ANALYSIS OF VARIABLE BENEFIT PLANS

by

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a Project submitted in partial fulfillment
of the requirements for the degree of
Master of Science
in the Department
of
Statistics and Actuarial Science

SIMON FRASER UNIVERSITY

Summer 2010

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Degree: Master of Science

Title of Project: Analysis of Variable Benefit Plans

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Date Approved: July 08, 2010

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Abstract

The operational characteristics of a target benefit plan with fixed annual contributions and variable benefit accruals based on an aggregate funding requirement are studied both analytically and by simulation under the assumption of a constant valuation rate and log-normal returns. The distribution of the pension entitlement at retirement is compared under three different parameter sets for asset returns. The performance of the target benefit plan is then compared to a DC benchmark, both in terms of the proximity of the retirement benefits to the targeted benefit level and in terms of intergenerational equity. Finally, two practical modifications to the benefit policy are considered and their effect on performance is assessed.

Keywords: Pension Plan; Target Benefit Plan; Simulation; Intergenerational Equity; Gaussian Process; Aggregate Cost Method; Forecast Valuation Method

Acknowledgments

Returning to university ten years after completing my undergraduate degree has been an enormously rewarding but also challenging journey and I owe a deep debt of gratitude to a number of people who helped me navigate this path.

First and foremost, I wish to thank my supervisor, Dr. Gary Parker, who saw my potential and actively encouraged me to pursue a graduate degree. I am grateful for his patience, flexibility and gentle guidance both in my coursework and as I explored my research interests. His utmost professionalism combined with a rare kindness made the last two years a cherished experience for me, and his dedication to students and to actuarial science serves as an inspiration for my future pursuits.

I would also like to thank the members of my examining committe, Dr. Yi Lu and Dr. Cary Chi-Liang Tsai, for their valuable feedback on my work.

I am indebted to the faculty and staff of the Department of Statistics and Actuarial Science for the enormous support they provided throughout my graduate studies. I am especially grateful to the Chair of the Department, Dr. Richard Lockhart, and to Dr. Charmaine Dean for giving me equal measures of encouragement, assistance and inspiration.

I would like to thank my colleagues and fellow graduate students for their friendship over the last two years, with special mention to Jervyn Ang, Joslin Goh and Alisha Albert-Green for their generous assistance as I reclaimed faded memories of statistical theory and learned about exciting new technologies.

Last but not least, I wish to thank my mother for instilling in me an abiding love of learning, and my husband for honouring my curiosity and desire to learn. Without their love and support I would not have been able to come this far.

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Introduction

1.1 Variable Benefit Plans

Variable benefit plans are funded group pension arrangements where the benefit payable to plan members varies with plan experience. Variable benefit plan designs exist largely in pension plans maintained for employees of specific union groups and tend to cover participants working for several employers within a sector ("multi-employer plans"). One example of such a plan is the Pulp and Paper Industry Pension Plan which provides benefits to members of the Communications, Energy and Paperworkers Union of Canada and to members of Pulp, Paper and Woodworkers of Canada, who work for specific employers within BC.

In Canada, the name "negotiated cost pension plan" has also been used to describe variable benefit plans, reflecting the feature that employer contributions to such plans are usually set by collective bargaining (that is, they are "negotiated" between the employers and the union) and are fixed for the duration of the collective agreement. At the same time, benefits payable to plan members are calculated by reference to a formula that involves the member's years of service and possibly wages. Since the economic and demographic experience of the plan fluctuates each year yet contributions to the plan remain fixed, the benefit formula must be adjusted from time to time to balance liabilities with plan assets. As a result, the same level of benefits cannot be guaranteed for all cohorts of members.

Negotiated cost pension plans resemble traditional defined contribution plans in that the employer's obligation is limited to the annual contributions set in the collective agreement. Employers are not legally obligated to automatically make up any funding shortfalls that may arise: plan deficits are instead addressed by negotiating an increase in employer contributions for a limited time (presumably in exchange for concessions by the union in other areas) or by decreasing plan benefits. Unlike traditional defined benefit plans, negotiated cost plans may reduce not only future accruals but past accruals as well, although the trustees of such plans generally consider reduction of accrued benefits to be undesirable and will use it only as a measure of last resort.

Clearly, under these plans plan members bear all of the investment risk; however, this risk is pooled among all active (and possibly retired) members instead of being borne individually as is the case under a DC plan. To the extent that intergenerational transfers reduce the risk borne by each individual cohort, these plans may contribute to social welfare without placing undue burden on the employers.

In the hopes of balancing employers' and employees' needs in an era of volatile market returns, the Ontario Expert Commission on Pensions has recently advocated the creation of more such plans, which it called "jointly trusteed target benefit plans": "jointly trusteed" referring to a governance model where employees have significant representation, and "target benefit" referring to the combination of fixed contributions with a targeted (but not guaranteed) benefit formula. In addition, the recent report of the Alberta/BC Joint Expert Panel on Pension Standards encouraged the development of separate pension standards applicable to what they called "specified contribution target benefit plans", "where the contributions are fixed by collective agreement or a similar method but benefits are provided based on a formula". Both reports stressed the importance of helping plan members and fiduciaries understand the risks taken, and called on the actuarial profession to develop more sophisticated tools and methods that facilitate the appropriate funding and management of these plans. This project contributes to the work called for by the Ontario Commission and the Joint Panel by exploring, through simulation, the risk characteristics of target benefit plans from the member's point of view and by identifying the shortcomings of certain simple designs over the long term.

It should be noted that while the term "target benefit plan" has come to describe this particular plan design in a Canadian context, it refers to an entirely different plan design in the United States. Although the use of the same term for two different concepts is unfortunate, we will continue to use the emerging Canadian terminology here.

1.2 Literature Review

Actuarial literature is rather scarce on the topic of target benefit plans. Kryvicky's 1981 paper provides an exhaustive exploration of funding issues concerning negotiated cost plans. Employing a wide range of deterministic projections, he investigated the key drivers of contribution and benefit levels. Curiously, there has been little discussion of negotiated cost plans in the actuarial literature since Kryvicky's early work despite significant advances in the stochastic modeling of both traditional defined contribution and defined benefit plans over the last two decades. Much of the work done to date regarding defined benefit plans, such as the derivation of the moments of the fund value and the contribution rate (Dufresne (1989)), the identification of optimal funding strategies (Huang and Cairns (2006)) and optimal strategies for the amortization of gains and losses (Cairns and Parker (1997), Haberman (1994)) would be worth translating to the target benefit plan context; however it is not trivial to do so. One of the basic assumptions in the defined benefit literature is that benefit payments expressed in constant dollars do not change, which is clearly violated in target benefit plans, necessitating new analytical approaches.

In the meantime, the concept of a "collective defined contribution plan" that pools investment risk across an infinite number of cohorts has made its appearance in the economics literature. In his 2008 paper, Gollier derived optimal investment strategies in a two-asset universe for a social security system following a target benefit plan design using utility theory. In the same work, he also quantified the impact of intergenerational transfers in such a plan relative to an optimally-invested defined contribution plan benchmark. Our work differs in two ways: we discard utility theory in favour of more traditional performance measures used in practice, and we focus more on the implications of benefit policy choices than on the asset mix selected.

1.3 Outline

Chapter 2 provides a brief primer on the operation of target benefit plans ("TBPs") with a strong emphasis on the benefit policy. A simplified target benefit plan design is then proposed for study and the performance criteria are identified.

Chapter 3 sets out the stochastic model for the analysis of this simplified target benefit plan. A rudimentary asset model consisting of a single fund with a constant proportion of fixed income and equity holdings is defined and calibrated. The benefit policy is then translated into mathematical terms.

In Chapter 4 a recursive relationship for the annual benefit accrual is derived in terms of the target benefit and the investment gain/loss each year, and this relationship is exploited to estimate the distribution of the annual accruals by simulation.

In Chapter 5 the simulation is extended to the pension entitlement at retirement and the distribution of the retirement pension is estimated under three different parameter sets. The performance of the target benefit plan is then compared to a DC benchmark.

Chapter 6 describes the impact of two modifications to the simple TBP design proposed in Chapter 2: the addition of a provision for adverse deviations and the use of an open group instead of a closed group valuation basis for setting the benefit each year.

Finally, our findings are summarized in Chapter 7.

Chapter 2

Operation of Target Benefit Plans

2.1 Key Features of TBPs in Practice

Target benefit plans make up a "substantial portion of the membership of private sector plans in Alberta and British Columbia" (Canada, Alberta/British Columbia (2008)) and equivalent designs are found in several large multi-employer plans in Ontario as well (Canada, Ontario (2008)).

The most prominent feature of such plans is fixed employer contributions each year (in dollar terms, or as a percentage of pay) although increases may be negotiated from time to time. Contributions in respect of all members are commingled in a pension trust fund and are invested in a combination of fixed income and equity vehicles, with bond holdings as high as 70 or 80 percent being common. Often, the target level of benefits is not explicitly identified in the plan documents but may be interpreted as the benefits promised at plan inception. Current benefits are often expressed as an annual pension starting at retirement, calculated according to a formula that multiplies a member's years of service by either a flat dollar amount (e.g. \$150) or a percentage of the member's wages (e.g. 2%).

The actual benefit level fluctuates with plan experience. At each valuation date, the plan actuary assesses whether the negotiated contributions can support benefits at the currently promised level by comparing plan assets with plan liabilities. Plan liabilities are generally valued using both an accrued benefit method (either traditional unit credit or projected unit credit, depending on whether the benefits are wage-linked or not) and an aggregate method which takes into account both past and future contributions and benefit accruals. Valuations are performed on both going-concern and solvency bases; however, target benefit plans have recently benefited from temporary moratoria on solvency funding requirements in several Canadian jurisdictions (Shilton (2007)).

If the valuations reveal a surplus, the plan trustees may continue to provide benefits at the current level and leave the surplus in the plan as a "cushion" to protect the fund from adverse experience in the future. Alternatively, the trustees may choose to spend some or all of the surplus by increasing the benefit. If the plan has a deficit, current pension regulations require that action be taken to address the shortfall over a reasonable period of time, usually 15 years or less (Shilton (2007)). This action may include increasing contributions, reducing benefits, or a combination of these two. It should be noted that, since the employer has no legal obligation to fund any deficits in these plans, increases in contributions must be explicitly negotiated each time between the employer and the employees.

The adequacy of the plan assets and contributions to support the current level of benefits must be assessed by law at least once every three years, although valuations and projections are conducted for plan trustees' internal use more frequently. The statutory filings are based on deterministic valuations. In order to monitor the risk exposure of plan members, the statutory valuation is generally supplemented by deterministic scenario testing and possibly by stochastic analyses, but the results of these are not required to be shared with regulators.

2.2 Benefit Policy

As noted above, if the plan actuary determines that the current level of benefits is not sustainable and a sufficient increase in contributions cannot be negotiated, plan benefits will need to be reduced. On the other hand, if the plan develops a significant surplus, benefits may be increased. In either case, plan trustees have considerable flexibility in determining exactly how the benefits are to be adjusted without much statutory guidance or limitations. In other words, they are in full control of the benefit policy. Some of the key decisions to be made by plan trustees when contemplating the adjustment of benefits are:

- **The number of cohorts of new entrants to take into account in the aggregate valuation when assessing the sustainability of benefits.**

  If no new cohorts are taken into account, this is equivalent to performing a closed group valuation, comparing the current level of the fund plus the present value of future contribution to be made by *members who are in the plan as of the valuation date* against the present value of all future benefits payable to those same members. At the other extreme, one may consider all future cohorts, comparing the current level of the fund plus the present value of future contributions to be made by *all current and future members* (essentially in perpetuity) against the present value of all future benefits payable to such members. It should be noted that open group valuations based on an infinite number of cohorts are an interesting theoretical construct, but they are not frequently used, possibly due to regulators' preference for completing the funding of a particular benefit level in finite time. In practice, the number of future cohorts considered in statutory valuations of target benefit plans is around ten.

- **The group of members to whom the adjusted benefit applies.**

  This could include active members only, or both active and retired members.

- **The years of service to which the adjustment applies.**

  This could be future service only, past service only, or both past and future service. Note that reducing *future* accruals is an option available to all plans, including traditional DB plans. Furthermore, DB plans may reduce *past service* benefits if plan assets are not sufficient on plan wind-up. However, the reduction of past service benefits while the plan is ongoing is only available to target benefit plans.

- **The imposition of a floor and/or cap on the benefit payable.**

  Imposing a floor is equivalent to guaranteeing a minimum benefit, which tends to increase plan costs by necessitating the infusion of cash in excess of the fixed contributions from time to time. The timing and size of these cash infusions is a function of the investment policy of the plan and the level at which the floor is set relative to the target benefit. With a floor that is close to the target benefit and an investment policy that creates significant mismatch risk between the plan assets and liabilities, the pattern of total cash contributions (being the sum of the fixed annual contributions and the random additional cash infusions) may be just as unattractive to employers as the contribution obligations under a traditional DB plan.

  By contrast, imposing a cap means designating a maximum benefit and refraining from spending any additional surplus on further benefit improvements. In theory such a provision may reduce the employer's long term cost, especially if excess funds can be withdrawn from time to time. However, due to current legislative barriers to the withdrawal of surplus from ongoing plans, the resulting excess funds could, in practice, be "trapped" in the pension plan.

- **The establishment and maintenance of an explicit contingency fund to soften the impact of adverse plan experience.**

  Pension actuarial standards in Canada require the inclusion of a provision for adverse deviations in all funding valuations, including those for both traditional defined benefit plans and target benefit plans. Generally accepted actuarial practice with respect to such a provision has been to increase plan liabilities by using a valuation interest rate assumption that is lower than the actuary's "best estimate" assumption by some small margin (0.5%-1.0%, depending on the mismatch between plan assets and liabilities). The plan's funded status is then determined by comparing plan assets to the resulting higher, more conservative liability. While all funding valuations include such a provision, few actuaries quantify and explicitly report the resulting "cushion" in the plan liabilities. Several commentators have criticized this practice as not being sufficiently transparent. In the case of target benefit plans, where an understanding of the risks facing the plan and of the measures taken to mitigate those risks are of paramount significance, such transparency is indispensable. As a result, the report of the Alberta/BC Joint Expert Panel on Pension Standards included several recommendations regarding the determination, use and reporting of the provision for adverse deviations in target benefit plans (Canada, Alberta/British Columbia (2008), pp. 148-150), calling on the actuarial profession to develop standards in this important area. Until such standards are put into practice (or until regulators impose their own requirements), actuaries and plan trustees have considerable flexibility in determining the appropriate size and ideal operation of their contingency funds.

The trustees often consult the plan actuary to quantify the impact of these decisions. While there is no one-size-fits-all solution and the options selected will always reflect the specific group's needs and preferences, certain objectives are generally shared by all trustees of target benefit plans. These objectives can be summarized as follows:

1. Maintain a reasonable expectation of receiving benefits at or near the target level.

2. Maintain intergenerational equity. This means balancing the interests of currently active and retired members, and weighing these against the interests of future cohorts of new entrants. Ideally, intergenerational transfers of risk would result in a reduction in the volatility of benefits for all cohorts. However, it is worth noting that Gollier's findings in an idealized two-asset universe with infinite horizon indicate that the primary benefit of pooling investment risks among cohorts may be manifested through a higher expected benefit payment rather than through a reduction in variability of benefits (Gollier (2008)).

3. Minimize fluctuations in benefits to retired members. Due to their longer investment horizons and generally higher income, active plan members are often viewed as being able to carry more risk with respect to their employer-funded pensions than retired members who have limited time to make up for losses and usually rely on fixed pension income. As a result, trustees try to avoid substantial reductions in benefits already in payment and focus instead on adjusting the benefit accruals of active members, whenever possible.

4. Minimize the probability of ruin, that is, the probability of the fund being exhausted prematurely.

Note that objectives 2 and 3 are in conflict: keeping retired members' benefits stable means that the investment and other risks associated with these members are transferred wholly or in part to active members. Given the demographic profile of mature pension plans where retired members may account for over 50% of plan liabilities, this can represent a substantial risk transfer. Active members may of course willingly accept this additional risk, since they can expect to benefit from a similar transfer of risks to new generations of active members once they retire. It should, however, be recognized that the only true way to keep retired members' benefits unchanged without significant intergenerational transfers is to remove these liabilities from the plan altogether; that is, annuitize benefits at retirement.

2.3 Example of a Simple TBP

We now describe a simple target benefit plan to be studied.

- Without loss of generality, we set the target benefit (expressed as an annual pension at retirement) to be $100 times the member's years of service.

- Annual contributions in respect of each active member are determined as the level cost of the target benefit for a single member over his working life, that is, the member's level-dollar Entry Age Normal cost.

- Contributions in respect of each active member are deposited in the pension fund, which is invested in a combination of Canadian long bonds (80%) and Canadian equities (20%). The assets of all members are commingled and individual accounts are not maintained.

- Valuations are performed annually to calculate the benefit accrual rate that can be afforded by the plan, taking into account plan experience during the previous year. The valuation method used is the Aggregate method: the benefit accruals applicable to the most recent year and all future years are set so that the sum of the fund and the present value of future contributions to the plan are exactly equal to the sum of the accrued benefits and the present value of future benefits.

- Each valuation is conducted on a closed group basis (i.e. only the members in the plan as of the valuation date are taken into consideration and no allowance is made for future new entrants).

- The actual benefit accruing each year is the same for all active members participating in the plan during that year.

- The annual pension a member is entitled to receive commencing on her retirement date is equal to the sum of the benefit accruals during each year of the member's active service. Consequently, all retirees in the same cohort receive the same benefit, whereas retirees in different cohorts will receive different benefits.

- Benefits payable to a member become fixed as of the member's date of retirement (i.e. retired members are completely insulated from plan experience).

- In order to avoid massive intergenerational transfers, all benefits are annuitized at retirement.

Note that using this approach, past years' benefit accruals are not changed explicitly; however, future benefit accruals may become negative which is equivalent to a reduction of the accrued benefits.

In practice, benefits are not adjusted quite so frequently or as rigidly. In fact, the procedure to be followed when considering changes to benefits may range from being defined in complete detail in a legally binding document to being completely ad-hoc; however, even ad-hoc adjustments should be consistent in their application over time.

Therefore, it is instructive to explore the operation of a well-defined (albeit simplified) benefit adjustment mechanism.

2.4 Performance Criteria

Our goal is to assess how well this particular design achieves the common objectives listed in section 2.2 by looking at:

- The distribution of retirement benefits payable to each cohort over a 200-year horizon relative to the target benefit.
- The difference in the distribution of retirement benefits payable to cohorts 1, 10, 35 and 100 years apart.
- The probability of ruin over the 200-year horizon.

Given that target benefit plans are multi-cohort equivalents of traditional DC plans, we compare the performance of our simplified target benefit plan to a DC benchmark based on the same fixed annual contributions.

Chapter 3

Model and Assumptions

3.1 Modeling Plan Assets

3.1.1 Assumptions and Methodology

We employ a simple stochastic model for asset returns. We denote the interest rate earned on the pension fund during the period $[t, t+1)$ by i_{t+1}. The corresponding force of interest is assumed to be constant over $[t, t+1)$ and is denoted by δ_{t+1}. We assume that this force of interest follows a discrete autoregressive process of order one centered on some value μ, so that the force of interest applicable to the fund during the period $[t, t+1)$ can be expressed as

$$\delta_{t+1} = \mu + \phi(\delta_t - \mu) + \sigma\epsilon_{t+1}, \quad t = 0, 1, 2, ..., \tag{3.1}$$

where ϕ is the autoregressive coefficient, σ is the local standard deviation of the process, and the ϵ's are independent standard normal random variables. We choose the initial value $\delta_0 = \mu$ so that the force of interest applicable over the first year is simply $\delta_1 = \mu + \sigma\epsilon_1$.

When $\phi = 0$, the force of interest reduces to a White Noise process centered on μ:

$$\delta_{t+1} = \mu + \sigma\epsilon_{t+1}. \tag{3.2}$$

Using these assumptions implies that the force of interest is normally distributed, which in turn means that the annual rate of return on the fund, $i_t = e^{\delta_t} - 1$, follows a shifted lognormal distribution.

Note that instead of modeling the force of interest applicable to the bond portion and the equity portion of the portfolio separately, we use only a single random variable for the force of interest on the combined portfolio here. More sophisticated approaches incorporating multivariate time series models are often applied when part of the research objective is to find an optimal portfolio: see MacDonald and Cairns (2006) for a DC plan example, or Gollier (2008) for an example in a collective DC plan context. However, in our case the asset allocation is given and we are not searching for an optimal portfolio, therefore the use of the simpler, univariate model is justified.

In fact, our simple model has many precedents, especially among studies of DB pension schemes. Both Dufresne (1989) and Haberman (1992) have modeled the rate of return i_t on DB plan assets using a single i.i.d. random variable. Subsequently, Haberman (1994) considered first and second-order discrete autoregressive processes for the force of interest on a DB fund. Cairns and Parker (1997) incorporated both of these approaches, first deriving results using the i.i.d. assumption for i_t, and then deriving new results by applying the AR(1) assumption to δ_t. We continue in this tradition here and consider both the autoregressive model of order one and its special case with $\phi = 0$.

3.1.2 Model Estimation

To calibrate our asset model, we start with the total annual rates of return on Canadian long bonds (i_t^{LB}) and on Canadian equities (i_t^E) for the years 1960-2009 published by the Canadian Institute of Actuaries. For each year t in our sample, we calculate the rate of return on our hypothetical portfolio using the relationship

$$i_t^P = 0.8 \cdot i_t^{LB} + 0.2 \cdot i_t^E \tag{3.3}$$

and we convert the resulting rate of return to a constant force of interest applicable during the year. The original data set is reproduced in Appendix A. The equivalent continuously compounded returns for the constructed portfolio are shown in Figure 3.1.

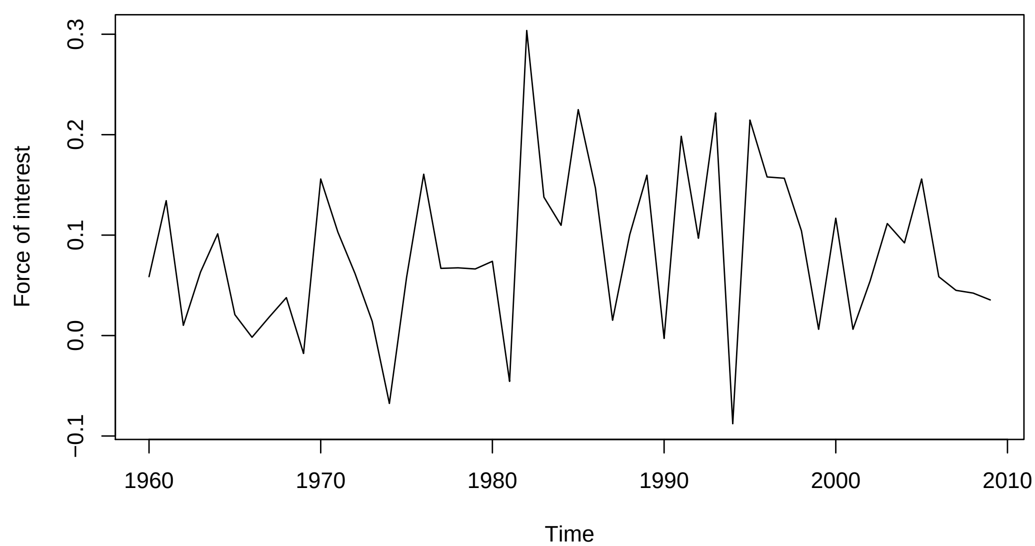


Figure 3.1: Investment returns on constructed portfolio

We then use the constructed force of interest series to calibrate our AR(1) model, applying the Yule-Walker method (Box and Jenkins (1976), p. 243) to estimate μ, ϕ and σ. The resulting parameter estimates are summarized in Table 3.1.

Table 3.1: Empirical estimates of asset model parameters

Parameter	Estimated Value	Standard Error
μ	0.083	0.010
ϕ	-0.094	0.140
σ	0.078	n/a

Note that the estimate of ϕ is statistically indistinguishable from zero, so the most parsimonious choice is in fact a White Noise model. Consequently, the base parameter set used in our asset model is $\mu = 0.08, \sigma = 0.08, \phi = 0$.

We also consider other, nonzero values for ϕ to explore whether autocorrelated returns result in better or worse TBP performance. Specifically, we construct two additional parameter sets: a low-correlation set fixing ϕ at 0.2 (ϕ_{low}) and a high-correlation set fixing ϕ at 0.6 (ϕ_{high}). We then calculate the corresponding local volatility parameters (σ_{low} and σ_{high}) which satisfy the condition that the long-term volatility of each of the resulting processes be equal. The long-term volatility of the White Noise process is simply equal to its local volatility, whereas the long-term volatility of an AR(1) process with parameters (μ, ϕ, σ) is

$$\sigma_{LT}^2 = \frac{\sigma^2}{1-\phi^2}. \tag{3.4}$$

The resulting parameter sets are summarized in Table 3.2.

Table 3.2: Parameter sets chosen for asset model

	μ	ϕ	σ
Base case	0.08	0	0.080
Low correlation	0.08	0.2	0.078
High correlation	0.08	0.6	0.064

3.2 Other Model Assumptions

In our model, we also make the following important assumptions:

1. All actuarial assumptions are realized exactly, except for investment returns.

2. Contributions are received and benefit payments are made at the beginning of the year.

3. The interest rate assumption used for valuing the present value of benefits (known as the "valuation rate") is fixed. In the discussion that follows, this rate is expressed as a force of interest denoted by δ.

4. $E[1 + i_t] = E[e^{\delta_t}] = e^{\delta}$, meaning the average accumulation of the fund over a 1-year period is equal to the accumulation based on the valuation rate δ.

5. The same number of new entrants join the plan each year.

6. Members join the plan at age 30 and retire at age 65. There are no decrements prior to age 65.

7. The benefit payment made to each retiring member at age 65 is the actuarial present value of the member's benefits accrued over the previous 35 years, where the present value is calculated using the constant valuation rate δ and the 1983 Group Annuity Mortality table (GAM83).

Assumptions 1-4 are often made in the study of DB plans, whereas Assumption 5 is somewhat weaker than usual. Given no preretirement decrements and a service length of 35 years, a constant number of new entrants (as required by Assumption 5) will ensure that the plan membership reaches a stationary state after at most 35 years no matter what the initial demographic profile is; however, traditionally stationarity is required from the outset. We will refer to this stronger, traditional assumption of a population that is stationary from plan inception as Assumption 5a.

3.3 Notation

At this point, we introduce some notation to accommodate both the specific features of our simple TBP and the assumption of an initially unstable demographic profile. We let:

- b^* be the target annual benefit accrual, that is, \$100,
- C be the fixed annual employer contribution in respect of each active member,
- $n(x,t)$ be the number of members aged x in the plan at time t, including members of the newest cohort that enter at time t, $x \geq 30$, $t \geq 0$.

Under Assumption 5, we have $n(30,t) = n$ for $t = 0, 1, 2, ...$ Under Assumption 5a this becomes $n(x,t) = n(30,t) = n$ for $30 \leq x \leq 65$, $t = 0, 1, 2, ...$ Our variables of interest are:

- $F(t)$, the market value of plan assets at time t,
- $B(t)$, the annual benefit accruing to each active member during the period $[t-1,t)$ determined based on the valuation at time t, and
- $Pen(t)$, the annual pension entitlement of a member retiring immediately after time t.

We also define:

- $TPmt(t)$ as the total benefit payments made from the plan to retiring members immediately after time t,
- $PVAB(t)$ as the actuarial present value of benefits accrued by all members as of time t.
- $TCon(t)$ as the total plan contributions received immediately after time t, and
- $PVFC(t)$ as the present value at time t of all future contributions to the plan in respect of currently active members.

Mathematically, the last five variables can be expressed as follows:

$$Pen(t) = \sum_{i=0}^{34} B(t-i), \tag{3.5}$$

$$TPmt(t) = n(65,t) \cdot Pen(t) \cdot \ddot{a}_{65} = n(65,t) \cdot \ddot{a}_{65} \cdot \sum_{i=0}^{34} B(t-i), \tag{3.6}$$

$$PVAB(t) = \sum_{x=31}^{65} n(x,t) \cdot \left(\sum_{i=0}^{x-31} B(t-i) \right) {}_{65-x|}\ddot{a}_x, \tag{3.7}$$

$$TCon(t) = C \cdot \sum_{x=30}^{64} n(x,t) \tag{3.8}$$

and

$$PVFC(t) = C \cdot \sum_{x=30}^{64} n(x,t) \cdot \ddot{a}_{\overline{65-x}|}. \tag{3.9}$$

Note that when we start the plan with more than one cohort present at $t = 0$, equations (3.5) to (3.7) require $B(t)$ to be defined for $t < 1$. If the plan provides benefits only for members' service after the plan inception date (referred to as prospective implementation) then $B(t) = 0$ for $t < 1$. On the other hand, if the plan provides past service credits (retrospective implementation) then it might be reasonable to set these credits at the target benefit level so that $B(t) = b^*$ for $t < 1$.

Finally, in section 2.3 we stated that the annual contribution in respect of each member is equal to the level-dollar Entry Age Normal cost of the target benefit. Therefore, we have:

$$C = \frac{35b^*{}_{35|}\ddot{a}_{30}}{\ddot{a}_{\overline{35}|}}. \tag{3.10}$$

3.4 The Benefit Policy Equation

The benefit policy laid out in section 2.3 requires that benefit accruals be set at each valuation date so as to satisfy an aggregate funding requirement. Specifically, at each time t, the current value of the fund plus the present value of future contributions to the plan must equal the present value of benefit payments to be made from the plan. The benefit payments to be made from the plan have three components:

1. the benefits that have already accrued in respect of service up to time $t-1$ and which remain to be paid as of time t;
2. the benefit accrual in respect of the period $[t-1, t)$; and
3. the benefits to be accrued in respect of future service after time t.

Denote the present value of these components as of time t by $PVFB_1(t)$, $PVFB_2(t)$ and $PVFB_3(t)$, respectively. Then our benefit policy requires that at each time $t = 1, 2, ...$

$$F(t) + PVFC(t) = PVFB_1(t) + PVFB_2(t) + PVFB_3(t). \tag{3.11}$$

Let us look at each of $PVFB_1(t)$, $PVFB_2(t)$ and $PVFB_3(t)$ in detail. $PVFB_1(t)$ is the present value at time t of benefits accrued by members present in the plan as of time t in respect of service up to time $t-1$, that is

$$PVFB_1(t) = \sum_{x=32}^{65} n(x,t) \cdot \left(\sum_{i=1}^{x-31} B(t-i) \right) {}_{65-x|}\ddot{a}_x.$$

In the absence of pre-retirement decrements $n(x,t) = n(x-1,t-1)$ for $31 < x \leq 65$ and ${}_{65-x|}\ddot{a}_x = {}_{65-x+1|}\ddot{a}_{x-1} \cdot e^{\delta}$. Therefore,

$$\begin{aligned}
PVFB_1(t) &= \sum_{x=32}^{65} n(x-1,t-1) \cdot \left(\sum_{i=1}^{x-31} B(t-1-(i-1)) \right) {}_{65-x+1|}\ddot{a}_{x-1} \cdot e^{\delta} \\
&= \sum_{u=31}^{64} n(u,t-1) \cdot \left(\sum_{j=0}^{u-31} B(t-1-j) \right) {}_{65-u|}\ddot{a}_u \cdot e^{\delta} \\
&= \sum_{u=31}^{65} n(u,t-1) \cdot \left(\sum_{j=0}^{u-31} B(t-1-j) \right) {}_{65-u|}\ddot{a}_u \cdot e^{\delta} \\
&\quad - n(65,t-1) \cdot \left(\sum_{j=0}^{34} B(t-1-j) \right) \ddot{a}_{65} \cdot e^{\delta} \\
&= PVAB(t-1) \cdot e^{\delta} - TPmt(t-1) \cdot e^{\delta}. \qquad (3.12)
\end{aligned}$$

$PVFB_2(t)$ is the present value at time t of benefits accrued in respect of the period $[t-1,t)$. That is,

$$PVFB_2(t) = \sum_{x=31}^{65} n(x,t) \cdot B(t) {}_{65-x|}\ddot{a}_x. \qquad (3.13)$$

$PVFB_3(t)$ is the present value at time t of benefits to be accrued in respect of future service. The choice of the future benefit accrual rate to be applied here has some important consequences. One reasonable practice might be to set all future accruals after time t to be the same as the accrual rate applicable during $[t-1,t)$. The rationale for this approach is that, should investment experience exactly match the assumptions, the members present in the plan at time t could continue to support the most recent benefit accrual rate $B(t)$ for the remainder of their working lives without relying on the contributions of new entrants. Using this approach,

$$PVFB_3(t) = \sum_{x=30}^{64} n(x,t) \cdot (65-x) \cdot B(t) \cdot {}_{65-x|}\ddot{a}_x. \tag{3.14}$$

Applying equations (3.12) – (3.14), equation (3.11) can be rewritten as

$$\begin{aligned} F(t) + PVFC(t) =& PVAB(t-1) \cdot e^{\delta} - TPmt(t-1) \cdot e^{\delta} \\ &+ n(30,t) \cdot 35 \cdot B(t) \cdot {}_{35|}\ddot{a}_{30} \\ &+ \sum_{x=31}^{65} n(x,t) \cdot (65-x+1) \cdot B(t) \cdot {}_{65-x|}\ddot{a}_x. \end{aligned} \tag{3.15}$$

The formula for $B(t)$, as determined based on plan experience up to time t, then emerges as:

$$B(t) = \frac{F(t) + PVFC(t) - PVAB(t-1) \cdot e^{\delta} + TPmt(t-1) \cdot e^{\delta}}{d(t)}, \tag{3.16}$$

where

$$d(t) = n \cdot 35 \cdot {}_{35|}\ddot{a}_{30} + \sum_{x=31}^{65} n(x,t) \cdot (65-x+1) \cdot {}_{65-x|}\ddot{a}_x$$

is a weighted sum of deferred annuity factors that determines how long a period the impact of each year's plan experience is spread out over. In the absence of pre-retirement decrements,

$$d(t) = n \cdot 35 \cdot \ddot{a}_{65} \cdot e^{-35\delta} + \ddot{a}_{65} \sum_{x=31}^{65} n(x,t)(65-x+1)e^{-(65-x)\delta}.$$

We will refer to equation (3.16) as the **benefit policy equation** and to $d(t)$ as the **spread factor**.

Chapter 4

Annual Benefit Accruals

Before we can begin to examine the distribution of $Pen(t)$, it is instructive to look at the distribution of its building blocks, the annual benefit accruals. Looking at equation (3.16), we would expect the benefit accrual rates applicable to consecutive years t and $t+1$ to be highly correlated since the values they are based on are highly correlated themselves. To understand this correlation structure better, we first derive an exact recursive relationship for $B(t)$.

4.1 Recursive Form

We start by writing the recursive equations applicable to each of the key variables in equation (3.16). For the market value of plan assets, assuming contributions and benefit payments are made at the beginning of the year, we have:

$$F(t+1) = [F(t) + TCon(t) - TPmt(t)] \cdot e^{\delta_{t+1}}, \tag{4.1}$$

where plan assets accumulate during $[t, t+1)$ at a rate of δ_{t+1} compounding continuously. For the present value of future contributions, we have:

$$PVFC(t+1) = [PVFC(t) - TCon(t)] \cdot e^{\delta} + C \cdot n(30, t+1) \cdot \ddot{a}_{\overline{35|}}. \tag{4.2}$$

Finally, the recursion equation for the present value of accrued benefits takes the form

$$PVAB(t+1) = [PVAB(t) - TPmt(t)] \cdot e^{\delta} + PVFB_2(t+1), \tag{4.3}$$

where $PVFB_2(t+1)$ is the present value of new benefits being earned by active members during $[t, t+1)$, as defined by equation (3.13). Detailed derivations of the recursion equations for $PVFC(t)$ and $PVAB(t)$ are shown in Appendices C and D. Using equations (4.1) – (4.3), we get the following expression for the benefit accrual rate:

$$B(t+1) = B(t) + \frac{[F(t) + TCon(t) - TPmt(t)]\,(e^{\delta_{t+1}} - e^{\delta})}{d(t+1)} + \frac{[b^* - B(t)]\,d(0)}{d(t+1)}. \tag{4.4}$$

The numerator of the second term in equation (4.4) is the difference between the actual investment return on the fund during the period $[t, t+1)$ and the anticipated return based on the valuation rate δ, that is, the investment gain relative to the valuation rate. We denote this random quantity by $G(t+1)$:

$$G(t+1) = [F(t) + TCon(t) - TPmt(t)]\,(e^{\delta_{t+1}} - e^{\delta}), \quad t = 0, 1, 2, ...$$

Then equation (4.4) can be written as:

$$B(t+1) = B(t) + \frac{G(t+1)}{d(t+1)} + \frac{[b^* - B(t)] \cdot d(0)}{d(t+1)}. \tag{4.4a}$$

Note that a positive value of $G(\cdot)$ means that actual returns exceeded assumed returns during the year, that is, the plan had an investment gain relative to our assumptions. This gain translates into an increase in the benefit accrual rate. A negative value of $G(\cdot)$ signifies an investment loss relative to our assumptions and has the effect of reducing the benefit accrual rate.

The third term in equation (4.4a) is an adjustment for new cohorts entering the plan at time $t+1$. It is expected that over the 35 years of each new cohort's working life, the employer will make annual contributions of C for each member in that cohort. Note that the contribution rate was set at a level that is sufficient to fund a target benefit of b^* per year of service over each member's working life. If the benefit accrual rate that is in force when the new cohort enters is lower than the target benefit accrual b^*, the future contributions to be made by the employer in respect of these new members will "overfund" their benefit. This overfunding (which is certain to materialize if investment assumptions are met over the next 35 years) represents a hitherto unaccounted-for revenue stream which translates into higher benefit accruals for all cohorts present.

Similarly, if a new cohort enters the plan when the benefit accrual rate happens to be higher than b^*, the contributions to be made in respect of this new cohort will not be sufficient to fund their benefit accruals. The anticipated underfunding of this cohort's benefits by the employer will be a drain on plan assets shared by all current members and will translate into a decrease in the benefit accruals for all cohorts present at time $t+1$.

Finally, if a new cohort enters when the benefit accrual rate in force happens to be exactly b^*, this new cohort's future benefits will be exactly funded by the employer's future annual contributions, so these members represent neither a drain on, nor a boon to, existing members. Consequently, no change will be made to the benefit accrual rate.

Next, we rewrite equation (4.4a) in the form of a general autoregressive process of order one fluctuating around some long-term mean value. Subtracting b^* from both sides and letting $\rho(t+1) = 1 - d(0)/d(t+1)$, we get

$$B(t+1) - b^* = \rho(t+1)\left[B(t) - b^*\right] + \frac{G(t+1)}{d(t+1)}. \tag{4.5}$$

After 35 years the plan membership reaches a stationary state, so that $d(t+1) = d(35) = d$ and $\rho(t+1) = \rho(35) = \rho$ for $t \geq 35$. At this point, equation (4.5) becomes

$$B(t+1) - b^* = \rho\left[B(t) - b^*\right] + \frac{G(t+1)}{d}, \quad t \geq 35. \tag{4.5a}$$

The behaviour of $B(t)$ is therefore driven by two components: the autocorrelation coefficient ρ and the behaviour of the random sequence $G(t+1)/d$, whose terms function here as the "innovations" or "random shocks". For reasonable values of δ used in practice (between 4% and 12%), we have $0.9 < \rho < 1$, which, in the context of traditional autoregressive models (assuming i.i.d. normal innovations), would be sufficient to ensure that the distribution of $B(t)$ converged to some stationary state. If the innovations violated either the normality or the independence assumption, the distribution of $B(t)$ would be much more difficult to characterize analytically. We therefore examine the distribution of $G(t)$ next.

4.2 Distribution of the Investment Gain

The investment gain on the fund in respect of the period $[t-1, t)$ relative to the valuation assumption δ is G(t):

$$G(t) = \left[F(t-1) + TCon(t-1) - TPmt(t-1)\right](e^{\delta_t} - e^{\delta}), \quad t = 1, 2, ...$$

Suppose the δ_t are i.i.d. normal random variables with mean μ and variance σ^2, and the valuation rate δ is set to $\mu + \sigma^2/2$ according to Assumption 4. Let H_{t-1} denote the history of the plan up to and including time $t-1$. Specifically, H_{t-1} includes information about the values of the random series $\{F\}$, $\{B\}$, $\{TCon\}$, $\{TPmt\}$ and $\{\delta\}$. We can obtain the expected value of $G(t)$ by first conditioning on H_{t-1} and taking expectations as follows:

$$
\begin{aligned}
E[G(t)] &= E\left\{E\left[G(t)|H_{t-1}\right]\right\} \\
&= E\left\{[F(t-1) + TCon(t-1) - TPmt(t-1)] \cdot E\left[e^{\delta_t} - e^{\delta}|H_{t-1}\right]\right\} \\
&= E\left\{[F(t-1) + TCon(t-1) - TPmt(t-1)] \cdot 0\right\} \\
&= 0.
\end{aligned}
$$

Using the same argument, the covariance between $G(t)$ and $G(t+h)$ can be shown to be zero whenever $h > 0$ under the white noise assumption for δ_t,

$$
\begin{aligned}
Cov[G(t), G(t+h)] &= E[G(t)G(t+h)] \\
&= E\left\{E\left[G(t)G(t+h)|H_{t+h-1}\right]\right\} \\
&= E\Bigg\{G(t) \cdot [F(t+h-1) + TCon(t+h-1) - TPmt(t+h-1)] \\
&\quad \cdot E\left[e^{\delta_{t+h}} - e^{\delta}|H_{t+h-1}\right]\Bigg\} \\
&= 0.
\end{aligned}
$$

By contrast, obtaining the variance of $G(t)$ by analytical means is not a trivial exercise:

$$\begin{aligned} Var[G(t)] &= Var\left\{E\left[G(t)|H_{t-1}\right]\right\} + E\left\{Var\left[G(t)|H_{t-1}\right]\right\} \\ &= 0 + E\left\{\left[F(t-1) + TCon(t-1) - TPmt(t-1)\right]^2 \cdot \left(e^{\sigma^2} - 1\right) e^{2\mu+\sigma^2}\right\} \\ &= \left(e^{\sigma^2} - 1\right) e^{2\mu+\sigma^2} \cdot E\left\{\left[F(t-1) + TCon(t-1) - TPmt(t-1)\right]^2\right\}. \end{aligned}$$

The expected value on the last line is complicated by the fact that the value of F depends on past values of $TPmt$. At the same time, the value of $TPmt$ depends on past values of B which also depend on past values of F, creating a feedback loop where $F(t-1)$ influences $G(t)$, which influences $B(t)$, which influences $TPmt(t)$ which influences $F(t+1)$. The resulting correlation structure between F, $TPmt$, B and G is difficult to characterize so we abandon the analytical approach and use simulation to study the distribution of G instead.

Our simulation considers 10,000 sample paths for δ_t, $t = 1, ..., 200$. Key features of the empirical distribution of $G(t)$ using the base parameter set (i.e. white noise model for δ_t) are shown in Figures 4.1 and 4.2. The simulation indicates that the distribution of G(t) is nearly symmetric around its mean (zero), with slightly more negative observations (losses) than positive ones (gains). This is a direct consequence of e^t being a convex function of t.

The dispersion of $G(t)$, as characterized by the distance between quantiles, increases with time until $t = 38$. During this time the membership is being built up to its stationary size and the fund is growing, resulting in larger and larger investment gains and losses in dollar terms. Shortly after the plan membership reaches its stationary state, $G(t)$ also settles down to what appears to be a stationary distribution based on the quantile plots of Figure 4.1. Notably the interquartile range (IQR), which is a robust statistic representing the dispersion of the distribution, becomes quite stable after 70 or 80 years – roughly one full generation after the plan membership reaches

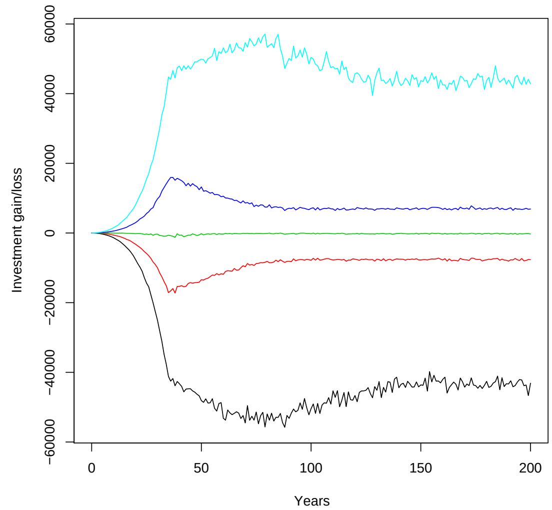


Figure 4.1: 5th, 25th, 50th, 75th and 95th percentiles of the empirical distribution of $G(t)$

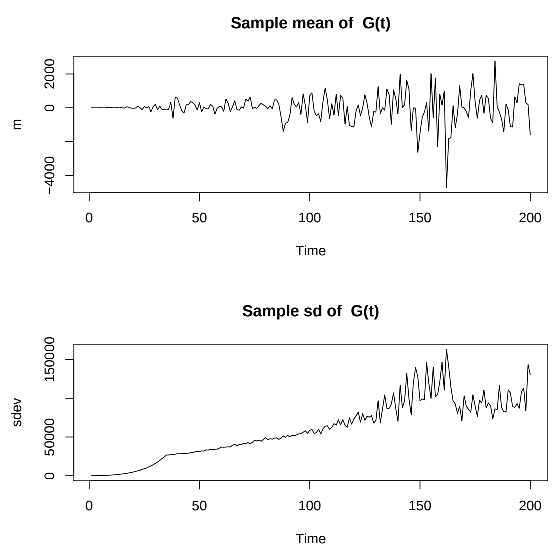


Figure 4.2: Sample mean and standard deviation of $G(t)$

its stationary state.

At the same time, the sample standard deviation continues to grow seemingly without bound, driven by extreme scenarios representing long runs of consecutive returns either above or below the assumed return. Although the frequency of such extreme scenarios may be misrepresented in our sample due to sampling variation, it is reasonable to expect that the longer our time horizon the more of these extremes will occur. Therefore, even if the IQR remains stable, the greater and greater frequency of the "outliers" (and the more extreme values they each take) will keep increasing the standard deviation of $G(t)$ even after $t = 80$. Consequently the random sequence $\{\mathrm{G}\}$ does not have the same distribution even when $t > 70$.

In addition to the problem of $G(t)$ not being identically distributed at each time-point t, the normality assumption of the classical Box-Jenkins model appears to be violated as well. The Q-Q plots in Figures 4.3 and 4.4 show that while the distribution of $G(t)$ appears to be reasonably close to normal within the second and third quartiles, the tails of the distribution differ markedly. As t increases, the fit gets better and better in the middle of the distribution with deviations being more and more severe in the tails. Consequently, the assumption of normality may be reasonable for predictions concerning quantities within the second and third quartiles, but will most likely lead to problems in the estimation of the central moments of the distribution which are very sensitive to outliers. Reliable estimates of the variance are, therefore, available neither theoretically nor by simulation.

4.3 Distribution of the Benefit Accruals

In section 4.1 we established that the annual benefit accruals $B(t)$ take the form

$$B(t+1) - b^* = \rho(t+1)\,[B(t) - b^*] + \frac{G(t+1)}{d(t+1)},$$

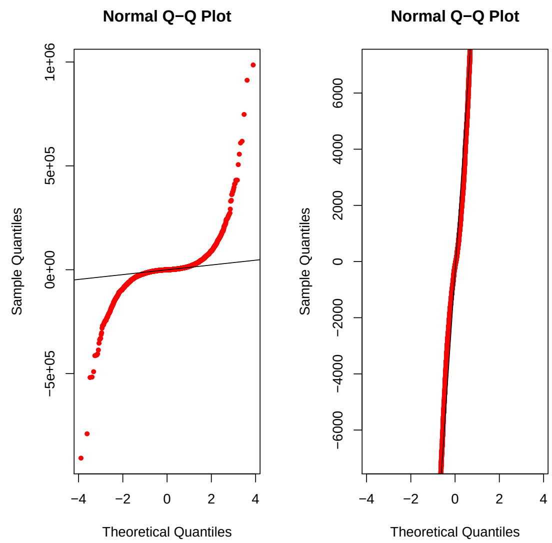


Figure 4.3: Q-Q plot of the empirical distribution of $G(70)$
(entire distribution on left, 2nd and 3rd quartiles on right)

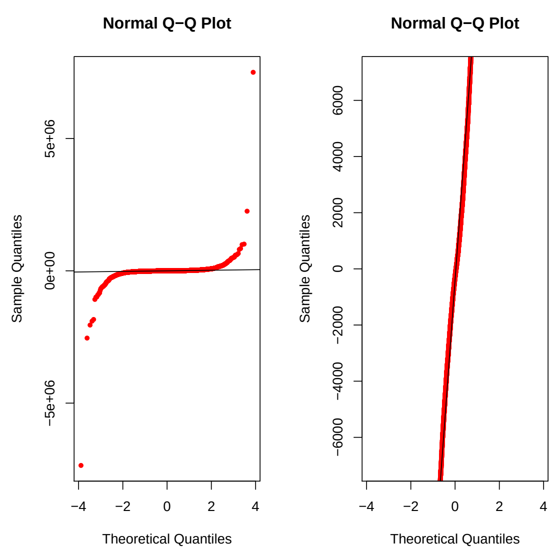


Figure 4.4: Q-Q plot of the empirical distribution of $G(200)$
(entire distribution on left, 2nd and 3rd quartiles on right)

where the innovations $G(t{+}1)/d(t{+}1)$ are non-normal and have variance that increases with time. We define the investment experience adjustment, $I(t)$, that is incorporated in the benefit accrual rate $B(t)$ as

$$I(t) = \frac{G(t)}{d(t)} = \frac{[F(t-1) + TCon(t-1) - TPmt(t-1)] \cdot (e^{\delta_t} - e^{\delta})}{d(t)} \tag{4.6}$$

so that

$$B(t+1) - b^* = \rho(t+1)\,[B(t) - b^*] + I(t). \tag{4.7}$$

As noted in section 4.2, the non-normality and heteroscedasticity of $I(t)$ even under a simple white noise assumption for δ_t means that $B(t)$ does not satisfy the conditions of a Box-Jenkins-type autoregressive model. This makes the study of the unconditional (i.e. asymptotic) distribution of $B(t)$ difficult. We can, however, comment on some properties of the distribution of $B(t)$ for $t > 0$, conditional on some earlier value $B(t-s)$, $s = 1, ..., t$. We rewrite equation (4.7) in its conditional form:

$$B(t) = b^* + \prod_{i=t-s+1}^{t} \rho(i)(B(t-s) - b^*) + \sum_{i=t-s+1}^{t-1} \prod_{j=i+1}^{t} \rho(j)I(i) + I(t) \quad \text{for } t > 1. \tag{4.8}$$

For $t = 1$, we simply have $B(1) = b^* + I(1)$. If the plan membership was stationary from the start (Assumption 5a), equation (4.8) would simplify to

$$B(t) = b^* + \rho^{t-s}(B(t-s) - b^*) + \sum_{i=1}^{t} \rho^{t-i} I(i). \tag{4.8a}$$

Applying $E[I(t)] = E[G(t)/d(t)] = 0$, we get

$$E[B(t)|B(t-s)] = b^* + \prod_{i=t-s+1}^{t} \rho(i)(B(t-s) - b^*) \tag{4.9}$$

which under Assumption 5a reduces to

$$E[B(t)|B(t-s)] = b^* + \rho^{t-s}(B(t-s) - b^*). \tag{4.9a}$$

That is, the expected value of the annual benefit accrual applicable to year t, conditional on the benefit accrual s periods earlier, is the target benefit accrual plus the remaining portion of the geometrically decaying difference between the benefit accrual in force s periods earlier and the target accrual rate. In other words, the benefit accrual rate is expected to be "pulled back" to the target accrual rate each year in the future. This is a desirable feature of the plan design.

We note that in the early years of the plan, while the membership is being built up, $\rho(t)$ is less than the ultimate level of ρ so the benefit accrual rate is pulled back to the target level faster. Once the plan membership reaches its stationary state, the factor ρ is locked in at a level relatively close to 1. Consequently, in a mature plan the decay is quite slow and the benefit accrual rate may take a long time to return to the target level even if the investment return assumptions are exactly met in the future.

In the special case of $s = t$ and $B(0) = b^*$ we have

$$E[B(t)|B(0)] = b^*, \quad t = 0, 1, 2, ... \tag{4.10}$$

meaning the expected value of the accrual rate at any future point in time, conditional on the starting value at plan inception, is the target benefit accrual rate b^*. This is consistent with the intention of our plan design.

The conditional variance of $B(t)$ given $B(t-s)$ for $1 < s < t-2$ can also be derived from equation (4.8) using the fact that

$$Cov[G(t), G(t+h)] = d(t)d(t+h)Cov[I(t), I(t+h)] = 0, \quad h > 0.$$

We then get

$$Var[B(t)|B(t-s)]$$
$$= \sum_{i=t-s+1}^{t-1} \left(\prod_{j=i+1}^{t} \rho(j) \right)^2 Var[I(i)|B(t-s)] + Var[I(t)|B(t-s)]. \tag{4.11}$$

From equation (4.6) we can see that the variability in $I(t)$ is driven by the variability in $F(t-1)$ and e^{δ_t}, and to a smaller extent by $TPmt(t-1)$. Since $B(t-s)$, $s \geq 1$, has only a small influence, if any, on $TPmt(t-1)$ and a very small influence on $F(t-1)$, it is reasonable to ignore this effect and use the approximation $Var[I(t)|B(t-s)] \approx Var[I(t)] = \sigma_I^2(t)$ for all $t > 1, s = 1, ..., t$. Then

$$Var[B(t)|B(t-s)] \approx \sum_{i=t-s+1}^{t-1} \left(\prod_{j=i+1}^{t} \rho(j) \right)^2 \sigma_I^2(i) + \sigma_I^2(t) \tag{4.12}$$

which, under Assumption 5a, becomes

$$Var[B(t)|B(t-s)] \approx \sum_{i=t-s+1}^{t} \rho^{2(t-i)} \sigma_I^2(i). \tag{4.12a}$$

In the special case $s = t$, equation (4.12a) reduces to

$$Var[B(t)|B(0)] \approx \sum_{i=1}^{t} \rho^{2(t-i)} \sigma_I^2(i). \tag{4.12b}$$

From section 4.2 we know that the variance of $G(t)$, and hence the variance of $I(t)$, is an increasing function of t. Let $\alpha(t)$ denote the difference in the variance of $I(t)$, $\alpha(t) = \sigma_I^2(t) - \sigma_I^2(t-1) > 0$ for $t > 1$. Then the conditional variance of $B(t)$ given the starting (target) benefit accrual at plan inception is

$$Var[B(t)|B(0)] \approx \sigma_I^2(1)\frac{1-\rho^{2t}}{1-\rho^2} + \sum_{i=2}^{t} \rho^{2(t-i)} \sum_{j=2}^{i} \alpha(j), \quad t > 1. \tag{4.13}$$

The behaviour of this quantity as a function of t depends on the rate of change in $\alpha(t)$, which is beyond the scope of this project.

We continue our exploration of the distribution of $B(t)$ by simulation. Using the same 10,000 scenarios of investment returns as in section 4.2, Figure 4.5 illustrates the development of the mean and standard deviation of the empirical distribution over time. We observe that the sample mean of $B(t)$ is near the theoretical expected value of 100 at first. As t increases, more extreme scenarios lead to the sample mean diverging from the expected value.

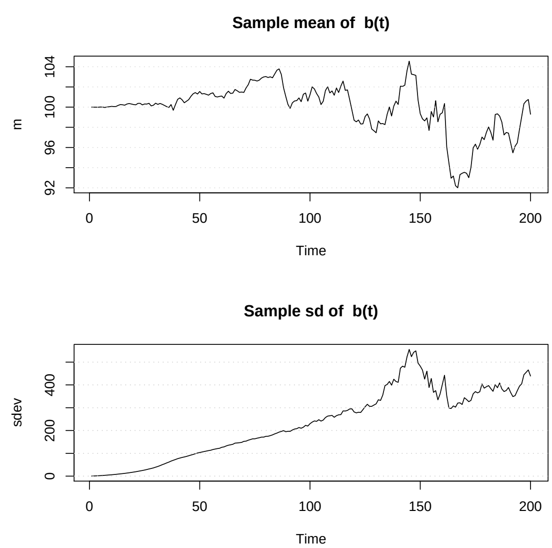


Figure 4.5: Sample mean and standard deviation of $B(t)$

The sample standard deviation, which is very sensitive to outliers, increases linearly for a while but then changes erratically from year to year as more and more extreme values appear. Even if we were to ignore the simulated values beyond $t = 130$ (where observations from the right tail of the distribution begin to severely disrupt the linear pattern), the estimated standard deviations are still too high relative to the mean, rendering this plan design unacceptable from a practical standpoint. A key question is then, how reliable are the simulated standard deviations as measures of the dispersion of $B(t)$?

Figure 4.6 shows key quantiles of the empirical distribution of $B(t)$. We make a number of observations.

- First, the distribution of B(t) gets more and more dispersed over the first 35 years as new cohorts enter. At $t = 35$, the distribution is still nearly symmetric with the second and third quartiles spanning the range [60,130]. Once the plan membership becomes stationary and benefit payments begin, most of the quantiles shift down while the highest quantiles continue to expand upward. The distribution becomes more and more positively skewed with a greater and greater proportion of the simulated values falling below the theoretical expected value of 100 (roughly 75% of the observations by $t = 100$) and a growing number of very large values.

- Each of the quantiles of the empirical distribution reaches a relatively stable long-term level at some point during our 200-year horizon. Generally, the lower the quantile, the quicker this level is reached: around $t = 60$ for the 25th percentile, $t = 90$ for the 50th percentile, $t = 120$ for the 75th percentile and $t = 150$ for the 95th percentile. The 5th percentile is an exception to this rule: the very low levels of $B(t)$ associated with this quantile tend to occur when the fund value $F(t)$ is negative, bringing about unusually unstable behaviour.

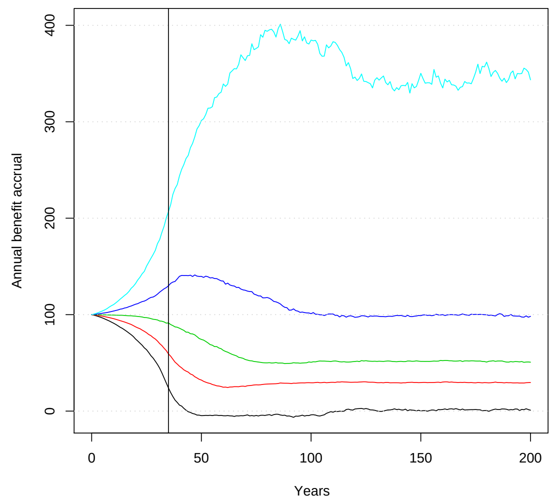


Figure 4.6: 5th, 25th, 50th, 75th and 95th percentiles of the empirical distribution of $B(t)$

- As the quantiles settle into their long-term values, the IQR stabilizes at roughly 65 (from 25 to 90). Although the standard deviation of the distribution continues to increase due to some rare but very large values, the practical dispersion (as measured by the IQR) actually remains remarkably stable from $t = 35$ onwards.

- The long-term median value of $B(t)$ is about 50, or one-half the target benefit accrual. It is questionable whether plan members would be satisfied knowing that 50% of the time they can expect to see benefit accruals lower than this value.

The explanation for the gradual downward shift in the distribution of $B(t)$ is relatively simple yet interesting. A string of lower-than-assumed investment returns (i.e. $\delta_t < \delta$) can have the effect of reducing $B(t)$ sharply in the matter of a few years, often to the point where $B(t)$ actually becomes negative (a rather undesirable situation implying that the accrued benefits are reduced and no new benefits are being earned). By the end of our 200-year horizon, 7,755 of our 10,000 simulation runs encounter a negative benefit accrual rate at least once; some as early as $t = 32$, some as late as $t = 200$, with most of the "first-negative" observations occurring around $t = 40$. Although it is possible for $B(t)$ to recover after such a catastrophic decline, it can take quite a long time, especially compared to how quickly $B(t)$ was depleted. As time goes by, more and more of the scenarios will have encountered at least one string of bad returns, so we are likely to find $B(t)$ stuck at a corresponding low level more and more often as t increases.

The reason it takes the benefit accruals a long time to recover after a sharp drop is that the string of lower-than-assumed investment returns which led to the drop in $B(t)$ also left the fund depleted. Indeed, there is very strong linear correlation between the value of the fund $F(t)$ at a given point in time and the size of the benefit accrual rate $B(t)$ in effect, with large values of the fund corresponding to larger accruals and small values of the fund corresponding to small or even negative accruals. The sample correlations of these two quantities at each time point during our 200-year horizon are shown in Figure 4.7, with an example of a single year's data ($t = 50$) shown in Figure 4.8 for emphasis.

This high positive correlation between $B(t)$ and $F(t)$ biases the movement of $B(t+1)$ downward in two ways: it makes small values of $B(t)$ more persistent and it makes large values of $B(t)$ more transitory. In particular, whenever the accrual rate $B(t)$ is at a very low level, $F(t)$ is likely to be also quite small. Then the same excess return over the course of the next year (i.e. $e^{\delta_{t+1}} - e^{\delta} > 0$) will result in a smaller investment gain $G(t+1)$ in dollar terms when applied to this small fund than if it was applied to an average- or large-sized fund. Since the investment experience adjustment $I(t+1)$ spreads this smaller gain over the same base (using the spread

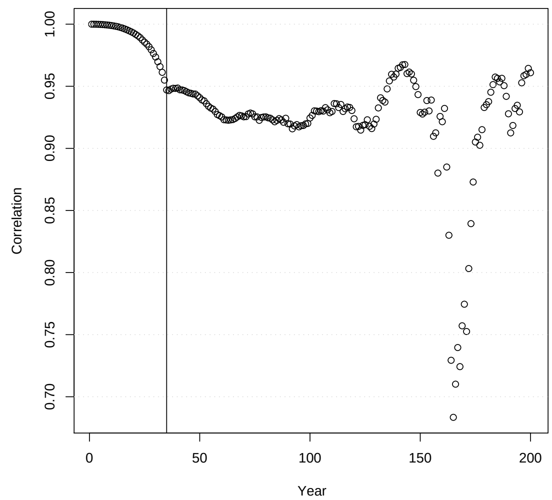


Figure 4.7: Sample correlation between $B(t)$ and $F(t)$

factor $d(t+1)$), the result is a relatively smaller dollar increase in the accrual rate than if we had started with a larger value of $B(t)$ at the beginning of the year.

Similarly, the same shortfall in the investment return relative to our assumption (i.e. $e^{\delta_{t+1}} - e^{\delta} < 0$) will generally lead to a smaller downward adjustment in $B(t+1)$ whenever $B(t)$ is small (and therefore probably paired with a small fund value) than when $B(t)$ is larger (and hence most likely paired with a larger fund). To summarize, once $B(t)$ becomes small, the investment experience adjustments $I(t+1)$ will also tend to be small, so the benefit accrual cannot easily get out of this lower range.

The opposite is true for large values of $B(t)$. Since above-average benefit accrual rates tend to be paired with above-average fund values, the corresponding investment experience adjustments $I(t+1)$ will also tend to be larger each year, leading to larger annual fluctuations in the benefit accrual rate. That is, when $B(t)$ is large, it can easily drop back down to average size and from there to a small size within the matter

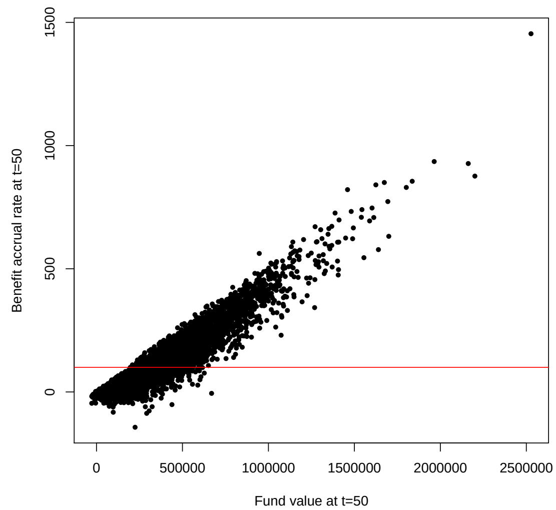


Figure 4.8: Scatterplot of $B(50)$ against $F(50)$ exhibiting strong correlation (target benefit level accrual marked in red)

of only a few years, whereas it is much more difficult to go back up to average size or beyond once $B(t)$ and the corresponding fund are small.

In other words, an upward movement from a lower value of $B(t)$ has a lower probability than a downward movement of equal size from a higher starting value. Consequently, as time passes more and more scenarios end up at lower values of $B(t)$ and the distribution of $B(t)$ shifts down. The evolution of the empirical distribution of $B(t)$ over time suggests that the rate of this downward shift eventually decreases and the median reaches an “equilibrium state”.

Chapter 5

Retirement Benefits

5.1 Distribution of the Pension Entitlement

Recall that the pension entitlement of a member retiring at time t was defined as the sum of the annual benefit accruals in each of the member's years of service:

$$Pen(t) = \sum_{i-0}^{34} B(t-i).$$

Since we only have limited information about the distribution of $B(t)$, we cannot say much about the distribution of $Pen(t)$ other than:

- The expected value of $Pen(t)$ given the starting value of the benefit accruals, $B(0) = b^*$, is equal to $35b^*$.

- The variance of $Pen(t)$ is greater than the sum of the variances of the benefit accruals that make up $Pen(t)$ because the covariances of the benefit accruals are positive.

Beyond these simple observations, we must turn to our simulation for answers.

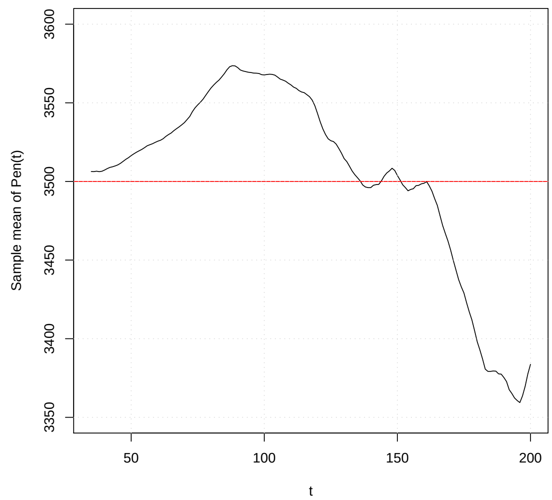


Figure 5.1: Sample mean of $Pen(t)$

Figures 5.1 and 5.2 show the development of the sample mean and standard deviation of $Pen(t)$ over time starting at $t = 35$ when the first benefit payment is made to the youngest cohort. The figures are based on the same 10,000 investment return scenarios we considered earlier. As expected, the sample mean of $Pen(t)$ remains relatively close to the targeted pension amount of 3500. On the other hand, the standard deviation increases rapidly. At first the increase is linear, but later on (after about $t = 130$) the pattern changes as the underlying variability of the annual benefit accruals $B(t)$ begins to change rapidly from year to year. Even if we were to ignore the more uncertain estimates of the standard deviation beyond $t = 130$, the variability of the pension benefit appears to be extremely high, with the coefficient of variation reaching 2.5 by year 130. As before, the key question is how much of this variability is driven by extreme values versus the bulk of the distribution.

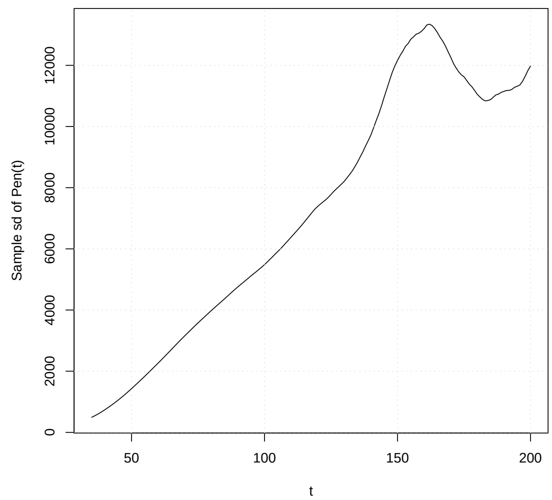


Figure 5.2: Sample standard deviation of $Pen(t)$

The answer can be found in the quantile plots of Figure 5.3. We observe that the distribution of $Pen(t)$ evolves in a way very similar to the distribution of $B(t)$. The distribution of the payments made to the first cohort is slightly skewed to the right, with the median value (\$3440) being slightly below the long-term mean (\$3500) marked by a black horizontal bar in Figure 5.3. A histogram of $Pen(35)$ in Figure 5.4 illustrates this in more detail. As time goes by, the positive skewness already present in the distribution of $Pen(35)$ becomes amplified as most of the quantiles begin to shift down while the highest quantiles continue to expand. By $t = 115$, more than 75% of the sample points lie below the theoretical expected value.

As with $Ben(t)$, the quantiles of the empirical distribution of $Pen(t)$ settle down to a stable long-term level at some point during our 200-year horizon. Once again, the lower quantiles generally reach this level more quickly: around $t = 85$ for the 25th percentile, $t = 105$ for the 50th percentile, $t = 150$ for the 75th percentile and

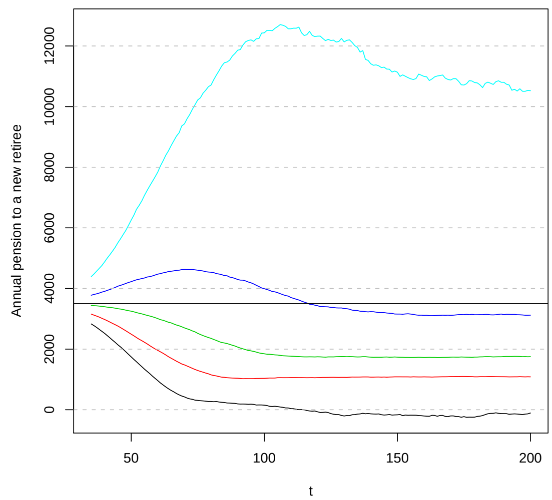


Figure 5.3: 5th, 25th, 50th, 75th and 95th percentiles of the empirical distribution of $Pen(t)$

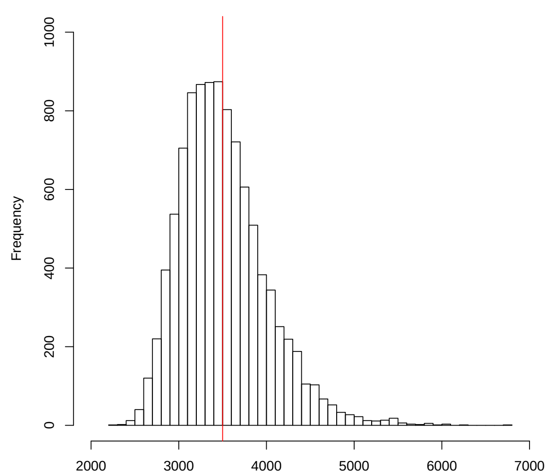


Figure 5.4: Histogram of $Pen(35)$

$t = 200$ for the 95th percentile. Note that the time horizon for reaching the stationary distribution is longer for $Pen(t)$ than it was for $Ben(t)$.

The interquartile range of $Pen(t)$ stabilizes at roughly 2070, spanning values of $Pen(t)$ from approximately 1080 to 3150. As with $B(t)$, the IQR remains remarkably stable even as the standard deviation continues to grow due to more and more extreme values in the right tail. We note that the long-term median value of the pension payable is around \$1730, which is just under one-half of the targeted benefit of \$3500. In fact, as noted above, the target level of the benefit is only reached about one-quarter of the time. This is not an acceptable level of performance by most practical standards. The dismal results are driven by the same mechanism outlined in section 4.3, namely the downward pressure on the distribution of the benefit accrual rate arising from the high correlation between the benefit accruals and the fund value.

Finally, we estimate the probability of ruin, that is, the probability that the fund value drops below zero at any point during our 200-year horizon. Out of the 10,000 scenarios in our simulation, the fund value is depleted at least once in 3,979 scenarios, or roughly 40% of the time. This is a rather high probability of ruin that would generally be found unacceptable in practice.

5.2 Distribution of the Pension Entitlement - Alternative Parameter Sets

We are now interested in exploring how the distribution of $Pen(t)$ changes if we assume that the annual investment returns δ_t are correlated. Specifically, we assume δ_t follows an AR(1) model with either a low or high autocorrelation coefficient. The local volatility of the process is adjusted so that the long-term volatility of δ_t matches that of our base case.

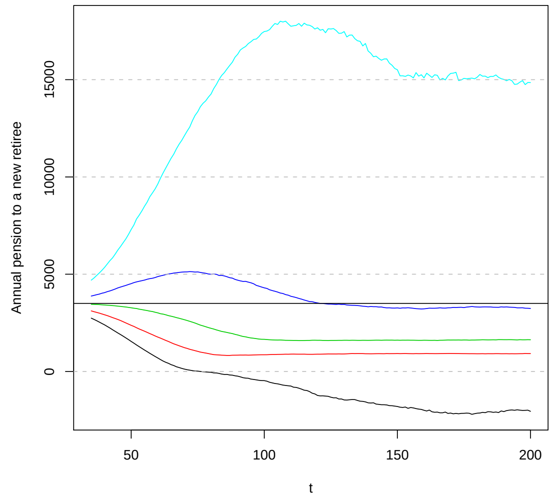


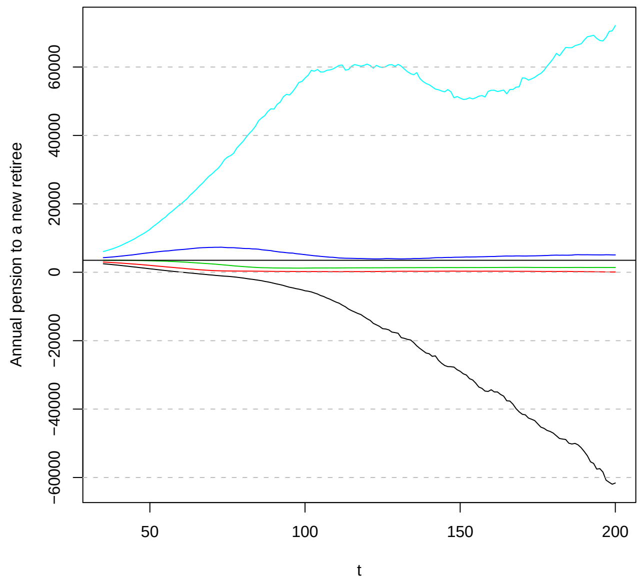
Figure 5.5: 5th, 25th, 50th, 75th and 95th percentiles of the empirical distribution of $Pen(t)$ with low-correlation parameter set

Key quantiles of the empirical distribution of $Pen(t)$ for both the low-correlation case ($\phi = 0.2$ and $\sigma = 0.078$) and the high-correlation case ($\phi = 0.6$ and $\sigma = 0.064$) are shown in Figures 5.5 and 5.6, respectively. Robust estimates of the central location and dispersion of the distributions at the end of our 200-year horizon are shown in Table 5.1 under all three parameter sets.

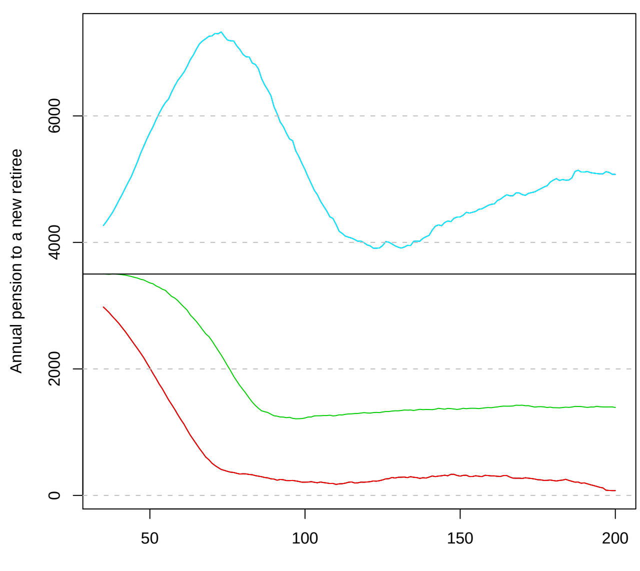
Table 5.1: Summary statistics of the distribution of $Pen(200)$ under three different parameter sets

	Sample median	Sample IQR
Base parameter set	1730	2070
Low-correlation parameter set	1631	2321
High-correlation parameter set	1390	5000

In the low-correlation case, the general development of the distribution of $Pen(t)$



(a) 5th, 25th, 50th, 75th and 95th percentiles



(b) 25th, 50th and 75th percentiles only

Figure 5.6: Percentiles of the empirical distribution of $Pen(t)$ with high-correlation parameter set

over time is very similar to that under the base case. The distribution starts out slightly skewed to the right at $t = 35$ and the skewness becomes more pronounced over time as the bulk of the observations shift downward and more extreme observations accumulate in the right tail. The distribution reaches a stationary state in roughly the same timeframe as under the base case and this ultimate distribution is remarkably similar to the stationary distribution of $Pen(t)$ under the base case, with only a slightly lower median benefit and slightly higher IQR. Since this median value is far below the target benefit level, our TBP with the low-correlation assumption for the investment returns also fails the first performance criterion.

The greatest drawback of including a low level of correlations appears to be that the probability of ruin increases to 58% compared to 40% under the base case. This is not surprising, since autocorrelation in δ_t means that the investment returns in consecutive years will generally be more closely clustered than under the white noise model, thereby increasing the probability of a string of lower-than-assumed returns leading to ruin over any given time horizon.

While the performance of the TBP under the base case and the low-correlation case is quite similar, the high-correlation case presents quite a different picture. When the autocorrelation coefficient is high, consecutive values of δ_t will be even closer to each other. Instead of being a rarity, a five-year sequence of below-average returns will occur quite frequently, allowing the fund to be depleted sooner and more frequently. It is not surprising then that the estimated probability of ruin in the high-correlation case is 87%, more than double the probability of ruin under the base case. On the other hand, persistent above-average returns may lead to the fund reaching astronomical levels. As the fund value fluctuates widely, so do the annual benefit accruals, leading to much more extreme values in the tails of the distribution of $Pen(t)$ than under the base case. Interestingly, while the 5th and 95th quantiles of the $Pen(t)$ reach truly extreme levels ($\pm$ 60,000) the second and third quartiles behave somewhat similarly to the base case (see Figure 5.6(b)). The median of $Pen(t)$ drops gradually until it reaches its long-term level at 1390, barely 40% of the target benefit. The IQR, although somewhat stabilized after 200 years, is very high at 5000. Under the assumption of a high autocorrelation coefficient in the AR(1) process for δ_t, the TBP fails the first performance criterion worse than ever.

5.3 The DC Benchmark

In a traditional defined contribution plan each member has an individual account into which contributions are deposited. The account is invested in some portfolio of assets and the accumulated balance of the fund at retirement is paid out to the member either as a lump sum or is converted into a guaranteed stream of payments by purchasing an annuity. Given fixed annual contributions, the accumulated balance at retirement is a function of the investment returns earned during the member's working life. It is a random quantity whose distribution is based on the distribution of fund returns.

From the employer's perspective there is no difference between our simple target benefit plan and a traditional DC plan: the employer's obligation is limited to the fixed annual contributions. From the employees' perspective, the two plan designs are also similar in that the retirement benefit ultimately received is a random quantity based on the investment returns of the fund. The key difference is that in a traditional DC plan the "fund" (that is, each DC account) covers only a single cohort with one participant ($n = 1$), whereas in a target benefit plan the fund commingles the assets and experience of many cohorts, each with $n \geq 1$.

To compare the performance of our TBP against a DC benchmark, we will compare key properties of the distribution of the benefits received by different cohorts of members such as the mean, standard deviation and representative percentiles. To ensure that the comparison is fair, the benefit entitlements calculated under both plan designs must be based on the same set of assumptions. For the DC benchmark, this means:

- The annual contribution made to the DC plan is equal to the annual contribution C made in respect of each member under the TBP.
- The DC account is invested according to the same asset mix as the TBP and is rebalanced annually. The continuously compounded rate of return on the DC account during the period $[t-1, t)$ is, therefore, δ_t.
- Contributions are received and the benefit is paid out at the beginning of the year.
- The member joins the DC plan at age 30 and retires at age 65. There are no decrements prior to retirement.
- The DC account value is converted to an annuity at retirement based on the constant valuation rate δ and the 1983 Group Annuity Mortality table.

We define the following variables:

- $F_{DC}(t)$, the final fund value under the DC benchmark for a member who enters the plan at time $t-35$ and retires at time t, and
- $Pen_{DC}(t)$, the annuitized retirement benefit payable under the DC benchmark to a member retiring at time t.

Unlike under our TBP, the benefit payable under the DC benchmark only depends on the final value of the DC account at retirement and not on any interim values:

$$Pen_{DC}(t) = \frac{F_{DC}(t)}{\ddot{a}_{65}}. \tag{5.1}$$

Since the annuity conversion basis is fixed, the distribution of $Pen_{DC}(t)$ is the same as the distribution of $F_{DC}(t)$, except that the standard deviation is adjusted by a factor of $1/\ddot{a}_{65}$. To get an idea of the distribution of $F_{DC}(t)$, consider the final fund value at time t as the sum of the accumulated values of each year's contributions:

$$F_{DC}(t) = \sum_{i=0}^{34} C \cdot exp\left(\sum_{j=0}^{34-i} \delta_{t-j}\right). \tag{5.2}$$

According to our asset model δ_t is normally distributed; therefore each term in the outer sum above is lognormally distributed. Consequently $F_{DC}(t)$ is the sum of 35 lognormal random variables with significant correlations between them: the first two terms in the sum have 34 years of investment return history in common, the second and third terms have 33 years in common, etc., with additional correlations arising when δ_t is modeled as an autoregressive process. Since a closed form expression for the distribution of the sum of several correlated lognormal random variables does not exist, we rely on simulation to approximate the distribution of the benefit payable from the DC benchmark and compare this empirical distribution against the empirical distribution of benefits payable from our target benefit plan.

Figure 5.7 shows the evolution of the sample mean of $Pen_{DC}(t)$ over the period $t = 35, ..., 200$ based on the same 10,000 scenarios of investment returns as were used for our target benefit plan. The evolution of the sample mean of $Pen(t)$ from our target benefit plan is superimposed as a blue dashed line for comparison. We note that the sample mean of $Pen_{DC}(t)$ remains close to the target benefit level of 3500 throughout the entire time horizon. By contrast, the sample mean of $Pen(t)$ has wider departures from the asymptotic mean but follows a much smoother path year-to-year.

Figure 5.8 shows the sample standard deviation of the simulated values of $Pen_{DC}(t)$ over the same period. Unlike the standard deviation of $Pen(t)$ under our TBP which increased steeply in t, the standard deviation of $Pen_{DC}(t)$ is quite stable, hovering

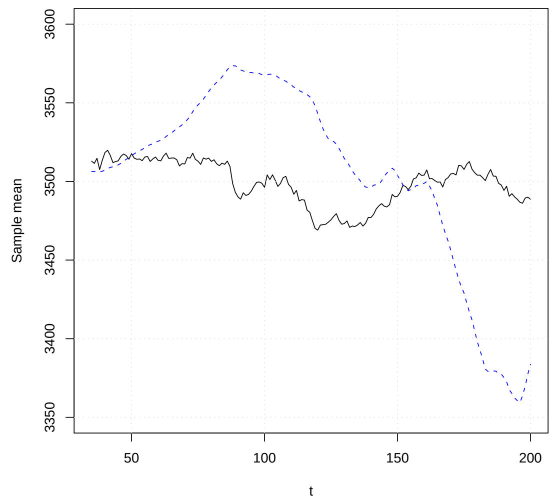


Figure 5.7: Sample mean of $Pen_{DC}(t)$ in solid black, and $Pen(t)$ in dotted blue

near 1300 over $t = 35, ..., 200$.

Key quantiles of the distribution of $Pen_{DC}(t)$ are shown in Figure 5.9. Unlike the quantiles of $Pen(t)$ under the target benefit plan, the quantiles of $Pen_{DC}(t)$ hardly change at all, suggesting that the pension entitlements of all the cohorts are identically distributed. This is to be expected, since the formula for $Pen_{DC}(t)$ given in equation (5.1) always has 35 lognormally distributed terms and the parameters of each of those terms are the same for all values of t, depending only on the parameters of δ_t.

Stability in the distribution of $Pen_{DC}(t)$ over time should not be mistaken to mean that $Pen_{DC}(t)$ remains at the same level on each simulation path. On the contrary, each simulation path sees $Pen_{DC}(t)$ moving around randomly within the distribution. To illustrate, we choose one scenario of investment returns (from δ_1 to δ_{200}) and superimpose the corresponding sample path of $Pen_{DC}(t)$ over the background of stable quantiles as a dashed black line in Figure 5.10(a) . We see that on this particular

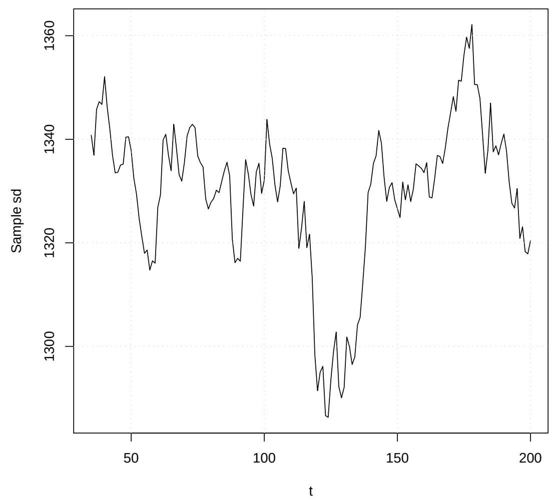


Figure 5.8: Sample standard deviation of $Pen_{DC}(t)$

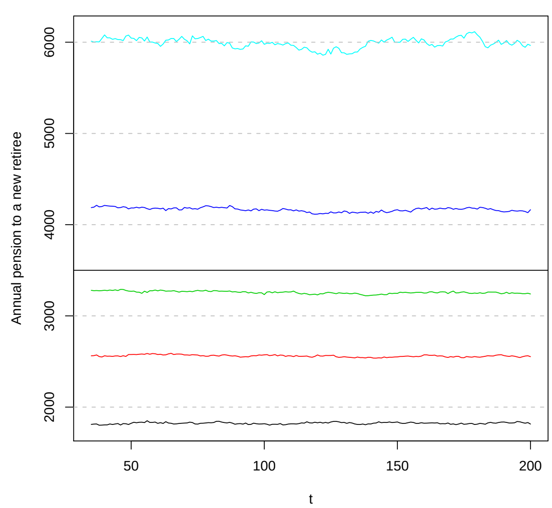
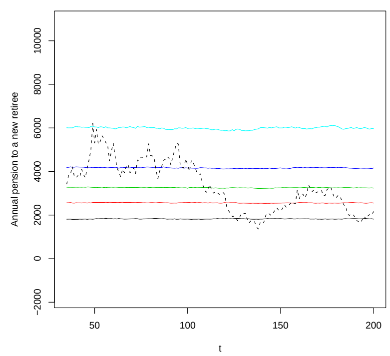
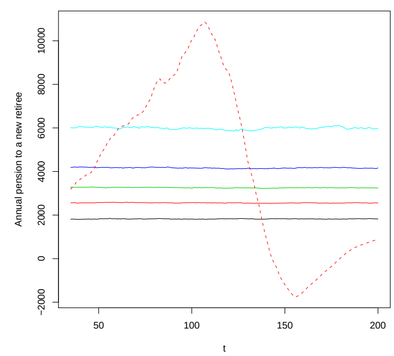


Figure 5.9: 5th, 25th, 50th, 75th and 95th percentiles of the empirical distribution of $Pen_{DC}(t)$



(a) Path of $Pen_{DC}(t)$



(b) Path of $Pen(t)$

Figure 5.10: Corresponding sample paths of $Pen_{DC}(t)$ and $Pen(t)$, superimposed over the quantiles of $Pen_{DC}(t)$

path $Pen_{DC}(t)$ starts off near 3500 at $t = 35$, wanders upward reaching as high as the 95th percentile of the distribution by $t = 50$, stays above the median for quite some time, then wanders down below the 5th percentile by $t = 140$, then back up to the median and so on. The path of $Pen_{DC}(t)$ is reminiscent of a random walk, moving mostly in small increments with frequent changes in direction.

By contrast, the corresponding sample path of $Pen(t)$ from our target benefit plan is shown in Figure 5.10(b) as a dashed red line against the same backdrop. Two things are immediately noticeable about this latter path: it makes wider departures and is quite smooth compared to the path of $Pen_{DC}(t)$. On the red path more steps are taken in the same direction before a reversal; in other words, $Pen(t)$ appears to have more "momentum" than $Pen_{DC}(t)$. This is because each value of $Pen(t)$ carries within it the investment experience of *all* prior years but $Pen_{DC}(t)$ only carries information from the last 35 years. As time passes and each new year's investment experience is incorporated into the new pension entitlement under the TBP, prior experience fades away only slightly. By contrast, a full year's information is completely discarded from

Table 5.2: Summary statistics of the distribution of $Pen_{DC}(200)$ and $Pen(200)$

	$Pen_{DC}(200)$	$Pen(200)$
Sample mean	3489	3384
Sample standard devation	1320	11976
Sample median	3240	1730
Sample IQR	1611	2070
Probability of benefit being lower than target	0.58	0.78

$Pen_{DC}(t)$ whenever a new year's information is incorporated. The shorter "memory" of $Pen_{DC}(t)$ allows it to take sharper "turns", keeping its values less dispersed.

Summary statistics of the distributions of $Pen_{DC}(200)$ and $Pen(200)$ are given in Table 5.2. These distributions (at $t = 200$) are considered to be representative of the distributions of $Pen_{DC}(t)$ and $Pen(t)$ once the IQR has stabilized, which we shall refer to as the "stable" distributions of $Pen_{DC}(t)$ and $Pen(t)$.

The superiority of the DC plan over the target benefit plan is immediately obvious from Table 5.2. The pension entitlement under the DC benchmark is considerably "closer" to the targeted benefit in every sense: not only is the distribution centered around the correct value but its dispersion around this central value is also far smaller.

We next compare the intergenerational equity of the two plan designs.

5.4 Intergenerational Equity

As noted in Section 2.2, the maintenance of intergenerational equity is one of the shared objectives of trustees of target benefit plans. This means ensuring that each generation's rewards are proportionate to their risks. Ideally two cohorts that share similar risks should have similar benefits; however, assessing just how "similar" the benefits of two cohorts are is not as easy as it might appear. Specifically, to determine how similar the benefits of two cohorts (say, one retiring at time t and the other retiring at time $t-h$) are, it is not appropriate to compare the entire distribution of $Pen(t)$ against the entire distribution of $Pen(t-h)$. Instead, one ought to compare the distribution of the *change* in the benefit along the same path over time.

Since members' assets are kept separate in a traditional DC plan, there are no opportunities for intergenerational risk transfers and consequently each member's DC plan benefit is related only to his own risk. Therefore, we will use the "similarity" of pension benefits payable to two different cohorts under the DC plan as our benchmark for intergenerational equity. We define $D^h_{DC}(t)$ as the percentage change between the pension benefits payable to a cohort retiring at time t versus a cohort retiring h years earlier at time $t-h$ under the DC plan:

$$D^h_{DC}(t) = \frac{Pen_{DC}(t) - Pen_{DC}(t-h)}{Pen_{DC}(t-h)} \qquad h{>}0,\ t \geq 35{+}h. \tag{5.3}$$

A positive value of $D^h_{DC}(t)$ corresponds to an increase in pensions over time and a negative value of $D^h_{DC}(t)$ corresponds to a decrease in pensions. The theoretical distribution of $D^h_{DC}(t)$ is known and its quantiles can be calculated in a straightforward manner. The empirical distributions of $D^h_{DC}(t)$ for $h = 1$, 10, 35 and 100 are shown in the four panels of Figure 5.11. We make the following observations:

- Clearly, the distribution of the *change* in the pension is not the same for all values of h: the closer together two cohorts are the narrower the distribution of the change is around zero. This is reasonable, since two cohorts closer together share more of the same investment history and therefore their benefits should be more similar.

- The empirical distribution of $D^h_{DC}(t)$ appears to be the same from the very beginning for all values of h. In fact, one could show that the empirical distributions shown here are consistent with the theoretical distribution.

- The median of the change in pensions stays at zero under our DC plan for all values of h, suggesting that the benefit is equally likely to move up or down when comparing any two cohorts.

Summary statistics of the distribution of $D^h_{DC}(t)$ are presented in Table 5.3. Since the distribution of $D^h_{DC}(t)$ under the DC plan is not very skewed, the information communicated by the standard deviation and the IQR is very similar.

Table 5.3: Summary statistics of the percentage change in pensions payable to cohorts 1, 10, 35 and 100 years apart under the DC plan

	$D^1_{DC}(200)$	$D^{10}_{DC}(200)$	$D^{35}_{DC}(200)$	$D^{100}_{DC}(200)$
Sample mean	0.3%	4%	13%	14%
Sample sd	8%	30%	62%	62%
Sample median	0%	0%	0%	0%
Sample IQR	11%	38%	71%	69%

To assess the intergenerational equity of our TBP, we let

$$D^h(t) = \frac{Pen(t) - Pen(t-h)}{Pen(t-h)} \qquad h>0,\ t \geq 35+h \tag{5.4}$$

and compare it against $D^h_{DC}(t)$. Figure 5.12 shows the 5th, 25th, 50th, 75th and 95th percentiles of the empirical distribution of $D^h(t)$ for $h = 1, 10, 35, 100$ for a target benefit plan under the base parameter set (i.e. white noise assumption for δ_t).

Note that unlike the distribution of $D^h_{DC}(t)$ which is stationary from the outset, the empirical distribution of $D^h(t)$ undergoes a shift over time until it reaches a more-or-less stationary state (again, we will refer to this as the "stable" distribution of $D^h(t)$). In each of the four panels this stable distribution is more dispersed than

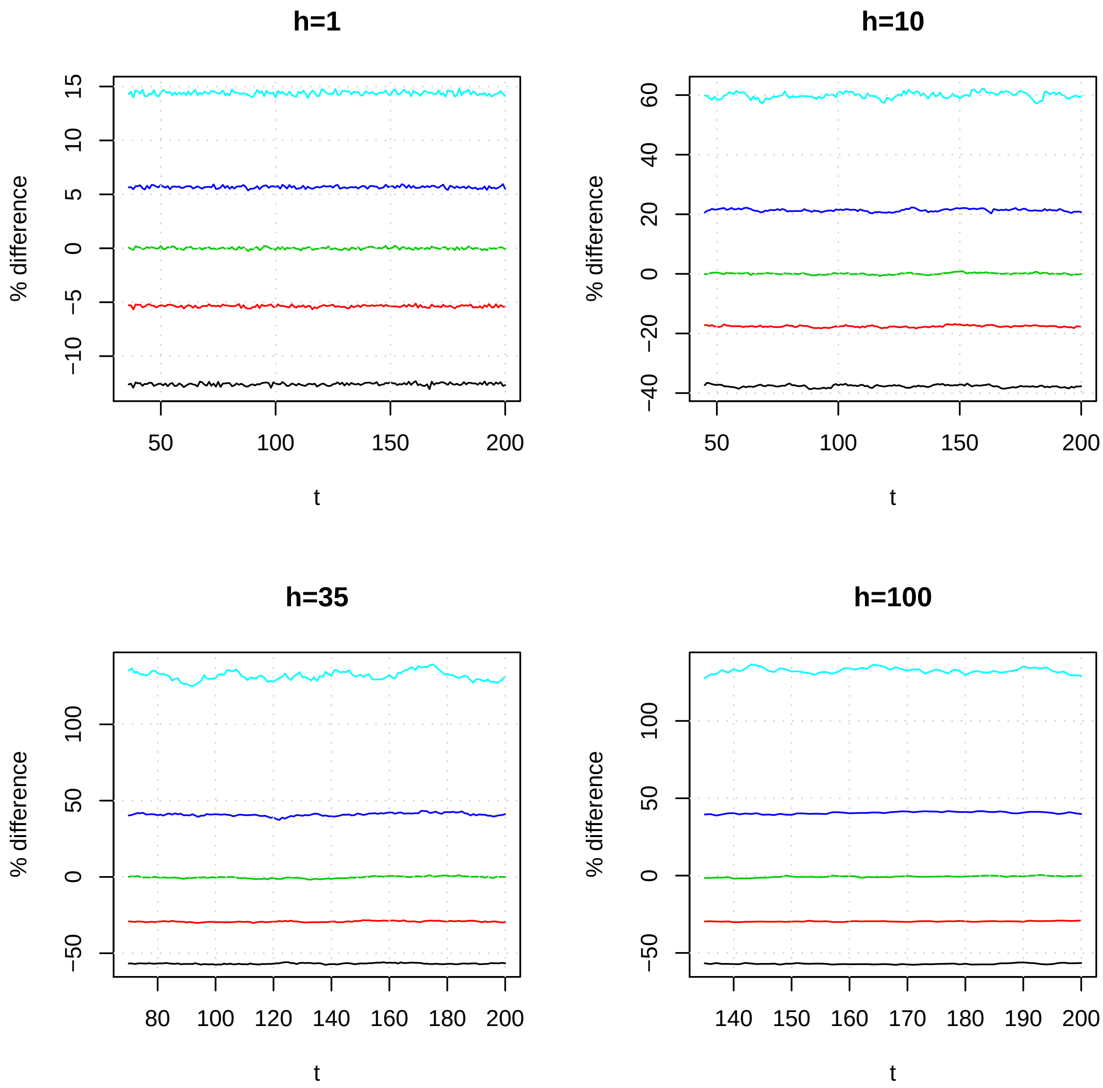


Figure 5.11: 5th, 25th, 50th, 75th and 95th percentiles of $D^h_{DC}(t)$, the percentage difference in DC pensions to members retiring at time t versus $t-h$

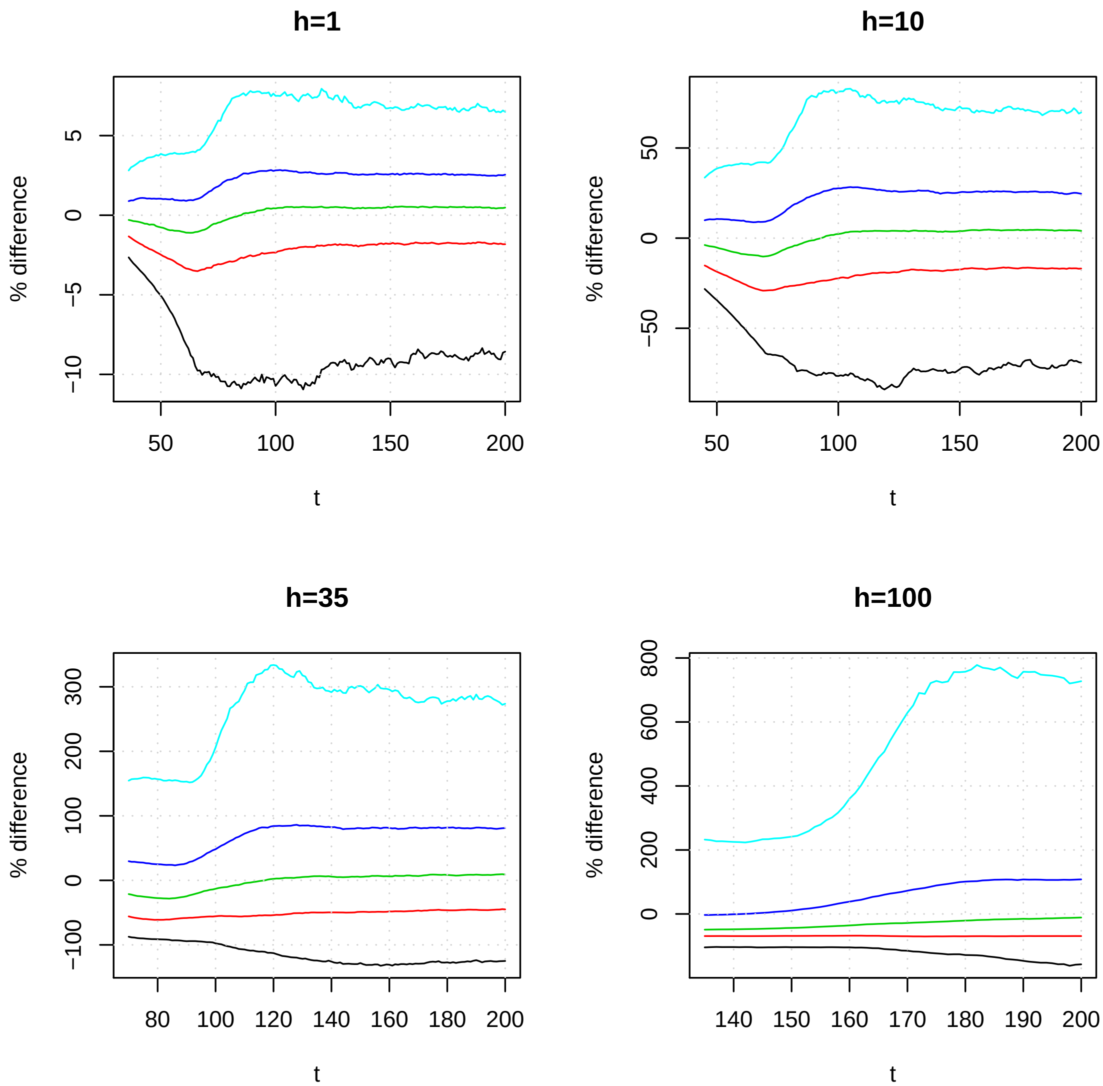


Figure 5.12: 5th, 25th, 50th, 75th and 95th percentiles of $D^h(t)$, the percentage difference in pensions to members retiring at time t versus $t-h$

the distribution applicable at earlier time points, suggesting that although this plan design may appear more equitable at the outset, ultimately the variability of the benefits payable to different cohorts tends to increase.

We also observe that in the first panel (when $h = 1$, that is, comparing cohorts 1 year apart) the stable distribution is slightly skewed to the left. In the second panel, for $h = 10$, the stable distribution is more or less symmetric, meaning the pension is equally likely to increase or decrease when comparing cohorts 10 years apart. In the third panel, for $h = 35$ corresponding to cohorts one full generation apart, the stable distribution becomes positively skewed. Finally, for $h = 100$ the stable distribution has very strong positive skewness, suggesting some very large values occur in the right tail.

Summary statistics of the distributions at the end of our 200-year horizon are shown in Table 5.4 for $h = 1$, 10, 35 and 100. The statistics reported include both the traditional measures of location and scale (the sample mean and standard deviation) as well as their more robust counterparts (the median and the IQR).

Table 5.4: Summary statistics of the percentage change in pensions payable to cohorts 1, 10, 35 and 100 years apart under the TBP

	$D^1(200)$	$D^{10}(200)$	$D^{35}(200)$	$D^{100}(200)$
Sample mean	1%	8%	9%	708%
Sample sd	151%	699%	1056%	76338%
Sample median	0.5%	4%	9%	-11%
Sample IQR	4%	41%	126%	177%

Looking at only Table 5.4 first, we make the following observations:

- Clearly, the standard deviation is not a very informative measure of scale here, due to the very long tails of the distribution.
- The mean and the median are quite similar for h=1, 10 and 35, but are very

different for h=100. This is consistent with the more severe skewness of the distribution observed for $h = 100$ in Figure 5.12.

- The median values of the percentage change in pension benefits payable to cohorts 1 and 10 years apart suggests a reasonable level of intergenerational equity, with pension amounts drifting only slightly on average (by 0.5% and 4%, respectively). The shift is more pronounced for non-overlapping cohorts.

- The narrow interquartile range (IQR) for cohorts 1 year apart confirms the intergenerational equity of this plan design: since these cohorts shared 34 of their 35 years of membership in the plan, their "experience" must have been similar and therefore it is sensible that their benefits should be very similar too.

- The farther apart the cohorts are, the less time they would have spent together in the plan and the less "experience" they will have shared, so their benefits ought to be less closely related. The higher variability in the change in benefits is confirmed by the expanding IQRs as h increases. For cohorts that have no overlap in service (that is when $h \geq 35$), we do not expect to see any relationship between the pensions payable, and indeed there is virtually none, as indicated by the very large IQRs.

Next, we compare the entries in Table 5.4 with the corresponding entries for the DC benchmark in Table 5.3. Using the more robust estimates (i.e. the median and the IQR) as our basis for comparison, we conclude that the performance of the TBP is close to the DC benchmark in terms of ensuring equity between cohorts 1 or even 10 years apart. When comparing cohorts a full generation apart (i.e. cohorts that have no overlap in service) the variability of the change in the pension entitlement increases sharply under both the target benefit plan and the DC benchmark. The variability is greater under the TBP, and new generations of members may find this unattractive since it means that the benefit they may ultimately receive could be further from the benefit payable to current retirees. However, they may be willing to take on this extra risk knowing that it carries more upside potential than downside risk, as more than half the time they can expect to see higher benefits than the previous generation. Therefore the TBP performs at least as well as the DC benchmark in terms of intergenerational equity.

The performance of the TBP under the low-correlation assumption set is similar with respect to intergenerational equity. Plots of the key percentiles of $D^h(t)$ for $h = 1$, 10, 35 and 100 are included in Appendix F. These plots show a stationary distribution developing over time. Although the tails of the distributions are heavier in the low-correlation case than in the base case, the second and third quartiles are quite similar, suggesting that the TBP exhibits a reasonable level of intergenerational equity even when some positive autocorrelation is present in the annual investment returns.

Chapter 6

Benefit Policy Modifications

In light of the poor results achieved by our simple target benefit plan relative to the DC benchmark regarding the proximity of the benefit to the target, we explore some modifications to the benefit policy with the goal of improving performance over the long term.

6.1 Provision for Adverse Deviations

The first modification we investigate is the addition of a provision for adverse deviations. From a practical standpoint, provisions for adverse deviations are required to be included in all valuations prepared for funding purposes, as dictated by section 1740 of the Standards of Practice of the Canadian Institute of Actuaries as of 2009. Generally accepted actuarial practice is to include such a provision by reducing the valuation rate by some margin (often in the range of 50-100 basis points) relative to the actuary's best estimate assumption. We follow the same approach here, so that our new valuation rate (δ') is 75 basis points lower than the expected return on plan assets (δ). The benefit policy equation then becomes

$$B(t) = \frac{F(t) + PVFC'(t) - PVAB'(t-1) \cdot e^{\delta'} + TPmt(t-1) \cdot e^{\delta'}}{d'(t)}, \tag{6.1}$$

where $PVFC'(t)$, $PVAB'(t)$ and $d'(t)$ are defined as before, but using a valuation rate of δ' instead of δ. Consequently, the benefit accrual in the first year becomes

$$B(1) = b^{*\prime} + \frac{G'(1)}{d'(1)} \tag{6.2}$$

and the recursive formula for $B(t)$ changes to

$$B(t+1) = B(t) + \frac{G'(t+1)}{d'(t+1)} + \frac{[b^{*\prime} - B(t)] \cdot d'(0)}{d'(t+1)}, \quad t \geq 1, \tag{6.3}$$

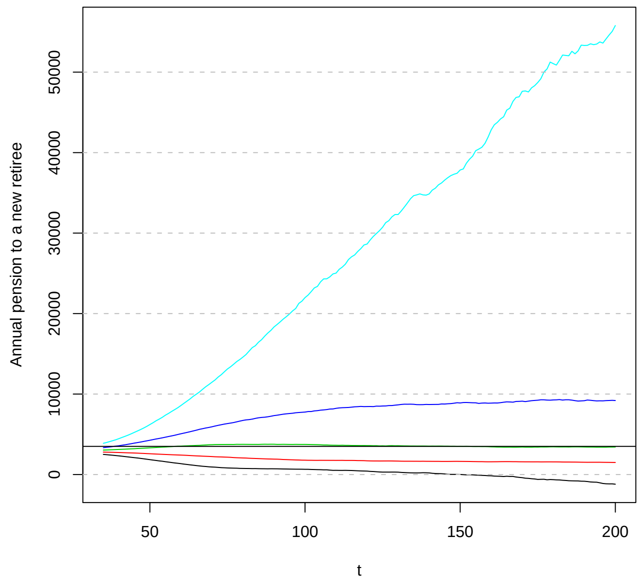
where $b^{*\prime}$ is the benefit accrual level corresponding to the level-dollar Entry Age Normal cost of C using δ',

$$b^{*\prime} = \frac{C\ddot{a}_{\overline{35}|}}{35_{35|}\ddot{a}_{30}} \tag{6.4}$$

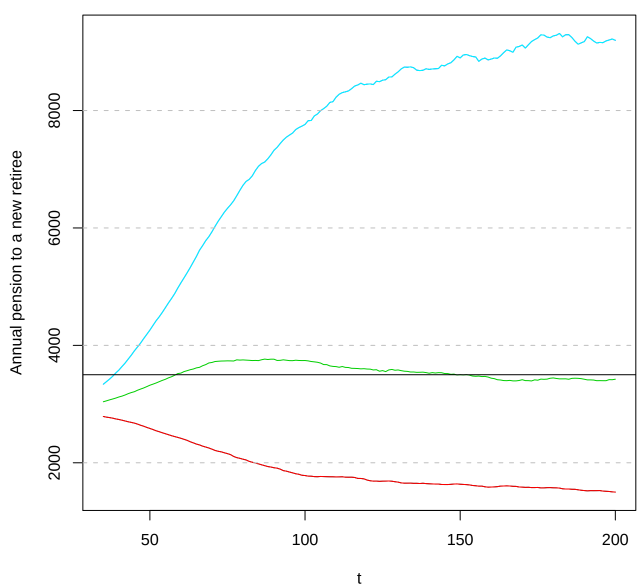
and $G'(t+1)$ is the investment gain during the period $[t, t+1)$ relative to our reduced valuation rate:

$$G'(t+1) = [F(t) + TCon(t) - TPmt(t)] \cdot (e^{\delta_{t+1}} - e^{\delta'}). \tag{6.5}$$

The evolution of the distribution of $Pen(t)$ over our 200-year horizon using a margin of 75 basis points in the valuation rate is shown in Figure 6.1. We note that the distribution is far more skewed to the right than without the margin, which is not surprising since we expect to see more gains than losses relative to our reduced valuation rate. Focusing on the second and third quartiles, we see that the median of $Pen(t)$ is now at 3424, meaning a member's pension entitlement is as likely to exceed the target as it is to fall short. We also note that after 200 years of plan



(a) 5th, 25th, 50th, 75th and 95th percentiles



(b) 25th, 50th and 75th percentiles only

Figure 6.1: Percentiles of the empirical distribution of $Pen(t)$ with 75 bp margin

operation the distribution of $Pen(t)$ has not yet reached a stationary state: all of the quantiles are still shifting, especially the most extreme ones. However, by the end of 200 years the IQR has stabilized at about 7100. This is about 2.5 times wider than the IQR under the base case meaning the pension benefits are far more dispersed, but they are dispersed around a much higher central value that is quite close to our target. Therefore, the modified plan provides pensions at or above the target more often than our simple TBP. The question then arises, does it do so at the expense of intergenerational equity?

The answer is no. On the contrary, the provision for adverse deviations results in considerable stability in the benefits payable to different cohorts. Figure 6.2 shows the 5th, 25th, 50th, 75th and 95th percentiles of the distribution of $D^h(t)$ for $h = 1, 10, 35, 100$ for our modified target benefit plan including a 75 basis point margin in the valuation rate. Not only are the distributions less dispersed but they are also remarkably stable from the outset, whereas they took quite some time to reach stability under our simple TBP.

Finally, we note that the probability of ruin is significantly reduced by including a provision for adverse deviations: the fund value is depleted at least once during the 200-year horizon in 2,469 of our 10,000 scenarios, compared to 3,979 scenarios under the simple TBP. In summary, the modified benefit policy including a provision for adverse deviations outperforms our base case in every respect, and does so without increasing plan costs.

6.2 Open Group Valuation

The second benefit policy modification we investigate concerns the number of future cohorts to acknowledge in the calculation of the benefit accrual rate at each valuation date. Under our simple TBP, we only acknowledged members present in the plan as

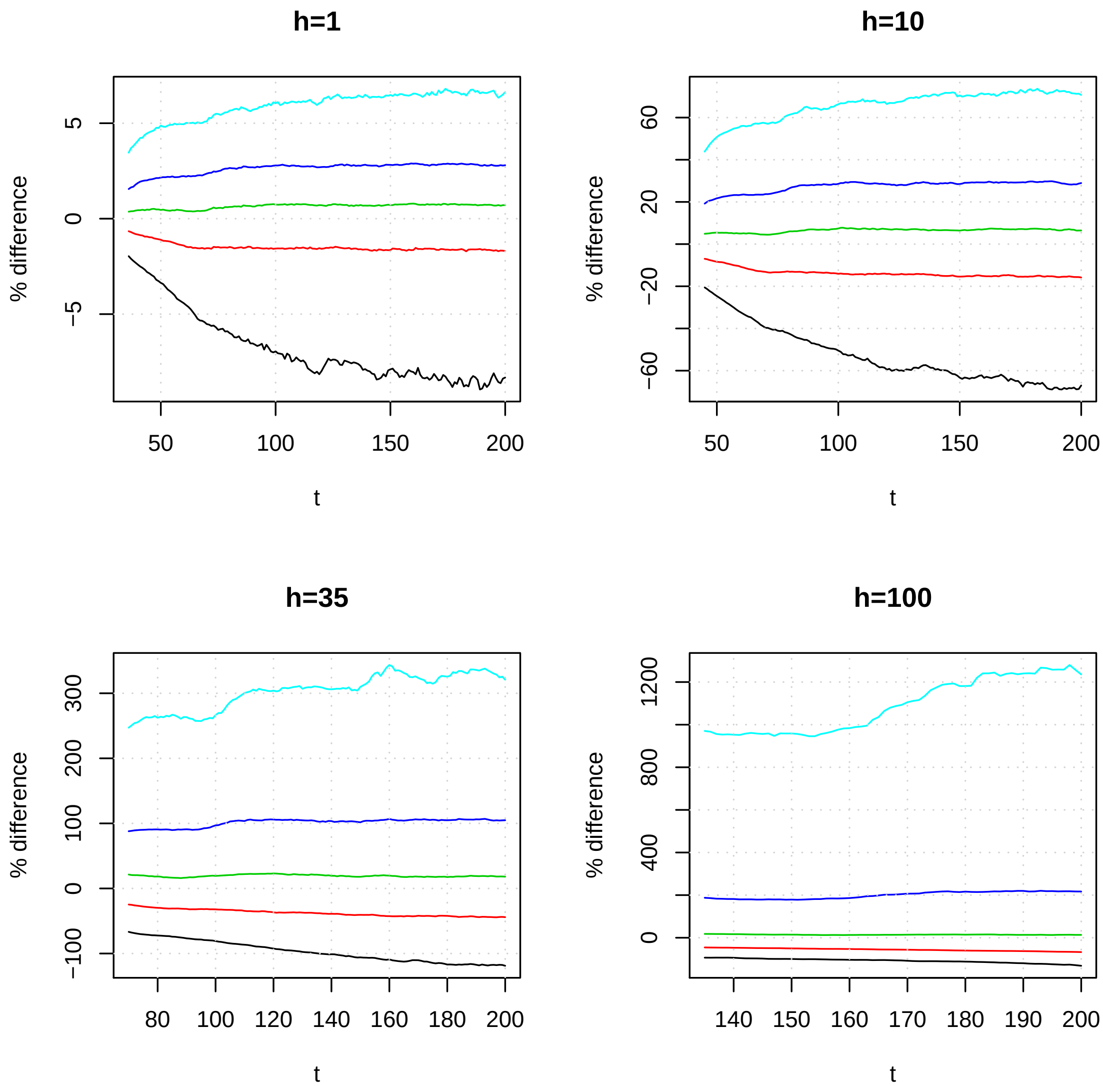


Figure 6.2: 5th, 25th, 50th, 75th and 95th percentiles of $D^h(t)$, the percentage difference in pensions to members retiring at time t versus $t-h$, with 75bp margin

of the date of the valuation. This included in the valuation at time t the cohort that just entered at time t but no future cohorts. Then at the following valuation (at time $t+1$) the newest cohort that just entered at $t+1$ was also taken into account. At each valuation date an adjustment was made to the benefit accrual rate depending on whether the benefit accrual rate in force in the prior year happened to be above or below the target benefit rate, to equalize any over- or under-contributions expected in respect of the newest cohort just recognized.

In practice, we usually know that a plan will have new entrants and sometimes there is an allowance made for this in the valuation of a TBP. Suppose at each valuation at time t, we acknowledged the future entry of k cohorts at times $t+1, ..., t+k$. The corresponding benefit policy equation would be

$$B(t) = \frac{F(t) + PVFC(t) + PVFC''_k - PVAB(t-1)e^{\delta} - TPmt(t-1)e^{\delta}}{d(t) + d''_k}, \tag{6.6}$$

where $PVFC(t)$, $PVAB(t)$, $TPmt(t)$ and $d(t)$ are as previously defined, $PVFC''_k$ is the present value of the future contributions in respect of the k new cohorts anticipated to enter at times $t+1, ...t+k$, and d''_k is the present value of an annual pension of 35 payable to the same k new cohorts. We have

$$PVFC''_k = \sum_{i=1}^{k} nC\ddot{a}_{\overline{35|}}e^{-\delta i} = nC\ddot{a}_{\overline{35|}} \cdot a_{\overline{k|}} \tag{6.7}$$

and

$$d''_k = 35n \cdot {}_{35|}\ddot{a}_{65} \sum_{i=1}^{k} e^{-\delta i} = 35n \cdot {}_{35|}\ddot{a}_{65} \cdot a_{\overline{k|}}. \tag{6.8}$$

Using this method, the benefit accrual in the first year is

$$B(1) = b^* + \frac{G(1)}{d(1) + d''_k} \tag{6.9}$$

and subsequent years' accruals are adjusted recursively as follows:

$$B(t+1) = B(t) + \frac{G(t+1)}{d(t+1) + d''_k} + \frac{[b^* - B(t)]\, d(0)e^{-k\delta}}{d(t) + d''_k}, \quad t \geq 1. \tag{6.10}$$

Note that this is almost exactly the same as our original recursion in equation (4.4a) with only three small modifications:

- each year's investment gain (or loss) is spread out over more cohorts,
- the anticipated over- or under-contribution, $(b^* - B(t)) \cdot d(0)$ is also spread out over more cohorts, and
- there is an extra discount factor in the numerator of the third term to reflect the fact that the newest cohort being recognized is the one entering k years from the valuation date.

If we were to recognize all possible future cohorts right from the very first valuation, $PVFC''_k$ and d''_k would form perpetuities:

$$PVFC''_\infty = \sum_{i=1}^{\infty} nC\ddot{a}_{\overline{35|}} e^{-\delta i} = \frac{nC\ddot{a}_{\overline{35|}}}{e^\delta - 1}, \tag{6.7a}$$

$$d''_\infty = 35n \cdot {}_{35|}\ddot{a}_{65} \sum_{i=1}^{\infty} e^{-\delta i} = \frac{35n \cdot {}_{35|}\ddot{a}_{65}}{e^\delta - 1}. \tag{6.8a}$$

In this case, the benefit accrual applicable in the first year would become

$$B(1) = b^* + \frac{G(1)}{d(1) + d''_\infty} \tag{6.9a}$$

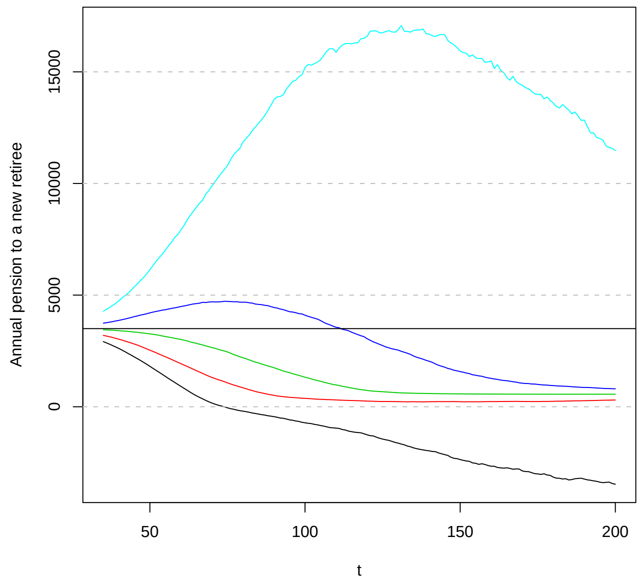
and the recursion equation for subsequent years' accruals would be

$$B(t+1) = B(t) + \frac{G(t+1)}{d(t+1) + d''_\infty}, \qquad t \geq 1. \tag{6.10a}$$

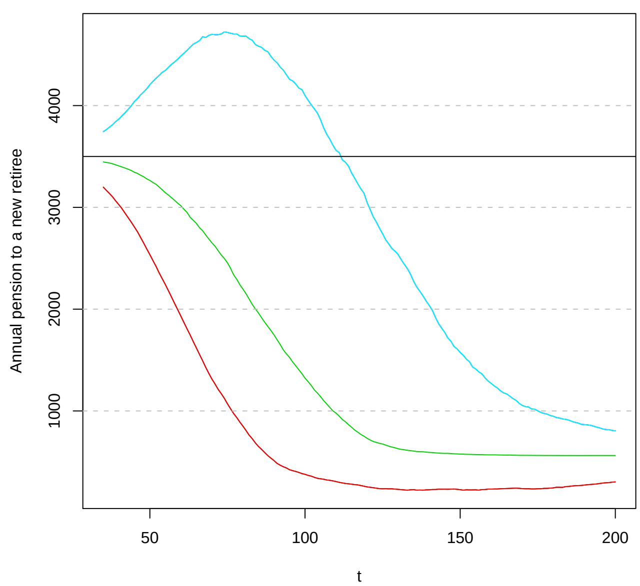
Note the conspicuous lack of a new cohort adjustment in equation (6.10a). This is not surprising, since if the entry of all possible future cohorts is recognized at $t = 1$, there are no other "new" cohorts left to be recognized at future valuations, so no further adjustments are required.

Now let us examine the performance of such an open valuation with $k = \infty$. Figure 6.3 shows the development of the distribution of $Pen(t)$ under this modified benefit policy. The performance appears to be the worst one yet: although the IQR is very narrow (approximately 500), the median level of benefits is extremely low (562, or less than 20% of our target). Also, the probability of ruin is extremely high with the fund being completely depleted at least once during our 200-year horizon in 7,770 of our 10,000 scenarios.

The main reasons for this poor performance are the higher spread factor $d(t) + d''_\infty$ and the lack of a new-cohort adjustment that would pull $B(t)$ back towards the target b^*. This is therefore not a recommended policy modification, although we note that performance improves considerably when combined with a provision for adverse deviation.



(a) 5th, 25th, 50th, 75th and 95th percentiles



(b) 25th, 50th and 75th percentiles only

Figure 6.3: Percentiles of the empirical distribution of $Pen(t)$, full open group valuation

Chapter 7

Conclusions

In this study, we examine the operational characteristics of a simple target benefit plan with fixed annual contributions and variable benefit accruals that are determined each year based on an aggregate funding requirement. Under our simple design, retired members' benefits are shielded from all investment risk and are annuitized at the date of retirement. A constant number of new entrants each year is assumed, building up to a stationary population. An AR(1) model is used to project investment returns on the pension fund and the model is calibrated using Canadian data. In addition to the market-consistent parameter set derived from historical data, which turns out to be a white noise model, two other parameter sets are considered with different levels of correlations between annual returns.

The benefit policy is specified and is translated into a recursion for the annual benefit accrual rate, taking the form of an autoregressive process with non-normal, heteroscedastic innovations. The distributional properties of the benefit accruals are estimated by simulation and a growing stability is observed in the empirical distribution over a 200-year horizon.

The distribution of the total pension entitlement is also studied by simulation.

Empirical results indicate that this distribution changes over time with more and more observations at lower values as time progresses. This gradual downward shift is due to the high correlation between the annual benefit accruals and the fund value. At the same time as the bulk of the distribution shifts downward, the right tail continues to expand due to more and more extreme values appearing every year, increasing dispersion and skewness. When the asset returns are independent year to year, the changes in the distribution of the pension entitlements slow down over the 200-year horizon and a more-or-less stationary distribution is achieved. This near-stationary distribution is highly skewed, so neither the mean nor the standard deviation are practically meaningful summary statistics. Instead, their more robust counterparts (the median and the interquartile range) are used to capture key features of the distribution. The median pension entitlement is significantly below the target level of pension and the interquartile range is relatively wide. Similar results are obtained when the investment returns exhibit a low level of positive correlation; however, when the correlation between consecutive years' returns is higher, the distribution of the pension entitlements becomes quite widely dispersed and fails to stabilize within the 200-year horizon. Probabilities of ruin are quite high even under the white noise model and get progressively worse as the correlation between annual returns increases.

The performance of the target benefit plan is then compared against a DC benchmark with the same annual contributions based on two criteria: the proximity of the pension entitlement at retirement to the target benefit, and intergenerational equity. The DC benchmark is clearly superior with respect to the first criterion. To assess intergenerational equity, a new variable, $D^h(t)$, is introduced which is the percentage change in pension entitlements between the cohort retiring at time t and the cohort retiring h years earlier at time $t-h$, along the same simulation path. The distribution of $D^h(t)$ under the target benefit plan is comparable to the distribution of $D^h(t)$ under the DC benchmark for cohorts up to 10 years apart, but the variability of benefits actually increases relative to the DC benchmark for non-overlapping cohorts ($h \geq 35$). Once again, the higher the correlation of the investment returns, the worse the target benefit plan's performance is relative to the DC benchmark.

The performance of the TBP improves sharply in every respect when a provision for adverse deviations is added in the form of a 75 basis point margin in the valuation interest rate. By contrast, changing to an open group valuation impairs performance, at least in the context of an aggregate funding requirement.

Appendix A

Asset Data for Model Estimation

The following data were used in subsection 3.1.2 to construct portfolio returns for the simplified TBP, from which the parameters of the asset model were estimated. The figures shown below are percentage rates of change/return on long-maturity Canadian bonds (labeled "Cdn Long Bonds") and Canadian stocks (labeled "Cdn Equity") on a market basis assuming purchase on December 31 of the previous year and sale on December 31 of the current year, including reinvested dividends, coupons or payments.

Source: CIA Report on Canadian Economic Statistics, 1924-2009.

Year	Cdn Long Bonds	Cdn Equity	Year	Cdn Long Bonds	Cdn Equity
1960	7.1	1.78	1985	25.26	25.07
1961	9.78	32.75	1986	17.54	8.95
1962	3.05	-7.09	1987	0.45	5.88
1963	4.26	15.6	1988	10.45	11.08
1964	6.97	25.43	1989	16.29	21.37
1965	0.96	6.68	1990	3.34	-14.8
1966	1.55	-7.07	1991	24.43	12.02
1967	-2.2	18.09	1992	13.07	-1.43
1968	-0.8	22.45	1993	22.88	32.55
1969	-2.01	-0.81	1994	-10.46	-0.18
1970	21.98	-3.57	1995	26.28	14.53
1971	11.55	8.01	1996	14.29	28.35
1972	1.11	27.38	1997	17.45	14.98
1973	1.71	0.27	1998	14.13	-1.58
1974	-1.69	-25.93	1999	-7.15	31.71
1975	2.82	18.48	2000	13.64	7.41
1976	19.02	11.02	2001	3.92	-12.57
1977	5.97	10.71	2002	10.09	-12.44
1978	1.29	29.72	2003	8.06	26.72
1979	-2.62	44.77	2004	8.46	14.48
1980	2.06	30.13	2005	15.05	24.13
1981	-3.02	-10.25	2006	3.22	17.26
1982	42.98	5.54	2007	3.3	9.83
1983	9.6	35.49	2008	13.65	-33
1984	15.09	-2.39	2009	-4.26	35.05

Appendix B

Mortality Table

1983 Group Annuitant Mortality Table, Males (Table 18-A)

Source: SOA Committee on Annuities (1983). Development of the 1983 Group Annuity Mortality Table. *Transactions of the Society of Actuaries*. 35. 880.

Age	Rate	Age	Rate	Age	Rate	Age	Rate
20	0.000377	45	0.002183	70	0.02753	95	0.234086
21	0.000392	46	0.002471	71	0.030354	96	0.248436
22	0.000408	47	0.00279	72	0.03337	97	0.263954
23	0.000424	48	0.003138	73	0.03668	98	0.280803
24	0.000444	49	0.003513	74	0.040388	99	0.299154
25	0.000464	50	0.003909	75	0.044597	100	0.319185
26	0.000488	51	0.004324	76	0.049388	101	0.341086
27	0.000513	52	0.004755	77	0.054758	102	0.365052
28	0.000542	53	0.0052	78	0.060678	103	0.393102
29	0.000572	54	0.00566	79	0.067125	104	0.427255
30	0.000607	55	0.006131	80	0.07407	105	0.469531
31	0.000645	56	0.006618	81	0.081484	106	0.521945
32	0.000687	57	0.007139	82	0.08932	107	0.586518
33	0.000734	58	0.007719	83	0.097525	108	0.665268
34	0.000785	59	0.008384	84	0.106047	109	0.760215
35	0.00086	60	0.009158	85	0.114836	110	1
36	0.000907	61	0.010064	86	0.12417		
37	0.000966	62	0.011133	87	0.13387		
38	0.001039	63	0.012391	88	0.144073		
39	0.001128	64	0.013868	89	0.154859		
40	0.001238	65	0.015592	90	0.166307		
41	0.00137	66	0.017579	91	0.178214		
42	0.001527	67	0.019804	92	0.19046		
43	0.001715	68	0.022229	93	0.203007		
44	0.001932	69	0.024817	94	0.217904		

Appendix C

Derivation of Recursion for $PVFC(t)$

The formula for $PVFC(t)$ is given in equation (3.9). Then at time $t+1$ we have

$$
\begin{aligned}
PVFC(t+1) &= C \sum_{x+1=30}^{64} n(x+1, t+1) \cdot \ddot{a}_{\overline{65-(x+1)}|} \\
&= C \cdot n(30, t+1) \cdot \ddot{a}_{\overline{35}|} + C \sum_{x+1=31}^{64} n(x+1, t+1) \cdot \ddot{a}_{\overline{65-(x+1)}|} \\
&= C \cdot n(30, t+1) \cdot \ddot{a}_{\overline{35}|} + C \sum_{x=30}^{63} n(x, t) \cdot \left(\ddot{a}_{\overline{65-x}|} - 1\right) e^{\delta} \\
&= C \cdot n(30, t+1) \cdot \ddot{a}_{\overline{35}|} + Ce^{\delta} \sum_{x=30}^{63} n(x, t) \cdot \ddot{a}_{\overline{65-x}|} - Ce^{\delta} \sum_{x=30}^{63} n(x, t).
\end{aligned}
$$

Adding and subtracting $Ce^{\delta} n(64, t)$ on the right side, we get

$$PVFC(t+1) = C \cdot n(30, t+1) \cdot \ddot{a}_{\overline{35}|} + PVFCT(t)e^{\delta} - TCon(t)e^{\delta}.$$

Appendix D

Derivation of Recursion for $PVAB(t)$

From section 3.3 we have

$$PVAB(t) = \sum_{x=31}^{65} n(x,t) \cdot \left(\sum_{i=0}^{x-31} B(t-i) \right) {}_{65-x|}\ddot{a}_x.$$

Then at time $t+1$,

$$\begin{aligned}
PVAB(t+1) &= \sum_{x+1=31}^{65} n(x+1,t+1) \left(\sum_{j=0}^{x+1-31} B(t+1-j) \right) {}_{65-(x+1)|}\ddot{a}_{x+1} \\
&= \sum_{x+1=31}^{65} n(x+1,t+1) \cdot B(t+1) \cdot {}_{65-(x+1)|}\ddot{a}_{x+1} \\
&\quad + \sum_{x+1=32}^{65} n(x+1,t+1) \left(\sum_{j=1}^{x+1-31} B(t+1-j) \right) {}_{65-(x+1)|}\ddot{a}_{x+1}.
\end{aligned}$$

The first term is the present value at time $t+1$ of the new benefits accrued by all plan members during the period $(t, t+1)$, that is $PVFB_2(t+1)$. Also, in the absence of pre-retirement decrements, we have $n(x,t) = n(x+1, t+1)$ and ${}_{65-(x+1)|}\ddot{a}_{x+1} = {}_{65-x|}\ddot{a}_x \cdot e^{\delta}$ for $31 \le x \le 64$. Therefore,

$$\begin{aligned} PVAB(t+1) &= PVFB_2(t+1) + \sum_{x+1=32}^{65} n(x,t) \left(\sum_{j=1}^{x+1-31} B(t+1-j) \right) {}_{65-x|}\ddot{a}_x \cdot e^{\delta} \\ &= PVFB_2(t+1) + \sum_{x=31}^{64} n(x,t) \left(\sum_{i=0}^{x-31} B(t-i) \right) {}_{65-x|}\ddot{a}_x \cdot e^{\delta}. \end{aligned}$$

Adding and subtracting $\sum_{i=0}^{34} n(65,t) B(t-i) \ddot{a}_{65} \cdot e^{\delta}$ from the right side, we get:

$$\begin{aligned} PVAB(t+1) &= PVFB_2(t+1) + e^{\delta} \sum_{x=31}^{65} n(x,t) \left(\sum_{i=0}^{x-31} B(t-i) \right) {}_{65-x|}\ddot{a}_x \\ &\quad - e^{\delta} \sum_{i=0}^{34} n(65,t) B(t-i) \ddot{a}_{65} \\ &= PVFB_2(t+1) + PVAB(t)e^{\delta} - TPmt(t)e^{\delta}. \end{aligned}$$

Appendix E

Derivation of Recursion for $B(t)$

From section 3.4 we have

$$B(t) = \frac{F(t) + PVFC(t) - PVAB(t-1) \cdot e^{\delta} + TPmt(t-1) \cdot e^{\delta}}{d(t)}$$

where

$$d(t) = n \cdot 35 \cdot \ddot{a}_{65} \cdot e^{-35\delta} + \ddot{a}_{65} \sum_{x=31}^{65} n(x,t)(65 - x + 1)e^{-(65-x)\delta}$$

in the absence of pre-retirement decrements. Then at $t + 1$ we have

$$\begin{aligned}
B(t+1) &= \frac{F(t+1) + PVFC(t+1) - PVAB(t) \cdot e^{\delta} + TPmt(t) \cdot e^{\delta}}{d(t+1)} \\
&= \frac{1}{d(t+1)} \{[F(t) + TCon(t) - TPmt(t)]e^{\delta_{t+1}} \\
&\quad + [PVFC(t) - TCon(t)]e^{\delta} + Cn\ddot{a}_{\overline{35|}} \\
&\quad - [PVAB(t-1) - TPmt(t-1)]e^{2\delta} - PVFB_2(t)e^{\delta} \\
&\quad + TPmt(t)e^{\delta}\}
\end{aligned}$$

$$\begin{aligned}B(t+1) = \frac{1}{d(t+1)}\{&[F(t) + TCon(t) - TPmt(t)]\left(e^{\delta_{t+1}} - e^{\delta}\right)\\
&+ F(t)e^{\delta} + TCon(t)e^{\delta} - TPmt(t)e^{\delta}\\
&+ PVFC(t)e^{\delta} - TCon(t)e^{\delta} + Cn\ddot{a}_{\overline{35|}}\\
&- PVAB(t-1)e^{2\delta} + TPmt(t-1)e^{2\delta} - PVFB_2(t)e^{\delta}\\
&+ TPmt(t)e^{\delta}\}.\end{aligned}$$

Note that $F(t)e^{\delta} + PVFC(t)e^{\delta} - PVAB(t-1)e^{2\delta} + TPmt(t-1)e^{2\delta} = B(t)d(t)e^{\delta}$ and $Cn\ddot{a}_{\overline{35|}} = 35b^{*}n\ddot{a}_{65}e^{-35\delta}$ so we have

$$B(t+1) = \frac{G(t+1)}{d(t+1)} + \frac{B(t)d(t)e^{\delta}}{d(t+1)} - \frac{PVFB_2(t)e^{\delta}}{d(t+1)} + \frac{35b^{*}n\ddot{a}_{65}e^{-35\delta}}{d(t+1)}.$$

Focusing on the second and third terms, we have

$$\begin{aligned}&B(t)d(t)e^{\delta} - PVFB_2(t)e^{\delta}\\
&= B(t)35n\ddot{a}_{65}e^{-35\delta}e^{\delta} + B(t)\ddot{a}_{65}\sum_{x=31}^{65} n(x,t)(65-x+1)e^{-(65-x)\delta}e^{\delta}\\
&\quad - B(t)\ddot{a}_{65}\sum_{x=31}^{65} n(x,t)e^{-(65-x)\delta}e^{\delta}\\
&= B(t)35n\ddot{a}_{65}e^{-34\delta} + B(t)\ddot{a}_{65}\sum_{x=31}^{64} n(x,t)(65-x)e^{-(65-x-1)\delta}\\
&= B(t)\ddot{a}_{65}\sum_{x=30}^{64} n(x,t)(65-x)e^{-(65-x-1)\delta}.\end{aligned}$$

In the absence of pre-retirement decrements, $n(x,t) = n(x+1, t+1)$ for $30 \leq x \leq 64$. Let $y = x+1$ and change the index of summation from x to y, so

$$\begin{aligned} B(t)d(t)e^{\delta} - PVFB_2(t)e^{\delta} &= B(t)\ddot{a}_{65} \sum_{y=31}^{65} n(y, t+1)(65 - y + 1)e^{-(65-y)\delta} \\ &= B(t)d(t+1) - B(t)d(0). \end{aligned}$$

Then

$$\begin{aligned} B(t+1) &= \frac{G(t+1)}{d(t+1)} + \frac{B(t)d(t+1)}{d(t+1)} - \frac{B(t)d(0)}{d(t+1)} + \frac{35b^* n\ddot{a}_{65}e^{-35\delta}}{d(t+1)} \\ &= \frac{G(t+1)}{d(t+1)} + B(t) - \frac{B(t)d(0)}{d(t+1)} + \frac{b^* d(0)}{d(t+1)} \\ &= B(t) + \frac{G(t+1)}{d(t+1)} + \frac{[b^* - B(t)]d(0)}{d(t+1)}. \end{aligned}$$

Appendix F

$D_h(t)$ under Low-Correlation Parameter Set

The figures on the following page show percentiles of the empirical distributions of $D_h(t)$, the percentage difference in pensions to members retiring at time t versus $t-h$, for selected values of h under the low-correlation parameter set (i.e. with $\phi = 0.2$ in the AR(1) model for δ_t).

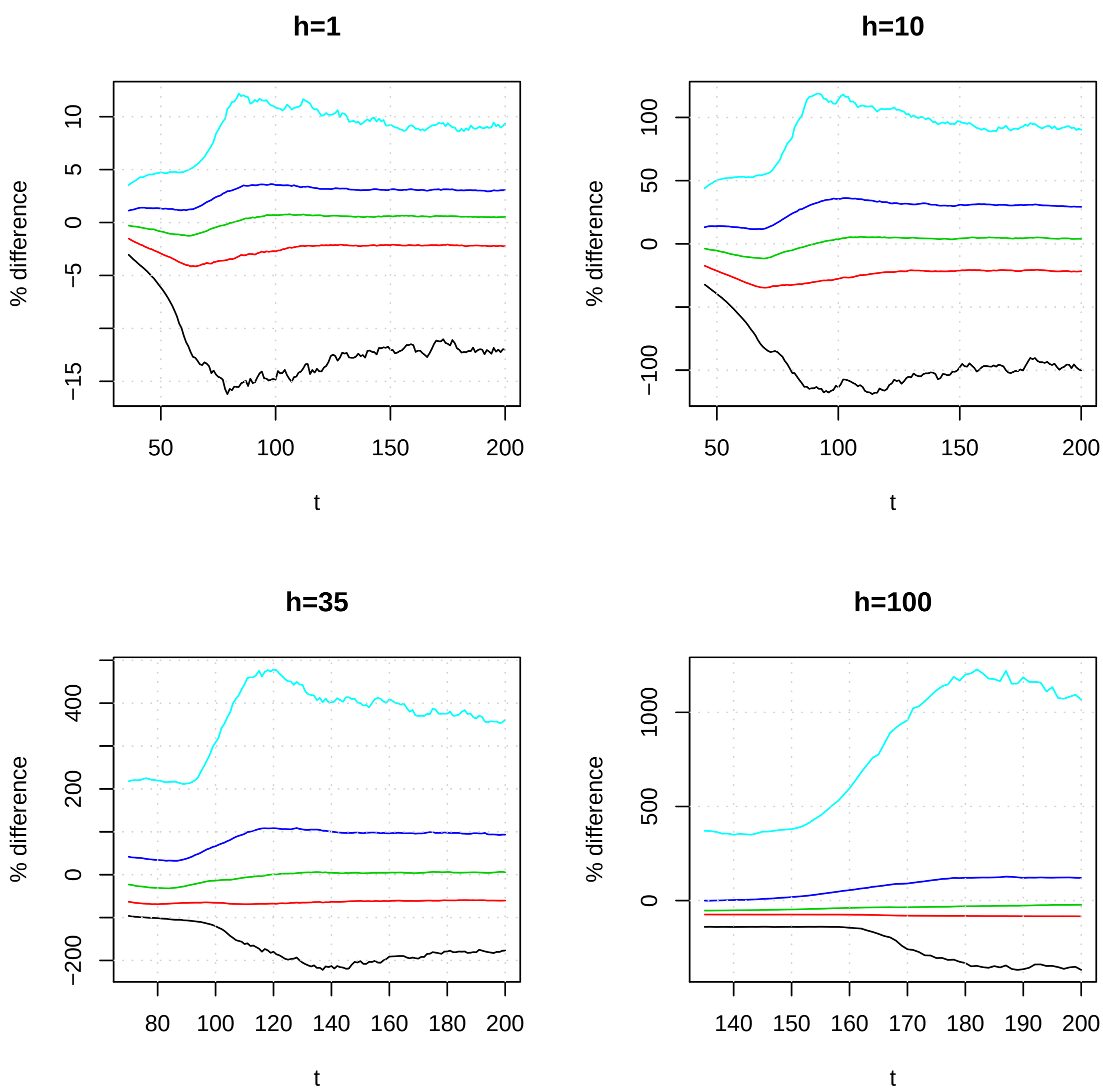


Figure F.1: 5th, 25th, 50th, 75th and 95th percentiles of $D^h(t)$ under the low-correlation parameter set

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