including the experience and anticipation of pain, thirst, hunger, and smell. There is evidence to suggest that these functions are under the control of different areas within the cerebellum. (p. 374)

Schmahmann and Pandya (1997) proposed the “dysmetria of thought hypothesis” by analogy to the motor dysmetria that often accompanies cerebellar lesions. According to Schmahmann (2004), “[T]here is a universal cerebellar impairment, namely, dysmetria. This includes dysmetria of movement—ataxia, and dysmetria of thought and emotion—the cerebellar cognitive affective syndrome” (p. 375, Table 3).

Accordingly, something similar to motor dysfunction may apply to diverse areas of cognition. Application of the term “dysmetria” to both motor and non-motor cognition exhibits a desire to find a common ground for explaining the role of the cerebellum in all forms of cognition, the universal cerebellar transform. The meaning of dysmetria is clarified as follows by Schmahmann and Pandya (1997):

> [T]he role of the cerebellum in cognition, affect, and autonomic function is viewed to be one of modulation rather than generation, i.e., *the cerebellum serves as an oscillation dampener, maintaining function steadily around a homeostatic baseline and smoothing out performance* (Schmahmann, 1996). . . . It may also transpire that in the same way as the cerebellum regulates the rate, force, rhythm, and accuracy of movements, so it may regulate the speed, capacity, consistency, and appropriateness of mental or cognitive processes. (p. 55, italics added)

Now, in their presentation of dysmetria, Schmahmann and colleagues do not determine that dysmetria of movement belongs to the anterior cortex, whereas dysmetria of thought belongs to the posterior cortex. In fact, it appears that they refer dysmetria of thought and dysmetria of movement both to the anterior cortex. They do not directly interpret the function of the posterior cerebrocerebellar loops.

It should be noted that my framework in this dissertation is different from the framework of Schmahmann and colleagues. On the basis of Fuster's (2003) cognitive network theory, I have emphasized the primordial functional divide of anterior cortex and posterior cortex, action and perception, respectively. According to Fuster (2003), there is a vast quantitiy of research in cognitive neuroscience to support this division. Given that dysmetria of thought, has perceptual and conceptual components I cannot delegate dysmetria of thought entirely to the anterior cortex. Within my framework it is necessary to interpret dysmetria of thought with respect also to posterior cerebrocerebellar loops connecting cerebellum and association areas of the parietal cortex. I do not see any problem with this stance, given the uniformity of cerebellar histology (which implies, in some sense, uniformity of cerebellar function), and the existence of these posterior cerebrocerebellar closed-loop circuits. Moreover, the dysmetria of thought hypothesis was developed on the basis of clinical data. There is no reason not to suppose that it involves a dysfunction of posterior cerebrocerebellar loops instead of, or in addition to, anterior cerebrocerebellar loops.

In consequence, I take the italicized phrase in the above quotation to be the essence of dysmetria, in both anterior and posterior cortices. The question now is whether dysmetria of movement, in this sense, is a failure of schematization with respect to the anterior cortex and whether dysmetria of thought, in the same sense, is a failure of schematization with respect to the posterior cortex. If this question can be answered in the affirmative, then the CSH, as far as possible, is aligned with the dysmetria hypothesis.

With regard to action, the cerebellum as an "*oscillation dampener, maintaining function steadily around a homeostatic baseline and smoothing out performance*" seems to imply the temporally coherent evolution of acts from procedures, which, as explained in Chapter 4 (pp. 166-167), is a role of the cerebellum with respect to the anterior cortex. Failure of this function means that the procedure is not adequately present as a "*homeostatic baseline*," and individual acts resolve erratically from the procedure rather than "*smoothing out performance*" of the procedure.

Analogous reasoning works to explain schematization as an expression of dysmetria of thought with respect to the posterior cortex. Dysmetria of thought as an "*oscillation dampener*" or a regulator of the "speed, capacity, consistency . . . of mental or cognitive processes" seems to involve a temporal component.

However, the “*homeostatic baseline*” may be regarded in terms of temporal stability rather than temporal coherence. After all, schematization focuses attention on essential properties, and the essential properties of the concept have only one way to resolve to actual properties of the percept, moment by moment. In addition, the schematized concept and percept have fewer properties attended to than the unschematized concept and percept, and there is less potential for variability, moment by moment, as to which properties are the focus of attention. In other words, schematization may act either to stabilize the values of properties in the percept or to stabilize the particular properties that are attended to. Consequently, the cerebellum that schematizes concepts and percepts facilitates temporal stability of the percept, “*smoothing out* [perceptual] *performance*.” On the other hand, the “*homeostatic baseline*” may reflect the learned ability to perceive the essence of a given situation, and the ability not to get lost in a morass of distracting detail. A circle is a circle is a circle, or so it is perceived—a “*homeostatic baseline*”—no matter the incidental context in which it is embedded.

I have made some creative interpretations, but it is clear that a failure of schematization is possibly an expression of dysmetria. This new argument for the plausibility of the CSH is that schematization, therefore, is supported by the same set of results, primarily lesion studies, which have led to the dysmetria of thought hypothesis. In other words, cerebellar lesions are associated with dysmetria of thought and therefore also associated with a dysfunction of schematization in the posterior cortex.

5.9 Selective Attention as a Functional Role of the Cerebellum

The dysmetria of thought hypothesis and the cerebellar cognitive affective syndrome is an illuminating approach to the universal cerebellar transform. However, dysmetria of thought, as originally presented, is still concerned with the temporal evolution of cognition, which suggests the anterior cortex. So are there theories, other than dysmetria, that can isolate the role of the posterior cortex? An extensive search of the literature uncovered only one such candidate: selective attention in perception. This is particularly promising, because schematization was identified with attention in Section 4.5.

Akshoomoff and Courchesne (1992) conducted experiments with patients with cerebellar lesions. They investigated two types of attention. Firstly, in focused attention participants must concentrate on one modality, and respond to stimuli in this modality, ignoring all other stimuli. Secondly, in selective attention, participants have to shift attention quickly and voluntarily between modalities. (According to the classification of types of types of attention by Sohlberg and Mateer,1989, reviewed in Chapter 4, the second type of attentional task should more properly be referred to as “alternating attention.”)

The researchers found no difference between cerebellar and non-cerebellar participants with respect to focused attention, but with respect to “selective” attention, performance of the cerebellar patients was significantly impaired. They *were* able to shift attention, but it took longer. The authors claimed that their experimental method isolated the attentional role of the cerebellum from its role in motor control. In other words, it isolated the role of the cerebellum with respect to the posterior cortex. According to Akshoomoff and Courchesne (1992),

> It is an interesting possibility that if the neocerebellum affects the control of shifts of attention, it may do so in a fashion comparable to, but relatively independent of, its role in the control of movement. The neocerebellum may, perhaps, affect the rapid, smooth, and precise performance of cognitive operations as it does motor operations. . . . Perhaps in the cognitive realm, this structure helps us to effortlessly shift from one domain of thought to another. (p. 737)

Courchesne et al. (1994) come closer to explaining the meaning of cerebellar activity with respect to the posterior cortex in conceptual terms. Accordingly,

> It seems possible, therefore, that within the domain of sensory processing, the cerebellum . . . performs operations that optimize the neural signal-to-noise conditions in whichever systems (e.g., brainstem, thalamic, cerebral, and hippocampal) will be involved in processing such events. Selective attention may involve an analogous anticipatory enhancement of signal (the desired, to-be-attended information) relative to noise (the information to be ignored). (p. 860)

They continue,

> We . . . propose that the cerebellum optimally shifts excitability thresholds in neurons likely to be used in any sensorimotor or mental action. That is, the cerebellum adjusts responsiveness in whatever neural array or network is anticipated to be needed to attain a prescribed goal (the goal perhaps being prescribed by cerebral cortical or other subcortical systems). Optimal preparation of neural networks needed to achieve such goals may require the cerebellum to implement a succession of precisely timed and selected changes in the pattern or level of neural activity in diverse networks. The patterns chosen and the time implemented hinge on the goals, current context, and anticipated intervening events. (p. 861)

In this proposal, the authors are approaching the cognitive network interpretation of attention discussed in Chapter 4. The “goals” in this respect do not have to be relevant to motor behaviour in the external world, but “prescribed by cerebral cortical or other subcortical systems." If the "goal" is concerned with recognition of a particular concept or percept, and if the cerebellum differentially adjusts the excitability of cognitive networks in order to achieve this goal, then the result is to direct attention to those aspects of perception that facilitate recognition. Another way of describing this process is to say that a concept or percept is schematized.

Now, Courchesne et al.'s (1994) experimental method involves the shifting of attention between *different* modalities. However, image-based geometrical reasoning remains in the *visual* modality, and schematization is accomplished by selective attention to certain properties in this modality.

Townsend et al. (1999) discuss shifting attention within the *visual* modality according to visual attention cues. Cerebellar patients were less efficient in redirecting visual attention. Attentional shift, here, however, involves alternative spatial locations, rather than looking within an object to recognize a shape in the exclusion of incidental properties. It is still not exactly the schematization described in Section 4.5. Moreover, once again, the type of attention required is alternating attention rather than selective attention as such. According to the model of Sohlberg and Mateer (1989), selective attention is prior to alternating attention on the hierarchical scale of attentional process. However, even if the cerebellum is responsible for the jump from selective attention to alternating attention, there is no guarantee that the cerebellum is also responsible for selective attention per se.

Fortunately, research by Allen et al. (1997) really does seem to isolate selective attention with respect to the properties of visual percepts (and therefore visual concepts). This study was conducted by FMRI brain imaging techniques on healthy patients. In order to isolate neural activity due to non-motor cognition, the researchers conducted three experiments: (1) a visual attention task not requiring motor activity, (2) a motor activity task, and (3) a task combining visual attention and motor activity. The visual attention task is particularly relevant:

> During the Attention task, circles, squares, or triangles in red, green, or blue were presented one at a time at a single spatial location in the center of foveal vision (*17*). This task tested the ability to attend selectively to targets (squares or red shapes) within a visual dimension (form or color). Subjects were instructed to silently count each target stimulus, which required attention to visual stimuli in the absence of a motor response. (p. 1940)

In other words, the subjects had to concentrate attention either on the shape of the stimulus or on its colour.

According to Allen et al. (1997),

> We found evidence of a classic double dissociation in structure and function between areas of the cerebellum: Visual attention activates one anatomic location within the cerebellar cortex, whereas motor performance activates a distinctly different location. Moreover, attention activation can occur independently of motor involvement. (p. 1940)

The region of the cerebellum activated for attention was distinct from the region of the cerebellum activated for motor behaviour. The authors continue,

> Our results demonstrate that such cerebellar preparatory influences can occur independently of motor involvement. In the Attention task, attention to sensory information alone was sufficient to activate the cerebellum, and engagement of the motor system was not necessary to produce cerebellar activation. Cerebellar attention activation occurred even though no motor learning was required; no motor response selection, error detection, or error correction was required; no imagined motor action was required; and no guidance of motor systems was required. In sum, these findings are contrary to the expectation of traditional theories of the cerebellum as a motor control system (*6*). (p. 1941)

Now, shape and colour are quite different properties, requiring activation of distinct parts of the visual system, whereas the schematization for geometry largely concerns shape alone. Nevertheless, the paradigm of Allen et al. (1997) really does seem to refer to selective attention, the ability to filter distracting information and focus only on certain aspects of a situation. The authors, therefore, come very close to identifying the cerebellum as a mechanism for schematization in the posterior cortex that would support my proposed schematic nature of image-based geometrical reasoning.

Haarmeier and Thier (2007) critique the evidence of Allen et al. (1997) and others that the cerebellum has a role in attention because they argue that the role of the cerebellum in attention has been confounded with demands for occulomotor, motor, and/or working memory that typically accompany attention. However, they do not demonstrate that the cerebellum does *not* have a role in attention. Moreover, Allen et al. controlled carefully for motor involvement, and regions of the cerebellum known to contribute to occulomotor activity are phylogenetically ancient and quite distinct from those regions in the cerebellum implicated by Allen et al. in attention.

The possibility that Allen et al. (1997) confounded attention with working memory is another issue. It is possible that the cerebellar activity they detected may have been related to working memory or procedural learning (C. E. MacLeod, personal communication, May, 2010). With respect to the former, as discussed in Chapter 4, the neurophysiological model of Knudson (2007) identifies working memory with focused attention. Focused attention is lower than selective attention in the hierarchical scale of Sohlberg and Mateer (1989), but nevertheless, it *is* a type of attentional mechanism. With respect to the latter, it is true, as discussed earlier in this chapter, that the cerebellum has been identified as contributing to procedural learning. Actually, because of the counting involved, I would rather identify the confounding feature of the paradigm as sequencing, which is also discussed as a functional role of the cerebellum earlier in this chapter. In any case, the possibility remains that the cerebellar activity detected by Allen et al. was concerned either with procedural learning or sequencing rather than selective attention per se.

It is a difficult problem: to design an experimental paradigm that isolates selective attention from all other cognitive processes. The experiment of Schweizer et al (2007) appears to accomplish the task of isolating the functional role of the cerebellum in attention from all other cognitive tasks, although the type of attention concerned is not selective attention. The researchers investigate the role of the cerebellum in the “attentional blink.”

According to Schweizer et al. (2007), the attentional blink paradigm has been used extensively in cognitive neuroscience to investigate attention. If objects are presented to vision rapidly and serially, the participant may not be able to notice two identical objects that are presented close together. This is the attentional blink. In line with the dichotomy I have pursued in this dissertation between the anterior cortex and the posterior cortex, between action and perception, the attentional blink appears to be entirely a visual, posterior-cortex phenomenon. The attentional blink is unconscious, which appears to rule out any question of executive control and the prefrontal anterior cortex. According to the authors, an explanation of the attentional blink is that attention is captured by the first stimulus followed by a brief period of around half a second before attention can be re-allocated to process the second stimulus.

Schweizer et al. (2007) investigated the attentional blink in normal, control subjects and in subjects with cerebellar lesions. Subjects were presented with a rapid sequence of stimuli. The authors found that the attentional blink was heightened in cerebellar patients, as measured by their decreased ability to identify repeated stimuli within the typical period of the attentional blink, around 500 ms. However, they also found that the typical attentional blink period of around 500 ms was not extended in cerebellar patients. The authors concluded that there is a cerebellar role in the attentional blink, but its effect is within ~400-500 ms. Furthermore,

> The results of the current AB [attentional blink] experiment provide evidence for a role of the cerebellum in attention independent of motor impairment at any level. . . . Thus, impairment cannot easily be attributed to simultaneous demands for motor or action planning. . . . All stimuli were presented in central fixation, thus impairment cannot be due to a dysmetria of saccades. . . . None of the patients had any extracerebellar lesions. (p. 3073)

Schweizer et al. (2007) noted that there was no impairment of the ability to recognize a stimulus the first time that it occurs. The only impairment in cerebellar patients was in identification of the stimulus when it was presented rapidly after the first stimulus. According to the authors this is evidence that cerebellar lesions do not affect selective attention. However, we need to be careful in identifying exactly what form of attention is involved. According to the hierarchical classification of Sohlberg and Mateer (1989), the type of attention involved in identifying the stimulus the first time it occurs is focused attention and not selective attention—there is no question of picking out the stimulus from among distractors.

The same point with respect to selective attention is at issue in the research of Gottwald et al. (2003). The authors used an experiment to measure ability for selective attention among cerebellar patients. Five stimuli were presented in random order, among which were two target stimuli. Participants had to depress a response key as rapidly as possible upon detection of the target stimuli. There was no significant difference between the cerebellar patients and healthy control participants. Once again, however, there were no distractors, indicating that there was no selective attention, according to the definition of Sohlberg and Mateer (1989).

Although Schweizer et al. (2007) seem to have isolated the role of the posterior cortex and the cerebellum in attention, their research does not specifically identify selective attention. My hypothesis is that the cerebellum is concerned with selective attention and the posterior cortex. Allen et al. (1997) come very close to directly substantiating the CSH, although there are potentially confounding factors.

What experimental paradigm could identify (or not) a cerebellar involvement in the posterior cortex with regard to selective attention specifically for the schematic nature of image-based geometrical reasoning, without potentially confounding factors? Selective attention in image-based geometrical reasoning would require focused attention on a particular aspect of a geometrical diagram in the face of distracting information from within the diagram. My suggestion is that a participant should be instructed in advance to attend to the triangle within a diagram, which consists of a triangle embedded among incidental configurations. To substantiate the hypothesis, there should be concurrent neural activation in the posterior parietal cortex and the lateral cerebellar cortex. To falsify the hypothesis, concurrent neural activation in these brain regions would not be present. There would also be activity in prefrontal regions of the anterior cortex, as the executive decision is made to look for the triangle. There may also be activity in the anterior cortex with respect to eye movement as a triangle is identified in a specific region of the diagram. However, there would seem to be no question within this paradigm of the issue of procedural learning or sequencing. Given the perception-action loop and connections between posterior cortex and anterior cortex via the cerebellum, there is no question that parieto-cerebellar activity would not also initiate anterior cortex activity. There seems to be no question either that the executive decision to search for the triangle would precede the parieto-cerebellar activity that would mark the CSH.

The challenge of my proposed experimental paradigm would be in untangling confounding influences. It may depend on a careful analysis of the time series of neural activation. A hint that this is the case is given by Knudson (2007), who writes,

> [E]ach time we make a saccadic eye movement to a new location, our sensitivity to stimuli at that location increases tens of ms before the eyes move (Shepard et al. 1986). Thus, *orienting eye movements and spatial attention are functionally linked (although separable) in the brain*. (p. 67, italics added)

Thus, perhaps counter to intuition, the neural activity corresponding to selective attention may occur slightly before the neural activity corresponding to eye movement. Intuition would also tell us that neural activity corresponding to the executive decision to search for the triangle should precede the neural activity corresponding to selective attention, but is this really the case? In the end, it makes no difference. As long as there is a distinct window of parieto-cerebellar activity that is temporally distinct from potentially confounding activity within the anterior cortex, the CSH would at least be substantiated, if not confirmed.

A further challenge depends on the requirement for high temporal resolution of the measured neural activity. High temporal resolution is possible with EEG of the cerebral cortex, but for reasons pertaining to signal-to-noise ratios for EEG there are practical difficulties in reliably measuring cerebellar activity (i.e., electromylogenetic artifacts, impedance problems due to hair length, boundary problems, and so on); measurement of cerebellar activity can be more reliably measured with hemodynamic methods such as FMRI, which, however, lack high temporal resolution (S. R. Campbell, personal communication, January, 2009).

This challenge remains to be resolved. At least in principle, however, I can state that the CSH is a scientific hypothesis that can be substantiated or refuted. An experimental paradigm similar to the one suggested in the above paragraphs should identify selective attention accompanied by parieto-cerebellar activity in healthy patients, and a corresponding deficit in selective attention in cerebellar patients.

The (qualified) empirical support for the notion that the cerebellum directs selective attention is one of my last arguments for the plausibility of a functional role of the cerebellum in schematization. It is as close as the existing research comes in providing direct empirical substantiation of the CSH.

5.10 Summary of the Arguments

Scattered throughout the preceding pages are a number of arguments in support of the CSH, which is a proposal concerning the functional role of the cerebellum. The aim of this summary section is to set the CSH in context and to gather the arguments together in one place.

Firstly, with respect to scope, it should be reiterated that the CSH refers only to the functional role of cerebrocerebellar loops connecting the lateral zone of the cerebellar cortex with association areas of the parietal cortex. It says nothing about the classical understanding of cerebrocerebellar connectivity, and its variations, and neither does it say anything about reciprocal connections between the cerebellum and the prefrontal cortex. These alternative, perhaps more mainstream, ways of approaching the functional role of the cerebellum are concerned with motor behaviour or its analogues in higher cognitive function. The CSH concerns perception alone.

Perhaps these alternative ways of understanding the functional role of the cerebellum also have their place in mathematics education. After all, within the framework of embodied cognition, particularly as understood by Lakoff and Núñez (2000), or even Varela et al. (1991), cognition is embodied in an organism's gross physical interaction with the world. This perspective is addressed in Section 1.4, in which I explain my reasons for adopting the radical approach to embodied cognition, in which cognition is embodied primarily in the electrical and chemical activity of the brain.

Nevertheless, cognition as embodied in motor behaviour clearly recalls the motor cerebellum, as discussed earlier in this chapter. It should be noted that mathematical ideas can indeed by embodied in motor behaviour—we walk in straight lines and turn in circles (S. R. Campbell, personal communication, June, 2009). This level of embodiment of mathematical ideas is the foundation of the cognitive metaphor approach of Lakoff and Núñez (2000). The cerebellum, according to the traditional interpretation of its role, is responsible for the smooth, efficient execution of these movements.

On the other hand, embodiment of straight lines and circles with respect to the posterior cortex and the CSH is embodiment at the level of neural activity, resulting in *perception* of straight lines and circles, a higher cognitive function. Embodiment at the level of neural activity in the posterior cortex is my primary concern. The motor cerebellum, I am sure, has its place in the theory of embodied cognition. However, it lies outside of the scope of this dissertation.

I am concerned with the spatial aspects of geometrical reasoning and learning, as manifested in their neural correlates in the cognitive networks of the posterior cortex, within the framework of a radical embodied cognition. Specifically, I am concerned with schematic perception and inferences from geometrical diagrams. The cerebellum *does* connect reciprocally with the regions of the cerebral cortex that are primarily responsible for these high-level cognitive networks. The CSH concerns only these connections. It proposes that the extensional concepts of the posterior cortex are schematized, or made intensional, so that the essential properties of these concepts are emphasized and their incidental properties are de-emphasized.

The arguments in support of the CSH are scattered throughout this long chapter. I close this chapter by bringing together and summarizing these arguments.

5.10.1 Structural Arguments

The structural arguments in favour of the plausibility of the CSH depend on the phylogenesis of the cerebellum, its connections with the cerebral cortex, and its histology. These are indirect, circumstantial grounds for believing the CSH. Without these grounds, however, the hypothesis would be untenable.

Argument from phylogenesis. Some cerebellar and related structures are of very recent phylogenetic origin. Researchers maintain that these structures co-evolved with association areas of the cerebral cortex. There is reason to believe that just as the role of association areas of the cerebral cortex is higher cognitive function, the role of phylogenetically recent cerebellar and related structures is also higher cognitive function. Of course, this argument says nothing about the nature of higher cognitive functions that are facilitated by the cerebellum.

Argument from structural connections. Cerebrocerebellar loops connect the phylogenetically recent cerebellar and related structures to association areas of the posterior cortex, including BA 39 in the inferior parietal cortex. Conceptual thought is associated with this region of the cerebral cortex. There is reason to believe, therefore, that these cerebrocerebellar loops are implicated in conceptual thought. Earlier in this chapter, I did discuss the alternative, classical interpretation of cerebrocerebellar connectivity, and gave my reasons for emphasizing the closed-loop understanding.

Argument from uniformity of cerebellar histology. The cerebellar cortex has a remarkably uniform histology, despite the fact that different regions of the cerebellar cortex connect to various regions of the cerebral cortex that play widely disparate functional roles in cognition. In some sense, the cerebellum must undertake the same role no matter where in the cerebral cortex it connects, because of the uniformity of its histology. This functional role is the universal cerebellar transform. Smooth, efficient motor behaviour of the anterior cortex corresponds to smooth, efficient perception (and conception) in the posterior cortex—schematization.

5.10.2 Substantiating Arguments

The second set of arguments for the CSH concern ways in which other theories of the cerebellum can be interpreted in terms of the CSH. And then all the research that substantiates these theories also substantiates the CSH. I have chosen two theories to concentrate on in this respect, dysmetria of thought and the functional role of the cerebellum in selective attention, although the presentation earlier in this chapter may suggest others. (For example, aspects of the theory of sequencing, discussed above, strongly recall the CSH.)

Argument by interpretation of dysmetria of thought. Dysmetria of thought represents failure of the universal cerebellar transform with respect to non-motor cognition. Lesions of phylogenetically recent cerebellar structures result in dysmetria of thought, producing cognitive dysfunctions that are consistent with an impaired ability to schematize concepts and percepts. (These same lesions do not result in motor dysfunction.) Dysmetria of thought is interpreted therefore as an inability to schematize concepts and percepts, implying that an aspect of the universal cerebellar transform is schematization of concepts and percepts. Empirical evidence to support dysmetria of thought can be conscripted to substantiate the CSH.

Argument from the role of the cerebellum in attention. Schematization may be interpreted as a variety of selective attention. Studies have shown that the cerebellum has a functional role in selective attention. These studies directly substantiate the CSH. Of course, these studies are not perfect, and there may be confounding factors. A confirmation of the role of the cerebellum in selective attention requires further empirical substantiation.

5.10.3 “Seeing the General in the Particular”

There is one final argument that is based on the theoretical framework and the structural similarities assumed to subsist between the neurophysiological and psychological domains. Schematization of concepts and percepts is my reinterpretation of the phrase “seeing the general in the particular” (Mason & Pimm, 1984). This cognitive function is necessary for image-based geometrical reasoning, specifically schematic inferences from geometrical diagrams. *There must be an objective correlate for “seeing the general in the particular.”* The content of much of this dissertation is the pursuit of just such an objective correlate. This final argument attests to its existence. Its nature is addressed in this chapter.

5.11 Conclusion

The CSH is falsifiable and therefore scientific. Like any scientific hypothesis, it is “true” only to the extent of its explanatory power. I believe the hypothesis is not only plausible, but has substantial explanatory power. Further substantiation can be sought by empirical methods. The research of Allen et al. (1997) on selective attention, for example, can be adapted to investigate directly the cerebellar contribution to the schematic nature of geometrical reasoning. If there is an appropriate cerebellar response in healthy participants when engaging the schematic nature of geometrical diagrams, then the hypothesis is substantiated. On the other hand, impairment of the ability for schematic perception and schematic inferences in patients with appropriately located lesions of the cerebellum would also substantiate the hypothesis. These experiments may also refute the hypothesis.

▲►▼◄▲►▼◄▲►▼◄

This chapter has reviewed research on the cerebellum in an attempt to establish the CSH as one way to understand the functional role of the cerebellum. I am going to assume now that the hypothesis has been established. The argument in the next chapter tackles the implications for mathematics education of the proposed functional role of the cerebellum.

CHAPTER 6: MATHEMATICS EDUCATION AND THE CEREBELLUM

I request the reader bear in mind that the aim and contribution of this dissertation lies in the middle ground between education and the neurosciences: in the utlization of the results and theories of one to inform and constrain the results and theories of the other. Ideas in mathematics education inspired development of the CSH, and in turn the CSH should inspire further ideas with regard to mathematics education. By engaging the middle ground between education and the neurosciences my research has resulted in a thesis in *educational neuroscience*.

The development of the theoretical framework in Chapters 3 and 4 culminated in a reinterpretation of “seeing the general in the particular” within the context of Bergson’s theory of duration and cognitive network theory. In my revised interpretation, the phrase “seeing the general in the particular” refers herein to schematization of concepts and percepts, which is a fundamental aspect of mathematical reasoning, specifically the schematic nature of image-based reasoning in geometry. The arguments in Chapter 5 help support the CSH. The aim of the present chapter is to consider some key implications of the CSH for mathematics education.

The CSH applies to concepts and percepts of the posterior cortex. Schematization of concepts and percepts has a natural extension to procedures and acts of the anterior cortex, and indeed some reference to the anterior cortex is necessary for the arguments in Chapter 5. However, my main goal is to understand the schematic nature of image-based geometrical reasoning, which primarily concerns perception rather than action (see p. 109, n. 11) and hence concerns the posterior cortex. In effect, I want to emphasize that the posterior cortex is equally important as the anterior cortex for higher cognitive functions, in particular, image-based reasoning in geometry. In addition, the role of the anterior cortex in temporal integration is more difficult to analyze than the role of the posterior cortex in spatial integration. Consequently, the main thrust of this dissertation involves concepts and percepts of the posterior cortex and the CSH in the form that the CSH was presented in the previous chapter.

Development of the cognitive facility in students to “see the general in the particular” is referred to as *cerebellar learning*. It contrasts with *cerebral learning*, in which concepts are apprehended, but not with the full clarity that permits efficient and accurate image-based geometrical reasoning. This distinction is illustrated with reference to the Angles in a Circle Theorem in the Prologue, and it is dealt with more fully below. The CSH is the main hypothesis of my thesis. A secondary hypothesis is decontextualization: cerebellar learning, the ability to form schematic percepts, concepts, and inferences from geometry diagrams, is facilitated provided that image-based geometrical ideas are presented in a format that is decontextualized—in other words, as starkly and purely as possible. As illustrated in the Prologue, a decontextualized environment is provided, for example, by dynamic geometry software such as Geometer’s Sketchpad.

I have claimed that image-based reasoning in geometry, and perhaps mathematical reasoning in general, depends crucially on the co-schematization of concepts and percepts, as detailed in Chapter 4. Cerebellar learning implies facilitating student access to these schematized constructs, in other words raising their awareness of the schematic nature of geometrical diagrams. My secondary hypothesis is that students more readily schematize concepts if these concepts are already decontextualized—i.e., schematized as far as possible in the manner of their presentation.

On the other hand, the notion of decontextualization appears to contradict established theories of mathematics education such as constructivism and situated learning. However, decontextualization is just one relatively small aspect of the geometrical learning environment. I do not claim that it is a panacea. There may well be affective, social, or motivational factors that outweigh the benefits of decontextualization.

6.1 "Seeing the General in the Particular"

Mason and Pimm's (1984) seminal paper on generic examples is the source of the phrase "seeing the general in the particular," which I have sprinkled throughout this dissertation. My reinterpretation of the phrase is that it is equivalent to schematization, where attention is sharply focused on those aspects of a geometrical diagram that are essential to the problem at hand. Of course, my view may be quite different from the original intentions of the authors, and so it is necessary to qualify my remarks in this section.

In the first place, the cases cited by the authors are from everyday life or from algebra, with no mention of geometry. The paradigm example is that of $2N$, which is made to stand for the equivalence class of even numbers in proofs that require manipulation of an even number. Mason and Pimm (1984) describe $2N$ as "a particular non-specific even number" (p. 283). It is a *generic example* in the sense that it carries the weight of generality in the particular. However, the authors rightly point out the potential for confusion in that letters in algebra can assume different roles—markers for genericity, as in this case, or variables.

In Handscomb (2005) I point out that image-based reasoning is a necessary component of geometrical arguments at the high-school level. The formal geometry of Hilbert (1899/1971) is not the goal of the high-school curriculum. In other words, students must, unavoidably, use information gleaned from geometrical diagrams in their arguments. For example, the straight line meets the circle at zero, one, or two points, and this piece of information is available to students in no other way that by experimenting with diagrams. On the other hand, the diagrams that students use necessarily contain *particular* lines, circles, triangles, and so on, even though they are aiming to prove general theorems. In this sense, the lines, circles, and triangles in student diagrams are "particular non-specific" figures. In other words, they are generic examples in geometry.

Although my argument is transposed to geometry from algebra, it seems that it is very much in accord with the original intent of Mason and Pimm (1984). Moreover, there is no potential confusion in geometry, unlike algebra, because of the different roles that letters can assume. To my mind, the notion of generic example in geometry is more powerful and transparent than it is in algebra. Take the paradigm example in algebra, mentioned above—its genericity is enforced by use of the letter N. There is no such enforcement in geometry, with the typical static geometric illustrations of student textbooks. Instead, students must learn a new way of perceiving their geometrical diagrams: they must learn to "see the general in the particular." Acquisition of this skill is what I mean by "cerebellar learning."

The whole thrust of this dissertation is to discover an objective equivalent to this psychological function in terms of neural structure and activity, and then to establish some implications for mathematics education. There is an analysis of my ideas with respect to "seeing the general in the particular" in Section 4.3, culminating in a new interpretation of Bergson's (1896/2005) cone of duration. As mentioned above, I refer to "seeing the general in the particular" also as "schematic perception." And then, in a long argument running through Chapter 5, I develop the hypothesis that a role of the cerebellum is to facilitate schematization. The argument finishes in Chapter 5 with evidence that the cerebellum may be concerned with selective attention—schematic perception may be regarded as a form of selective attention. In their development of the notion of generic example, Mason and Pimm (1984) write,

> A generic example is an actual example, but one presented in such a way as to bring out its intended role as the carrier of the general. This is done by means of *stressing and ignoring various key features*, of attempting to structure one's perception of it. Different ways of seeing lead to different ways of knowing. (p. 287, italics added)

Incredibly, “stressing and ignoring various key features” is just the definition of attention that is given in Section 4.1.

“Schematic perception” and “seeing the general in the particular” may be very similar psychological functions—they may indeed refer to equivalent ideas, except that I have transposed Mason and Pimm’s (1984) phrase to geometry from algebra. In Section 7.2, I associate schematic perception with abstraction, which belongs to the cerebellum, whereas generalization is a psychological function, which, I maintain, belongs to the cerebral cortex. The framework and method of this dissertation is quite different from that of Mason and Pimm (1984). However, my hijacking of the phrase “seeing the general in the particular” has kept as close as possible to its original sense. I have thereafter gone much further than the original intent by giving it a neuroscientific basis. I have kept the phrase “seeing the general in the particular” because it is intrinsically evocative and because it situates my research close to a seminal paper in mathematics education.

Mason and Pimm (1984) raise the interesting idea that much mathematical teaching relies on the teacher’s presentation of generic examples. Indeed, all examples given by a teacher to demonstrate a mathematical technique may be regarded as particular but generic. The student should perceive the general method in the particular example. The authors ask an important question with regard to the technique of teaching by generic examples:

> How can you expose the genericity of an example to someone who sees only its specificity? Apart from stressing and ignoring, and repeating the general statement over and over, *how can the necessary act of perception, of seeing the general in the particular*, be fostered? (p. 287, italics added)

Since the ability to "see the general in the particular" derives from what I refer to as cerebellar learning, the authors are asking how cerebellar learning can be facilitated. I hypothesize herein that cerebellar learning in geometry can be fostered by decontextualizing geometrical diagrams. Decontextualization is discussed later in this chapter, but firstly the distinction must be clarified between cerebral learning and cerebellar learning.

6.2 Cerebral Learning and Cerebellar Learning

From the perspective of this dissertation, there are two aspects to the learning of geometry: cerebral and cerebellar. The line between the two is illustrated in the Prologue, with the Angles in a Circle Theorem, and now this section continues that discussion. *Cerebral learning* refers to the formation of cognitive networks of extensional concepts in the cerebral cortex. Thus, children come to recognise triangles, squares, and circles. *Cerebellar learning* refers to the establishment of the cerebellar response to schematize the cerebral extensional concepts. Thus, children learn to perceive triangles, circles, and squares in the full abstraction of their mathematical essence, in other words to schematize geometrical diagrams. Hence, the distinction between cerebral and cerebellar learning is significant for mathematics education theory.

Cerebellar learning is the ability to schematize concepts. Without cerebellar involvement there is potential for confused reasoning. Where does the triangle concept begin and where does it end? A host of associations may be unconnected to the mathematical idea of “triangle,” associations which are inherent in the very nature of the massively interconnected cognitive network hierarchy in the posterior cortex. This network of associations is the lifeblood of poetry but the enemy of mathematics. The triangle may be red, it may be pleasing, it may be pointing off the page, and so on, and all these incidental associations may lead to inefficient, and even inaccurate, geometrical reasoning. With cerebellar learning, on the other hand, the triangle is perceived for just what it is—a mathematical triangle—and no more. The concept has been purified, and the triangle is perceived *generically*. Image-based geometrical reasoning can now proceed efficiently and accurately.

Nevertheless, cerebral learning must come prior to—or at least concurrently with—cerebellar learning, for otherwise there is no “raw material,” as it were, for the cerebellum to purify and schematize. Cerebellar learning is not about discovering triangles, which is the concern of the cerebral cortex not the cerebellum. The student may know much about triangles even without the cerebellum. After cerebellar learning, the student may, potentially, look within a diagram and perceive nothing but the mathematical triangle, “seeing the general in the particular,” making a schematic inference.

This dissertation makes no claim to defend a general theory of geometrical learning. Consciousness, affect, motivation, language, and social interaction are not considered. These factors are important for any comprehensive theory of learning, but they are aspects of the learning of *any* topic. For image-based geometrical reasoning, on the other hand, cerebellar learning is important. Mathematics educators should seek ways to promote the cerebellar learning of their students.

6.3 Decontextualization

Given the physiological structure of the cerebellum and its connections with the cerebral cortex, my hypothesis is that decontextualization is a way in which mathematical ideas may be presented in order to facilitate cerebellar learning, the growth of the ability to "see the general in the particular," to make schematic inferences from geometrical diagrams.

Presentation of a mathematical concept in a way that minimizes the number of incidental properties in the percept (and concept)—even before cerebellar schematization—is referred to as *decontextualization* of the mathematical concept. When the concept is decontextualized, it means that the percept into which the concept resolves is constrained by external means, by presentation of the concept as starkly as possible, bare of distractors. Then, because concept and percept have the same range, constraint of the percept propagates backwards through the cognitive network hierarchy to restrict the potential of the concept. Decontextualization occurs, for example, in the way that situations in Euclidean geometry are typically presented—the diagrams are plain line drawings on a white background.

Note that decontextualization can only ever be relative. Incidental properties are always present. Given the starkest black-line figure on a white background, it necessarily has a certain size, orientation, and so on. However, it is certainly decontextualized relative to a colourful figure situated in a busy background. On the latter point, given that learning of a particular concept is the goal, decontextualization of that concept refers to presenting it—as far as possible—“without context.” In other words, complex mathematical situations, which contain the target concept as a component, are still contextualizing the target concept, even if the context is fully mathematical.

It is worthwhile reiterating here a point that was first made in the discussion of Bergson’s cone (or cylinder) of duration, in Section 4.3: *the geometrical diagram is not the percept*. The percept is already a psychological construct, although its objective correlate is a physical cognitive network. Neither the percept, nor its physiological correlate, is the diagram “out there” in the external world. Different ways of presenting the *same* mathematical concept, in diagrams or otherwise, may give rise to percepts with different qualities.

Imagine these two cases: (A) a colourful geometrical diagram is situated in context, whether that context is completely irrelevant or at least relevant in the sense of being part of a “real world” problem; (B) a stark, black-line geometrical diagram is drawn on a plain, white background. The properties of percepts resulting from situations A and B differ. The former is likely to contain many more incidental properties than the latter. In case B, the geometrical diagram has been decontextualized, and therefore the percept is likely to be partially schematized already because of the manner of presentation. Because of equality of range, the concept, too, has fewer potential properties. This is a necessary assumption: representation of geometrical concepts in the external world in a manner that is relatively decontextualized yields concepts and percepts that are partially schematized.

My hypothesis is that decontextualization of the percept, or equivalently decontextualization of the concept, facilitates cerebellar schematization of the concept. In other words, if geometrical diagrams are presented with few incidental distractions then it is easier for the student to gain the facility to “see the general in the particular,” to schematize, and to reason effectively with these geometrical ideas. Moreover, this hypothesis refers to cerebellar *learning*, implying longer-term changes in the student’s cognitive abilities. It does not just mean that the student is less confused in the moment, with a specific geometrical situation, but that the student gains the facility to reason more clearly about similar geometrical situations at future times.

On the other hand, the efficacy of decontextualization appears to be a relatively easy point to accept on purely intuitive grounds, provided that the reader agrees that concepts must be schematized for clear, accurate geometrical image-based reasoning, in other words that schematic inferences from geometrical diagrams are necessary. To the extent to which incidental properties are confused with geometrical concepts, the real import of a geometrical argument is obscured. If concepts are decontextualized in presentation, then they have already, to an extent, been schematized. It seems reasonable to suppose that decontextualized concepts, to the degree that they have already been shorn of incidental properties, are more readily schematized by students, facilitating cerebellar learning of these concepts.

In practical terms, what does decontextualization mean for the classroom? Cerebellar learning is quite different from cerebral learning. Assume that the students already know about triangles, squares, and circles, even perhaps the Angles in a Circle Theorem in its various manifestations. They have learned these concepts extensionally, but their reasoning can be inefficient and even inaccurate because of the various associations, incidental properties, that tag along with the geometrical concepts. If the teacher simply wants the students to get on with some geometrical reasoning, then he or she gives the students examples that are decontextualized. With relatively little practice, students grasp the geometrical essences in the examples, and they are able to succeed with real geometrical reasoning. If the examples are contextualized, on the other hand, then the students' grasp of the geometrical essence, their ability to make schematic inferences, is retarded, making it more difficult for them to employ geometrical reasoning successfully. Presentation of complex, contextualized mathematical situations may prevent some students from reaching the point where they can begin any geometrical reasoning.

Now, especially at the high-school level, geometrical problems may well be presented in ways that are already relatively decontextualized in the sense of being presented starkly and symbolically as black lines on a white background. In this case, my argument is that concepts should be presented simply at first, with gradually increasing complexity. After all, as I argue in Handscomb (2005, 2006), geometrical reasoning proceeds in a serious of stages, in which various aspects of the diagram are focused on in turn. A difficulty may arise because the geometrical meaning of one aspect of the diagram may be lost: *as far as this aspect is concerned, the rest of the diagram is incidental*. As far as possible, therefore, geometrical concepts should be presented in isolation at first. There is nothing new in this approach, per se, except that it can be identified as an aspect of cerebellar learning.

The Angles in a Circle Theorem is a particularly interesting case, because even when it is presented in total isolation, as simply as possible, it contains a wealth of incidental properties that can confuse students. As explained in the Prologue, mathematics teachers may teach it, to all intents and purposes, as *five separate theorems* rather than as a single theorem, which is very inefficient. In order for students to perceive the essence of the theorem in its various manifestations, they may need to spend some time working with dynamic geometry software such as Geometer's Sketchpad.

These are preliminary suggestions for pedagogy. The aim of this dissertation, after all, is to clarify just one aspect of the theory of geometry education. Moreover, the decontextualization hypothesis is just that—a hypothesis. It requires empirical substantiation.

Now, the notion of decontextualization flies in the face of some established mathematics education theory. For example, the situated learning theory of Lave and Wenger (1991) advocates contextualizing mathematical concepts. Admittedly, there may be affective or other reasons for making mathematics relevant to students' lives by contextualizing mathematical concepts, and these reasons may outweigh the benefits of decontextualization, even assuming that the decontextualization hypothesis is subsequently substantiated.

Nevertheless, no matter how geometry is embedded in real world examples, or otherwise contextualized, *the geometrical structure must be extricated*. My hypothesis is that this extrication is easier, and therefore geometrical reasoning is more within the reach of students, if the geometrical concepts are already partially schematized by means of decontextualization. Other factors aside, my hypothesis is that decontextualization facilitates geometrical learning, provided that the goal of “learning” is direct apprehension of geometrical structure, making schematic inferences from geometry diagrams.

The following section examines the decontextualization hypothesis from two perspectives: firstly, arguments against it from within constructivism and situated learning theory; secondly, arguments supporting it from other sources in mathematics education research.

6.4 Decontextualization and Mathematics Education Theory

There is substantial research in mathematics education both for and against the idea of decontextualization. This section discusses some of that research. The literature needs to be approached carefully, however, because of the distinction between cerebral and cerebellar learning. It may be that a view ostensibly opposed to decontextualization is really applicable to cerebral rather than cerebellar learning.

The last point is reasonable, I believe. I am not using it to sidestep evidence that appears to discredit the decontextualization hypothesis. The distinction between cerebral learning and cerebellar learning is based on my understanding of the relationship between the cerebellum and cerebral cortex. It is a way in which the neurosciences “constrain and inform” mathematics education theory. It arises from within the internal logic of the research method of educational neuroscience.

6.4.1. Decontextualization, Situated Learning Theory, and Constructivism

Two big ideas in mathematics education theory are situated learning and constructivist learning. Both ideas are, apparently, to varying degrees, antipathetic to the decontextualization hypothesis. The seminal work for situated learning is Lave and Wenger (1991). According to the authors,

> Learning viewed as a situated activity has as its central defining characteristic a process that we call *legitimate peripheral participation*. By this we mean to draw attention to the point that learners inevitably participate in communities of practitioners and that the mastery of knowledge and skill requires newcomers to move toward full participation in the sociocultural practices of a community. (p. 29, authors’ italics)

Lave and Wenger challenge the idea that knowledge is abstract or general, in the sense of being acquired out of context and then applied in a variety of particular situations (pp. 37-38). In other words, learning is necessarily contextual.

According to Anderson et al. (2000), the examples often cited in support of the situated nature of learning are Lave’s (1988) description of shoppers who can calculate supermarket discounts but not equivalent paper-and-pencil problems, and Carraher, Carraher, and Schliemann’s (1985) description of Brazilian street children who can perform calculations on the street but not in the classroom. Anderson et al. write,

> Even if these claims are valid and generalizable beyond the specific anecdotes that have been cited, they demonstrate *at most* that particular skills practiced in real-life situations do not generalize to school situations. They assuredly do not demonstrate that arithmetic procedures [and concepts] taught [abstractly] in the classroom cannot be applied to enable a shopper to make price comparisons or a street vendor to make change. (Situated learning section, para. 3, authors' italics)

In other words, the situated learning argument proceeds from the truth of a proposition to the truth of its converse.

Anderson et al. (2000) admit that there is much psychological evidence that knowledge gained in one situation cannot be generalized easily to other situations. However, there is much evidence also of knowledge transfer between situations, and, moreover, "[k]nowledge does not have to be taught in the precise context in which it will be used, and grave inefficiencies in transfer can result from tying knowledge too tightly to specific, narrow contexts" (Situated learning section, para. 10).

Note that in the brief discussion above, no attempt is made to distinguish between cerebral learning and cerebellar learning. If situated learning theory concerns cerebral learning only, then situated learning theory has nothing to say about cerebellar learning and decontextualization.

Turning to constructivist theory in education (e.g., Confrey & Kazak, 2006), one of its important claims is that effective learning must take place in complex learning situations or must concern the "big picture" (Anderson et al., 2000). Accordingly,

> [C]onstructivists recommend, for example, that children learn all or nearly all of their mathematics in the context of complex problems (e.g., Lesh & Zawojeski, 1992). This recommendation is put forward without any evidence as to its educational effectiveness.
>
> There are two serious problems with this approach, both related to the fact that a complex task will call upon a large number of competences. First, . . . a learner who is having difficulty with many of the components can easily be overwhelmed by the processing demands of the complex task. Second, to the extent that many components are well mastered, the student will waste a great deal of time repeating these mastered components to get an opportunity to practice the few components that need additional effort. (Anderson et al., 2000, Constructivism section, Claim 3 subsection, paras. 1-2)

These two problems appear to be close to the issues that were discussed in the previous section on decontextualization: students may find it difficult to perceive geometrical essences when the geometry is embedded in a complex context; students' ability to perceive mathematical essences, to make schematic inferences from geometry diagrams, may develop when the context is made gradually more complex.

Nevertheless, as with situated learning, it is uncertain whether constructivism really applies to cerebellar learning. If not, then the constructivist ideas about complex learning situations are not pertinent to my discussion.

6.4.2. Decontextualization in Other Areas of Mathematics Education Theory

The remainder of this section discusses evidence that appears to favour the decontextualization hypothesis. I have tried to select studies that focused primarily on conceptual rather than procedural aspects of mathematical reasoning. It is only fair to remark, however, that once again it is unclear whether the learning referred to is cerebral or cerebellar. In addition, the research cited below largely concerns algebraic reasoning rather than image-based reasoning in geometry. My whole thesis is concerned with schematization of geometrical diagrams, and so research cited below is not directly relevant. However, it provides at least circumstantial evidence in favour of the decontextualization hypothesis.

Kaminski et al. (2008) experimented in teaching the concepts of group theory to students by means of concrete (i.e., contextualized) representations and by means of abstract (i.e., decontextualized) representations. The learning of their subjects was enhanced with the abstract representations. According to the authors, "concrete information may compete for attention with deep to-be-learned structure (*6-8*). Specifically, transfer of conceptual knowledge is more likely to occur after learning a generic instantiation than after learning a concrete one (*7*)" (p. 454). Moreover,

> These results indicate that learning one, two, or three concrete instantiations resulted in little or no transfer, whereas learning one generic instantiation resulted in significant transfer. If transfer from multiple instantiations depends on whether the learner abstracts and aligns the common structure from the learned instantiations (*1, 4*), then transfer failure suggests that participants may have been unable to recognize and align the underlying structure. (p. 455)

The authors suggest that concrete and generic instantiations have different learning benefits, and that an ideal learning environment would combine both:

> If a goal of teaching mathematics is to produce knowledge that students can apply to multiple situations, then presenting mathematical concepts through generic instantiations, such as traditional symbolic notation, may be more effective than a series of 'good examples,' This is not to say that educational design should not incorporate contextualized examples. What we are suggesting is that grounding mathematics deeply in concrete contexts can potentially limit its applicability. Students might be better able to generalize mathematical concepts to various situations if the concepts have been introduced with the use of generic instantiations. (p. 455)

It seems fairly clear that the experimental paradigm of Kaminski et al. concerns knowledge that is at least partially conceptual rather than procedural. In addition, it is possible that the abstract representations may have enabled the participants to “see” the abstract group structure in its full purity, implicating cerebellar learning.

Koedinger et al. (2008) found that simpler problems in algebra benefit from grounded (i.e., contextualized) representations whereas more complex problems benefit from abstract (i.e., decontextualized) representations. They concluded that it is not necessarily the case that one type of presentation is better than the other. Their results show that for simpler problems, students perform better with word problems, but for complex problems, students perform better with symbolic representations.

It would appear that the relative ease with which young students solve simple algebra problems when they are contextualized—i.e., presented verbally—provides a counter example for the decontextualization hypothesis. However, perhaps it is the case that with younger students and with simpler problems, affective (or at least non-cerebellar) elements of the learning situation dominate. In any case, it should be noted that Koedinger et al. (2008) are mainly concerned with algebraic learning, and therefore, with the anterior cortex (see Chapter 7).

Approaching the same idea from a different perspective, the effect on mathematical reasoning of strictly irrelevant information is investigated in Bana and Nelson (1978). According to the authors,

> Such irrelevant data are generally referred to as ‘noise’ or ‘distractors.’ Skemp (1971) maintains that ‘the greater the noise, the harder it is to form the concepts’ (p. 29). Dienes (1963) and Biggs (1968) also discuss the role of noise in concept formation. The Piagetian studies provide numerous examples where young children’s centrations on irrelevant perceptual cues prevent them from developing specific concepts (Inhelder & Piaget, 1964; Piaget, 1952, 1969; Piaget & Inhelder, 1963; Piaget, Inhelder, & Szeminska, 1960). (p. 55)

In their own study, Bana and Nelson investigated three types of distractors: situational (i.e., real-world contextual) distractors, colour-attribute distractors, and spatial-numerical distractors. Their study showed that children who attended to distractors did less well in problem solving than those who did not. The authors ask, “Is it possible that the more realistic the problem the more difficult it will be because of the likelihood of irrelevancies?” (p. 60).

Hegarty and Kozhevnikov (1999) distinguish between two types of visual representation of problems:

> [S]chematic representations . . . primarily encode the spatial relations described in a problem and pictorial representations . . . primarily encode the visual appearance of the objects or persons described. (p. 688)

(“Schematic” here does not correspond necessarily with the use of “schematic” in this dissertation.) The authors clarify,

> Visual imagery refers to a representation of the visual appearance of an object, such as its shape, color, or brightness. Spatial imagery refers to a representation of the spatial relationships between parts of an object and the location of objects in space or their movement. (p. 685)

It is clear, therefore, that visual imagery refers largely to the incidental properties of a visual representation, whereas spatial imagery refers to the essential properties. The authors

> showed that visual-spatial representations can be reliably classified into one of these types and that the types are differentially related to problem-solving success. Use of schematic representations is positively related to success in mathematical problem solving, whereas use of pictorial representations is negatively related to success in mathematical problem solving. (p. 688)

Once again, as with the other research cited, it is unclear whether the research of Hegarty and Kozhevnikov (1999) refers to cerebellar learning or cerebral learning. *It is interesting to note that a distinction which can go unnoticed psychologically becomes quite clear when viewed physiologically*. This point perfectly illustrates an advantage of the research method of educational neuroscience over more conventional methods of research in mathematics education.

6.5 Conclusion

The cognitive network model of the neural correlates of cognition is hierarchical, starting from the bottom level, at interface of organism and world, and leading through layers of increasingly complex association to concepts and procedures of the utmost generality. Concepts are established in a manner that is naturally extensional rather than intensional. The posterior-anterior dichotomy clarifies the distinction between concepts and procedures and the relationship between them: concepts offer spatial indeterminacy, whereas procedures offer temporal indeterminacy. Procedures resolve through the anterior hierarchy to temporally determinate acts, and concepts resolve through the posterior hierarchy to spatially determinate percepts. In the reverse direction, acts are assimilated to procedures, and percepts are assimilated to concepts. Procedures can be “perceived” and concepts can be “acted” by means of the reciprocal operation of the perception-action loop.

The cognitive network theory, however, involves only *cerebral* learning and reasoning. Extensional concepts (and procedures), I argue, are naturally indistinct, and therefore insufficient for geometry, which requires clarity and purity. According to the CSH, the cerebellum interacts with the cerebral cortex in such a way as to focus attention on those properties of concepts that throw their geometrical essence into relief. In this way, the *extensional* concept becomes *intensional*, and thereby appropriate for the schematic inferences of image-based geometrical reasoning. With respect to the concepts and percepts of the posterior cortex, *cerebellar learning* enables “the general to be seen in the particular.”

The present chapter has investigated the implications of cerebellar learning for mathematics education, specifically image-based geometrical reasoning. For precision, the discussion is carefully delimited. Firstly, cerebral learning of extensional concepts (and procedures) is ignored. Only cerebellar learning is discussed with respect to any *new* implications for mathematics education theory. Secondly, the ongoing dialogue in mathematics education with respect to the relative benefits of procedural learning or conceptual learning is disregarded, although it is addressed briefly in Chapter 7. Thirdly, although the theory has obvious extensions to cognitive networks of the anterior cortex, the entire focus is image-based geometrical reasoning and schematization of geometrical structure, which is the domain of the posterior cortex.

Within these constraints, my hypothesis is that decontextualization of geometrical concepts is a way to facilitate cerebellar learning of these geometrical concepts. In other words, my hypothesis is that,in order to promote clarity of geometrical thinking, geometrical situations should be presented starkly, without distracters, without colours, and as far as possible without any incidental features. The geometrical thinking of novices may otherwise become inexact and confused because they have yet to develop the necessary insight for perceiving geometrical structure in all its purity.

The situated learning theorists and constructivists, on the face of it, would be opponents. However, it is uncertain that their ideas really apply to cerebellar as well as cerebral learning. As I explained at the beginning of this chapter, I am not attempting with this statement to evade evidence ostensibly opposing the decontextualization hypothesis. My thesis exists in the middle ground between education and the neurosciences, and in this middle ground the distinction between cerebral learning and cerebellar learning is a reality (assuming the CSH) .

In any case, situated learning theory and constructivism offer valuable insights that are widely applicable, though perhaps not universal. I concur with Mighton (2007), who writes, “Too often in the math wars we tend to throw out the good with the bad and we swing wildly back and forth between competing trends” (p. 62). Everything has its place. The truth, I believe, is multifaceted, and I have attempted to shine a light on one of those facets that may have hitherto remained obscure.

▲►▼◄▲►▼◄▲►▼◄

The main presentation is now complete. In summary, (1) “seeing the general in the particular” is a cognitive function that is necessary for the schematic nature of image-based geometrical reasoning, (2) the cerebellum may act to facilitate “seeing the general in the particular,” and (3) decontextualization may enable the cerebellum to perform its role more effectively. It remains, in the final chapters, to look toward potential implications of the theory, to review the dissertation as a whole, and to identify its contributions and limitations.

CHAPTER 7: POTENTIAL IMPLICATIONS AND FUTURE DIRECTIONS

The discussion in Chapters 3 to 6 has been focused on presenting the CSH and on developing a secondary hypothesis that decontextualization may facilitate some aspects of image-based geometrical reasoning. The goal of the dissertation was reached by the end of the preceding chapter. However, there are several other implications for mathematics education that, while peripheral to the main line of reasoning, seem to offer interesting directions for future research. These other implications are outlined in this chapter. *The reader should understand that these are suggestions for future research rather than fully worked out positions*. While I have limited the discussion so far to image-based reasoning in geometry, this chapter refers to mathematical reasoning more generally.

Cerebral cortex and cerebellum represent the two dimensions of the argument presented in the previous chapters. With respect to the cerebral cortex, and particularly the clear division between anterior cortex and posterior cortex, there may be a new approach to the distinction between procedural reasoning and conceptual reasoning and the various theories of concept formation. On the other hand, the physiological divide between cerebellum and cerebral cortex suggests a way to clarify relationships between the various theories of mathematical abstraction.

7.1 Mathematics Education and the Cerebral Cortex

Some interesting implications for mathematics education follow from the absolute division in cognition between action and perception, which is instantiated physiologically by the rift that is the Rolandic fissure, separating anterior cortex and posterior cortex. This primordial divide, I suggest, may correspond to the distinction between procedural reasoning and conceptual reasoning, which has long been a topic of investigation by mathematics educators.

The perception-action loop connects the hierarchies of perception and action at every level, implying that procedural reasoning and conceptual reasoning are intimately related. Moreover, the portion of the perception-action loop that links procedures of the anterior cortex to concepts of the posterior cortex suggests that theories of concept formation in mathematics education, such as APOS, have a physiological correlate. On the other hand, the very existence of the *reciprocal* connection from concepts to procedures suggests that the reverse may be possible, and that procedures may emerge from concepts.

The previous paragraph illustrates almost perfectly the method of educational neuroscience. In other words, a psychological theory of concept formation in mathematics education inspires the search for a physiological correlate, and the physiological correlate in turn suggests a modification of the theory of concept formation.

7.1.1 Procedural Reasoning versus Conceptual Reasoning

An individual interfaces with the world objectively by means of perception *of* an external world (posterior cortex) and action *on* an external world (anterior cortex). Activity of the cognitive network structures created at this interface reverberates upwards through the cognitive network hierarchy. As discussed in Section 4.1 the centripetal, bottom-up flow of neural activity is accompanied by a top-down, centrifugal flow. The opposing flows are simultaneous and set up a kind of “standing wave” of cognition. Specifically, in the posterior cortex, concepts resolve to percepts and percepts are assimilated into concepts. The two are linked, metaphorically, through Bergson’s cone (or cylinder) of duration.

The major thrust of the argument throughout this dissertation concerns the posterior cortex. However, in order to broaden the picture with regard to the nature of mathematical reasoning, it is necessary also to refer to activity of the anterior cortex.

While mathematical concepts—numbers and spatial configurations—are perceived in the posterior cortex, mathematical reasoning is more than just perception of mathematical objects. Arithmetic and algebra, for example, consist largely of mathematical procedures, and procedures belong to the anterior cortex.

Reciprocal perception-action loops link analogous levels of the cognitive network hierarchies of the anterior cortex and posterior cortex. At the lowest level, the perception-action loop passes through the external world, becoming, really, an act-movement-sensation-percept loop. At higher levels, however, cognitive networks of the action hierarchy directly modify corresponding cognitive networks of the perception hierarchy, and vice versa.

The fuzzy, indistinct extensional concepts of the cerebral cortex have to be schematized, made crisp and pure, in order to be of use for image-based geometrical reasoning. Chapter 5 argues for the hypothesis that the cerebellum is responsible for this schematization. Assume for now that the cerebellum is operating in this way, that concepts (and indeed procedures) have been schematized to be of use in mathematics, and focus purely on the distinction between concepts and procedures as they are instantiated in cognitive networks of the posterior cortex and anterior cortex, respectively.

The cognitive network of the procedure in the anterior cortex resolves to a sequence of cognitive networks correlative to acts, moment by moment. These acts, as they discharge, influence the percepts that are resolving in the posterior cortex, by means of the perception-action loop. The sequence of percepts has resolved from a sequence of concepts, each of which has the same range as its actualization in an individual percept. This sequence of concepts is in turn influenced by the procedure via the higher-level perception-action loop. The mathematical procedure, in this sense, is not experienced as a single integrated concept in the posterior cortex. This is referred to as *procedural reasoning*, when coherent activity in the anterior cortex dominates cognition. An example of procedural reasoning would be solving an equation, perceived as a series of algebraic manipulations.

On the other hand, the posterior cortex may dominate cognition, resulting in *conceptual reasoning*. The concept resolves to a percept that influences the resolving act in the anterior cortex. A stable percept over a sequence of cognitive moments would imply a stable act over the same sequence, and so there is no coherent procedure that resolves to an integrated sequence of different acts. An example of conceptual reasoning would be perception (and recognition) of a geometrical figure.

I propose that algebra, at least as typically taught at the secondary level, primarily involves procedural reasoning, whereas geometry, at least as understood herein, primarily involves conceptual reasoning (see p. 109, n. 11). Algebra is about the manipulation of symbols, much as concrete objects are manipulated in the solution of a puzzle. Image-based reasoning in geometry, on the other hand, is about seeing into a static diagram and perceiving the relationships between its components. Nevertheless, procedural reasoning must involve the posterior cortex—otherwise the algebraic symbols could not be perceived—and conceptual reasoning must involve the anterior cortex—otherwise the geometer would be catatonic. For everyday life, and for mathematics, too, normal activity involves a blend of procedural and conceptual reasoning. I would like to suggest that for mathematics, in particular, it is possible to identify certain forms of reasoning that are *predominantly* procedural or *predominantly* conceptual, although the distinction requires careful delimitation.

The distinction between procedural reasoning and conceptual reasoning has long been a topic for mathematics education research. Different names have been coined for the two types of reasoning over the years, but these different names all serve to identify similar ideas. Skemp (1976/2006) referred to *instrumental understanding* and *relational understanding*. Instrumental understanding is, metaphorically, the ability to reach a location by following directions. On the other hand, relational understanding is like having a map so that various paths can be followed to the goal. These two ideas may be compared to procedural reasoning and conceptual reasoning, respectively. Skemp prefers relational understanding as a pedagogical goal because relational understanding involves a few general principles (i.e., one map), whereas instrumental understanding involves a multiplicity of rules (i.e., different sets of directions for different locations).

Hiebert and Lefevre (1986) distinguish between procedural knowledge and conceptual knowledge. Although, their use of “procedural” and “conceptual” correspond approximately to the use of the terms herein, conceptual knowledge appears to be a somewhat broader idea. Accordingly,

> Conceptual knowledge is characterized most clearly as knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network. (pp. 3-4)

Whereas,

> Procedural knowledge . . . is made up of two distinct parts. One part is composed of the formal language, or symbol representation system, of mathematics. The other part consists of algorithms, or rules, for completing mathematical tasks. (p. 6)

Conceptual knowledge, therefore, involves a whole web of related ideas rather than a single concept. Likewise, procedural knowledge includes the symbol system, whereas the issue of symbol systems (or more generally language) is not discussed herein. Interestingly, the authors claim that conceptual knowledge must be learned with meaning, whereas procedural knowledge may be learned with or without meaning, depending on whether it is related to conceptual knowledge. On the other hand, both types of knowledge are important, and

> Linking conceptual knowledge and procedural knowledge has many advantages. Usually the advantages are claimed for procedural knowledge. Procedural knowledge that is informed by conceptual knowledge results in symbols that have meaning and procedures that can be remembered better and used more effectively. A closer look reveals theoretical advantages for conceptual knowledge. Procedural knowledge provides a formal language and action sequences that raise the level and applicability of conceptual knowledge. (p. 16)

It is interesting that Hiebert and Lefevre (1986) assign, perhaps, equal importance to conceptual knowledge and procedural knowledge. The same balance can be observed in the ideas of Rittle-Johnson et al. (2001):

> We define *procedural knowledge* as the ability to execute action sequences to solve problems. This type of knowledge is tied to specific problem types and therefore is not widely generalizable. . . In contrast to procedural knowledge, we define *conceptual knowledge* as implicit or explicit understanding of the principles that govern a domain and of the interrelations between units of knowledge in a domain. This knowledge is flexible and not tied to specific problem types and is therefore generalizable. (pp. 346-347, authors' italics)

The authors cite theories in mathematics education that claim concepts take precedence and other theories in mathematics education that claim procedures take precedence. Rittle-Johnson et al., however, support a bidirectional, iterative approach, where conceptual development reinforces procedural development, and vice versa:

> Children’s conceptual and procedural knowledge develop iteratively. Rather than development of one type of knowledge strictly preceding development of the other, conceptual and procedural knowledge appear to develop in a hand-over-hand process. Gains in one type of knowledge support increases in the other type. (p. 360)

The notion of an iterative, to-and-fro growth of knowledge, conceptual and procedural, recalls the perception-action loop.

Mighton (2007) argues the same point when he claims, “Rules and concepts are often hard to separate; even in cases where the distinction is clear, mastery of the rules can help induce understanding of concepts as much as understanding of concepts supports mastery of rules” (p. 172). Mighton is trying to redress what he perceives as a bias against procedural learning in mathematics education. Constructivist ideas in education (Chapter 6) often emphasize the importance of learning concepts before procedures. His point is that conceptual understanding can follow procedural practice, but that “[m]any teachers and educators have trouble recognizing that following rules involves thought, because they are convinced that students must discover mathematical concepts in order to understand them” (p. 167).

In practice, according to Rittle-Johnson et al. (2001), either conceptual knowledge or procedural knowledge may develop first, depending on the types of stimulus that children receive in their learning environments. I propose that image-based geometrical reasoning tends towards a dominance of conceptual knowledge (or reasoning), whereas arithmetical and algebraic learning tend towards a dominance of procedural knowledge (or reasoning). The authors' point, however, is well taken, and their view "eliminates fruitless arguments about whether conceptual or procedural knowledge generally precedes the other" (p. 347).

Presmeg (1986, 1992) distinguishes verbal-logical thinking and visual-pictorial thinking in mathematics. It is tempting to over-simplify by equating these two types of mathematical reasoning with procedural reasoning and conceptual reasoning, respectively. However, it seems to be clear that Presmeg's ideas are in the same vein of thought and are expressing, again, a certain perspective on a deep distinction. Presmeg's account differs from others, however, in that she suggests that some people are visualizers, who prefer to reason with images. Visualizers are contrasted with non-visualizers, who prefer a verbal-logical approach to mathematical reasoning. And this distinction cuts across all mathematical domains, arithmetic, algebra, and geometry.

The distinction between visualizers and non-visualizers may have its neural interpretation in the distinction between posterior cortex and anterior cortex. In any case, Presmeg (1986, 1992) argues that image-based reasoning is important for abstraction and generalization in mathematics, and therefore teaching of mathematics that emphasizes imagery—i.e., prioritizing the posterior cortex—may have significant benefits.

7.1.2 Concept Formation

The distinction between procedural reasoning and conceptual reasoning is important for theories of concept formation in mathematics education. While procedural reasoning is dominant, procedures are resolving to acts, mathematical objects are being manipulated, and the manipulations are perceived through the perception-action loop. However, the procedure, as a whole, it is assumed, exists as a cognitive network structure. By means of the perception-action loop at the level of procedures and concepts, the procedure may enter perception as a concept. In a single cognitive moment, the procedure is “seen” and understood as a whole. Dominance thereby passes to conceptual reasoning, at least momentarily. On the other hand, while conceptual reasoning is dominant, the concept may evoke, by means of the higher-level perception-action loop, a procedure in the anterior cortex. And then this procedure may resolve to a sequence of acts as the dominant mode of reasoning becomes procedural.

Readers may recognize, in the argument of the previous paragraph, an important theme in mathematics education research: the *reification* of concepts (Sfard, 1991) or the *encapsulation* of concepts (Czarnocha et al., 1999). The cognitive network understanding of image-based geometrical reasoning provides an alternative way to engage these ideas. Below I discuss the APOS theory of mathematical concept formation (e.g., Czarnocha et al., 1999). The description of APOS below is largely taken from Handscomb (2005).

APOS stands for *action-process-object-schema*. The *action* consists of a physical or mental transformation of a physical or mental object according to some external instructions. The paradigm example is that of the natural numbers, where the *action* corresponds to counting a collection of items, reaching, for example, 4 beans.

When the student can perform the action mentally, or think about it without performing it down to the last detail, then the action has been interiorized to a *process*. Using the example of natural numbers, the student is able to operate efficiently by counting any collection of items, and the identity of the items themselves should slip into the background. Instead of 4 beans, for example, it could be 4 anything, and the action of counting becomes detached from the items themselves.

When the student can see the process as a totality, then the student has *encapsulated* the process to an *object*. Note that even after the process has been encapsulated, the student must be able to operate with the process or the object, as required. The number 4 now exists, for example, as a concept, even though the student is still able to count 4 items.

Finally, the student groups together a coherent collection of actions, processes, and objects into a *schema* for the concept in question. In other words, the number 4 is packaged along with the other natural numbers and counting actions and processes into the natural numbers schema. Note that alongside the development of the concepts of the numbers themselves, operations such as addition, subtraction, and so on, have developed from primitive to more abstract, complete forms. These operations, too, are part of the schema.

The ideas of APOS theory can be partially reinterpreted in cognitive network terms. It seems clear that the “action” of APOS theory is already temporally extended, and would correspond therefore to a procedure rather than to an act. Then, it would seem to be in accord with the intent of APOS theory if the APOS “process” is a cognitive network association of such “actions,” in such a way that the “process” is general in contrast to the specificity of an individual “action.” “Actions” and “processes” are both procedures, of respectively lesser and greater specificity.

The encapsulation of APOS theory may be understood as “perception” of a procedure as a concept by activation of the perception-action loop that operates between higher levels of the cognitive network hierarchies of the anterior and posterior cortices. In other words, the temporally structured procedure is perceived instantly as a concept. Of course, the vastly extended cognitive network of the concept does not suddenly spring into existence, as a whole, but rather the concept is formed as certain strategic associations are made between lower-level cognitive networks that were already in existence. The existence of these lower-level cognitive networks of the posterior cortex have been facilitated by lower levels of the perception-action loop between the anterior and posterior cortices.

It natural and reasonable within the cognitive network theory for the reciprocal influence to occur from posterior cortex to anterior cortex, at all levels of the perception-action loop. In other words, at the level of concepts and procedures, an analogous process to encapsulation operates in the reverse direction, whereby a mathematical concept can initiate a mathematical procedure in the anterior cortex. There is no place for this reverse process in APOS theory.

All mathematical concepts begin, according to APOS theory, from actions. However, it seems much less natural for image-based geometrical concepts to be formed beginning with action or process. It *is* possible to devise ways in which geometrical ideas can be encapsulated with the APOS model: a simple way of encapsulating a geometrical idea would be through the procedure of actually constructing an object representing the idea. Encapsulating a geometrical concept in this way, however, seems to be a poor substitute for immediate Gestalt recognition of a geometrical shape. Tall et al. (2000) put it as follows:

> In geometry, although there are many processes involved, including the formal processes of geometric construction, we hypothesize that the main focus is initially on *objects*. This leads to a sequence of development from teasing out the properties of the objects, making verbal descriptions, thinking about relationships, verbalizing inferences, formulating verbal proofs, leading to a broad development after the fashion described by Van Hiele (1986). (p. 236, authors' italics)

The authors are right. The reverse process, from concept to procedure should be admitted. The other side of the concept-formation coin is procedure formation, which occurs naturally via the perception-action loop, whereby concepts become procedures in the anterior cortex. The child, who knows full well what a circle looks like, learns how to construct a circle with a pair of compasses.

According to Tall (1999), “Dubinsky and his co-workers have made an impressive effort to formulate everything in action-process-object language. However, the urge to place this sequence to the fore leads to a description that, to me, soon becomes over-prescriptive” (p. 113). He continues, “The brain observes objects, and what seem to be primitive mathematical and logical concepts in ready-made brain modules. This seriously questions a rigid Action-Process-Object-Schema strategy in every curriculum" (p. 114). Rhetorically, he asks, "Is it providing a service to necessary diversity in human thought by restricting the learning sequence to one format of building mathematical actions, mathematical processes and mathematical objects?" (p. 117). I must agree.

The characteristics of the anterior cortex and posterior cortex may offer some clues with regard to the distinction between procedural reasoning and conceptual reasoning. It may also suggest ways in which the APOS theory may be augmented to account for the primacy of the posterior cortex for image-based geometrical reasoning. The next section discusses generalization and abstraction with respect to the cerebral cortex and the cerebellum.

7.2 Theories of Generalization and Abstraction in Mathematics Education

Mathematical reasoning requires precision and clarity, a quality of image-based reasoning that is delivered by the cerebellum, according to the CSH. The cerebellum schematizes the fuzzy, indistinct extensional concepts of the cerebral cortex by eradicating incidental properties, resulting in purified mathematical concepts. This sounds very much like "abstraction," or at least one sense of the term. This section suggests ways in which abstraction, and also generalization, as understood in the mathematics education community, may be interpreted in the neurological terms of this dissertation.

Firstly, let me clarify my interpretation of the meaning of “generalization.” Generalization is related to abstraction and is often confused with abstraction (Michelmore, 2002). While abstraction, or at least one form of it, may concern the cerebellum and intensional concepts, generalization may be understood as the process by which extensional concepts are formed in the cerebral cortex.

Firstly, in psychological terms, a very reasonable definition of generalization is extension of the domain of a concept (Michelmore, 2002). For example, a generalization of the arithmetical structure of the natural numbers is the concept of ring and a generalization of the arithmetical structure of the real numbers is the concept of field. Equivalently, the natural numbers are a particular ring and the real numbers are a particular field. One must be careful, however, because *when the “domain is extended” the concept changes*. In other words, the concept “natural numbers” (with respect to their arithmetical structure) and the concept “ring” are quite different. In like manner, the image-based triangle concept and image-based triangle percept are related as general and particular—the “domain” of a particular triangle is extended to encompass all triangles. The triangle concept may be regarded as a generalization of the triangle percept. (Remember that concepts and percepts, in my scheme are objects of the same type, differing merely in degree.) On a higher level, the concept “polygon” is a generalization of the concept “triangle” or equivalently, the triangle is a particular polygon.

The psychological interpretation of generalization has a precise neurophysiological correlate in the cognitive network structure of the posterior cortex. Generalization may be regarded, objectively, as the association of lower level cognitive networks into higher level networks. Generalization, therefore, is extensional, belonging more naturally to the cerebral cortex, whereas abstraction is intensional, belonging more naturally to the cerebellum (according to the CSH).

It seems simplistic to associate the cerebral cortex with generalization and the cerebellum with abstraction. However, here, as elsewhere in this chapter, the neuroscience suggests ways for clarifying the relationships between ideas. I do not claim to have plumbed the depths of these deepest of topics, but merely to suggest fruitful ways of looking at them and then hypothesizing about them. The veracity of such hypotheses, of course, requires empirical support.

Piaget (e.g., Piaget, 1929/1960; Piaget & Inhelder, 1948/1963) is the father of constructivism (see Confrey & Kazak, 2006, e.g., pp. 308-309). He also wrote extensively on various forms of abstraction. Dubinsky (1991) explains that Piaget's thought on abstraction is contained in many locations in his corpus, and it underwent development over a long period. A clear understanding of Piaget's views on abstraction is difficult to obtain from primary sources, and so I use instead Dubinsky's own analysis and summary of Piaget's work.

According to Dubinsky (1991), Piaget identified three kinds of abstraction. Firstly,

> *Empirical abstraction* derives knowledge from the properties of objects (Beth & Piaget, 1966, pp. 188-189). We interpret this to mean that it has to do with experiences that appear to the subject to be external. The knowledge of these properties is, however, internal and is the result of constructions made internally by the subject. According to Piaget, this kind of abstraction leads to the extraction of common properties of objects and extensional generalizations, that is, . . . the passage from the specific to the general. (p. 97, author's italics)

Piaget seems to mean “abstraction” here in the sense of forming concepts rather than refining or purifying concepts. The “extraction of common properties of objects” should be taken, therefore, to mean that empirical abstraction is understood in an extensional sense, in the comparison of different objects, and in the relationship between different objects because of their common properties. Empirical abstraction, when understood in this way, refers to the process of concept formation in the posterior cognitive network hierarchy, the assimilation of percepts into a concept. Mitchelmore (2002) writes, “Empirical abstraction undoubtedly occurs in the early stages of the development of many . . . *spatial* concepts, such as triangle, symmetry, and circle” (p. 159, italics added). Empirical abstraction can reasonable be interpreted as a form of generalization.

Secondly,

> *Pseudo-empirical abstraction* . . . teases out properties that the actions of the subject have introduced into objects (Piaget, 1985, pp. 18-19). Consider, for example the observation of a 1-1 correspondence between two sets of objects which the subject has placed in alignment (*ibid*., p. 39). Knowledge of this situation may be considered empirical because it has to do with the objects, but it is their configuration in space and relationships to which this leads that are of concern and these are due to the actions of the subject. Again, of course, understanding that there is a 1-1 relation between these two sets is the result of internal constructions made by the subject. (Dubinsky, 1991, p. 97, author’s italics)

Pseudo-empirical abstraction, then, results from actions on objects, but it is still “empirical because it has to do with the objects.” My interpretation of this passage is that pseudo-empirical abstraction belongs to the action hierarchy, but in the sense of actions that are perceived through the perception-action loop. It is still, therefore, empirical abstraction, but empirical abstraction on actions that become perceived. It is a process in which the anterior cortex has a larger role.

Thirdly, and lastly,

> [*R*]*eflective abstraction* is drawn from what Piaget (1980, pp. 89-97) called the *general coordinations* of actions and, as such, its source is the subject and it is completely internal. . . . This kind of abstraction leads to a very different sort of generalization which is constructive. (Dubinsky, 1991, p. 97, author's italics)

The term "general coordinations of actions" recalls the idea of procedure. To clarify,

> [R]eflective abstraction differs from empirical abstraction in that it deals with action as opposed to objects and it differs from pseudo-empirical abstraction in that *it is concerned, not so much with the actions themselves, but with the interrelationships among actions*, which Piaget (1976, p. 300) called 'general coordinations.' (Dubinsky, 1991, p. 99, italics added)

This seems to point fairly clearly to the notion of procedure. Moreover,

> Whatever is thus [reflectively] 'abstracted' is projected onto a higher plane of thought (1985, pp. 29-31) where other actions are present as well as more powerful modes of thought.
>
> It is at this point that the real power of reflective abstraction comes in. (Dubinsky, 1991, p. 99)

I suggest that the projection is not necessarily to "higher planes of thought," but rather projection from the anterior cortex to the posterior cortex via the perception-action loop, and the procedure is encapsulated thereby in a concept. According to Czarnocha et al. (1999), reflective abstraction is the inspiration for APOS theory.

My interpretation of Piaget, therefore, via Dubinsky (1991), is that empirical abstraction is the process of concept formation in the posterior cortex, (i.e., generalization), pseudo-empirical abstraction is the process of concept formation on actions that become perceptible in the posterior cortex, and reflective abstraction is perception directly, through the perception-action loop, of procedures of the anterior cortex.

According to Dubinsky (1991),

> [T]hese different kinds of abstraction are not completely independent. The actions that lead to pseudo-empirical and reflective abstraction are performed on objects whose properties the subject only comes to know through empirical abstraction. On the other hand, empirical abstraction is only made possible through assimilation schemas which were constructed by reflective abstraction (Piaget, 1985, pp. 18-19). . . . This mutual interdependence can be roughly summarized as follows. Empirical and pseudo-empirical abstraction draws knowledge from objects by performing (or imagining) actions on them. Reflective abstraction interiorizes and coordinates these actions to form new actions and, ultimately new objects . . . . Empirical abstraction then extracts data from these new objects through mental actions on them, and so on. (p. 98)

Dubinsky, here, is interpreting the relationship between the forms of abstraction as a feedback cycle which proceeds from lower to higher levels of mathematical abstraction. This is the full-blown APOS theory, which cycles through actions, processes, and objects.

The reader may note that all forms of abstraction identified by Piaget are concerned with concept formation in the posterior cortex. There is nothing to suppose that these forms of abstraction in any way apply to schematizing, or purifying, mathematical concepts (and procedures). With respect to empirical abstraction and pseudo-empirical abstraction, although Piaget (as interpreted by Dubinsky, 1991) is not explicit, we may assume that the concepts formed thereby are extensional rather than intensional—in other words, they are assimilations by means of the relationship of similarity between many particulars.

Skemp’s notion of abstraction also is extensional. He writes,

> *Abstracting* is an activity by which we become aware of similarities . . . among our experiences. *Classifying* means collecting together our experiences on the basis of these similarities. An *abstraction* is some kind of lasting change, the result of abstracting, which enables us to recognize new experiences as having the similarities of an already formed class. . . . To distinguish between abstracting as an activity and abstraction as an end-product, we shall . . . call the latter a *concept*. (Skemp, 1986, cited in Mitchelmore, 2002, p. 158; Skemp's italics, Mitchelmore's ellipsis)

Mitchelmore and White (2004) claim, "As students relate together situations which were previously conceived as disconnected . . . they form new ideas . . . . In a sense, these new ideas *embody* the similarities recognized" (p. 332, authors' italics). Explicitly, this is identified with Paiget's ideas: "The process of similarity recognition followed by embodiment of the similarity in a new idea is an *empirical abstraction* process" (p. 332, authors' italics).

The abstractions of Skemp (Michelmore, 2002) and Michelmore and White (2004) may both be regarded as forms of generalization, as defined above. However, as with Piaget, although it seems to be the case that the neuroscience perspective brings clarity, there are fine points and subtleties in these authors' psychological descriptions of abstraction, and my attempts to "simplify" should be regarded strictly as provisional.

As mentioned in the previous section, in the discussion of APOS theory, Tall (1999) argues that the procedure-to-concept rule of APOS theory is too prescriptive. There may be an alternative route for concept formation, for geometrical concepts, in particular. This is the origin of the procept idea, which is placed here rather than following the description of APOS theory in the previous section because of the references in the current section to various forms of abstraction. Tall (1995) supposes that there are distinct sequences of concept construction, one in geometry, which follows the development proposed by van Hiele (1986), and another in arithmetic and algebra, based on symbols that represent both process and concept. The former develops via empirical abstraction; the latter, in Tall's interpretation, develops by pseudo-empirical abstraction at first, followed by reflective abstraction, allowing the mind to go back and forth between the "process" resulting from pseudo-empirical abstraction and the object resulting from reflective abstraction. The latter is called a *procept* by Gray and Tall (1991, 1994). The idea is developed fully also in Tall (1999).

Tall et al. (2000) distinguish various types of object construction, which result in perceived objects, platonic objects, procepts, and defined objects:

> Perceived objects arise through empirical abstraction, and more sophisticated platonic objects may be later constructed through reflective abstraction. Procepts arise first through pseudo-empirical abstraction from actions on real-world objects and then by higher level reflective abstraction on the resulting conceived objects that [are] represented by symbols enabling us to pivot between process and concept. Defined objects occur by reflective abstraction from the properties of perceived or conceived objects (including both platonic and proceptual). . . . Platonic objects, procepts, and defined objects are all categories of conceived objects. (p. 239)

In my opinion, the platonic objects of Tall et al. are the perceived objects after they have undergone schematization. The defined objects and the symbolic aspect of procepts lie outside the scope of the current discussion.

The "abstraction" processes described so far seem to result in extensional concepts, or "generalizations." An exception, perhaps, is the process that results in the platonic objects of Tall et al. (2000). What about abstraction that is *intensional*, resulting in a schematic concept with the level of purity required for mathematical reasoning?

Mitchelmore and White (1995, 2004) refer to the concepts formed by empirical abstraction as *abstract-general*. The authors claim, “Empirical [i.e., abstract-general] concepts are often rather fuzzy and difficult to define” (Mitchelmore & White, 2004, p. 333)—exactly! They contrast abstract-general concepts with *abstract-apart* concepts, the latter being mathematical concepts produced by formal definition.

There is a similar distinction between concept image and concept definition in Tall and Vinner (1981):

> We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures. (p. 152, authors’ italics)

Although there is no reference to concept formation here, the concept image seems to correspond to the concept that results from an association of cognitive networks. In fact, the authors make explicit reference to neurophysiology:

> Sensory input excites certain neuronal pathways and inhibits others. In this way different stimuli can activate different parts of the concept image. . . . .
>
> We shall call the portion of the concept image which is activated at a particular time the *evoked concept image*. (p. 152, authors’ italics)

It seems the authors are referring to the mechanism of selective attention discussed in Chapters 4 and 5. The concept image, it may be assumed, is the unschematized concept whose neural correlate is a cognitive network in the cerebral cortex, with all its associations. Tall and Vinner recognize that the concept image is insufficient for mathematical reasoning and that another form of concept is needed. They write,

> We shall regard the *concept definition* to be a form of words used to specify that concept. It may be learned by an individual in a rote fashion or more meaningfully learnt and related to a greater or lesser degree to the concept as a whole. (p. 152, authors' italics)

Both the abstract-apart concept of Mitchelmore and White (1995, 2004) and the concept definition of Tall and Vinner (1981) are attempts to produce schematized, pure concepts by means of formal definition. However, formal definition of the mathematical concept does not necessarily lead to actual apprehension of the schematic concept itself. It is possible that the formal definition can *prompt* cognition of the schematic concept, but the two are essentially dissimilar experiences. The student needs to *see* the triangle as a mathematical triangle, devoid of incidental properties, rather than be told that the triangle is a polygon with three sides.

A century ago, Godfrey (1910) called this the "geometrical 'eye'" (p. 197). He refers to developing students' "power of seeing geometrical properties detach themselves from a figure" (p. 197). Fischbein (1993) suggests something similar:

> I do not intend to affirm that the representation we have in mind, when imagining a geometrical figure, is devoid of any sensorial quality (like color) except space properties. But I affirm that, while operating with a geometrical figure, we act *as if no other quality counts*. (p. 143, author's italics)

Harel and Tall (1991) write, "An abstraction process occurs when the subject focuses attention on specific properties of a given object and then considers these properties in isolation from the original" (p. 39). According to Sierpinska, "I am used to thinking of abstraction as a dual mental activity whereby some aspects of the object of thought are ignored while other [sic] are highlighted" (Boero et al., 2002, pp. 129-130). These authors are all referring to *seeing* the schematic concept rather than defining it formally, or more accurately, attending to those properties of the percept that are essential.

However, for more detailed consideration of the schematic concept and its significance for mathematics we must go back more than 2,300 years, to Aristotle. According to Mendell (2004), mathematical objects for Aristotle are created by abstraction, or "removal" (*ta ex aphaireseôs*):

> In the *Analytics* . . . Aristotle begins with a particular geometrical perceptible figure. What is removed is its particularity and all that comes with this, including its being perceptible. What is left then is a universal of some sort. (Section 7.1, para. 2)

It is interesting that Aristotle requires removing even the object's perceptibility, because then a "concept" would exist in pure abstraction without a percept. It is possible to understand this to mean that there is no attentional focus on any single percept subnetwork of the cognitive network of the concept. In other words, there is no distinct percept that resolves from the concept. In my understanding, however, perceptibility is by default an essential property, and so the percept is retained. I wish to stop short of the full Aristotelian rigor.

Mendell (2004) writes also of Aristotle's notion of precision (*akribeia*). Sciences that have more properties removed are more precise. Mathematics is the most precise of all the sciences. To describe the cerebellum as instrument of mathematical precision, as I would like to with the CSH, would be a deliberately Aristotelian use of the term "precision."

With regard to Aristotle, I briefly discussed Lear's (1982) interpretation of the *qua* operator in Handscomb (2005). This discussion is paraphrased in the following. Lear writes,

> Let *b* be a [physical object] and let '*b qua F*' signify that *b* is being considered *as an F*. Then a property is said to be true of *b qua F* if and only if *b* is an *F* and its having that property follows of necessity of its being an *F*. (p. 168, author's italics)

In other words, "door *qua* rectangle" means that the door is being considered as a rectangle. By this means, the mathematician places himself behind a "veil of ignorance" (p. 168), in which he can say nothing about the properties of *b* that do not follow from necessity of its being an *F*. Those properties of *b* that do follow of necessity are the *essential* properties of *b qua F*, whereas those properties of *b* that do not follow of necessity are *incidental* properties of *b qua F* (pp. 168-169). My usage of the terms "essential" and "incidental" should also be regarded, therefore, as Aristotelian. Behind the "veil of ignorance," the percept is schematized to its essential properties.

Most interpretations of "abstraction" refer to concept formation in an extensional sense, where the concept associates similarities between particular objects and assimilates them to a single cognitive network. I would prefer to regard this as generalization. Some treatments, including Aristotle's, interpret "abstraction," on the other hand, in an intensional sense, where concepts result from the removal of incidental properties from a particular object. The latter sense is closest to the idea of schematic concept. However, even when concepts appear to be defined intensionally by a process of Aristotelian abstraction, the implication is that the concept is somehow a reduced version of a percept, and the percept is somehow a facsimile representation of a real external object. The treatment herein is a little different. Firstly, the “real” external object is often implied by the percept (although not always, because the percept may be imagined, as in a dream or in conscious imagination), but as far as cognition is concerned all that matters is the subjective percept and its objective neural correlate in the cognitive network. Secondly, *the concept is not a reduced version of the percept, but is equal in range with the percept, and the two are schematized in tandem*. The metaphor to bear in mind is the narrowing of Bergson’s cone (or cylinder) of duration, by means of which reasoning becomes more precise, in the Aristotelian sense. Because of confusion in the literature with the terms “generalization” and “abstraction” I have declined to assign them a technical meaning. Instead, I refer to “concept formation” in the extensional sense and “schematization” in the intensional sense, as appropriate.

▲►▼◄▲►▼◄▲►▼◄

It is interesting to note that the various potential directions for future research outlined in this chapter touch on much of the existing research in mathematics education. The various notions of the distinction between procedural reasoning and conceptual reasoning are manifestations of a single underlying principle that is instantiated physiologically by the fissure dividing anterior cortex from posterior cortex. Moreover, this perspective naturally leads to a critique of theories of concept formation in which procedures always precede concepts. For image-based geometrical reasoning, in particular, it seems likely that visual concepts are prior to procedures. On the other hand, the role of the cerebellum in schematization, in accordance with the CSH, is a new approach to the notion of mathematical abstraction. Various notions of abstraction can be understood either from the cognitive network perspective within the cerebral cortex, in other words as generalizations, or from the cerebellar perspective as schematization. A physiological divide, between cerebral cortex and cerebellum, can potentially clarify the discussion of abstraction in the literature of mathematics education.

CHAPTER 8: REFLECTION AND CRITIQUE

In this final chapter, it is time to evaluate the contributions and limitations of the dissertation. In addition, some suggestions can be made with respect to empirical substantiation of my largely theoretical research.

8.1 Contributions

The contribution of this dissertation to the field of mathematics education, specifically mathematics educational neuroscience, mainly consists in the working through of the following line of thought: (1) the idea that effective image-based geometrical reasoning is necessarily schematic, (2) the hypothesis that a functional role of the cerebellum is to facilitate schematic perception and inferencing, and (3) the hypothesis that decontextualization of mathematical concepts is a valuable pedagogical strategy. The innovations are reviewed chapter by chapter.

The main contribution of Chapter 4 is the property analysis, from which follows (1) the distinction between essential and incidental properties, (2) clarification of the extensional nature of cerebral concepts, (3) the notion that concepts and percepts have equality of range, and (4) the definition of schematization as constraining concepts and percepts to the essential properties. These ideas in themselves are not novel (except perhaps the notion of equality of range). Their consequence is a new way of understanding Mason and Pimm's (1984) phrase, "seeing the general in the particular." Bergson's cone of duration is reinterpreted to provide an evocative metaphor for schematization.

Many of the arguments for the functional role of the cerebellum in Chapter 5 have been extracted from the literature on the cerebellum. However, the emphasis on perception and the posterior cortex is new, as is the hypothesis itself that the cerebellum is responsible for schematizing concepts and percepts of the posterior cortex. In the literature, the emphasis is overwhelmingly directed toward action-based cognition and the anterior cortex. Ideas such as dysmetria of thought and the role of the cerebellum in selective attention come close to the CSH. Schematization of concepts and percepts, a variety of selective attention, can incorporate these theories in a coherent framework. The CSH requires further empirical substantiation.

I propose a new distinction in Chapter 6 between cerebral learning and cerebellar learning. Cerebral learning refers to the establishment of new cognitive networks for concepts (and procedures) in the cerebral cortex. These concepts are indistinct and extensional. Cerebellar learning requires input from the cerebellum to schematize these fuzzy, extensional concepts, making them pure and intensional. This distinction relies on the CSH.

With regard to consequences of the CSH for the classroom, Chapter 6 presents the decontextualization hypothesis. The decontextualization hypothesis proposes that geometrical concepts should be presented in ways that are decontextualized in order to facilitate cerebellar learning. In practical terms, if a teacher wishes to encourage student engagement in mathematical reasoning, then mathematical situations pertaining to image-based geometrical reasoning should be presented as starkly as possible, without distractors. This does not entail that more contextualized and situated “scaffolding” would be inappropriate. The decontextualization hypothesis requires further empirical substantiation.

I propose a number of potential implications of the theory for mathematics education in Chapter 7. Firstly, the cognitive network model of the cerebral cortex, and its division into action and perception, may shed light on the distinction between procedural reasoning and conceptual reasoning and on theories of concept formation. In particular, Tall’s (1999) critique that the APOS theory of Czarnocha et al. (1999) is less appropriate for geometrical reasoning than for algebraic or arithmetical reasoning may now be realized in terms of the perception-action loop and the symmetry of the anterior cortex and posterior cortex.

Secondly, I develop the idea of cerebellar schematization as a new perspective on abstraction in Chapter 7. It may shed light on some theories of abstraction that are prevalent in mathematics education literature. In particular, it seems that Piaget’s three forms of abstraction—empirical, pseudo-empirical, and reflective—refer primarily to cerebral rather than cerebellar processes. Cerebral processes concern generalization, whereas cerebellar processes concern abstraction. The ideas in Chapter 7 represent directions for future research rather than established positions..

8.2 Limitations

It is apparent that a number of new ideas, analyses, and perspectives are offered throughout the dissertation. In some of them I have more confidence than others. The cognitive network approach appears to be eminently reasonable, and according to Fuster (2003) it is supported by an immense quantity of empirical evidence. The arguments based on a property analysis within cognitive network theory, however, represent a new theoretical orientation, which should be tested empirically and specifically. It follows that the property analysis arguments throughout are speculative.

The arguments I present for the CSH are not conclusive. They may support its plausibility, but the CSH itself remains an unsubstantiated hypothesis. In Chapter 5 I suggest some preliminary ideas for an experimental paradigm that would lead to substantiation or refutation of the CSH.

Decontextualization to facilitate cerebellar learning is an intuitively reasonable idea, and there appears to be (qualified) support for it from mathematics education research. However, it is also an unsubstantiated hypothesis, which requires further work.

Throughout the dissertation, theories of the cerebral cortex and the cerebellum are united with research in mathematics education by means of a theoretical framework drawing on embodied cognition and the philosophy of duration. It was not possible to be fully comprehensive with regard to the literature of any of these areas. I have tried to ensure that the background literature cited is important and seminal. However, the reader should be aware that there is an immense quantity of additional literature, particularly in cognitive neuroscience, much of it directly or indirectly relevant to the discussion herein.

Even with respect to mathematics education, the research cited herein is far from comprehensive. There is no account of the influence of language, consciousness, affect, or society on mathematics education theory. All of these areas are undoubtedly of great educational significance, but all were deliberately passed over for now.

8.3 Summary

To complete this chapter on the contributions and limitations of my dissertation I will return to and reiterate some comments that I made in Chapter 1. From the perspective of mathematics education alone there may be concerns that my research does have the depth and coverage that would be required of a work existing solely within the field of mathematics education.

On the other hand, from the cognitive neuroscience perspective, my research obviously does not achieve the full nuanced detail that would be required of a work in cognitive neuroscience per se. My background research and knowledge is necessarily limited. My understanding of the functional relationships between brain structures may appear simplistic.

Nevertheless, my research is not a work in mathematics education alone, and neither does it belong entirely to cognitive neuroscience. Rather, it exists in the middle ground between the two, in which results, conclusions, and theories on one side inform and constrain results, theories, and conclusions on the other. In this respect, my research is a distinct contribution to educational neuroscience. To my knowledge, no other work has attempted to accomplish anything similar, a coherent, albeit limited, integration and synthesis of the fields of mathematics education and cognitive neuroscience. I would hope that my readers have been able to place themselves alongside me, at least temporarily, in the new field of educational neuroscience.

▲►▼◄▲►▼◄▲►▼◄

The big picture of this dissertation is a new understanding of concepts and percepts. They have the same range in that they share the same properties. Concepts have greater potential. Concepts of the cerebral cortex are extensional. Concepts schematized by the cerebellum are intensional. Schematized concepts have no greater potential than schematized percepts. In reinterpreting the meaning of “seeing the general in the particular,” mathematical ideas are perceived directly, unmasked and austerely beautiful. Mathematics educators must seek ways to develop students’ ability to “see the general in the particular.”

REFERENCE LIST

Abbie, A. A. (1934). The projection of the forebrain on the pons and cerebellum. *Proceedings of the Royal Society of London, Series B*, *115*, 504-522.

Ackermann, H., Mathiak, K., & Riecker, A. (2007). The contribution of the cerebellum to speech production and speech perception: clinical and functional imaging data. *Cerebellum*, *6*, 202–213.

Akshoomoff, N. A., & Courchesne, E. (1992). A new role for the cerebellum in cognitive operations. *Behavioral Neuroscience*, *106*(5), 731-738.

Allen, G., Buxton, R. B., Wong, E. C., & Courchesne E. (1997). Attentional activation of the cerebellum independent of motor involvement. *Science*, *275*(5308), 1940-1943.

Allen, G., McColl, R., Barnard, H., Ringe, W. K., Fleckenstein, J., & Cullum, C. M. (2005). Magnetic resonance imaging of cerebellar-prefrontal and cerebellar-parietal functional connectivity. *NeuroImage*, *28*, 39-48.

Altman, J., & Bayer, S. A. (1997). *Development of the cerebellar system in relation to its evolution, structure, and functions*. New York: CRC Press.

Anderson, J. R., Reder, L. M., & Simon, H. A. (2000). Applications and misapplications of cognitive psychology to mathematics education. *Texas Educational Review*. Retrieved December 16, 2008 from http://act-r.psy.cmu.edu/papers/misapplied.html

Apps, R., & Garwicz, M. (2005). Anatomical and physiological foundations of cerebellar information processing. *Nature Reviews: Neuroscience*, *6*, 297-311.

Bana, J., & Nelson, D. (1978). Distractors in nonverbal mathematical problems. *Journal for Research in Mathematics Education*, *9*(1), 55-61.

Battista, M. T., & Borrow, C. Van Auken (1997). Shape Makers: A computer microworld for promoting dynamic imagery in support of geometric reasoning. In J. Dossey, J. O. Swafford, M. Parmantie, & A. E. Dossey (Eds.), *Proceedings of the 19th annual conference of the North American chapter of the International Group for the Psychology of Mathematics Education, Vol. 2* (pp. 571-578). Bloomington-Normal, Illinois. (ERIC Document Reproduction Service No. ED420495)

Ben-Yehudah, G., Guediche, S., & Fiez, J. A. (2007). *The Cerebellum*, *6*, 193-201.

Bengtsson, S. L., Ehrsson, H. H., Forssberg, H., & Ullén, F. (2005). Effector-independent voluntary timing: Behavioural and neuroimaging evidence. *European Journal of Neuroscience*, *22*, 3255-3265.

Bergson, H. L. (1944). *Creative evolution* (A. Mitchell, Trans.). New York: Random House. (Original work published 1907)

Bergson, H. L. (1960). *Time and free will* (F. L. Pogson, Trans.). New York: Harper and Brothers. (Original work published 1889)

Bergson, H. L. (2005). *Matter and memory* (N. M. Paul & W. S. Palmer, Trans.). New York: Zone Books. (Original work published 1896)

Bergson, H. (1999). *An introduction to metaphysics* (T. E. Hulme, Trans.). Indianapolis, IN: Hackett Publishing Company. (Original work published 1903)

Bjaalie, J. G., & Brodal, P. (1997). Cat pontocerebellar network: numerical capacity and axonal collateral branching of neurons in the pontine nuclei projecting to individual parafloccular folia. *Neuroscience Research*, *27*, 199-210.

Black, J., & Greenough, W. (1986). Induction of pattern in neural structure by experience: Implications for cognitive development. In M. E. Lamb, A. L. Brown, & B. Rogoff (Eds.), *Advances in developmental psychology. Vol. 4* (pp. 1-50). Hillsdale, NJ: Lawrence Erlbaum.

Bloedel, J. R., & Bracha, V. (1997). Duality of cerebellar motor and cognitive functions. In J. D. Schmahmann (Ed.), *The cerebellum and cognition* (*International review of neurobiology*), Vol. 41 (pp. 613-634). San Diego, CA: Academic Press.

Boero, P., Dreyfus, T., Gravemeijer, K., Gray, E., Hershkowitz, R., Schwartz, B., Sierpinska, A., & Tall, D. (2002). Abstraction: Theories about the emergence of knowledge structures. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th annual conference of the International Group for the Psychology of Mathematics Education: Vol. 1* (pp. 113-138). Norwich, England. (ERIC Document Reproduction Service No. ED476065)

Bower, J. M. (1997). Is the cerebellum sensory for motor's sake, or motor for sensory's sake: The view from the whiskers of a rat. In C. E. de Zeeuw, P. Strata, & J. Voogd (Eds.), *The cerebellum: From structure to control* (pp. 463-498). Amsterdam: Elsevier.

Brodal, P., & Bjaalie, J. G. (1997). Salient anatomic features of the cortico-ponto-cerebellar pathway. *Progress in Brain Research*, *114*, 227-249.

Brown, J. W. (2000). *Mind and nature: Essays on time and subjectivity*. Chichester, England: John Wiley & Sons Inc.

Brown, J. W. (2002). *The self-embodying mind: Process, brain dynamics and the conscious present*. Barrytown, NY: Station Hill.

Bruer, J. T. (1997). Education and the brain: A bridge too far. *Educational Researcher*, *26*(8), 4-15.

Bruer, J. T. (1999). In search of . . . brain-based education. *Phi Delta Kappan*, *80*(9), 648-657.

Bruer, J. T. (2006). On the implications of neuroscience research for science teaching and learning: Are there any? *CBE-Life Sciences Education*, *5*, 104-110.

Butterworth, B. (1999). *What counts: How every brain is hardwired for math*. New York: The Free Press.

Byrnes, J. P. (2001). *Minds, brains, and learning: Understanding the psychological and educational relevance of neuroscientific research*. New York: Guilford Press.

Campbell, S. R. (1993). *The embodied mind and worlds without ground* (unpublished).

Campbell, S. R. (1998). *Preservice teachers' understanding of elementary number theory: Qualitative constructivist research situated within a Kantian framework for understanding educational inquiry*. Doctoral thesis, Simon Fraser University, Burnaby, British Columbia, Canada.

Campbell, S. R. (2001). Enacting possible worlds: Making sense of (human) nature. In J. F. Matos, W. Blum, S. K. Houston, & S. P. Carreira (Eds.), *Modelling and mathematical education* (pp. 3-14). Chichester, England: Horwood Publishing.

Campbell, S. R. (2003). Reconnecting mind and world: Enacting a (new) way of life. In S. J. Lamon, W. A. Parker, & S. K. Houston (Eds.), *Mathematical Modelling: A Way of Life: International Conference on the Teaching of Mathematical Modelling and Applications (11th: 2003: Milwaukee, Wisconsin)* (pp. 245-253*)*. Chichester, England: Horwood Publishing.

Campbell, S. R. (2006). Defining mathematics education neuroscience. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education: Vol. 2* (pp. 442-449). Mérida, México.

Campbell, S. R. (2010). Embodied minds and dancing brains: New opportunities for research in mathematics education. In B. Sriraman & L. English (Eds.), *Theories of Mathematics Education* (pp. 309-331). New York: Springer-Verlag.

Campbell, S. R., & Handscomb, K. (2007, April). An embodied view of mind-body correlates. Paper presented to the American Educational Research Association: Brain, Neuroscience, and Education SIG. Chicago, IL, U.S.A.

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1985). Mathematics in the streets and in schools. *British Journal of Developmental Psychology*, *3*, 21-29.

Christou, C., Mousoulides, N., Pittalis, M., & Pitta-Pantazi, D. (2005) Problem solving and problem posing in a dynamic geometry environment. *The Montana Mathematics Enthusiast*, *2*(2), 125-143. Retrieved March 31, 2009, from http://www.math.umt.edu/tmme/vol2no2/TMMEv2n2a5.pdf

Clower, D. M., West, R. A., Lynch, J. C., & Strick, P. L. (2001). The inferior parietal lobule is the target of output from the superior colliculus, hippocampus, and cerebellum. *The Journal of Neuroscience*, *21*(16), 6283-6291.

Cohen, L., & Dehaene, S. (1995). Number processing in pure alexia: The effect of hemispheric asymmetries and task demands. *Neurocase*, *1*, 121-137.

Cohen, L., & Dehaene, S. (1996). Cerebral networks for number processing: Evidence from a case of posterior callosal lesion. *Neurocase*, *2*, 155-174.

Confrey, J., & Kazak, S. (2006). A thirty-year reflection on constructivism in mathematics education in PME. In A. Gutiérrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education, Past, Present, and Future* (pp. 305-345). Rotterdam, Netherlands: Sense Publishers.

Cordes, D., Haughton, V. M., Arfanakis, K., Wendt, G. J., Turski, P. A., Moritz, C. H., Quigley, M. A., & Meyerand, M. E. (2000). Mapping functionally related regions of brain with functional connectivity MR imaging. *American Journal of Neuroradiology*, *21*, 1636-1644.

Courchesne, E., Townsend, J., Akshoomoff, N. A., Saitoh, O., Yeung-Courchesne, R., Lincoln, A. J., James, H. E., Haas, R. H., Schreibman, L., & Lau. L. (1994). Impairment in shifting attention in autistic and cerebellar patients. *Behavioral Neuroscience*, *108*(5), 848-865.

Czarnocha, B., Dubinsky, E., Prabhu, V., & Vidakovic, D. (1999). One theoretical perspective in undergraduate mathematics education research. In O. Zaslavsky (Ed.), *Proceedings of the 23rd annual conference of the International Group for the Psychology of Mathematics Education: Vol. 1* (pp. 95-110). Haifa, Israel.

Dehaene, S. (1992). Varieties of numerical abilities. *Cognition*, *44*, 1-42.

Dehaene, S. (1996). The organization of brain activations in number comparison: Event-related potentials and the additive-factors method. *Journal of Cognitive Neuroscience*, *8*(1), 47-68.

Dehaene, S. (1997). *The number sense: How the mind creates mathematics*. New York: Oxford University Press.

Dehaene, S., & Cohen, L. (1995). Towards an anatomical and functional model of number processing. *Mathematical Cognition*,*1*, 83-120.

Dehaene, S., & Cohen, L. (1997). Cerebral pathways for calculation: Double dissociation between rote verbal and quantitative knowledge of arithmetic. *Cortex*, *33*, 219-250.

Dehaene, S., Izard, V., Pica, P., & Spelke, E. (2006). Core knowledge of geometry in an Amazonian indigene group. *Science*, *311*, 381-384.

Dehaene, S., Molko, N., Cohen, L., & Wilson, A. J. (2004). Arithmetic and the brain. *Current Opinion in Neurobiology*, *14*, 218-224.

Dehaene, S., Piazza, M., Pinel, P., & Cohen, L. (2004). Three parietal circuits for number processing. In J. I. D. Campbell (Ed.), *Handbook of mathematical cognition* (pp. 433-453). New York: Routledge.

Depraz, N., Varela, F. J., & Vermesch, P. (2000). The gesture of awareness: An account of its structural dynamics. In M. Velmans (Ed.), *Investigating phenomenal consciousness: New methodologies and maps* (pp. 121-136). Philadelphia, PA: J. Benjamins Publishing Company.

Desmond, J. E., & Fiez, J. A. (1998). Neuroimaging studies of the cerebellum: Language, learning and memory. *Trends in cognitive Sciences*, *2*(9), 355-362.

DeYoe, E. A., & Van Essen, D. C. (1988). Concurrent processing streams in monkey visual cortex. *Trends in Neuroscience*, *11*(5), 219-226.

Dow, R. S. (1974). Some novel concepts of cerebellar physiology. *Mount Sinai Journal of Medicine*, *41*, 103-119.

Dow, R. S. (1988). Contribution of electrophysiological studies to cerebellar physiology. *Journal of Clinical Neurophysiology*, *5*(4), 307-323.

Doyson, J. (1997). Skill learning. In J. D. Schmahmann (Ed.), *The cerebellum and cognition* (*International review of neurobiology*), Vol. 41 (pp. 273-294). San Diego: Academic Press.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95-123). Dordrecht, The Netherlands: Kluwer.

Dum, R. P., & Strick, P. L. (2003). An unfolded map of the cerebellar dentate nucleus and its projections to the cerebral cortex. *Journal of Neurophysiology*, *89*, 634-639.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics*, *24*(2), 139-162.

Fiez, J. A., Petersen, S. E., Cheney, M. K., & Raichle, M. E. (1992). Impaired non-motor learning and error detection associated with cerebellar damage. *Brain*, *115*, 155-178.

Fiez, J. A., Raife, E. A., Balota, D. A., Schwarz, J. P., & Raichle, M. E. (1996). A positron emission tomography study of the short-term maintenance of verbal information. *The Journal of Neuroscience*, *16*(2), 808-822.

Fitting, M. (2007, February 26). Intensional logic. In E. N. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy (Spring 2009 Edition)*. Retrieved April 11, 2009 from http://plato.stanford.edu/archives/spr2009/entries/logic-intensional/

Fowler, D., & Robson, E. (1998). Square root approximations in Old Babylonian mathematics: YBC 7289 in context. *Historia Mathematica*, *25*(4), 366-378.

Frank, B., Schoch, B., Richter, S., Frings, M., Karnath, H. –O., & Timmann, D. (2007). Cerebellar lesion studies of cognitive function in children and adolescents—limitations and negative findings. *The Cerebellum*, *6*, 242-253.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht, The Netherlands: D. Reidel.

Friedman, M. (2000). Geometry, construction, and intuition in Kant and his successors. In G. Sher & R. Tieszen (Eds.), *Between logic and intuition* (pp. 186-218). Cambridge, UK: Cambridge University Press.

Friston, K. J. (1994). Functional and effective connectivity in neuroimaging: A synthesis. *Human Brain Mapping*, *2*, 56-78.

Fuster, J. M. (2003). *Cortex and mind: Unifying cognition*. New York: Oxford University Press.

Fuster, J. M. (2006). The cognit: A network model of cortical representation. *International Journal of Psychophysiology*, *60*, 125-132.

Gao, J-H., Parsons, L. M., Bower, J. M, Xiong, J., Li, J., & Fox, P. T. (1996). Cerebellum implicated in sensory acquisition and discrimination rather than motor control. *Science*, *272*, 545-547.

Gilbert, C. D., & Sigman, M. (2001). The neural basis of perceptual learning. *Neuron*, *31*, 681-697.

Glickstein, M. (2007). What does the cerebellum really do? *Current Biology*, *17*(19), 824-827.

Glickstein, M., & Doron, K. (2008). Cerebellum: Connections and functions. *Cerebellum*, *7*, 589-594.

Glickstein, M., Strata, P., & Voogd, J. (2009). Cerebellum: History. *Neuroscience*, *162*, 549-559.

Godfrey, C. (1910). The Board of Education circular on the teaching of geometry. *Mathematical Gazette*, *5*, 195-200.

Gottwald, B., Mihajlovic, Z., Wilde, B., & Mehdorn, H. (2003). Does the cerebellum contribute to specific aspects of attention? *Neuropsychologia*, *41*, 1452-1460.

Gray, E., & Tall, D. (1991). Duality, ambiguity and flexibility in successful mathematical thinking. In F. Furinghetti (Ed.), *Proceedings of the 15th annual conference of the International Group for the Psychology of Mathematics Education: Vol. 2* (pp. 72-79). Assisi, Italy. (ERIC Document Reproduction Service No. SE055792)

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education*, *26*(2), 115-141.

Greaves, M. (2002). *The philosophical status of diagrams*. Stanford, CA: CSLI.

Greenough, W., & Black J. (1992). Induction of brain structure by experience: Substrates for cognitive development. In M. Gunner & C. Nelson (Eds.), *Developmental behavioral neuroscience. Minnesota symposium on child psychology: Vol. 24* (pp. 155-200), Hillsdale, NJ: Lawrence Erlbaum.

Greenough, W. T., Black, J. E., & Wallace, C. S. (1987). Experience and brain development. *Child Development*, *58*(3), 539-559.

Grill-Spector, K., & Malach, R. (2004). The human visual cortex. *Annual Review of Neuroscience*, *27*, 649-677.

Haarmeier, T., & Thier, P. (2007). The attentive cerebellum—myth or reality? *The Cerebellum*, *6*, 177-183.

Habas, C., & Cabanis, E. A. (2007). Cortical projections to the human red nucleus: Complementary results with probabilistic tractography at 3 T. *Neuror91996adiology*, *49*, 777-784.

Hadamard, J. (1996). *The mathematician's mind: The psychology of invention in the mathematical field*. Princeton, NJ: Princeton University Press. Original work published 1945.

Hanakawa T., Dimyan M. A., & Hallett M. (2008). Motor planning, imagery, and execution in a distributed motor network: a time-course study with functional MRI. *Cerebral Cortex*, *18*, 2775–2788.

Handscomb, K. (2005). *Image-based reasoning in geometry*. Master's thesis, Simon Fraser University, Burnaby, British Columbia, Canada.

Handscomb, K. (2006). Principles of conceptualization for image-based reasoning in geometry. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th annual conference of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 418-420). Mérida, México.

Handscomb, K. (2007). Enactive cognition and Spinoza's theory of mind. In T. Lamberg & L. R. Wiest (Eds.), *Proceedings of the 29th annual conference of the North American chapter of the International Group for the Psychology of Mathematics Education*. Lake Tahoe, Nevada. Retrieved April 20, 2009 from http://www.allacademic.com//meta/p_mla_apa_research_citation/1/9/3/6/7/pages193673/p193673-1.php

Harel, G., & Tall, D. (1991). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics*, *11*(1), 38-42.

Hayek, F. A. (1976). *The sensory order: An enquiry into the foundations of theoretical psychology.* Chicago: University of Chicago Press. (Original work published 1952)

Heath, R. G. (1977). Modulation of emotion with a brain pacemaker: Treatment for intractable psychiatric illness. *Journal of Nervous and Mental Diseases*, *165*(5), 300-317.

Heath, T. (1956). *The thirteen books of Euclid's elements* (2nd ed., Vols. 1-3). New York: Dover. (Original work published in 1908)

Hebb, D.O. (1964). *The organization of behavior: A Neuropsychological theory.* New York: John Wiley & Sons.

Hegarty, M., & Kozhevnikov, M. (1999). Types of visual-spatial representations and mathematical problem solving. *Journal of Educational Psychology*, *91*(4), 684-689.

Hilbert, D. (1971). *Foundations of Geometry* (2nd ed.) (L. Unger, Trans.). La Salle, IL: Open Court. (Original work published 1899)

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory J analysis. In. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum Associates.

Hollander, B. (1920). In search of the soul and the mechanism of thought, emotion, and conduct. Vol. 1: The history of philosophy and science from ancient times to the present day. London: Kegan Paul, Trench, Trubner & Co., Ltd.

Holmes, G. (1939). The cerebellum of man. *Brain*, *62*, 1-30.

Hubbard, E. M., Piazza, M., Pinel, P., & Dehaene, S. (2005). Interactions between number and space in parietal cortex. *Nature Reviews: Neuroscience*, *6*, 435-448.

Husain, M., & Nachev, P. (2006). Space and the parietal cortex. *Trends in Cognitive Sciences*, *11*(1), 30-36.

Inhoff, A. W., Diener, H. C., Rafal, R. D., & Ivry, R. B. (1989). The role of cerebellar structures in the execution of serial movements. *Brain*, *112*, 565-581.

Ito, M. (1993). Movement and thought: Identical control mechanisms by the cerebellum. *Trends in Neuroscience*, *16*(11), 448-450.

Ito, M. (2006). Cerebellar circuitry as a neuronal machine. *Progress in Neurobiology*, *78*, 272-303.

Ito, M. (2008). Control of mental activities by internal models in the cerebellum. *Nature*, *9*, 304-313.

Jackiw, N. (1991, 2009). *The Geometer's Sketchpad (computer software, multiple versions)*. Emeryville, California: Key Curriculum Press.

James, W. (1950). *The principles of psychology, Vol. 1.* New York: Dover Publications, Inc. (Original work published 1890)

Kaminski, J. A., Sloutsky, V. M., & Heckler, A. F. (2008). The advantage of abstract examples in learning math. *Science*, *320*, 454-455.

Keele, S. W., & Ivry, R. (1990). Does the cerebellum provide a common computation for diverse tasks? A timing hypothesis. *Annals of the New York Academy of Sciences*, *608*, 179-211.

Kelly, R. M., & Strick, P. L. (2003). Cerebellar loops with motor cortex and prefrontal cortex of a nonhuman primate. *Journal of Neuroscience*, *23*, 8432-8444.

Kilner, J. M., Paulignan, Y., & Broussaoud, D. (2004). Functional connectivity during real vs imagined visuomotor tasks: An EEG study. *NeuroReport*, *15*(4), 637-642.

Kim, S. –G., Ugurbil, K., & Strick, P. L. (1994). Activation of a cerebellar output nucleus during cognitive processing. *Science*, *265*, 949-951.

Knudson,E. I. (2007). Fundamental components of attention, *Annual Review of Neuroscience*, *30*, 57-78.

Koechlin, E., Ody, C, & Kouneiher, F. (2003). The architecture of cognitive control in the human prefrontal cortex. *Science*, *302*, 1181-1185.

Koedinger, K. R., Alibali, M. W., & Nathan, M. J. (2008). Trade-offs between grounded and abstract representations: Evidence from algebra problem solving. *Cognitive Science*, *32*, 366-397.

Kosslyn, S. M. (1988). Aspects of a cognitive neuroscience of mental imagery. *Science*, *240*(4859), 1621-1626.

Kosslyn, S. M., Ganis, G., & Thompson, W. L. (2001). Neural foundations of imagery. *Nature Reviews: Neuroscience*, *2*, 635-642.

Kourtzi, Z., & DiCarlo, J. J. (2006). Learning and neural plasticity in visual object recognition. *Current Opinion in Neurobiology*, *16*, 152-158.

Krienen, F. M., & Buckner, R. L. (2009). Segregated fronto-cerebellar circuits revealed by intrinsic functional connectivity. *Cerebral Cortex*, *19*, 2485-2497.

Lachaux, J-P., Rodriguez, E., Martinerie, J., & Varela, F. J. (1999). Measuring phase synchrony in brain signals. *Human Brain Mapping*, *8*, 194-208.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books

Lashley, K. S. (1988). In search of the engram. In J. A. Anderson & E. Rosenfeld (Eds.), *Neurocomputing: Foundations of research* (pp. 59-63). Cambridge, MA: The MIT Press. (Original work published 1950)

Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life*. Cambridge, England: Cambridge University Press.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge, England: Cambridge University Press.

Lear, J. (1982). Aristotle's philosophy of mathematics. *The Philosophical Review*, *91*(2), 161-192.

Leggio, M. G., Tedesco, A. M., Chiricozzi, F. R., Clausi, S., Orsini, A., & Molinari, M. (2008). Cognitive sequencing impairment in patients with focal or atrophic cerebellar damage. *Brain*, *131*, 1332-1343.

Leiner, H. C., & Leiner, A. J. (1997). How fibers subserve computing capabilities: Similarities between brains and machines. In J. D. Schmahmann (Ed.), *The cerebellum and cognition* (*International review of neurobiology*), Vol. 41 (pp. 535-553). San Diego, CA: Academic Press.

Leiner, H. C., Leiner, A. L., & Dow, R S. (1986). Does the cerebellum contribute to mental skills? *Behavioral Neuroscience*, *100*(4), 443-454.

Leiner, H.C, Leiner, A. L., & Dow, R. S. (1987). Cerebro-cerebellar learning loops in apes and humans. *The Italian Journal of Neurological Sciences*, *8*, 425-436.

Leiner, H. C., Leiner, A. L, & Dow, R. S. (1989). Reappraising the cerebellum: What does the hindbrain contribute to the forebrain? *Behavioral Neuroscience*, *103*(5), 998-1008.

Leiner, H. C., Leiner, A. L., & Dow, R. S. (1991). The human cerebro-cerebellar system: Its computing, cognitive, and language skills. *Behavioral Brain Research*, *44*, 113-128.

Leiner, H. C., Leiner, A. L., & Dow, R. S. (1993). Cognitive and language functions of the human cerebellum. *Trends in Neurosciences, 16*(11). 444-447.

Lutz, A., Lachaux, J-P., Martinerie, J., & Varela, F. J. (2002). Guiding the study of brain dynamics by using first-person data: Synchrony patters correlate with ongoing conscious states during a simple visual task. *Proceedings of the National Academy of Sciences*, *99*(3), 1586-1591.

Lutz, A., & Thompson, E. (2003). Neurophenomenology. *Journal of Consciousness Studies*, *10*(9-10), 31-52.

MacLeod, C. E. (2000). *The cerebellum and its part in the evolution of the hominoid brain*. Doctoral thesis, Simon Fraser University, Burnaby, British Columbia, Canada.

MacLeod, C. E. (2005). Cerebellar anatomy and function: From the corporeal to the cognitive. In S. T. Parker, J. Langer, & C. Milbrath (Eds.), *Biology and knowledge revisited* (pp. 145-177). Hillsdale, NJ: Lawrence Erlbaum.

MacLeod, C. E., Zilles, K., Schleicher, A., Rilling, J. K., & Gibson, K. R. (2003). Expansion of the neocerebellum in Hominoidea. *Journal of Human Evolution*, 44, 401-429.

McClelland, J. L., & Rumelhart, D. E. (1987). *Parallel distributed processing. Explorations in the microstructures of cognition. Vol. 2: Psychological and biological models* [Electronic version]. Cambridge, MA: MIT Press.

Mandler, J. M. (1988). How to build a baby: On the development of an accessible representational system. *Cognitive Development*, *3*, 113-136.

Mandler, J. M. (1992). How to build a baby: II. Conceptual primitives. *Psychological Review*, *99*(4), 587-604.

Manni, E., & Petrosini, L. (2004). A century of cerebellar somatotopy: a debated representation. *Nature Reviews: Neuroscience*, *5*, 241-249.

Marrades, R. & Gutiérrez, A. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics*, *44*, 87-125.

Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics*, *15*, 277-289.

Mendell, H. (2004, March 26). Aristotle and mathematics. In E. N. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy (Winter 2008 Edition)*. Retrieved April 11, 2009 from http://plato.stanford.edu/archives/win2008/entries/aristotle-mathematics/

Merleau-Ponty (2005). *Phenomenology of perception* (C. Smith, Trans.). New York: Routledge. (Original work published 1945)

Merleau-Ponty, M (2006). *The structure of behavior* (A. L. Fisher, Trans.). Pittsburgh, PA: Duquesne University Press. (Original work published 1942)

Middleton, F. A., & Strick, P. L. (1994). Anatomical evidence for cerebellar and basal ganglia involvement in higher cognitive function. *Science*, *266*(5184), 458-461.

Middleton, F. A., & Strick, P. L. (1997). Dentate output channels: motor and cognitive components. In C. E. de Zeeuw, P. Strata, & J. Voogd (Eds.), *The cerebellum: From structure to control* (pp. 553-566). Amsterdam: Elsevier.

Middleton, F. A., & Strick, P. L. (1998). Cerebellar output: Motor and cognitive channels. *Trends in Cognitive Sciences*, *2*(9), 348-354.

Mishkin, M., Ungerleider, L. G., & Macko, K. A. (1983). Object vision and spatial vision: Two cortical pathways. *Trends in Neuroscience*, *6*, 414-417.

Mitchelmore, M. C. (2002). The role of abstraction and generalisation in the development of mathematical knowledge. *EARCOME 2002 Proceedings: Vol. 1* (pp. 157-167). (ERIC Document Reproduction Service No. ED466962)

Mitchelmore, M., & White, P. (1995). Abstraction in mathematics: Conflict, resolution and application. *Mathematics Education Research Journal*, *7*, 50-68.

Mitchelmore, M., & White, P. (2004). Abstraction in mathematics and mathematics learning. In M. J. Hønes & A. B. Fuglestad (Eds.), *Proceedings of the 28th annual conference of the International Group for the Psychology of Mathematics Education: Vol. 3* (pp. 329-336). Bergen, Norway.

Mighton, J. (2007). *The end of ignorance: Multiplying our human potential.* Toronto, Canada: Alfred A. Knopf Canada.

Milner, A. D., & Goodale, M. A. (2006). *The visual brain in action* (2nd. ed.). Oxford, England: Oxford University Press.

Molinari, M., Chiricozzi, F. R., Clausi, S., Tedesco, A. M., De Lisa, M., & Leggio, M. G. (2008). Cerebellum and detection of sequences, from perception to cognition. *The Cerebellum*, *7*, 611-615.

Molinari, M., & Leggio, M. G. (2007). Cerebellar information processing and visuospatial functions. *The Cerebellum*, *6*, 214-220.

Molinari, M., Petrosini, L., & Grammaldo, L. G. (1997). Spatial event processing. In J. D. Schmahmann (Ed.), *The cerebellum and cognition* (*International review of neurobiology*), Vol. 41 (pp. 217-230). San Diego: Academic Press.

Molinari, M., Petrosini, L., Misciagna, S., & Leggio, M. G. (2004). Visuospatial abilities in cerebellar disorders. *Journal of Neurology, Neurosurgery and Psychiatry*, *75*, 235-240.

Mountcastle, V. B. (1997). The columnar organization of the neocortex. *Brain*, *120*, 701-722.

Nathan, P. W., & Smith, M. C. (1982). The rubrospinal and central tegmental tracts in man. *Brain*, *105*, 223-269.

Netz, R. (1999). The shaping of deduction in Greek mathematics: A study in cognitive history. Cambridge, England: Cambridge University Press.

Netz, R. (2004). *The works of Archimedes translated into English, together with Eutocius' commentaries, with commentary, and critical edition of the diagrams. Volume I: The two books On the Sphere and the Cylinder*. Cambridge, UK: Cambridge University Press.

Otzas, E. (2003). Neuronal tracing. *Neuroanatomy*, *2*, 2-5.

Ozdemir, A., & Black, P. M. (2005). Mapping of the human visual cortex. *Neurosurgery Quarterly*, *15*(2), 65-71.

Paquier, P. F., & Mariën, P. (2005). A synthesis of the role of the cerebellum in cognition. *Aphasiology*, *19*(1), 3-19.

Passingham, R. E. (1975). Changes in the size and organisation of the brain in man and his ancestors. *Brain Behavior and Evolution*, *11*, 73-90.

Pasupathy, A. (2006). Neural basis of shape representation in the primate brain [Electronic version]. In S. Martinez-Conde, S. L. Macknik, L. M. Martinez, J.-M. Alonso, & P. U. Tse (Eds.), *Progress in Brain Research: Vol 154* (pp. 293-313).

Paulin, M. G. (1993). The role of the cerebellum in motor control and perception. *Brain, Behavior and Evolution*, *41*, 39-50.

Paulin, M. G. (2005). Evolutionary origins and principles of distributed neural computation for state estimation and movement control in vertebrates. *Complexity*, *10*(3), 56-65.

Paulin, M. G., Hoffman, L. F., & Assad, C. (2001). A model of cerebellar computations for dynamical state estimation. *Autonomous Robots*, *11*, 279-284.

Petersen, S. E., Fox, P. T., Posner, M. I., Mintun, M., & Raichle, M. E. (1989). Positron emission tomographic studies of the processing of single words. *Journal of Cognitive Neuroscience*, *1*(2), 153-170.

Piaget, J. (1960). *The child's conception of the world* (J. Tomlinson & A. Tomlinson, trans.). London: Routlege & K. Paul. (Original work published 1929)

Piaget, J., & Inhelder, B. (1963). *The child's conception of space* (F. J. Langdon & J. L. Lunzer, Trans.). London: Routledge & Kegan Paul. (Original work published in 1948)

Prescott, J. W. (1971). Early somatosensory deprivation as ontogenetic process in the abnormal development of the brain and behavior. In E. I. Goldsmith, & J. Moor-Jankowski (Eds.), *Medical Primatology 1970* (pp. 356-375), Basel, Switzerland: S. Karger.

Presemeg, N. C. (1986). Visualization in high school mathematics. *For the Learning of Mathematics*, 6(3), 42-46.

Presemeg, N. C. (1992). Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics. *Educational Studies in Mathematics*, *23*, 595-610.

Quartz, S. R., & Sejnowski, T. J. (1997). The neural basis of cognitive development: A constructivist manifesto. *Behavioral and Brain Sciences*, *20*, 537-596.

Radford, L. (2008). *Semiotics in mathematics education*. Rotterdam, The Netherlands: Sense Publishers.

Radford, L., Bardini, C., Sabena, C., Diallo, P., & Simbagoye, A. (2005). On embodiment, artifacts, and signs: A semiotic-cultural perspective on mathematical thinking. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th annual conference of the International Group for the Psychology of Mathematics Education: Vol. 4* (pp. 113-120). Melbourne, Australia. (ERIC Document Reproduction Service No. ED496955)

Raichle, M. E., Fiez, J. A., Videen, T. O., MacLeod, A. –M., K., Pardo, J. V., Fox, P. T., & Petersen, S. E. (1994). Practice-related changes in human brain functional anatomy during nonmotor learning. *Cerebral Cortex*, *4*, 8-26.

Ramnani, N. (2006). The primate cortico-cerebellar system: Anatomy and function. *Nature Reviews: Neuroscience*, *7*, 511-522.

Richter, S., Aslan, B., Gerwig, M., Wilhelm, H., Kramer, S., Todika, O., Schoch, B., Dimitrova, A., Gizewski, E. R., Thilmann, A. F., & Timmann, D. (2007). Patients with chronic focal cerebellar lesions show no cognitive abnormalities in a bedside test. *Neurocase*, *13*, 25-36.

Riesenhuber, M., & Poggio, T. (2000). Models of object recognition. *Nature Neuroscience* Supplement, *3*(11), 1199-1204.

Riesenhuber, M., & Poggio, T. (2002). Neural mechanisms of object recognition. *Current Opinion in Neurobiology*, *12*, 162-168.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, *93*(2), 346-362.

Roland, P. E., & Gulyas, B. (1994). Visual imagery, and visual representation. *Trends in Neurosciences*, *17*(7), 281-287.

Roland, P. E., & Gulyas, B. (1995). Visual memory, visual imagery, and visual recognition of large field patterns by the human brain: Functional anatomy by positron emission tomography. *Cerebral Cortex*, *5*, 79-93.

Roth, W., & Hwang, S. (2006). Does mathematical learning occur in going from concrete to abstract or in going from abstract to concrete? *Journal of Mathematical Behavior*, *25*, 334-344.

Rumelhart, D. E., McClelland, J. L., & The PDP Research Group (1987). *Parallel distributed processing: Explorations in the microstructures of cognition. Vol. 1: Foundations* [Electronic version]. Cambridge, MA: MIT Press.

Ryle, G. (1963). *The concept of mind*. London: Hutchinson. (Original work published 1949)

Schmahmann, J. D. (1991). An emerging concept: The cerebellar contribution to higher function. *Archives of Neurology*, *48*, 1178-1187.

Schmahmann, J. D. (1996). From movement to thought: Anatomic substrates of the cerebellar contribution to cognitive processing. *Human Brain Mapping*, *4*, 174-198.

Schmahmann, J. D. (1997). Rediscovery of an early concept. In J. D. Schmahmann (Ed.), *The cerebellum and cognition* (*International review of neurobiology*), Vol. 41 (pp. 3-27). San Diego, CA: Academic Press.

Schmahmann, J. D. (1998). Dysmetria of thought: Clinical consequences of cerebellar dysfunction on cognition and affect. *Trends in Cognitive Sciences*, *2*(9), 362-371.

Schmahmann, J. D. (2004). Disorders of the cerebellum: Ataxia, dysmetria of thought, and the cerebellar cognitive affective syndrome. *Journal of Neuropsychiatry and Clinical Neuroscience, 16*(3), 367-378.

Schmahmann, J. D., & Pandya, D. N. (1989). Anatomical investigation of projections to the basis pontis from posterior parietal association cortices in rhesus monkey. *The Journal of Comparative Neurology*, *289*, 53-73.

Schmahmann, J. D., & Pandya, D. N. (1990). Anatomical investigation of projections from thalamus to the posterir parietal association cortices in rhesus monkey. *The Journal of Comparative Neurology*, *295*, 299-326.

Schmahmann, J. D., & Pandya, D. N. (1995). Prefrontal cortex projections to the basilar pons in rhesus monkeys: Implications for the cerebellar contribution to higher function. *Neuroscience Letters*, *199*, 175-178.

Schmahmann, J. D., & Pandya, D. N. (1997). The cerebrocerebellar system. In J. D. Schmahmann (Ed.), *The cerebellum and cognition* (*International review of neurobiology*), Vol. 41 (pp. 31-60). San Diego, CA: Academic Press.

Schmahmann, J. D., & Sherman, J. C. (1998). The cerebellar cognitive affective syndrome. *Brain*, *121*, 561-579.

Schweizer, T. A., Alexander, M. P., Cusimano, M., Stuss, D. T. (2007). Fast efficient visuotemporal attention requires the cerebellum. *Neuropsychologia*, *45*, 3068-3074.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1-36.

Sinclair, N. (2002). Initiating problem posing through exploration with Sketchpad. In D. Mewborn, P. Sztajn, D. White, H. Wiegel, R. Bryant, & K. Nooney (Eds.), *Proceedings of the 24th annual conference of the North American chapter of the International Group for the Psychology of Mathematics Education* (pp. 795-796). Columbus, Ohio.

Skemp, R. R. (2006). Relational understanding and instrumental understanding. *Mathematics Teaching in the Middle School*, *12*(2), 88-95. (Original work published in *Mathematics Teaching*, December 1976)

Sohlberg, M. M., & Mateer, C. A. (1989). *Introduction to cognitive rehabilitation: theory and practice*. New York: Guilford Press.

Spinoza, B. (1996). *Ethics* (E. Curley, Ed. & Trans.). London: Penguin Books. Original work published 1677.

Stein, J. P., Miall, R. C., & Weir, D. J. (1987). The role of the cerebellum in visual guidance and movement. In M. Glickstein, C. Yeo, & J. Stein (Eds.), *Cerebellum and neuronal plasticity* (pp. 175-191). New York: Plenum.

Strick, P. L., Dum, R. P., & Fiez, J. A. (2009). Cerebellum and nonmotor function. *Annual Review of Neuroscience*, *32*, 413-434.

Schweizer, T. A., Alexander, M. P., Cusimano, M., Stuss, D. T. (2007). Fast efficient visuotemporal attention requires the cerebellum. *Neuropsychologia*, *45*, 3068-3074.

Silvester, J. R. (2001). *Geometry: ancient and modern.* Oxford: Oxford University Press.

Sultan, F., & Glickstein, M. (2007). The cerebellum: Comparative and animal studies. *The Cerebellum*, *6*, 168-176.

Tall, D. (1995). Mathematical growth in elementary and advanced mathematical thinking. In L. Meira & D. Carraher (Eds.), *Proceedings of the 19th annual conference of the International Group for the Psychology of Mathematics Education: Vol. 1* (pp. 61-75). Recife, Brazil. (ERIC Document Reproduction Service No. SE057135)

Tall, D. (1999). Reflections on APOS theory in elementary and advanced mathematical thinking. In O. Zaslavsky (Ed.), *Proceedings of the 23rd annual conference of the International Group for the Psychology of Mathematics Education*: *Vol. 1* (pp. 111-118). Haifa, Israel.

Tall, D., Thomas, M., Davis, G., Gray, E., & Simpson, A. (2000). What is the object of the encapsulation of a process? *Journal of Mathematical Behavior*, *18*(2), 223-241.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151-169.

Tavano, A., Grasso, R., Gagliardi, C., Triulzi, F., Bresolin, N., Fabbro, F., & Borgatti, R. (2007). Disorders of cognitive and affective development in cerebellar malformations. *Brain*, *130*, 2646-2660.

Thach, W. T. (1996). On the specific role of the cerebellum in motor learning and cognition: Clues from PET activation and lesion studies in man. *Behavioral and Brain Sciences*, *19*, 411-431.

Thompson, E., & Varela, F. J. (2001). Radical embodiment: Neural dynamics and consciousness. *Trends in Cognitive Sciences*, *5*(10), 418-425.

Timmann, D., & Daum, I. (2007). Cerebellar contributions to cognitive functions: A progress report after two decades of research. *The Cerebellum*, *6*, 159-162.

Townsend, J., Courchesne, E., Covington, J., Westerfield, M., Harris, N. S., Lyden, P., Lowry, T. P., & Press, G. A. (1999). Spatial attention deficits in patients with acquired or developmental cerebellar abnormality. *Journal of Neuroscience*, *19*(13), 5632-5643.

Ungerleider, L. G., & Mishkin, M. (1982). Two cortical visual systems. In D. J. Ingle, M. A. Goodale, & R. J. W. Mansfield (Eds.), *Analysis of Visual Behavior* (pp. 549-586). Cambridge, MA: MIT Press.

van Hiele, P. (1986). *Structure and insight: A theory of mathematics education*. Orlando, FL: Academic Press.

Vandervert, L. R. (1997). The evolution of Mandler's conceptual primitives (image-schemas) as neural mechanisms for space-time simulation structures. *New Ideas in Psychology, 15*(2), 105-123.

Vandervert, L. R. (1999). A motor theory of how consciousness within language evolution led to mathematical cognition: The origin of mathematics in the brain. *New Ideas in Psychology*, *17*, 215-235.

Vandervert, L. R. (2003). How working memory and cognitive modeling functions of the cerebellum contribute to discoveries in mathematics. *New Ideas in Psychology*, *21*, 15-29.

Varela, F. J. (1995). Resonant cell assemblies: A new approach to cognitive functions and neuronal synchrony. *Biological Research*, *28*, 81-95.

Varela, F. J. (1996). Neurophenomenology: A methodological remedy for the hard problem. *Journal of Consciousness Studies*, *3*(4), 330-349.

Varela, F. J. (1999). The specious present: A neurophenomenology of time consciousness. In J. Petitot, F. J. Varela, B. Pachoud, & J-M. Roy (Eds.), *Naturalizing phenomenology: Issues in contemporary phenomenology and cognitive science* (pp. 266-329). Stanford, CA: Stanford University Press.

Varela, F., Lachaux, J-P., Rodriguez, E., & Martinerie, J. (2001). The brainweb: Phase synchronization and large-scale integration. *Nature Reviews: Neuroscience*, *2*, 229-239.

Varela, F. J., & Shear, J. (1999). First-person methodologies: What, why, how? *Journal of Consciousness Studies*, *6*(2-3), 1-14.

Varela, F. J., & Thompson, E. (2003). Neural synchrony and the unity of mind: A neurophenomenological perspective. In A. Cleeremans (Ed.), *The unity of consciousness* (pp. 266-287). New York: Oxford University Press.

Varela, F. J., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. Cambridge, MA: The MIT Press.

Vinberg, J., & Grill-Spector, K. (2008). Representation of shapes, edges, and surfaces across multiple cues in the human visual cortex. *Journal of Neurophysiology*, *99,* 1380-1393.

Voogd, J., Feirabend, H. K. P., & Schoen, J. H. R. (1990). Cerebellum and precerebellar nuclei. In G. Paxinos (Ed.), *The human nervous system* (pp. 321-386). San Diego: Academic Press.

Wallis, G., & Bülthoff, H. (1999). Learning to recognize objects. *Trends in Cognitive Science*, *3*(1), 22-31.

Welker, W. (1986). Comparative study of cerebellar somatosensory representations: the importance of micromapping and natural stimulation. In M. Glickstein, C. H. Yeo, & J. Stein (Eds.), *Cerebellum and neuronal plasticity* (pp. 109-118). Plenum: New York.

Weisstein, E. W. (2005a). Pasch's Axiom. Retrieved May 24, 2010, from Wolfram Research Web site: http://mathworld.wolfram.com/PaschsAxiom.html

Weisstein, E. W. (2005b). Continuity Axioms. Retrieved May 24, 2010, from Wolfram Research Web site: http://mathworld.wolfram.com/ContinuityAxioms.html

Whitehead, A. N. (1967). *Science and the modern world*. New York: The Free Press. (Original work published 1925)

Whitehead, A. N. (2007). *An enquiry concerning the principles of natural knowledge*. New York: Cosimo Classics. (Original work published 1919)

Whiting, B. A., & Barton, R. A. (2003). The evolution of the cortico-cerebellar complex in primates: Anatomical connections predict patterns of correlated evolution. *Journal of Human Evolution*, *44*, 3-10.

Wolpert, D. M., Miall, R. C., & Kawato, M. (1998). Internal models in the cerebellum. *Trends in Cognitive Sciences*, *2*(9), 338-347.