USING SUBORDINATION TO TEACH AND LEARN MATHEMATICS

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THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

In the
Secondary Mathematics Education Program
Faculty of Education

SIMON FRASER UNIVERSITY

Summer 2011

APPROVAL

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Degree: **Master of Science**

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ABSTRACT

The practice tasks we find in mathematics are often problematic, both for student motivation and for student learning. Often, nothing is learned because the student is involved in mechanistic repetition. Drawing on the idea that only awareness is educable, Dave Hewitt argues for an approach to practice that avoids thoughtless repetition by shifting the focus of the activity. His subordination tasks focus attention on a more engaging task, while still enabling practice. This action research study involves developing subordination tasks for principles of mathematics 10 students and investigates whether such tasks can be designed and implemented across the curriculum, and used in every class. Six tasks were created and implemented. An analysis of the results shows that in addition to the three characteristics of subordination defined by Hewitt, tasks need to consider the classroom milieu, be accessible, engaging, and relevant, practice an appropriate skill, and provide immediate feedback.

For my husband, Terry,
who has always given me
unconditional love, patience and support.

For my children, Matthias and Nathalie,
whose smiles and laughter light up my life.

ACKNOWLEDGEMENTS

I would like to express my deepest appreciation to Dr. Nathalie Sinclair for her patience, encouragement and guidance from the beginning of this journey. I would also like to thank Dr. Peter Liljedahl for his insights and support. My thanks also go out to Dr. David Pimm who introduced me to the article that became the focus of my research.

I would like to thank my fellow classmates of the Mathematics Education Program. In particular, I am grateful to Dan Kamin who encouraged me to keep going, helped to edit parts of my first draft, and gave me advice whenever I needed it.

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BACKGROUND

I have discussed with teachers from several schools in many school districts about what it means for students to practice their subject. In a language class, not only do students learn vocabulary and grammatical skills through fill-in the blank worksheets, but they also practice the non-native language by writing paragraphs and creating skits to hone their oral and communication skills. In English class, students do not just learn literary definitions and the structure of a sentence, but they read novels and write paragraphs and essays and discuss, debate and reflect about the themes, conflicts and characters that they encounter. In science classes, students read about biological, chemical, and physical processes, but they are also given the opportunity to carry out experiments to explore and discover. However, when I consider a mathematics class, the majority of the students practice the concepts by copying out questions from a textbook or worksheet and imitating what they saw the teacher do on the board. In many classrooms, there is little discussion and almost no exploration by the students as they do a multitude of the same types of questions over and over again with only a slight variation of increasing difficulty. This present system of mathematics education reminds me of *A Mathematician's Lament* by Paul Lockhart (2002). I believe that it is time to break this traditional mold observed in many mathematics classrooms and entertain new methods of practicing mathematical concepts.

When I graduated with my teaching degree in 1997, I was introduced to the idea of learning and practicing by problem solving, yet I never had the opportunity to see it firsthand. I did not know what this method of teaching looked like and when I got my first teaching job, I did what I had been taught and had seen in high school: the old chalk and talk. I would teach a lesson by showing my students the exact steps they should take to solve a problem, deviating from my path to introduce an additional method only if I believed that both methods had equal value. I felt that by doing so, I was giving my students a choice in their mathematical problem solving. This, I believed, made mathematics less prescriptive. There was not only *one* right pathway, but a choice. Little did I know that since I was creating both or all the choices for my students, there really was little consideration or thinking on their part. What was really happening was my asking them to show a preference in the way they wanted to solve the problem. When I asked my more experienced colleagues about the teaching methods they employed, I seemed to be conforming to the majority.

RATIONALE

As I become a more experienced teacher, I begin to realize that even though I think a lesson has gone well, many students will still have difficulty remembering the concept a year, a month, a week and even a day later. This lack of retention is most likely not caused by a lack of practice but by the mechanism in which they learned the mathematics. Since students have little opportunity to create their own understanding by connecting with their prior knowledge or to reflect and communicate their thoughts and ideas when they encounter new problems, they feel the need to memorize rules, and when they have forgotten the rule, they feel that they can not reason out the problem because they do not know how to. Indeed, there are some names and conventions in mathematics that need to be memorized and some that do not. However, if everything is taught by pure memorization, with no connection to why it is mathematically important, it may not stick and hence, be soon forgotten. Practice is important but what I am suggesting is that there may be a better way to introduce and practice mathematical concepts so that they can be remembered a little longer.

During my post-graduate studies, I encountered an intriguing article written by Dave Hewitt (1996), "Mathematical Fluency: the Nature of Practice and the Role of Subordination" where he uses the notion of subordination[1] to teach and learn mathematics. His method works by getting students to give less conscious attention to the concept to be mastered but instead, subordinates the learning by getting students to focus on a different task. I am attracted to his description of practice because it seems to accord an important place to the role of practice in mathematical learning while still working closely on developing understanding and meaning. Because the technique of subordination is less radical than other reform-type strategies, such as open-ended projects described by Boaler (2002) and structured problem solving (Stigler and Hiebert, 1999), I am more likely to adopt it into my own practice as I am not ready to completely change my teaching pedagogy.

The purpose of tasks used in the mathematics classroom is usually to learn something new, to practice something just learnt and to review. Hewitt asserts that subordination can be used to learn a new concept, but while students are learning something new, they can actually be practicing too. Practice is important for something that is new to become something that is so familiar that no conscious attention is given to it (Hewitt, 1996). For example, when I think about the first time I used chopsticks, I concentrated on where I had to place my fingers so that the two sticks would stay together while at the same time, being able to move so that I could pick up some morsel

[1] **Subordinate** means to place lower in order or to make secondary.

of food. I knew how I needed to hold the chopsticks even though I had not mastered it yet. I also knew when I did it correctly because I could see the consequences of my actions, grasping the piece of food. I did not practice intentionally, but my chopstick practicing was subordinate to eating, and it was not long before I was not thinking about how to move my fingers but deciding which piece of delicious food I wanted to choose. Subordination can be used to describe the process and practice that I went through to learn to use chopsticks. I had a larger goal in mind, that of eating, and my practice happened while in pursuit of that goal. Practicing mathematical concepts can be carried out similarly. I have seen that familiar, repetitive exercises may not benefit all students and my plan is to develop the notion of subordination, exploring how it can be used to influence students' learning of mathematics.

In Chapter 2, I begin by looking at literature that compares and contrasts traditional and reform methods of teaching and learning, before focusing on the purpose of mathematical tasks and different types of practice exercises, including subordination tasks. Chapter 3 consists of a detailed description of the theoretical framework and the methodology used to conduct the research. Following this, I describe the tasks that were implemented along with the results and an analysis of each task in Chapter 4. The common themes that emerge from this analysis are discussed in Chapter 5. Finally, in Chapter 6, I present some conclusions regarding the design and implementation of subordination tasks for teaching and learning mathematics.

Chapter 2 : LITERATURE REVIEW

Designing tasks for practice that has the features of subordination is little understood and as stated in the introduction, will be the focus of this thesis. Almost all tasks in a mathematics classroom are directed at practicing some mathematical concept but it is the teacher's philosophies and beliefs that dictate the delivery method and hence, the type of practice the students engage in. To better understand the diverse methods and where subordination fits in, this chapter begins with an analysis of the research regarding the much criticized but still prevalent traditional method of practicing mathematics such as routine, drill, mechanical and rote exercises. In contrast, different views of teaching and learning with understanding will be chronologically reviewed next. This is followed by an explanation of what is meant by a mathematical task. What and how students learn depends on the kind of work they do in classrooms, and the tasks they are asked to participate in determines the type of thinking in which they will engage (Doyle, 1988). Afterwards, since practice is the unifying theme in this thesis, the role of practice in mathematics is examined followed by a review of some non-conventional practice methods. Finally, the last section examines the subordination of teaching to learning, particularly how this notion can be incorporated into practicing mathematical concepts.

USING TRADITIONAL METHODS

A supporter of traditional teaching methods may view knowledge as discrete, hierarchical, sequential and fixed where a student learns best by working through a textbook. Although these traditional ideologies have been criticized (Dewey, 1933; Doyle, 1988; Skemp 1976) and have even led to mathematics education reform spearheaded by the National Council of Teachers of Mathematics (NCTM), many practitioners continue to use these methods (Boaler, 2002; Doyle, 1988; Jacobs, Hiebert, Givvin, Hollingsworth, Garnier & Wearne, 2006; Skemp, 1976). Schiefele and Csikszentmihalyi (1995) found that

> the most preferred instruction activities of senior high school teachers are lecture, discussion, and seatwork assigned from the textbook. The least preferred activities are use of hands-on or manipulative material, use of computers, working in small groups and completing supplemental worksheets (p. 179).

Researchers attempt to understand the advantages of traditional tasks and reasons why teachers choose this method over more student-centered activities.

Boaler (2002) spent three years examining the experiences of students and mathematics teachers from two very different English schools: Amber Hill, which used mainly traditional methods and Phoenix Park, which had a more progressive approach. She enlisted these two schools to learn more about how teaching strategies and tasks affect students' learning and perceptions of mathematics. Even though this thesis focuses on practice, these two schools with dichotomous teaching methods show a larger scope of how practice fits into the classroom ethos. Amber Hill's approach will be described here and Phoenix Park's method will be described later in the next section. At Amber Hill, the mathematics lesson began with the teacher presenting a concept at the board in a clear and structured way but with no discussion on why this method was chosen or when and why it worked. Students were encouraged to memorize these procedures so that they could be recalled later. The pace of the lessons was rather fast in order to cover as much material as possible to satisfy the demands of the national curriculum. After the lesson, the students completed textbook exercises to practice what they had just witnessed, spending the majority of their time engaged in short, procedural and closed questions.

Some students admitted that they did not even need to think about what they were doing when working on this traditional practice. When students encountered more difficult open questions at the end of a set, the teacher would break the questions into small atomistic parts and guide students through any mathematical decision-making (Boaler, 2002).

Amber Hill illustrates the teacher-telling and student-doing method. Some educators favour this traditional practice because activity flow is smooth and deemed easier to teach and understand (Doyle, 1988; Skemp, 1976) which is favourable given the limited instructional time teachers have to fulfill an over-burdened syllabi (Skemp, 1976; Boaler, 2002). And, even though the routine, recurring exercises, which Doyle (1988) calls “familiar work,” can become quite difficult, the outcomes are predictable. Students feel success which in turn, builds self-confidence. There is little ambiguity about what to do, how to do it and little risk that things will go wrong. This *rules without reasons* practice leads to what Skemp describes as “instrumental understanding”. In contrast, novel work or activities that bring about “relational understanding” (Skemp, 1976) is lengthier and requires students to “assemble information and operations from several sources in ways that have not been laid out explicitly in advance by the teacher” (Doyle, 1988, p. 173). The activity flow of teaching for meaning is slow (Brownell, 1947; Doyle, 1988; Skemp, 1976) and bumpy and since the outcomes are not predictable and there is a higher risk that answers will be incorrect, cognitive and emotional demands are higher for both the students and the teacher. Stein (1987) confirms that, “it is much easier to teach the execution of an algorithm than the ability to analyze” (p. 2). In addition, he points out that assessing the skill of executing an algorithm is easier than assessing novel work.

Another advantage of individual textbook or worksheet practice is that it helps the teacher to better manage the classroom environment (Dewey, 1933; Doyle, 1988) since tasks are easily initiated and students experience immediate success. “In particular, familiar work provides a tangible structure and a clear program of action that is accessible to nearly all students” (Doyle, 1988, p.178). A large amount of work is accomplished because students seldom encounter tasks with which they are required to struggle with meaning and when they do, the teacher usually comes to their rescue by breaking down the problem and the solution into a series of steps and procedures, similar to what the Amber Hill teachers did. Others (Stein, 1987; Schoenfeld, 1988; Doyle, 1988) also found that mathematics teachers commonly provided students with detailed structure to help them solve problems. A classroom ethos that encompasses traditional work is easier to manage.

“Automatic skill and quantity of information are educational ideals which pervade the whole school” (Dewey, 1933, p. 53) particularly since most adults, including current mathematics teachers, were instructed in rule-based environments (Battista, 1994). Hence, familiar work continues to dominate classrooms thereby allowing the current mathematics curriculum to be self-perpetuating (Battista, 1994). Moreover, the school system influences teacher practices as textbooks and state, provincial or national standardized exams also encourage instrumental understanding (Battista, 1994). Success on standardized tests are an important part of the school environment (Boaler, 2002) because it is one way that schools can be compared, as administrators and teachers are under the scrutiny of parents and outsiders. Therefore, teachers develop tasks that will be similar to the ones that students will encounter on these tests (Boaler, 2002; Doyle, 1988;

Skemp, 1976). At Phoenix Park, Boaler (2002) reported that the mathematics teachers used open-ended projects up until the middle of grade 10 but then switched to a more procedural format as students prepared for the national examinations. Even though the teachers of the school believed in using a problem-based method of learning, they did not believe that this method would produce success on the exams. Since the majority of standardized mathematics tests contain closed questions that emphasize procedures rather than sense-making (Battista, 1994; Boaler, 2002), the teaching and learning environment often becomes one where students watch teachers work through mathematics procedures and then mimic teachers as they practice the algorithms.

Traditional school mathematics relies on what some have called a telling pedagogy, pedagogy of control or the transmission model of instruction (O'Brien, Stewart & Moje, 1995). "In this type of setting, discourse is often characterized by teacher 'telling' or by the asking and evaluation of 'right-answer' questions, tasks that require memorization of facts or algorithms" (Turner, Meyer, Cox, Logan, DiCintio & Thomas, 1998, p. 733). After being told and shown what to do, students engage in similar exercises from a worksheet or textbook. "The notion behind such an exercise is that practice makes perfect, or perhaps more appropriately, repetition makes perfect" (Hewitt, 1996, p. 29). Stigler and Hiebert (1999) observe that a common practice in many United States classrooms is that after demonstrating a procedure, the teacher gives the students a worksheet containing forty problems which emphasizes terms and procedures. Boaler and Greeno (2000) notice that students seem to accept this lack of variety in mathematics lessons, not because they enjoy the lessons, but because they think that is the way mathematics is. However, mathematical tasks do not have to be the monotonous rote exercises traditionally presented as many researchers (Hewitt, 1996; Rohrer, 2009; Watson & Mason, 2004, 2005) study mathematical learning and how variances in the types of practice exercises can influence students' learning of mathematics.

TEACHING AND LEARNING WITH UNDERSTANDING

> Understanding is crucial because things learned with understanding can be used flexibly, adapted to new situations, and used to learn new things. Things learned with understanding are the most useful things to know in a changing and unpredictable world. (Hiebert et al., 1997, p. 1)

In contrast to Amber Hill described above, Phoenix Park (Boaler 2002) used a reform approach to develop and practice concepts. Rather than using textbooks, teachers designed open-ended projects that generally lasted for two to three weeks. Students "were not given specified paths through their activities; they were merely introduced to starting questions or themes and expected to develop these into extended pieces of work" (Boaler, 2002, p. 51) on their own. The teacher did not spend much time at the board telling students what to do but instead, facilitated discussions and negotiations as students worked. The Phoenix Park teachers chose open and rich activities that provided differentiated opportunities, had different access points for different students and enabled students to work on them at different mathematical levels. Students did not practice rote-style questions of the same type from a textbook but when they finished their project or became bored with their work, the teacher would invent extensions for them or offer other ideas for them to work on. Teachers refrained from giving students the exact information they needed to know but encouraged independence and responsibility as they pointed out places where they could search for information. The students at Phoenix Park were engaged in activities and projects in which the need for certain mathematical methods became apparent only after the students' need for them emerged rather than the teacher ordering and structuring the curriculum for them.

The type of practice engaged by the Phoenix Park students, where they develop purpose and meaning to mathematical concepts, stands in stark contrast to the traditional practices described earlier. In the early 1900's, Dewey (1933) criticized repetitive exercises.

> Sheer imitation, dictation of steps to be taken, mechanical drill, may give results most quickly and yet strengthen traits likely to be fatal to reflective power. The pupil is enjoined to do this and that specific thing, with no knowledge of any reason except that by doing so he gets his result most speedily; his mistakes are pointed out and corrected for him; he is kept at pure repetition of certain acts till they become automatic (Dewey, 1933, p. 51).

He believed in the significance of play and related it to how one could train thought. Brownell (1957) continued studying the place of meaning in mathematics teaching and found that teachers who taught "meaningful arithmetic actually paid dividends. For one thing, it protected the children from the absurd mistakes commonly made under other programs of instruction" (p.259). He described a meaningful arithmetic experience as one which made sense mathematically to the student. A student "behaves meaningfully with respect to a quantitative situation when he knows what to do arithmetically and when he knows how to do it; and he possesses arithmetical meanings when he understands arithmetic as mathematics" (p. 257). The psychologist, Skemp (1976), helped practitioners consider two perspectives of understanding that was briefly introduced above: *relational understanding*, which is knowing both what to do and why, and *instrumental understanding*, which is possessing a rule and having the ability to use it.

Practice activities that encourage relational understanding create more connections which help students adapt to new tasks, remember more, and aid students' long-term learning of mathematics. As will be discussed later, subordination tasks actually force these connections as students unconsciously use new techniques to carry out a task.

In 1989, the National Council of Teachers of Mathematics (NCTM), a group of mathematics educators in North America, called for a de-emphasis of manual repetitive arithmetic found in traditional practice exercises. They were in favor of students' discovering their own knowledge and making sense of the mathematics. They published a series of documents, referred to as the *Standards* (1989, 2000), which was an important benchmark articulating a new way of teaching that emphasized:

- problem solving
- individual and group work
- assigning of challenging tasks that require several days in which to develop a solution and which may include exploring alternative solution methods
- real-life connections
- use of technology
- use of reasoning skills
- use of different representations such as physical objects and drawings to model and solve problems
- the opportunity for students to actively participate in mathematical communication during lessons to present and explain strategies.

This was part of a reform movement made in an attempt to replace "the current obsolete, mathematics-as-computation curriculum with a mathematics curriculum that genuinely embraced conceptual understanding, reasoning, and problem solving as the fundamental goals of instruction" (Battista, 1994, p. 463). Draper (2002) reported that students who learned in this *Standards* environment learned more than students in the conventional classroom because learning took place in a student-centered classroom that deemphasized rote memorization of isolated skills and facts. "Students are more engaged in student-controlled versus teacher-controlled learning activities. A useful distinction to make is that whole group instruction tends to be perceived by students as relatively teacher-controlled, whereas small group and individual instruction are perceived as relatively student-controlled" (Shernoff, Csikszentmihalyi, Schneider & Shernoff, 2003, p. 160).

Classrooms need to support the development of mathematical thinking and understanding. Mason and Johnston-Wilder (2006) describe the learning and teaching environment, regardless of teaching style, as the milieu. "The milieu includes:

- the classroom ethos and the characteristic ways of working
- the degree to which learners and teachers are responsible for ensuring learning
- the classroom organization (social structure, resources and so on)" (p. 35).

They emphasize that these features are interdependent and maintain that in order for learners to educate their awareness and train their behaviour, teachers need to develop a milieu that fosters and sustains mathematical thinking.

> Learners need to develop a stance in which problems are seen as challenges to work on and not as tests of memory but at the same time they need to practise techniques so that these become automatic and internalized (p. 36).

Furthermore, as part of the didactical contract (Brousseau, 1997), teachers need to develop engaging tasks that are neither too easy nor too difficult while learners need "to take the initiative and to treat mathematics as a constructive and creative activity" (p. 40). One of the five dimensions that Hiebert et al. (1997) looked at that shapes the learning environment is the nature of the learning tasks. (The other four dimensions which will not be described in detail here are the role of the teacher, the social culture of the classroom, the kind of mathematical tools that are available and the accessibility of mathematics for every student.) They believe that instruction, and consequently the tasks, can create genuine problems for learners that will provide them with opportunities to explore, reflect, communicate and make meaning out of mathematics. How students spend their class time and where they place their attention during these times of practice is determined by the task that the teacher asks them to complete. It is this placement of attention as the learner carries out the practice that is a key component of subordination which will be discussed later.

In 2001, Flewelling and Higginson developed *A Handbook on Rich Learning Tasks* (2001) in the hopes of producing a vision statement for mathematics educators in Canada that focused on the three fundamental components of learning: the student, the teacher and the learning tasks on which both focus. Compared to traditional tasks, characteristics of rich tasks included:

- addressing relatively many learning outcomes
- providing an opportunity to use a broad range of skills
- encouraging more thinking, reflecting and use of imagination
- allowing for demonstration of a wide range of performance
- providing enrichment within the task
- encouraging the use of a wide variety of teaching and learning strategies
- encouraging a greater engagement of students and teachers (Flewelling and Higginson, 2001)

It may be difficult to address all the characteristics of rich learning tasks defined by Flewelling and Higginson but, as will be seen, subordination tasks contain many of them.

It has been reviewed above that if teachers want to spend more time instructing for relational understanding and helping students to make sense of mathematics, they will need to change the way they implement the curriculum. However, despite the existence of the NCTM *Standards* for almost twenty years, Battista (1994) and Jacobs et al (2006) found that, while teachers are cognizant of the current ideas, they continue to use the kind of traditional teaching that has been around for most of the past century. Changing teacher practice is a long and difficult process even if teachers are willing to implement reform (Battista, 1994). What teacher may not know is that mathematical thinking does not imply that teaching methods need to be akin to those of Phoenix Park. In favour of sense-making in mathematics but also believing in effective practice, researchers and educators (Hewitt, 1996; Watson and Mason, 2004; Mason and Johnston-Wilder 2006) have studied and developed activities and tasks that aid students to retain skills beyond a relatively short period of time. Some of these will be discussed in the next section.

MATHEMATICAL TASKS

> One finds consistent recommendations for the exposure of students to meaningful and worthwhile mathematical tasks, tasks that are truly problematic for students rather than simply a disguised way to have them practice an already-demonstrated algorithm. In such tasks, students need to impose meaning and structure, make decisions about what to do and how to do it, and interpret the reasonableness of their actions and solutions (Stein, Grover & Henningsen, 1996, p. 456).

In order to better comprehend tasks that exhibit the notion of subordination in mathematics, the general features of mathematical tasks will be looked at first. Stein et al (1996, p. 460) define a *mathematical task* “as a classroom activity, the purpose of which is to focus students’ attention on a particular mathematical idea.” These tasks can include the calculations that are to be performed, the mental images and diagrams to be developed and discussed, and the symbols to be manipulated (Watson and Mason, 2005). Stein et al’s conception of mathematical task is similar to Doyle’s (1983) notion of *academic task* in that it includes attention to what students are expected to produce, how they are expected to produce it and with what resources. Doyle believes that the types of tasks that teachers choose “influence learners by directing their attention to particular aspects of content and by specifying ways of processing information” (p. 161). He identifies four general categories of academic tasks based on the cognitive operations required to complete a task. They are: memory tasks, procedural or routine tasks, comprehension or understanding tasks, and opinion tasks. Students adjust their strategies for selecting and processing information depending on what they are asked to do. “For instance, they can be asked to

- recognize or reproduce information previously encountered,
- apply an algorithm to solve problems,
- recognize transformed versions of information from texts or lectures, or
- select from among several procedures those which are applicable to a particular type of problem” (Doyle and Carter, 1984, p. 130) .

Depending on what the teacher wants students to focus on, the teacher will ultimately decide on the appropriate task. If students are asked to practice prescribed procedures from sets of exercises, they will think that mathematics is about following directions to move symbols around but if teachers want students to think that doing mathematics means solving problems, then most of the students’ time should be spent solving problems (Hiebert et al, 1997).

Mason and Johnston-Wilder (2006) summarize a framework, that can be used to think about the purpose of a lesson, called See-Experience-Master devised by Floyd et al. (1981). This framework indicates that the “purpose of a task will include at least one of the following:

- exposing learners to something new (seeing)
- extending learners’ experience of something that is becoming familiar (experiencing)
- practicing behaviors to develop further competence and facility (mastering)” (Mason and Johnston-Wilder, 2006, p. 30).

They argue that this framework allows learners' expectations to be broadened which will in turn permit them to be more open to changes and challenges. "Tasks can be structured so that the old is constantly being used to encounter the new" (Mason and Johnston-Wilder, 2006, p. 65) which is an important aspect of developing subordination tasks. Another framework that they and Hewitt (2010) describe that is related to the power of subordination is Tahta's notion (1981) that activities have an inner and outer meaning. The *outer* task is what the task asks the learner to do explicitly while the *inner* task refers "to any mathematical powers that learners may find themselves using, as well as to the mathematical themes, topics, terms and techniques that they may encounter and the awarenesses that may be invoked" (Mason and Johnston-Wilder, 2006, p. 31). Hewitt (2010) further explains that "the inner meaning concerns the significant awarenesses and skills developed through focused engagement with the outer activity" (p. 10) which is another vital feature of subordination tasks that will be described at the end of this literature review.

PRACTICE IN MATHEMATICS

Unlike riding a bicycle or playing a piano where the skill remains stable over a long period of time, understanding a concept and being fluent in a technique tend to fade if not used (Mason and Johnston-Wilder, 2006). Therefore, even if learners practice a mathematical technique until it seems that it has been integrated into their repertoire of skills, they may still forget it (Li, 1999). Mathematics teachers and textbooks typically describe practice as repetitive exercises that repeat similar computations over and over again, albeit with progressively more difficult questions (Hewitt, 1996; Li, 1999; Watson and Mason, 2004; Mason and Johnston-Wilder, 2006). On the one hand, there has been evidence that students who are diligent with large amounts of routine practice, such as those from East Asia, often place at the top in international assessments (Li, 1999). “If we insist that students should not undertake routine practice and solve problems until they have understood the related concepts, they may lose any opportunities to be engaged in learning” (p. 35). However, Li also acknowledges that there are problems with mathematics taught in this manner such as being dull, boring and offering little understanding. Stein et al (1996) found in their study of mathematical tasks in reform classrooms that “mathematical educators agree that some practice on routine skills, as well as thoughtful inquiry into complex problems, is needed” (p. 472). What needs to be shown is that practice tasks do not have to be the traditional exercises often found in classrooms. Three types of practice exercises that offer work on procedures but are different from traditional exercises are discussed below in order from most similar to most different. The first is the most similar to heavy repetitive practice found in textbooks but alters the way these questions are presented. The second type of exercises concentrates on the order between consecutive questions and focuses on variation in practice exercises. The third practice strategy is quite different from traditional tasks but still involves learners practicing by generating examples of a mathematical entity with certain properties.

Order of Practice Problems

Rohrer (2009) has found that the arrangement of sets of mathematics problems can influence student performance. A typical mathematics textbook in British Columbia contains a lesson followed by a set of practice problems. Usually, this practice set is organized in an arrangement known as “*blocked practice*, a group of consecutive problems devoted to the immediately preceding lesson" (Rohrer, 2009, p.4). A blocked practice set might include a variety of procedural problems, word problems and so forth but there is often heavy repetition that consists of several problems of the same type on the same topic, referred to as an *overlearning* strategy (p.6). Rohrer comments that studies show no support for heavy repetition since students who complete three problems or nine problems of the same type produce very similar test results. An alternative arrangement to blocked practice is *mixed review*. This is where "a group of problems is drawn from many different lessons and ordered so that problems requiring the same skill or concept do not appear consecutively" (p.5). For example, rather than work ten problems on the same topic in one session, a student would work on ten problems that covered several concepts from several lessons. "Blocked practice requires students to know *how* to perform a procedure" (p.10) while mixed practice emphasizes knowing *which* procedure is appropriate. This ability to discriminate is more challenging but mixed practice is more favourable since problems in the real world require people to decide which strategy they will use. Another strategy for presenting practice problems involves the spacing or timing between which a particular type of problem is completed. For example, practice problems for a particular topic can be divided across two sessions separated by a week. Spacing has been shown to improve subsequent test scores since it works by reducing the rate of forgetting but there have not been many spacing experiments involving mathematics so its benefits to mathematics are not fully known. In summary, "spacing provides review that improves long-term retention and mixing improves students' ability to pair a problem with the appropriate concept or procedure"

(p. 4) which are practice methods that can both be easily adopted. However, this type of practice is still very similar to the traditional exercises typically used.

Variation in Practice Exercises

As Rohrer (2009) has found, most textbook exercises are organized as blocked practice. See Figure 2.1 for an example from a selection of textbook questions on ratio.

Reduce to simplest terms:

a) $\frac{4}{12}$ b) $\frac{36}{12}$ c) $\frac{240}{300}$ d) 5:5 e) $ab{:}ab$ f) $\frac{6ab}{3b}$

Figure 2.1 Selection of textbook questions
(Watson and Mason, 2004, p. 107)

Although there is some variation from question to question with the numbers getting larger and a change from numerical to variable, Watson and Mason (2004) point out that the students concentrate on finding the common factors to cancel rather than on any concept to do with ratio. They find that practice exercises are often sequentially different in an arbitrary way and learners progress through these questions in a start-stop fashion. In contrast, they believe that teachers can purposefully make systematic variations in practice exercises within blocked practice so that as students progress from question to question there is a "journey of conceptual development through being exposed to a variety of different forms" (p. 108). Consider the following exercise in Figure 2.2.

Multiply each of the terms in the top row by each of the terms in the bottom row in pairs:

$x – 1$	$x + 1$	$x + 2$	$x + 3$
$x – 1$	$x + 1$	$x + 2$	$x + 3$

Figure 2.2 Multiplying binomials exercise
(Watson and Mason, 2004, p. 108)

“This appears to offer enough similarity to encourage fluency and some awareness of, and control over, change which may allow learners to get a sense of underlying structure while doing the examples” (p. 108). The exercise appears simple and tedious but the change is systematic and with not too many dimensions changing at once, teachers can pose or students may wonder, “what changes and what stays the same?” (p.108). Rather than the focus being on fluency, speed and accuracy, learners contemplate relationships and consider effects of changes in one particular aspect (Watson and Mason, 2006). They use the term *dimensions of possible variation* to describe the associated ranges of change experienced and observed by the learner in these types of exercises. They construe that if teachers can create an exercise that provokes learners to conjecture, predict and reflect, the practice can stimulate exploration and conceptual learning.

Teacher-Initiated Learner-Generated Examples

Believing that mathematics is a constructive activity, Hazzan and Zazkis (1997) and Watson and Mason (2005) describe a practice strategy in which learners are asked to construct their own examples of mathematical objects. By getting “plenty of experience with examples before applying a given algorithm” (Watson and Mason, 2005, p.9), the learner can use exemplification to see the general through the particular. Watson and Mason identify several benefits of learner-generated examples. For one, students can rehearse newly learned techniques as if it were a practice exercise by making up their own examples. And, if the operation is an inverse to something they already know, they can check it. Also, students can choose the range of their numbers, influencing the level of difficulty. Second, example creation can provide an arena for conceptual learning if students apply a conceptual or systematic approach to make discoveries. Third, examples created by each student can be compiled to generate practice exercises for the whole class. Lastly, by getting students to generate their own examples, they can become more familiar with different objects and “the properties that distinguishes them, so that in the future more examples might come to mind from which to choose” (p.22). In addition, Hazzan and Zazkis notice that tasks that involve learner-generated examples ask students to provide an example of a mathematical object with certain properties which forces them to concentrate and practice a particular concept. Also, students have not pre-learned an algorithm to create the examples and the example they devise is not unique which provides an endless amount of examples to practice. Teacher-initiated learner-generated examples can be applied at all grade levels with a variety of topics. It is seen as an effective pedagogical strategy that promotes active engagement and can be used as a rich practice exercise.

“Practice is clearly required for something new to become something which is known so well that it can be used when little or no conscious attention is given to it” (Hewitt, 1996, 29). As is often said, practice makes perfect (Hewitt, 1996; Li, 1999). In addition to the well-known traditional repetitive exercises, educators and researchers (Hazzan and Zazkis, 1997; Watson and Mason, 2004, 2005; Rohrer, 2009) suggest some alternative methods of structuring practice as described above. They know that it is necessary for students to achieve some level of numerical and algebraic fluency as a foundation for subsequent mathematical progress. Hewitt finds that if learners are willing, they will practice things until they have become mastered but what is important is *where* learners place their attention when engaging in such activities. Watson and Mason’s dimensions of possible variation and learner-generated examples are also concerned with where learners place their attention. Some of their practice tasks can be regarded using the lens of subordination and will be discussed next.

SUBORDINATION OF TEACHING TO LEARNING

> It is precisely because a child may engage in tasks beyond the skills they currently possess, that they become good at those skills, and become so skilled that they can do them with little or no conscious attention being given to the skills themselves (Hewitt, 1996, p. 28).

Caleb Gattegno (1970) originally coined the notion subordination of teaching to learning which literally means placing teaching at a lower priority to learning. He first developed and presented the approach for the teaching of mathematics but then extended it to other teachable areas. He believed that it yielded acceleration of learning and a radical transformation in the classroom as “the intelligent use of the powers of the mind” (p.10) of both teachers and students were involved. Subordination forces techniques to become automated by reducing the amount of attention needed to carry them out since another feature of the task is being focused upon. Other researchers and educators (Hewitt, 1996; Watson and Mason, 2005; Mason and Johnston-Wilder, 2006) have referred to this notion in their development of practice exercises. Hewitt’s ideas will be looked at in more detail since they encapsulate a broader sense of subordination in which the other practice exercises fall into.

Armed with the belief that students can achieve mathematical fluency with a more economic use of time and effort that does not involve traditional repetitive exercises, Hewitt (1996, 2001) refines the approach of subordination and explicitly explains the role it has in helping students develop mathematical fluency. Before examining how subordination fits into practice, what Hewitt means by arbitrary and necessary in mathematics will be defined first. *Arbitrary* in the mathematics curriculum refers to

> those things which cannot be known by a student without that student being informed…The arbitrary concerns names and conventions which have been established within a culture and which need to be adopted by students if they are to participate and communicate successfully within this culture (Hewitt, 2001, p.44).

In contrast, the *necessary* are “those things which someone can work out and know to be correct without being informed by others…[such as] properties and relationships” (p. 44). Although informing students of the arbitrary is important, it is only of “value if the learning of names and conventions is not seen as an end in itself, but as being purposeful in supporting work on what is necessary” (p.44). Flewelling and Higginson (2001) also indicate that “traditional skills should be taught when they are needed to allow a student to successfully engage in a rich learning task” (p. 14). Hewitt believes that it is the teacher’s role to provide suitable tasks to help students successfully memorize the arbitrary and educate their awareness so that what is necessary need not be told.

As stated, a teacher can choose to inform students of what is arbitrary or necessary. Tasks can include chanting and reciting until the thing to be memorized is ingrained into the students’ minds (though there may still be the job of helping students connect it to its properties) or completing many traditional exercises where the process is continually repeated. Other memorisation techniques include visual and aural imagery and association such as the mnemonic SOHCAHTOA to help students memorize the trigonometric ratios ‘sine’, ‘cosine’ and ‘tangent’. However, the lack of connection made to concepts makes memorizing difficult which is why teachers usually need to inform the students again and again. Alternatively, Hewitt suggests that subordination can offer a way in which students can practice the arbitrary while engaging in meaningful mathematical activity (Hewitt, 2001). It requires the teacher to artificially create a pedagogic task which forces awareness, meaning that “engagement in the task is likely to result in a student becoming aware of a property which was pre-determined by the teacher” (Hewitt, 2001, p.48). As defined by Hewitt (1996), the features of subordination in such a task include:

- possessing the skill to accomplish the task (the skill may be an existing necessity or created through the “rules” of a task)
- being able to see the consequences of the actions of the skill on the task at the same time as using the skill
- not needing to be knowledgeable about the skill (concept) in order to accomplish the task

To paraphrase the third point above which will be explained in more detail in Chapter 3, “one does not have to possess a good understanding of [the concept] to be able to engage in the task of practicing [the concept]” (Hewitt, 1996, p. 30). The important aspect of subordination is where the learner is placing his/her attention when carrying out the task. To clarify the features stated above, a simple example of a task taken from Hewitt (1996) will be presented here with a more in-depth look at the theoretical framework of subordination presented in the next chapter. The task is based on the old computer game of *Space Invaders* and is developed to practice knowing the value of digits in a number. Students begin by entering the following number in a calculator:

52846173

> The task is to *zap* the digit ‘1’ (turn it into a zero), whilst keeping all the other digits as they are. The only operation allowed is a single subtraction. Next, the digit ‘2’ has to be *zapped*, then the digit ‘3’, etc., until all the digits have disappeared. (In some cases, such as with ‘5’, the digit may disappear rather than being turned into a zero.) (Hewitt, 1996, p. 30)

The main task here is to zap digits (make them turn into zero or disappear) in numerical order until all of them have disappeared. Thus, “the practicing of place value is subordinate to the task of *zapping* digits in numerical order" (p. 30). The only skills the learners need are to know the digits from '0 to 9' and the subtraction button on the calculator. Consequently, they do not need to possess a good understanding of place value to engage in the task of practicing place value. They can also see the consequence of any subtraction they make by looking at the calculator's display screen which will inform them of what to try next time. Hewitt (2001) describes this practice acquired through subordination as "*practice through progress* – practicing one part of the curriculum whilst progressing in another part of the curriculum" (p. 49). The teacher only tells those things which must be told while "everything else comes from the awareness of the students, including the processes involved" (Hewitt, 1996, p. 34). Hewitt argues that those things which are arbitrary will be learned quicker if they are practiced using a subordination task than if they are practiced in isolation with the attention remaining purely on the practicing such as what is found in traditional exercises. In addition, this

> type of practice is effective in helping something become fluent, which classically is what is required with the adoption of the arbitrary. The nature of the arbitrary is such that it is required to be adopted and used with fluency rather than be the focus of attention and conscious consideration. Thus, practicing the arbitrary whilst progressing with the necessary is particularly powerful, since effective practise of the arbitrary takes place whilst time is not 'wasted' doing this, since simultaneously attention is with the mathematics of the necessary (Hewitt, 2001, p. 49).

The notion and features of subordination along with how to develop tasks that exemplify subordination will be developed further in the Theoretical Framework and Methodology chapter.

CONCLUSION

Based on the aforementioned research, it appears that too much time in the mathematics classroom is spent on traditional, repetitive exercises trying to get students to accept and adopt names and conventions. Consequently, less time is spent working on understanding mathematics and allowing students to work out concepts for themselves which for many is a crucial part of learning mathematics. Educators and researchers (Stein et al, 1996) acknowledge that working on procedures does not necessarily need to be viewed as negative. In fact, students need to eventually be able to use and understand the proper names and conventions if they want to communicate with the rest of the mathematics community. What teachers need to consider is the effectiveness of the tasks that they use to introduce and practice routine skills and the conventions of mathematics.

One method proposed by Hewitt (1996) uses the notion of subordination, a mechanism that works by forcing students to practice some skill while working and focusing on some other mathematical task. Others (Kieran, 1979; Boaler and Humphreys, 2005; Wagner, 2003) have also devised tasks that exhibit the features of subordination. Such generalized findings suggest that the use of subordination within learning should be further investigated as a teaching tool for mathematics. This thesis concentrates particularly on Hewitt’s use of the notion of subordination to design tasks that facilitate practicing of a skill while the focus of attention is on some other more, engaging task. I use Hewitt’s features of subordination to design tasks, considering which mathematical concepts can be learned by subordination, examining the requirements in developing an effective subordination task and observing student engagement while working on these types of tasks.

Chapter 3 : THEORETICAL FRAMEWORK AND METHODOLOGY

RESEARCH QUESTIONS

Subordination gives learners an alternative way of practicing mathematical concepts by shifting the focus of the activity. The idea of practice by subordination is probably familiar to many but there has not been much research regarding this mechanism, nor do teachers realize that they may already be employing this notion when they ask students to complete a particular task.

This thesis aims to explore two different aspects of subordination as it pertains to mathematics. The first relates to the designing of subordination tasks and forms the major part of my research as I intend to answer the following questions: Which mathematical teaching contexts are more amenable to the use of subordination? Particularly, can it be used in every class and does it depend on the content involved or the stage of the content development? What challenges and tensions are faced when trying to create and implement subordination tasks? What features constitute a good subordination task? The second subsidiary aspect regards student engagement. Specifically, what are students doing when participating in subordination tasks?

THEORETICAL FRAMEWORK

The theoretical framework for this research is based on the work of Dave Hewitt (1996, 2001). His model arose from his interest of students' acquiring of numerical and algebraic fluency. He believed that an alternative approach could achieve this fluency with a more economical use of students' time and effort compared to traditional repetitive exercises. To better understand Hewitt's model, I first highlight traditional practice using a common exercise of solving linear equations that he describes. A common section of a textbook described by Hewitt (1996) "begins with the following:

> You can add equal numbers to both sides.
> You can subtract equal numbers from both sides.
> You can multiply both sides by the same number.
> You can divide both sides by the same number, (not 0)." (p. 31)

After these instructions, there are several examples such as these:

Example 1: $x + 10 = 17$

Subtract 10 from both sides.

$x = 7$

Example 2: $13x - 20 = 6x + 8$

Subtract 6*x* from both sides.

$7x - 20 = 8$

Add 20 to both sides.

$7x = 28$

Divide both sides by 7.

$x = 4$

Practice exercises follow these examples under the heading of "Solve these". Here is a sample to get a sense of the exercises:

1. $a - 7 = 14$
2. $\frac{1}{4}(g - 2) = 6$
3. $2(5 + x) - 3(6 - x) = 42$

   (Hewitt, 1996)

In this task of solving equations, students place their attention on the four operations, given at the beginning, which are used to solve an equation. Hewitt asserts that

> attention is begin placed on carrying out the repetition. Those things which are subordinated during this repetition (use of letters, syntax of formal algebraic notation, multiplying out brackets, order of operations, etc.) are assumed to be already known and understood (Hewitt, 1996, p. 32).

As described in Chapter 2, students use traditional exercises to repeat the processes of a particular concept over and over again to achieve fluency. In order to undertake the practice, it is assumed that students already know the mathematics to carry it out.

However, if they do not know the mathematics then they may not be able to initiate the practice. "This is the classic Catch 22 [Heller, 1962] of traditional exercises – if someone doesn't understand already, then they won't be able to begin; if someone does understand already, then there is little need for them to begin" (Hewitt, 1996, p. 31). In contrast to this traditional exercise, Hewitt proposes a different activity (described little later in this chapter) that uses subordination of teaching for learning. Central to the role of subordination is the idea of awareness and where learners place their attention while focusing on the activity. Instead of focusing on the repetition of practice like the exercise described above, he suggests students place their focus on a task. According to Hewitt, a task is a subordination task if it follows the definition outlined below. "A skill, A, is subordinate to a task, B, only if the situation has the following features" (p. 31):

1) A is required to do B.
2) The consequences of the actions of A on B are observable at the same time as making those actions.
3) It is not necessary to be knowledgeable about, or be able to do, A in order to understand the task, B.

In order to better comprehend how these three features are applied to a task, I present four tasks, one non-mathematical and three mathematical. These tasks demonstrate "that those things which are given conscious attention may not be learned as well as those things which have conscious attention taken away from them through their subordination to a task" (Hewitt, 1996, p. 34).

Non-mathematical Example of the Use of Subordination

I begin with a non-mathematical example of how a task exhibits the features of subordination using Hewitt's (1996) description of a situation in which he is learning to sail. When sailing, the angle of the boat and sail are dependent on three elements: the rudder, the rope and the body position. He knows how to adjust each of these elements independently but if he focuses on any one alone, he forgets the other two and upsets the boat. Consequently, rather than focusing on any one, he needs to keep his attention only on the sail to ensure that it does not flap in the wind. Therefore, concentrating solely on the sail, he is able to make finer adjustments to the rudder, rope and body position and attend to “the *consequences* of any movements in these three, rather than the movements themselves” (p. 31). Thus, his focus is away from where his learning is taking place. It is not just a withdrawal of attention from an area in which he is learning but a placement of attention on a task which subordinates the skills he is trying to learn. “The angle of the sail is at a *higher subordinate level* than the rudder, rope and body position” (Hewitt, 1996, p. 30). How subordination is used can be summarized by the following flow chart, Figure 3.1, which is modeled after the one given in Hewitt (2001).

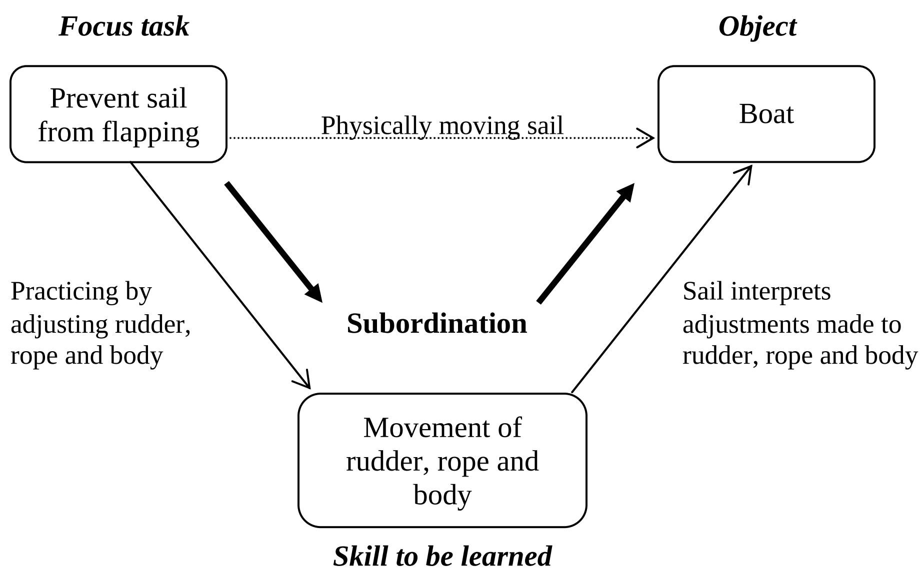


Figure 3.1 Subordination flow chart for sailing

This flow chart shows that in order to prevent the sail from flapping, he must adjust the rudder, rope and body together, which is then interpreted by the sail to move the boat.

Mathematical Examples of Using Subordination

From Hewitt's research (1996, 2001), I generalize that there are three types of mathematical concepts that can be practiced using subordination in reference to the arbitrary: notation, definitions and conventions. I provide an example of a task gathered from Hewitt's research to demonstrate how subordination is used for each element.

Using subordination to introduce notation

"Think of a number" is a mathematical practice activity designed by Hewitt (1996) that subordinates algebraic notation to the task of finding a number. He verbally describes, to the students, a series of operations which is carried out on an unknown number before giving the final result, whereby the students then need to state what the original number was. He begins with one-step and two-step operations. For example, "I am thinking of a number and then add three to it, multiply by two, and get 14. What is my number?" (Hewitt, 1996, p.32). Because this activity is begun verbally, after working for a while and when the students know what to do to work out his number, he deliberately makes the list of operations too long for the students to remember and so they need to have a method of writing the operations down. For example, he gradually writes "the following on the board, being careful to write the symbols associated with the words [he] is saying, at the same time as saying them:

$$6\left(\frac{2(x+3)-5}{3}+72\right)=100$$

> *OK. So, …let me see… I am thinking of a number* (writes x on the board) *… I add three* (writes +3) *… then I'm going to … multiply by* (writes brackets round the expression so far) *… two* (writes 2 in front of the brackets) *… then I am going to take away five* (writes –5) *… then I am going to divide by* (writes a line underneath the expression so far) *… three* (writes 3 below the line) *... then I'm going to …* (… writes + following on from the division line …) *add… 72* (writes 72 after the addition sign) *… then I'm going to multiply by* (writes brackets round the expression so far) *… ummm … six* (writes 6 in front of the brackets) *… and I get* (writes = to the right of the expression so far) *... umm … 100* (writes 100 to the right of the equals sign)" (Hewitt, 1996, p.32).

This may be the first time that some students have encountered the algebraic notation but he attempts to keep the focus of attention on the task which is to find his number. The purpose of introducing the notation here is to remind students of what was done to his number since he had originally not allowed the students to write anything down.

> Thus, the notation, which has been met for the first time by many students, becomes immediately subordinate to the task of finding [his] number. Although the notation may be new, the task of finding [his] number is not, and so the *task* is understandable without the need to be able to interpret the notation (Hewitt, 1996, p. 32).

Continuing with the lesson, the following equation with operations inverse to the above equation, is gradually written on the board below the first one:

$$\frac{3\left(\frac{100}{6}-72\right)+5}{2}-3=x$$

In order to find the number, learners practice writing the original expression and the inverse expression using algebraic notation. See Figure 3.2 for a visual depiction of how subordination is used to learn algebraic notation.

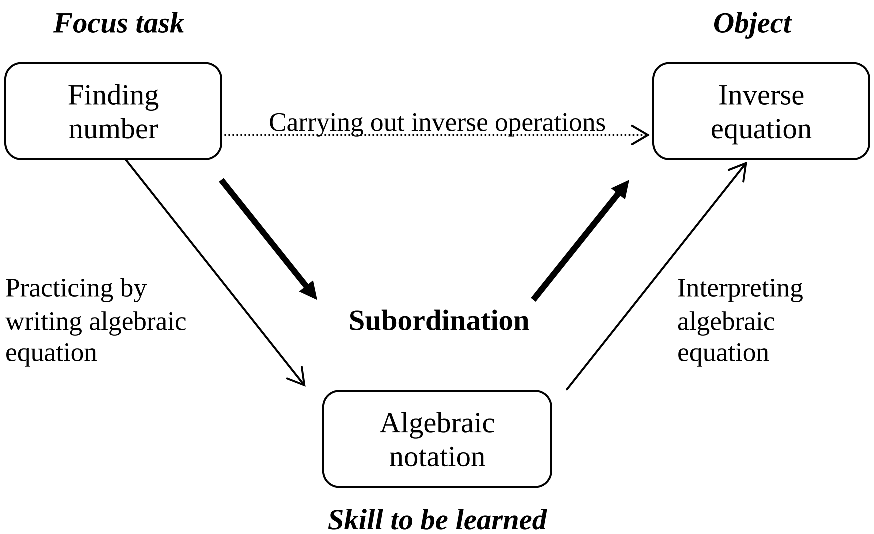


Figure 3.2 Subordination flow chart for *Think of a Number*

In the above example, "the object of the lesson is to help students articulate the inverse operations involved in getting back to [his] number and be able to write the expressions involved in 'correct' algebraic notation" (Hewitt, 2001, p. 47). Eventually, the students are able to find the "number" regardless of how difficult the equations have become. "They subordinate the notation to this task (since [he has] constructed the situation so that they have no other choice), and as a consequence become fluent in interpreting and writing formal algebraic notation" (Hewitt, 1996, p. 33). Kieran (1979) found similar results when she introduced brackets in the order of operations. "It is only by creating in the student's mind a <u>need</u> for such a notation that he will accept it and thereby make use of it" (p. 133). Besides the syntax of formal algebraic notation, Hewitt (1996) mentions other "knowings" which are required to carry out the task successfully. These include the use of a letter in an equation and the order of operations that are also being subordinated to the task of finding the number. The latter will be the focus of one of the tasks in the next chapter.

Using subordination to practice definitions

Here is a task designed by Hewitt (2001) that shows how subordination can be used to practice using words that are associated with the properties of triangles such as ‘equilateral’, ‘isosceles’, ‘right-angled’, ‘acute’, and ‘obtuse’. Students work in pairs where one person holds copies of triangles in Figure 3.3 and the other person has drawings of the target shapes in Figure 3.4.

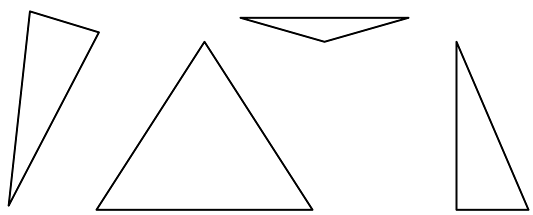


Figure 3.3 Starting triangles

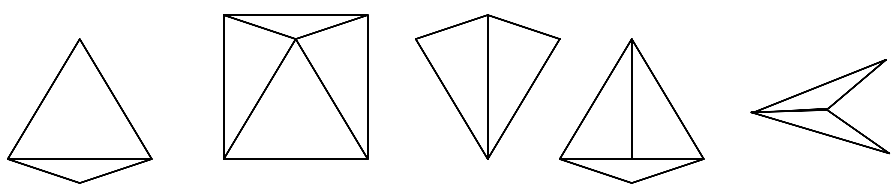


Figure 3.4 Target shapes

The student holding the target shapes has to get his/her partner to create these shapes using the starting triangles and following these “strict rules of communication:

a) no pointing, either directly (i.e. physically pointing) or indirectly (i.e. “the one next to you” by either party is allowed);

b) only the person with the sheet of target shapes is allowed to speak;

c) all references to triangles must start with one or more of the following words: equilateral, isosceles, right-angled, acute-angled, obtuse-angled.

These artificially created rules are designed to ‘force’ the practice of naming whilst attention is with a different task, in this case creating the target shapes” (Hewitt, 2001, p. 48). The names of the shapes have been met before but the task provides a method in which they are further practiced and reinforced as they are subordinate to the main task of creating the target shapes. In order for Student 2 to draw the target shape, Student 1 must say the names of the triangles which Student 2 interprets by moving the actual triangles. See Figure 3.5 for the visual depiction.

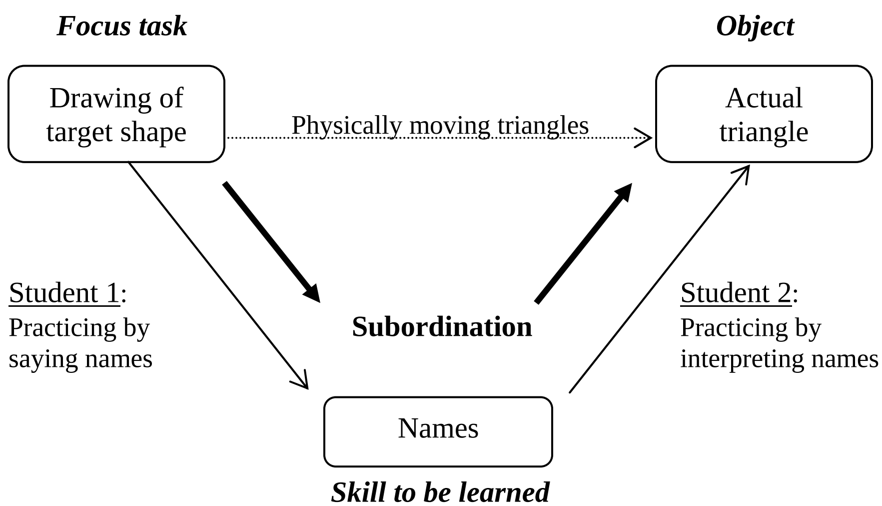


Figure 3.5 Subordination flow chart for naming triangles

The naming of the shapes has a direct effect on the main task. As Student 1 names the triangle he wants Student 2 to pick up, Student 2 interprets what Student 1 says and either picks up the intended triangle or a different one; thus giving feedback to Student 1. This task forces learners to pay attention to the mathematical properties of the triangles while the arbitrary components, the names, are being practiced (Hewitt, 2001). This practice exercise is important because even though the words have been introduced, in order for them to be used with second nature, the words need to be used again and again. The discussion encourages this practice particularly since it is amongst peers.

Using subordination to introduce conventions

In the previous chapter, *Zapping Digits* demonstrated how the conventions of place value can be practiced using subordination. To further illustrate how subordination can aid students in the learning of other mathematical conventions, another task will be described here. In the task *Do we meet*? (Hewitt, 2001), learners practice the arbitrary convention of vector notation but also learn about transformations and symmetry while concentrating on trying "to meet up". The teacher and student begin by choosing two different starting positions on a grid. The student makes the first move by stating a vector that is subsequently drawn on the grid. After observing the student's move, the teacher then makes a move according to an unsaid rule that is connected to the student's movement. This exchange continues back and forth with the student trying to move to a place where he will meet the teacher. See Figure 3.6 for an example of a game with the teacher's unsaid rule being a 90º rotation clockwise of the student's vector.

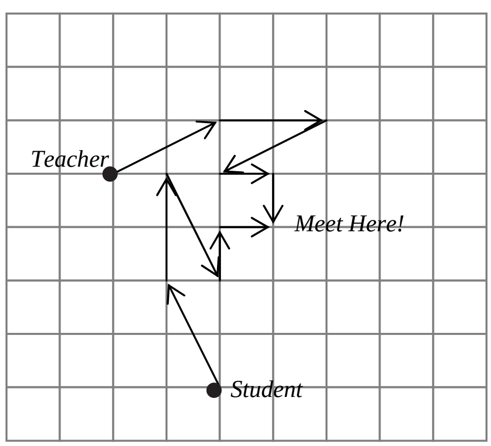


Figure 3.6 A game of *Do we meet*? (Hewitt, 2001, p. 49)

In this game, the student is focused on finding out the teacher's rule and producing a vector for himself that will land him in the same spot as the teacher. The vector that the student calls out has a direct effect on where the teacher will move. And, each time the teacher moves, it gives a clue as to what the unsaid rule may be; thus keeping the focus of attention on the task of trying to meet up by producing an appropriate vector. After watching a game or two with the teacher, the students can engage in this task in pairs creating their own rules for movement. In order for Student 1 to meet up with Student 2 (teacher's role), Student 1 must use vector notation which Student 2 interprets by drawing a line to move Student 1's location and then drawing her own line according to the unsaid rule. See the flow chart in Figure 3.7 which shows how subordination is used.

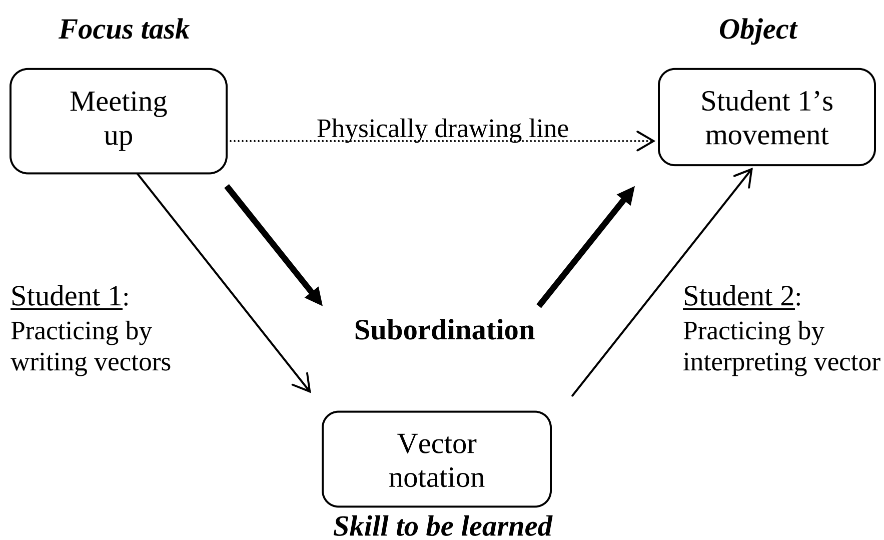


Figure 3.7 Subordination flow chart for *Do we meet*?

This activity forces the practice of the arbitrary conventions of vector notation while progress is made with transformations and symmetry. As stated before, this is what Hewitt (2001) refers to as "practice through progress – practicing one part of the curriculum whilst progressing in another part of the curriculum" (p. 49). Besides practicing vector notation, this game can be adapted so that students practice the mathematical convention of the coordinate system.

All the tasks described above, both mathematical and non-mathematical, demonstrate how subordination is used to practice a particular skill while the focus of attention is on a more central task. As long as the three features of subordination are satisfied, the task is considered to be a subordination task. Note that in Hewitt's characterisation of subordination tasks, there is no explicit mention of how students might differ in the way they experience these tasks. In reference to the mathematical tasks, subordination can offer an effective way to practice those things that are arbitrary, while attention is with the mathematics of the necessary. These pedagogic tasks have all been artificially created to ‘force’ awareness meaning that engagement in the task will likely result in the learner becoming aware of a property that was pre-determined by the teacher (Hewitt, 2001). As Kieran (1979) articulates in reference to Skemp (1976), if “*relational understanding* means knowing which rule to use and why it works, the student must first be aware of the necessity of such a rule” (p. 133). In other words, the task needs to demand or require the particular skill in order to achieve its outcome.

Engagement

The focus of the thesis is on the design of subordination tasks but I am also interested in students’ experiences of these types of activities. For one, I want to look at student engagement while working on subordination tasks. Since Hewitt does not say much about how he conceptualizes engagement, I need to draw on other sources. One way to look at the depth of engagement is to examine the perceived flow of time. Larson and von Eye (2006) directly study the perceived flow of time and depth of engagement.

Others (Shernoff et al., 2003; Meyer and Turner, 2006) look at flow of time and student engagement from the perspective of flow theory. Csikszentmihalyi (cited from Shernoff et al, 2003) describes flow as “a state of deep absorption in an activity that is intrinsically enjoyable, as when artists or athletes are focused on their play or performance” (p. 160). “Flow theory is based on a symbiotic relationship between challenges and skills needed to meet those challenges” (Shernoff et al, 2003, p. 160). Flow is not applicable in its entirety but during flow there is a sense of loss of time which suggests that students are highly involved in the task (Meyer and Turner, 2006). Another way to study student engagement is by assessing concentration, interest, enjoyment, challenge/skill conditions and instructional method (Shernoff et al., 2003).

The above research (Larson and von Eye, 2006; Shernoff et al., 2003; Meyer and Turner, 2006) regarding engagement substantiates Hewitt’s (1996) desire of using subordination to practice something with little conscious attention devoted to it and leads to the second experience I want to look at. Hewitt (1996) seems to suggest subordination tasks should not feel like practice since students are more highly involved in the task. In other words, if students are involved and engaged, they should not feel like they are practicing. I want to see whether this holds true for the students.

METHODOLOGY

General Setting

The research study was conducted at a secondary school in Richmond, British Columbia in the school year 2008-2009. This middle class urban high school contains approximately 1200 students from grades 8 to 12 with a variety of cultural backgrounds. The study was carried out on a full year Principles of Mathematics 10 course. At this time, there were three courses offered at the grade 10 level: Principles of Mathematics 10, Applications of Mathematics 10 and Essentials of Mathematics 10. Each of these courses has a curriculum that is mandated by the provincial government along with a comprehensive provincial exam written in June. Because the principles of mathematics stream is usually required by the majority of college and university programs, and parents and students want to leave their options open to attending these types of post-secondary institutions, the majority of the students (87% this year) enroll in Principles of Mathematics 10 as opposed to only 13% (this year) in Essentials of Mathematics 10 and none in Applications of Mathematics 10. Consequently, the Principles of Mathematics 10 classes contain students that have very mixed mathematical ability with many students struggling with the more abstract material and faster paced course.

Classroom Structure and Instruction

The physical layout of my classroom is classified as semi-traditional. Students sit in groups of three which all face the front of the classroom. This arrangement allows them to see the front where the teacher usually stands but also permits them to engage in conversation with each other since their desks are adjacent to each other. This layout is not specific for my research as I have used a similar layout for the last thirteen years. My typical instruction can also be labeled as traditional interspersed with some non-traditional activities once or twice per unit. By traditional, I mean that I stand in front of the classroom introducing new concepts by presenting notes and formulae for students to copy. After each mini idea or concept is taught and I have demonstrated several examples, students try several similar practice questions on their own while I walk around helping students who are having difficulty, checking answers and answering questions. This back and forth lesson of my instructing and students' copying and practicing usually takes approximately 30 to 45 minutes of a 75-minute class. The first 15 minutes is usually spent taking up old homework or working on review questions from last day's concept and the last 15 minutes is set aside for students to start their homework and practice the concept they just learned.

Participants

At the beginning of the year, I informed the students from my two Principles of Mathematics 10 classes that I was working on my thesis and asked them to participate in my study. Participation in the research was completely optional and students wishing to participate submitted a consent form signed by their guardian and themselves. Participant's ages ranged from 14 to 16 as there were accelerated grade 9's and repeating grade 11 students as well as the typical 15 year old for a mathematics 10 course. This grade level was chosen because the mathematics curriculum at this level is often found to be one of the turning points for many students as more arbitrary notations and conventions are introduced and students find it difficult to connect with the abstractness of algebra (Wagner and Parker, 1993). The first task was presented to the class I perceived to be mathematically stronger but quieter (which I call Class 2). After this first task, I used the subsequent subordination tasks with the comparably weaker, but more verbose, class (which I call Class 1). I was hoping that Class 1 would provide more feedback because of their comparative verbosity.

Task Creation

Before creating the tasks, I needed to consider and assess which mathematical concepts I believed were appropriate and amenable to having a subordination task developed for them. As I sifted through the curriculum and considered the mechanism of subordination, I realized that the part of the mathematics curriculum that dealt with those things which cannot be known without being informed would be more suitable. Hewitt (1999) defines these things as arbitrary. “The nature of the arbitrary is such that it is required to be adopted and used with fluency rather than be the focus of attention and conscious consideration” (Hewitt, 2001, p. 49). As described in the theoretical framework, these arbitrary aspects include notation, definitions and conventions. It was difficult deciding for which concepts to create a task since I realized that more of the curriculum concentrated on the necessary of mathematics rather than the arbitrary. Therefore, I decided to choose topics that I deemed difficult for students and they would enjoy more if practiced in an alternative way. I began by attending to the skill I want practiced and then considering an activity that students would focus on but would require the particular skill to carry out. Task creation was governed by Hewitt’s definition and the features of a subordination task (1996). As stated before, these included:

- possessing the skill to accomplish the task (the skill may be an existing necessity or created through the “rules” of a task)
- being able to see the consequences of the actions of the skill on the task at the same time as using the skill
- not needing to be knowledgeable about the skill (concept) in order to accomplish the task

Based on my interpretation of his definition and the skills that I wanted practiced, I created and developed tasks acquired from a variety of resources including internet, print material and fellow colleagues. In the end, six tasks were developed that encompassed six different topics from four different units in the Principles of Mathematics 10 curriculum. The six topics were, in the order that they were introduced: radicals, exponential laws, equation solving with literal coefficients, function notation, domain and range, and equation of a line in slope *y*-intercept form. The tasks were set up according to the theoretical framework of subordination described earlier. Four were used to introduce topics and two were used for review practice. Table 3.1 summarizes the task developed, the skill practiced in each task, the purpose of the task and the skill's arbitrary aspect category.

Task	**Topic**	**Purpose**	**Category**
1. Stacking Squares	Equivalent radicals	Introduce	Convention
2. Expression Scattergories	Exponential laws	Review	Convention
3. What's My Number?	Equation solving	Review	Convention
4. Function Factory	Function notation	Introduce	Notation
5. What's My *x*?	Domain and range	Introduce	Definition
6. Algebra versus the Cockroaches	Slope y-intercept form	Introduce	Convention

Table 3.1 Six tasks developed to test subordination

Research Procedure

The procedure that best describes the design of this study is action research. According to Mills (as cited in Creswell, 2008), action research allows the teacher-as-researcher to gather information about how to teach and how well students learn in the teaching/learning environment. This research design method is appropriate for my thesis as I reflect on my teaching practice and investigate a new idea (that of subordination) in an attempt to find an alternative to using traditional practice exercises in mathematics.

Reasons for doing so were explained in the introduction and I believe that the notion of subordination will have a positive change in the mathematics classroom. As I test my ideas of subordination, gather data and reflect on each task's success or failure of being a subordination task, I try new situations in subsequent tasks to uncover when and how subordination tasks can be used. The data collection techniques that I use align with Mill's "Three E's" (Cresswell, 2008): experiencing (observations and fieldnotes), enquiring (questionnaire) and examining (documents, audiotapes, videotapes, and fieldnotes). During this inquiry, I become the researcher in my own classroom but at the same time, act as a participant who can legitimize certain aspects of learner's mathematical activity and implicitly sanction others (Yackel and Cobb, 1996). Although this action research is carried out individually and is self-reflective, I informally share my experiences and tasks with a school colleague who teaches the same course in order to gain additional feedback.

As stated earlier, this thesis focuses mainly on the tasks themselves so the tasks that I develop will form part of the data to be analyzed. The six tasks were administered to the Principles of Mathematics 10 students from October 2008 to April 2009, interspersed with traditional and non-traditional lessons throughout the year. Two tasks involved whole class participation while others were completed individually or in small groups. During each task, fieldnotes were taken as I walked around the classroom observing the students. For the three tasks that involved my being a main participant in the whole class activity, the lessons were video taped so that I could review them later. I knew that I would not be able to take detailed notes while I worked with the students and

I would most likely not remember all the classroom events and interactions. Consequently, at the end of each class, I also made self-reflections of the lesson.

To assess students' engagement with subordination tasks, I made informal observations and collected data from the students. For the first task, the students wrote a letter describing what they had experienced in the class. From this open-ended task, I was hoping the students would mention something about their involvement with the task in terms of interest, enjoyment, challenge and/or whether they felt like they were practicing. The results from the letter writing activity were interesting but I decided to get students to complete a short questionnaire that had more directed questions for the remainder of the lessons. Therefore, for the second task, the two questions specific to my study from the questionnaire (see Appendix 1) were:

1. What did you learn today?
2. Did you enjoy doing the practice today? Explain why or why not.

The responses from the students were too general and I wanted to guide them more towards reflecting about their experience with the task. I decided to change the questionnaire (see Appendix 2) for the remaining four tasks. In the second questionnaire, to assess engagement, I replaced the two questions above with:

1. Did class seem long or short today? Explain.
2. Did you feel like you were "practicing" mathematical concepts? Explain.

As stated previously, the depth of engagement influences learners' perceived flow of time which was the purpose of question 1. The reason for the second question was that if students did not feel like they were practicing, then they may have been enjoying the task, leading to a feeling of engagement. In addition, I asked students a general open-ended question about the activity in hopes of obtaining responses that would suggest their interest and enjoyment level.

3. What did you like and dislike about today's activity?,

The responses to the questionnaires were compiled for each task and grouped according to similar themes along with any unique and interesting comments.

The tasks, fieldnotes, audiotape transcript, videotape transcripts, reflections and students' responses were compiled to create the analysis by task in the next chapter. In Chapter 5, I provide an analysis and summary of the common themes that emerge.

Chapter 4 : RESULTS AND ANALYSIS BY TASK

This chapter presents the results and analysis of each task in the order that I introduced them to my students. For each task, I begin by providing some background that includes students' prior knowledge coming into the task and how I normally teach the concept. In the rationale section, I describe any difficulties that I encounter with this method of teaching, suggest possible reasons for these problems and propose some recommendations. Next, I explain how the task was designed before describing the task itself and how it exhibits the features of subordination. I use Hewitt's (1996) description of subordinate tasks, which was stated in the theoretical framework, to analyze each task. It is restated here for convenience in Figure 4.1.

> A skill, A, is subordinate to a task, B, only if the situation has the following three explicit features:
>
> 1) A is required to do B.
> 2) The consequences of the actions of A on B is observable at the same time as making those actions.
> 3) It is not necessary to be knowledgeable about, or be able to do, A in order to understand the task, B.

Figure 4.1 Three explicit features of subordination tasks (Hewitt, 1996, p. 31)

Following the observations, I re-analyze each task to determine if the task still exhibits the features of subordination after its implementation and if so, identify which features propel it to be a good subordination task. In addition, I briefly describe any challenges and tensions that were faced while creating and implementing the subordination tasks, particularly those that are in conflict with the particular aspects of subordination.

TASK 1 – STACKING SQUARES[2]

Background

One of the prescribed learning outcomes of Principles of Mathematics 10 is being able to "perform operations on irrational numbers of monomial and binomial form, using exact values" (Ministry of Education, 2006, p. 35). To the teacher and textbook authors, this means being able to simplify radicals and radical expressions. Although students have calculated square roots and cube roots using a calculator in the past, this will be the first time they attempt to find equivalent radical expressions or to simplify square roots. Normally, I teach simplifying square roots using the textbook method. This involves finding perfect squares that divides the number underneath the radical sign. Here is a typical outline of how I demonstrate simplifying $\sqrt{72}$.

1) Find the largest perfect square that divides the number (72 in this case) under the radical symbol.

$$\sqrt{72}$$

2) Write this number (36) and its quotient (2) under two separate radicals.

$$\sqrt{72} = \sqrt{36}\sqrt{2}$$

3) Find the square root of the perfect square ($\sqrt{36}$ = 6) and write it in front of the other radical.

$$\sqrt{72} = 6\sqrt{2}$$

[2] Title is based on the activity by NCTM Illuminations http://illuminations.nctm.org/LessonDetail.aspx?id=L622 and David Wagner's article, 'We have a problem here: 5 +20 = 45?'

After my instructions and students' trying several similar examples on their own, I give them more questions to practice from the textbook.

Rationale

I chose this concept after finding a task I considered would take the tedium away from the traditional practice usually associated with simplifying radicals. Most students are successful at mastering the above procedure for simplifying single irrational numbers. However, the method assumes that students know many of their perfect squares by memory and how the arithmetic operations work with radicals. From my personal teaching experience, these assumptions give rise to problems that students encounter when applying this technique such as:

1) being unable to identify *any* perfect squares that divides the number,
2) being unable to find the *largest* perfect square that divides the number so that the radical is not completely simplified, and
3) misunderstanding the convention and/or concept of the radical and writing an incorrect radical such as $36\sqrt{2}$ or $18\sqrt{2}$ or something similar.

The first two problems are not misconceptions regarding the concept of irrational numbers but are a result of a student's difficulty with factoring. Therefore, I will only address the third problem since this error is related to a misunderstanding of the convention of radical notation or the process of simplifying a square root. Mathematics 10 students so far have a limited notion of the concept of radicals as also acknowledged by Balakrishnan (2008). They are most familiar with the operation of the square root and regard it purely as a process. For example, they see $\sqrt{9}$ = 3 and $\sqrt{72}$ = 8.485. In order to move beyond this view and perform operations with radicals, learners need to see radicals as mathematical objects. This is in line with Kieran's (1981) research regarding the interpretation of the equality sign as an operator symbol to “do something” in elementary school and then as a symbol for equivalence in high school when working with algebraic equations. Another explanation for the misunderstanding of the radical convention or the process of simplifying may be caused by confusion in using the algorithm, such as which number comes out of the radical sign. Therefore, when students encounter questions that involve operations with radicals later, they are unsure of what to do because they do not understand what a radical actually represents. Learning by imitation, rules and memorization often limits a student’s ability to apply their skills when they encounter unfamiliar situations (Dewey, 1933; Brownell, 1947; Skemp 1976; Doyle 1988).

> Skills which are learned mechanically, with a minimum of meaning, quickly deteriorate. To keep them alive, one must practice them ceaselessly. However, the conditions of life afford little opportunity for continuous practice, and once the unremitting drill of the school is withdrawn, the skills suffer (Brownell, 1947, p. 261).

To contribute to this confusion, students are learning about radicals in isolation instead of using their prior knowledge to develop any ideas they may already have about radicals. Students should be encouraged to evoke their concept image (Tall and Vinner, 1981) of square root to help them understand how to simplify radicals. A concept image comprises “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (Tall and Vinner, 1981, p. 152). It can be changed as students learn more about the concept. When encountering new information, Tall (1989) professes that learners can only make sense of it by making new connections to what they already know. If this cannot be done or is not allowed to happen, the new properties and processes are not assimilated into the concept image and will need to be memorized. Therefore, considering where learners are and their readiness

(Mason and Johnston-Wilder, 2006), I propose an activity that will help students make more connections to the concept of radicals. They will learn about radicals as mathematical objects while focusing on another more engaging task.

Task Development

One of my professors of my graduate studies recommended David Wagner's *Stacking Squares* (NCTM Illuminations, 2003) activity to me when I approached him with the idea of using subordination for teaching. The main idea of the task is to find stacks of squares that will produce the exact same height as the given initial square. The initial square represents an unsimplified whole radical and the stacked squares represent mixed radicals that are equivalent to the initial square. The intent of the activity is to help students explore the meaning of square roots before they see the traditional algorithm for simplifying square roots. Although the task does not necessarily lead to finding the simplest radical, finding equivalent radical expressions can lead to a discussion about the simplest radical.

I am attracted to the activity because it is open-ended and encourages communication, reasoning and using multiple forms of representations, processes that are recognized as important in *Principals and Standards for School Mathematics* (NCTM, 2000). The task also calls for visual and concrete models which I believe will make the practice more interesting, enhance mathematical understanding, and aid in recall. In the typical simplifying radical algorithm that I usually employ, students learn only to write the *one* radical expression in simplest form. However, the *Stacking Squares* activity encourages students to find several equivalent square root expressions, allowing discussion about equivalent radical expressions along with the idea of the simplest or smallest radical. The activity will also help students make a connection between their invented methods and the traditional algorithm.

Task Description: Outline of the lesson plan

This investigation is implemented at the beginning of the unit on radicals with the intention that students explore the meaning of square roots before they see the traditional algorithm for simplifying radicals. There are two learning objectives of *Stacking Squares*:

1) investigate ways of finding equivalencies that involve square roots

2) make and test conjectures about connections between geometric and numeric representations of squares and between whole numbers and their square roots

There are also two pre-requisites for this task: finding the area of a square and knowing the relationship between the side length of a square and the area of a square. With students organized into groups of two or three, each group is provided with one copy of the “Playing with Squares” activity sheet (see Appendix 3). Figure 4.2 provides a brief description of what the task entails.

The 45 cm^2 square is the exact same height as the two stacks of squares beside it. The squares in the stack on the left have areas of 5 cm^2 and 20 cm^2. Each of the three squares in the stack on the right has an area of 5 cm^2. For this assignment, the area of any square should be a natural number, when measured in square centimeters. Find stacks of squares that would be the exact same height as a square with area 72 cm^2.

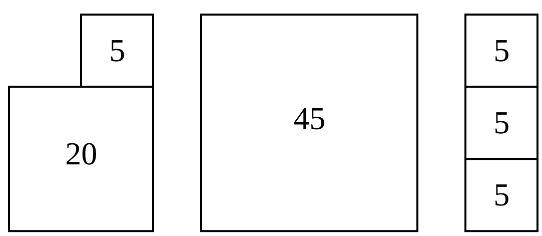


Figure 4.2 Playing with Squares

Grid paper, ruler, scissors, and markers are made available but to prevent from leading the students on, none of the materials are emphasized. The teacher needs to permit students to work on the task for at least 30 minutes to create “stacks”, encourage students to find more than two stacks, and ensure everyone in the group understands how to create each stack they find.

After a sufficient amount of time is provided for the students to create stacks, the teacher holds a class discussion to consider the different stacks that the students create. Students share the different methods that they use to create their stacks and attempt to find something that is common to all the methods. If time is available, students can draw posters to display the stacks they create and the method(s) they use to create them. The teacher should try to help students make connections between:

- the side length of each square of the stack and the side length of the large square
- the sum of the area of the individual squares and the area of the original square
- how all the different stacks are equivalent

The site http://illuminations.nctm.org/LessonDetail.aspx?id=L622 (NCTM Illuminations, 2003) provides other questions that can be posed. There should also be a discussion about the idea of the simplest or smallest radical since this is ultimately what students will need to do on exams. After the activity, either in the same class or in subsequent classes, the teacher can help students make a connection between their invented methods and the traditional algorithm. As an extension, students can find the side length of cubes given the volume of rectangular prisms.

Why *Stacking Squares* is a Subordination Task

The purpose of the task is for students to learn the skill of writing equivalent radical expressions, leading to a better understanding of the arbitrary convention of the radical notation. Therefore, referring to Figure 4.1, the skill, A, that students are practicing is writing equivalent square root expressions using exact values. The task, B, is to find stacks of squares that give the same *height* as a square with the target area of 72 cm^2. *Stacking Squares* satisfies the three basic requirements of being a subordination activity because

1) writing equivalent square root expressions using exact values is required to create stacks of squares that give the same height as the larger square.

2) once a stack of squares is found in exact values, the height can be immediately self-checked using a calculator to see if it matches the height of the large square

3) the students know that their task is to find stacks of squares of equivalent height to the large square though they may not know how to write equivalent radical expressions.

Figure 4.3 below depicts how subordination is used to practice writing equivalent square root expressions. In order to create a stack of squares, students need to use square roots to create equivalent expressions which are used to match the large square.

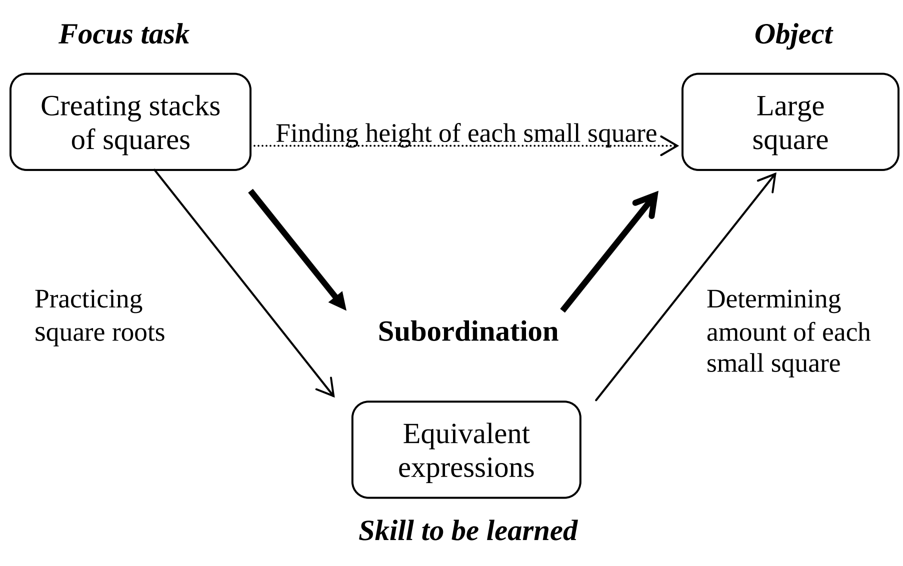


Figure 4.3 Subordination flow chart for *Stacking Squares*

Results and Analysis

Observations

This task was carried out with Class 2. My focus for the students was getting them to find as many stacks as they could, having every group member understand how to find a stack and ensuring they knew how to check their solutions. From my glance around the classroom, all the students were engaged in the task from the beginning. None chose to use the grid paper but they all confined themselves to drawing squares on regular lined paper. About five minutes into the task, seven students were already off-task and needed encouragement to keep working at it. At the 10-minute mark, five students were off-task.

As I walked around, I observed that the most common technique for finding stacks was to use a ratio method that was based on the provided example. For instance, they saw that in the example, the small square of 5 could be obtained by dividing the large square of area 45 by “9”. This gave them the three squares of area 5, as shown in Figure 4.2 and again in the top left diagram of Figure 4.4 below. Analogously, they divided the large square of area 72 by “9” as well and got 8. Upon checking the side length of three squares of area 8 with a calculator, they discovered that it was the same length as the large square of area 72, as shown in the bottom left of Figure 4.4. To obtain the stack with two squares, students applied a similar technique. The 5-square was obtained as described above. Then, they noticed that if the value of 5 was multiplied by “4”, they got a value of 20 for the 20-square. (See top right of Figure 4.4.) Therefore, mimicking how they obtained the 5- and 20-square, the students multiplied 8 by “4” to obtain 32 for the 32-square. (See bottom right of Figure 4.4.) When they checked the sum of the side length of the 8-square and the 32-square using their calculator, they got the same height as the 72-square.

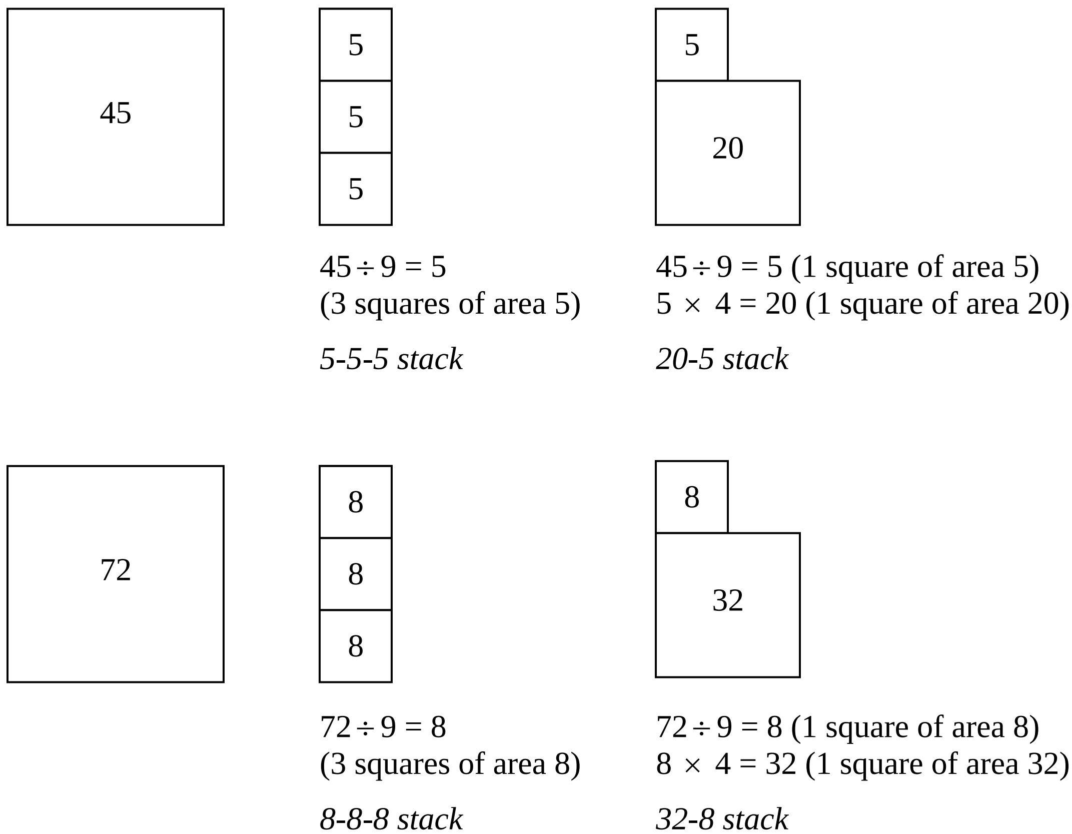
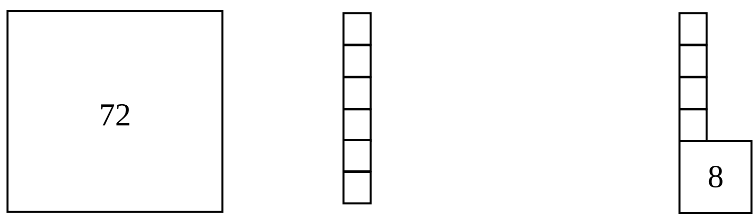


Figure 4.4 Using ratios to find stacks

What students were doing here was essentially multiplying the large square by $\frac{1}{9}$ to obtain the small square of 8 and $\frac{4}{9}$ to obtain 32. Unfortunately, the students did not notice this and hence, using ratios in their manner was limiting – this is something some of Wagner's students had also encountered – since it only produced the two stacks shown in Figure 4.4. Therefore, I encouraged students to apply other techniques. In time, four groups discovered more stacks by dividing the area of the large square by the same number twice. Here's a clip from the audiotape that demonstrated how one student thought about it.

> Ok, we wanted to do it another way since we had to do it another way. Umm, so I was thinking, ok, if you want to divide it (referring to the square with area 72) by something more than just 3, right so I divided by 6. Then, 6, so then that would equal like 12. Then, 12 you would want to divide it by 6 again so you want to divide it into six blocks which equals 2 so then it said no stacks with the exact same height so you put it together which is 4, [I mean] which is 8. (Brenda, Class 2)

Here, the student demonstrated how to obtain six blocks of area 2, and then at the very end, she also came up with four blocks of area 2 and one block of area 8. See Figure 4.5.



$72 \div 6 \div 6 = 2$
(6 squares of area 2)

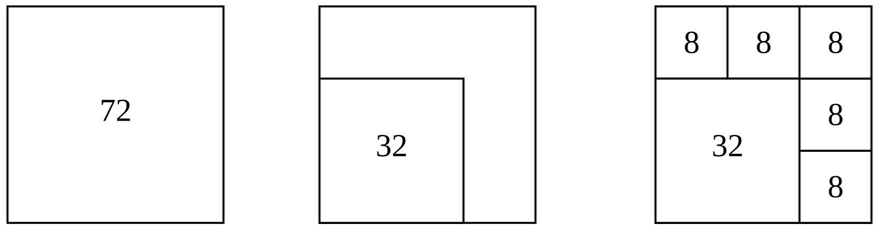
2-2-2-2-2-2 stack

(4 squares of area 2)
(1 square of area 8)

8-2-2-2-2 stack

Figure 4.5 More stacks

To make sure that students understood why their stacks worked and that they were not guessing, I continuously asked students to check their solutions. To do this, all of them used a calculator to compare decimal values. I stressed the use of the calculator as only a checking tool and reminded them that the areas, however, had to be exact values or whole numbers. The concept of exact values impeded the progress of some students and a couple of them were confused that they were allowed to use a calculator to obtain decimal values since I had said that the areas needed to be whole numbers. Sometimes, there also seemed to be confusion in comparing heights and areas. Besides using the calculator to check, one student used areas. This only worked with the 32-8 stack. They placed the 32-square into the 72-square, as shown in the middle diagram of Figure 4.6.



Square of area 32
fits into large square.

Five squares of area 8
needed to fill in space.

Figure 4.6 Student's method to check 32-8 stack

Then, they figured out how many squares were needed to fill the empty space. This required five squares with area 8. See the right diagram of Figure 4.6. When they summed up all the values, they obtained the area of the large square, 72.

I gave the students a total of 30 minutes to "practice" finding equivalent square roots for the 72-square and to summarize their findings to share. Based on a poll of the groups, here is the number of groups out of 11 who were able to create different stacks, though not necessarily the same stacks.

0 stacks created – 2 groups
2 stacks created – 5 groups
more than 2 stacks created – 4 groups

Some students had copied from the other groups and some groups had only one or two members in the group who had developed and discovered the stacks but the above numbers represent the outcome as a group. Out of ten possible different stacks with the exact same height as a square with an area of 72, where the order of the squares was insignificant, these were the stacks found collectively as a class:

- 8-8-8
- 32-8
- 18-18

- 2-2-2-2-2-2
- 8-2-2-2-2
- 8-8-2-2
- 18-2-2-2
- 18-8-2
- 32-2-2

More students found stacks of squares that had equal areas than those with mixed areas. Also, only one student was able to discover the last four stacks in the list above. The only stack that was not found was the 50-2 stack.

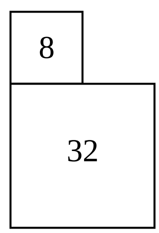
As students shared, two students brought up the use of perfect squares to find stacks. From my informal observations, at least three other groups had unknowingly used perfect squares when they divided by the same number twice, similar to Brenda quoted above, but none of them acknowledged using perfect squares. Since most students were unaware of this method, I used perfect squares to demonstrate how to create the two common stacks, 8-8-8 and 32-8. Other stacks, such as 18-18 and 2-2-2-2-2-2, were also compared to show how they were equivalent. To close the lesson, I asked the students to consider two questions: 1) how to determine the quantity of each type of square area in the stack and 2) how to find a way to create mixed area squares, since most students had not found any.

Analysing Engagement

In the next class, I asked students from both classes (Class 1 participated in a normal lesson as described above in the background and Class 2 participated in the subordination task) to write a letter to a member of another class, to describe what they had done the previous lesson. The purpose of this activity was to determine whether students would mention the subordination task as being interesting or engaging. In Class 1, none of the students described the lesson with any words one would characterize as qualifying the experience. All of them stated that they “did” or “learned about” radicals and 20 out of 23 of them mentioned simplifying radicals as the main task. 17 of them simply recalled the algorithm of simplifying radicals like the one described in the background. In Class 2, out of the 21 students who handed in the letter, one student described the Stacking Squares task as an “interesting assignment” (Ariel), another student mentioned that she “played around with boxes” (Susan) and one student referred to it as a “game” (Janine). Eleven other students called it an “activity” which is important since no one in Class 1 had characterized note-taking as an activity. Learners tend to not view note-taking as an activity since there is less thinking or effort involved. One student wrote in his letter that, “After doing this activity, we realized we learned a new topic without notes” (Roland). Therefore, the use of the words “play”, “game”, and “activity” suggest to me that the students viewed the task as something that required more concentration; thus, these students were more engaged than the note-takers. Two students mentioned that they were “challenged” (Brenda and Randy) to find stacks of squares which may indicate that the subordination task was engaging. Shernoff et al (2003) found that students who participated in “activities that contained high challenges but also required higher developed skills also reported greater interest, concentration and enjoyment in the activity” (p. 167), measures that were related to student engagement. *Stacking Squares* was challenging for the majority of the students, which was the reason that many of them became engaged in the task, but it also became the same reason that many of them stopped engaging in the task. *Stacking Squares* was not accessible to six students, causing the disengagement.

Is This a Good Subordination Task?

I compared the three characteristics of subordination tasks I had outlined before implementing *Stacking Squares* with the actual activity that occurred in the classroom. The first characteristic was the only one compromised. Writing square root expressions were not required to create stacks of squares. Instead, the majority of the students used the concept of ratios to create stacks. Furthermore, the task itself did not require students to leave their findings in radical form and so, students used square roots to create stacks of squares but did not use the square root symbol to record their findings. For example, instead of writing $\sqrt{8}+\sqrt{32}$, students drew the following diagram:



In fact, all of the students drew diagrams instead of using any mathematical notation. However, the convention of the notation was just one more step beyond the drawing as students needed to interpret the height of the stack arithmetically instead of geometrically. Students demonstrated awareness of using square roots by using the square root button on their calculator to check their stack heights. However, the resulting decimal value, instead of exact value, may have also attributed to the students' seeing the square root as a process to be calculated rather than as an object. This is similar to what Hewitt (2009) found with his students when working with addition and subtraction and what Gray and Tall (1994) describe as the ambiguity between process and concept.

Therefore, in order to satisfy the first characteristic of subordination tasks, I need to rephrase the skill, A, that the students were practicing. Instead of *writing* equivalent square root expressions using exact values, the skill becomes *finding* equivalent square root expressions. Thus, finding square root expressions is immediately subordinate to the task of finding stacks of equivalent height. The second and third characteristics of subordination tasks were satisfied as stated. The calculator may have caused the problem of process but the students were able to use it to obtain immediate feedback regarding whether their stacks matched the height of the larger square or not; thus, satisfying the second feature of subordination tasks. Thirdly, the students did not need to know how to write equivalent radical expressions but as described above, they were still able to find equivalent stacks.

A feature of subordination tasks that has emerged from *Stacking Squares* is the use of prior knowledge. This may be included in Hewitt's (1996) third feature of being able to understand the task without actually being knowledgeable about the concept to be learned. One student mentioned in her letter that she needed to use her prior knowledge in order to understand how to find area and side length of a square in order to accomplish the task. Subordination tasks are dependent on students using prerequisite skills and therefore, students should already possess them and be very comfortable using them before participating in the task.

As stated in the previous chapter, Hewitt (1996) asserts that subordination tasks offer a "way to practice something, helping it to become known and used with little conscious attention – something which is an important aspect of mastery" (p. 34). This is confirmed by Roland above who wrote that he realized he had learned a new topic without notes. Another student reported that, “This activity was a visual example of how to simplify radicals” (Dana) seeming to signify that she had learned something unexpectedly. Both these students’ comments also support the use of the subordination activity in connecting stacked squares with radical expressions. However, the unforeseen approach of applying ratios caused students to prematurely terminate their practice with five groups not finding more than two stacks. In addition, using ratios meant that they did not discover the use of perfect squares which was a sub-skill that I wanted students to unveil. For the two groups who discovered no stacks, *Stacking Squares* ceased being a subordination task for them. The task was too challenging and they became disengaged in the task. Csikszentmihalyi (cited from Shernoff et al, 2003) reports that the balance of challenge and skill is fragile and with high challenges and low skills, anxiety develops which disrupts the flow. Therefore, subordination tasks, particularly the task, B, where students are placing their attention, need to be set at a level that is attainable by all students. Another consideration is that the task was used to *introduce* the concept of simplifying irrational numbers but it may have been better suited as a *practice* exercise as part of seatwork. Both of these ideas will be further analyzed in the next chapter.

To summarize, *Stacking Squares* is a subordination task that is useful in introducing the concept of simplifying irrational numbers depending on the readiness of the students. In Chapter 5, I will further discuss the challenges I faced in its design and implementation, particularly my using the task for introducing a concept, my mispredicting the attainability of the task, and students’ use of unexpected methods.

TASK 2 – EXPRESSION SCATTERGORIES

Background

Students are usually introduced to the exponential laws for integral exponents in mathematics 9. For my students, this means that it has been approximately four months since their last encounter with them when they prepared for their mathematics 9 final exam. In principles of mathematics 10, they typically need to do some review before learning to apply these same laws for variables with rational exponents. The exponential laws that students should already be familiar with are:

Exponent Law for Multiplication:	$a^m \times a^n = a^{m+n}$
Exponent Law for Division:	$a^m \div a^n = a^{m-n}$
Power of a Power Law:	$(a^m)^n = a^{m \times n}$
Power of a Product Law:	$(ab)^m = a^m b^m$
Power of a Quotient Law:	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

My usual review consists of going over the laws with the students prior to giving them several practice questions that progress from easy (use of one law) to difficult (use of several laws in one question). As students work on these problems, I walk around checking their simplified expressions, answering questions and helping those who are stuck. Afterwards, selected students write up their answers on the board and we analyze the ones that are troublesome together as a class.

Rationale

Students appear to be proficient at simplifying expressions with exponents as they follow along. They only encounter difficulties when different or a variety of exponential laws are required in successive questions, such as those found on exams. Mixing questions in this manner requires students to identify the appropriate strategy based on the problem and this discrimination is difficult because students are accustomed to the block practice found in textbooks where a group of problems are devoted to the same concept (Rohrer, 2009). For instance, the Mathpower 10 textbook (1998) section for reviewing the exponential laws contains practice for all five laws but the questions within each “Simplify” subheading are devoted to only one or two laws. Once the learner figures out which law to apply, he no longer has to think since all the questions in that subsection will require the same law. Another reason that students may encounter difficulties with simplifying exponential expressions has to do with the presence of both numbers and variables in the expression. Often, the rules are misapplied in these algebraic expressions because students have not comfortably integrated arithmetic and algebra and therefore, when they apply their memorized rules, they do so inconsistently (Lee and Wheeler, 1989; Wong 1997). For instance, when asked to simplify the expression $(5x^3)^2$, the coefficient and the variable are both supposed to be raised to a power of two to obtain $25x^6$. Because the power of a power law states to multiply the exponents (in the above example, it is three by two), some students erroneously also multiply the coefficient five by two to obtain $10x^6$. Or sometimes, because the assumed exponent of one is not written on the coefficient, it is ignored to obtain $5x^6$. Most likely, these same students would not have applied the exponential rule incorrectly if the question had simply been $(5)^2$ or $(x^3)^2$ but with the inclusion of numbers and variables together, students seem to view algebra as a system which uses essentially arbitrary rules (Pycior, 1984). Wong (1997) found similar results in her study of grade 10 mathematics students who applied different procedures depending on whether the expression contained only numbers, only letters or a combination of numbers and letters. Instead of understanding what exponents mean, students memorize rules and then often misapply them in the various situations. These traditional laws are taught “without a solid foundation of conceptual understanding and … [therefore] many of them have the rules all mixed up – and no sense of the reasonableness of their answers” (Boaler and Humphreys, 2005, p. 40). Problems seem to be compounded when negative exponents are involved in the question or are obtained in the answer. However, my goal here is not to change their processes and procedures for working with algebra. My aim is to develop a more engaging and challenging practice method for students to gain more confidence using the exponential laws.

Task Development

As I had mentioned above, the exponential laws were not new concepts to my students. Therefore, instead of designing a task to introduce a topic like in Task 1, I needed to develop a task that was going to review the laws. Unfortunately, I could not find any practice activities that had the hint of subordination like I did for the first task. Consequently, I had to develop my own task and wanted to create a practice exercise that was more engaging and interactive than the review I usually asked my students to do. I spent much time trying to invent an activity that simplified expressions but was unsuccessful. In the end, I think I generated something richer and more rewarding.

The essence for this task, *Expression Scattergories*, came from a board game that I used to play called *Scattergories* by Milton Bradley. In this game, a 20-sided die is first rolled to obtain a key letter. Then, the players have three minutes to fill in their answer sheet with responses for 12 categories. Answers must fit the category and must begin with the key letter rolled. Sample categories include “a boy’s name”, “insects” or “TV shows.” When the time is up, the players compare answers and a point is awarded for each response that is unique compared to the other player(s). After several rounds with different letters, the player with the highest score wins. For my activity, instead of a letter, I would give the students a simplified expression such as $9x^6$. Then, they would have a limited amount of time to conceive as many equivalent expressions as they could. Expressions would then be compared and each unique expression would earn the player a point. I assumed that most students enjoyed playing games and this friendly competition would help engage and sustain work on the task. “The pleasure that learners get from playing games serves to engage them in situations where they can practice the use of technical language when describing what they are doing or seeing” (Mason and Johnston-Wilder, 2006, p. 63). In *Expression Scattergories,* they are practicing the language of exponential laws. After the game, a discussion of why the first expression is simplest can ensue.

The interesting characteristic about this activity is that instead of simplifying expressions, students are generating equivalent expressions. Watson and Mason (2005) call these ‘learner-generated examples’. As stated in the literature review, they and Hazzan and Zazkis (1997) claim that this is an effective teaching strategy that promotes active engagement in mathematics. It is also more cognitively demanding since it is more difficult to ‘construct’ an object than to ‘execute’ a given property on an object. The technique that I employ is also one of task reversal which Hazzan and Zazkis assert is a powerful strategy. Using a learner-generated task helps fulfill a side goal of mine, demonstrating that there is not always just one correct answer in mathematics, a strong belief with many students. Boaler and Greeno (2000) find this to be a popular point of view with several AP Calculus students. As one student puts it, “There’s only one right answer and you can, it’s not subject to your own interpretation or anything it’s always in the back of the book right there. If you can't get it you're stuck" (p.9). This one task may not change my students' perception of mathematics but it may depict mathematics in a different light.

Task Description: Outline of the lesson plan

Expression Scattergories can be presented to the students immediately after reviewing the exponential laws. The main learning objective of the task is for students to practice writing equivalent expressions involving integral exponents. In order to be successful or to even participate, the students need to already know some of the exponential laws which are repeated here:

- multiplication: $a^m \times a^n = a^{m+n}$
- division: $a^m \div a^n = a^{m-n}$
- power of a power: $(a^m)^n = a^{m\times n}$
- power of a product: $(ab)^m = a^m b^m$
- power of a quotient: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Before the students play on their own in groups of three, the teacher should do an example on the board with the class to demonstrate the activity. Here are some possible simplified expressions that can be used for the class example or the game itself:

1) $16x^4y^2$ 2) $9x^6$ 3) $8x^3y^{12}$ 4) $\frac{3x}{2y}$ 5) $\frac{x^4}{16y^2}$

The teacher begins by writing a simplified expression on the board after which, the students have a set amount of time to write as many equivalent expressions as possible. When the time is up, students compare their expressions with each other in their small group, making sure that each expression is legitimate. One point is earned for every expression that is different from their partners, meaning no one else in the group has written the same expression. After one round is complete, students tally their scores to see who wins the round and then, the teacher presents a new expression. The champion is the person who earns the most points in all the rounds

When carrying out the class example, the teacher should stress the importance of creating unique expressions in order to gain more points. As the activity progresses, points can be awarded only for expressions that use different exponential rules. For example, $9x^6 = \frac{9x^5}{x^{-1}} = \frac{9x^4}{x^{-2}} = \frac{9x^3}{x^{-3}} = \frac{9x^2}{x^{-4}}$ could be considered the same type of expression since the same exponential rule is continuously used. Therefore, only one point would be awarded if a student wrote any one of these. This restriction may force students to use the exponential laws in more interesting ways, perhaps even use more than one law in each expression. However, this constraint would only be introduced if the teacher felt it was necessary to challenge the class and the learners were ready for it.

Why *Expression Scattergories* is a Subordination Task

Expression Scattergories follows the three features outlined by Hewitt in Figure 4.1. The skill, A, that is being practiced is using the exponent laws. The task, B, is to find as many equivalent expressions as possible. In terms of the definition,

1) the students need to use the exponent laws to create equivalent expressions.
2) once the students have found several expressions, they use their partner(s) to check for accuracy.
3) the students may not be proficient at using the exponential laws but they do understand that their task is to create as many equivalent expressions as possible.

Figure 4.7 below depicts how subordination is used to practice the exponential laws.

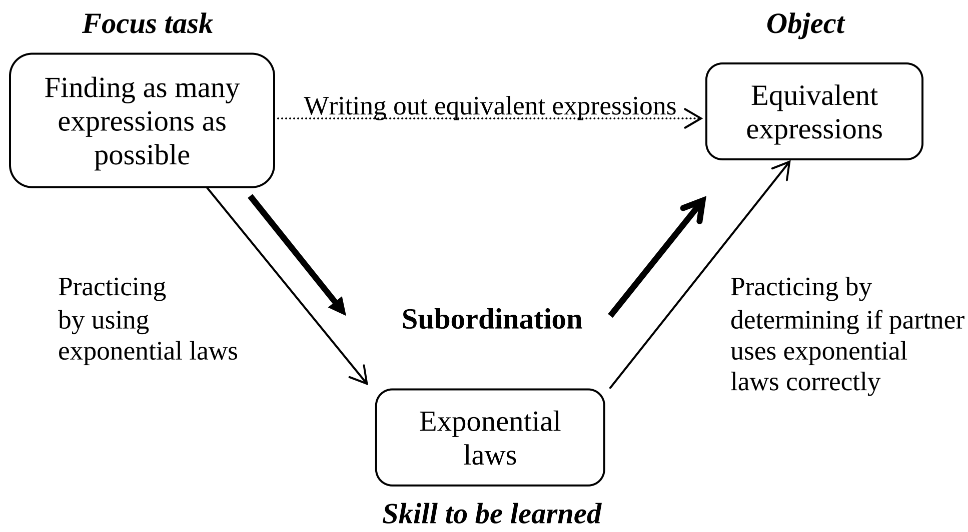


Figure 4.7 Subordination flow chart for *Expression Scattergories*

In order to find as many expressions as possible, students must use the exponential laws to create equivalent expressions. *Expression Scattergories* forces students to use and practice the exponential laws while their focus and attention is on creating as many equivalent expressions as possible.

Results and Analysis

Observations

After reintroducing the exponential laws to Class 1, I demonstrated how to play the game with a class example using the expression $16x^4y^2$. I showed the students a couple of ways to write equivalent expressions and then gave them one minute to see what other type of equivalent expressions they could create in this little time. When time was up, volunteers shared their results which I recorded on the whiteboard. Here were the first four responses in the order that they were shared:

$$\frac{x^4y^2}{16^{-1}}, \frac{16}{x^{-4}y^{-2}}, \frac{16y^2}{x^{-4}}, \frac{16x^2y^1}{x^{-2}y^{-1}}$$

I relentlessly kept asking students if anyone had created something that was different than the above in order to determine which exponential laws were applied. One student then came up with $\frac{8 \times 2}{x^{-4}y^{-2}}$. I felt that the students were still not maximizing the potential use of the exponential laws since they were only employing the division law. Therefore, before playing the next round, I wanted to demonstrate how to diversify the type of expressions they could create by using exponential laws they may not have considered. I did this by prompting students to fill in the blank of the following expressions:

$$(8x^2y^2) \times \square \quad \text{and} \quad (4x^2y)^{\square}$$

Students filled in the blanks easily and I stressed the importance of writing expressions similar to these. To end the class demonstration, I showed students how to tally points. By a show of hands, I discovered there were no unique expressions since at least two students had come up with each of the expressions recorded on the board. Therefore, I again stressed the importance of trying to create unique expressions to gain points and checking each other's work to make sure it was correct to prevent group members from gaining points undeservingly.

I put the students into groups of my choosing to try to place a strong student in each group to act as a good checker. Then, for the next question, I gave them a very simple expression, $9x^6$, to try within their small groups. Because they had not generated many expressions during the short time which was offered in the class example, I gave them approximately three minutes this time while I walked around to observe. All the students participated eagerly except for one student who did not understand what to do and I had to re-explain the instructions to him. At the end of three minutes, there were only two students who had stopped working. As students compared answers, I noticed that the expressions resulted from typical uses of the multiplication, division and power laws, except for one student who came up with $\sqrt[3]{729x^{18}}$ which she had learned from her tutor but was able to apply to this situation. I highlighted a group of similar expressions that many students had created,

$$\frac{9x^5}{x^{-1}}, \frac{9x^4}{x^{-2}}, \frac{9x^3}{x^{-3}}, \ldots$$

indicating that it was clever that they saw a pattern but challenged them to construct different types of expressions in the next round. While comparing and checking solutions, they were all engaged except for one student, different from the first, who was socially disconnected from her group.

I gave them $8x^3y^{12}$ for their third expression and again, all the students participated. I noticed a few students covering up their answers to prevent their partners from peeking and others glancing over to see what their partners were writing. When comparing solutions, students took longer this time because they had created more expressions from having picked up on techniques utilized by their group members from the second round. Students wrote from five to twenty different expressions, much more than the first two times as well. Some students needed me to check their answers to see if they were actually correct, expressing concerns of doing it wrong or getting points when they should not be. Some expressions were incorrect and my concern for these students caused me to stop the game prematurely so I could help them more individually. I assigned students seatwork to complete for the rest of the class.

Analyzing Engagement

As students participated in this activity, I noticed their competitiveness as described above in the observations. When they compared expressions, there were emphatic comments such as, “I didn’t get any of those!” Their behaviour suggests that they were actively engaged in the game. From the activity questionnaire (See Appendix 1), 16 out of 23 students enjoyed the task, four thought it was “alright” and three did not enjoy it. Interest and enjoyment are good indicators of student engagement (Shernoff et al., 2003). The most common reason, acknowledged by ten students, for enjoying *Expression Scattergories* was that it was more interactive than normal lessons and they got to talk to each other. Here are some responses from the questionnaire as to why they enjoyed the practice.

> “I enjoyed today’s activity mostly because it was more interactive than before” (Aaron).
>
> “Yes, because it wasn’t just doing notes. I enjoyed trying to find things out for myself” (Cari).
>
> “Yes I did, it was more enjoyable rather than just sitting there and listening” (Cecilia).
>
> “I enjoyed the practice today because it was a lot more involved with the people at my group” (Ellen).

Shernoff et Al’s study reaffirms that students report higher levels of engagement during group work and individual work compared to listening to lectures. In this subordination task, students got to do both. They first worked individually creating expressions on their own and then talked to their partners to compare and check expressions. In addition, this task was a game and as stated before, games are effective in engaging learners. Six other students commented on not having to write notes or the lesson was better than what they normally did (which was taking notes). Here is how one student put it.

> I enjoyed the practice today. I learned a lot more than just writing notes and I was able to understand the concept better than just writing notes. When I just write notes, I only copy them down and often do not remember the information. (Collin)

Ironically, two of the three students, who had stated that they did not enjoy the activity, would have preferred taking notes. Taking notes is easier and less risky which is why "students and teachers are willing to trade the benefits of challenge seeking (competence, pride, efficacy and enjoyment) for the safety of avoiding mistakes and appearing competent" (Turner and Meyer, 2004, p. 311). In the other class who had taken notes, approximately the same percentage of students stated that they enjoyed the lesson and used words like relaxed, easy to follow, short and concise. These adjectives described the lesson itself which had been shorter than normal. In contrast, students in Class 1 used words to describe their own participation in the lesson.

Is This a Good Subordination Task?

Expression Scattergories is a task that provides students with a method for practicing the exponential laws. Students need to be able to apply at least one of the exponential rules to create equivalent expressions, in hopes of creating the most expressions in their group. Learners may not necessarily know how to use the exponential rules proficiently but they still are aware and focused on creating expressions. The exponential laws are subordinate to the task of writing equivalent expressions. This satisfies the first and third features of subordination tasks outlined earlier. To fulfill the second feature, students need to obtain feedback as to whether or not their expressions are equivalent. I had attempted to place a mathematically strong student in each group to check the expressions but unfortunately, I had over-estimated students' abilities of using the exponential laws. Many students were still in the learning phase and were unable to check the accuracy of other students' work because of their own limited use of the exponential rules. Therefore, because students were unable to receive immediate or accurate feedback about their expressions, this task cannot be called a true subordination task unless this problem is rectified. Regardless of subordination, students may not learn anything from the practice if they do not know what they are doing is correct or not. Hewitt (2004, 2010) explains that without appropriate feedback, students may opt out of the activity or may continue with it but have no sense of whether they are doing something relevant or whether they are completing the original task. One way to address this vital and required feature of subordination tasks and make *Expression Scattergories* a powerful task for practicing the exponential laws is to use technology. Feedback and the use of technology in subordination tasks will be discussed in the next chapter.

During the task, students most likely knew that they were doing practice but for many of them, their focus was not on the exponential laws. Based on their responses from the questionnaire (see Appendix 1), eleven students specifically stated that they learned to express exponential equations in different or unique ways while only four students actually mentioned learning about exponents or exponential rules. This conforms to Hewitt's (1996, 2001) desire of immediately subordinating something which is to be learned (in this case, the arbitrary exponential laws) with another task (in this example, creating equivalent expressions) being the focus of attention. The second question on the questionnaire specifically asked students if they liked the practice but what they may not have been aware of was the quantity of questions they were practicing. In only two rounds with two expressions, one student created up to 17 unique expressions which, in a textbook, would mean that they completed at least 17 questions in six minutes. This number only focuses on the unique expressions, disregarding other expressions they may have created that were similar to their group members. Therefore, more than 17 questions were likely “practiced”. Other students did not conceive as many expressions but it did not imply that they did not have a good practice session. It only indicated that they did not create many unique expressions. From my observations, most of the students created at least ten expressions all together, in addition to checking their partner’s expressions.

Expression Scattergories was an effective subordination task but in order for it to have been even more valuable, I should have allowed the students to play more rounds and to try more varied expressions. This afterthought was validated by one student who stated that she would have liked more time and two students who wanted more variety in the questions. My concern for the students’ interest in the task clouded my ability to focus on the main purpose of the task, practicing the exponential laws. I have realized that students still need to spend quite a bit of time with subordination tasks in order to master the skill that is being taught. Another important point to mention about *Expression Scattergories* is that although the appropriate skill was subordinated, the students played around with different variations of an expression rather than simplifying them which was my original intention. This implies that students may end up being good at the former without knowing how to do the latter but it is the latter that will be tested.

TASK 3 – WHAT’S MY NUMBER?

Background

Previous to engaging in this next task, the students have spent three weeks solving equations that involve variables on both sides of the equal sign. Now, they will learn to solve equations that contain more than one variable. For instance, given the formula for the perimeter of a rectangle, $P = 2(l + w)$, a typical question is to solve for l or in other words, rearrange the equation to isolate l. The Mathpower 10 (1998) textbook that the students use terms these "equations with literal coefficients" (p. 192), coefficients that are represented by letters instead of numbers. To teach this lesson, I normally present several questions, like the perimeter of a rectangle, to work out with the students. I highlight the operations acting on the variable to be isolated and then demonstrate the use of inverse operations to solve the equation. Afterwards, I produce more problems or choose more from the textbook for students to practice on their own.

Rationale

"Failure to solve equations correctly is common among many high school and early college students" (Adi, 1978, p. 204). Based on my experiences, I have found that students often have difficulty deciding which operation or the order of operations to use to solve an algebraic equation. Most students recognize that they have to "do the opposite" but they often do not carry out these inverse operations in the correct order. This is made more difficult when solving equations that possess literal coefficients because the presence of more than one variable makes the equations abstract and learners cannot check their solutions. Teachers assume that students can understand the syntax of formal algebraic notation and look at an equation and know the order of operations (Hewitt, 1996). However, students find learning formal algebraic notation challenging (Gray and Tall, 1994; Van Amerom, 2003). The traditional repetitive exercises that students practice may not help them learn to solve equations either. "There is no mechanism to help these procedures become something which is carried out *without* conscious attention, other than repetition. And repetition takes up considerable time, leads to boredom and lack of progress, and is mainly concerned with short term success" (Hewitt, 1996, p. 34). In these traditional exercises, students place their attention on the four arithmetic operations and the mathematical conventions such as algebraic notation are assumed to be already known and understood. If "relational understanding" (Skemp, 1976) is to occur, a student must see a need for mathematical notation in order to accept, adopt and use it (Kieran, 1979; Hewitt, 1996). The mechanism of subordination can create a need for mathematical notation as students work on another mathematical task that hopefully is more interesting than the repetitive exercises they normally encounter.

Task Development

Adi (1978) investigated two traditional solving equation techniques and discovered that the learners' developmental characteristics influenced with which one they would be more successful. The two techniques are the "reversal" or "cover-up" technique and the "formal" technique of reciprocity or compensation. To differentiate between the two techniques, observe the following examples by Adi (1978).

Example – Solve: $14 - \frac{15}{7 - x} = 9$

(1) *Cover-up or reversal method*:

14 minus what equals 9? (5)
15 divided by what equals 5? (3)
7 minus what equals 3? (4)
Solution: 4

(2) Formal method:

Multiply both sides of the given equation by (7 – *x*),

$14\,(7 - x) - 15 = 9\,(7 - x)$

$$98 - 14x - 15 = 63 - 9x$$

$$83 - 14x = 63 - 9x$$

Add ($14x$) to both sides,

$$83 = 63 + 5x$$

Subtract (63) from both sides,

$$20 = 5x$$

Divide both sides by 5,

$$4 = x$$

The reversal method or something similar is a common technique that learners utilize when they begin to solve equations (Adi, 1978). This logical, non-memorized method works well for many students, so much so that when they are introduced to the formal method, they are resistant to use the latter. According to Flavell (cited from Adi, 1978) "inversions is expected to be developmentally simpler than the application of compensations [the formal method], since the class groupings of concrete operations develop before the relational groupings" (p. 204). Hewitt's (1996) "think of a number" subordination activity, which was described in Chapter 3, considers learners' developmental characteristics operating at the concrete operational level using reversals/inversions (Adi, 1978) and creates a situation which brings them to a place where they require formal operations. I extend the "think of a number" activity to lead students from the equations of one unknown with numerical coefficients to those with literal coefficients. The task, which I call *What's My Number?*, reveals that the same technique for solving equations with numerical coefficients can be applied to equations with literal coefficients. When I feel that students can solve equations with numerical solutions, I give them a sequence of operations that involve literal coefficients. For example, the task should help students transfer the techniques for solving $3x - 5 = 7$ to solving $ax - b = c$. I want students to be more comfortable using algebraic notation and interpreting the order of operations required to solve any algebraic equation. Instead of using a traditional lesson of teacher-telling and student-memorizing, W*hat's My Number?* follows Hewitt's (1996) "think of a number" activity. I only tell those things which must be told, such as the operations, while "everything else comes from the awareness of the students, including the processes involved in solving an equation" (p.34). This situation requires the students to interpret the order of operations that are acting on the variable to figure out the number. Initially, the solutions are numerical values but by the end of the task, students will hopefully have a new concept image of solving algebraic equations that involve solutions with literal coefficients.

Task Description: Outline of the lesson plan

Unlike *Expression Scattergories*, W*hat's My Number?* can be implemented at the beginning of the lesson without review. Students do not need to know formal algebraic notation to accomplish the task of finding the original number but they do need to know how to use the basic arithmetic operations. The learning objective of the task is to make students become aware of inverse operations and the order they are to be performed to solve for a number.

As the teacher reads the following statements and ask students to find the number, students may use pencil and paper to write down information if they want to. Here are some sample questions:

"I'm thinking of a number. If I …"

1) … add three and then times by two, I get 16. What's my number?
2) … subtract four, times five, I get 15. What's my number?

3) … times three, add eight and divide by two, I get 10. What's my number?

4) … subtract one, take the reciprocal, and times by 15, I get 3. What's my number?

After each question, the teacher needs to make sure the students can obtain the answer. The fourth statement above may cause some difficulty. Therefore, it may be a good time to pause here to see how students are recording their operations – are they writing steps down symbolically, in words or not at all? It is important to identify what is being said verbally and make a connection with how it can be written symbolically. Since students should have worked on this in previous grades, it should not take long. Then, the next important step is to have a discussion on what needs to be done to find the number. As the teacher writes the inverse equation that is used to find the number, the term inverse operation can be introduced here if the teacher feels inclined to, though it is not necessary to accomplish the task. What is required is to use the symbolic method to record the operations and then call attention to the order of operations needed to find the number. After this discussion, the teacher can proceed to the next set of four questions such as the ones provided here.

5) … subtract four, divide by three and add six, I get 8. What's my number?

6) … subtract three and times by two, I get y. What's my number?

7) … times by two, add $2w$, I get P. What's my number?

8) … times by $4a$, subtract xy, I get 5. What's my number?

Question 5 is provided as an easy check. The last three questions introduce the use of literal coefficients so longer pauses may be needed as students take more time to find the number. At the end of the activity, the teacher needs to emphasize that regardless of numerical or literal coefficients, the order of operations needed to find the number remains the same.

Why *What's My Number* is a Subordination Task

What's My Number? is similar to Hewitt's (1996) "think of a number" activity described in Chapter 3. However, instead of focusing on the syntax of formal algebraic notation, the skill, A, (see Figure 4.1) that I want students to learn is to use the correct inverse order of operations to solve an equation. The task, B, is to find the number. The activity satisfies the requirements of being a subordination activity because

a) the students need to be able to use order of operations to find the number.

b) once the students find the number, they can substitute the value back into the series of steps to see if it satisfies the equation. (I also provide them with the correct solution.)

c) the students know that their task is to find the number although they may not be aware that they are using inverse order of operations to do this.

Refer to Figure 4.8 to observe how subordination is used in *What's My Number?*.

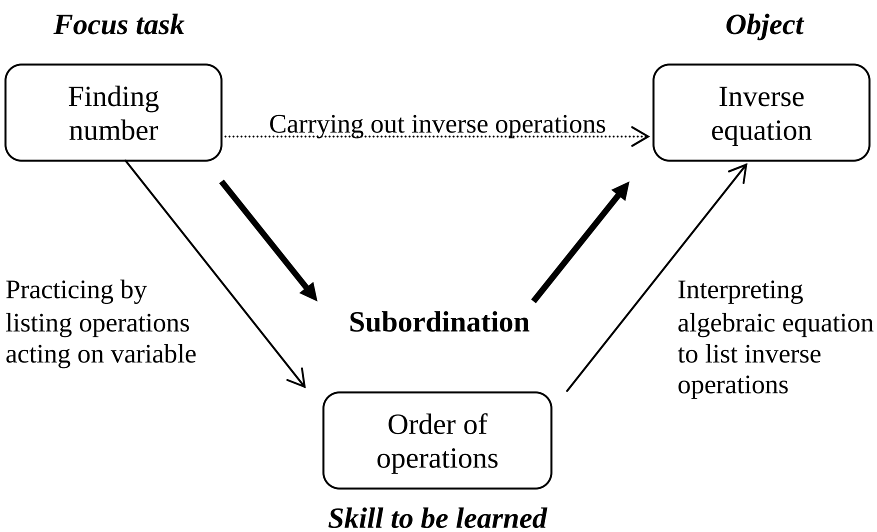


Figure 4.8 Subordination flow chart for *What's My Number*?

As students focus on finding the number, they interpret the order of operations acting on the variable to deduce the inverse operations that will be used to create the inverse equation. When working with literal coefficients, this will actually be the solution.

Results and Analysis

Observations

What's My Number? was presented to Class 1. From my informal observations, all the students were involved in the activity from the beginning. After the third question, most students were confident with how to find the number with the awareness that they were using inverse operations, even though I never mentioned that they were using them. I could see that some students wrote single numbers but there were also many students who wrote steps down symbolically to help them remember the operations. The majority of them used x to represent the number they were finding. However, when I read question number four where one of the steps required taking the reciprocal, most students could not find the number using their symbolic method. They were now in a situation where, if they could figure out the inverse of this operation, they would be able to find my number. I purposely gave them an operation that was atypical of the four operations they were used to. At this stage, I offered to write on the board what I was doing to my number. I used words and symbols and emphasized the relationship between the two, as shown below. (The right-hand column outlines the steps I took but what I wrote on the board was consistently built on the same expression to arrive at the equation in the last step.)

Verbal Steps	Symbolic Steps
- my number	x
- minus 1	$x - 1$
- take the reciprocal	$\frac{1}{x-1}$
- times by 15	$15\left(\frac{1}{x-1}\right)$
- get 3	$15\left(\frac{1}{x-1}\right) = 3$

Afterwards, I asked the students what they would do to solve for my number. I wrote the following on the board, first verbally and then symbolically.

Verbal Solution	Symbolic Solution
- start with 3	$\rightarrow\ 3 = x$
- divide by 15 $\rightarrow \frac{1}{5}$	$\rightarrow\ \frac{3}{15} = x$
- take reciprocal $\rightarrow \frac{5}{1}$	$\rightarrow\ \frac{15}{3} = x$
- add one $\rightarrow$ 6	$\rightarrow\ \frac{15}{3} + 1 = x$ $\rightarrow\ 6 = x$

After going over the solution to question 4, I gave the students a similar example (question 5 above) to observe how many would adopt this approach. Many students tried this method and we discussed how they not only did the “opposite” but they also worked backwards.

The next set of questions, numbers 6 to 8, involved literal coefficients. The students showed surprise by the inclusion of variables but many seemed to still be able to apply the correct order of inverse operations. However, those that volunteered to provide a solution did not seem confident with their answer. There seemed to be an understanding that these types of questions were more difficult as one student expressed, “This is so hard. Does it mean our answers are going to have variables in it" (Paul)? This comment suggests that perhaps it is not so much the mechanics of solving equations with literal coefficients but the discomfort of ending up with something that has letters in it that makes these types of questions difficult. However, based on a show of hands, approximately half of the class was able to isolate the variable in question number eight. This meant that they were beginning to apply the concept that they had learned, that of interpreting the order of operations and using inverse operations in reverse order to isolate the variable. I suggest that the subordination task may have alleviated some of the discomfort by having students concentrate on finding the number as opposed to worrying about notation or conventional methods which is usually stressed by the teacher.

Analyzing Engagement

From my observations, I noticed that the entire class participated in this activity, particularly in the beginning when the questions were simpler. Even though I had not portrayed the task as a game, there seemed to be a little bit of competition, as students tried to find the number before their peers. This was displayed by their eagerness to be the first to raise their hands when they figured out the number. The challenge of trying to be the first engaged many of the students. Based on the questionnaire (see Appendix 2) that the students filled out after the task, Table 4.1 summarizes the responses of whether students felt the class to be short, same as normal or long.

	Short	Normal	Long
Class 1 (n = 24)	14 (58%)	1 (4%)	9 (38%)
Class 2 (n = 23)	6 (26%)	4 (17%)	13 (57%)

Table 4.1 Results of questionnaire for Task 3

The students who found the class to appear short commented on the activity being easy, fun or interactive.

> "Extremely short. The concept was extremely easy to grasp" (Aaron).
>
> "Short, because of the interactiveness of the lesson" (Clayton).
>
> "Very short! I was having fun" (Maraya).

This contrasts Class 2 students' comments. Many felt class to be short because there were few notes but lots of time for homework. Here is a response representative of the majority of these students.

> "Short, because we had a few necessary notes, and time to do our homework" (Joanie)!

Responses from Class 1 for why the class seemed long included there not being time for homework and math/school is supposed to feel long. Class 2 students felt that class seemed long because there were lots of examples and class was last period Friday afternoon. Larson and von Eye (2006) found that students experienced a faster sense of time passing when they were actively engaged in tasks. With the subordination task, many students were actively engaged in the whole class activity indicated by the raising of their hands. Most students from Class 2 found the class to appear long but it is difficult to tell if this was because of the note-taking or the day of the week.

When asked what they liked about the activity, many students from Class 1 commented that the concept was easy to understand, that the lesson was an activity (as opposed to notes) and they enjoyed working together. According to Shernoff et al. (2003), the lack of lecturing and the presence of whole class group work and discussion contribute to higher levels of engagement. Most comments for discontentment with the lesson were not relevant to the subordination task but were remarks about the lack of homework time. In contrast, overwhelmingly many students from Class 2 liked their lesson because there were not many notes which provided them with more time to complete their homework in class. However, there were some complaints by Class 2 that the lesson only contained notes without an activity. The presence of an activity seems to make the class time appear shorter and hence, more engaging, particularly if the task is fun and interactive. Students' comments about the lack or presence of homework time suggest that subordination tasks take longer to implement than the traditional presentations of lectures and notes. This will be analyzed further in the next chapter.

Is This a Good Subordination Task?

What's My Number? satisfies the three characteristics of being a subordination task described earlier. First, students needed to use the correct order of inverse operations to find the number or expression. Second, I gave the students feedback as to whether or not their solution was correct by providing them with the solution or getting students to volunteer their solutions. They could also self-check their numerical solutions by reiterating the operations with their number. Lastly, although interpreting the order of operations from the mathematical notation may have been a new skill, the task of finding the number was not. The discussion about notation and order of operations was carried out after the students had already done some work on finding the numbers. Therefore, the task of finding the number or expression was still understandable without the need to be able to interpret the inverse order of operations. Students practiced using inverse order of operations to solve equations but this skill was not the focus of attention. Determining the inverse order of operations was immediately subordinate to the task of finding the number. As implied by Hewitt (1996), working on subordination tasks should not feel like practice. This is supported by the students in Class 1, who felt that class was short or that time had passed quickly and may have experienced a loss of self-consciousness (Meyer and Turner, 2006). However, their responses to the second question on the questionnaire,

Did you feel like you were "practicing" mathematical concepts? Explain.

seemed to suggest otherwise. Only four students replied "no" but two of them were confused and did not understand the lesson. One student said that he felt like he was playing a game and the other said that "the problems were more logic-oriented than methods and rules to follow to gain answers" (Aaron). This last comment does not particularly pertain to my thesis but it is interesting to note that the student did not believe that logic was a part of mathematics. The remaining 20 students all said yes but they seemed to have attributed "practicing" mathematical concepts as anything that involved numbers, variables, and equations. Students may have misinterpreted the question and were thinking about what they were doing as opposed to how they felt about what they were doing. Therefore, the information gathered from this question is inconclusive as to whether students felt like the subordination task was a practice exercise or not.

I designed *What's My Number?* as a subordination task to make a connection between solving equations with numerical coefficients to equations with literal coefficients. What I discovered is that interpreting the inverse order of operations needed to solve an equation is the convention that needs to be learned. It is irrelevant that an equation has literal coefficients since these type of equations are only extensions of those with numerical coefficients. The process of learning how to interpret and subsequently solve equations is the purpose of this task. The subordination activity helped to make the skill be acquired in a more interactive and hence, more enjoyable lesson.

TASK 4 – FUNCTION FACTORY

Background

Students have been using the *x*-*y* notation for purposes such as solving equations and for graphing functions for the last few years. Therefore, students are familiar with inputs and outputs and the concept of a function. What they probably have never encountered before is the formal definition or notation of a function. To introduce function notation, I start by making a connection between the *x*-*y* notation and the $f(x)$ notation by showing students how an equation can be written in both of these ways. For example, I write

$$y = 3x - 5 \text{ and } f(x) = 3x - 5,$$

and state that by looking at the notation, we can see that $f(x)$ is the same as *y* but is only another way to write the output. I briefly talk about the definition of a function and then demonstrate how function notation works – the numerical value in the brackets replaces *x* in the expression and is then evaluated. Afterwards, I give them several questions to practice evaluating a function such as the following:

If $f(x) = 3x - 4$, find:
a) $f(2)$ b) $f(-2)$ c) $f(0.5)$

Other questions involve graphing and finding the inputs of functions. When I feel comfortable that students can work with functions, I assign questions for homework similar to the ones we have been working on.

Rationale

Most students accept the arbitrary function notation but do not understand why it is used or needed because “at this stage, the property [of a function] is not necessarily something of which students are aware…A definition only introduces this link; it does nothing to establish it” (Hewitt, 2001, p.47). Sometimes, students comment “What’s the point of using this new notation when the old one works?” or “Why do we need this notation if $f(x)$ just represents y? Why can’t we just use y?” Because students see no need or purpose for function notation, it is difficult for them to make a connection with it, fully comprehend its power as a mathematical notation and use it correctly in future mathematical work when it is not the focus of a lesson. As stated before in *Expression Scattergories*, students find learning formal algebraic notation challenging (Gray and Tall, 1994; Van Amerom, 2003). The problem may be more due to the learners’ difficulties in interpreting and using someone else’s notation as opposed to developing their own symbols (Hewitt, 2009). I have found from my previous years of teaching that after only a couple of months later, many students have already forgotten how to interpret function notation. Some students think that $f(x)$ means f times x. This misunderstanding is not uncommon since mathematics has notoriously been seen to contain conventions that serve a dual purpose (Gray and Tall, 1994; Hewitt, 2009).

Hewitt (1999) describes names, labels and conventions as *arbitrary*, something that can only be known to be true by being informed by external means such as the teacher or a book. Consequently, my aim for this task is to assist students to memorize a new convention. However, instead of introducing function notation at the beginning of the lesson, I emphasize the properties of a function first. Then, when the students are consciously aware of the appropriate properties of a function will I introduce the associated word.

> The word then has an awareness to which it can become glued – there is a hook on which the name can be hung. The general principle here is only to introduce a word when the associated properties are already established and form the focus of students' attention (Hewitt, 2001, p. 48).

By my purposefully creating a need for function notation, students can make a better connection between the notation and its related properties and use it to support future mathematical work. "The function concept has been suggested as a fundamental idea to underpin the whole curriculum" (Akkoç and Tall, 2005, p. 1) and being able to read and use the notation is vital for success in future mathematical activities.

Task Development

With the help of one of my professors from my graduate studies, I developed the activity "Function Factory" whereby students try to deduce the job of each factory as it takes in inputs and produces outputs. Initially, I begin with factories that take single numerical inputs and produce single numerical outputs. After I feel that students are comfortable with this process, I progress to a compound factory that takes inputs from two other factories to create its outputs. I make students aware that the identification of each factory is confusing and writing out the factory's function in words is tedious. I try to create a need for symbolism and introduce function notation to help students differentiate the factories and keep track of which input they use to produce each output. "The formal $f(x)$ notation … condenses a great deal of information very efficiently but causes difficulty even for advanced students" (Wagner and Parker, 1993). However, using a subordination task to introduce the properties of a function first will make introducing function notation seem natural.

Task Description: Outline of the lesson plan

The learning objective of *Function Factory* is to develop an understanding of function notation. With only pencil and paper, students need to find the relationship between two given numbers, the input and output. It is the fact that there is a relationship that is important so the operations between the two numbers should not be too difficult nor too numerous. The teacher begins by drawing a factory to represent the function (see Figure 4.9) and explaining that the factory takes in materials (inputs) to create products (outputs).

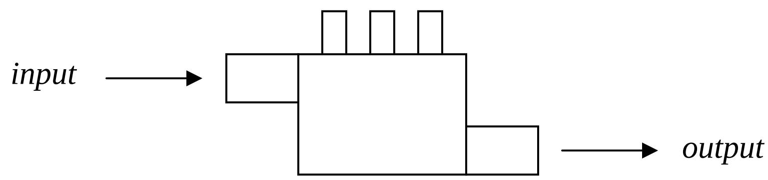


Figure 4.9 Factory drawing to represent the function

Inputs and outputs of each factory are written down, with pauses in between each pair, to allow students to spot a pattern and identify the job of each factory. Here are some examples that outline the inputs and outputs of each factory and a basic direction to help students identify the factories' jobs.

1) **Factory 1**
   - input 2, output 7
   - input 5, output 10
   - input -8, output -3

   What is the job of this factory? Write it in words.
   (Answer: takes inputs and adds 5 to create outputs)

2) **Factory 2**
   - input 2, output 6
   - input 5, output 15
   - input -8, output -24

   What is the job of this factory? Write it in words.
   (Answer: takes inputs and multiplies them by 3 to create outputs)

3) **Factory 3**
- input 2, output 5
- input 5, output 11
- input -8, output -15

What is the job of this factory? Write it in words.
(Answer: takes inputs and multiplies them by 2 and adds 1 to create outputs)

4) **Factory 4**
- input 2, output 13
- input 5, output 25
- input -8, output -27

What is the job of this factory? Write it in words.
(Answer: takes outputs from Factory 1 and Factory 2 as inputs and adds them together to create outputs)

(As a side note, other inputs besides 2, 5 and -8 could have been used.) After students have determined Factory 4's job, the teacher emphasizes how it takes a long time to write out the job in words and asks how one can identify the factories and their jobs without actually writing Factory 1, Factory 2, etc. This is when the function notation can be introduced to specify factories that take in only one input and factories that take in inputs from other factories. Here is the function notation for the four factories given above:

1) $f(x) = x + 5$
2) $g(x) = 3x$
3) $h(x) = 2x + 1$
4) $j(x) = f(x) + g(x)$ or $j(x) = 4x + 5$ or $j(x) = 2h(x) + 3$

After this discussion, more factories can be drawn up whereby students have the opportunity to record their findings using the new notation.

Why *Function Factory* is a Subordination Task

Function Factory is similar to Hewitt's (1996) "think of a number" activity in terms of the way subordination is used to introduce the mathematical idea. The convention is introduced only after students see a need for it. Based on Figure 4.1, the skill, A, that I want students to learn is to be able to interpret and use function notation. The task, B, is to find the factory's job. The activity does not strictly follow Hewitt's definition as described in Figure 4.1 because the skill A is not directly stated in the requirements as written. However, I believe the task still satisfies the features of being a subordination activity. What has changed is that requirements 1) and 3) do not require the skill A, function notation, to be introduced to find the factory's job but the concept of a function is needed. Here are the features rephrased to fit the task:

1) the students need to be able to use the concept of a function to find the factory's job.

2) once the students believe they have identified the factory's job, they can check if this "function" satisfies all the number pairs.

3) the students concentrate on finding the factory's job but they may not be aware that they are working with functions.

Refer to Figure 4.10 to observe how subordination is used in the *Function Factory* task.

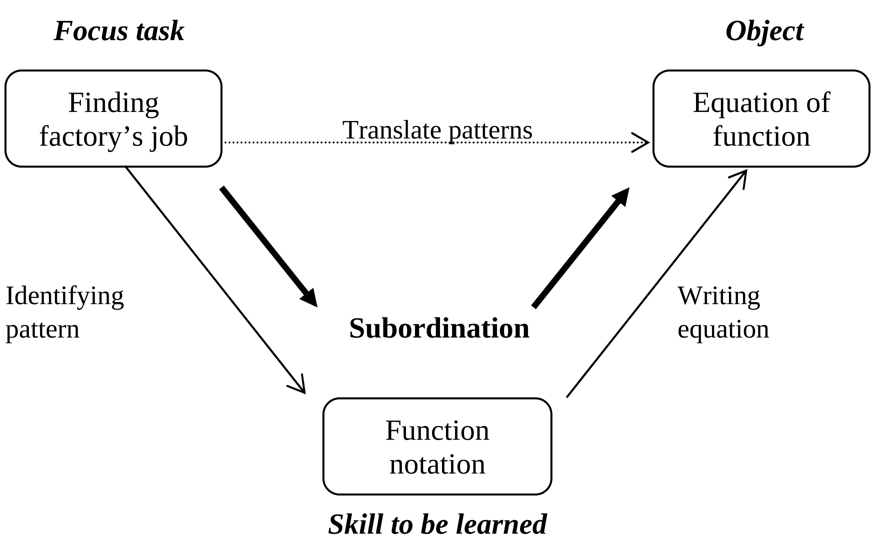


Figure 4.10 Subordination flow chart for *Function Factory*

What is different about this activity compared to the previous tasks is that the skill to be learned is introduced only as the need arises, but then, is subsequently used in the practice afterwards. As students focus on finding the factory's job, they need to identify and translate the patterns that they see into an equation of a function using function notation, after they have been introduced to this new convention.

Results and Analysis

Observations

All the students were involved in the activity from the beginning but unlike some of the previous activities, I got students to copy down what I was writing on the board. For example, students drew Factory 1 and then wrote the factory's job in words underneath the factory as shown in Figure 4.11.

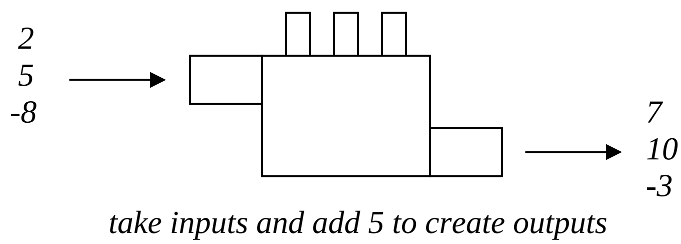


Figure 4.11 Factory 1

I am not exactly sure why I chose to ask students to draw the diagrams but I wanted students to keep a record of what they were doing. However, looking back, it was probably unnecessary because students could see what was happening on the board and it ended up being more time consuming than I had planned. I could not give them many examples and it became boring for some as it felt like a traditional lesson. By the third factory, the students were able to quickly identify the factory's job even though it was a two-step operation. Therefore, I decided to give them the compound factory that took outputs from two other factories to be used as its inputs. From my observations, the students appeared to understand the concept of taking inputs and creating outputs and so, the next step was to get them to think about how to identify the different factories without actually calling them Factory 1, 2, etc. I began by asking students how they could describe the factories' jobs using algebraic notation to make it shorter. I was not sure what to expect so was pleasantly surprised when one student suggested using subscripts (x_1 and x_2) to show the inputs being added to create the output of Factory 4. I was impressed by his consideration to use subscripts as part of the mathematical notation because it showed his connection to previous mathematical conventions but I questioned whether the use of *x*'s would be appropriate if we wanted to identify these inputs also as outputs coming from Factory 1 and 2. I realized that the students would probably not come up with the function notation since it was something that was invented long time ago, needed to be provided (Hewitt, 2001), and the purpose of the subordination activity was to introduce it when it was necessary. Therefore, I identified the first factory as "*f*", stating "*f* for factory" and indicated that the inputs that it took would be placed in brackets beside the *f*. This is what I wrote on the board underneath "takes inputs and adds 5 to create outputs" for Factory 1:

$$f(x) = x + 5$$

Since the notation was built around the context of the factory with which they were already familiar, the students seemed to accept this notation without any difficulty as there were no complaints or questions. When I proceeded to the next factory, I stated that we could not use *f* again because that was the name of the first factory so a student voluntarily came up with *g*. No one disagreed with this naming and I reaffirmed that it made sense alphabetically. In the end, we came up with $g(x) = 3x$ and $h(x) = 2x + 1$ for Factory 2 and 3, respectively. Lastly, we needed to conceive a notation to describe what was happening in Factory 4. I asked students if a new variable were necessary in this situation given Factory 4's inputs were actually from the other factories. A student immediately suggested $f + g$ which I expanded to $f(x) + g(x)$. Students had no questions regarding the notation and at this, I gave them additional factories to determine the function and practice using function notation.

Analyzing Engagement

The use of visuals to introduce a concept was different from how I normally taught a concept and what the students were accustomed to and so, the entire class seemed to be engaged in this activity from the beginning. However, even though the task was similar to *What's My Number?*, in the sense that they were both whole class activities and the concepts were related to solving equations (finding the "number" in *What's My Number?* and deducing the "operations" to lead to the number in this task), comparatively more students found this lesson to be longer than shorter, in contrast to the former task. See Table 4.2 for the results from the questionnaire (see Appendix 2) that the students filled out after the task, regarding whether students felt the class to be short, same as normal or long.

	Short	Normal	Long
Class 1 (n = 25)	9 (36%)	5 (20%)	11 (44%)
Class 2 (n = 21)	8 (38%)	4 (19%)	9 (43%)

Table 4.2 Results of questionnaire for Task 4

Students from Class 1 responded that the class felt short because the activity was easy, fun and/or interactive. Here is a comment representative of these students: “Short because it was interactive and fun” (Denise). The majority of the students felt class to be long because there was lots of classwork and the concepts were simple. Here is a comment indicative of this feeling: “Long because we did notes the whole class” (Ivan). I think that more students found this task to be longer than the previous one because I had made the students copy down the notes. Note-taking and lectures are low challenge and skill activities which is exemplified by low engagement and results in making time seem to pass slower or stay the same if it is a typical experience (Larson and von Eye, 2006). Shernoff et al. (2003) also found that lecture activities scored lower in interest, concentration and enjoyment which are subcomponents of engagement. It is interesting to note that students’ reactions to the same activity are different. One student found the class to be fun and interactive while the other student found the class to be long.

When asked what they liked about the activity, students mentioned liking the use of pictures, factories, the whiteboard and class discussion. One student also commented on liking the “connections to day to day life” in reference to the factories. These comments suggest that many students are visual learners and notice changes to the instructional method. Subordination tasks are different from traditional tasks to varying degrees and students were recognizing these changes. The replies regarding what they disliked did not have to do with the activity at all but the fact that there were many notes and lots of homework.

I wanted to study the effects of subordination but felt uncomfortable with not providing equally engaging activities for Class 2. Therefore, I also used an activity that I considered to be engaging and fun, to introduce the concept of function to them. For their activity, students collected data by blowing up a balloon, measuring the diameter of the balloon and the time it took to deflate, and then plotting this data on a graph. This task was supposed to demonstrate an example of a function. The balloon activity was not a subordination task because the students focused on collecting data to be graphed instead of anything to do directly with operations and function notation. After the activity, I gave them some short notes on the definition of a function and function notation. About the same percentage of students in Class 2 indicated that the class felt short, normal or long. (See Table 4.2.) The disparity came in the reasons that they gave. Here are various responses from some Class 2 students as to why the class seemed short:

> "Short, time flew by and we got a lot accomplished" (Brenda).
> "Short once we got going into the notes" (Randy).

The students who felt class to be long found the lesson confusing or commented on the balloon activity making it seem long. Most students in this class had nothing to say about what they liked or disliked about the lesson. The data from the students would make it seem that the subordination task did not differ – at least in the eyes of the students – from the balloon one. However, there were three students from Class 2 who articulated a difference that is particularly relevant to subordination. Here are their comments.

> "The balloon graph seemed pointless to me" (Ariel).
> "No explanation about why we need the functions" (Matthias).
> "Last part was easy, subject came out of nowhere" (Roger).

These comments contrast the student in Class 1's response who had said that she had made a connection with the factories analogy. There was a relevance to the subordination task that the balloon activity did not have. The subordination activity was able to establish a link between the factory and the notation while the balloon activity seemed like a pointless task. These comments do not represent all the students and had I wanted to assess their understanding of function in relation to the task, I should have asked both classes a more pointed question about this connection.

Participation in an activity does not necessarily make class time appear shorter as students from both classes feel that the activity made the class seem longer. What I am cognizant of is that the activity needs to be obviously relevant to the concept being learned. Otherwise, it will seem meaningless and boring to the students. The task also has to contain high challenges and require higher developed skills in order to elicit interest, concentration and enjoyment from the students (Shernoff et al., 2003).

Is This a Good Subordination Task?

Function Factory is an activity that maintains the integrity of being a subordination task even though it does not conform to the three features defined by Hewitt (1996). As previously explained, the skill that I wanted students to learn was the use of function notation but before introducing this to the students, they needed to become comfortable working with functions. Therefore, the use of “function notation” is not directly stated in the first and third features defined for subordination tasks but instead, is replaced by the use of “functions”. With this in place, the three features defined above were satisfied. I will reiterate them here, elaborating slightly. First, the students used the concept of a function to determine how one quantity (the input) was related to another quantity (the output) for each factory. Second, the students could immediately check to see if the function they discovered worked for all the number pairs by inserting the inputs into the function to see if it produced the corresponding outputs.

Finally, the students did not need to know the definition of a function in order to determine the factory's job. However, when the need arose for function notation, I introduced it to the students. Afterwards, they were permitted to use this mathematical abbreviation, if they wanted, to describe previous and subsequent factories without the use of words. The important aspect of this subordination task is that students were already practicing something before I introduced the notation and after I introduced the notation, they were still working on the same thing. As described earlier, the students' focus and awareness was on the task of finding the factories' jobs and when the function notation was introduced, they were already familiar with the properties of a function. Therefore, introducing function notation seemed to be a natural and normal process. Similar to Hewitt's (1996) *Think of a Number* activity when he introduced those conventions which must be told (the algebraic notation in his case), I introduced the function notation when the students required it. Everything else came from the awareness of the students, including the processes of finding the output.

Function Factory is a subordination task that introduces students to function notation. My implementation did not involve as much practice as some of the other tasks but my main focus was to introduce the students to the need for function notation and I believe the activity achieved that. After creating several examples with the factories, including a factory that took outputs from other factories as its inputs, I created a need to distinguish each factory's job in a simpler form and hence, introduced function notation. I never actually assessed whether the students understood the need for function notation nor directly ask them why we used $f(x)$ instead of y but the students did not seem perturbed by this arbitrary convention. The notation seemed sensible and nonburdensome as it emerged naturally out of the activity while students concentrated on finding the output or input of the factory. Hewitt (1996) argues that this type of practice is particularly effective in helping students become fluent in something, particularly when that something is arbitrary. “The nature of the arbitrary is such that it is required to be adopted and used with fluency rather than be the focus of attention and conscious consideration” (Hewitt, 2001, p. 49). Thus, when students were introduced to the arbitrary function notation, they were already progressing with what was necessary, working with functions. This is especially powerful and useful since time is not wasted and the arbitrary and necessary are practiced simultaneously.

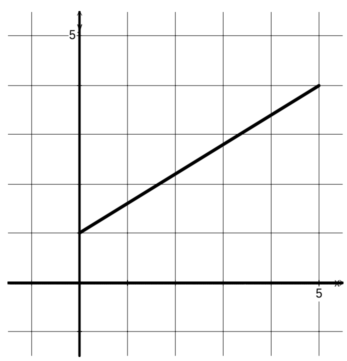
TASK 5 – WHAT’S MY *x*?

Background

The introduction of functions in mathematics 10 is usually accompanied by the sub-concepts of domain and range. At this level, students are asked to identify the domain and range of graphs of relations. To introduce these two sub-concepts, I normally start by presenting students with different graphs of relations and helping them identify the domain and range, progressing from domains and ranges that involve only integers, to a restricted real number system and then eventually to all the real numbers. This lesson is carried out in a teacher-led discussion as I reveal the graphs step-by-step and prompt students about the possible x and y-values in each graph. Afterwards, students practice identifying the domain and range for several different but similar relations. The hope is that if they are exposed to as many graphs as possible, they will be able to correctly identify the domain and range of any graph.

Rationale

After having taught domain and range for several years using the above instructional method, I notice that many students find these two function concepts difficult to identify graphically (and algebraically as well, but since the prescribed learning outcome for Principles of Mathematics 10 is only to identify the domain and range graphically, I will only regard the graphical aspects of domain and range.) It is difficult to pinpoint exactly why these concepts are challenging but I notice that students often have difficulty interpreting the graph and determining the values in the real number system where the graph lies. For instance, in the following graph,



the domain is $0 \leq x \leq 5$ but many students often write {0, 1, 2, 3, 4, 5}, forgetting that non-integer values are also part of the domain. This leads me to consider another problem in regards to domain and range, the notation that we expect students with which to write their answer. As stated before with the *Function Factory* task, interpreting and using someone else's notation is seen as challenging (Hewitt, 2009). I observe that students can often verbally describe the domain and range, but when asked to write it using formal algebraic notation, they are unable to do so. Van Amerom (2003) has found that reasoning and symbolizing appear to develop as independent capabilities. For example, students may be able to describe the domain of the graph above as "x-values from 0 to 5" but be unable to write the formal notation.

I have also come to realize that my step-by-step teacher-guided approach that is probably similar to the ones used by many other teachers (Battista, 1994; Doyle, 1983, 1988; Schoenfeld 1988; Boaler, 2002), may be doing more harm than good. The detailed structure and paths of guidance may give students mathematical confidence but the problem is that "mere imitation … would not give rise to thinking; if we could learn like parrots by simply copying the outward acts of others, we should never have to think; nor should we know, after we had mastered the copied act, what was the meaning of the thing we had done" (Dewey, 1933, p. 208). Breaking the process of finding the domain and range into a series of steps may be beneficial but the mimicking and memorization technique does not give students a connection to the concept and so, they do not understand it or quickly forget it. My approach also reaffirms students' belief that "mathematics consists of different rules for different types of problems" (Erlwanger, 1973) as different types of graphs require the domain and range to be written differently. Hence, this has spurred me to create a subordination task to show students how to identify the domain and range of graphs and record it using mathematical notation.

Task Development

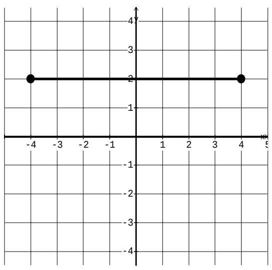
I collaborated on *What's My x*? with a colleague at my school who had faced similar challenges of teaching domain and range. We felt that instead of our telling the students how to find the domain and range in a step-by-step format, we would get the students to carry out the steps on their own using a guided handout (see Appendix 4). The idea is similar to Watson and Mason's (2004) use of dimensions of variation in mathematical exercises as described in Chapter 2. Learners work on an exercise that provides systematic change as they progress through each question. Each question varies slightly from the one before which forces the learner to reflect on the task set before them. “By constructing tasks for learners which provoke conjecture and modification, learners are called upon to use and develop their natural powers of mathematical sense making” (Watson and Mason, 2004, p.111-112). This will stimulate learners to explore rather than to complete the worksheet as quickly as possible. If the students get to think and attempt to work out the problems themselves, they can remember the concepts better because they develop more of an ownership of what they learn. As indicated by one student who played *Expression Scattergories*, she enjoyed finding things out for herself. Reflection is required in two parts (Part III and Part VI) of the worksheet, where students justify and explain what they are doing. For this activity, I chose not to carry it out in a group format because I wanted to see if individual or pair activities that used a worksheet could also be amenable to subordination.

Task Description: Outline of the lesson plan

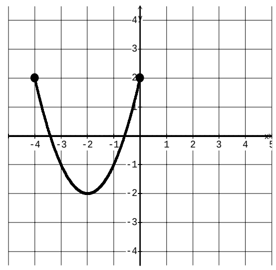
The learning objective for the task, *What’s My x*?, is to differentiate between domain and range, identify the values that satisfy the domain and range and begin to use mathematical notation to write it. In order to carry out this task, students need to be able to read graphs, particularly being able to differentiate the *x*- and *y*-axis with ease. No formal instructions are needed as students can carry out most of the worksheet on their own and the terms domain and range are not introduced until the end of the task. The full worksheet can be found in Appendix 4 but I will provide a description of the task implementation here.

To begin, students complete Part I where they list all possible integer values for *x* and *y* that satisfy the graph. The purpose is to get students to start thinking about the range of values using integers. Here are some sample questions from Part I.

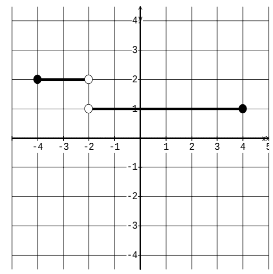
1.



3.

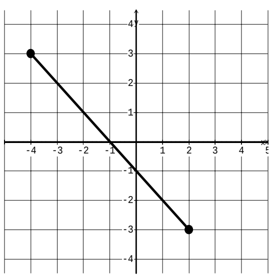


5.

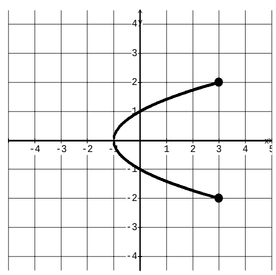


In Part II, students consider values that are not integers. They are asked to list two possible non-integral values each for *x* and *y* that are found in the graphs such as the ones below.

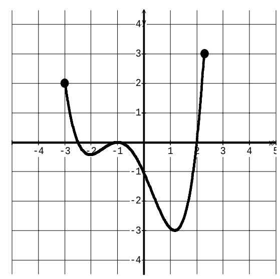
7.



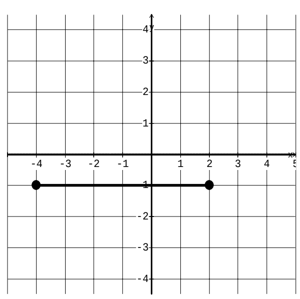
8.



9.



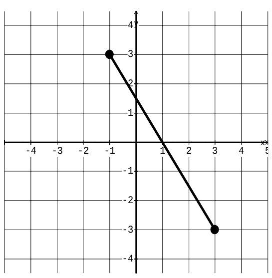
Finally, in Part III, students are asked to think about how many possible values there can be for the domain. They do this by considering the following graph,



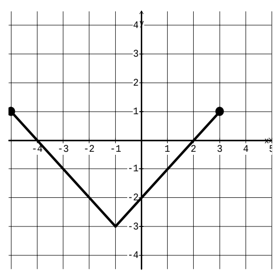
and are asked to indicate how many total possible *x* and *y* values exist, and to justify and explain their reasoning. I assume that students will do this verbally so the next question asks them to think of a shorter way to show their solution to get them to start using mathematical notation. At this point, there can be a discussion on the first three parts to give students some feedback regarding the values they have come up with.

Next, students proceed to Parts IV and V on their own where they are asked to identify the smallest and largest possible values for the domain and range. The purpose of Part IV is to get the students to think about the outside limits of the domain and range values. See below for some examples.

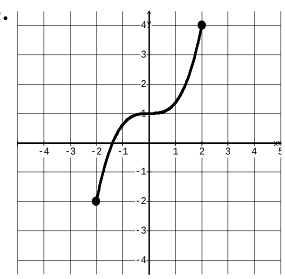
11.



12.

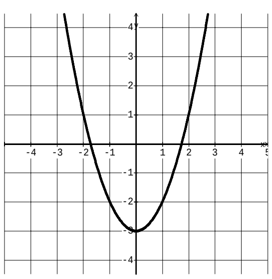


15.

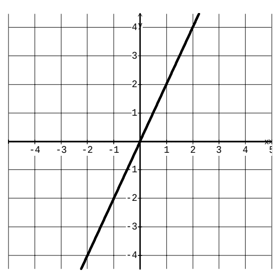


Part V gets students to consider infinity by investigating the following graphs.

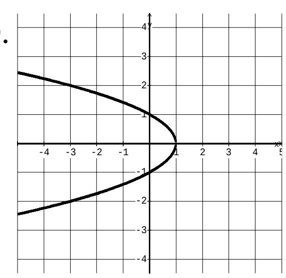
17.



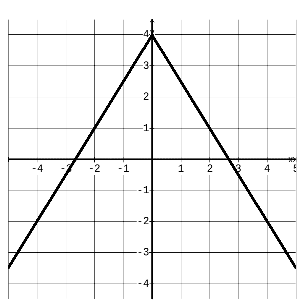
18.



19.



To end the investigation, students complete Part VI where they are asked to identify all the possible *x* and *y* values for the following graph and to explain how to find them.



In addition, students need to think about how to represent these values using mathematical notation.

After going over the answers for Parts IV to VI as a class and having a discussion on ranges of values and infinity, the teacher can introduce the terms domain and range.

Why *What's My x*? is a Subordination Task

In the lesson there are three concepts that I want students to learn. First, I want them to be able to differentiate between domain and range. Second, I want the students to be able to read the graph to identify the domain and range. And thirdly, I want them to be able to use correct mathematical notation to write the domain and range. The subordination task does not explicitly work on all three but indirectly touches on all of them. In particular, the task works on skill, A, reading a graph to identify the significant values that lead to the domain and range. The task, B, is to identify the possible *x* and *y*-values that satisfy the question posed for each graph. The activity satisfies the requirements, based on Figure 4.1, of being a subordination activity because

1) the students need to be able to read the graph in order to identify the possible *x* and *y*-values that satisfy the question posed for each graph.

2) once the students believe they have identified the *x* and *y*-values, they can check to see if the value(s) or the values in the range satisfy the graph by visually confirming whether their chosen value(s) is actually on the graph.

3) the students do not need to know how to identify the domain and range in order to find the *x* and *y*-values that satisfy the question posed for the graph.

Similar to *Function Factory*, the skill is only partially stated in the features 1) and 3) above. The terms domain and range are not introduced until the end of the task. As students progress from Part I to Part VI, they move from integers to rational numbers to real numbers, all the while concentrating on naming the values from the graph that satisfy the conditions posed in the question. As they focus on listing these numbers, they are actually finding values that fit the description of domain and range. Students work on reading the graph, but they do not know what the domain and range are until I introduce it to them. Refer to Figure 4.12 to observe how subordination is used in *What's My x*?. As students focus on listing the *x*- and *y*-values that satisfy the conditions, they need to read and interpret the graph to write the domain and range using mathematical notation.

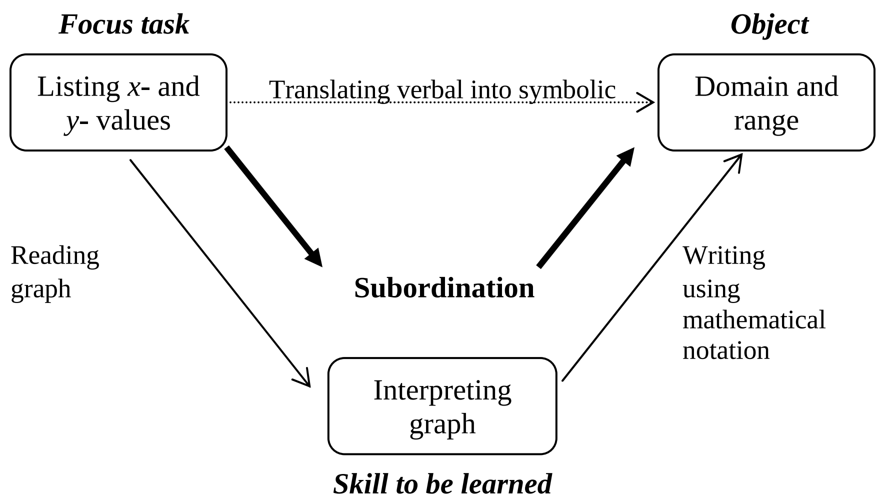


Figure 4.12 Subordination flow chart for *What's My x*?

Results and Analysis

Observations

The students diligently worked singly, in pairs, or in threes, on Part I of the worksheet. When they got to Part II, many students had difficulty because they did not know what type of numbers to write if they were not permitted to write integral values. Some students were comfortable writing decimal numbers ending in 0.5. However, since these type of numbers were the only ones that were chosen, I decided to have a class discussion to talk about other decimal numbers, such as those ending in 0.1, 0.2, 0.3, etc.

After this discussion, most students were able to answer Part III by stating that there were many possible *x*-values but only one possible *y*-value. When the students had to think of a shorter way to show this explanation, they knew to use mathematical symbols but were unsure of how to do this. I prompted them to use inequality symbols but for some reason, students were still confused by which inequality symbol they needed to use. Inequality symbols are not new to mathematics 10 students but they are a notation that often causes confusion. We had a short discussion about using inequalities to write the domain/range, but because we would talk about it more in the next two sections, I did not emphasize it as much here. The students seemed to accept the notation as a natural method for showing their solution. The students found Part IV easy because they only needed to identify the smallest and largest possible values for *x* and *y* but Part V was again more difficult since some students did not know how to interpret the smallest or largest value when there was no end. Hence, we had a discussion on the concept of positive infinity and negative infinity, when to use all real numbers to describe the domain and range and how to apply inequalities to describe the domain/range when it was half-closed. This led to Part VI where students considered these new ideas.

I found that my expectations and my teaching of the mathematical notation to describe domain and range, were different than the teaching of function notation in Task 4. In *Function Factory*, function notation was completely new to the students, while in this task, the notation and vocabulary that the students should have been using were not new. They had encountered all the notation and vocabulary necessary to describe domain/range in previous mathematics courses and it was my responsibility to help the students elicit this knowledge. Although students were not proficient at using mathematical notation to describe the domain and range, most of them were able to describe it using words.

Analyzing Engagement

The lesson was different from my traditional lesson because instead of my giving students techniques and rules to memorize, the students had to figure things out on their own using the guided worksheet. I am not sure what it is about worksheets but students seem to enjoy them more than lectures and textbook work as I noticed that all the students were engaged in *What's My x*? from the start. Most worked in small groups of twos or threes but a couple of students did work independently. As stated before, Shernoff et al. (2003) find that students report higher engagement when working in groups or individually compared to listening to a lecture. As to whether students felt the class to be short, normal or long, see Table 4.3 for the results from the questionnaire (see Appendix 2) that the students filled out after the task.

	Short	Same as Normal	Long
Class 1 (n = 24)	9 (38%)	5 (21%)	10 (42%)
Class 2 (n = 23)	4 (17%)	12 (52%)	7 (30%)

Table 4.3 Results of questionnaire for Task 5

Because I thought that the worksheet would help students understand domain and range better than my usual instructional method, the worksheet was completed by both classes. However, my analysis is based only on my observations and data from Class 1. Responses from Class 1 students about why the class seemed short include understanding the lesson, working together, doing an activity rather than notes and having little explanation from the teacher. These results and comments seem to coincide with Larson and von Eye's (2006) findings. They state that "high engagement leads to more intense focus …and, if the challenge–ability match is within the person's comfort range, a faster sense of time in passing" (p. 121). For the students who felt that class seemed long, they found the lesson boring, confusing, difficult to understand and containing too many images and examples. "If, however, the challenge–ability match is not within the person's repertoire but stretches his or her abilities beyond reasonable limits, the complexity may be perceived as overwhelming and time may appear to pass slowly" (p. 121). It is interesting to note that Class 2 had half of its students feel that the class seemed normal because they were used to working on notes.

When asked what they liked about the class, five students commented that it was easy or not too difficult, three liked working with graphs and two liked the discussion. One comment that pertains to the subordination task was a student "liked doing the equations/work as we learned about it" (Clayton). This is in essence how subordination tasks work. As students focus on the task, they learn the skill because they need to use it to accomplish the task. Seven students disliked the task because it was confusing and some did not understand the work.

> "I don't like reading graphs without points because it's hard to read" (Dina).
>
> "It was hard and it seemed like everyone understood except me" (Nathalie).

Since reading graphs was the focus of the activity, the subordination task broke down for Dina because she could not work on the task. This is a similar occurrence to what happened in *Stacking Squares* and will be discussed in the next chapter.

Is This a Good Subordination Task?

When I created the activity, I believed *What's My x*? to be a subordination task that introduced students to the concept of domain and range by forcing them to interpret a graph to identify *x*- and *y*-values asked for in the questions. The students were never asked to find the domain and range specifically but were forced only to identify significant *x*- and *y*-values that would lead to the domain and range. Except for being able to obtain good immediate feedback, I felt the subordination task satisfied the three features that were established earlier.

However, after implementing the subordination task, I recognized that there were several problems. First, the skill that students practiced was not directly connected to the skill I wanted learned, writing the domain and range using mathematical notation. Although identifying significant values for the domains and ranges were useful, it was not as instrumental to leading to the use of mathematical notation as I had hoped. Also, being able to answer the questions posed for each graph did not necessarily mean that the student could later identify the domain and range. Another problem with the subordination task was the students' inability to read the graph. As stated earlier, one student could not read the graph when no precise points were marked. If she was unable to do so, then she would not be able to identify the *x* and *y*-values which was the task I assumed everyone could do. Hence, she was not able to complete the task and was not using subordination. The last problem in the task was that students did not get immediate feedback. The students could observe if their chosen value(s) were on the graph but the problem was that some students did not know how to read the graph correctly.

Although I felt that scaffolding the worksheet was useful in helping students focus on the pertinent parts of the graph, I now believe that this helped make the activity a nonsubordination task. The students may have learned to read graphs but they were not practicing to find the domain and range. In order for an activity to be a subordination task, students need to be using the specific skill or a very related skill to complete the task. To recall, in *Expression Scattergories*, I wanted students to learn how to simplify expressions but the skill they practiced was using the exponential laws, a direct skill required to simplify expressions anyway. For *What's My x*?, interpreting the graph was a step before using mathematical notation to identify the domain and range and this identification did not occur until after the subordination task.

TASK 6 – ALGEBRA VERSUS THE COCKROACHES[3]

Background

It is in mathematics 10 where students usually first encounter the properties of a linear function. According to the prescribed learning outcomes (BC Ministry of Education, 2006), it is expected that students will:

- determine the x- and y-intercepts and the slope of the graph of a linear function, given its equation
- sketch the graph of a linear function given its equation in the form $y = mx+b$
- determine the equation of a line, given information that uniquely determines the line

After teaching students how to calculate and find the slope and the x- and y-intercepts in previous lessons, I introduce students to the equation of the line in slope y-intercept form, $y = mx+b$. My normal practice is to give students approximately nine linear equations, with different slopes and y-intercepts, that are already in slope y-intercept form, and ask them to graph them on grid paper. Once all the lines are graphed, I ask them to identify any relationships they observe between the equations and the graphs, and what the literal

[3] Title is based on the activity obtained from hotmath.com using the URL http://hotmath.com/games.html

symbols *m* and *b* from the equation could represent on the graph. Eventually, students learn that *m* represents the slope of the line and *b* represents the *y*-intercept of the line. After this activity, I present them with notes and several examples where they do the reverse and identify the equation of a line from a graph.

Rationale

Some students have difficulty graphing the nine lines but more students have problems identifying any correlation between the graphs and the equations until I reveal it to them or encourage them to look at slopes and intercepts. Nonetheless, I believe that my method of instruction described above is a good introduction to the equation of a line in slope *y*-intercept form because it allows students to use their own awareness to make a discovery as opposed to receiving an explanation from the teacher (Hewitt, 2001), if they are able to accomplish it. The limitation is that these nine questions are restrictive and may not provide the quantity of examples to convince the students that this correlation between form and graph is consistent. In addition, students find this activity boring and tedious by the fifth graph. Because time is spent graphing the lines, less time is devoted to identifying any relationship between the graphs and the equations. Their focus and attention is often with graphing neat lines rather than the more important goal of identifying relationships. The learners "become absorbed in the *doing* of a task – carefully drawing lines, colouring shapes, cutting out and so on – they are being active, they are engaging in activity, but they may not be learning what is intended" (Mason and Johnston-Wilder, 2004, p. 29). My intended goal is for students to learn how to write the equation of a line in slope *y*-intercept form and not concentrate on graphing. Therefore, I want to develop a task that emphasizes the equation.

Task Development

I had wanted to use technology for at least one of my subordination tasks because I knew that it could provide the quick feedback that I had desired for the other tasks. In addition, “the computer can be a powerful resource for the notion of subordinating certain names and conventions, since there always needs to be a way in which a student communicates with the computer” (Hewitt, 2001, p. 49). I found the on-line computer game, *Algebra versus the Cockroaches,* to be a very engaging mathematical activity that I thought my students would enjoy. In addition to amusing sound effects, the game offers mature graphics for a fifteen year old. The premise behind the game is to exterminate the cockroach(es) by drawing a line. Instead of physically drawing the line, the player has to identify the slope and *y*-intercept of each line and insert these values into the correct positions in the equation. The computer then draws the line that is entered and the player can see whether or not it coincides with the line that the cockroaches are walking. If the equation is correct, then the cockroach(es) are exterminated. One of the best things about the activity is that the teacher does not have to go around to check each student’s equations. One may say that the activity is no different from blocked practice but when students are engaging in this task, they may not be conscious of the skill they are practicing.

Task Description: Outline of the lesson plan

The learning objective for *Algebra versus the Cockroaches* is to write the equation of a line in slope *y*-intercept form, $y = mx+b$. In order to achieve this, the prerequisite is being able to read the graph of a line to identify the slope and *y*-intercept.

In order to carry out this task, students need a computer that has access to the internet. No pre-instruction is needed except to perhaps remind students about how to find the slope and *y*-intercept of a line. To login to the game, students enter the site http://hotmath.com/games.html (hotmath.com, n.d.). The game is best played singly or in pairs. To begin, the players choose a weapon from a rocket, laser gun, spray can or shoe. The goal of the game is for the students to exterminate the cockroaches which they do by identifying the slope and *y*-intercept of the line that the cockroaches are walking and inserting these values into the appropriate blanks of the equation,

$$y = \_\_\_x + \_\_\_.$$

Students do not need to be introduced to the form of the line before they play the game since the equation is displayed on the bottom of the screen for the students to fill in.

The game begins with one single cockroach walking back and forth in a straight line. As time elapses and the cockroaches are not exterminated, more and more cockroaches appear on the screen, up to a maximum of 16 cockroaches, at which time the game will end and the round will need to be restarted. The game progresses through the following nine rounds with the different category of lines.

Round 1) horizontal lines
Round 2) lines through the origin
Round 3) lines not through the origin with positive whole number slopes
Round 4) lines not through the origin with positive rational number slopes
Round 5) lines not through the origin with negative whole number slopes
Round 6) lines not through the origin with negative rational number slopes
Round 7) all types of lines with different slopes
Round 8) all types of lines with different slopes but speed is important
Round 9) all types of lines with different slopes but speed is very important

There are five equations that need to be written to pass each round and students can make as many attempts as time will allow. One can say that the rounds tend to increase in level of difficulty and if students are able to reach Round 7, then they have covered all the different types of lines and their understanding of writing the equation of a line in slope y-intercept form is quite comprehensive. (It is important to reach Round 7 because at this level, the different types of lines are mixed up.) Round 9 is very difficult as cockroaches appear very quickly.

Because it is sometimes difficult to see where the cockroaches are walking, students can choose to utilize provided hints to help them visualize the line and/or identify the slope. Here are the three hints in the order they are presented as "hint" is clicked :

1) Two points are highlighted with integer coordinates to help visualize the triangle used to identify the slope.
2) A triangle is drawn using the two coordinate points from hint 1.
3) The slope of the line is the: $\frac{\text{rise}}{\text{run}}$.

Note that none of the hints actually indicate what the solution is. At any point during the game, results can be obtained which provide:

- the Round which was reached
- the number of equations/lines that were presented
- the number of attempts
- the number of correct equations entered by the student
- the number of times that time ran out (16 cockroaches are born)

After playing the game, there needs to be a small discussion about the equation of the line in slope *y*-intercept form to check students' understanding.

Why *Algebra versus the Cockroach* is a Subordination Task

Algebra versus the Cockroaches is a subordination task that uses technology to force students to practice writing the equation of a line in slope *y*-intercept form. The skill, A, that I want students to learn is to write the equation of the line in slope *y*-

intercept form, $y = mx+b$. The task, B, is to exterminate the cockroaches. The game satisfies the requirements of being a subordination activity because

1) the students need to be able to write the equation of the line in order to exterminate the cockroach.

2) once the students have written the equation, they can immediately see the graph of the line that corresponds with their equation.

3) the students know that their task is to exterminate the cockroaches even though they may not know how to write the equation of the line.

Figure 4.13 below shows how the task subordinates writing the equation of the line in slope *y*-intercept form with the task of exterminating the cockroaches.

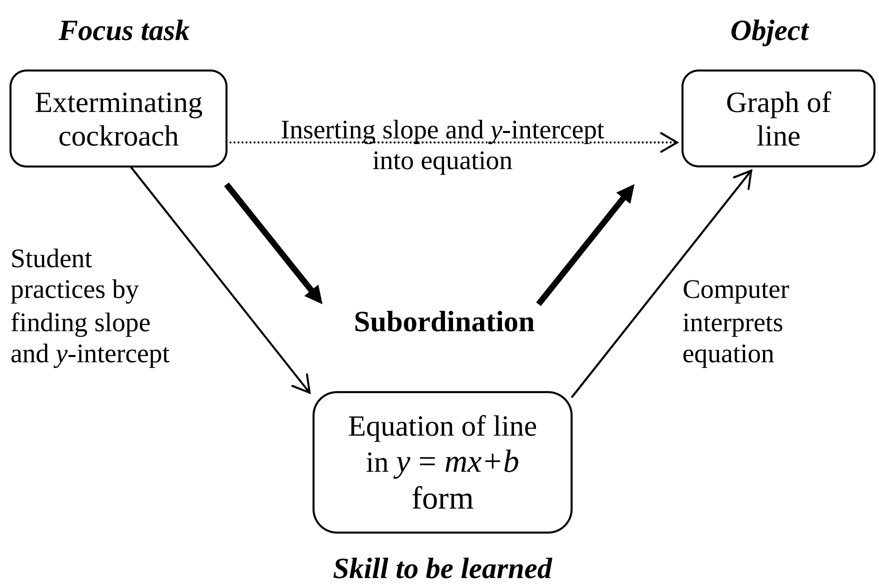


Figure 4.13 Subordination flow chart for *Algebra vs. the Cockroaches*

As students focus on eliminating the cockroach, they need to identify the slope and *y*-intercept to be inserted appropriately into the equation of the line in slope *y*-intercept form.

Results and Analysis

Observations

There was an initial few minutes of getting into the game but, once the students understood what to do, they were all very engaged. All the students worked collegially together in groups of two or three without much pausing, distractions or getting off-task, except for a couple of girls who got bored about ten minutes into the game at Round 3. There was a feeling of competition in the air as students checked what round others were on during play. This occurred mostly amongst the boys.

Many students, with the majority of them being girls, used the hints in order to clarify the line in which the cockroaches walked. The hints did not prove to aid the students' speed through the rounds though. These students only made it to Round 5 or 6. Most of the students wanted to reach the BONUS Round and I could tell by their complaining that they did not want to stop playing even though I had given them a 5 minute warning to tell them that they had to end the game soon. After 25 minutes of "practicing," here is how many students reached each round:

Round 5 – 3 students
Round 6 – 2 students
Round 9 (Bonus Round) – 20 students

Given that there are five questions in each round, to reach the Bonus Round, students would have had to complete at least 40 questions. When I checked their statistics, the average number of questions that the 25 students attempted was around 38. Students inevitably wrote some of the equations incorrectly but that did not stop them from trying again as depicted by their number of attempts. If an equation was written incorrectly and time ran out, the round would have to be restarted from the beginning.

Analyzing Engagement

Except for some computer glitches, all the students started playing the game immediately when they got to the computer lab. As mentioned before, many students competed against each other to see who could finish first which helped to engage many of them, particularly the male students. All 25 students stated that they felt the class was short with none saying that they felt it was long even though one student said that it was fun at first but got boring or repetitive. Nine students explained that the class felt short because the game was fun while four mentioned going to the computer lab. Here are some of the students' reasons why the class felt short:

> "Short because we played a game and didn't care about time" (Everett).
>
> "Short, because whenever you're having fun, time flies" (Ivan)!!
>
> "Short, because we played a game on a computer, which we don't normally do" (Nathalie).
>
> "We had a lot to do which made time pass faster" (Terry).

Sackett, Meyvis, Nelson, Converse, & Sackett, (2010) found that participants who felt that time flew rated the task as more enjoyable, challenging, engaging and fun. One of the more interesting observations made from analyzing the data is that this was the only task that a particular five students, who normally felt that the subordination tasks were long, stated that the lesson was short this time. These five mentioned playing a game, having fun and working in pairs. When students were asked what they liked or disliked about the activity, 15 students stated that they liked how they played a game, three liked using computers, and three students mentioned that they liked working in pairs or a group. One student said she liked how they got to work at their own pace. Like what was mentioned in *Expression Scattergories*, playing games is an engaging task. I find that if a game is well-constructed, the player will be absorbed in the game, become engaged and lose track of time.

Is This a Good Subordination Task?

In *Algebra versus the Cockroaches*, the students practice writing the equation of a line in slope *y*-intercept form while attending to the task of exterminating the cockroaches. The activity has all the characteristics of being a subordination task as outlined above. First, the students need to be able to write the equation of the line in order to kill the cockroach. Although students only need to fill in the blanks with the slope and *y*-intercept, they need to pay attention to where they are clicking on the screen and to insert the slope and *y*-intercept into the correct blank. Second, once the students have written the equation, they can immediately see the graph that corresponds to their equation. The students get immediate feedback on whether their equation is correct or not. Similar to Hewitt's model of subordination within sailing (Hewitt, 1996), the students can see the graph of the equation that they enter relative to the marching cockroach. If the equation is incorrect, they can hopefully use the graph that they see on the screen to influence the next equation that they write. Here is one student's comment to support this. "I can easily know if my answer is wrong and I can correct it right away" (Denise). Either the slope, the *y*-intercept or both need to be changed so that I say that they are subordinate to the graph of the line relative to the walking cockroach. The use of a computer definitely makes the process of obtaining feedback much easier and will be discussed in the next chapter. Finally, the students understand that their task is to exterminate the cockroaches even though they may not write the equation of the line correctly the first time. This is what the subordination task is trying to accomplish – to get the students to write the correct equation of a line in slope *y*-intercept form while they are concentrating on a different task, that of eradicating the cockroaches. The students' incorrect answers are just as important as the correct answers because that is where they get to practice and adjust their writing of the equation of the line in relation to the feedback that they obtain.

The students did an excellent job of practicing writing the equation of the line in slope *y*-intercept form as many practiced at least 40 questions in only 25 minutes. In response to the question of whether they felt like they were practicing, all except two students felt like they were still practicing mathematical concepts because they had to find slope or write equations. One of the students who had said no gave the reason "because it was in the form of a game" (Angela). As stated before, the other students may have a limited view of what practicing mathematics means but it does not seem to affect their engagement with the task. Many students made comments to show that although they were having fun, they were aware that they were "practicing".

> "The cockroach game was fun and it taught us math" (Keith).
>
> "Short because the class was fun and we learned at the same time" (Matthias).
>
> "I like the game because I'm learning while enjoying" (Denise).
>
> "Ya because the game, to advance in the rounds you had to know the math" (Everett).

These comments are also relevant to subordination because even though the students know they are learning math, they are concentrating on the task at hand.

> "With the graph you wanted to get it right so you tried hard" (Matthias).
>
> "You had to solve the questions very quickly or you lose the game" (Noah).

Regardless of whether or not the students gave conscious attention to the mathematics, the students focused very intently on exterminating the cockroaches and required mathematics to do this.

Algebra versus the Cockroaches is an excellent subordination task that offers a way to practice writing the equation of the line in slope *y*-intercept form as students focus on another task, that of eliminating the cockroaches. The game format of the task helped to make the practice very engaging and the use of computers helped to give feedback promptly. I felt that this last task, in my group of six, was the most successful which will be discussed further in the next chapter.

Chapter 5 : ANALYSIS SUMMARY

Through the course of this research, several themes emerged regarding subordination tasks. These themes contribute to the success or failure of subordination tasks and are related both to the teacher and the students. In this chapter, I split the themes depending on whether it was encountered during the design of the tasks or during their implementation. Exploring these themes will provide a better understanding of subordination of teaching to learning within mathematics and lead to the characteristics of good subordination tasks described in the next chapter.

DESIGN

"One of the most critical responsibilities for a teacher is setting appropriate tasks" (Hiebert et al, 1997, p.30). Through my research, I have discovered that searching for the ideal task is neither necessary nor a good use of my time. Mason and Johnston-Wilder (2004) claim that it is more effective to develop a way to create tasks or adapt already existing tasks. I find that this is true since less energy is expended to alter something than to invent something from scratch. For example, Task 3 – *What's My Number?* was adapted from Hewitt's (1996) *Think of a Number* activity and took the least amount of time. Task 2 – *Expression Scattergories* was the only task that I designed and not having much experience designing subordination tasks, took me the longest to create.

Designing tasks that exhibit the three features of subordination is not easy. I constantly have to keep envisioning what students will be focused on while they work on the task, and if they are indeed, utilizing the skill that I want them to learn. If the students are concentrating on the skill to be learned, then the task is not a subordination task. While I am aware of this main feature of subordination, three other design-related themes emerged. They are as follows: deciding on the purpose of the task and student readiness, using the task to practice a different skill, and mispredicting the attainability of the task.

Purpose of Task and Student Readiness

As described in the literature review, Mason and Johnston-Wilder (2004) summarize a framework that gives a way of thinking about the purposes of a lesson, including seeing, experiencing or mastering. Subsequently, subordination tasks can be designed for any one of these purposes which I reduce to introducing (seeing) and practicing/reviewing (experiencing or mastering). When to use the task is contingent on students' stage of development or familiarity with the concept. For example, many students found Task 1 – *Stacking Squares* to be difficult. It was implemented near the beginning of the year, when students were just getting to know one another, and it may have been better to use the subordination task after students had already learned how to simplify radical expressions. Using the task as practice to find equivalent radical expressions may have alleviated some of the slow-start behavior that was exhibited by some students. On the other hand, although Wagner (2003) had successfully used the task one month after he had already taught his classes how to add and simplify radical expressions, he recognized the potential for rich connections if he had used it beforehand.

When I was developing and reflecting on the purpose of the subordination tasks, I had originally thought that it would be more interesting and rewarding for students if the tasks were used to introduce concepts. I surmised that students would retain the skill or technique longer if it were introduced right from the beginning of its conception. Hewitt (1996, 2001) seemed to have used most of his tasks in this manner and did not give the impression that subordination tasks depended on the students' particular stage of concept development or mathematical readiness. However, *Stacking Squares* demonstrates that the success of subordination tasks do depend on student readiness. If students are not prepared to successfully accept a subordination task to introduce a concept, it should be saved and utilized for practice instead. For the six tasks I developed, I categorize their purposes as introducing and/or practicing a concept, allowing the teacher to decide when to best use the task.

Task 1 – Stacking Squares → introduce or practice
Task 2 – Expression Scattergories → practice
Task 3 – What's My Number? → introduce or practice
Task 4 – Function Factory → introduce
Task 5 – *What's My x*? → not a subordination task
Task 6 – Algebra Vs the Cockroaches → introduce or practice

If students are willing to be involved in challenge and/or exploration, a subordination task is a good vehicle to use to introduce a concept, particularly mathematical notation such as function notation in Task 4 – *Function Factory*. The intention is that the task will help the notation stick when students use it in future mathematics courses. However, if students are unwilling to be challenged, it is best to use the subordination task for review.

Task Practices a Different Skill

Two of the tasks I designed are subordination activities but they do not subordinate the exact skill I desire. In Task 2 –*Expression Scattergories*, the subordination task acts as a bridge to transfer the concept the students learn in the subordination activity to the skill I ultimately want my students to learn. In this task, students practice using the exponential laws when they are writing equivalent expressions, the reversal of what I want. However, the skill of simplifying exponential expressions I want them to achieve still requires the use of the exponential laws. As stated before by Hazzan and Zazkis (1997), inverse tasks can be effective pedagogical exercise. The problem is that one might be very good at the former without knowing how to do the latter. *Stacking Squares* presents a similar situation since students again are finding equivalent radical expressions rather than the simplified radical expression that is desired. Regardless, both tasks still force students to practice mathematical concepts that are very much related to the intended skill. In Task 5 – *What's My x*?, students focus on identifying the significant *x*- and *y*-values that lead to the domain and range. These numbers are an important part of the domain and range but the desired skill of identifying the domain and range is never explicitly asked for. The concept practiced does not provide a good link to the intended skill. Therefore, *What's My x*? is decided to not be a subordination task. Subordination tasks may not practice the exact concept desired but the skill that is practiced should be directly connected or another task needs to be developed.

Mispredicting the Attainability of the Task

When designing tasks, teachers make assumptions about what students can achieve. However, there is often a disconnection between what teachers believe students can do and what students can actually do, causing a problem in devising effective tasks (Mason and Johnston-Wilder, 2004). For instance, when I chose *Stacking Squares* for students to learn about simplifying radicals, I thought that students would easily be able to find stacks of squares that gave the same height as the target square. Although all the students made some attempt, there were two groups who were unable to find any equivalent stacks; thus, they never found any equivalent radical expressions and forfeited the skill that I was trying to get them to practice. For them, the subordination task broke down. In *What's My x*?, I thought that sequencing the type of numbers (from integers to real numbers) would help students identify the significant values. Although they were able to recognize the values when they were integers in Part I, some students had difficulty determining any rational numbers in Part II because they could not interpret the graph. To rephrase what one student had admitted, she did not like reading graphs without points. If she could not read the graph, she could not work on the task and this rendered the practice useless. Mason and Johnston-Wilder (2004) assert that "the most important feature of a task is that it contains some challenge without being overly taxing" (p. 52). On the other end of the spectrum, tasks can also not be trivial or suitably simplified. Otherwise, they may be seen as not worthwhile to the learner. Some students found *Function Notation* too easy and got bored with it. Subordination tasks need to be designed at a level that is attainable by all students but are challenging enough to be engaging and to maintain interest level throughout the activity.

IMPLEMENTATION

When implementing a task, teachers need to be sensitive to learner readiness and adapt the lesson when and where needed. As I carried out the subordination tasks, five themes related to implementation emerged. First, the learning and teaching environment have to accommodate the change in teaching style. This change in the milieu affects my sensitivity to the students' behaviour and the students' role in the classroom. Second, students need to have some problem solving heuristics in their repertoire to initiate and sustain work on subordination tasks. Third, teachers need to be aware that students may use unexpected methods to carry out the task. Another theme concerns the second vital feature of subordination tasks, feedback. Finally, subordination tasks do not necessarily need to be engaging but engaging tasks sustain activity and make it not feel like practice.

Classroom Milieu

My classroom and the classrooms that my students have been accustomed to, for at least the last two years, have been traditional learning environments where the instructor teaches “knowledge directly as if it were a cultural fact” (Brousseau, 1997, p. 227). Learners copy notes from the teacher and tend to work independently. On the other hand, subordination tasks tend to require students to be active participants in the classroom where they may work in pairs, small groups or as a whole class. With the use of subordination tasks, three changes in the classroom milieu became apparent: my sensitivity to the students’ reactions to the lesson, classroom ethos and classroom organization.

Sensitivity to Students’ Reactions to Task

Brousseau (1997) describes the relationship between the mathematics teacher and the mathematics student as the didactical contract. Formulated within this contract is for the teacher to ensure learning in the classroom and thus, most teachers are sensitive to students’ reactions to the activities they are asked to perform. Because subordination tasks were new to both me and my students, I became even more sensitive to students’ reactions than usual. I had more invested in these activities than in my regular lessons. Therefore, I wanted students to do well, affecting how I implemented the tasks and how I responded to any difficulty the students may have had. In *Expression Scattergories,* the activity was progressing well but because the students received inadequate feedback (which will be discussed shortly) each round took longer than expected. Consequently, I felt some students getting bored and restless and therefore, I stopped the game prematurely after only two rounds. I did not realize I had done this until I read the comments from the questionnaire. Two students commented that it would have been better to play longer. Here is one student's comment,

> I enjoyed doing the practice today because I learnt it last year but I completely forgot. Since now I reviewed I remember how. If we did more, it would be more understandable (Ivan).

Only three out of 25 students actually said that they did not enjoy the practice. The interesting thing about my reaction is that I probably would not have stopped giving them exercise problems from the textbook if students were bored. As students worked on *What's My x*?, I felt a need to help them even though they were already working on a guided worksheet. Herbst (2002) made similar findings when his analysis of a teacher's work of engaging students in producing two-column proofs in geometry placed conflicting demands on teachers due to the didactical contract. It is probably this sensitivity to students' ways of working which originally encouraged me to give students a guided worksheet. Being sensitive to students' reactions is important but in order for the subordination tasks to do its job, learners must still get enough practice.

Change in the Classroom Ethos and Ways of Working

The use of subordination tasks brought change to the classroom ethos and ways of working. Some students found subordination tasks more challenging than the normal routine of copying notes they had been experiencing for at least the last two years. Those who did not enjoy participating in the subordination tasks tended to be those students who had become dependent on the teacher-telling-student presentations of mathematics classes and found a change in the presentation style difficult to cope with (Mason and

Johnston-Wilder, 2004). Even though the students played a game in *Expression Scattergories*, two out of 25 students commented that they would have rather taken notes. “Learners can become habituated to certain ways of working and may resent radical change” (p. 38). Boaler (2002) found similar results at Phoenix Park. After having encountered traditional presentations of mathematics for the previous eight years, some students at Phoenix Park objected to the open-ended work, preferring workbooks and structure. My Class 2 was a quiet class with many students who wanted structure, direction and teacher instruction, and I felt that they did not respond as well to a change in the lesson style compared to Class 1. They were not a class who misbehaved but many students, even the capable ones, for some reason seemed unwilling to think of ways to work on their own but waited for me to instruct them as displayed by their non-start behaviour in *Stacking Squares*. They preferred having a straight path towards the answer, desiring a result more than the process of getting there. This was similar to Sinclair’s (2001) ‘track-takers’ who “preferred having an established, obligatory route that was quick and straight – like a railway track” (p. 29). There was generally also less participation in class discussions with Class 2 which was often an integral part of the subordination task. Subordination tasks also typically took longer than the traditional lesson. *What’s My Number*? took almost the entire class, something the students disliked, as indicated by their comments on not having time to work on homework. In contrast, several students from the class, who had the traditional lesson, liked the fact that there was plenty of homework time.

Students are not the only ones who need to adapt to the change in the classroom ethos. My role as the teacher requires me to keep consonant a learning environment that fosters and sustains mathematical thinking. This is difficult since “it is easy for practices to drift and turn into routines that no longer create the ethos originally intended” (Mason and Johnston-Wilder, 2004, p. 38). In *Function Factory*, my intention was to make the lesson interactive but because I felt that the students needed to have everything recorded in their notebook, I made them copy what I was writing on the board. Although this was not the intention, I had unwittingly reverted back to a typical lesson but with more class participation and use of pictures. This was the only task out of the last four where more students felt class was long rather than short. Subordination tasks often require more interaction, engagement and mathematical thinking. This means that the traditional classroom ethos needs to be adapted if these skills are to be encouraged.

Classroom Organization

The third change in the milieu involved the classroom organization. The subordination tasks required students to work independently, in pairs, in small groups and as a whole class. Many students enjoyed working in small groups and having class discussions as indicated by the questionnaire comments in every task. However, from facial expressions and posture, I could tell that some students did not know how to work in small groups or were uncomfortable with their group as some students did all the work and others would just sit back and watch. This was particularly true for *Stacking Squares*. *What’s My Number?* and *Function Factory* both required students to work individually and as a large group back and forth. This worked well except it was usually the same students who participated. In *Expression Scattergories*, three out of 25 students commented that the activity would have been better if they were allowed to switch groups. I had created the groups instead of allowing the students to choose their own. The stigma of working with peers they were unfamiliar with may have stymied their mathematical thinking in the subordination activity but I believe it is important to work with a variety of people. “Learning to be a member of a mathematical community means taking ownership of the goals and accepting the norms of social interaction” (Hiebert et al, 1997, p. 43). The teacher needs to build a community of mathematical practice in which students feel comfortable reflecting, communicating and sharing and where the teacher can teach and students can study (Hiebert et al., 1997; Lampert, 2001). Yackel and Cobb’s (1996) analysis of sociomathematical norms indicate that it is critical for teachers to establish a classroom environment that stresses the importance of a mathematical community. It is important to do this early on in the year so that the rest of the year can be devoted to building mathematical understanding.

Problem Solving Heuristics

The lack of problem solving heuristics may have hindered students’ progress in at least one of the subordination tasks. Lampert (2001) provides a clear and elegant account of establishing a classroom culture where the routine setting supports students to reason mathematically, make conjectures, argue respectfully and work collaboratively. This type of environment can be built up as the teacher provides students with rich mathematical problems to engage in throughout the year. As students learn to work with more and varied problems, they develop a bank of problem solving methods that they can apply. *Stacking Squares* was completed near the beginning of the year when the students had not done very much problem solving yet. Therefore, they did not have a large repertoire of problem solving heuristics to draw upon, causing many students to display slow-start behaviour. Except for the last task, Task 6 – *Algebra versus the Cockroaches*, there was always at least one student who made a comment about the task being confusing or difficult to understand. While this is probably true with any lesson, the problem is often caused by students' trying to get that one correct answer. Boaler and Greeno (2000) made similar findings during their interviews with AP Calculus students. As one student put it, "There's only one right answer and you can, it's not subject to your own interpretation or anything it's always in the back of the book right there. If you can't get it you're stuck" (p.9). The role of subordination tasks is many, least of which is getting the one correct answer. *Stacking Squares* and *Expression Scattergories* contained many solutions. *What's My Number?* and *Function Factory* had correct answers but the goal was learning a convention, inverse order of operations and function notation, respectively. *Algebra versus the Cockroaches* was the only task that had one correct answer but it was also the only task where all the students enjoyed it and could get themselves unstuck. Mason and Johnston-Wilder (2006) state that "learners make progress when they become aware that what matters about a task is the method and the thinking involved, rather than the specific numbers" (p. 10). The nature of subordination tasks dictates that some prior skills are necessary, but if the teacher tells students what technique to use, then the subordination task will be compromised.

Unexpected Methods

When I created the subordination tasks, there was a particular skill that I wanted students to discover, experience or practice in each task. In *Stacking Squares*, I wanted students to discover that the heights of each stack of squares were related to perfect squares and could be written with equivalent radical expressions. However, instead of applying perfect squares, all the students initiated the activity by using ratios. Most were successful at discovering two stacks but confined only to this viable approach, they were unable to discover other stacks. Wagner (2003) found that one of the groups in his class used the same approach and encountered the same problem. He extends its potential connection to radical arithmetic though none of the students were able to relate it on their own. Mason and Johnston-Wilder (2004) point out that merely doing a task is not enough to guarantee that students will work on the intended learning outcome. As described earlier, the "difference between how the learners and the teacher sees the task is the major problem in trying to devise effective tasks" (p. 29). Students bring with them their own powers of sense-making and they may stress something different than what the teacher planned. To avoid this problem, teachers sometimes give instructions for a task that are more directive such as what I did for *What's My x*? When this happens, students carry out the instructions unproblematically but are less likely to encounter the intended ideas behind the task. "The more precisely the teacher structures the task, the less likely the learners are to get anything out of doing it" (Mason and Johnston-Wilder, 2004, p. 34).

Feedback

Feedback is what moves the learning forward as students react to what they are given (Wiliam, 2007). In subordination tasks particularly, the consequence of the actions of the skill on the task needs to be observable at the same time as making those actions. This feedback was difficult to achieve in some of the tasks, causing interruption or even termination of student activity. In *Expression Scattergories*, students were not able to get appropriate feedback from their peers for the equivalent expressions they wrote. Students were able to create many expressions but unfortunately, the expressions were sometimes incorrect and none of the students in the group were aware of this. *What's My x*? held similar problems. Unless I went around to check each individual's work, the students did not know if they had identified the correct values for each graph. Both of these tasks had feedback that reminds me of Hewitt's (2004, 2010) *Ball and Box Activity* where the thrower is blindfolded and attempts to throw a ball into a box. When there is none or poor feedback, the thrower randomly tosses the ball in any direction the next time. In fact, "the research on feedback shows that much of the feedback that students received has, at best, no impact on learning and can actually be counterproductive" (Wiliam, 2007). For *Expression Scattergories*, I had placed a student with good mathematical ability in each group, thinking that he/she could give appropriate feedback to his/her group members. For some groups, this was sufficient but for others, I was incorrect in my assessment of students' abilities and they were unable to determine if the expressions they created were indeed correct. Immediate feedback is necessary for students to succeed in completing subordination tasks because the feedback they get is immediately applied in the next attempt. Hewitt (2001) suggests using a computer to do the necessary work of feedback which frees the student to work on making sense of the mathematics. The use of technology will also free the teacher from checking every student's work, giving more time to help students make sense of the mathematics.

Engagement

As described in the previous chapter, feedback moves the learning forward. However, tasks need to attract students to initially work on them. Tasks need to be challenging but not trivial and exercise the familiar. With these qualities in place, "they are much more likely to be engaging and to yield a sense of satisfaction than tasks that concentrate entirely on practicing some technique" (Mason and Johnston-Wilder, 2004,

p. 55). To look at engagement, I analyzed students' responses from the questionnaires, particularly their perceived flow of time during the subordination tasks. More students found *Function Factory* and *What's My x?* to be longer than shorter. The reason students found these tasks to be long was there were lots of work and the concepts were simple. Both Larson and von Eye (2006) and Shernoff et al. (2003) attribute this low engagement to low challenge and skill. On the other hand, all the students felt *Algebra versus the Cockroaches* to be short. The reason given by most of the students was that the game was fun. As stated before, game playing can be engaging (Mason and Johnston-Wilder, 2004) if the challenges are high but attainable. The majority of the students also enjoyed the other game, *Expression Scattergories*, but for different reasons. They liked how it was more interactive than normal and they got to talk to each other. Engagement was less dependent on instructional strategy than the type of activity. Students' engagement level with a task also did not affect whether or not the task was a subordination activity. Though some students felt that tasks were long and boring, such as *Function Factory*, the task still succeeded in accomplishing what was intended, learning function notation.

My research has made me aware of issues both in the design and implementation of subordination tasks. The ones that stand out in particular are: deciding on the purpose of the task and student readiness, using the task to practice a different skill, changing classroom ethos, and obtaining good feedback. Considering these features, I offer characterizations of a good subordination task in the next chapter.

Chapter 6 : CONCLUSIONS

SUMMARY

I began this research with Hewitt's (1996) notion of subordination of teaching to learning. His three characteristics of subordination seemed to be the crux of creating a successful subordination task. As I designed six tasks to study this notion and implemented them with my mathematics 10 students, I realized that there was more to subordination than those three features. I experienced challenges in design such as deciding the purpose of the task, not being able to subordinate the exact concept I wanted students to learn, and mispredicting the attainability of the task. Then, when I implemented the tasks, there were other elements I needed to attend to including: the classroom milieu, students' ability to problem solve, students' use of unexpected methods, feedback and engagement level. These issues with creating and implementing have made me realize that I cannot only look at the mathematical teaching content but also must attend to where students are at and how they may differ in their experiences of the task. This means that I need to consider the subjective features as well as the three objective features of subordination when designing tasks. In the following section, I outline the additional characteristics of a good subordination task, which I arrived at by analysing the design and implementation of the tasks presented in Chapter 4.

CHARACTERISTICS OF GOOD SUBORDINATION TASKS

Good is a relative term but I have deemed several factors necessary to create a successful subordination task. Most of these features were introduced briefly in Chapter 4. Besides the three features described by Hewitt (1996), in order to design and implement a good subordination task or to alter an existing task to increase its subordination factor, it needs to have the following characteristics: be accessible, engaging, and relevant, introduce an arbitrary skill or practice a necessary procedure, and provide adequate and immediate feedback. The first three are pertinent to how students view the task and are important to the implementation of any activity, but I will describe how subordination tasks particularly require these features. The last two are objective features and are notably related to the design of subordination tasks.

Accessible

A task is not a subordination task for a student unless it is accessible. If students are unable to start or require too much explanation before beginning, the subordination task will be compromised and ineffective as an introduction to the concept. *Stacking Squares* was a little too difficult for some students while the skill level required for *What's My Number?* was desirable. However, this does not mean that *Stacking Squares* is a poor subordination task. When to implement any task depends on the classroom ethos and learner readiness. If the task is designed to introduce a convention and students are not ready, it may be more successful used as practice to review the convention.

Engaging

I have found that one of the useful purposes of subordination tasks in mathematics is for practicing a concept. However, as I have discovered from *Expression Scattergories*, students still need sufficient practice from the activity in order for it to be effective. This means that in order for students to sustain this practice and reap the benefits of subordination, the task to be focused upon needs to be engaging. It should offer a challenge that draws the students in but not too challenging that they cease trying. Larson and von Eye (2006) stress the importance of the challenge-ability match. *Stacking*

Squares was challenging and worked well for some students but not for others. *Function Factory* was too easy and many students got bored (though the task was still successful). Activities that involved a game, such as *Expression Scattergories*, or had a goal, such as *What's My Number*?, had more interaction and hence, more engagement, than *What's My x*?, a worksheet activity. Tasks that required small group or whole class involvement were also engaging. *Algebra versus the Cockroaches* was the most engaging task and many students did not feel like they were practicing. Much of it had to do with playing a game that was fun but also the use of a computer. Computers appeal to most adolescents and combined with the game aspect, attracts the learner into its imaginary world.

Relevant

Although relevance is a minor feature, tasks that are relevant to students make an impression on them. Both *Function Factory* and *Algebra versus the Cockroaches* had at least one student mention that they felt what they were doing was relevant. If a student can relate a task to day-to-day activities or some other way, then the task has more meaning and students have more of an inclination to partake in the activity. Subordination tasks that are used to assist students to memorise the arbitrary, such as function notation or algebraic notation, have the ability to help students see its relevance, particularly if the convention emerges naturally from the activity.

Arbitrary and Necessary

I found that the subordination tasks were more successful at introducing and assisting students memorize what Hewitt (2001) calls the arbitrary, such as names and conventions. Although I did not create a task that introduced names, Hewitt's (2001) triangle naming activity seemed impressive. I felt that *Function Factory* was successful in introducing function notation. It is an arbitrary mathematical convention and as stated before, can only be known if the teacher informs the students. Instead of the teacher strictly providing the arbitrary, the subordination task stresses the properties first and then the teacher provides the associated name or notation when it is needed. *Expression Scattergories* and *What's My Number*? involved exponential laws and inverse order of operations, respectively. I felt that these conventions involved the necessary part of mathematics which meant that students could figure it out on their own. Both of these tasks were appropriately used as practice for experience or mastery.

Adequate and Immediate Feedback

A successful subordination task requires adequate and immediate feedback so that the learner knows if any adjustments need to be made to what they are doing in preparation for the next attempt. The whole class activities, *What's My Number*? and *Function Factory*, provided adequate feedback because I would tell the students the answer, though this did not give them an opportunity to try it again if they got it incorrect. *Algebra versus the Cockroaches* dispensed very good feedback because the computer would draw the line that the students had typed in. If it were incorrect, they could see how close their line was to the walking line of cockroaches. It never told the students the exact error but the students could see if there was an error in their slope or *y*-intercept by looking at the graph. The computer can probably give better feedback than the teacher because the computer can give immediate individual feedback to each student or student pair where as the teacher can only provide the correct answer or guide everyone towards the right answer in a whole class activity.

FINAL REFLECTIONS

As a result of this study, I better understand how subordination of teaching for learning can be utilized in mathematics. I have found that subordination tasks are useful for introducing mathematical concepts that are considered arbitrary and reviewing concepts that are necessary. However, when to use a subordination task partly depends on where the students are in their learning of the convention. For example, Hewitt (1996) used the *Think of a Number* activity to introduce notation while I used the same activity to make students review inverse order of operations to solve an equation. A subordination task can be used for more than one purpose, introduction or practice, and for learning different mathematical concepts. I would not suggest using a subordination task for every class since students can tire from the amount of activity, reflection and communication required to make it successful. However, as described in the previous chapter, changes in the classroom milieu and development of good mathematical reasoning will help subordination tasks become more fruitful. Students need to be comfortable with each other, with the teacher and sometimes, even with the change in teaching pedagogy.

During my study, I gave both classes the same quizzes and tests. This did not compromise my research but if I had wanted to assess the students' retention of concepts taught with subordination tasks, it would have been more useful if I had adapted my assessment to better match the subordination tasks. For example, the class who had experienced *Expression Scattergories*, could have been given a quiz that asked them to create five equivalent expressions using the five exponential laws. Regardless, the subordination tasks did not have a negative or positive impact on student learning. However, one thing I did discover from comparing the two classes was that students tended to view the subordination activities as more favourable and engaging. This was indicated by the word “fun” that was often used to describe the tasks. Although students in the class who did not participate in the subordination task seemed to be engaged in their lesson as well, they were not usually active participants and the word “fun” was never used to describe note-taking. They usually had more negative than positive comments regarding their lessons.

As a result of this study, I have become more sensitive to the type of tasks that I present my students. The paradox provided by Catch 22 (Heller, 1961) causes me to reflect and wonder why students need to practice if they already understand, and if they do not understand, how will they practice? For me, the use of subordination tasks remedy this situation a bit. When introducing a concept, the initial properties are stressed first with the name or convention emerging naturally from the activity. In addition, subordination tasks are used for experience and mastery. If students already think they understand, the practice will solidify their understanding while providing them with an alternative way of looking at a concept since they are not concentrating on the concept itself but on an alternative task. If they do not understand, then the task can provide a good way to reinforce a concept, showing students why it is necessary as opposed to forcing them to memorize it again and again. I have realized that I often spend too much time telling students to memorize something when in fact, they should be developing it on their own within a suitable pedagogically designed task.

Practice exercises do not have to be the boring, repetitive traditional practice that we always allude to when we think about mathematics. Practice can be engaging and relevant activities where learners acquire new skills while recalling the familiar. Focus of attention can be on some other, more intriguing task, as students practice the arbitrary and necessary of mathematics. Subordination tasks embody these qualities and provide students and teachers with something meaningful to do.

APPENDICES

APPENDIX 1

ACTIVITY QUESTIONNAIRE 1

1. What did you learn today?

2. Did you enjoy doing the practice today? Explain why or why not.

APPENDIX 2

ACTIVITY QUESTIONNAIRE 2

1. Did class seem long or short today? Explain.

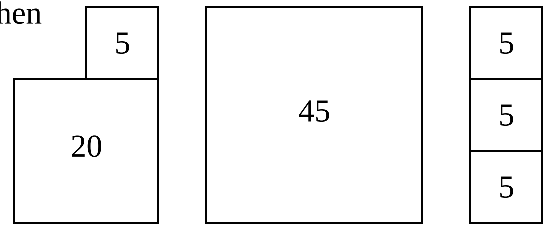
2. Did you feel like you were “practicing” mathematical concepts? Explain.

3. What did you like and dislike about today’s activity?

APPENDIX 3

PLAYING WITH SQUARES

The 45 cm^2 square is the exact same height as the two stacks of squares beside it. The squares in the stack on the left have areas of 5 cm^2 and 20 cm^2. Each of the three squares in the stack on the right has an area of 5 cm^2. For this assignment, the area of any square should be a natural number, when measured in square centimeters.



Stacking Squares

- Find stacks of squares that would be the exact same height as a square with area 72 cm^2.
- Do any squares exist that can have no stacks that are the exact same height? Explain.
- Explain how to find the stacks that would match a given square in height.

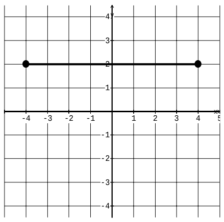
Add a Dimension

- How would these explorations work for cubes instead of squares?

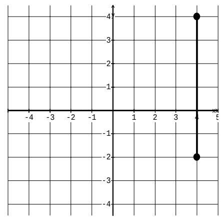
APPENDIX 4

DOMAIN AND RANGE INVESTIGATION

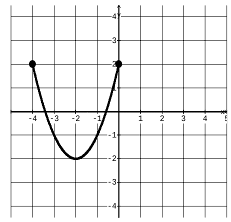
PART I: List all possible integer values for *x* and *y* in each graph below.

1.


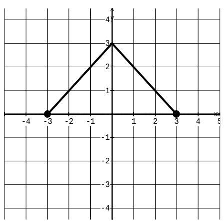
X: ________________
Y: ________________

2.


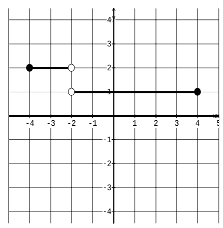
X: _________________
Y: _________________

3.


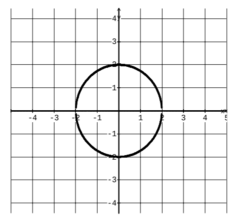
X: _________________
Y: _________________

4.


X: _________________
Y: _________________

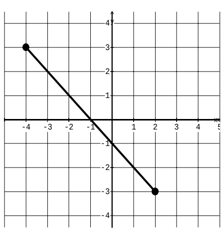
5.


X: _________________
Y: _________________

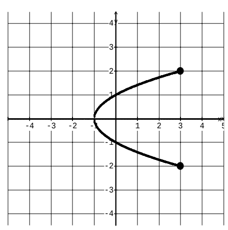
6.


X: _________________
Y: _________________

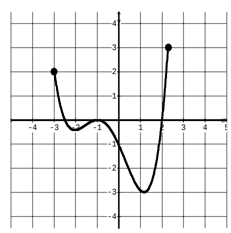
PART II: For each graph, list two possible values for *x* and *y* that are **NOT** integers.

7.


X: _______&_______
Y: _______&_______

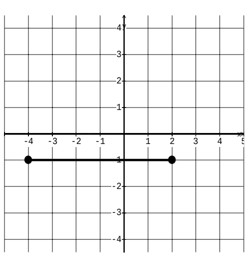
8.


X: _______&_______
Y: _______&_______

9.


X: _______&_______
Y: _______&_______

PART III: Given parts I and II above, how many total possible *x* and *y* values does the following graph have? Justify your answer and explain your reasoning below.

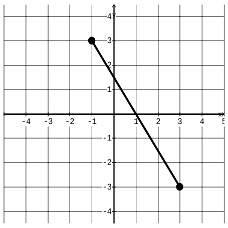
10.


____
____
____
____
____

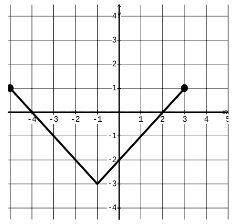
Can you think of a shorter way to show your solution?

____

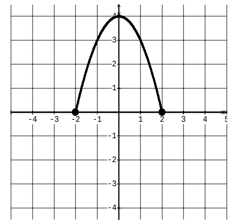
PART IV: List the smallest and largest possible values for both *x* and *y* in each graph below.

11. 

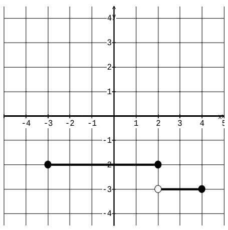
X: ______&______
Y: ______&______

12. 

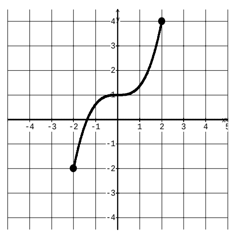
X: ______&______
Y: ______&______

13. 

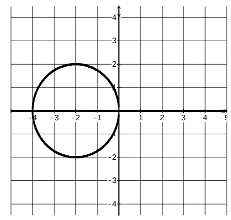
X: ______&______
Y: ______&______

14. 

X: ______&______
Y: ______&______

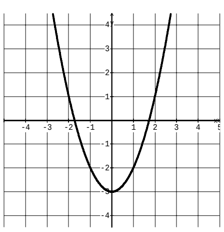
15. 

X: ______&______
Y: ______&______

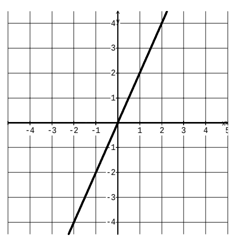
16. 

X: ______&______
Y: ______&______

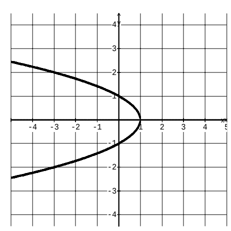
PART V: Find the smallest and largest possible values for both *x* and *y* in each graph below. How are these different than the examples from Part IV? Explain your answer below.

17. 

X: ______&______
Y: ______&______

18. 

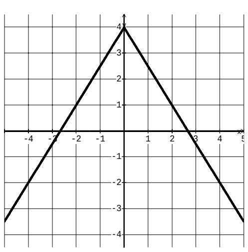
X: ______&______
Y: ______&______

19. 

X: ______&______
Y: ______&______

____
____

PART VI: Explain what all of the possible *x* and *y* values are for the following graph as well as <u>**HOW**</u> you know what they are.

20. 

____
____
____
____
____

How can you represent this solution using mathematical notation?

____

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