STRING THEORY, DUAL THEORIES AND D-BRANES

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B.Sc. University of Calgary, 1997

M.Sc. Simon Fraser University, 1999

THESIS SUBMITTED IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN THE DEPARTMENT

OF

PHYSICS

SIMON FRASER UNIVERSITY

Fall 2004

APPROVAL

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Degree: Doctor of Philosophy

Title of Thesis: String Theory, Dual Theories and D-Branes

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Date Approved: October 22nd, 2004

Abstract

In the context of the Anti-de Sitter / Conformal Field Theory correspondence we consider the Berenstein-Maldacena-Nastase (BMN) sector of the large-N Super Yang-Mills theory and demonstrate explicitly the correspondence of four-impurity operators therein to known states in string theory on the pp-wave background obtained as a Penrose limit of AdS space. In the corresponding gauge theory we calculate matrix elements of the dilatation operator in the BMN operator basis. These matrix elements are found to coincide with those of the light-cone string Hamiltonian, which is computed using the string field theory vertex in the pp-wave background. Our results are in agreement with others' results obtained using gauge-theory three-point functions.

We next solve perturbative superstring theory on the Nappi-Witten background, obtaining the bosonic and fermionic spectra, and find that supersymmetry can be preserved in the Penrose limit. Our results indicate that the high-energy sector of little string theory, being holographically dual to the string theory which we solve, retains a supersymmetric spectrum. We perform a semiclassical analysis of strings in the Nappi-Witten metric and find that the relationship between energy and momentum coincides with the known result for a flat background.

In the context of Vacuum String Field Theory (VSFT), we put forth some ideas as to how a distinction might be made between 'background' D-branes, which are encoded explicitly in the formulation of split-string field theory, and 'string-field' D-branes, which correspond to solitonic lump solutions.

We use the geometrical surface-state formulation of VSFT to investigate tachyon fluctuations about certain lump solutions, called sliver states, and thereby calculate their tensions. We perform this analysis both with and without a background B-field, and are able to reproduce the standard string-theory results for the ratios of D-brane tensions.

We investigate tachyon fluctuations about another state known as the butterfly. As would be expected for a D-brane, the equation of motion derived for the tachyon field corresponds to the requirement that the quadratic term in the string-field action vanish on-shell. We begin a calculation of the tension of the butterfly and conjecture that this too will coincide with the standard D-brane expression.

Dedication

To Charlotte; I miss you.

Acknowledgements

Professor Viswanathan has been a superb supervisor, and has become a wonderful friend during my time at SFU. I have learned a great deal from him, and wish to thank Vish sincerely for his guidance, insight, generosity and patience. Actually, Vish has very little of this last quantity; the others more than make up for it.

My other collaborators also deserve thanks. Partha, Rash, Taejin and Yi have been enjoyable and inspiring to work with.

I would like to thank my parents for all the tangible and intangible ways in which they have interacted with me; I have come to realise that parents like mine are rare.

Lastly, I would like to thank Una for all her affection, understanding (though not necessarily of String Theory) and friendship.

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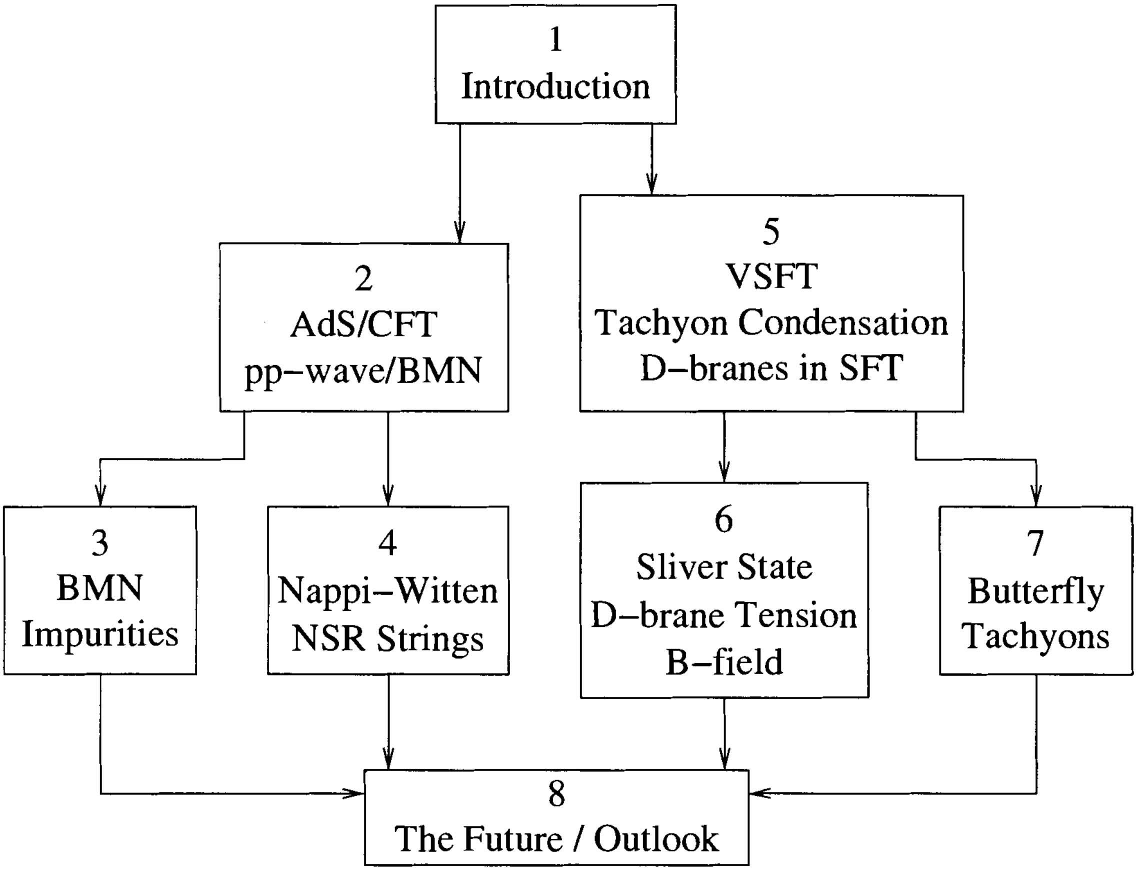

Figure 1: thesis flow diagram

Chapter 1

Strings and Duality

1.1 Introduction

String theory has now been around for many years, having first been put forward in the late 1960's as a theory of strong interactions. Despite being replaced in this regard in the early 1970's in favour of phenomenologically successful QCD, string theory due to its ever-increasing richness and depth has remained an active field of study. The motivation since that time has been the hope that it will lead eventually to a consistent 'theory of everything', by which is usually meant a unification of quantum field theory and gravity. Very early on it was realised that string theory contained in its spectrum a massless spin-two particle, which could be interpreted as the graviton. With the advent in the 1970's of superstring theory by Wess and Zumino and the GSO projection of Gliozzi, Sherk and Olive, a description of fermions was brought to light and the problem of tachyons set to rest.[1] The use of earlier ideas of Kaluza and Klein led to compactification and acceptance that the critical dimension of ten or 26 was not necessarily in conflict with nature. It was found in the 1980's that the number of ways in which string theory could be formulated was limited, and type I, IIA, IIB, and the two heterotic theories emerged as the five separate string theories which seemed to exhaust the possibilities. A beautiful field theory of strings was constructed by Witten in 1986, which although technically difficult in terms of computation, provided an underlying conceptual framework for string theory and paved the way for ideas about background independence and the origin of geometry. More recent and more rapid progress in the 1990's showed that string theory is not, after all, a theory of strings alone; Polchinski's discovery of D-branes enabled non-perturbative studies of string theory and led to the application of non-commutative geometry to D-brane worldvolumes. Maldacena formulated the AdS/CFT correspondence, bringing the idea of holographic duality to centre stage in string theory and showing that gauge theories and string theories were not entirely different entities, but shared the same degrees of freedom. Not only were the five known string theories found to be related by an intricate web of dualities, but it is conjectured and believed that all five-string theories are also dual to eleven-dimensional supergravity, and that all of these six are specific limits of the parent M-theory. Progress in the last few years has been spurred on by these developments; Vacuum String Field Theory (VSFT) was formulated following the conjecture by Sen in 1999 that D-branes could decay via tachyon condensation,

[1]Seldom is there only one solution to a problem; tachyons are not considered a 'problem' in Vacuum String Field Theory, and fermions may appear in other ways [1].

while another recent accomplishment has been the investigation of AdS/CFT beyond the supergravity limit using the Penrose limit and Berenstein, Maldacena and Nastase's identification of large R-charge operators in Yang-Mills theory with specific string states in the dual theory. It has thus come to pass that the visionary ideas of 't Hooft in the early 1970's about string and gauge theory duality have found concrete expression, and that the gauge theory of QCD, the original usurper of the strong interactions from strings, may indeed be contained in a string theory. In concluding this brief historical account, the reader is referred to the the classic text by Green, Schwarz and Witten [2] and to the more modern exposition by Polchinski [3].

This thesis documents contributions to string theory made by the author in collaboration with Taejin Lee, R. Parthasarathy, R. C. Rashkov, K. S Viswanathan, and Y. Yang. In chapter 2 we overview the ideas of AdS/CFT and specifically the Penrose limit of this correspondence in which string theory may be solved, leading to BMN gauge theory and the identification of impurity operators as the duals of string states. Chapter 3 is based on [4] in which we investigate four-impurity operators within this framework and compare with results from string field theory. In chapter 4 we solve string theory in the Nappi-Witten background, which is a Penrose limit of an NS5-brane metric; this theory is expected to be dual to a sector of Little String Theory. This work is contained in [5]. In chapter 5 we shift gears somewhat and present an overview of Vacuum String Field Theory. In introducing string-field D-branes we exhibit some of our ideas from [6] on representing explicit D-brane backgrounds before embarking on an analysis of D-brane tension using the sliver state in chapter 6, based on [7]. Chapter 7 follows with an investigation of tachyonic fluctuations about the butterfly state, based on [8]. Since we began above with a brief summary of the past, we conclude the thesis with a look toward the future in chapter 8, where we discuss where some of the developments related to work in this thesis might lead.

In the remainder of chapter 1 we get some exercise with our notation and conventions and provide some short summaries of string/gauge dualities, AdS/CFT, BMN gauge theory and Little String Theory, readying the reader for chapters 2,3 and 4. We then summarise some aspects of Cubic String Field Theory and comment on Vacuum String Field Theory, in preparation for chapters 5, 6 and 7.

1.2 Large-N Gauge Theory and Strings

It was 't Hooft who first perceived that there should be a connection between non-Abelian gauge theory and strings. Perturbative field theory involving an expansion in powers of the coupling constant was understood as a way of dealing with interactions, but 't Hooft had the ingenious idea of considering a gauge theory with, say, $U(N)$ symmetry and for large N performing an expansion in powers of $1/N$. [9, 10]

It was identified that in a large-N gauge theory the quantity $\lambda = g_{\rm YM}^2 N$, now called the 't Hooft coupling, was the proper quantity to treat perturbatively in the usual way; in writing down Feynman diagrams the number of loops corresponds to the power of λ. Each diagram requires a minimum-genus surface on which it can be drawn without overlapping propagators, and the genus h of this surface corresponds to a factor of N^{-2h}. This comes about as follows. Examination of the Yang-Mills Lagrangian for fields in the adjoint representation reveals that in Feynman diagrams, each propagator will bring with it a factor of $g_{\rm YM}^2 = \lambda/N$, while each vertex will produce the inverse, N/λ. It is also clear that each contraction of gauge indices produces a factor of N. Since the adjoint representation is a matrix field, we think of the propagator in the standard 'double-line' notation, where each line corresponds to one index. Following such a line around a given diagram, eventually we will be led back to the beginning and write a factor of N since such a closed loop represents an index contraction. Now, we view each of these closed index loops as tracing around the edge of a single face of a simplical-complex decomposition of a Riemann surface with F faces in total. We identify the propagators as the edges which border these faces, and denote their number by E. Finally, the discretised surface will have V vertices which simply correspond to the interaction vertices. The factor associated with such a diagram is, according to the above discussion,

$$\Big(\frac{N}{\lambda}\Big)^V \Big(\frac{\lambda}{N}\Big)^E N^F = N^\chi \lambda^{E-V} \tag{1.1}$$

where $\chi \equiv V + F - E$ is the Euler number of the Riemann surface. For closed, oriented surfaces, $\chi = 2 - 2h$, and we see that a given quantity C in large-N Yang-Mills theory may be written as a double expansion

$$C \sim N^c \sum_{h=0}^{\infty} N^{-2h} \sum_{l=0}^{\infty} C_{h,l} \lambda^l \tag{1.2}$$

where $C_{h,l}$ are the expansion coefficients. The power of N in front depends on the quantity being considered; a vacuum amplitude and a correlator will have different values of c. In summary, we see in eqn.(1.2) that large-N gauge theory may be formulated as a double perturbative expansion: a loop expansion in λ and a genus expansion in $1/N^2$.

Now, this begs comparison with the case of string theory. Heuristically, one can see that given some complicated mesh of propagators and vertices forming a Feynman diagram, it will correspond in the above-mentioned way to a Riemann surface of some genus, and this might bring to mind a string interaction diagram in which the same surface represents the worldsheet. A genus-one scattering diagram in gauge theory would correspond to a genus-one worldsheet endowed with vertex operators in string theory, and this identification would persist for higher-loop contributions in the gauge theory at the same genus.

1.2.1 AdS/CFT

This intuitive picture of associating a worldsheet of genus h with a mesh of gauge-theory propagators at genus h is compelling, but it it non-trivial to promote it to a precise relationship. This was done in 1997 by Maldacena [11] by considering string theory in a specific context. He considered a background of D3-branes and came to the conclusion that type IIB string theory on $AdS_5 \times S^5$ is precisely dual to 4-dimensional $\mathcal{N} = 4$ Super Yang Mills theory, which is a conformal field theory [11, 12, 13]. The worldsheet theory is interacting as the background is curved; the worldsheet coupling of the string theory is l_s/R where R is the AdS radius and l_s is the string scale. The relation to the gauge theory is

$$\left(\frac{R}{l_s}\right)^4 = \lambda. \tag{1.3}$$

We see from this that the AdS/CFT correspondence is a strong/weak duality and that the perturbative region of each theory is mapped to the the strong-coupling regime of the other. This makes the still-conjectured duality between strings on AdS and Super Yang Mills theory very interesting and potentially powerful, but also difficult to study.

The first tests of AdS/CFT were performed by considering the low-energy supergravity limit of string theory, suggested in [12]. Correlators of supergravity fields may be projected onto the boundary of the anti-de Sitter space and related to correlators in the Yang Mills theory, which may be thought of as living on this projective boundary. Since the SYM theory is a conformal theory, correlators can be found exactly and compared with those found from the string theory.

More recently it has been possible to study the AdS/CFT correspondence beyond the supergravity limit; in [14] Berenstein, Maldacena and Nastase identified a sector of the Yang Mills gauge theory dual to string theory on a Penrose limit of AdS space.

1.2.2 Plane-Waves and BMN Gauge Theory

The notion of a tangent space underlies Riemannian geometry; identifying a point, one may construct a tangent space at that point and express the metric in terms of a curvature expansion about this flat space. The natural extension of this concept of a tangent space and curvature expansion is the Penrose limit. To obtain the Penrose limit of a geometry, one first identifies a geodesic and then a similar curvature expansion may be performed about this line [15]. The resulting metric is known as a plane-wave metric, and in some cases a 'plane-parallel wave' or pp-wave metric [16]. It has been shown that string theory is generically soluble in pp-wave backgrounds when formulated in the light-cone gauge [17, 18, 19, 20, 21, 22, 23] and this led to the realisation by Berenstein, Maldacena and Nastase that string modes in these backgrounds could be identified precisely with a certain sector of Yang Mills theory. The limit involved on the Yang Mills side of the correspondence is slightly different from the usual 't Hooft large-N limit.

Berenstein, Maldacena and Nastase [14] considered a particular Penrose limit of $AdS_5 \times S^5$. Since under the full AdS/CFT duality string theory on this space is dual to the full $\mathcal{N} = 4$, $d = 4$ Super Yang Mills theory, this limit on the string side must correspond to a specific 'BMN limit' of the Yang Mills theory. The S^5 symmetry manifests itself as an $SO(6)$ R-symmetry in the gauge theory, and the so-called BMN sector on the gauge theory side was found to consist of operators with large charge under some $U(1)$ subgroup of this R-symmetry. These BMN operators are dual to string states. Anomalous conformal dimensions on the gauge side translate into light-cone energies in the string theory; impurities inserted into these operators correspond to added oscillator degrees of freedom.

This leap in understanding led to greatly increased interest in AdS/CFT, as prior progress had largely been limited to the supergravity limit. Now, proper string states could be considered and curvature corrections to the Penrose limit on the string side could be expressed as $1/J$ corrections in the dual gauge theory, where J is the R-charge and is dual to the angular momentum.

In chapter 2 we will explain AdS/CFT and the BMN limit in further detail, making use of BMN operators in chapter 3 in our investigation and comparison with string field theory results.

1.2.3 Little String Theory

As mentioned in the last subsection, string theory may be solved in general in pp-wave backgrounds, and as a starting point anti-de Sitter space (or the anti-de Sitter supergravity solution) is not the only choice. In particular, another background of interest is the NS5-brane background, whose Penrose limit is the Nappi-Witten metric.

Little String Theory is a six-dimensional theory which sports 'stringy' properties such as T-duality and a Hagedorn density of states. Although it is not known how to formulate such a six dimensional theory directly, since there is no known string theory with a critical dimension of six, LST does have a holographic description: String theory on the NS5-brane background is dual to LST on the NS5-brane worldvolume.

Taking a Penrose limit on the string side of this duality brings the NS5-brane metric to the Nappi-Witten form, while on the LST side it isolates the high-energy sector of the theory. The holographic description of LST motivates our calculation in chapter 2 of the string spectrum in the Nappi-Witten background in chapter 4. The aforementioned high-energy sector of LST must have a spectrum dual to this, and must also retain the supersymmetry, since we find that this is unbroken in the pp-wave limit.

1.3 Strings and String Fields

Most of the progress in understanding string theory has been made using worldsheet formulations of the theory; the dynamics of a string are governed by a two-dimensional theory living on its worldsheet. This 2-d conformal field theory then leads to amplitudes for configurations on Riemann surfaces of vertex operators, defining S-matrix elements. As is well recognised, this is rather different from the procedure used to perform S-matrix calculations in field theories. Generically, in a field theory there is a quadratic part to the action whence can be derived a propagator and then the higher-order terms in the action are used to define interaction vertices. The resulting Feynman rules are used to calculate scattering amplitudes. In string theory we start in some sense with the Feynman rules, or even with just the propagator defined by some initial and final states at either end of a two-dimensional worldsheet, and introduce interactions in a consistent way by representing physical states with vertex operators and connecting them with such surfaces. The conformal symmetry makes possible explicit calculations since only conformally inequivalent surfaces are physically distinct.

In this procedure one can see several drawbacks from the point of view of a fundamental formulation. One shortcoming is that this construction is perturbative; the Feynman diagrams constructed with Riemann surfaces and vertex operators will not capture non-perturbative properties. Of course, it may be possible to define a theory solely by its Feynman rules and thereby obtain a construction free from any non-perturbative physics, but given the remaining drawbacks of a worldsheet formulation mentioned below, this is not desirable. To mind come D-branes (solitons), dualities, vacuum structure and background geometry as physical properties which may not be understood or considered without at least acknowledging that there exists some sort of theory for which the worldsheet calculation framework constitutes the 'Feynman rules'.

Another deficiency in the worldsheet formulation of string theory is that one begins with a background geometry. This may be something like flat space or a plane wave where string theory is soluble or it may be a more complicated curved background like anti de Sitter space where the worldsheet theory is highly non-linear, but changes to the background are supposed to be encoded in the graviton modes that arise in the string spectrum. Although the background should be a solution to the appropriate low-energy supergravity theory, it is not otherwise derivable from the string theory point of view. It has not been possible to formulate a worldsheet description of string theory in some more general sense which is not dependent on this geometry. Consistency is obtained via the following: String theory at low energies corresponds to supergravity, and solutions to supergravity may be found. These are then suitable backgrounds on which to formulate perturbative string theory. At least in principle, one must be able to describe explicitly how a fundamental theory gives rise to and interacts with the background geometry non-perturbatively.

Finally, combining the above two points, a second-quantised theory of strings is desirable in order to describe multi-string states. String theory is defined on a space of single string states. This is insufficient since coherent states of strings should underlie supergravity background geometries, and D-branes should be expressible in a concrete way in terms of string solitons.

Cubic string field theory was introduced by Witten nearly 20 years ago [24] and has been successful in rectifying the above shortcomings of the worldsheet formulation.

1.3.1 Cubic String Field Theory

In this section we provide a brief summary and introduction to Witten's open string field theory, setting out our notation and conventions. The theory was first introduced in [24] and a supersymmetric version was then given in [25]. In these two papers Witten initially gave an abstract algebraic formulation which led Gross and Jevicki to devise an explicit operator construction [26, 27].

Let $\mathcal{A}$ be the space of string fields. We will refer to elements of $\mathcal{A}$ in an abstract way for the moment, and afterwards discuss more explicit formulations. $\mathcal{A}$ is taken to be a graded algebra with grading G, the ghost number. The associative product on $\mathcal{A}$,

$$\star : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}, \tag{1.4}$$

is called the star product. Under this product the grading is additive. There is a differential Q on $\mathcal{A}$ which on single-string states acts as the BRST operator of usual 2-d worldsheet string theory which computes the cohomology of the Virasoro algebra;

$$Q : \mathcal{A} \rightarrow \mathcal{A}. \tag{1.5}$$

Q is of course nilpotent, $Q^2 = 0$, and is a derivation with respect to the star product;

$$Q(\Psi \star \Phi) = Q\Psi \star \Phi + (-)^{G_\Psi} \Psi \star Q\Phi, \tag{1.6}$$

and $G(Q) = 1$. There is an operation called integration on $\mathcal{A}$,

$$\int : \mathcal{A} \rightarrow \mathbb{C}, \tag{1.7}$$

which is distributive with respect to addition. The integration is graded-cyclic,

$$\int \Psi \star \Phi = (-)^{G_\Psi G_\Phi} \int \Phi \star \Psi, \tag{1.8}$$

and the integral of a Q-exact state vanishes,

$$\int Q\Psi = 0, \tag{1.9}$$

as does the integral of any state of ghost number other than 3;

$$\int \Psi = 0 \text{ when } G_\Psi \neq 3. \tag{1.10}$$

The action is given by

$$S = -\frac{1}{2}\int \left(\Psi \star Q\Psi + g\frac{2}{3}\Psi \star \Psi \star \Psi\right) \tag{1.11}$$

where g is the dimensionless string coupling. The term with the differential Q is to be thought of as a kinetic term, while the cubic term encodes the interactions. The action is invariant under the gauge transformation

$$\delta\Psi = Q\Lambda + \Psi \star \Lambda - \Lambda \star \Psi \tag{1.12}$$

for any string field $\Lambda \in \mathcal{A}$. To see this, one can 'integrate by parts,' since Q is a derivation and $\int Q = 0$. The action (1.11) gives rise to the string-field equation of motion

$$Q\Psi + \Psi \star \Psi = 0. \tag{1.13}$$

Although this equation of motion was presented in 1986 by Witten, we remark that still no explicit classical solution has been found; the difficulty arises from the complicated structure of the Q operator.

Point particles classically are objects which may occupy any point x in a manifold M. A point-particle field theory involves functions on this space, so that a field is a function on M. Since a classical string may be described by its embedding x where

$x : [0, \pi] \rightarrow M$, it is natural to define a string field as a functional on the space of such embeddings $\mathcal{E}$. In fact, we must define a string field to be a map from the space of matter and ghost embeddings to the complex numbers. In this way, a string field is a functional of the worldsheet fields. In order to regain conformal invariance in the usual way on-shell, we must also accommodate the other worldsheet fields, the ghosts; A string field is then a functional $\Psi[x; c; b]$, where x is the string embedding and c and b are the ghost fields. We will in this section omit Lorentz indices and write simply x when referring to the embedding function whose values are $x^\mu(\sigma)$, with $\sigma \in [0, \pi]$. This structure may be extended to superstring field theory [25] but we will not develop any need for it. In the calculations we perform in chapter 5 and beyond, will not need to deal explicitly with the ghost degrees of freedom, so that we may think of a string field as solely a function of the string embedding, $\Psi : \mathcal{E} \rightarrow \mathbb{C}$. In the present discussion, we retain the ghosts for completeness, thus

$$\Psi : \mathcal{E} \otimes \mathcal{G}_{c,b} \rightarrow \mathbb{C}, \tag{1.14}$$

where $\mathcal{G}_{c,b}$ denotes the space of ghost embeddings for the c and b fields. In addition, it has been found convenient to bosonise the ghosts, replacing the fermionic ghost coordinates b and c with a single bosonic coordinate ϕ. This bosonisation procedure and many other aspects of the ghosts are exhibited in [28].

This explicit representation of the string-field algebra [29, 30] may be concretely written as follows. Integration is represented as

$$\int \Psi = \int \Psi[y] \prod_{0 \leq \sigma \leq \pi/2} \mathrm{d}x(\sigma)$$

$$\text{where } y(\sigma) \equiv \begin{cases} x(\sigma) & \text{for } 0 \leq \sigma \leq \pi/2 \\ x(\pi - \sigma) & \text{for } \pi/2 \leq \sigma \leq \pi. \end{cases}, \tag{1.15}$$

with the star product given by

$$\Psi_1 \star \Psi_2[z] = \int \Psi_1[x]\Psi_2[y] \prod_{\pi/2 \leq \sigma \leq \pi} \left[\delta(x(\sigma) - y(\pi - \sigma))\mathrm{d}x(\sigma)\mathrm{d}y(\pi - \sigma)\right], \tag{1.16}$$

$$\text{where } z(\sigma) \equiv \begin{cases} x(\sigma) \text{ for } 0 \leq \sigma \leq \pi/2 \\ y(\sigma) \text{ for } \pi/2 \leq \sigma \leq \pi \end{cases}.$$

The identity field is

$$I[x] = \prod_{0 \leq \sigma \leq \pi/2} \delta(x(\sigma) - x(\pi - \sigma)).. \tag{1.17}$$

In the above it is conspicuous that a midpoint has been chosen in order to deal with the two halves of the string. Although this formulation of the star product is not invariant with respect to reparametrisations of the embedding, it has been claimed [24, 25] that the Q operator takes care of this and produces a theory which is indeed insensitive to our choice of midpoint [31]. This was verified explicitly in the operator formulation which we discuss shortly. We refer the reader also to [32, 33] for a further-refined Moyal-product formulation of the star product, and also to [34].

Although we have not included ghosts explicitly in the above, the bosonised ghost ϕ behaves almost exactly like the matter embedding x with one exception; in the star product, there is a 'ghost insertion' at the midpoint. Witten's original formulation of the star product involves a field theory on a surface which is flat near three boundaries which are endowed with a string-embedding boundary condition (much like the surface states we define later in chapter 5). The integrated curvature of this surface provides an additional ghost factor, and when the surface is 'contracted' so that the string boundaries coincide, that factor appears concentrated at the midpoint. The reader is referred to the original paper by Witten [24] for details of this construction, and to [28] for details of bosonised ghosts and the midpoint insertion.

Construction of a continuum expression of the Q operator is very tedious and probably is not the best way to investigate the theory. In considering quantisation, anomalies must be understood and controlled so that the string-field algebra is faithfully represented; a continuum formulation such as we have written is unwieldy for this purpose and a operator description, first constructed by Gross and Jevicki [26, 27] has been found to be more appropriate.

We begin by defining the state space; to every string functional Ψ we associate a state $|\Psi\rangle$. We may use the position-embedding single-string basis $\{|x\rangle\}$ where $\langle x|y\rangle \equiv \delta[x-y]$ to relate the two, so that

$$|\Psi\rangle \equiv \int \mathrm{D}x\Psi[x]|x\rangle \quad , \quad \Psi[x] = \langle x|\Psi\rangle. \tag{1.18}$$

The integration operation corresponds to taking the inner product with the identity state,

$$\int \Psi = \langle I|\Psi\rangle. \tag{1.19}$$

So far this amounts to a translation of notation, but in the state formulation we may now express operations using oscillators a and $a^\dagger$ familiar from string theory in the following way.

The string field Ψ is a functional of the embedding, and it can just as easily be considered as a function of the Fourier components of this embedding; $\Psi[x,c,b] = \Psi[\{x_n, c_n, b_n\}]$, where

$$x(\sigma) = x_0 + \sqrt{2}\sum_{n=1}^{\infty} x_n \cos n\sigma \tag{1.20}$$

and

$$c_\pm(\sigma) = \sum_{n=-\infty}^{\infty} c_n e^{\pm in\sigma}, \tag{1.21}$$

$$b_\pm(\sigma) = \sum_{n=-\infty}^{\infty} b_n e^{\pm in\sigma}. \tag{1.22}$$

The ghost number is given by

$$G = \frac{1}{2}(c_0 b_0 - b_0 c_0) + \sum_{n=1}^{\infty}(c_{-n}b_n - b_{-n}c_n) \tag{1.23}$$

and the Virasoro generators are

$$L_n^x = L_{-n}^x{}^\dagger = \sum_{m=1}^{\infty}\sqrt{m(n+m)}a_m^\dagger \cdot a_{n+m} + \frac{1}{2}\sum_{m=1}^{n-1}\sqrt{m(n-m)}a_m \cdot a_{n-m} + \sqrt{n}a_0 \cdot a_n \tag{1.24}$$

and

$$L_n^{b,c} = \sum_m (n+m) b_{n-m} c_m. \tag{1.25}$$

The BRST charge is given by

$$Q = c_0(\mathcal{L}_0 - 1) + \sum_{n=1}^{\infty}(c_{-n}\mathcal{L}_n - \mathcal{L}_{-n}c_n) \tag{1.26}$$

where the combined zero-central charge Virasoro generator is defined by

$$\mathcal{L}_n = L_n^x + \frac{1}{2}L_n^{b,c}. \tag{1.27}$$

In the following discussion of basis and vertex states, we disregard the ghost modes. Although they may easily be included in the formulation, and indeed must be, we may simplify the discussion somewhat by leaving them out, and as we mentioned above, this structure will not be required for our purposes. The interested reader is referred to the papers [26, 27] for 'ghostly' details. The coordinate and momentum modes may be written in terms of oscillators, so that

$$\begin{aligned} \hat{x}_n = \frac{i}{\sqrt{2n}}(a_n - a_n^\dagger) \quad &, \quad \hat{p}_n = \sqrt{\frac{n}{2}}(a_n + a_n^\dagger) \quad , \quad n \neq 0 \\ \hat{x}_0 = \frac{i}{2}(a_0 - a_0^\dagger) \quad &, \quad \hat{p}_n = a_0 + a_0^\dagger. \end{aligned} \tag{1.28}$$

The string-field basis state may now be written

$$|x\rangle = \exp\left[-\frac{1}{2}(x|E^{-2}|x) - i\sqrt{2}(a^\dagger|E^{-1}|x) + \frac{1}{2}(a^\dagger|a^\dagger)\right]|\Omega\rangle, \tag{1.29}$$

where we have used the vector notation $|x)$ and $|a)$ to denote quantities with components x_n and $a_n^\dagger$ with $n = 0...\infty$, and the matrix E is given by

$$E_{mn}^{-1} = \delta_{mn}\sqrt{n} + \delta_m\delta_n\sqrt{2}. \tag{1.30}$$

In the above expression, $|\Omega\rangle$ is the vacuum state. The commutation relation $[a_m, a_n^\dagger] = \delta_{mn}$ along with consequent relations (the upper label p denotes an exponent)

$$[a_m^\dagger, a_n^p] = -p\delta_{mn}a_n^{p-1} \tag{1.31}$$

$$[a_m^\dagger, e^{\lambda a_n}] = -\lambda\delta_{mn}e^{\lambda a_n} \tag{1.32}$$

$$[a_m^\dagger, e^{\lambda a_n}] = -\lambda\delta_{mn}e^{\lambda a_n} \tag{1.33}$$

and

$$[a_m^\dagger, e^{(a|\lambda|a)}] = -\sum_n(\lambda_{mn} + \lambda_{nm})a_n e^{(a|\lambda|a)}, \tag{1.34}$$

may be used to show that $|x\rangle$ is an eigenstate of the $\hat{x}$ operator with eigenvalue x, the embedding function. Thus

$$\hat{x}|x\rangle = x|x\rangle. \tag{1.35}$$

Inner products correspond to a combination of star products and integration,

$$\int \Psi_1 \star \Psi_2 \star \cdots \Psi_n = \langle V_n|\Psi_1\rangle|\Psi_2\rangle \cdots |\Psi_n\rangle, \tag{1.36}$$

where the vertex squeezed state $|V_n\rangle \equiv V_n \bigotimes_{i=1}^{n} |\Omega\rangle_i$ is an element of the product Fock space of n strings. The operator representation of the star product $\star$ we denote by $*$. The identity state is given by

$$|I\rangle \sim e^{-\frac{1}{2}(a^\dagger|C|a^\dagger)}|\Omega\rangle, \tag{1.37}$$

where $C_{mn} = (-)^m \delta_{mn}$. We have already seen that $V_1 = I$. V_2 is given by

$$V_2 = e^{-(a^{\dagger 1}|C|a^{\dagger 2})}, \tag{1.38}$$

where C is the same as in eqn.(1.37), and now upper indices on the oscillators indicate in which single-string space they act. Gross and Jevicki also derived higher vertex states [26, 27], such as

$$V_3 = e^{-\frac{1}{2}(a^{\dagger i}|U^{ij}|a^{\dagger j})} \tag{1.39}$$

where the 'Neumann matrices' U^{ij} are given by (for an analysis of these matrices, see [35])

$$U^{11} = U^{22} = U^{33} = \frac{1}{3}(C + U + \bar{U}) \tag{1.40}$$

$$U^{12} = U^{23} = U^{31} = \frac{1}{6}(2C - U - \bar{U}) + \frac{i}{2\sqrt{3}}(U - \bar{U}) \tag{1.41}$$

$$U^{21} = U^{32} = U^{13} = \frac{1}{6}(2C - U - \bar{U}) - \frac{i}{2\sqrt{3}}(U - \bar{U}) \tag{1.42}$$

and

$$U = (2 - EYE^{-1} + E^{-1}YE)(EYE^{-1} + E^{-1}YE)^{-1} \tag{1.43}$$

$$Y = -\frac{1}{2}C + \frac{\sqrt{3}}{2}X \tag{1.44}$$

$$\bar{U} = CUC. \tag{1.45}$$

These matrices have the properties

$$CX = -XC, \qquad (1-Y)E(1+U) = 0, \tag{1.46}$$

$$C^2 = 1, \qquad (1+Y)E^{-1}(1-U) = 0, \tag{1.47}$$

$$Y^2 = 1. \tag{1.48}$$

Finally, X is defined by

$$\begin{aligned}
X_{nm} &= \frac{i}{\pi}(-)^{(m-n-1)/2}(1-(-1)^{m+n})\Big(\frac{1}{m+n} + \frac{(-1)^m}{m-n}\Big) \quad , \quad m \neq n \neq 0, \\
X_{0m} &= -X_{m0} = \frac{i\sqrt{2}}{\pi m}(-)^{(m-1)/2}(1-(-1)^m) \quad , \quad m \neq 0, \\
X_{nm} &= 0 \quad , \quad m = n,
\end{aligned} \tag{1.49}$$

and satisfies

$$X = X^T = -CXC \quad , \quad X^2 = 1. \tag{1.50}$$

All higher vertex states may be expressed in terms of V_3, since it may also be used to compute string-field star products,

$$\langle \Psi_1 * \Psi_2 | = \langle V_3 | \Psi_1 \rangle | \Psi_2 \rangle. \tag{1.51}$$

Such products are associative and can thus be used to build up and vertex state $|V_n\rangle$.

Squeezed states are especially simple to work with in the operator formalism. To go back and forth between the functional and state formulation for such states, we may use the following correspondence. A squeezed state of the form

$$|\Psi\rangle = \exp\left[-\frac{1}{2}(a^\dagger|S|a^\dagger)\right]|\Omega\rangle, \tag{1.52}$$

is described by the string functional

$$\Psi[x] = \exp\left[-\frac{1}{2}(x|E^{-1}\frac{1-S}{1+S}E^{-1}|x)\right]. \tag{1.53}$$

Clearly this relationship can easily be inverted. The string wave functional of the ground-state is obtained by setting $S = 0$,

$$\Omega[x] = \exp\left[\frac{1}{2}(x|E^{-2}|x)\right]. \tag{1.54}$$

In the operator language, the action (1.11) is written

$$S = -\frac{1}{2}\langle\Psi|Q\Psi\rangle - \frac{g}{3}\langle\Psi|\Psi * \Psi\rangle. \tag{1.55}$$

The preceding analysis and formulae apply to a purely bosonic formulation in which the ghost degrees of freedom have been ignored. One may not simply disregard the ghosts, and therefore this would not constitute a valid formulation of string field theory were it not for a famous conjecture by Sen [36].

1.3.2 Sen's Conjecture; Tachyons and Vacuum String Field Theory

Bosonic string theory has a tachyon mode, and traditionally this has been viewed as a severe drawback of the theory, signaling a misidentification of the vacuum. This viewpoint has been significantly broadened by Sen's conjecture [36] that unstable D-brane configurations collapse into stable configurations via condensation of the tachyon. In particular, the D25-brane background of open bosonic string theory is expected to condense into a non-perturbative tachyon vacuum [37, 38]. The study of string theory near this properly identified tachyon vacuum has been termed vacuum string field theory, and it has a number of properties which make it much simpler to deal with than regular cubic string field theory. In VSFT the usual BRST operator Q, whose cohomology classes represent physical states in the full theory, is replaced by a new operator $\mathcal{Q}$ which operates only on ghost degrees of freedom and has trivial cohomology [39]. Therefore there are no perturbative physical states in VSFT, and moreover due to this simplification it is possible to find classical solutions to the string-field equation of motion [40, 41, 42, 43]. Vacuum String Field Theory [43] has undergone significant development since Sen's conjecture, with more recent papers dealing with the so-called rolling tachyon field. In Chapter 5 we will review VSFT in more detail and in Chapters 6 and 7 we will deal specifically with two solutions to VSFT, the Sliver and the Butterfly.

Chapter 2

AdS/CFT Penrose/BMN

In this chapter we review briefly the ideas behind the AdS/CFT correspondence and the Penrose/BMN limit. In an effort to make this thesis self-contained, this is meant to prepare the reader for the treatment of four-impurity BMN operators in chapter 3 and the quantisation of NSR strings on a pp-wave in chapter 4.

2.1 Maldacena's Conjecture

Working with black brane solutions in supergravity and D-branes in string theory Maldacena was led to his conjecture [11] that $SU(N)$ $\mathcal{N}=4$ $d=4$ Super Yang Mills theory was exactly dual to superstring theory on $AdS_5 \times S^5$. We here present a summary and a few details of this correspondence. For a full treatment, the reader is encouraged to enjoy the original papers by Maldacena [11] and Witten [12] and to consult the comprehensive review [13].

AdS/CFT is an example of a holographic duality. It had been understood for some time that any quantum theory which includes gravity should exhibit holography duality [44, 45]. One way to motivate this is via the Bekenstein entropy bound [46]. In a gravitational theory, a black hole may form, and typically will have an entropy given by the area of its event horizon. In black hole formation, involving the collapse of some quantity of matter, the entropy is not expected to decrease. This means that the region inside the event horizon could not have had higher entropy than the area of its boundary, and this is expected to apply to any particular volume in such a theory; the number of degrees of freedom in a gravitational theory scales not with the volume considered, but with the area of the boundary. It is thus conjectured that all the degrees of freedom of a gravitational theory can be understood as 'living on the boundary'. Before Maldacena's conjecture, a concrete example of this was not known. Anti-de Sitter space has a projective, flat boundary at spatial infinity. The statement can therefore be made that any quantum gravity theory on AdS_{d+1} should be expressible in terms of a boundary theory on d-dimensional Minkowski space.

Let us describe a little about the Yang-Mills theory. $\mathcal{N} = 4$ is the maximal number of supercharges for the case $d = 4$. In addition to the gauge fields the theory has four fermion fields and six scalars, all in the adjoint representation. The Lagrangian is completely determined by supersymmetry. There is an $SU(4)$ R-symmetry which rotates both the six scalars and the four fermions. The theory is exactly conformal, and the conformal group in $d = 4$ is the finite-dimensional $SO(4, 2)$.

If a string theory is to be dual to this theory, it must share these symmetries; the easiest way to ensure this is as follows. The string theory must be in ten dimensions, so it is natural to begin with the background $AdS_5 \times S^5$. The isometry group of AdS_5 is $SO(4,2)$, matching the conformal group on the boundary, and the isometry group of S^5 corresponds to the Yang-Mills R-symmetry, since $SU(4) \sim SO(6)$. Remaining is the $U(N)$ symmetry, which may be accommodated in the string theory using an appropriate background of N D-branes. The supersymmetry must also map in some way to the string theory, and we remark here that the introduction of D-branes breaks (at least, depending on geometry) half of the supersymmetry. A Type II string theory has both left- and right-moving worldsheet degrees of freedom, and the supersymmetry is split between these independent fields. The insertion of a D-brane introduces the possibility of open strings, for which the left- and right-movers are no longer independent, implying that the maximal supersymmetry is half of what it was with no D-brane.

At low energies, string theory is represented by supergravity, and Dp-branes then correspond to black p-brane solutions. Black p-branes are called black because they possess event horizons. They are sources for RR-form potentials; a p-brane couples to a $p+1$-form potential $A^{(p+1)}$, with a $p+2$-form field strength $F^{(P+2)} = \mathrm{d}A^{(p+1)}$.

A specific supergravity solution with N units of RR-flux sources by a flat black p-brane may be found by beginning with the ten-dimensional supergravity action, imposing flat geometry in p dimensions and imposing spherical symmetry in the remaining $10-p$ directions. The condition

$$\int_{S^{8-p}} {}^*F^{(p+2)} = N, \tag{2.1}$$

where the integration is over a spatial $8-p$-sphere centred on the p-brane, is the condition that the p-brane sources N units of RR flux.

After solving the supergravity equations of motion, the resulting metric contains an extended singularity, which is the black p-brane itself. This singularity may or may not be hidden by an event horizon, as determined by parameters which may be adjusted. The case where the singularity and horizon coincide is called the extremal case, and in particular preserves half the supersymmetry, whereas deviations from the extreme case preserve less. This is commensurate with the above remarks on the supersymmetry-halving by D-branes.

Now, in string theory, a single Dp-brane is a source for a single unit of flux for a $p+1$-form gauge potential; a stack of N coincident D-branes will produce both the N units of $p+1$-form flux needed to match the black p-branes, and the $U(N)$ symmetry needed to match the Yang-Mills symmetry. This D-brane configuration carries the same charge and exhibits the same supersymmetry as the black p-brane considered above. The conceptual leap is to see that the low-energy gravitational black p-brane and the string description as a stack of D-branes are two complementary descriptions of the same physical object.

Considering N coincident D3-branes, the near-horizon geometry is then $AdS_5 \times S^5$ and the low-energy dynamics on the worldvolume is described by a $U(N)$ gauge theory with $\mathcal{N}=4$ supersymmetry. The radius of curvature of AdS depends on N, so that when N is large the radius of curvature is also large, and the flat-space limit appears when $N \to \infty$. The perturbative field-theory description of the object is valid when $g_s N$ is small, so that the field theory is weakly coupled but the radius of curvature, being small, means that the low-energy gravitational p-brane description is inapplicable. Conversely, when the radius of curvature is very large compared with the string scale, $g_s N$ is large, and the p-brane description is a good one. In this regime, the gauge theory is strongly coupled and so does not admit a perturbative description. The correspondence is thus between the strongly-coupled gauge theory and the near-horizon gravitational theory of the black brane.

In this way, Super Yang-Mills theory is supposed to provide a non-perturbative description of string theory on an AdS background. Since the holographic principle tells us that the boundary theory should encode all the physics in the bulk, including gravitational effects, any excitation including black holes is expected to be mirrored in the boundary theory, so long as the space is asymptotically AdS.

Since it has not been possible to verify the AdS/CFT correspondence directly, different special cases have been considered and elucidated. Initially, most analyses were carried out in the supergravity limit; there is a well-defined prescription to relate a field theory on AdS to the conformal boundary theory. Suppose ϕ is a supergravity field in the bulk AdS space. There will be a partition function in which ϕ is the integration variable,

$$Z_{AdS} = \int \mathrm{D}\phi e^{-S[\phi]}. \tag{2.2}$$

Since AdS has a boundary, this is more carefully written as a function of the boundary field ϕ_0,

$$Z_{AdS}[\phi_0] = \int_{\phi_0} \mathrm{D}\phi e^{-S[\phi]}. \tag{2.3}$$

In the boundary CFT, there will be some operator $\mathcal{O}$ which couples to the above boundary field as a source, so that the CFT generating functional is

$$W_{\mathrm{CFT}}[\phi_0] = -\log\langle e^{\int \mathcal{O}\cdot\phi}\rangle_{\mathrm{CFT}}. \tag{2.4}$$

The correspondence states that these are to be identified:

$$W_{\mathrm{CFT}}[\phi_0] \equiv Z_{AdS}[\phi_0]. \tag{2.5}$$

This identification has been used extensively to compare correlators in the AdS and Yang-Mills theories. Although AdS/CFT is a strong/weak duality, two- and three-point functions on the gauge side are fixed completely by conformal invariance and so its being strongly coupled is not a hindrance. Other tests have been performed involving identification of multiplets and states on either side, other symmetries of the spectrum, and Wilson loops. More recently, it has become possible to investigate AdS/CFT beyond the supergravity limit.

2.2 The BMN Limit

The AdS/CFT correspondence is supposed to be an exact duality, and it is highly desirable to gain understanding by subjecting it to tests beyond the supergravity limit. In doing so, not only do we build our confidence in the duality itself, which after all is still a conjecture, but we understand theories and structures better on either side.

As discussed briefly in section 1.2.2, Berenstein, Maldacena and Nastase identified a specific limit of AdS/CFT in which impurity operators on the Yang-Mills side could be identified with string states [14].

On the string side of the correspondence, a Penrose limit of AdS space is taken, producing a plane-wave background[47]. On the Yang-Mills side this corresponds to a specific sector of the theory in which both N and J, the R-charge, are taken to infinity, holding N/J^2 fixed. This is known as the BMN double-scaling limit[48, 49, 50, 51, 52, 53, 54]. Nice reviews have been given in [55] and [56]. To leading order in $g_{\rm YM}^2 N/J^2$ the light-cone gauge energy of a string state is given by [22]

$$E_{\rm LC}/\mu = \Delta - J \tag{2.6}$$

where μ is the mass parameter of the AdS space, Δ is the scaling dimension of the gauge-theory operator corresponding to the string state, and J is its R-charge. Writing the exact correspondence in terms of operators, the light-cone string Hamiltonian is given by

$$\mathcal{H}_{\rm LC}/\mu = D - J \tag{2.7}$$

where D is the dilatation operator in the Yang-Mills theory.

On the Yang-Mills side, one may write down a perturbative expansion in the effective t 'Hooft coupling $\lambda' = g_{\rm YM}^2 N/J^2$ and in the genus-counting parameter $g_2 = J^2/N$. The gauge theory sports an $SO(6)$ R-symmetry group with fields ϕ_m transforming in the vector representation. One picks two of these, say ϕ_5 and ϕ_6, and defines $Z = \phi_5 + i\phi_6$ and $\bar{Z} = \phi_5 - i\phi_6$, which have plus and minus unit R-charge respectively. The operators in correspondence with oscillator string states in this limit are the BMN operators, which are products of traces of powers of Z, sprinkled with 'impurities' which consist of the other fields [14, 57, 58, 59, 60, 61, 62, 63]. We do not try to list all relevant papers here; they are numerous. In the BMN limit, these powers are taken to be large and these operators form a basis in which one can investigate the dilatation operator D and obtain information about the light-cone string Hamiltonian. Anomalous dimensions of these operators correspond to light-cone string energies [14, 64, 62, 65]. Finding anomalous conformal dimensions amounts to diagonalising the dilatation operator, and this is made non-trivial by operator mixing [51] when non-planar and one-loop contributions are considered. Such investigations may also be carried out by considering two-point functions [49, 63, 66, 67].

Here we review the plane-wave limit and the BMN construction in more detail. We leave a description of how to find the conformal dimensions of operators, using the quantum-mechanical system approach [52], for the next chapter, where we will use the gauge theory to derive interaction vertices in string field theory.

2.2.1 Plane-wave Geometry

A Penrose limit may be taken using any geodesic, and so is not unique. The specific pp-wave limit used in the BMN correspondence is the Penrose limit associated with a geodesic trajectory around an equator of the S^5, corresponding to a string moving with large angular momentum in this direction. The large angular momentum in S^5 on the string will corresponds to large $SO(6)$ R-charge on the gauge side. The metric of $AdS_5 \times S^5$ may be written

$$\mathrm{d}s^2 = R^2\big[-\cosh^2\rho \mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\rho \mathrm{d}\Omega^2_{3(AdS)} + \cos^2\theta \mathrm{d}\psi^2 + \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\Omega'^2_{3(S)}\big] \quad (2.8)$$

where the coordinates t and ρ belong to AdS space and ψ and θ belong to S^5. R is the common radius of both AdS space and the sphere. To expand about the trajectory

$\theta = 0$, with $\psi(t)$, we first introduce the coordinates

$$\tilde{x}^{\pm} = \frac{1}{2}(t \pm \psi) \tag{2.9}$$

and then perform a scaling

$$x^+ = \tilde{x}^+ \quad , \quad x^- = R^2 \tilde{x}^- \quad , \quad \rho = \frac{r}{R} \quad , \quad \theta = \frac{y}{R}. \tag{2.10}$$

Taking the limit of large R and writing the metric in terms of $x^{\pm}$, r and y,

$$\mathrm{d}s^2 = -4\mathrm{d}x^+\mathrm{d}x^- + \mathrm{d}\mathbf{r}^2 + \mathrm{d}\mathbf{y}^2 + (\mathbf{r}^2 + \mathbf{y}^2)\mathrm{d}x^{+2} \tag{2.11}$$

where $\mathbf{r}$ and $\mathbf{y}$ are 4-d vectors. Combining $\mathbf{r}$ and $\mathbf{y}$ into one 8-d vector $\mathbf{z}$, and introducing the mass parameter μ by rescaling once more

$$x^+ \to \mu x^+ \quad , \quad x^- \to \frac{x^-}{\mu} \tag{2.12}$$

we obtain

$$\mathrm{d}s^2 = -4\mathrm{d}x^+\mathrm{d}x^- + \mu^2\mathbf{z}^2\mathrm{d}x^{+2} + \mathrm{d}\mathbf{z}^2. \tag{2.13}$$

The metric in eqn.(2.13) is that of a plane-wave and it will be this to which we refer when naming 'the pp-wave limit'. It represents the metric which a particle or string 'sees' when it is boosted to high angular momentum, traveling in the ψ direction. Now, the context of the above discussion was in standard $AdS_5 \times S^5$, but it applies as well to the supersymmetric extension of this space. In the supersymmetric case, the above Penrose limit will give rise to a three-form field strength proportional to the mass parameter μ,

$$F_{1234} = F_{5678} \propto \mu. \tag{2.14}$$

The energy and angular momentum generators in the present limit are

$$E = i\partial_t \quad \text{and} \quad J = -i\partial_\psi. \tag{2.15}$$

The momenta are defined by $p^\pm \equiv \partial_{x^\pm}$, so that

$$2p^- = -p_+ = i(\partial_t + \partial_\psi) = E - J \tag{2.16}$$

$$2p^+ = -p_- = \frac{i}{R^2}(\partial_t - \partial_\psi) = \frac{E+J}{R^2}. \tag{2.17}$$

$+$ and $-$ 'indices' on p are raised and lowered in the usual way. On the CFT side, E will correspond to Δ, the conformal dimension of an operator in Super Yang-Mills theory, while J will be the R-charge.

String theory may be solved in light-cone gauge on the pp-wave background. The string action in the metric (2.13) is given by

$$S = \frac{1}{2\pi\alpha'}\int \mathrm{d}t \int_0^{2\pi\alpha' p^+} \mathrm{d}\sigma \big[\frac{1}{2}\dot{z}^2 - \frac{1}{2}z'^2 - \frac{1}{2}\mu^2 z^2 + i\bar{\psi}(\Gamma\cdot\partial + \mu\Gamma^{1234}\psi)\big], \tag{2.18}$$

where z has eight-components, representing the transverse degrees of freedom, and ψ is an $SO(8)$ Majorana spinor. Quantisation of this action gives the light-cone Hamiltonian,

$$\mathcal{H}_{\mathrm{LC}} = \sum_{-\infty}^{\infty} N_n \sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}}. \tag{2.19}$$

Here, N_n are the occupation numbers, including both bosons and fermions. The $n < 0$ modes are left-movers while the modes with $n > 0$ are right-movers. We have not shown the details of this quantisation, but it is analogous to the quantisation we will perform in chapter 4. In terms of the pp-wave quantities defined above , we have

$$\mathcal{H}_{\mathrm{LC}} = -p_+ = 2p^-. \tag{2.20}$$

There is a constraint that the total momentum on the string vanish;

$$P = \sum_{-\infty}^{\infty} n N_n = 0. \tag{2.21}$$

When only the zero modes are excited, the above produces the massless supergravity spectrum, as one would expect. Taking the limit $\mu \to 0$ reduces the metric (2.13) to a flat-space geometry, and the above to a flat-space spectrum.

Now, we may write

$$2p^- = \Delta - J = \sum_n N_n \sqrt{1 + \frac{4n^2 R^4}{\mu^2 (\Delta + J)^2 \alpha'^2}}. \tag{2.22}$$

Since AdS/CFT requires $R^4 = \pi g_{\rm YM}^2 N \alpha'^2$, in the large-$J$ limit we have

$$\mathcal{H}_{\rm LC} = \mu \sum_n N_n \sqrt{1 + \frac{4\pi g_{\rm YM}^2 n^2 N}{\mu^2 J^2}}. \tag{2.23}$$

2.2.2 Classical Solutions

Gubser, Klebanov and Polyakov [68] performed a semi-classical analysis of string solitons. In this paper they consider string theory in the full $AdS_5 \times S^5$ background. Although the resulting worldsheet theory is rather non-linear, classical solutions corresponding to particularly symmetric string motions may be found. These classical solutions sport conserved quantities such as energy, angular momentum and spin, and these may be considered as large quantum numbers. It is then possible to relate them to operators in the super Yang Mills theory. The original analysis of [68], which contained calculations of three distinct string configurations in $AdS_5 \times S^5$, has since been extended in various ways [69, 70, 71, 71]. There are now very many papers on the subject, and we do not try to give a comprehensive list here. In the next section we go into some detail as to how perturbative string excitations are mapped to SYM operators. When looking at classical solutions on the string side, one may pick out in particular solutions with some maximal spin to energy ratio (the leading Regge trajectory). Typically some intuition is used for this, and a 'once-folded' string configuration is analysed (the closed string looks like a propeller). We perform a brief classical analysis along these lines in section 4.5, relating the energy, angular momentum and spin in a classical treatment and comparing the relation with that obtained by quantising the theory in a pp-wave background.

2.2.3 Impurity Operators and String Excitations

The BMN limit consists of the following. A Penrose limit is taken on the string side, giving the geometry (2.13), while on the gauge side $g_{\rm YM}^2$ is held fixed and small. J^2/N is held fixed, while the large-N limit is taken. $\Delta - J$ is kept fixed and finite, which is to say that operators are considered which have large R-charge, but fixed 'defect charge'.

To derive a the prescription for associating light-cone string states with gauge-theory operators, BMN began with the identification

$$\frac{1}{\sqrt{JN^J}}\mathrm{tr}[Z^J] \quad \leftrightarrow \quad |0,p_+\rangle_{\rm LC}. \tag{2.24}$$

As mentioned earlier, the quantity Z is defined by

$$Z \equiv \phi_5 + i\phi_6, \tag{2.25}$$

so that we have chosen the $SO(2)$ subgroup of $SO(6)$ which rotates the 5-6 plane to be the R-charge $SO(2)$. The operator $\mathrm{tr}[Z^J]$ is the only operator with $\Delta - J$=0, and so uniquely should correspond to the string ground-state. The normalisation factor $1/\sqrt{JN^J}$ is obtained by the consideration of two-point functions. The next step is the representation of string excitations, with which the spectrum can be built from the ground state $|0,p_+\rangle$. This can be done by inserting impurities into the state $\mathrm{tr}[Z^J]$. Available for this purpose are the other four scalar fields, $\phi_i|_{i=1}^4$, four derivatives of Z in those directions, D_iZ, and the eight positive-J components of the gaugino field, $\chi_+^a|_{a=1}^8$. On the string side, excited states are built using the bosonic and fermionic transverse oscillators $a^{\dagger i}_n|_{i=1}^8$ and $\psi^b{}_n|_{b=1}^8$. The identification between string excitations and impurities is then as follows:

$$a^{\dagger i} \leftrightarrow D_iZ \quad i = 1..4 \tag{2.26}$$

$$a^{\dagger i} \leftrightarrow \phi_{i-4} \quad i = 5..8 \tag{2.27}$$

$$\psi^a \leftrightarrow \chi_+^a \quad a = 1..8. \tag{2.28}$$

This identification is fine as it stands for the zero modes, but it remains to specify in general how the oscillator numbers (which we have not written above) are to be represented in the Yang-Mills operators. We introduce the notation

$$\mathcal{O}_p = \mathrm{tr}[Z^p] \tag{2.29}$$

so that the Yang-Mills operator we identified above in eqn.(2.24) is $\mathcal{O}_J$. To represent an operator with h impurities, let us write

$$\mathcal{O}^{1,2,\ldots,h}_{p_1,p_2,\ldots,p_h} = \mathrm{tr}\big[\prod_{n=1}^{h} \varsigma_n Z^{p_n}\big] \tag{2.30}$$

where the ς represent impurities, bosonic or fermionic; each is either a ϕ, DZ or χ. We then define an h-impurity momentum basis state by

$$|n_1, n_2, \ldots, n_{h-1}; \rho\rangle \equiv$$
$$\rho^{\frac{1-h}{2}} \sum_{\sigma\in S_{h-1}} (-)^{\sigma'} \int_{\sum x_r=\rho} \mathrm{d}^h x e^{\frac{2\pi i}{\rho}\sum_{m=1}^{h} n_{\sigma(m)} \sum_{n=1}^{m} x_n} \mathcal{O}^{1,\sigma(2,\ldots,h)}(x_1, x_2, \ldots, x_h). \tag{2.31}$$

This expression warrants some explanation. The total R-charge of the operator has been set to J_0, so that the number of fields inside the trace is $J_0 + h$. The normalisation factor in front is again found via consideration of two-point functions. With J (and J_0) large, we have replaced the discrete p_i with continous coordinates $x_i = p_i/J$. The integration domain is an h-dimensional generalisation of a sort of triangle, and is a natural choice for a Fourier transform; it may be thought of as an integration over a simple $h-1$-dimensional cube of side length $\rho = J_0/J$ in the coordinates x_1, x_1+x_2, $\ldots$, $x_1+x_2+\cdots+x_{h-1}$, so that n_m are the momenta with respect to these coordinates. The summation is taken so that we include all operators with cyclically inequivalent impurity orderings, which will be different under the trace. We are free to put the number '1' impurity first inside the trace. σ' denotes the cyclicity of those impurities which are fermionic.

We can now state the duality clearly: The h-impurity momentum state defined in eqn.(2.31) is in correspondence with a string state with h excitations as set out in (2.26), (2.27) and (2.28), with the oscillator numbers $n_1, n_2, \ldots, n_{h-1}$ and $-\sum_m n_m$, so that the total oscillator number vanishes, satisfying the constraint (2.21).

Operators of the form (2.31) do not have well-defined conformal dimension to all orders. Written in a basis of such operators, the dilatation operator is diagonalised to one-loop and planar order. At higher loops and higher genera, there will be operator mixing. In chapter 3 we will investigate these operators in the case of four impurities and compute order-g_2 off-diagonal elements of the dilatation operator. These matrix elements are in correspondence with amplitudes for three-point functions in the dual theory. We make use of the string field theory cubic vertex to confirm this relationship.

Other analyses comparing super Yang-Mills matrix elements with the string field theory vertex have been performed in [72, 73, 74, 75]. Another method is that of considering the Yang-Mills operator as a spin-chain [76, 73, 77] and using the Bethe Ansatz to solve it [78].

Chapter 3

Four-Impurity Operators and BMN Correspondence

As discussed in the previous chapter, although the AdS/CFT correspondence represents a deep and important relationship between string and gauge theories, it has only been possible to test it in certain circumstances where the calculations are tractable. AdS/CFT has not been proven, but the evidence uncovered over the last few years has convinced most beyond doubt that it must be an exact and complete duality. The BMN correspondence has been a significant step in this understanding. The present chapter is based on the four-impurity BMN-limit calculations of [4].

3.1 Dilatation Operator and Impurity Basis

Here we investigate four-impurity BMN operators, using the dilatation operator to construct the string Hamiltonian for level-four states. The method we use here, called "BMN

Quantum Mechanics" has been used in the two-impurity[79] and three-impurity[80] cases. Since there are only four distinct impurity fields, more impurities cannot be considered without taking into account the combinatoric effects of repeated impurities; thus we complete the analysis of possible distinct-impurity states by considering the maximal four-impurity case. Operators with arbitrary numbers of impurities have been considered using Yang-Mills perturbation theory[67] in which matrix elements can be extracted from three-point functions. This method has been used for various combinations of scalar and vector impurities[81, 67, 59, 60, 61] and for the case of two fermion impurities[82]. Our results using BMN Quantum Mechanics will be found to agree with these perturbative Yang-Mills computations. Using the perturbative approach it was possible for the authors of [81] and [67] to consider an arbitrary number of impurities; from our experience with the four-impurity calculation, it seems tedious to consider higher-impurity states using the quantum-mechanical method. In contrast, it should be mentioned that since in the perturbative approach the three-point function is used to obtain matrix elements, the quantum-mechanical approach may lend itself more readily to the consideration of multitrace states with more than two traces.

We construct the string Hamiltonian by first diagonalising the dilatation operator at leading order and planar level, and then using this basis to write genus-one dilatation operator elements at one-loop order. The calculations, although analogous to two- and three-impurity cases, are very lengthy and tedious in comparison. The resulting Hamiltonian, expressed in terms of its matrix elements in the four-impurity state basis is expected to be in correspondence with the three-string light-cone interaction vertex calculated in String Field Theory. We calculate this vertex for level-four string states and find precise agreement with the gauge-theory calculation; the string interaction Hamiltonian of three string states with a total of four excitations is verified to correspond to the matrix dilatation operator on the gauge-theory side between the corresponding four-impurity BMN operators.

3.2 String Hamiltonian

We wish to find the string Hamiltonian, given by

$$H = \lim_{N\to\infty, N/J^2 \text{fixed}} (D - J), \tag{3.1}$$

in the basis of states of the form (2.31). As explained in [80, 79] this is not automatically Hermitian, so that H found in this way may not be directly interpreted as the string Hamiltonian. To remedy this, one begins by defining the inner product using the planar free theory,

$$\langle a|b\rangle \equiv \langle \mathcal{O}_a \mathcal{O}_b \rangle_{\text{free, planar}}. \tag{3.2}$$

H is not Hermitian with respect to this product, but with respect to the product defined by the full non-planar free correlator

$$\langle a|b\rangle_{g_2} \equiv \langle \mathcal{O}_a \mathcal{O}_b \rangle_{\text{free, full}} \equiv \langle a|S|b\rangle, \tag{3.3}$$

where S is Hermitian with respect to the original planar product. A new basis state may be defined by the non-unitary transformation

$$|\tilde{a}\rangle \equiv S^{-1/2}|a\rangle. \tag{3.4}$$

Now, a 'new' Hamiltonian $\tilde{H}$ may be defined by[83, 66, 84, 85]

$$\langle\tilde{a}|\tilde{H}|\tilde{b}\rangle \equiv \langle\tilde{a}|H|\tilde{b}\rangle_{g_2} \tag{3.5}$$

so that

$$\tilde{H} = S^{1/2} H S^{-1/2} \tag{3.6}$$

is Hermitian in the original basis and should correspond to the light-cone string Hamiltonian $\mathcal{H}_{\text{LC}}$, up to a possible unitary transformation. Now, since $\langle a|S|b\rangle = \langle a|b\rangle_{g_2}$, S is just the full non-planar mixing matrix for basis states. Expanding S and H in the genus-counting parameter g_2,

$$S = 1 + g_2\Sigma + \mathcal{O}(g_2^2) \quad , \quad H = H_0 + g_2 H_1 + \mathcal{O}(g_2^2), \tag{3.7}$$

and we may write matrix elements of the string Hamiltonian;

$$\langle a|\tilde{H}|b\rangle = \langle a|(1+\frac{1}{2}g_2\Sigma)H(1-\frac{1}{2}g_2\Sigma)|b\rangle = \langle a|H_0|b\rangle + g_2\langle a|\big(\frac{1}{2}[\Sigma, H_0] + H_1\big)|b\rangle \tag{3.8}$$

Here, H_0 is just the planar part of D_2, the dilatation operator at one-loop order while H_1 is the genus-one part of D_2. The dilatation operator is given by [48, 50, 51, 86, 87, 88, 89]

$$D = D_0 + \frac{g_{\text{YM}}^2}{16\pi^2} D_2 + \mathcal{O}(g_{\text{YM}}^4) \tag{3.9}$$

where

$$D_0 = \text{tr}\big(\phi_m \check{\phi}_m\big) \quad , \quad D_2 = - : \text{tr}\big([\phi_m, \phi_n][\check{\phi}_m, \check{\phi}_n] + \frac{1}{2}[\phi_m, \check{\phi}_n][\phi_m, \check{\phi}_n]\big) : . \tag{3.10}$$

Here, m and n are $SO(6)$ indices running from 1 to 6, $\check{\phi} = \delta/\delta\phi$ and the normal ordering symbol denotes that the enclosed ϕ-derivatives only act on fields outside it. For operators containing Z and ϕ_i fields (but no $\bar{Z}$ fields), the above expression for D_2 reduces to

$$D_2 = - : \mathrm{tr}\big([\phi_i, \phi_j][\check{\phi}_i, \check{\phi}_j] + 2[\phi_i, Z][\check{\phi}_i, \check{Z}] + \frac{1}{2}[\phi_i, \check{\phi}_j][\phi_i, \check{\phi}_j]\big) : \tag{3.11}$$

where i and j run only over impurity fields, that is from 1 to 4.

In the following we do not consider so-called 'boundary' terms in which the impurities are neighbours. In shuffling around impurities in an operator with J fields in a trace, the impurities will only end up next to each other on the order of $1/J$ of the time. As the number of fields in the operators is taken large, these terms become scarce; contributions from boundary terms will be suppressed by a factor of $1/J$. Taking the continuum limit, 'next to each other' loses its meaning and it is valid to neglect these boundary terms. Keeping the boundary terms makes diagonalisation of D very difficult in a discrete basis, but as in the continuum BMN limit they become unimportant we may safely neglect them.

With this in mind we neglect the first term in eqn.(3.11), which always produces these boundary states. Next, since the four impurities we consider are distinct, the third term may also be neglected, since it contains a double derivative of a single ϕ field. Of course, this term would have to be kept in the case of more than four impurities, or when some of the impurities are the same.

When dealing with the BMN correspondence, it is convenient to calculate using $U(N)$ as the gauge group, rather than $SU(N)$. The resultant error will be of relative order $1/N$ so that in the large-N limit this is a safe substitution. The fields are in the adjoint representation, and we do not write the generators or the group indices; ϕ denotes $\phi^a T^a$ where T^a are the generators.

It will be helpful to make note of the following contraction identities.

$$\mathrm{tr}[1] = N, \tag{3.12}$$

$$\mathrm{tr}[A\check{\phi}]\mathrm{tr}[B\phi] = \mathrm{tr}[AB] \qquad (\text{"fusion"}), \tag{3.13}$$

$$\mathrm{tr}[A\check{\phi}B\phi] = \mathrm{tr}[A]\mathrm{tr}[B] \qquad (\text{"fission"}), \tag{3.14}$$

and the complete contractions

$$\mathrm{tr}[Z^p\bar{Z}^q] = \delta_{pq}N^{p+1} + \mathcal{O}(N^{p-1}), \tag{3.15}$$

$$\mathrm{tr}[Z^p]\mathrm{tr}[\bar{Z}^qZ^r] = \delta_{p+r,q}p(r+1)N^{p+r} + \mathcal{O}(N^{p+r-2}), \tag{3.16}$$

$$\mathrm{tr}[Z^p\bar{Z}^qZ^r\bar{Z}^s] = \delta_{p+r,q+s}N^{p+r+1}(\min(p,q,r,s)+1) + \mathcal{O}(N^{p+r-1}) \tag{3.17}$$

which are easily obtained from the contraction identities; at each genus level, which is to say at each power of N, there is some number of ways of contracting all the Z fields with the $\bar{Z}$ fields one at a time, using the three identities (3.12)-(3.14).

3.3 Two Impurities

BMN quantum mechanics was first used to investigate two-impurity operators in [79]. Based on this paper, we here exhibit some details from such a two-impurity calculation, in preparation for the four-impurity case.

As we introduced in chapter 2, we use the notation

$$\mathcal{O}^{1,2,\ldots,h}_{p_1,p_2,\ldots,p_h} \equiv \mathrm{tr}[\phi_1 Z^{p_1}\phi_2 Z^{p_2}\ldots\phi_h Z^{p_h}] \tag{3.18}$$

for a single-trace operator with h scalar impurities.

The basis states for two impurity operators with l traces are given by

$$\mathcal{O}^{12}_{p_1p_2}\prod_{j=1}^{l-1}\mathcal{O}_{J_j} \tag{3.19}$$

for the case where both impurities share a trace, and by

$$\mathcal{O}^1_{p_1}\mathcal{O}^2_{p_2}\prod_{j=1}^{l-2}\mathcal{O}_{J_j} \tag{3.20}$$

for the case of trace-sundered impurities. We assign $J_0 \equiv p_1 + p_2$.

Using eqn.(3.11) we can compute the action of H_0 on these basis states. We find

$$H_0\mathcal{O}^{12}_{p_1p_2}\prod_j \mathcal{O}_{J_j} = -\frac{g_{\mathrm{YM}}^2 N}{8\pi^2}2(2\mathcal{O}^{12}_{p_1p_2} - \mathcal{O}^{12}_{p_1-1,p_2+1} - \mathcal{O}^{12}_{p_1+1,p_2-1})\prod_j \mathcal{O}_{J_j} \tag{3.21}$$

and

$$H_0\mathcal{O}^1_{p_1}\mathcal{O}^2_{p_2}\prod_j \mathcal{O}_{J_j} = 0. \tag{3.22}$$

The action of H_0 on the continuum states defined in chapter 2 may then be read off; we have

$$H_0\mathcal{O}^{12}_{x,r_0-x}\prod_j \mathcal{O}_{r_j} = -\frac{\lambda'}{8\pi^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathcal{O}^{12}_{x,r_0-x}\prod_j \mathcal{O}_{r_j}, \tag{3.23}$$

where $r_j = J_j/J$ and $x_i = p_i/J$ are thought of as continuum variables in the large-J limit.

We see here, and we will see again in the four-impurity case, that the dilatation operator, which is essentially the string Hamiltonian, becomes a second-derivative operator on the position-basis string states. This looks like the Hamiltonian for a simple non-relativistic system of two particles (one for each impurity) on a circle. The one-dimensional space is S^1 rather than $\mathbb{R}^1$ since the trace in which the impurities live is cyclic. The positions x are the distances between the particles, so that for h particles we have $h-1$ independent variables. These observations [52] will also be easy to see in the following four-impurity calculations.

The momentum basis state which solves this we read off from (2.31),

$$|n; r_0\rangle = \frac{1}{\sqrt{r_0}} \int_0^{r_0} \mathrm{d}x e^{\frac{2\pi i n x}{r_0}} \mathcal{O}^{12}(x, r_0 - x), \tag{3.24}$$

where there is only one term on the right since there are no other cyclically inequivalent orderings of the two impurities. We than have the eigenvalues of H_0 for our two basis states,

$$H_0|n; r_0\rangle = \frac{n^2}{r_0^2}|n; r_0\rangle, \tag{3.25}$$

$$H_0|r\rangle|s\rangle = 0, \tag{3.26}$$

where $|r\rangle$ and $|s\rangle$ are single-impurity states.

The matrix elements of H_1 and Σ may then be found in this basis and used to construct the string Hamiltonion as discussed in section 3.2. The off-diagonal elements of this Hamiltonian constitute interactions between different string states, and have been verified to coincide with string field theory calculations using the cubic vertex $|V_3\rangle$. In [80] this same method was used for three-impurity operators. We perform such a computation and comparison in the case of four impurities later in this chapter [4].

Now we turn our attention to H_0. The above procedure involving H_1 is at the level of a genus-one correction, as can be seen from eqn.(3.7), and the off-diagonal elements in H_1 may be used to find corrections to the anomalous dimensions. This was done to first and second order in [52] as follows. The correction to the anomalous dimension (or the string energy) for a state $|\alpha\rangle$ may be found using standard quantum-mechanical perturbation theory. Writing the H_0 eigenvalue of $|\alpha\rangle$ as $E^{(0)}$, the first-order correction is given by

$$E^{(1)}_{|\alpha\rangle} = \sum_{\beta\neq\alpha} \frac{\langle\alpha|H_1|\beta\rangle\langle\beta|H_1|\alpha\rangle}{\langle\alpha|\alpha\rangle(E^{(0)}_{|\alpha\rangle} - E^{(0)}_{|\beta\rangle})}. \tag{3.27}$$

In [52] this is computed for two-impurity states, along with the next-order correction $E^{(2)}$. We comment on these corrections in the four-impurity case in section 3.6.

3.4 Four-Impurity BMN Operators

Since there are six ϕ-fields and two are used to define Z and $\bar{Z}$, we can accommodate up to four impurities while keeping them distinct. In the present chapter we are interested in investigating four-impurity operators; there are five ways of spreading four impurities among different traces, so that l-trace four-impurity BMN operator basis elements may be written

$$\text{`4'}: \qquad \mathcal{O}^{1234}_{p_1p_2p_3p_4} \prod_{j=1}^{l-1} \mathcal{O}_{J_j} \tag{3.28}$$

$$\text{`31'}: \qquad \mathcal{O}^{123}_{p_1p_2p_3}\mathcal{O}^{4}_{p_4} \prod_{j=1}^{l-2} \mathcal{O}_{J_j} \tag{3.29}$$

$$\text{`22'}: \qquad \mathcal{O}^{12}_{p_1p_2}\mathcal{O}^{34}_{p_3p_4} \prod_{j=1}^{l-2} \mathcal{O}_{J_j} \tag{3.30}$$

$$\text{`211'}: \qquad \mathcal{O}^{12}_{p_1p_2}\mathcal{O}^{3}_{p_3}\mathcal{O}^{4}_{p_4} \prod_{j=1}^{l-3} \mathcal{O}_{J_j} \tag{3.31}$$

$$\text{`1111'}: \qquad \mathcal{O}^{1}_{p_1}\mathcal{O}^{2}_{p_2}\mathcal{O}^{3}_{p_3}\mathcal{O}^{4}_{p_4} \prod_{j=1}^{l-4} \mathcal{O}_{J_j} \tag{3.32}$$

where $\mathcal{O}_p$ simply indicates an operator with no impurities, $\mathrm{tr}[Z^p]$. We shall use the indicated labels '4', '31', '22', '211' and '1111' as a short-hand way of referring to these operators. It may be noted that the superscripts in the above, in addition to indicating which impurities are present, also denote the order of the impurities. In general, cyclically inequivalent impurity orderings within a trace produce distinct operators. In the case of two impurities there is only one possibility. We write $J_0 = p_1 + p_2 + p_3 + p_4$ so that $J = \sum_{j\geq 0} J_j$ is the total R-charge of the operator. Taking the continuum BMN limit $J \to \infty$ with $x_i = p_i/J$ and $r_i = J_i/J$, we now write string states corresponding to the above. According to the map discussed in chapter 2, these string states are

$$\text{`4'}: \qquad \frac{\sqrt{N^{J+4}}}{J}|x_1, x_2, x_3, x_4\rangle^{1234} \prod_{j=1}^{l-1} |r_j\rangle \tag{3.33}$$

$$\text{`31'}: \qquad \frac{\sqrt{N^{J+4}}}{J}|x_1, x_2, x_3\rangle^{123} |x_4\rangle^4 \prod_{j=1}^{l-2} |r_j\rangle \tag{3.34}$$

$$\text{`22'}: \qquad \frac{\sqrt{N^{J+4}}}{J}|x_1, x_2\rangle^{12} |x_3, x_4\rangle^{34} \prod_{j=1}^{l-2} |r_j\rangle \tag{3.35}$$

$$\text{`211'}: \qquad \frac{\sqrt{N^{J+4}}}{J}|x_1, x_2\rangle^{12} |x_3\rangle^3 |x_4\rangle^4 \prod_{j=1}^{l-3} |r_j\rangle \tag{3.36}$$

$$\text{`1111'}: \qquad \frac{\sqrt{N^{J+4}}}{J}|x_1\rangle^1 |x_2\rangle^2 |x_3\rangle^3 |x_4\rangle^4 \prod_{j=1}^{l-4} |r_j\rangle \tag{3.37}$$

These states are orthonormal generalisations of the discrete states. Their normalisations may be understood by examining their tree-level planar two-point functions[80] which are to be identified with their inner products. The discrete states (3.28)-(3.32) are orthogonal, and give Kronecker δ functions when two-point functions are computed. These Kronecker δ functions combine with factors of J which have been included in the continuum states produce δ-functions of x. There will be a factor of N^{J+4} (at the planar level) which has been absorbed into the continuum states. We have thus

$$
\begin{aligned}
{}^{1234}\langle x_1, x_2, x_3, x_4 | y_1, y_2, y_3, y_4\rangle^{1234} &= \delta(x-y) \\
{}^{4}\langle x_4|^{123}\langle x_1, x_2, x_3 | y_1, y_2, y_3\rangle^{123} |y_4\rangle^{4} &= \delta(x-y) \\
{}^{34}\langle x_3, x_4|^{123}\langle x_1, x_2 | y_1, y_2\rangle^{123} |y_3, y_4\rangle^{34} &= \delta(x-y) \\
{}^{4}\langle x_4|^{3}\langle x_3|^{12}\langle x_1, x_2 | y_1, y_2\rangle^{12} |y_3\rangle^{3}|y_4\rangle^{4} &= \delta(x-y) \\
{}^{4}\langle x_4|^{3}\langle x_3|^{2}\langle x_2|^{1}\langle x_1 | y_1\rangle^{1} |y_2\rangle^{2}|y_3\rangle^{3}|y_4\rangle^{4} &= \delta(x-y).
\end{aligned}
\tag{3.38}
$$

Other relative impurity orderings may or may not vanish at planar order; one must analyse the associated two-point function to determine this. We investgate other impurity orderings for the states we need in section 3.4.4. These states are operators in the Yang-Mills theory, in the limit of infinitely many fields. We will refer to them as string basis states, as they are in correspondence with states on the string side of the correspondence as well. In the following section we will use these to define the momentum basis states which are directly identifiable as string states via the mapping presented in section 2.2.3. As we discussed in section 3.3, the dilatation operator written in this basis will lead us to a string Hamiltonian governing the quantum mechanics of four particles on a circle.

3.4.1 Matrix Elements of H_0

Our first step is to operate with H_0 on the discrete basis (3.28)-(3.32). Operating with D_2 and identifying the terms which do not alter the number of traces, we find

$$
\begin{aligned}
H_0 \mathcal{O}^{abcd}_{p_a p_b p_c p_d} \prod_j \mathcal{O}_{J_j} = -\frac{g^2_{\mathrm{YM}} N}{8\pi^2} \Big(-2\mathcal{O}^{abcd}_{p_a, p_b, p_c, p_d} + \mathcal{O}^{abcd}_{p_a-1, p_b, p_c, p_d+1} + \mathcal{O}^{abcd}_{p_a+1, p_b, p_c, p_d-1} \Big) \prod_j \mathcal{O}_{J_j} \\
+ \text{(three other cyclic permutations of } abcd),
\end{aligned}
\tag{3.39}
$$

$$H_0 \mathcal{O}^{abc}_{p_a p_b p_c} \mathcal{O}^{d}_{p_d} \prod_j \mathcal{O}_{J_j} = -\frac{g_{\rm YM}^2 N}{8\pi^2}\big(-2\mathcal{O}^{abc}_{p_a,p_b,p_c} + \mathcal{O}^{abc}_{p_a-1,p_b,p_c+1} + \mathcal{O}^{abc}_{p_a+1,p_b,p_c-1}\big)\mathcal{O}^{d}_{p_d} \prod_j \mathcal{O}_{J_j}$$
$$+ \text{(two other cyclic permutations of } abc), \qquad (3.40)$$

$$H_0 \mathcal{O}^{ab}_{p_a p_b} \mathcal{O}^{cd}_{p_c p_d} \prod_j \mathcal{O}_{J_j} = -\frac{g_{\rm YM}^2 N}{2\pi^2}\big(-2\mathcal{O}^{ab}_{p_a,p_b} + \mathcal{O}^{ab}_{p_a-1,p_b+1} + \mathcal{O}^{ab}_{p_a+1,p_b-1}\big)\mathcal{O}^{cd}_{p_c p_d} \prod_j \mathcal{O}_{J_j}$$
$$+ (abcd \to cdab), \qquad (3.41)$$

$$H_0 \mathcal{O}^{ab}_{p_a p_b} \mathcal{O}^{c}_{p_c}\mathcal{O}^{d}_{p_d} \prod_j \mathcal{O}_{J_j} = -\frac{g_{\rm YM}^2 N}{4\pi^2}\big(-2\mathcal{O}^{ab}_{p_a,p_b} + \mathcal{O}^{ab}_{p_a-1,p_b+1} + \mathcal{O}^{ab}_{p_a+1,p_b-1}\big)\mathcal{O}^{c}_{p_c}\mathcal{O}^{d}_{p_d} \prod_j \mathcal{O}_{J_j}, \qquad (3.42)$$

$$H_0 \mathcal{O}^{a}_{p_a}\mathcal{O}^{b}_{p_b}\mathcal{O}^{c}_{p_c}\mathcal{O}^{d}_{p_d} \prod_j \mathcal{O}_{J_j} = 0. \qquad (3.43)$$

In the continuum limit these become (x_{123} denotes $x_1 + x_2 + x_3$, etc.)

$$H_0|x_1, x_2, x_3, r_0 - x_{123}\rangle^{1234}$$
$$= -\frac{\lambda'}{8\pi^2}\big(\partial^2_{x_1} + (\partial_{x_2} - \partial_{x_1})^2 + (\partial_{x_3} - \partial_{x_2})^2 + \partial^2_{x_3}\big)|x_1, x_2, x_3, r_0 - x_{123}\rangle^{1234} \qquad (3.44)$$
$$H_0|x_1, x_2, x_3\rangle^{123}|r_0 - x_{123}\rangle^4$$
$$= -\frac{\lambda'}{8\pi^2}\big(\partial^2_{x_1} + \partial^2_{x_2} + (\partial_{x_2} - \partial_{x_1})^2\big)|x_1, x_2, x_3\rangle^{123}|r_0 - x_{123}\rangle^4 \qquad (3.45)$$
$$H_0|x_1, x_2\rangle^{12}|x_3, r_0 - x_{123}\rangle^{34}$$
$$= -\frac{\lambda'}{8\pi^2}\big((\partial_{x_1} - \partial_{x_2})^2 + \partial^2_{x_3}\big)|x_1, x_2\rangle^{12}|x_3, r_0 - x_{123}\rangle^{34} \qquad (3.46)$$
$$H_0|x_1, x_2\rangle^{12}|x_3\rangle^3|r_0 - x_{123}\rangle^4 = -\frac{\lambda'}{4\pi^2}\partial^2_{x_1}|x_1, x_2\rangle^{12}|x_3\rangle^3|r_0 - x_{123}\rangle^4 \qquad (3.47)$$

In the above expressions, we have suppressed the factor $\prod_j |r_j\rangle$ since it is unaffected by H_0. We confirm that the cyclic symmetries in these expressions match the cyclic structures of the operators. Once again, H_0 acts on the states as four particles moving on a circle; they are the four impurities moving within the trace.

The above eigenvalue equations are solved by defining appropriate momentum-basis states[49, 63] as follows. Making use of the string-state formula (2.31) for the cases of four-, three- and two-impurity states, we write

$$
\begin{aligned}
|n_1, n_2, n_3; r_0\rangle &\equiv \frac{1}{\sqrt{r_0^3}} \\
&\times \sum_{(abc)} \int_{x_{123}<r_0} \mathrm{d}^3x e^{\frac{2\pi i}{r_0}(n_a x_1 + n_b x_{12} + n_c x_{123})} \mathcal{O}^{1,a+1,b+1,c+1}(x_1, x_2, x_3, r_0 - x_{123}), \quad (3.48)
\end{aligned}
$$

$$
\begin{aligned}
|n_1, n_2; r_0 - s\rangle^{123} |s\rangle^4 &\equiv \frac{1}{r_0 - s} \\
&\times \sum_{(ab)} \int_{x_{12}<r_0-s} \mathrm{d}^2x e^{\frac{2\pi i}{r_0-s}(n_a x_1 + n_b x_{12})} \mathcal{O}^{1,a+1,b+1}(x_1, x_2, r_0 - s - x_{12}) \mathcal{O}^4(s), \quad (3.49)
\end{aligned}
$$

$$
\begin{aligned}
|n_1; r_0 - s\rangle^{12} |n_2; s\rangle^{34} &\equiv \frac{1}{\sqrt{(r_0 - s)s}} \\
&\times \int_0^{r_0-s} \mathrm{d}x_1 \int_0^s \mathrm{d}x_2 e^{2\pi i(\frac{n_1}{r_0-s}x_1 + \frac{n_b}{s}x_2)} \mathcal{O}^{12}(x_1, r_0 - s - x_1) \mathcal{O}^{34}(x_2, s - x_2), \quad (3.50)
\end{aligned}
$$

$$
\begin{aligned}
|n; r_0 - s - t\rangle^{12} |s\rangle^3 |t\rangle^4 &\equiv \frac{1}{\sqrt{r_0 - s - t}} \\
&\times \int_0^{r_0-s-t} \mathrm{d}x e^{\frac{2\pi i}{r_0-s-t}nx} \mathcal{O}^{12}(x, r_0 - s - t - x) \mathcal{O}^3(s) \mathcal{O}^4(t), \quad (3.51)
\end{aligned}
$$

while zero-impurity states are normalised as $|r\rangle \equiv \frac{1}{\sqrt{r}}\mathcal{O}(r)$. Here, $\mathcal{O}(x)$ simply denotes the continuum version of the discrete operator basis (3.28)-(3.32). $\sum_{(abc)}$ denotes a summation in which *abc* takes on each of the six permutations of 123, and similarly for $\sum_{(ab)}$. That is, included are all cyclically inequivalent permutations of the impurities within each operator; the superscripts now indicate only which impurities are contained in each trace. Reading them off from the expressions (3.44)-(3.47) for H_0, the above momentum states have the energy eigenvalues

$$
\begin{aligned}
E_{\text{`}4\text{'}} &= \frac{\lambda'}{2}\frac{n_{123}^2+n_1^2+n_2^2+n_3^2}{r_0^2}, && (3.52)\\
E_{\text{`}31\text{'}} &= \frac{\lambda'}{2}\frac{n_{12}^2+n_1^2+n_2^2}{(r_0-s)^2}, && (3.53)\\
E_{\text{`}22\text{'}} &= \frac{\lambda'}{2}\left(\frac{n_1^2}{(r_0-s)^2}+\frac{n_2^2}{s^2}\right), && (3.54)\\
E_{\text{`}211\text{'}} &= \lambda'\frac{n^2}{(r_0-s-t)^2}, && (3.55)\\
E_{\text{`}1111\text{'}} &= 0. && (3.56)
\end{aligned}
$$

Again, these do not depend on whether or not the states $\prod |r_j\rangle$ are present; H_0 will not operate on such factors. These eigenvalues may be compared with the energies obtained by quantising string theory in the pp-wave background. Expansion of the square-root in eqn.(2.23) to order λ' yields the same energies when the same excitations are included, according to the map given in section 2.2.3. The momentum states (3.48)-(3.51) are orthonormal, so that

$$
\begin{aligned}
&\prod_{j=1}^{l}\langle r_j|m_1,m_2,m_3;r_0|n_1,n_2,n_3;s_0\rangle\prod_{j=1}^{l'}|s_j\rangle\\
&\qquad = \delta_{m_1,n_1}\delta_{m_2,n_2}\delta_{m_3,n_3}\delta_{l,l'}\delta(r_0-s_0)\sum_{\pi\in S_l}\prod_{k=1}^{l}\delta(r_{\pi(k)}-s_k) \qquad (3.57)
\end{aligned}
$$

and similarly for the remaining states (3.49)-(3.51). These properties are a consequence of the inner products at the end of section 3.4.

3.4.2 Matrix Elements of $H_\pm$

We must now calculate explicitly H_1 and Σ in the above H_0-eigenstate basis. For clarity, we follow the convention of writing $H_1 = H_+ + H_-$, where H_+ increases and H_- decreases the number of traces when acting on a basis state.

Upon writing out the matrix elements of H_+ and H_- we will notice that for matrix elements not involving a '4'-state, the calculation effectively reduces to the case of fewer impurities. The matrix elements between our '31', '211' and '1111' states therefore need not be considered further, since they correspond to the three-impurity case already studied in [80] or to the two-impurity calculation of [79]. Moreover, upon careful inspection we note that the '22-22' matrix element of H_1 simply involves two copies of the '2-2' element from the two-impurity case studied in [79].

We first calculate the action of H_1 on the discrete basis states (3.28)-(3.32) and then find the matrix elements of H_1 in the momentum basis (3.48)-(3.51). Analogous to the calculation of H_0, we operate with D_2 and identify the terms which increase or decrease the number of traces. We find

$$\begin{aligned}
H_+ \mathcal{O}^{abcd}_{p_a p_b p_c p_d} \prod_j \mathcal{O}_{J_j} = & \frac{-\lambda'}{8\pi^2} \Bigg[\sum_{i=1}^{p_1-1} \left(\mathcal{O}^{abcd}_{p_a-i-1,p_b,p_c,p_d+1} + \mathcal{O}^{abcd}_{p_a-i-1,p_b+1,p_c,p_d} - 2\mathcal{O}^{abcd}_{p_a-i,p_b,p_c,p_d} \right) \mathcal{O}_i \\
& + \sum_{i=0}^{p_1-1} \Bigg(\left(\mathcal{O}^{bcd}_{p_b,p_c+1,p_a-i-1} - \mathcal{O}^{bcd}_{p_b,p_c,p_a-i} \right) \mathcal{O}^{a}_{p_d+i} \\
& + \left(\mathcal{O}^{ad}_{i,p_d+1} - \mathcal{O}^{ad}_{i+1,p_d} \right) \mathcal{O}^{bc}_{p_b,p_c+p_a-i-1} \\
& + \left(\mathcal{O}^{bc}_{p_b+1,p_a-i-1} - \mathcal{O}^{bc}_{p_b,p_a-i} \right) \mathcal{O}^{ad}_{p_c+i,p_d} \\
& + \left(\mathcal{O}^{acd}_{i,p_c+1,p_d} - \mathcal{O}^{acd}_{i+1,p_c,p_d} \right) \mathcal{O}^{b}_{p_b+p_a-i-1} \Bigg) \Bigg] \prod_j \mathcal{O}_{J_j} \\
& + (\text{three other cyclic permutations of } abcd), \qquad (3.58)
\end{aligned}$$

$$H_-\mathcal{O}^{abcd}_{p_ap_bp_cp_d}\prod_j \mathcal{O}_{J_j}=\frac{-\lambda'}{8\pi^2}\Bigg[\sum_{i=1}^{l} J_i\big(\mathcal{O}^{abcd}_{p_a+J_i-1,p_b,p_c,p_d+1}-\mathcal{O}^{abcd}_{p_a+J_i,p_b,p_c,p_d}$$

$$+\mathcal{O}^{abcd}_{p_a+1,p_b,p_c,p_d+J_i-1}-\mathcal{O}^{abcd}_{p_a,p_b,p_c,p_d+J_i}\big)\prod_{j\neq i}\mathcal{O}_{J_j}\Bigg]$$

$$+(\text{three other cyclic permutations of } abcd). \tag{3.59}$$

$$H_+\mathcal{O}^{abc}_{p_ap_bp_c}\mathcal{O}^d_{p_d}\prod_j \mathcal{O}_{J_j}=\frac{-\lambda'}{8\pi^2}\Bigg[\sum_{i=1}^{p_a-1}\big(\mathcal{O}^{abc}_{p_a-i-1,p_b,p_c+1}+\mathcal{O}^{abc}_{p_a-i-1,p_b+1,p_c}-2\mathcal{O}^{abc}_{p_a-i,p_b,p_c}\big)\mathcal{O}^d_{p_d}\mathcal{O}_i$$

$$+\sum_{i=0}^{p_a-1}\Bigg(\big(\mathcal{O}^{cb}_{p_a-i-1,p_b+1}-\mathcal{O}^{cb}_{p_a-i,p_b}\big)\mathcal{O}^a_{p_c+i}\mathcal{O}^d_{p_d}$$

$$+\big(\mathcal{O}^{ca}_{p_c+1,p_a-i-1}-\mathcal{O}^{ca}_{p_c,p_a-i}\big)\mathcal{O}^b_{p_b+i}\mathcal{O}^d_{p_d}\Bigg)\Bigg]\prod_j \mathcal{O}_{J_j}$$

$$+(\text{two other cyclic permutations of } abc), \tag{3.60}$$

$$H_-\mathcal{O}^{abc}_{p_ap_bp_c}\mathcal{O}^d_{p_d}\prod_j \mathcal{O}_{J_j}=\frac{-\lambda'}{8\pi^2}\Bigg[\sum_{i=1}^{l} J_i\big(\mathcal{O}^{abc}_{J_i+p_a-1,p_b,p_c+1}-\mathcal{O}^{abc}_{J_i+p_a,p_b,p_c}$$

$$+\mathcal{O}^{abc}_{p_a+1,p_b,p_c+J_i-1}-\mathcal{O}^{abc}_{p_a,p_b,p_c+J_i}\big)\mathcal{O}^d_{p_d}$$

$$+\sum_{i=0}^{p_a-1}\Bigg(\mathcal{O}^{dbca}_{p_a-i-1,p_b,p_c,p_d+i+1}-\mathcal{O}^{dbca}_{p_a-i,p_b,p_c,p_d+i}$$

$$+\mathcal{O}^{dbca}_{p_d+i+1,p_b,p_c,p_a-i-1}-\mathcal{O}^{dbca}_{p_d+i,p_b,p_c,p_a-i}\Bigg)\mathcal{O}_{J_i}$$

$$+\sum_{i=0}^{p_d-1}\big(\mathcal{O}^{adbc}_{p_d-i-1,p_a+i,p_b,p_c+1}-\mathcal{O}^{adbc}_{p_d-i,p_a+i,p_b,p_c}$$

$$+\mathcal{O}^{abcd}_{p_a+1,p_b,p_c+i,p_d-i-1}-\mathcal{O}^{abcd}_{p_a,p_b,p_c+i,p_d-i}\big)\mathcal{O}_{J_i}\Bigg]\prod_{j\neq i}\mathcal{O}_{J_j}$$

$$+(\text{two other cyclic permutations of } abc). \tag{3.61}$$

$$H_+\mathcal{O}^{ab}_{p_ap_b}\mathcal{O}^{cd}_{p_cp_d}\prod_j \mathcal{O}_{J_j}=\frac{-\lambda'}{8\pi^2}\sum_{i=1}^{p_a-1}2\big(\mathcal{O}^{ab}_{p_a-i-1,p_b+1}-\mathcal{O}^{ab}_{p_a-i,p_b}\big)\mathcal{O}^{cd}_{p_cp_d}\mathcal{O}_i\prod_j \mathcal{O}_{J_j}$$

$$+\begin{pmatrix} a\leftrightarrow c \\ \\ b\leftrightarrow d\end{pmatrix}, \text{ then } +(a\leftrightarrow b), \text{ then } +(c\leftrightarrow d), \tag{3.62}$$

$$
\begin{aligned}
H_- \mathcal{O}^{ab}_{p_a p_b} \mathcal{O}^{cd}_{p_c p_d} \prod_j \mathcal{O}_{J_j} = \frac{-\lambda'}{8\pi^2} \Bigg[& \sum_{i=1}^{l} J_i \big(\mathcal{O}^{ab}_{J_i+p_a-1, p_b+1} - \mathcal{O}^{ab}_{J_i+p_a, p_b} \\
& + \mathcal{O}^{ab}_{p_a+1, p_b+J_i-1} - \mathcal{O}^{ab}_{p_a, p_b+J_i} \big) \mathcal{O}^{cd}_{p_c p_d} \\
& + \sum_{i=0}^{p_a-1} \big(\mathcal{O}^{cdba}_{p_c+1, p_d+i, p_b, p_a-i-1} - \mathcal{O}^{cdba}_{p_c, p_d+i, p_b, p_a-i} \\
& + \mathcal{O}^{cbad}_{p_a-i-1, p_b, p_c+i, p_d+1} - \mathcal{O}^{cbad}_{p_a-i, p_b, p_c+i, p_d} \big) \mathcal{O}_{J_i} \Bigg] \prod_{j \neq i} \mathcal{O}_{J_j} \\
& + \begin{pmatrix} a \leftrightarrow c \\ b \leftrightarrow d \end{pmatrix}, \text{ then } + (a \leftrightarrow b), \text{ then } + (c \leftrightarrow d). \qquad (3.63)
\end{aligned}
$$

$$
\begin{aligned}
H_+ \mathcal{O}^{ab}_{p_a p_b} \mathcal{O}^{c}_{p_c} \mathcal{O}^{d}_{p_d} \prod_j \mathcal{O}_{J_j} = \frac{-\lambda'}{8\pi^2} \Bigg[& -\frac{1}{2} \sum_{i=1}^{p_a-1} \big(\mathcal{O}^{ab}_{p_a-i, p_b} - \mathcal{O}^{ab}_{p_a-i-1, p_b+1} \\
& - \mathcal{O}^{ab}_{p_a-1, p_b-i+1} + \mathcal{O}^{ab}_{p_a, p_b-i} \big) \mathcal{O}^{c}_{p_c} \mathcal{O}^{d}_{p_d} \mathcal{O}_i \Bigg] \prod_j \mathcal{O}_{J_j} \\
& + (c \leftrightarrow d), \text{ then } + (a \leftrightarrow b), \qquad (3.64)
\end{aligned}
$$

$$
\begin{aligned}
H_- \mathcal{O}^{ab}_{p_a p_b} \mathcal{O}^{c}_{p_c} \mathcal{O}^{d}_{p_d} \prod_j \mathcal{O}_{J_j} = \frac{-\lambda'}{8\pi^2} \Bigg[& -\frac{1}{2} \sum_{i=1}^{l} J_i \big(\mathcal{O}^{ab}_{J_i+p_a, p_b} - \mathcal{O}^{ab}_{J_i+p_a-1, p_b+1} \\
& + \mathcal{O}^{ab}_{p_a, p_b+J_i} - \mathcal{O}^{ab}_{p_a+1, p_b+J_i-1} \big) \mathcal{O}^{c}_{p_c} \mathcal{O}^{d}_{p_d} \\
& - \sum_{i=0}^{p_c-1} \Big(\big(\mathcal{O}^{abc}_{p_a, p_c-i, p_b+i} - \mathcal{O}^{abc}_{p_a+1, p_c-i-1, p_b+i} \\
& + \mathcal{O}^{abc}_{p_a, p_b+i, p_c-i} - \mathcal{O}^{abc}_{p_a+1, p_b+i, p_c-i-1} \big) \mathcal{O}^{d}_{p_d} \\
& + \frac{1}{2} \big(\mathcal{O}^{dc}_{p_c-i, p_d+i} - \mathcal{O}^{dc}_{p_c-i-1, p_d+i+1} \\
& + \mathcal{O}^{dc}_{p_d+i, p_c-i} - \mathcal{O}^{dc}_{p_d+i+1, p_c-i-1} \big) \mathcal{O}^{ab}_{p_a p_b} \Big) \mathcal{O}_{J_i} \\
& - \sum_{i=0}^{p_a-1} \big(\mathcal{O}^{cba}_{p_a-i, p_b, p_c+i} - \mathcal{O}^{cba}_{p_a-i-1, p_b, p_c+i+1} \\
& + \mathcal{O}^{cba}_{p_a+i, p_b, p_c-i} - \mathcal{O}^{cba}_{p_a+i+1, p_b, p_c-i-1} \big) \mathcal{O}^{d}_{p_d} \mathcal{O}_{J_i} \Bigg] \prod_{j \neq i} \mathcal{O}_{J_j} \\
& + (c \leftrightarrow d), \text{ then } + (a \leftrightarrow b). \qquad (3.65)
\end{aligned}
$$

$$H_+ \mathcal{O}^a_{p_a} \mathcal{O}^b_{p_b} \mathcal{O}^c_{p_c} \mathcal{O}^d_{p_d} \prod_j \mathcal{O}_{J_j} = 0. \tag{3.66}$$

$$\begin{aligned} H_- \mathcal{O}^a_{p_a} \mathcal{O}^b_{p_b} \mathcal{O}^c_{p_c} \mathcal{O}^d_{p_d} \prod_j \mathcal{O}_{J_j} = & \frac{-\lambda'}{8\pi^2} \Bigg[\sum_{i=0}^{p_b-1} \big(\mathcal{O}^{ab}_{p_b-i,p_a+i} - \mathcal{O}^{ab}_{p_b-i-1,p_a+i+1} \\ & + \mathcal{O}^{ab}_{p_a+i,p_b-i} - \mathcal{O}^{ab}_{p_a+i+1,p_b-i-1} \big) \mathcal{O}^c_{p_c} \mathcal{O}^d_{p_d} \Bigg] \prod_j \mathcal{O}_{J_j} \\ & + (\text{11 other permutations of } abcd), \end{aligned} \tag{3.67}$$

As with the case of H_0, we now write the continuum forms. In order to save space, we will not write out all the permutations, instead using the convention that in the following the permutations must first be carried out, and then x_d set to $r_0 - x_{abc}$ and ∂_d set to zero. We will use $abcd = 1234$, so that x_1, x_2 and x_3 are the independent coordinates. Since they are unaffected by H_+, we have no need of writing the factors of $\prod_j \mathcal{O}(r_j)$ in the expressions for H_+.

We find:

'4'

$$\begin{aligned} & H_+ \mathcal{O}^{abcd}(x_a, x_b, x_c, x_d) = \frac{-\lambda'}{8\pi^2} \\ & \int_0^{x_a} \mathrm{d}y \big[(\partial_d - 2\partial_a + \partial_b) \mathcal{O}^{abcd}(x_a - y, x_b, x_c, x_d) \mathcal{O}(y) \\ & + (\partial_c - \partial_a) \mathcal{O}^{bcd}(x_b, x_c, x_a - y) \mathcal{O}^a(x_d + y) \\ & + (\partial_d - \partial_a) \mathcal{O}^{ad}(x_a - y, x_d) \mathcal{O}^{bc}(x_b + x_c + y) \\ & + (\partial_b - \partial_a) \mathcal{O}^{bc}(x_b, x_a - y) \mathcal{O}^{ad}(x_c + y, x_d) \\ & + (\partial_c - \partial_a) \mathcal{O}^{acd}(x_a - y, x_c, x_d) \mathcal{O}^b(x_b + y) \big] \\ & + (\text{three other cyclic permutations of } abcd) \end{aligned} \tag{3.68}$$

$$H_{-}\mathcal{O}^{abcd}(x_a,x_b,x_c,x_d)\prod_j \mathcal{O}(r_j)=\frac{-\lambda'}{8\pi^2}$$

$$\sum_{i=1}^{l} r_i(\partial_d-\partial_a)(\mathcal{O}^{abcd}(x_a+r_i,x_b,x_c,x_d)-\mathcal{O}^{abcd}(x_a,x_b,x_c,x_d+r_i))\prod_{j\neq i}\mathcal{O}(r_j)$$

$$+(\text{three other cyclic permutations of } abcd) \qquad (3.69)$$

'31'

$$H_{+}\mathcal{O}^{abc}(x_a,x_b,x_c)\mathcal{O}^{d}(x_d)=\frac{-\lambda'}{8\pi^2}$$

$$\int_0^{x_a}\mathrm{d}y\big[(\partial_c-2\partial_a+\partial_b)\mathcal{O}^{abc}(x_a-y,x_b,x_c)\mathcal{O}^{d}(x_d)\mathcal{O}(y)$$

$$+(\partial_b-\partial_a)\mathcal{O}^{cb}(x_a-y,x_b)\mathcal{O}^{a}(x_c+y)\mathcal{O}^{d}(x_d)+(\partial_c-\partial_a)\mathcal{O}^{ca}(x_c,x_a-y)\mathcal{O}^{b}(x_b+y)\mathcal{O}^{d}(x_d)\big]$$

$$+(\text{two other cyclic permutations of } abc) \qquad (3.70)$$

$$H_{-}\mathcal{O}^{abc}(x_a,x_b,x_c)\mathcal{O}^{d}(x_d)\prod_j \mathcal{O}(r_j)=\frac{-\lambda'}{8\pi^2}$$

$$\sum_{i=1}^{l} r_i(\partial_c-\partial_a)(\mathcal{O}^{abc}(x_a+r_i,x_b,x_c)-\mathcal{O}^{abc}(x_a,x_b,x_c+r_i))\mathcal{O}^{d}(x_d)\prod_{j\neq i}\mathcal{O}(r_j)$$

$$+\Big\{\int_0^{x_a}\mathrm{d}y\big[(\partial_d-\partial_a)\mathcal{O}^{dbca}(x_d+y,x_b,x_c,x_a-y)+(\partial_d-\partial_a)\mathcal{O}^{dbca}(x_d-y,x_b,x_c,x_a+y)\big]$$

$$+\int_0^{x_d}\mathrm{d}y\big[(\partial_c-\partial_d)\mathcal{O}^{adbc}(x_d-y,x_a+y,x_b,x_c)+(\partial_a-\partial_d)\mathcal{O}^{abcd}(x_a,x_b,x_c+y,x_d-y)\big]\Big\}$$

$$\times\prod_j \mathcal{O}(r_j)$$

$$+(\text{two other cyclic permutations of } abc) \qquad (3.71)$$

'22'

$$H_{+}\mathcal{O}^{ab}(x_a,x_b)\mathcal{O}^{cd}(x_c,x_d)=\frac{-\lambda'}{8\pi^2}$$

$$\int_0^{x_a}\mathrm{d}y 2(\partial_c-\partial_a)\mathcal{O}^{ab}(x_a-y,x_b)\mathcal{O}^{cd}(x_c,x_d)\mathcal{O}(y)$$

$$+(a\leftrightarrow c,b\leftrightarrow d),\ \text{then}+(a\leftrightarrow b),\ \text{then}+(c\leftrightarrow d) \qquad (3.72)$$

$$
\begin{aligned}
&H_-\mathcal{O}^{ab}(x_a,x_b)\mathcal{O}^{cd}(x_c,x_d)\prod_j \mathcal{O}(r_j) = \frac{-\lambda'}{8\pi^2} \\
&\sum_{i=1}^{l} r_i(\partial_a-\partial_b)(\mathcal{O}^{ab}(x_a,x_b+r_i)-\mathcal{O}^{ab}(x_a+r_i,x_b))\mathcal{O}^{cd}(x_c,x_d)\prod_{j\neq i}\mathcal{O}(r_j) \\
&+\int_0^{x_a} \mathrm{d}y\big[(\partial_c-\partial_a)\mathcal{O}^{cdba}(x_c,x_d,x_b,x_a) \\
&+(\partial_d-\partial_a)\mathcal{O}^{cbad}(x_a-y,x_b,x_c+y,x_d)\big]\prod_j \mathcal{O}(r_j) \\
&+(a\leftrightarrow c, b\leftrightarrow d),\ \text{then} + (a\leftrightarrow b),\ \text{then} + (c\leftrightarrow d)
\end{aligned} \tag{3.73}
$$

'211'

$$
\begin{aligned}
&H_+\mathcal{O}^{ab}(x_a,x_b)\mathcal{O}^{c}(x_c)\mathcal{O}^{d}_{x_d} = \frac{-\lambda'}{8\pi^2} \\
&\frac{1}{2}\int_0^{x_a} \mathrm{d}y(\partial_b-\partial_a)(\mathcal{O}^{ab}(x_a-y,x_b)+\mathcal{O}^{ab}(x_a,x_b-y))\mathcal{O}^{c}(x_c)\mathcal{O}^{d}(x_d)\mathcal{O}(y) \\
&+(c\leftrightarrow d),\ \text{then} + (a\leftrightarrow b)
\end{aligned} \tag{3.74}
$$

$$
\begin{aligned}
&H_-\mathcal{O}^{ab}(x_a,x_b)\mathcal{O}^{c}(x_c)\mathcal{O}^{d}(x_d)\prod_j \mathcal{O}(r_j) = \frac{-\lambda'}{8\pi^2} \\
&\frac{1}{2}\sum_{i=1}^{l} r_i(\partial_a-\partial_b)(\mathcal{O}^{ab}(x_a,x_b+r_i)-\mathcal{O}^{ab}(x_a+r_i,x_b))\mathcal{O}^{c}(x_c)\mathcal{O}^{d}(x_d)\prod_{j\neq i}\mathcal{O}(r_j) \\
&+\int_0^{x_c} \mathrm{d}y\big[(\partial_a-\partial_c)(\mathcal{O}^{abc}(x_a,x_c-y,x_b+y)+\mathcal{O}^{abc}(x_a,x_b+y,x_c-y))\mathcal{O}^{d}(x_d) \\
&+\frac{1}{2}(\partial_d-\partial_c)(\mathcal{O}^{dc}(x_c-y,x_d+y)+\mathcal{O}^{dc}(x_d+y,x_d+y,x_c-y))\mathcal{O}^{ab}(x_a,x_b)\big]\prod_j \mathcal{O}(r_j) \\
&+\int_0^{x_a} \mathrm{d}y(\partial_a-\partial_c)(\mathcal{O}^{cba}(x_a+y,x_b,x_c-y)-\mathcal{O}^{cba}(x_a-y,x_b,x_c+y))\mathcal{O}^{d}(x_d)\prod_j \mathcal{O}(r_j) \\
&+(c\leftrightarrow d),\ \text{then} + (a\leftrightarrow b)
\end{aligned} \tag{3.75}
$$

'1111'

$$
H_+\mathcal{O}^{a}(x_a)\mathcal{O}^{b}(x_b)\mathcal{O}^{c}(x_c)\mathcal{O}^{d}(x_d)\prod_j \mathcal{O}(r_j) = 0 \tag{3.76}
$$

$$H_{-}\mathcal{O}^a(x_a)\mathcal{O}^b(x_b)\mathcal{O}^c(x_c)\mathcal{O}^d_{x_d} = \frac{-\lambda'}{8\pi^2}$$
$$\int_0^{x_b} \mathrm{d}y(\partial_b - \partial_a)(\mathcal{O}^{ab}(x_b - y, x_a + y) + \mathcal{O}^{ab}(x_a + y, x_b - y))\mathcal{O}^d(x_d)\mathcal{O}^c(x_c)$$
$$+(\text{eleven other permutations of abcd}) \tag{3.77}$$

We point out that the elements not involving four-impurity states are very similar to those calculated in the study of three-impurity states in [80], the only difference being that there appears an additional single-impurity state which, like the zero-impurity states, is unaffected by H_1. As discussed in the beginning of this section, we need only consider elements involving '4' states. We now restrict our attention to the novel '4-4', '4-31' and '4-22' matrix elements, and express these in the momentum-state basis.

The '4-4' component of H_+ may be obtained from the four permutations of the first line of eqn.(3.68). Substituting momentum states and performing the derivatives, this may be written

$$\begin{aligned}
H_+\big|_{\text{'4-4'}}|m_1, m_2, m_3; r_0\rangle^{1234} = & \frac{-\lambda'}{8\pi^2} \sum_{n_1,n_2,n_3} \int_{x_{123}<r_0} \mathrm{d}^3x \\
& \Big\{ -(2n_1 + n_2 + n_3) \int_0^{x_1} \mathrm{d}s \frac{1}{r_0 - s} e^{\frac{2\pi i}{r_0 - s} m_{123} s} \\
& + (n_1 - n_2) \int_0^{x_2} \mathrm{d}s \frac{1}{r_0 - s} e^{\frac{2\pi i}{r_0 - s} m_{23} s} \\
& + (n_2 - n_3) \int_0^{x_3} \mathrm{d}s \frac{1}{r_0 - s} e^{\frac{2\pi i}{r_0 - s} m_3 s} \\
& + (n_1 + n_2 + 2n_3) \int_0^{r_0 - x_{123}} \mathrm{d}s \frac{1}{r_0 - s} \Big\} \\
& e^{\frac{2\pi i}{r_0}\left(D_1 x_1 + D_2 x_{12} + D_3 x_{123}\right)} \\
& \frac{-2\pi i}{\sqrt{(r_0 - s)^3 s r_0^3}} |n_1, n_2, n_3; r_0 - s\rangle^{1234} |s\rangle,
\end{aligned} \tag{3.78}$$

where $D_a = \frac{n_a}{r_0} - \frac{m_a}{r_0 - s}$. Here, we have not yet added states where the impurities $2, 3, 4$

are permuted along with the three momenta. We can express this component of H_+ in terms of the basis state (3.48) by adding these permutations. In this case, this involves adding five more terms, in which the impurities $2,3,4$ are permuted, along with the momenta n_1, n_2, n_3 and also the momenta m_1, m_2, m_3. We postpone this to save space. Rearranging so that the x-integrals are innermost, we have

$$
\begin{aligned}
\cdots = & \frac{-\lambda'}{8\pi^2} \sum_{n_1,n_2,n_3} \int_0^{r_0} \mathrm{d}s \frac{-2\pi i}{\sqrt{(r_0-s)^3 s r_0^3}} \\
& \Big\{ -(2n_1+n_2+n_3) e^{\frac{2\pi i}{r_0-s} n_{123} s} \int_s^{r_0} \mathrm{d}x_1 \int_0^{r_0-x_1} \mathrm{d}x_2 \int_0^{r_0-x_{12}} \mathrm{d}x_3 \\
& +(n_1-n_2) e^{\frac{2\pi i}{r_0-s} n_{23} s} \int_0^{r_0-s} \mathrm{d}x_1 \int_s^{r_0-x_1} \mathrm{d}x_2 \int_0^{r_0-x_{12}} \mathrm{d}x_3 \\
& +(n_2-n_3) e^{\frac{2\pi i}{r_0-s} n_3 s} \int_0^{r_0-s} \mathrm{d}x_1 \int_0^{r_0-x_1-s} \mathrm{d}x_2 \int_s^{r_0-x_{12}} \mathrm{d}x_3 \\
& +(n_1+n_2+2n_3) \int_0^{r_0-s} \mathrm{d}x_1 \int_0^{r_0-x_1-s} \mathrm{d}x_2 \int_0^{r_0-x_{12}-s} \mathrm{d}x_3 \Big\} \\
& \times e^{2\pi i \left(D_1 x_1 + D_2 x_{12} + D_3 x_{123}\right)} |n_1, n_2, n_3; r_0 - s\rangle^{1234} |s\rangle .
\end{aligned}
\tag{3.79}
$$

Performing the x-integrations we obtain

$$
\begin{aligned}
\cdots = & \frac{-\lambda'}{8\pi^2} \sum_{n_1,n_2,n_3} \int_0^{r_0} \mathrm{d}s \frac{1}{\sqrt{(r_0-s)^3 s r_0^3}} \Big[-(2n_1+n_2+n_3) e^{2\pi i s D_{123}} + (n_1-n_2) e^{2\pi i s D_{23}} \\
& +(n_2-n_3) e^{2\pi i s D_3} + (n_1+n_2+2n_3) \Big] \\
& \times \left[\frac{e^{-2\pi i s D_{123}} - 1}{D_{123} D_1 D_{12}} - \frac{e^{-2\pi i s D_{23}} - 1}{D_{23} D_1 D_2} + \frac{e^{-2\pi i s D_3} - 1}{D_3 D_2 D_{12}} \right] |n_1, n_2, n_3; r_0 - s\rangle^{1234} |s\rangle
\end{aligned}
\tag{3.80}
$$

to which we still must add the five permutations of $m_{1,2,3}$ and $n_{1,2,3}$. After adding these and simplifying, our final result for the '4-4' matrix element of H_+ is given by

$$\begin{aligned}
&\langle s|\langle m_1, m_2, m_3; r_0 - s|H_+|n_1, n_2, n_3; r_0\rangle \\
&= \frac{-\lambda'}{2\pi^4}\frac{1}{\sqrt{(r_0-s)^3 s r_0^3}} \sin\left(\frac{\pi s n_1}{r_0}\right) \sin\left(\frac{\pi s n_2}{r_0}\right) \sin\left(\frac{\pi s n_3}{r_0}\right) \sin\left(\frac{\pi s n_{123}}{r_0}\right) \\
&\times \frac{m_1 D_1 + m_2 D_2 + m_3 D_3 + m_{123} D_{123}}{(r_0 - s) D_1 D_2 D_3 D_{123}} \\
&\text{where } D_a = \frac{n_a}{r_0} - \frac{m_a}{r_0 - s}
\end{aligned} \tag{3.81}$$

where we have taken an inner product from the left. We have rewritten the definition of the D symbols here for convenience; in the following, we shall use the notations D_a and D_a^b differently in each case. We comment that this expression and the others that follow do not leap to the page quickly; what is described in the preceding paragraph requires significant time and care to calculate.

The other components of H_+ and H_- are calculated in similar fashion using the continuum forms (3.68)-(3.77). We continue to omit the product $\prod_j |r_j\rangle$ of zero-impurity states which we omitted in the above, with the understanding that this product may appear in each of the following states without affecting the calculation. The '4-4' element of H_- is found to be

$$\begin{aligned}
&\langle m_1, m_2, m_3; r_0|H_-|n_1, n_2, n_3; r_0 - s\rangle|s\rangle \\
&= \frac{\lambda'}{2\pi^4}\frac{1}{\sqrt{(r_0-s)^3 s r_0^3}} \sin\left(\frac{\pi s m_1}{r_0}\right) \sin\left(\frac{\pi s m_2}{r_0}\right) \sin\left(\frac{\pi s m_3}{r_0}\right) \sin\left(\frac{\pi s m_{123}}{r_0}\right) \\
&\times \left[\frac{m_{123}}{r_0 D_{123}}\left(\frac{1}{D_2 D_3} + \frac{1}{D_1 D_3} + \frac{1}{D_1 D_2}\right) + \frac{1}{r_0 D_{123}}\left(\frac{m_1}{D_2 D_3} + \frac{m_2}{D_1 D_3} + \frac{m_3}{D_1 D_2}\right)\right] \\
&\text{where } D_a = \frac{n_a}{r_0 - s} - \frac{m_a}{r_0}.
\end{aligned} \tag{3.82}$$

For the '4-31' elements of H_+ and H_- we find

$$
\begin{aligned}
&{}^{123}\langle m_1, m_2; r_0 - s|^4\langle s|H_+|n_1, n_2, n_3; r_0\rangle \\
&= \frac{-\lambda'}{2\pi^4}\frac{1}{\sqrt{(r_0-s)^2 r_0^3}} \sin\left(\frac{\pi s n_1}{r_0}\right)\sin\left(\frac{\pi s n_2}{r_0}\right)\sin\left(\frac{\pi s n_3}{r_0}\right)\sin\left(\frac{\pi s n_{123}}{r_0}\right) \\
&\times \left[\frac{m_1}{r_0-s}\left(\frac{1}{D_2^{-2}D_1^{-1}} + \frac{1}{D_{123}^{-12}D_2^{-2}}\right) + \frac{m_2}{r_0-s}\left(\frac{1}{D_2^{-2}D_1^{-1}} + \frac{1}{D_{123}^{-12}D_1^{-1}}\right)\right] \\
&\text{where } D_a^b = \frac{n_a}{r_0} + \frac{m_b}{r_0 - s}
\end{aligned}
\tag{3.83}
$$

and

$$
\begin{aligned}
&\langle m_1, m_2, m_3; r_0|H_-|n_1, n_2; r_0 - s\rangle^{123}|s\rangle^4 \\
&= \frac{\lambda'}{2\pi^4}\frac{1}{\sqrt{(r_0-s)^2 r_0^3}} \sin\left(\frac{\pi s m_1}{r_0}\right)\sin\left(\frac{\pi s m_2}{r_0}\right)\sin\left(\frac{\pi s m_3}{r_0}\right)\sin\left(\frac{\pi s m_{123}}{r_0}\right) \\
&\times \left[\left(\frac{1}{D_{12}^{-123}} - \frac{1}{D_{12}^{-12}} - \frac{m_{123}}{m_3 D_{12}^{-12}}\right)\left(\frac{1}{D_1^{-1}} + \frac{1}{D_2^{-2}}\right) - \frac{1}{m_3 D_{12}^{-123}}\left(\frac{m_1}{D_2^{-2}} + \frac{m_2}{D_1^{-1}}\right)\right] \\
&\text{where } D_a^b = \frac{n_a}{r_0 - s} + \frac{m_b}{r_0}.
\end{aligned}
\tag{3.84}
$$

Finally, the '4-22' elements are

$$
\begin{aligned}
&{}^{12}\langle m_1; r_0 - s|^{34}\langle m_2; s|H_+|n_1, n_2, n_3; r_0\rangle \\
&= \frac{-\lambda'}{2\pi^4}\frac{1}{\sqrt{(r_0-s) s r_0^3}} \sin\left(\frac{\pi s n_1}{r_0}\right)\sin\left(\frac{\pi s n_2}{r_0}\right)\sin\left(\frac{\pi s n_3}{r_0}\right)\sin\left(\frac{\pi s n_{123}}{r_0}\right) \\
&\times \left[\frac{m_1}{r_0-s}\frac{1}{D_2^2 D_3^{-2}}\left(\frac{1}{D_1^{-1}} + \frac{1}{D_{123}^{-1}}\right) + \frac{m_2}{s}\frac{1}{D_{123}^{-1}D_1^{-1}}\left(\frac{1}{D_2^2} - \frac{1}{D_3^{-2}}\right)\right] \\
&\text{where } D_a^1 = \frac{n_a}{r_0} + \frac{m_1}{r_0 - s} \text{ and } D_a^2 = \frac{n_a}{r_0} + \frac{m_2}{s}
\end{aligned}
\tag{3.85}
$$

and

$$
\begin{aligned}
&\langle m_1, m_2, m_3; r_0 | H_- | n_1; r_0 - s\rangle^{12} | n_2; s\rangle^{34} \\
&= \frac{\lambda'}{2\pi^4} \frac{1}{\sqrt{(r_0 - s) s r_0^3}} \sin\left(\frac{\pi s m_1}{r_0}\right) \sin\left(\frac{\pi s m_2}{r_0}\right) \sin\left(\frac{\pi s m_3}{r_0}\right) \sin\left(\frac{\pi s m_{123}}{r_0}\right) \\
&\times \left[\frac{m_1 - m_2}{m_{23}} \frac{1}{D_1^{-123} D_2^{-3}} + \frac{m_2 + m_{23}}{m_{23}} \frac{1}{D_1^{-1} D_2^{-3}} \right. \\
&\qquad \left. - \frac{m_{123} + m_3}{m_{23}} \frac{1}{D_1^{-1} D_2^{2}} + \frac{m_1 - m_3}{m_{23}} \frac{1}{D_1^{123} D_2^{2}} \right]
\end{aligned}
$$

where $D_1^b = \dfrac{n_1}{r_0 - s} + \dfrac{m_b}{r_0}$ and $D_2^b = \dfrac{n_2}{s} + \dfrac{m_b}{r_0}$. (3.86)

3.4.3 Matrix Elements of Σ

Here we consider matrix elements of Σ in the momentum-state basis; these are found by first calculating simple two-point functions in the original discrete basis and then transforming. For the present calculation, we need to find the matrix elements of Σ corresponding to those calculated in the previous section for H_+ and H_-. Since Σ is Hermitian in the momentum-state basis, we do not need to distinguish between trace-number increasing and decreasing parts (which other authors have labeled Σ_+ and Σ_-) since these are simply related by conjugation.

The '4-4' component of Σ is found by considering the correlator

$$\langle \bar{\mathcal{O}}^{1234}_{q_1,q_2,q_3,q_4-k} \bar{\mathcal{O}}_k \mathcal{O}^{1234}_{p_1,p_2,p_3,p_4} \rangle. \tag{3.87}$$

In this correlator, other relative impurity orderings contribute only at higher order in $1/N$. Consulting eqn.(3.17) we see that to genus-one order this is given by

$$N^{p_1+p_2+p_3+p_4+3} k(p_4 - k - 1) \delta_{p_1,q_1} \delta_{p_2,q_2} \delta_{p_3,q_3} \delta_{p_4,q_4}. \tag{3.88}$$

Taking the continuum limit, we replace p and q with x and y. We arrive at the expression (D_a are defined below)

$$\begin{aligned}\Sigma|n_1,n_2,n_3;r_0\rangle\Big|_4 = \int_{x_{123}<r_0} & \mathrm{d}^3x e^{2\pi i(D_1x_1+D_2x_{12}+D_3x_{123})} \\ & \Big\{\int_0^{r_0-x_{123}} \mathrm{d}s(r_0-x_{123}-s) \\ & + \int_0^{x_3} \mathrm{d}s(x_3-s)e^{\frac{2\pi i}{r_0-s}m_3s} \\ & + \int_0^{x_2} \mathrm{d}s(x_2-s)e^{\frac{2\pi i}{r_0-s}m_{23}s} \\ & + \int_0^{x_1} \mathrm{d}s(x_1-s)e^{\frac{2\pi i}{r_0-s}m_{123}s}\Big\} s|m_1,m_2,m_3;r_0-s\rangle|s\rangle, \end{aligned} \tag{3.89}$$

where the '4' indicates that we are only, for the moment, interested in the '4'-state part. Were we to include the zero-impurity states $\prod_j|r_j\rangle$ which we have been neglecting to write, we would obtain another term on the right-hand side of eqn.(3.89) corresponding to the splitting of one of these states into two such states. Bringing the s-integration outside, and performing the x-integrals, a procedure similar to that used to calculate the elements of H_+ and H_-, we obtain

$$\begin{aligned} & \langle s|\langle m_1,m_2,m_3;r_0-s|\Sigma|n_1,n_2,n_3;r_0\rangle \\ & = \frac{1}{\pi^4\sqrt{(r_0-s)^3sr_0^3}}\sin\left(\frac{\pi sn_1}{r_0}\right)\sin\left(\frac{\pi sn_2}{r_0}\right)\sin\left(\frac{\pi sn_3}{r_0}\right)\sin\left(\frac{\pi sn_{123}}{r_0}\right) \\ & \times\frac{1}{D_1D_2D_3D_{123}} \\ & \text{where } D_a = \frac{n_a}{r_0}-\frac{m_a}{r_0-s}. \end{aligned} \tag{3.90}$$

Using the same method, the '4-31' element may be found from the correlator

$$\langle\bar{\mathrm{O}}^{123}_{q_1,q_2,q_3}\bar{\mathrm{O}}^4_{q_4}\mathrm{O}^{1234}_{p_1,p_2,p_3,p_4}\rangle = N^{p_1+p_2+p_3+p_4+3}(\min(p_3,q_3,p_4,q_4)+1)\delta_{p_1,q_1}\delta_{p_2,q_2}\delta_{p_3+p_4,q_3+q_4} \tag{3.91}$$

giving

$$
\begin{aligned}
&{}^{123}\langle m_1, m_2; r_0 - s|^4\langle s|\Sigma|n_1, n_2, n_3; r_0\rangle \\
&= \frac{1}{\pi^4\sqrt{(r_0-s)^2 r_0^3}} \sin\left(\frac{\pi s n_1}{r_0}\right) \sin\left(\frac{\pi s n_2}{r_0}\right) \sin\left(\frac{\pi s n_3}{r_0}\right) \sin\left(\frac{\pi s n_{123}}{r_0}\right) \\
&\times \frac{1}{D_{123}^{-12} D_3 D_1^{-1} D_2^{-2}} \\
\text{where } & D_a^b = \frac{n_a}{r_0} + \frac{m_b}{r_0 - s},
\end{aligned}
\tag{3.92}
$$

and the '4-22' element from the correlator

$$
\langle \bar{\mathcal{O}}^{12}_{q_1,q_2} \bar{\mathcal{O}}^{34}_{q_3,q_4} \mathcal{O}^{1234}_{p_1,p_2,p_3,p_4} \rangle = N^{p_1+p_2+p_3+p_4+3} (\min(p_2, q_2, p_4, q_4) + 1) \delta_{p_1,q_1} \delta_{p_3,q_3} \delta_{p_2+p_4,q_2+q_4},
\tag{3.93}
$$

so that

$$
\begin{aligned}
&{}^{12}\langle m_1; r_0 - s|^{34}\langle m_2; s|\Sigma|n_1, n_2, n_3; r_0\rangle \\
&= \frac{1}{\pi^4\sqrt{(r_0-s) s r_0^3}} \sin\left(\frac{\pi s n_1}{r_0}\right) \sin\left(\frac{\pi s n_2}{r_0}\right) \sin\left(\frac{\pi s n_3}{r_0}\right) \sin\left(\frac{\pi s n_{123}}{r_0}\right) \\
&\times \frac{1}{D_{123}^{-1} D_1^{-1} D_3^{-2} D_2^{2}} \\
\text{where } & D_a^1 = \frac{n_a}{r_0} + \frac{m_1}{r_0 - s} \text{ and } D_a^2 = \frac{n_a}{r_0} + \frac{m_2}{s}.
\end{aligned}
\tag{3.94}
$$

3.4.4 Impurity Orderings

In the results in the previous section, other impurity orderings may be accommodated by considering appropriate permutations of the momenta. All orderings would be needed, for instance, in a decay calculation such as that done for three-impurity states in [80], or in a perturbative calculation of corrections to the conformal dimensions which are the eigenvalues of H_0, as in section 3.6. In a sum over final states in a decay calculation, or intermediate states in a perturbation calculation, all impurity orderings must be included. These permutations are found using the definitions of the momentum states (3.48)-(3.51). Let $|\rangle^{abcd}$ be any single- or multi-trace state containing the four impurities $abcd$. We wish to find the matrix element of some operator M between this state and the '4'-state,

$$^{abcd}\langle|M|n_1, n_2, n_3; r_0\rangle, \tag{3.95}$$

given that we already have this quantity for the case $abcd = 1234$;

$$M_{n_1,n_2,n_3} \equiv {}^{1234}\langle|M|n_1, n_2, n_3; r_0\rangle. \tag{3.96}$$

To emphasize that on the right-hand side we have included simultaneous permutations of 234 and n_1, n_2, n_3 in the definition of the basis state (3.48), let us temporarily write expression (3.95) as

$$^{abcd}\langle|M|n_1, n_2, n_3; r_0\rangle^{\underline{1234}}. \tag{3.97}$$

Clearly, since we retain contractions between like impurities, this is the same as

$$^{1234}\langle|M|n_1, n_2, n_3; r_0\rangle^{\underline{abcd}}. \tag{3.98}$$

Now, we have reduced the problem to learning how to manipulate the impurity order within the 'ket' state.

Since the permutations of $n_{1,2,3}$ are linked to the permutations of b, c, d, it is easy to accommodate all orderings with a '1' in the first position; simply setting $a = 1$ we have

$$|n_1, n_2, n_3; r_0\rangle^{1\underline{bcd}} = |n_{b-1}, n_{c-1}, n_{d-1}; r_0\rangle^{\underline{1234}}. \tag{3.99}$$

Now, considering instead $b = 1$, appearing in the definition of the basis state (3.48) will be the factor (it will be sufficient to consider only one of the six terms)

$$e^{n_1 x_1 + n_2 x_{12} + n_3 x_{123}} |x_1, x_2, x_3, r_0 - x_{123}\rangle^{a1cd} = e^{n_1 x_1 + n_2 x_{12} + n_3 x_{123}} |x_2, x_3, r_0 - x_{123}, x_1\rangle^{1cda} \tag{3.100}$$

which may be written with redefined $x_{1,2,3}$ as

$$e^{n_2 x_1 + n_3 x_{12} - n_{123} x_{123}} |x_1, x_2, x_3, r_0 - x_{123}\rangle^{1cda} \tag{3.101}$$

and we see that the necessary replacement to obtain this from the original state is $n_1 \to n_2$, $n_2 \to n_3$ and $n_3 \to -n_{123}$. This amounts to

$$|n_1, n_2, n_3; r_0\rangle^{\underline{a1cd}} = |n_2, n_3, -n_{123}; r_0\rangle^{\underline{1cda}}. \tag{3.102}$$

We can permute a '1' in the third or fourth position to the second by using eqn.(3.99) and then to the first position using eqn.(3.102). This gives

$$|n_1, n_2, n_3; r_0\rangle^{\underline{ab1d}} = |n_1, n_3, -n_{123}; r_0\rangle^{1\underline{bda}} \tag{3.103}$$

$$|n_1, n_2, n_3; r_0\rangle^{\underline{abc1}} = |n_1, n_2, -n_{123}; r_0\rangle^{1\underline{bca}}. \tag{3.104}$$

Making use of formulae (3.99), (3.102), (3.103) and (3.104) in eqn.(3.98), the different instances of eqn.(3.95) may now be written. For $\pi \in S_3$ and $abc = \pi(234)$,

$$^{1abc}\langle|M|n_1, n_2, n_3; r_0\rangle = M_{\pi(n_1,n_2,n_3)} \tag{3.105}$$

$$^{a1bc}\langle|M|n_1, n_2, n_3; r_0\rangle = M_{\pi(-n_{123},n_2,n_3)} \tag{3.106}$$

$$^{ab1c}\langle|M|n_1, n_2, n_3; r_0\rangle = M_{\pi(-n_{123},n_1,n_3)} \tag{3.107}$$

$$^{abc1}\langle|M|n_1, n_2, n_3; r_0\rangle = M_{\pi(-n_{123},n_1,n_2)}. \tag{3.108}$$

3.4.5 String Hamiltonian

Now we are in a position to calculate the string Hamiltonian $\tilde{H}$, assembling H_0, H_1 and Σ using eqn.(3.8). The '4-4' element of the genus-one correction to $\tilde{H}$ is given by

$$\tilde{H}_{\text{'4-4'}}\Big|_{g_2} = g_2\langle s|\langle m_1, m_2, m_3; r_0 - s|\left(\frac{1}{2}[\Sigma, H_0] + H_1\right)|n_1, n_2, n_3; r_0\rangle, \tag{3.109}$$

which may be calculated either using Σ and H_+ from eqn.(3.90) and eqn.(3.81), or Σ and H_- from the conjugate of eqn.(3.90) and eqn.(3.82). It may easily be verified that these give the same result, showing that $\tilde{H}$ is Hermitian as it should be, and serving as a check on the calculations. The result is

$$\begin{aligned}\tilde{H}_{\text{'4-4'}}\Big|_{g_2} &= \frac{\lambda' g_2}{4\pi^4}\frac{1}{\sqrt{(r_0-s)^3 s r_0^3}}\frac{(D_1)^2+(D_2)^2+(D_3)^2+(D_{123})^2}{D_1 D_2 D_3 D_{123}} \\ &\quad \times \sin(\frac{\pi s n_1}{r_0})\sin(\frac{\pi s n_2}{r_0})\sin(\frac{\pi s n_3}{r_0})\sin(\frac{\pi s n_{123}}{r_0}) \\ \text{where } D_a &= \frac{n_a}{r_0} - \frac{m_a}{r_0 - s}.\end{aligned} \tag{3.110}$$

Of course, the genus-zero '4-4' component is simply given by H_0 in eqn.(3.52). The '4-31' element is similarly obtained using Σ from eqn.(3.92) and $H_\pm$ from eqn.(3.83) or eqn.(3.84) to calculate

$$\begin{aligned}\tilde{H}_{\text{'4-31'}} &= g_2{}^{123}\langle m_1, m_2; r_0 - s|^4\langle s|\left(\frac{1}{2}[\Sigma, H_0] + H_1\right)|n_1, n_2, n_3; r_0\rangle \\ &= \frac{\lambda' g_2}{4\pi^4}\frac{1}{\sqrt{(r_0-s)^2 r_0^3}}\frac{(D_{123}^{-12})^2+(D_1^{-1})^2+(D_2^{-2})^2+(D_3)^2}{D_{123}^{-12}D_1^{-1}D_2^{-2}D_3} \\ &\quad \times \sin(\frac{\pi s n_1}{r_0})\sin(\frac{\pi s n_2}{r_0})\sin(\frac{\pi s n_3}{r_0})\sin(\frac{\pi s n_{123}}{r_0})\end{aligned} \tag{3.111}$$

$$\text{where} \qquad D_a^b = \frac{n_a}{r_0} + \frac{m_b}{r_0 - s}, \tag{3.112}$$

and the '4-22' element by using eqn.(3.94) and eqn.(3.85) or eqn.(3.86), giving

$$\begin{aligned}\tilde{H}_{\text{`4-22'}} &= g_2 {}^{12}\langle m_1; r_0 - s|^{34}\langle m_2; s|\big(\frac{1}{2}[\Sigma, H_0] + H_1\big)|n_1, n_2, n_3; r_0\rangle \quad (3.113)\\ &= \frac{\lambda' g_2}{4\pi^4}\frac{1}{\sqrt{(r_0-s)sr_0^3}}\frac{(D_{123}^{-1})^2 + (D_2^2)^2 + (D_3^{-2})^2 + (D_1^{-1})^2}{D_{123}^{-1}D_2^2D_3^{-2}D_1^{-1}}\\ &\quad\times \sin(\frac{\pi s n_1}{r_0})\sin(\frac{\pi s n_2}{r_0})\sin(\frac{\pi s n_3}{r_0})\sin(\frac{\pi s n_{123}}{r_0})\end{aligned}$$

$$\text{where} \quad D_a^1 = \frac{n_a}{r_0} + \frac{m_1}{r_0 - s} \text{ and } D_a^2 = \frac{n_a}{r_0} + \frac{m_2}{s}. \quad (3.114)$$

With the order-g_2 string Hamiltonian now in hand, we turn to the computation of the string field theory vertex with which it is expected to correspond.

3.5 Comparison with String-Field Vertex

Now, having completed our calculation of matrix elements on the Yang-Mills side, we would like to see if we can reproduce these results from the string side. We can perform the appropriate computations using string field theory.

The number of impurities of a BMN operator on the SYM side of the correspondence is identified with the number of oscillator excitations of the corresponding state on the string side. Light-cone String Field Theory in the plane-wave background has been developed in [90, 91, 92, 93, 94, 95, 96], and the Neumann coefficients necessary for computations have been found in [97] with further results in [98, 99].

We shall consider a three-string interaction, with a total of eight oscillator excitations distributed among the three strings. A multi-string state with $2k$ excitations is given by

$$|A\rangle = \prod_{j=1}^{2k} \alpha_{(r_j)m_j}^{I_j\dagger}|0\rangle \quad (3.115)$$

where r_j are the string numbers, I_j label the transverse AdS directions (i.e. the impurity coordinates), and m_j are the oscillator numbers. For our purposes, we set $k = 4$.

In the case of our '4-4' interaction, we consider the three-string state

$$|4{,}4\rangle = \alpha^{1\dagger}_{(1)n_1}\alpha^{2\dagger}_{(1)n_2}\alpha^{3\dagger}_{(1)n_3}\alpha^{4\dagger}_{(1)-n_{123}}\alpha^{1\dagger}_{(3)m_1}\alpha^{2\dagger}_{(3)m_2}\alpha^{3\dagger}_{(3)m_3}\alpha^{4\dagger}_{(3)-m_{123}}|0\rangle_{(1)}\otimes|0\rangle_{(2)}\otimes|0\rangle_{(3)}, \quad (3.116)$$

where excitations are absent for string number two; it corresponds to a zero-impurity state. In [58] it is shown how to calculate the interaction vertex V_3, as discussed in section 1.3.1, between the strings in this state. We find

$$\langle 4{,}4|H_3\rangle = \frac{\alpha_{(1)}\alpha_{(2)}\alpha_{(3)}}{2}\sum_{j=1}^{4}\mathcal{N}_j \quad (3.117)$$

where

$$\mathcal{N}_1 = \left(\frac{\omega_{(1)n_1}}{\mu\alpha_{(1)}} + \frac{\omega_{(3)m_1}}{\mu\alpha_{(3)}}\right)\tilde{N}^{(1,3)}_{-n_1,m_1}\tilde{N}^{(1,3)}_{n_2,m_2}\tilde{N}^{(1,3)}_{n_3,m_3}\tilde{N}^{(1,3)}_{-n_{123},-m_{123}}, \quad (3.118)$$

$$\mathcal{N}_2 = \left(\frac{\omega_{(1)n_2}}{\mu\alpha_{(1)}} + \frac{\omega_{(3)m_2}}{\mu\alpha_{(3)}}\right)\tilde{N}^{(1,3)}_{n_1,m_1}\tilde{N}^{(1,3)}_{-n_2,m_2}\tilde{N}^{(1,3)}_{n_3,m_3}\tilde{N}^{(1,3)}_{-n_{123},-m_{123}}, \quad (3.119)$$

$$\mathcal{N}_3 = \left(\frac{\omega_{(1)n_3}}{\mu\alpha_{(1)}} + \frac{\omega_{(3)m_3}}{\mu\alpha_{(3)}}\right)\tilde{N}^{(1,3)}_{n_1,m_1}\tilde{N}^{(1,3)}_{n_2,m_2}\tilde{N}^{(1,3)}_{-n_3,m_3}\tilde{N}^{(1,3)}_{-n_{123},-m_{123}}, \quad (3.120)$$

$$\mathcal{N}_4 = \left(\frac{\omega_{(1)-n_{123}}}{\mu\alpha_{(1)}} + \frac{\omega_{(3)-m_{123}}}{\mu\alpha_{(3)}}\right)\tilde{N}^{(1,3)}_{n_1,m_1}\tilde{N}^{(1,3)}_{n_2,m_2}\tilde{N}^{(1,3)}_{n_3,m_3}\tilde{N}^{(1,3)}_{n_{123},-m_{123}}, \quad (3.121)$$

where the string frequencies are $\omega_{(r)m} = \sqrt{m^2 + \mu^2\alpha^2_{(r)}}$. The $\alpha_{(r)}$ are the fractions of outgoing light-cone momentum carried by each string, and in the present case these are $\alpha_{(1)} = 1 - s$, $\alpha_{(2)} = s$ and $\alpha_{(3)} = -1$. The Neumann coefficients for the plane-wave geometry are given by [97], and we display them here for convenience $(m, n > 0)$:

$$
\begin{aligned}
\tilde{N}^{(r,s)}_{0,n} &= \tilde{N}^{(r,s)}_{0,-n} = \frac{1}{\sqrt{2}} \bar{N}^{(r,s)}_{0,n} \\
\tilde{N}^{(r,s)}_{\pm m,\pm n} &= \frac{1}{2}\big(\bar{N}^{(r,s)}_{m,n} - \bar{N}^{(r,s)}_{-m,-n}\big) \\
\tilde{N}^{(r,s)}_{\pm m,\mp n} &= \frac{1}{2}\big(\bar{N}^{(r,s)}_{m,n} + \bar{N}^{(r,s)}_{-m,-n}\big) \\
\bar{N}^{(r,s)}_{0,n} &= \frac{1}{2\pi}(-1)^{s(n+1)} s_{(s)n} \sqrt{\frac{|\alpha_{(s)}|}{\alpha_{(r)}\omega_{(s)n}(\omega_{(s)n} + \mu\alpha_{(s)})}} \\
\bar{N}^{(r,s)}_{\pm m,\pm n} &= \pm\frac{1}{2\pi} \frac{(-1)^{r(m+1)+s(n+1)} s_{(r)m} s_{(s)n}}{\alpha_{(s)}\omega_{(r)m} + \alpha_{(r)}\omega_{(s)n}} \\
&\quad \times \sqrt{\frac{|\alpha_{(r)}\alpha_{(s)}|(\omega_{(r)m} \pm \mu\alpha_{(r)})(\omega_{(s)n} \pm \mu\alpha_{(s)})}{\omega_{(r)m}\omega_{(s)n}}} && (3.122) \\
s_{(1)m} &= s_{(2)m} = 1 \qquad s_{(3)m} = 2\sin\big(\pi m \frac{\alpha_{(1)}}{\alpha_{(3)}}\big) && (3.123)
\end{aligned}
$$

Expanding to leading order in $1/\mu$, the Neumann matrices become (for the cases we need)

$$
\tilde{N}^{(r,3)}_{m,n} = \frac{(-1)^{r(m+1)+n}\sin(\pi n s)}{2\pi\sqrt{\alpha_{(r)}}\big(\frac{m}{\alpha_{(r)}} - n\big)} + \mathcal{O}\big(\frac{1}{\mu^2}\big). \tag{3.124}
$$

Substituting into our expression (3.117) for the '4-4' string amplitude, we obtain

$$
\begin{aligned}
\langle 4,4|H_3\rangle &= \frac{1}{2}\frac{s}{1-s}\frac{(D_1)^2 + (D_2)^2 + (D_3)^2 + (D_{123})^2}{D_1 D_2 D_3 D_{123}} \\
&\quad \times \sin\big(\frac{\pi s n_1}{r_0}\big)\sin\big(\frac{\pi s n_2}{r_0}\big)\sin\big(\frac{\pi s n_3}{r_0}\big)\sin\big(\frac{\pi s n_{123}}{r_0}\big) \\
\text{where } D_a &= \frac{n_a}{r_0} - \frac{m_a}{r_0 - s}. && (3.125)
\end{aligned}
$$

Apart from normalisation, this is in complete agreement with the '4-4' matrix element (3.110) of the string Hamiltonian $\tilde{H}$ calculated on the SYM side. It is an impressive and satisfying result that the long calculation leading to the matrix element on the Yang-Mills side of the correspondence is indeed able perfectly to reproduce the interaction found using the cubic string field theory vertex. Similar agreement and satisfaction is to be found in the other elements which follow. A calculation similar to the above leads to

$$\begin{aligned}
\langle 4,3,1|H_3\rangle &= \frac{s(1-s)}{2}\Bigg\{\left(\frac{\omega_{(3)n_1}}{\mu\alpha_{(3)}}+\frac{\omega_{(2)0}}{\mu\alpha_{(2)}}\right)\tilde{N}^{(3,2)}_{-n_1,0}\tilde{N}^{(3,1)}_{n_2,m_1}\tilde{N}^{(3,1)}_{n_3,m_2}\tilde{N}^{(3,1)}_{-n_{123},-m_{12}}\\
&+\left(\frac{\omega_{(3)n_2}}{\mu\alpha_{(3)}}+\frac{\omega_{(1)m_1}}{\mu\alpha_{(1)}}\right)\tilde{N}^{(3,2)}_{n_1,0}\tilde{N}^{(3,1)}_{-n_2,m_1}\tilde{N}^{(3,1)}_{n_3,m_2}\tilde{N}^{(3,1)}_{-n_{123},-m_{12}}\\
&+\left(\frac{\omega_{(3)n_3}}{\mu\alpha_{(3)}}+\frac{\omega_{(1)m_2}}{\mu\alpha_{(1)}}\right)\tilde{N}^{(3,2)}_{n_1,0}\tilde{N}^{(3,1)}_{n_2,m_1}\tilde{N}^{(3,1)}_{-n_3,m_2}\tilde{N}^{(3,1)}_{-n_{123},-m_{12}}\\
&+\left(\frac{\omega_{(3)-n_{123}}}{\mu\alpha_{(3)}}+\frac{\omega_{(1)-m_{12}}}{\mu\alpha_{(1)}}\right)\tilde{N}^{(3,2)}_{n_1,0}\tilde{N}^{(3,1)}_{n_2,m_1}\tilde{N}^{(3,1)}_{n_3,m_2}\tilde{N}^{(3,1)}_{n_{123},-m_{12}}\Bigg\}, \qquad (3.126)
\end{aligned}$$

which upon substitution of the expanded Neumann matrices (3.124) leads to

$$\begin{aligned}
\langle 4,3,1|H_3\rangle &= \frac{1}{2}\sqrt{\frac{s}{(1-s)}}\frac{(D^{-12}_{123})^2+(D^{-1}_1)^2+(D^{-2}_2)^2+(D_3)^2}{D^{-12}_{123}D^{-1}_1D^{-2}_2D_3}\\
&\times\sin(\frac{\pi s n_1}{r_0})\sin(\frac{\pi s n_2}{r_0})\sin(\frac{\pi s n_3}{r_0})\sin(\frac{\pi s n_{123}}{r_0})\\
\text{where}\qquad D^b_a &= \frac{n_a}{r_0}+\frac{m_b}{r_0-s}. \qquad (3.127)
\end{aligned}$$

Again, this is in perfect agreement with the gauge-theory result (3.111). Finally, we calculate the '4-22' interaction

$$\begin{aligned}
\langle 4,2,2|H_3\rangle &= \frac{s(1-s)}{2}\Bigg\{\left(\frac{\omega_{(3)n_1}}{\mu\alpha_{(3)}}+\frac{\omega_{(1)m_1}}{\mu\alpha_{(1)}}\right)\tilde{N}^{(3,2)}_{-n_1,m_1}\tilde{N}^{(3,1)}_{n_2,-m_1}\tilde{N}^{(3,2)}_{n_3,m_2}\tilde{N}^{(3,2)}_{-n_{123},-m_2}\\
&+\left(\frac{\omega_{(3)n_2}}{\mu\alpha_{(3)}}+\frac{\omega_{(1)-m_1}}{\mu\alpha_{(1)}}\right)\tilde{N}^{(3,2)}_{n_1,m_1}\tilde{N}^{(3,1)}_{-n_2,-m_1}\tilde{N}^{(3,2)}_{n_3,m_2}\tilde{N}^{(3,2)}_{-n_{123},-m_2}\\
&+\left(\frac{\omega_{(3)n_3}}{\mu\alpha_{(3)}}+\frac{\omega_{(2)-m_2}}{\mu\alpha_{(2)}}\right)\tilde{N}^{(3,2)}_{n_1,m_1}\tilde{N}^{(3,1)}_{n_2,-m_1}\tilde{N}^{(3,2)}_{-n_3,m_2}\tilde{N}^{(3,2)}_{-n_{123},-m_2}\\
&+\left(\frac{\omega_{(3)-n_{123}}}{\mu\alpha_{(3)}}+\frac{\omega_{(2)-m_2}}{\mu\alpha_{(2)}}\right)\tilde{N}^{(3,2)}_{n_1,m_1}\tilde{N}^{(3,1)}_{n_2,-m_1}\tilde{N}^{(3,2)}_{n_3,m_2}\tilde{N}^{(3,2)}_{n_{123},-m_2}\Bigg\}, \qquad (3.128)
\end{aligned}$$

and find

$$\langle 4,2,2|H_3\rangle = \frac{1}{2}\sqrt{\frac{1-s}{s}}\frac{(D_{123}^{-1})^2+(D_2^2)^2+(D_3^{-2})^2+(D_1^{-1})^2}{D_{123}^{-1}D_2^2D_3^{-2}D_1^{-1}}$$
$$\times \sin(\frac{\pi s n_1}{r_0})\sin(\frac{\pi s n_2}{r_0})\sin(\frac{\pi s n_3}{r_0})\sin(\frac{\pi s n_{123}}{r_0})$$
$$\text{where} \quad D_a^1 = \frac{n_a}{r_0}+\frac{m_1}{r_0-s} \text{ and } D_a^2 = \frac{n_a}{r_0}+\frac{m_2}{s}; \tag{3.129}$$

agreement with eqn.(3.113) obtains.

We see that the string Hamiltonian $\tilde{H}$ calculated on the Yang-Mills side for four-impurity BMN states reproduces the light-cone string field vertex between four-excitation string states.

3.6 Anomalous Dimension Corrections

In section 3.3 we mentioned that corrections to the anomalous Yang-Mills dimensions, or what is the same, the string energies, may be found via standard non-degenerate perturbation theory as in eqn.(3.27). Here, in the case of four impurities, we may use eqn.(3.27) to write, for example, the first-order correction to the anomalous dimension of the '4' state. The summation is then over the various '4', '31', and '22' intermediate states:

$$\begin{aligned}
\frac{\lambda'}{2}E^{(1)}_{|m_1,m_2,m_3;r_0\rangle} = & \int_0^{r_0} \mathrm{d}s \sum_{n_1,n_2,n_3} \langle m_1,m_2,m_3;r_0|H_-|n_1,n_2,n_3;r_0-s\rangle|s\rangle \\
& \times \frac{\langle s|\langle n_1,n_2,n_3;r_0-s|H_+|m_1,m_2,m_3;r_0\rangle}{\frac{m_{123}^2+m_1^2+m_2^2+m_3^2}{r_0^2} - \frac{n_{123}^2+n_1^2+n_2^2+n_3^2}{(r_0-s)^2}} \\
& + \int_0^{r_0} \mathrm{d}s \sum_{n_1,n_2}\sum_{a=1}^{4} \langle m_1,m_2,m_3;r_0|H_-|n_1,n_2;r_0-s\rangle^{1234\backslash a}|s\rangle^a \\
& \times \frac{{}^a\langle s|^{1234\backslash a}\langle n_1,n_2;r_0-s|H_+|m_1,m_2,m_3;r_0\rangle}{\frac{m_{123}^2+m_1^2+m_2^2+m_3^2}{r_0^2} - \frac{n_{12}^2+n_1^2+n_2^2}{(r_0-s)^2}} \\
& + \int_0^{r_0} \mathrm{d}s \sum_{n_1,n_2}\sum_{a<b} \langle m_1,m_2,m_3;r_0|H_-|n_1;r_0-s\rangle^{1234\backslash ab}|n_2;s\rangle^{ab} \\
& \times \frac{{}^{ab}\langle n_2;s|^{1234\backslash ab}\langle n_1;r_0-s|H_+|m_1,m_2,m_3;r_0\rangle}{\frac{m_{123}^2+m_1^2+m_2^2+m_3^2}{r_0^2} - \frac{n_1^2}{(r_0-s)^2} - \frac{n_2^2}{s^2}}, \qquad (3.130)
\end{aligned}$$

where the backslash denotes exclusion. We note that intermediate states differ in their impurity orderings now, with the matrix elements obtained using the results of section 3.4.4. The matrix elements involved are given by equations (3.81) - (3.86). Although it was possible in [52] to perform such summation and integration for the two-impurity case, here it seems very tedious.

3.7 Discussion

We have used the dilatation operator on the gauge-theory side of the correspondence in the basis of four-impurity BMN operators to derive an expression for the corresponding Hamiltonian on the string side. Four-impurity operators, as in the two- and three-impurity cases, may be considered to have distinct impurities, and this fact has been used to simplify the calculations. Nevertheless, to obtain the gauge-theory matrix elements and transform to the momentum-state string basis requires many tedious pages of calculation. This is rewarded by the precise agreement between these and the matrix elements obtained directly from the string field theory vertex in the plane-wave background. Our results are also in agreement with the perturbative gauge theory analysis in [67, 81]. One could follow the analysis presented in [80] and calculate decay widths using our matrix elements of $\tilde{H}$, although it seems that only special cases may be treated analytically.

Our analysis gives further evidence of the correspondence of BMN operators to string oscillator states, and demonstrates the use of BMN Quantum Mechanics for the maximal number of four different scalar impurities. It would be interesting to understand in detail what the effect of repeated impurities would be, and to extend our calculations explicitly to consider more than four. In this case, there will be more possible contractions of the gauge-theory operators, since there will no longer be a unique contraction of the impurity fields. These calculations could be compared against the arbitrary impurity-number calculations of [67, 81].

Chapter 4

String Theory in Nappi-Witten Background

4.1 Introduction

Of recent interest have been consistent six-dimensional non-local theories which do not contain gravity, but which exhibit stringy properties such as T-duality and a Hagedorn density of states [100]. These so-called Little String Theories (LSTs) [101, 102, 103, 104] are thus interesting examples of theories which are 'in-between' field theories and string theories, and are expected to shed light on both the interpretation of non-local field theory and string theory. A very complete review of all the main points of little string theories has been given in [105]. We here mention a few basic properties; for further detail the reader is referred to that review and references therein.

Generically, little string theory is defined by first considering some background of

NS5-branes in type-II string theory. Taking a limit in which the string coupling g_s goes to zero while the string mass M_s is held fixed leads to a free theory in the bulk, decoupled from a theory living on each NS5-brane; this is the LST [106, 107]. The string scale $M_s = 1/l_s$ is the only defining parameter for the LST, and is important in the following way. NS5-branes are obtained from M-theory 5-branes by compactifying on a transverse circle S^1. The resultant string theory with NS5-branes exhibits T-duality with respect to this circle, and an NS5-brane wrapping the circle of radius R will map in the dual theory to an NS5-brane wrapping a circle of radius $1/RM_s^2$. Now, from either of these theories can be defined a little string theory by taking the limit mentioned above, and the LST will inherit the T-duality from the string theory. This is an indication that the resulting theory will be non-local on the scale of M_s, and it is somewhat surprising that a non-gravitational theory can exhibit such T-duality.

In type-II theories, NS5-brane solutions break half of the supersymmetry. Ten-dimensional supersymmetry, dimensionally reduced to a six dimensional world-volume, will result in a chiral theory with $(2,2)$ supersymmetry. The NS5-brane solution may break this in either of two ways, resulting in either $(2,0)$ or $(1,1)$ supersymmetry for type IIA or IIB string theory, respectively. Considering N NS5-branes, at low energies LST must reduce to a superconformal $(2,0)$ theory on N M5-branes for the first case, or a $(1,1)$ $U(N)$ gauge theory in the second, with $G_{\text{YM}} = 1/M_s$.

LST has a holographic description [108], which turns out to be useful for explicit computations of, for instance, the spectrum and the density of states. This description involves first considering a background of M5-branes, in the so-called linear dilaton form,[108] in which g_s goes as the exponential of some coordinate ϕ. $\phi \to \infty$ is then the 'boundary,' where the limit $g_s \to 0$ obtains, and a holographic LST may be constructed in this way. In particular, the string theory may be used to calculate the spectrum of states in the LST, and, for example, correlation functions in the $(2,0)$ LST may be obtained from supergravity.

Although many theories are expected to be holographic, the AdS/CFT correspondence is still the most tested and trusted example. String theory on $AdS_5 \times S^5$ is dual to $d = 4$, $\mathcal{N} = 4$ SYM theory. Taking the Penrose limit of the 'left-hand side' of this duality is equivalent to selecting a specific sector of large-dimension and large-charge operators in the SYM theory [14]. As explained in chapter 1, string theory on an NS5-brane background corresponds to LST on the NS5-brane worldvolume. Generally it is not possible to solve string theory exactly in an NS5-brane background, but solubility obtains in the pp-wave limit. It was first identified in [109] that taking a Penrose limit in this case will reduce the NS5-brane metric to the Nappi-Witten form [110, 111, 112, 113], and this in turn corresponds to the high-energy sector of the LST [114, 115].

With this correspondence in mind, we wish in the present paper to consider type-II string theory in the Nappi-Witten background. The bosonic case has been analysed in detail in [20], and we here treat the full supersymmetric theory. In light-cone gauge, the theory is found to be completely soluble, and the Hamiltonian is calculated, giving the spectrum. The high-energy sector of the LST corresponding to the parent theory on the NS5-brane metric is expected to share this spectrum. In particular, this means that at least this sector of the LST must be supersymmetric.

In the following section, we summarise some aspects of the NS5-brane geometry of interest and its pp-wave limit. In section 4.3 we perform the above computations and obtain the spectrum. In section 4.5, we compare the resulting energy-spin relation with that obtained by semi-classical methods along the lines of [69].

4.2 Type IIA NS5-brane background and pp-wave limit

In this section, we briefly review some important aspects of NS5-brane background geometries. For further detail, in the context of little string theory, the reader is referred to the comprehensive review [105]. As we mentioned, the type IIA NS5-brane background may be described as an M-theory 5-brane transversely compactified on S^1. From [116, 108], we see that the supergravity background corresponding to N coincident extremal M5-branes is

$$ds^2 = H^{-1/3}\left[\mathrm{d}\mathbf{x}^2 + H\left(\mathrm{d}x_{11}^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2\right)\right] \tag{4.1}$$

with

$$H = 1 + \sum_{n=-\infty}^{\infty} \frac{N l_p^3}{\left[r^2 + (x_{11} - nR_{11})^2\right]^{3/2}}, \tag{4.2}$$

where the 11^{th} dimension is the compact S^1 with radius R_{11}, and l_p is the 11-dimensional Planck length. $\mathbf{x}$ is six-dimensional. Defining new scaled coordinates by

$$U = r/l_p^3, \quad y_{11} = x_{11}/l_p^3, \quad R_{11} = l_p^3/l_s^2, \tag{4.3}$$

the metric becomes

$$ds^2 = l_p^2 \tilde{H}^{-1/3}\left[\mathrm{d}\mathbf{x}^2 + \tilde{H}\left(\mathrm{d}y_{11}^2 + \mathrm{d}U^2 + U^2\mathrm{d}\Omega_3^2\right)\right] \tag{4.4}$$

with

$$\tilde{H} = l_p^6 + \sum_{n=-\infty}^{\infty} \frac{N}{\left[U^2 + (y_{11} - n/l_s^2)^2\right]^{3/2}}, \tag{4.5}$$

where $l_s \sim \frac{1}{m_s}$ is the 'coupling constant' for the theory on the brane, which is decoupled from the bulk theory.

For $U \ll \frac{1}{l_s^2}$, the summation in eqn.(4.5) may be approximated by the $n = 0$ term; in this case the metric (4.4) assumes the form of $AdS_7 \times S^4$. on which string theory is dual to the six-dimensional (2,0) SCFT. [108]

Conversely, for $U \gg \frac{1}{l_s^2}$, the summation in eqn.(4.5) effectively becomes an integration [108, 107]. The metric of N coincident extremal NS5-branes is then obtained; (up to a conformal factor and without the $\mathrm{d}y_{11}^2$ term)

$$\mathrm{d}s_{\mathrm{str}}^2 = \mathrm{d}\mathbf{x}^2 + A(U)\left(\mathrm{d}U^2 + U^2\mathrm{d}\Omega_3^2\right), \tag{4.6}$$

$$e^{2\Phi} = g_s^2 A(U) \tag{4.7}$$

with

$$A(U) = l_p^6 + \frac{N l_s^2}{U^2}. \tag{4.8}$$

We see that the energy scale $\sqrt{N} l_s$ is important here. The background possesses an asymptotically flat region $U \gg \frac{\sqrt{N} l_s}{l_p^3}$ connected to a semi-infinite flat tube $\frac{1}{l_s^2} \ll U \ll \frac{\sqrt{N} l_s}{l_p^3}$, with the topology of $\mathbb{R}^+ \times S^3 \times \mathbb{R}^6$. In this case, rather than a SCFT, the limiting theory on the brane will be little string theory [108].

We now turn our attention to the pp-wave limit of the above geometry. The near-horizon limit of the NS5-brane metric (4.6) can be written in the linear dilaton form

$$\mathrm{d}s^2_{\mathrm{str}} = Nl_s^2\left(-\mathrm{d}\tilde{t}^2 + \cos^2\theta \mathrm{d}\psi^2 + \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 + \frac{\mathrm{d}U^2}{U^2}\right) + \mathrm{d}x_\mu^2, \tag{4.9}$$

where $\mathbf{x} = (t, x_\mu)$, $\mu = 1, \ldots, 5$, and $t = \sqrt{N}l_s\tilde{t}$. Introducing new coordinates and keeping ϕ and x_μ unchanged,

$$\tilde{t} + \psi = u, \quad \tilde{t} - \psi = 2\frac{v}{Nl_s^2}, \quad U = \frac{\sqrt{N}l_s}{l_p^3}\exp(y/\sqrt{N}l_s), \quad \theta = \frac{x}{\sqrt{N}l_s}. \tag{4.10}$$

By taking the large-N limit and keeping the rescaled coordinates fixed, [114]

$$\mathrm{d}s^2_{\mathrm{str}} = -2\mathrm{d}u\mathrm{d}v - \frac{x_i^2}{4}\mathrm{d}u^2 + \mathrm{d}x_i^2 + \mathrm{d}x_A^2, \tag{4.11}$$

$$B_{ij} = u\epsilon_{ij}, \tag{4.12}$$

where $x_A = (y, x_\mu)$ and $\mathrm{d}x_i^2 = \mathrm{d}x^2 + x^2\mathrm{d}\phi^2$. This is the Nappi-Witten background [111] which was studied in the context of a WZW model generalising Poincaré symmetry. Here we have also a time-dependent B-field. The gauge transformation

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu\lambda_\nu - \partial_\nu\lambda_\mu \tag{4.13}$$

with $\lambda_i = \frac{1}{2}u\epsilon_{ij}z^j$ takes the B-field to

$$B_{iu} = -\frac{1}{2}\epsilon_{ij}x^j. \tag{4.14}$$

We can relate the light-cone energy and momentum to the original energies and momenta as measured in the linear dilaton metric [114]. Recalling that $\partial_t = \frac{1}{\sqrt{N}l_s}\partial_{\tilde{t}}$,

$$\begin{aligned} 2p^- &= -i\frac{\partial}{\partial u} = -i\left(\frac{\partial}{\partial\tilde{t}} + \frac{\partial}{\partial\psi}\right) = \tilde{E} - J = \sqrt{N}l_sE - J, \\ 2p^+ &= -i\frac{\partial}{\partial v} = -\frac{i}{Nl_s^2}\left(\frac{\partial}{\partial\tilde{t}} - \frac{\partial}{\partial\psi}\right) = \frac{\tilde{E}+J}{Nl_s^2} = \frac{\sqrt{N}l_sE + J}{Nl_s^2}. \end{aligned} \tag{4.15}$$

We will study string theory in the $N \to \infty$ limit, with $E \sim \sqrt{N}$, $J \sim N$, while keeping $\sqrt{N} l_s E - J$ finite.

4.3 σ-model and spectrum

In this section we apply light-cone quantisation to the NSR string in the Nappi-Witten background. The worldsheet fields in the light-cone gauge are as follows. The bosonic fields are x^i with $i = 1, 2$, x^A with $A = 1...6$, u and v. The corresponding fermionic fields, which are real Majorana worldsheet spinors, are ψ^i, ψ^A, ψ^u and ψ^v. We use the conventions $\rho^\sigma = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$, $\rho^\tau = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\rho^3 = \rho^\sigma \rho^\tau$, $\eta^{\tau\tau} = -\eta^{\sigma\sigma} = -1$, $\epsilon^{\tau\sigma} = +1$, and $\bar{\psi} = \psi^\dagger \rho^\tau$, with the components of a spinor labeled as $\psi = \left(\begin{smallmatrix} \psi_- \\ \psi_+ \end{smallmatrix}\right)$.

We begin with the NSR action in a curved background [117, 118],

$$
\begin{aligned}
2\pi\alpha' S = \int \mathrm{d}^2\sigma \sqrt{g} \Big[& -\frac{1}{2} g^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu} + \frac{i}{2} \bar{\psi}^\mu \rho^\alpha D_\alpha \psi^\nu G_{\mu\nu} \\
& - \frac{1}{12} R_{\mu\nu\rho\sigma} \bar{\psi}^\mu \psi^\nu \bar{\psi}^\rho \psi^\sigma + \frac{k}{8\pi} \frac{1}{\sqrt{g}} \epsilon^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu B_{\mu\nu} \\
& - \frac{ik}{16\pi} \bar{\psi}^\mu \rho^\alpha \rho_3 \psi^\nu \partial_\alpha x^\lambda T_{\mu\nu\lambda} \Big],
\end{aligned} \tag{4.16}
$$

where $T = \mathrm{d}B$ is the field strength for B, and D denotes the covariant derivative defined by

$$
D_\alpha \psi^\mu = \partial_\alpha \psi^\mu + \Gamma^\mu_{\sigma\lambda} \partial_\alpha x^\sigma \psi^\lambda. \tag{4.17}
$$

This action is worldsheet supersymmetric for arbitrary k and arbitrary background. The existence of spacetime supersymmetry of course depends on the background, as well as the value of the parameter k. We will find in the Nappi-Witten background considered next that we must have $k = 4\pi$ to produce a supersymmetric spectrum.

The original Nappi-Witten background [111] is specified by the metric

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = dx^{A^2} + dx^{i^2} - 2du dv + \big(b - \frac{x^{i^2}}{4}\big)du^2 \tag{4.18}$$

and the B-field

$$B_{iu} = -\frac{1}{2}\epsilon_{ij}x^j. \tag{4.19}$$

In the above metric, we shall be interested in the case $b = 0$; it does no harm to consider $b \neq 0$ for now, and in fact the b-dependence will turn out to be inconsequential, merely adding a constant term to the Hamiltonian. Substituting this background, the action (4.16) becomes

$$\begin{aligned}2\pi\alpha' S = \int d^2\sigma\sqrt{g}\Big[& -\frac{1}{2}g^{\alpha\beta}\partial_\alpha x^{i,A}\partial_\beta x^{i,A} + g^{\alpha\beta}\partial_\alpha u\partial_\alpha v - \frac{1}{2}g^{\alpha\beta}\big(b - \frac{x^{i^2}}{4}\big)\partial_\alpha u\partial_\beta u \\ & + \frac{i}{2}\bar{\psi}^{i,A}\rho\cdot\partial\psi^{i,A} + \frac{i}{2}\frac{x^i}{4}\bar{\psi}^i\rho\cdot\partial u\psi^u - \frac{i}{2}\bar{\psi}^u\rho\cdot\partial\psi^v \\ & - \frac{i}{2}\frac{x^i}{4}\bar{\psi}^u\rho\cdot\partial u\psi^i - \frac{i}{2}\frac{x^i}{4}\bar{\psi}^u\rho\cdot\partial x^i\psi^u - \frac{i}{2}\bar{\psi}^v\rho\cdot\partial\psi^u \\ & - \frac{i}{2}\big(b - \frac{x^{i^2}}{4}\big)\bar{\psi}^u\rho\cdot\partial\psi^u - \frac{1}{\sqrt{g}}\frac{k}{8\pi}\epsilon^{\alpha\beta}\partial_\alpha x^i\partial_\beta u\epsilon_{ij}x^j \\ & - \frac{ik}{16\pi}\epsilon_{ij}\big(\bar{\psi}^i\rho^\alpha\rho_3\psi^j\partial_\alpha u + \bar{\psi}^j\rho^\alpha\rho_3\psi^u\partial_\alpha x^i + \bar{\psi}^u\rho^\alpha\rho_3\psi^i\partial_\alpha x^j\big)\Big]\end{aligned}$$

$$, \tag{4.20}$$

where we have used the index notation $x^{i,A}$ to indicate that the summation is to be taken over values of both i and A. Varying this action with respect to $\bar{\psi}^v$ and v leads to the equations of motion

$$\rho\cdot\partial\psi^u = 0 \qquad \text{and} \qquad \partial^2 u = 0 \tag{4.21}$$

so, as in [2] (pg.211) we may choose the light-cone gauge

$$\psi^u = 0 \qquad , \qquad u = u_0 + p^+\tau, \tag{4.22}$$

where we have now made the choice $l_s = 1$. Imposing this gauge, and also $g_{\alpha\beta} = \eta_{\alpha\beta}$, we obtain the light-cone gauge action

$$\pi S_{\rm LCG} = \int \mathrm{d}^2\sigma \Big[-\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha x^{i,A}\partial_\beta x^{i,A} - p^+\partial_\tau v + \frac{1}{2}(b - \frac{x^{i2}}{4})p^{+2} + \frac{i}{2}\bar{\psi}^{i,A}\rho\cdot\partial\psi^{i,A} + \frac{k}{8\pi}p^+\partial_\sigma x^i\epsilon_{ij}x^j - \frac{ik}{16\pi}p^+\epsilon_{ij}\bar{\psi}^i\rho^\tau\rho_3\psi^j\Big]. \quad (4.23)$$

This action may be transformed to a massless form when the constant k is chosen to be 4π. We remark further on this point in section 4.5.

The canonical momenta conjugate to the x and ψ fields are

$$\pi^{i,A} = \frac{1}{\pi}\partial_\tau x^{i,A} \quad , \quad \xi^{i,A} = -\frac{i}{2\pi}\bar{\psi}^{i,A}\rho_\tau. \quad (4.24)$$

The field u is gauged away, while the term $p^+\partial_\tau v$ may be dropped as it is a total time-derivative. The Hamiltonian is thus

$$p^+H = -\frac{i}{2}\bar{\psi}^{i,A}\rho_\sigma\partial_\sigma\psi^{i,A} + \frac{1}{2}(\partial_\tau x^{i,A})^2 + \frac{1}{2}(\partial_\sigma x^{i,A})^2 - \frac{kp^+}{8\pi}\left(\partial_\sigma x^i\epsilon_{ij}x^j - \frac{i}{2}\epsilon_{ij}\bar{\psi}^i\rho^\tau\rho_3\psi^j\right) - \frac{p^{+2}}{2}(b - \frac{x^{i2}}{4}). \quad (4.25)$$

Varying the light-cone-gauge action (4.23) to obtain the equations of motion,

$$\delta x^A \quad \longrightarrow \quad \partial^2 x^A = 0, \quad (4.26)$$

$$\delta x^i \quad \longrightarrow \quad (\partial^2 - \frac{p^{+2}}{4})x_i = \frac{k}{4\pi}p^+\epsilon_{ij}\partial_\sigma x^j, \quad (4.27)$$

$$\delta\bar{\psi}^A \quad \longrightarrow \quad \rho\cdot\partial\psi^A = 0, \quad (4.28)$$

$$\delta\bar{\psi}^i \quad \longrightarrow \quad \rho\cdot\partial\psi_i = \frac{k}{4\pi}p^+\epsilon_{ij}\rho^\tau\rho_3\psi^j. \quad (4.29)$$

The equations of motion for the i-labeled coordinates may be decoupled by defining $x^\pm = x^1 \pm ix^2$ and $\psi^\pm = \psi^1 \pm i\psi^2$, giving

$$(\partial^2 - \frac{p^{+2}}{4})x^\pm = \mp i\frac{kp^+}{4\pi}\partial_\sigma x^\pm \quad (4.30)$$

and

$$\rho \cdot \partial\psi^{\pm} = \mp i \frac{kp^{+}}{8\pi} \rho^{\tau} \rho_3 \psi^{\pm}. \tag{4.31}$$

Expanding in modes, we find that the frequencies for bosons and fermions match only when $k = 4\pi$. Adopting this choice, we have

$$\begin{aligned} x^A = x_0^A + p^A \tau + \frac{i}{2} \sum_{n>0} \Bigg(& \frac{1}{\sqrt{n}} a_n^A e^{-in(\tau-\sigma)} - \frac{1}{\sqrt{n}} a_n^{A\dagger} e^{in(\tau-\sigma)} \\ & + \frac{1}{\sqrt{n}} \tilde{a}_n^A e^{-in(\tau+\sigma)} - \frac{1}{\sqrt{n}} \tilde{a}_n^{A\dagger} e^{in(\tau+\sigma)} \Bigg) \end{aligned} \tag{4.32}$$

with $x_0^A = x_0^{A\dagger}$ and $p^A = p^{A\dagger}$, and

$$\begin{aligned} x^+ = x^{-\dagger} &= \sqrt{\frac{2}{p^+}} a_0^{+\dagger} e^{i\frac{p^+}{2}\tau} + \sqrt{\frac{2}{p^+}} a_0^- e^{-i\frac{p^+}{2}\tau} \\ &+ e^{i\frac{p^+}{2}\tau} \sum_{n>0} \left(\frac{a_n^{+\dagger}}{\sqrt{n + \frac{p^+}{2}}} e^{in(\sigma+\tau)} + \frac{a_n^-}{\sqrt{n - \frac{p^+}{2}}} e^{-in(\sigma+\tau)} \right) \\ &+ e^{-i\frac{p^+}{2}\tau} \sum_{n>0} \left(\frac{\tilde{a}_n^-}{\sqrt{n + \frac{p^+}{2}}} e^{in(\sigma-\tau)} + \frac{\tilde{a}_n^{+\dagger}}{\sqrt{n - \frac{p^+}{2}}} e^{-in(\sigma-\tau)} \right). \end{aligned} \tag{4.33}$$

In the fermionic sector, we have

$$\psi_-^A = d_0^A + \sum_{r>0} \left(d_r^A e^{-ir(\tau-\sigma)} + d_r^{A\dagger} e^{ir(\tau-\sigma)} \right) \tag{4.34}$$

$$\psi_+^A = \tilde{d}_0^A + \sum_{r>0} \left(\tilde{d}_r^A e^{-ir(\tau+\sigma)} + \tilde{d}_r^{A\dagger} e^{ir(\tau+\sigma)} \right) \tag{4.35}$$

and

$$e^{\pm i\frac{p^+}{2}\tau} \psi_-^{\pm} = d_0^{\pm} + \frac{1}{\sqrt{2}} \sum_{r>0} \left(d_r^{\pm} e^{-ir(\tau-\sigma)} + d_r^{\mp\dagger} e^{ir(\tau-\sigma)} \right) \tag{4.36}$$

$$e^{\mp i\frac{p^+}{2}\tau} \psi_+^{\pm} = \tilde{d}_0^{\pm} + \frac{1}{\sqrt{2}} \sum_{r>0} \left(\tilde{d}_r^{\pm} e^{-ir(\tau+\sigma)} + \tilde{d}_r^{\mp\dagger} e^{ir(\tau+\sigma)} \right), \tag{4.37}$$

where $d_0^A = d_0^{A\dagger}$ and $\tilde{d}_0^A = \tilde{d}_0^{A\dagger}$. The summation in the fermionic sector may be over either integer or half-integer values of r, corresponding to the Ramond and Neveu-Schwarz sectors; for half-integer values, the zero modes are to be omitted.

We impose canonical quantisation relations

$$\left[x^{i,A}(\sigma,\tau),\pi^{i',A'}(\sigma',\tau)\right] = i\delta(\sigma-\sigma')\delta^{i,A\ i',A'}, \tag{4.38}$$

$$\left\{\psi_{\pm}^{i,A}(\sigma,\tau),\xi_{\pm}^{i',A'}(\sigma',\tau)\right\} = i\delta(\sigma-\sigma')\delta^{i,A\ i',A'}, \tag{4.39}$$

which in terms of oscillators become:

$$\begin{aligned}
&[x_0^A,p^B] = \delta^{AB} \\
&[a_n^A,a_m^{B\dagger}] = [\tilde a_n^A,\tilde a_m^{B\dagger}] = \delta_{nm}\delta^{AB} \qquad \{d_r^A,d_s^{B\dagger}\} = \{\tilde d_r^A,\tilde d_s^{B\dagger}\} = \delta_{rs}\delta^{AB} \\
&[a_n^{\pm},a_m^{\pm\dagger}] = [\tilde a_n^{\pm},\tilde a_m^{\pm\dagger}] = \delta_{nm} \qquad \{d_r^{\pm},d_s^{\pm\dagger}\} = \{\tilde d_r^{\pm},\tilde d_s^{\pm\dagger}\} = \delta_{rs}
\end{aligned} \tag{4.40}$$

The mode expansions and commutation relations for the bosonic fields agree with those found in [20] for the case of the bosonic string.

The Hamiltonian may now be written

$$\begin{aligned}
p^+H = {} & \frac{p^{A2}}{2} + \frac{p^+}{2}(a_0^{+\dagger}a_0^+ + a_0^{-\dagger}a_0^-) - \frac{p^+}{2}(d_0^{+\dagger}d_0^+ + \tilde d_0^{+\dagger}\tilde d_0^+) \\
& + \sum_{n>0} n(a_n^{A\dagger}a_n^A + \tilde a_n^{A\dagger}\tilde a_n^A) + \sum_{r>0} r(d_r^{A\dagger}d_r^A + \tilde d_r^{A\dagger}\tilde d_r^A) \\
& + \sum_{n>0}(n+\frac{p^+}{2})(a_n^{+\dagger}a_n^+ + \tilde a_n^{-\dagger}\tilde a_n^-) + \sum_{n>0}(n-\frac{p^+}{2})(\tilde a_n^{+\dagger}\tilde a_n^+ + a_n^{-\dagger}a_n^-) \\
& + \sum_{n>0}(n+\frac{p^+}{2})(d_r^{+\dagger}d_r^+ + \tilde d_r^{-\dagger}\tilde d_r^-) + \sum_{n>0}(n-\frac{p^+}{2})(d_r^{-\dagger}d_r^- + \tilde d_r^{+\dagger}\tilde d_r^+),
\end{aligned} \tag{4.41}$$

where $r = n$ or $r = n + 1/2$ for the Ramond or Neveu-Schwarz sectors, respectively. As dictated by supersymmetry, both the zero point energies and divergent terms cancel among bosonic and fermionic sectors.

Defining bosonic and fermionic number operators $N_n = a_n^\dagger a_n + \tilde a_n^\dagger \tilde a_n$ and $N_r = d_r^\dagger d_r + \tilde d_r^\dagger \tilde d_r$, we have as our final expression (setting $p^A = 0$)

$$H = \frac{1}{p^+}\sum_{n\geq 0}\left(\sum_{A=1}^{6} n(N_n^A + N_r^A) + (n+\frac{p^+}{2})(N_n^+ + N_r^+) + (n-\frac{p^+}{2})(N_n^- + N_r^-)\right), \tag{4.42}$$

where the same convention is used for the index r as in the previous expression. Supersymmetry is clearly evident in this expression; this is significant in that it demonstrates that the corresponding states of the dual LST must also fall into supersymmetric multiplets.

Writing this in terms of the original energy and momentum with eqn.(4.15) we can relate the energy to the angular momentum. Defining $\tilde{J} = J/\sqrt{N}$ we have

$$\left(E - \frac{\bar{N}}{\sqrt{N}}\right)^2 = \left(\tilde{J} + \frac{\bar{N}}{\sqrt{N}}\right)^2 + 2\sum_n n\left[\sum_{A=1}^{6}(N_n^A + N_r^A) + N_n^+ + N_r^+ + N_n^- + N_r^-\right] \quad (4.43)$$

where

$$\bar{N} \equiv \frac{1}{4}\sum_n (N_n^+ + N_r^+ - N_n^- - N_r^-). \quad (4.44)$$

We emphasize here that $\bar{N}$ does not depend in the A-indexed modes. We see from this expression that states for which $\bar{N} = 0$ will exhibit the standard relation between energy and angular momentum, while states with $\bar{N} \neq 0$ shift E and J, 'trading' one for the other at order $1/\sqrt{N}$.

In the following section we shall compare this result with the relation obtained via semi-classical analysis of a rotating once-folded string [69].

4.4 Semi-classical Analysis

As we discussed in section 2.2.2, a semi-classical method can be used, which allows for more general computations than those done in the pp-wave limit [68]. In $AdS_5 \times S^5$, the semi-classical method has been used to first order, with a point-like string boosted in an S^5 direction, to reproduce the result obtained via a pp-wave computation such as in the previous section [69, 70]. Here, following this example, we apply the semi-classical method to the case of the NS5-brane and compare with our pp-wave results.

We start with the linear dilaton metric

$$\begin{aligned} \mathrm{d}s_{\mathrm{str}}^2 &= -\mathrm{d}t^2 + dx_5^2 + \frac{Nl_s^2}{U^2}\left(\mathrm{d}U^2 + U^2\mathrm{d}\Omega_3^2\right) \\ &= Nl_s^2\left(-\mathrm{d}\tilde{t}^2 + \frac{\mathrm{d}U^2}{U^2} + \mathrm{d}\Omega_3^2\right) + \mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega_4^2 \end{aligned} \tag{4.45}$$

where $t = \sqrt{N}l_s\tilde{t}$ and

$$\mathrm{d}\Omega_3^2 = \cos^2\theta\mathrm{d}\psi^2 + \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2 \tag{4.46}$$

$$\mathrm{d}\Omega_4^2 = \mathrm{d}\beta_1^2 + \cos^2\beta_1\left(\mathrm{d}\beta_2^2 + \cos^2\beta_2\left(\mathrm{d}\beta_3^2 + \cos^2\beta_3\mathrm{d}\beta_4^2\right)\right). \tag{4.47}$$

Using the bosonic string action

$$S = -\frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma\mathrm{d}\tau G_{\mu\nu}\left(\partial_\sigma x^\mu\partial_\sigma x^\nu - \partial_\tau x^\mu\partial_\tau x^\nu\right), \tag{4.48}$$

with the above metric, the equation of motion may be obtained; to find a classical solution, we make the Ansatz

$$\begin{aligned} &\rho = \rho(\sigma), U = U(\sigma), \\ &\tilde{t} = \kappa\tau, \beta_4 = \omega\tau, \psi = \nu\tau, \\ &\theta = \phi = \beta_{1,2,3} = 0, \end{aligned} \tag{4.49}$$

which describes a string stretching along the directions ρ and U, rotating around the ϕ and φ directions. The action (4.48) becomes

$$S = -\frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma\mathrm{d}\tau\left[N(\kappa^2 - \nu^2) + \rho'^2 - \omega^2\rho^2 + N\frac{U'^2}{U^2}\right], \tag{4.50}$$

and the resulting equations for ρ and U are

$$\rho'' + \omega^2 \rho = 0, \tag{4.51}$$

$$U'' - \frac{U'^2}{U} = 0, \tag{4.52}$$

with the constraint

$$T_{++} = T_{--} = -N\kappa^2 + \rho'^2 + \omega^2\rho^2 + N\frac{U'^2}{U^2} + N\nu^2 = 0. \tag{4.53}$$

The solution is

$$\rho'^2 = N(\kappa^2 - \nu^2)\cos^2\theta_0 - \omega^2\rho^2 \tag{4.54}$$

$$U'^2 = (\kappa^2 - \nu^2)U^2 \sin^2\theta_0 \tag{4.55}$$

where θ_0 is an integration constant.

We consider the once-folded string configuration, with the string split into four segments; for $0 < \sigma < \pi/2$, the function $\rho(\sigma)$ increases from 0 to its maximal value ρ_0. Then,

$$\rho'\left(\frac{\pi}{2}\right) = 0 \quad \Rightarrow \quad \rho_0 = \frac{\sqrt{N(\kappa^2 - \nu^2)\cos^2\theta_0}}{\omega}. \tag{4.56}$$

Calculating the energy and angular momentum,

$$\tilde{E} = -P_{\tilde{t}} = \frac{1}{4\pi\alpha'}\int_0^{2\pi} \mathrm{d}\sigma 2N\kappa = \frac{N\kappa}{\alpha'}, \tag{4.57}$$

$$\begin{aligned} S &= P_{\beta_4} = \frac{1}{4\pi\alpha'}\int_0^{2\pi} \mathrm{d}\sigma 2\omega\rho^2 \\ &= \frac{4\omega}{\pi\alpha'}\int_0^{\rho_0} \frac{\rho^2 \mathrm{d}\rho}{\sqrt{N(\kappa^2 - \nu^2)\cos^2\theta_0 - \omega^2\rho^2}} = \frac{N(\kappa^2 - \nu^2)\cos^2\theta_0}{2\alpha'\omega^2}, \end{aligned} \tag{4.58}$$

$$J = P_\psi = \frac{1}{4\pi\alpha'}\int_0^{2\pi} \mathrm{d}\sigma 2N\nu = \frac{N\nu}{\alpha'}. \tag{4.59}$$

The relationship between E, S and J is thus

$$E^2 = \frac{\tilde{E}^2}{N} = \frac{J^2}{N} + \frac{2\omega^2}{\alpha' \cos^2\theta_0} S. \tag{4.60}$$

Imposing the periodicity condition

$$2\pi = 4\int_0^{\rho_0} \frac{\mathrm{d}\rho}{\sqrt{N(\kappa^2 - \nu^2)\cos^2\theta_0 - \omega^2\rho^2}} = \frac{2\pi}{\omega}, \tag{4.61}$$

we find $\omega = 1$. Thus we obtain

$$E^2 = \tilde{J}^2 + \frac{2}{\alpha' \cos^2\theta_0} S. \tag{4.62}$$

In the case $\theta_0 = 0$ this expression for the energy of the string undergoing the motion described above is the same as in the case of a flat background [69].

Identifying the order $n = 1$ oscillator state with the spin S, as explained in ref. [69], we may apply our BMN formula (4.43) (in the bosonic sector) to obtain (reinstating α')

$$\left(E - \frac{\bar{N}_1}{\sqrt{N}}\right)^2 = \left(\tilde{J} + \frac{\bar{N}_1}{\sqrt{N}}\right)^2 + \frac{2}{\alpha'} S \tag{4.63}$$

where $\bar{N}_1$ refers to the $n = 1$ contribution to $\bar{N}$. In the large-N limit this coincides with the relation (4.62). The reader is also referred to [119, 120]. We conclude that the string theory in the pp-wave limit of the NS5-brane background is dual to a sector of LST which has states in correspondence to states of this free worldsheet theory.

4.5 Discussion

In the Penrose limit, we are considering energy of order $\sqrt{N}\alpha' l_s$ in the sector with $J \sim N$. In the $AdS_5 \times S^5$ case, the pp-wave light-cone worldsheet action contains a 'mass term'. The mass term $x^2/2$ makes it different from string theory in flat spacetime. In the present case of an NS5-brane, the action (4.23) contains both a mass term and antisymmetric B-field. These two terms can be removed from our action by the following field redefinition with the choice $k = 4\pi$, which also is the choice which ensures space-time supersymmetry. We redefine

$$Z = e^{i\sigma p^+/2}\left(x^1 + ix^2\right), \tag{4.64}$$

$$\Phi^+ = e^{i\sigma p^+/2}\left(\psi^1 + i\psi^2\right), \tag{4.65}$$

$$\Phi^- = e^{-i\sigma p^+/2}\left(\psi^1 - i\psi^2\right); \tag{4.66}$$

the action becomes

$$\begin{aligned} S = \frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\sqrt{g}\Big\{&-\frac{1}{2}g^{\alpha\beta}\left(\partial_\alpha x^A\partial_\beta x^A + \partial_\alpha Z\partial_\beta \bar{Z} - 2\delta_{\alpha\tau}p^+\partial_\beta v\right) \\ &+\frac{i}{2}\bar{\psi}^A\rho^\alpha\left(\partial_\alpha\psi^A\right) + \frac{i}{2}\left[\frac{1}{2}\bar{\Phi}^+\rho^\alpha\left(\partial_\alpha\Phi^+\right) + \frac{1}{2}\bar{\Phi}^-\rho^\alpha\left(\partial_\alpha\Phi^-\right)\right]\Big\}, \end{aligned} \tag{4.67}$$

with the boundary conditions

$$x^A(\sigma + 2\pi) = x^A(\sigma), \quad A = 1, \ldots, 6, \tag{4.68}$$

$$Z(\sigma + 2\pi) = e^{i\pi p^+} Z(\sigma), \tag{4.69}$$

$$\Phi^+(\sigma + 2\pi) = \pm e^{i\pi p^+}\Phi^+(\sigma), \tag{4.70}$$

$$\Phi^-(\sigma + 2\pi) = \pm e^{-i\pi p^+}\Phi^-(\sigma). \tag{4.71}$$

We end up with a free worldsheet theory very similar to that of NSR strings in flat spacetime. The difference in this case is that two of the fields have twisted boundary conditions. This is, of course, what we found in the direct calculation of previous sections.

Chapter 5

VSFT, Tachyon Condensation and D-Branes

Tachyon condensation, Sen conjectured, would begin with a D25-brane background and lead to a non-perturbative vacuum previously unidentified in string field theory. Analyses could then be made of the theory near this vacuum. For a thorough discussion of many aspects of VSFT, the review [43] is recommended to the reader. In this chapter we review some of the developments leading to the resulting vacuum string field theory. These developments underlie the work done in chapters 6 and 7 on D-brane tension and tachyon fluctuations.

5.1 Sen's conjecture and Vacuum String Field Theory

Sen in 1999 made the conjecture [36] that condensation of the tachyon field in bosonic string theory [121] would lead to some final state with D-branes. Interest in string field theory had waned somewhat in the years before, partly due to the difficulty of analysing a theory where although the action (1.11), gauge symmetry (1.12) and equation of motion (1.13) were known, no classical solution could be identified, obstructing the use of perturbation theory and providing no example to solidify intuition.

As we mentioned in section 1.3.2, the key to solubility in Sen's non-perturbative tachyon vacuum is the purely ghost nature of the BRST operator Q. Rastelli, Sen and Zwiebach [38] identified certain properties the tachyon vacuum is expected to have, and thereby were able to construct the pure-ghost BRST operator, which we will denote $\mathcal{Q}$. In particular, in addition to depending only on ghost fields, $\mathcal{Q}$ must have vanishing cohomology, $H^*(\mathcal{Q}) = 0$, and must be independent of background. That is to say, $\mathcal{Q}$ must be independent of the particular conformal field theory used to define physical states.

Given the vacuum BRST operator, the string-field equation of motion (1.13)

$$\mathcal{Q}\Psi + \Psi \star \Psi = 0 \tag{5.1}$$

admits solutions which are factorisable into matter and ghost parts. Assuming such a factorisation

$$\Psi = \Psi_g \otimes \Psi_m \tag{5.2}$$

with $\Psi_m[x]$ and $\Psi_g[c,b]$, the equation of motion (5.1) becomes

$$\mathcal{Q}\Psi_g + \Psi_g \star \Psi_g = 0 \tag{5.3}$$

$$\Psi_m = \Psi_m \star \Psi_m. \tag{5.4}$$

The solution to the ghost equation (5.3) is thought to be universal for all D-brane vacua [37, 38]. Although a solution is known, this universality means that many properties of D-brane vacuum solutions, for example ratios of D-brane tensions, are dependent only on the matter solution Ψ_m to eqn.(5.4). For this reason, the ghost sector of the theory [122, 123] will not play a prominent role in the remainder of this chapter.

5.2 Solution to the Matter Equation of Motion

Kostelecký and Potting first found a solution to the matter string-field equation of motion (5.4) algebraically in the form of a squeezed state [40]. Soon after, Rastelli, Sen and Zwiebach generalized the solution somewhat to have Gaussian dependence on the zero modes, producing localization in some number p of space-time directions [38]. They identified these states as Dp-branes.

In the operator formulation, the star product of two states is found by means of the three-vertex state $|V_3\rangle$, as in eqn.(1.51). In [40] a squeezed-state Ansatz was made and a form was found satisfying the requirement of star-squaring to itself, using this vertex operator and some clever techniques of changing squeezed-state bases. (progress has also been made with squeezed states in [124].) Here we exhibit the D-instanton state of [38].

Using the notation introduced in chapter 1, it is given by

$$|\Xi_{-1}\rangle = e^{-\frac{1}{2}(a^\dagger|S|a^\dagger)}|\Omega\rangle. \tag{5.5}$$

Here,

$$S = C\frac{1}{2Z}\left(1 + Z - \sqrt{(1+3Z)(1-Z)}\right), \tag{5.6}$$

$$Z = \frac{C}{3}(C + U + CUC) = CU_{11}. \tag{5.7}$$

The state $|\Xi_{-1}\rangle$ may be verified to satisfy the equation of motion.

5.3 Split-String Formulation

In [41, 28] Gross and Taylor develop a split-string formalism, first suggested by Witten[24], in which the string field is considered as a functional of the embeddings of the 'left' and 'right' halves of strings. This method proves useful due to a correspondence between full-string wave functionals and operators on the space of half-string embeddings. The star product on full-string functionals is thereby mapped to ordinary operator multiplication. In [41, 28] a specific mode expansion of the full- and half-string degrees of freedom is used, compatible with the Neumann boundary condition, and demonstrate a solution to the string-field equations of motion. This solution is interpreted as a D-instanton.

5.3.1 Half-Strings

In this section we introduce the split-string formalism of [41] and exhibit the correspondence between operators and functionals in such a way that it does not depend on a specific boundary condition or mode expansion. A string state $|\Psi\rangle$ is represented by a functional $\Psi[x]$ of the string embedding $x(\sigma)$. We can write the string functional in terms of the left and right halves of the string embedding function, defined by

$$l(\sigma) = x(\sigma), \quad r(\sigma) = x(\pi - \sigma), \quad \text{for } 0 \leq \sigma \leq \pi/2. \tag{5.8}$$

Half-string states are thus described by functionals of half-string embeddings $h(\sigma)$. Ghosts may be incorporated as in the full-string formalism, bosonised or not. As before, we shall not need to discuss the ghost sector.

We define the half-string spaces as the spaces spanned by these basis states, and the full string space by

$$\mathcal{H} \equiv \mathcal{H}_l \otimes \mathcal{H}_r \tag{5.9}$$

A priori this full-string space has no relation to the full-string space with which we started out, aside from containing the same states. The point of the half-string formulation is that we may now introduce a new inner product on this space which causes the $\star$ operation to take a simple form in terms of operators. We define the inner product between half-string states through

$$\langle h|h'\rangle \equiv \delta[h - h']. \tag{5.10}$$

We can express the fundamental string operations in terms of half-string functionals as follows: Integration is written

$$\int \Psi = \int \mathrm{D}h \Psi[h, h] \tag{5.11}$$

and the star product is given by

$$\Psi_1 \star \Psi_2[l, r] = \int \mathrm{D}h \Psi_1[l, h] \Psi_2[h, r]. \tag{5.12}$$

Then the identity is just a delta functional

$$I[l,r] = \delta[l-r]. \tag{5.13}$$

To every string functional $\Psi[x] = \Psi[l,r]$, we associate an operator $\hat{\Psi}$ on the space of half-string states,

$$\Psi[l,r] = \langle l|\hat{\Psi}|r\rangle \qquad \leftrightarrow \qquad \hat{\Psi} = \int \mathrm{D}l\mathrm{D}r|l\rangle\Psi[l,r]\langle r|. \tag{5.14}$$

The following correspondence then obtains between operations with string functionals and half-string operators.

$$\begin{aligned} \int \Psi \quad &\leftrightarrow \quad \mathrm{Tr}\ \hat{\Psi}; \\ \Psi_1 \star \Psi_2 \quad &\leftrightarrow \quad \hat{\Psi}_1\hat{\Psi}_2; \\ I \quad &\leftrightarrow \quad 1. \end{aligned} \tag{5.15}$$

Recalling eqn.(5.4), the matter part of the full-string wavefunction satisfies the equation of motion $\Psi \star \Psi = \Psi$. A full-string state satisfying this equation corresponds to a half-string operator satisfying the projection equation

$$\hat{\Psi}^2 = \hat{\Psi}. \tag{5.16}$$

Such projection operators in the split-string formalism can be classified by their rank r. A rank-one projection operator may be written $P_\chi = |\chi\rangle\langle\chi|$ where $|\chi\rangle$ is a half-string state. Corresponding to $\chi[h]$ is the full-string functional

$$P_\chi[l(\sigma), r(\sigma)] = \chi[l]\chi[r]. \tag{5.17}$$

Let us suppose that the rank one projection operator may be represented by a gaussian functional in the half-string basis and let us find its representation in the full-string basis. We write the half-string functional $\chi[h]$ as

$$\chi[h] = \exp\left(-\frac{1}{2}\int_0^{\pi/2} h(\sigma)M(\sigma,\sigma')h(\sigma')\mathrm{d}\sigma\mathrm{d}\sigma'\right). \tag{5.18}$$

Since the half-string embedding is related to the full-string embedding linearly, the corresponding full-string representation of the rank one projection operator will also be given by a gaussian functional,

$$\Psi[x] = \exp\left(-\frac{1}{2}\int_0^{\pi} x(\sigma)L(\sigma,\sigma')x(\sigma')\mathrm{d}\sigma\mathrm{d}\sigma'\right), \tag{5.19}$$

where L is symmetric under exchange of σ and σ'. We write the full-string functional as

$$\Psi[x] = \exp\left\{-\frac{1}{2}\left(\int_0^{\pi/2}\mathrm{d}\sigma + \int_{\pi/2}^{\pi}\mathrm{d}\sigma\right)\left(\int_0^{\pi/2}\mathrm{d}\sigma' + \int_{\pi/2}^{\pi}\mathrm{d}\sigma'\right) x(\sigma)L(\sigma,\sigma')x(\sigma')\right\}, \tag{5.20}$$

which may be written in terms of half-string degrees of freedom;

$$\begin{aligned}\Psi[l,r] = \exp\Bigg(-\frac{1}{2}\int_0^{\pi/2} &\mathrm{d}\sigma\mathrm{d}\sigma'\big[l(\sigma)L(\sigma,\sigma')l(\sigma') + r(\sigma')L(\pi-\sigma,\sigma')l(\sigma')\\ &+ l(\sigma)L(\sigma,\pi-\sigma')r(\sigma') + r(\sigma)L(\pi-\sigma,\pi-\sigma')r(\sigma')\big]\Bigg).\end{aligned} \tag{5.21}$$

Requiring that this represent a half-string projection operator of the form (5.19) and eliminating M gives the relations

$$L(\sigma,\sigma') = M(\sigma,\sigma') = L(\pi-\sigma,\pi-\sigma'), \tag{5.22}$$

$$L(\pi-\sigma,\sigma') = L(\sigma,\pi-\sigma') = 0, \tag{5.23}$$

for $0 \leq \sigma,\sigma' \leq \pi/2$. Discrete versions of these equations are solved in the sections which follow. The normalisation of a gaussian state may formally be obtained as follows. Using

$|\Psi\rangle = \int \mathrm{D}x \Psi[x]|x\rangle$ and the orthonormality of the full-string basis states, we find

$$|\Psi|^2 = \langle\Psi|\Psi\rangle = \int \mathrm{D}x \Psi[x]^2 = [\mathrm{Det}L]^{-1/2}. \tag{5.24}$$

5.3.2 Left and Right Half-String Modes

In chapter 1, for the full-string case, we introduced a mode expansion of the embedding (1.20) and considered the string field as a function of the Fourier modes x_n of the embedding, rather than of $x(\sigma)$. We proceed to do the same with half-string embeddings. This formulation was devised by Gross and Taylor in [41, 28], and we base the following discussion on these papers. We adopt the left- and right-string expansions

$$l(\sigma) = \sqrt{2}\sum_{n=0}^{\infty} l_{2n+1} \cos(2n+1)\sigma \tag{5.25}$$

$$r(\sigma) = \sqrt{2}\sum_{n=0}^{\infty} r_{2n+1} \cos(2n+1)\sigma, \tag{5.26}$$

satisfying Neumann boundary conditions. Recalling the expansion (1.20) for $x(\sigma)$, the full-string modes $\{x_n\}$ and half-string modes $\{l_{2n+1}, r_{2n+1}\}$ are related by

$$l_{2n+1} = x_{2n+1} + \sum_{m=0}^{\infty} X_{2n+1,2m} x_{2m}, \tag{5.27}$$

$$r_{2n+1} = -x_{2n+1} + \sum_{m=0}^{\infty} X_{2n+1,2m} x_{2m}, \tag{5.28}$$

where

$$\frac{\pi}{4} X_{2n+1,2m} = X_{2m,2n+1} = \begin{cases} \frac{(-)^{m+n}(2n+1)}{(2n+1)^2-4m^2} & , m \neq 0 \\ \frac{1}{\sqrt{2}}\frac{(-1)^n}{2n+1} & , m = 0 \end{cases}. \tag{5.29}$$

The inverse also we can write,

$$x_{2n+1} = \frac{1}{2}(l_{2n+1} - r_{2n+1}), \tag{5.30}$$

$$x_{2n} = \frac{1}{2}\sum_{m=0}^{\infty} X_{2n,2k+1}(l_{2m+1} + r_{2m+1}). \tag{5.31}$$

The matrix defined by

$$X \equiv \begin{bmatrix} 0 & X_{2n+1,2m} \\ X_{2m,2n+1} & 0 \end{bmatrix} \tag{5.32}$$

is symmetric and orthogonal

$$X = X^T = X^{-1}, \tag{5.33}$$

owing to the completeness relations

$$\sum_{n=1}^{\infty}(-)^n X_{2m+1,2n} = -\frac{1}{\sqrt{2}} X_{2m+1,0}, \tag{5.34}$$

$$\sum_{n=0}^{\infty} 4n^2 X_{2m+1,2n} X_{2n,2p+1} = (2m+1)^2 \delta_{mp}, \tag{5.35}$$

and will be convenient in what follows. When working with these matrices, care must be given to convergence properties; in particular, matrices we define here are not always associative.

5.3.3 Projection Operators

In section 5.3.1 we discussed the rank one operator of gaussian form in the split-string formulation without using a specific mode representation. Here we construct the rank one operator for the string with Neumann boundary condition, using the modes $\{l_{2n+1}, r_{2n+1}\}$

we defined in the previous section. We choose the half-string functional to be of squeezed-state form,

$$\chi[l] = \exp\left(-\frac{1}{2} l_{2n} M_{nm} l_{2m}\right), \tag{5.36}$$

which is a discretized version of eqn.(5.18). The full-string functional, analogous to eqn.(5.19), we write as

$$\Psi[x] = \chi[l]\chi[r] = \exp\left(-\frac{1}{2} x_n L_{nm} x_m\right). \tag{5.37}$$

Making use of eqn.(5.22), we find that L must satisfy

$$L_{2n+1,2m+1} = 2M_{nm}, \tag{5.38}$$

$$L_{2n,2m} = 2X_{2n,2k+1} M_{k,l} X_{2l+1,2m}, \tag{5.39}$$

$$L_{2n+1,2m} = L_{2m,2n+1} = 0. \tag{5.40}$$

These equations may be solved for L, giving the conditions under which the full-string functional (5.37) corresponds to a half-string projection operator. The requirements on L are of course

$$L_{2n,2m} = X_{2n,2k+1} L_{2k+1,2l+1} X_{2l+1,2m}, \tag{5.41}$$

together with eqn.(5.40).

Gross and Taylor showed that the solution of Kostelecký and Potting [40] and the extension made in [38] do indeed satisfy these requirements. The solution with zero-mode dependence in all space-time directions they identified as the D-instanton state $|\Xi_{-1}\rangle$, which we exhibited in section 5.2. The change-of-basis matrix X is identified with the matrix X which we defined there. The satisfaction of the string-field equation of motion depends only on the properties of the matrix X; specifically, X is orthogonal, symmetric, and anticommutes with C.

5.4 Explicit Background D-branes

In the present section, based on the work [6], we show that the split-string formalism is not limited to the case of a Neumann boundary condition and can be applied equally well using a Dirichlet boundary condition and corresponding mode expansion. Such a formulation would correspond to a string field in the presence of an explicit background D-brane. We find that solutions to the equations of motion of the kind shown in [41, 42] and in the previous section correspond in a simple way to solutions constructed using the Dirichlet expansion. Such solutions in this case represent additional D-branes within the D-brane background.

Our motivation is twofold. Firstly, as mentioned in the introduction, an explicit background D-brane built into the theory should allow easy representation of systems involving D-branes within other D-branes[41], such as the D1-D5 system[125]. Secondly, since in string theory the T-duality transformation interchanges Neumann and Dirichlet boundary conditions, we expect that the split-string formalism also may be useful when formulated with the Dirichlet boundary condition. We expand the string coordinate as

$$x(\sigma) = A + \frac{B-A}{\pi}\sigma + \sqrt{2}\sum_{n=1}^{\infty} x_n \sin n\sigma \tag{5.42}$$

where A and B are constants. We would expect the above Dirichlet mode expansion to be appropriate if we wish to consider two flat D-branes at positions A and B. For simplicity, we set $A = B = 0$.

We here emphasize the important point that any particular choice of mode expansion, such as the above, simply restricts the domain on which we subsequently consider string functionals. Generically the string field Ψ is a map on the space of all string embeddings, and certain features of $\Psi[x]$ might best be seen using a particular mode expansion. The point of view taken here is that while the string field will not *contain* a D-brane as a result of the above choice, it may be taken to represent strings in the presence of such a D-brane, treating it as a background.

The procedure which follows is exactly analogous to that of Gross and Taylor which we presented in he previous section, while using a Neumann boundary condition. As in that case, the x_n can be written in terms of oscillators,

$$x_n = \frac{i}{\sqrt{2n}}(a_n - a_n^\dagger). \tag{5.43}$$

We note that there is now no zero mode, in contrast to the Neumann case. Although we now use the same vector notation, in this section it signifies components $n = 1...\infty$.

We split the full string function $x(\sigma)$, satisfying Dirichlet boundary conditions at $\sigma = 0, \pi$ into its left and right halves as in eqn.(5.8). Here $l(\sigma)$ and $r(\sigma)$ obey Dirichlet boundary conditions at both ends, $\sigma = 0, \pi/2$. Mode expansions of the left and right halves of the string may be written

$$l(\sigma) = \sqrt{2}\sum_{n=1} l_{2n} \sin 2n\sigma, \quad r(\sigma) = \sqrt{2}\sum_{n=1} r_{2n} \sin 2n\sigma. \tag{5.44}$$

Relating the full- and half-string modes, we find

$$l_{2m} = x_{2m} + \sum_{n=0} X_{2m,2n+1} x_{2n+1}, \tag{5.45}$$

$$r_{2m} = -x_{2m} + \sum_{n=0} X_{2m,2n+1} x_{2n+1}, \tag{5.46}$$

where

$$\frac{\pi}{4} X_{2m,2n+1} = \frac{(-1)^{n+m} 2m}{(2n+1)^2 - (2m)^2}. \tag{5.47}$$

This can be inverted;

$$x_{2m} = \frac{1}{2}\left(l_{2m} - r_{2m}\right), \tag{5.48}$$

$$x_{2m+1} = \frac{1}{2} \sum_{n=1} X_{2m+1,2n}(l_{2n} + r_{2n}). \tag{5.49}$$

Analogous to the matrix X which we defined in section 5.3.2, we define a matrix of coefficients relating the modes,

$$X \equiv \begin{pmatrix} 0 & X_{2n+1,2m} \\ X_{2m,2n+1} & 0 \end{pmatrix}, = X^T = X^{-1}. \tag{5.50}$$

We have also the set of completeness relations

$$\frac{1}{(2m)^{2p}} \sum_{q=0} X_{2m,2q+1}(2q+1)^{2p} X_{2q+1,2n} = \delta_{mn} \tag{5.51}$$

$$\frac{1}{(2m+1)^{2p}} \sum_{q=1} X_{2m+1,2q}(2q)^{2p} X_{2q,2n+1} = \delta_{mn} \tag{5.52}$$

valid for $p \in \mathbb{Z}$. In section 5.3.2 we found that analogous completeness relations (5.34) and (5.35) held in the case with Neumann boundary condition, but for $p \in \mathbb{N}$.

5.4.1 Projection Operators in the Dirichlet case

We treat the Dirichlet half-string modes as we did the Neumann modes in section 5.3.2. We choose the half-string functional

$$\chi[l] = \exp\left(-\frac{1}{2} l_{2n} M_{nm} l_{2m}\right), \tag{5.53}$$

corresponding to a full-string functional

$$\Psi[x] = \chi[l]\chi[r] = \exp\left(-\frac{1}{2} x_n L_{nm} x_m\right). \tag{5.54}$$

Again using eqn.(5.22), we find the constraints on L,

$$L_{2n+1,2m+1} = X_{2n+1,2k} M_{2k,2l} X_{2l,2m+1}, \tag{5.55}$$

$$L_{2n,2m+1} = L_{2m+1,2n} = 0. \tag{5.56}$$

These conditions are seen to be identical in form to those for the case of Neumann boundary condition if the role of even indices is interchanged with that of odd indices. It follows from this observation that we can construct the following state as a solution to the projection equation, which is analogous to the D-instanton state $|\Xi_{-1}\rangle$.

$$|\Xi^D_{-1}\rangle = \exp\left\{-\frac{1}{2}(a^\dagger|S|a^\dagger)\right\}|\Omega\rangle \tag{5.57}$$

with

$$S = -C\frac{1}{2Z}\left(1 + Z - \sqrt{(1+3Z)(1-Z)}\right) \tag{5.58}$$

$$Z = \frac{C}{3}(C - U - CUC) \tag{5.59}$$

$$U = (2 - EYE^{-1} + E^{-1}YE)(EYE^{-1} + E^{-1}YE)^{-1} \tag{5.60}$$

$$Y = \frac{1}{2}C(1 + i\sqrt{3}X). \tag{5.61}$$

The state $|\Xi^D_{-1}\rangle$ is of a form very similar to the D-instanton state, which we now denote with a superscript N by $|\Xi^N_{-1}\rangle$. The state $|\Xi^D_{-1}\rangle$ is independent of the zero modes of the string function $x(\sigma)$, while $|\Xi^N_{-1}\rangle$ is not, being a localized state. In order to describe a Dp-brane background we need to construct a string functional which is independent of the $(p+1)$ zero modes of the string coordinates in the longitudinal directions. In [41] Gross and Taylor obtained such a functional from the D-instanton functional by shifting L to set all coefficients $L_{2j+1,n}$ to zero in $(p+1)$ of the longitudinal space-time dimensions. In contrast to $|\Xi^N_{-1}\rangle$, the state $|\Xi^D_{-1}\rangle$ is independent of the zero modes by construction. It seems we thus have two ways of describing a given Dp-brane; Representing the D-brane as a solitonic state string field state allows the investigation of its properties using string field theory, whereas formulating string field theory on a D-brane background would focus on the behaviour of string modes in the presence of a fixed D-brane.

As is well-known the T-dual transformation in string theory interchanges the Neumann boundary condition with the Dirichlet boundary condition. Hence, $|\Xi^N_{-1}\rangle$ and $|\Xi^D_{-1}\rangle$ should be related by some T-dual transformation. To understand what this state may describe, we suggest that the state $(|\Xi^N_{-1}\rangle)^{26}$ is a D-instanton state *in the background of a D25-brane.* Recall that we employ the Neumann boundary condition for every direction of the string coordinate when we construct the state $(|\Xi^N_{-1}\rangle)^{26}$. Let us then change the boundary condition to Dirichlet along $(p+1)$ directions. This is much like a T-dual transformation since the background D25-brane turns into a D$(25-p-1)$-brane. Thus, the state $(|\Xi^N_{-1}\rangle)^{25-p}\otimes(|\Xi^D_{-1}\rangle)^{p+1}$ describes a string state in the background of a D$(25-p-1)$-brane. We see that this state is localized in $(25-p)$ directions since it carries $(25-p)$ zero modes only, contained in $(|\Xi^N_{-1}\rangle)^{25-p}$.

5.4.2 Discussion

In string field theory, D-branes may be constructed as classical solutions to the equations of motion; all degrees of freedom are contained within the string field itself. One might call such a solution a 'string-field D-brane'. In introducing their split-string formulation, Gross and Taylor used a mode expansion corresponding to Neumann boundary conditions. One may interpret this as constructing the string field on the background of a D25-brane. We have shown here that the split-string formalism may be used equally well to describe a string field using a mode expansion corresponding to Dirichlet boundary conditions, which we interpret as an explicit Dp-brane background; in contrast to a string-field D-brane this would be a 'background D-brane'.

We have seen that classical D-brane solutions from the usual Vacuum SFT can be translated over and remain solutions to the projection equation. The main difference is that when using the Dirichlet prescription, there is no dependence on the zero-mode.

The paper [126] appeared somewhat after our thoughts on the curent discussion of Dirichlet conditions for string fields. In [126] the authors use boundary condition-changing twist fields to implement Dirichlet conditions and in much the same way as we have described interpreted the resulting state as a D-brane. They did not use an algebraic mode expansion, but instead performed their analyses using surface states, which we describe in what follows.

5.5 Surface States

In the surface state formulation of string field theory, also called the geometrical formulation, a state is represented by a Riemann surface. It is the geometry of this surface which determines the structure of the state. Although not all string-field states are surface states, it is believed that string states which are not expressible as surface states are not physically relevant. In the following, it will become clear that surface states form a sub-algebra of the string field algebra.

A surface state is defined by a combination of a two-dimensional surface and a conformal field theory. The conformal field theory and associated boundary conditions depend on the background and need not be specified to introduce the surface-state formalism; in this sense it is background-independent.

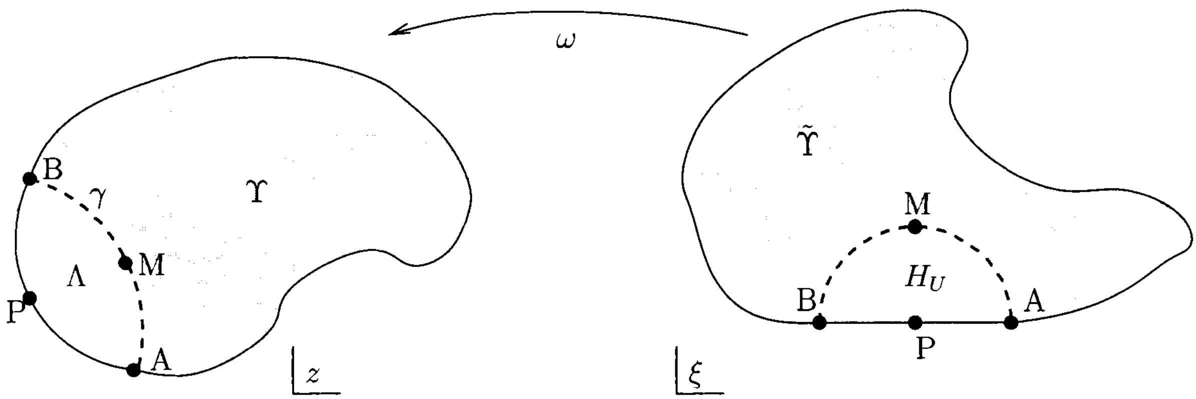

Figure 5.1: defining a surface state

Let the fields in the 2-d CFT be denoted collectively as φ. Referring to fig.5.1, we allow the surface state $|\Upsilon\rangle$ to be defined by the combination of the surface Υ and the unshaded 'local patch' Λ (or alternatively the mapping ω) in the following way. We consider the conformal field theory of φ on the surface Υ with boundary condition φ_0 on the 'string boundary' γ, which stretches from A through M to B. The boundary value $\varphi_0(\sigma)$ depends on the σ coordinate which is taken to parametrise γ such that $\sigma = 0$, $\pi/2$ and π correspond to points A, M and B. The boundary condition on the remainder of $\partial\Upsilon$ is taken to be the standard boundary condition of the conformal field theory; that is, it depends on the background. The analytic map ω takes the canonical upper half-disc H_U in the ξ plane to the local patch Λ in the z coordinate system, and the string boundary γ in the z plane is the image under ω of the upper half-circle $|\xi| = 1$. The quantity

$$\Phi = \int_{\varphi_0} \mathrm{D}\varphi e^{iS_{\mathrm{CFT}}[\varphi]} \tag{5.62}$$

is a functional of the string-boundary function φ_0. We identify $\Phi[\varphi_0]$ as the wavefunction associated with the surface Υ. At the 'puncture' P we insert a local operator ϕ. We define the state

$$|\phi\rangle \equiv \lim_{\xi\to 0} \phi(\xi)|0\rangle. \tag{5.63}$$

This state can also be expressed as a functional, analogous to $\Phi[\varphi_0]$ above; the integral is now over the field φ on the local patch surface H_U,

$$\phi[\varphi_0] = \int_{\varphi_0} \mathrm{D}\varphi e^{iS_{\mathrm{CFT}}[\varphi]}. \tag{5.64}$$

Next, we define the product of states as

$$\langle\Upsilon|\phi\rangle \equiv \langle\omega\circ\phi(0)\rangle_D, \tag{5.65}$$

where the one-point function is now computed on the surface $D = \Upsilon \cup \Lambda$ (the whole surface) in the z system. We have denoted by $\omega\circ\phi(0)$ the operator in the z system corresponding to ϕ in the ξ system at P. In the case that ϕ is a primary operator, this transformation is simple;

$$\omega \circ \phi(z) = \phi(\omega(\xi)) \left(\frac{\mathrm{d}\omega}{\mathrm{d}\xi}\right)^h , \tag{5.66}$$

where h is the conformal dimension of ϕ. States are thus defined in reference to the arbitrary local operator ϕ inserted at the puncture in the coordinate system in which the local patch is the canonical half-disc, and a state $|\Upsilon\rangle$ may be specified by means of $\langle\Upsilon|\phi\rangle$.

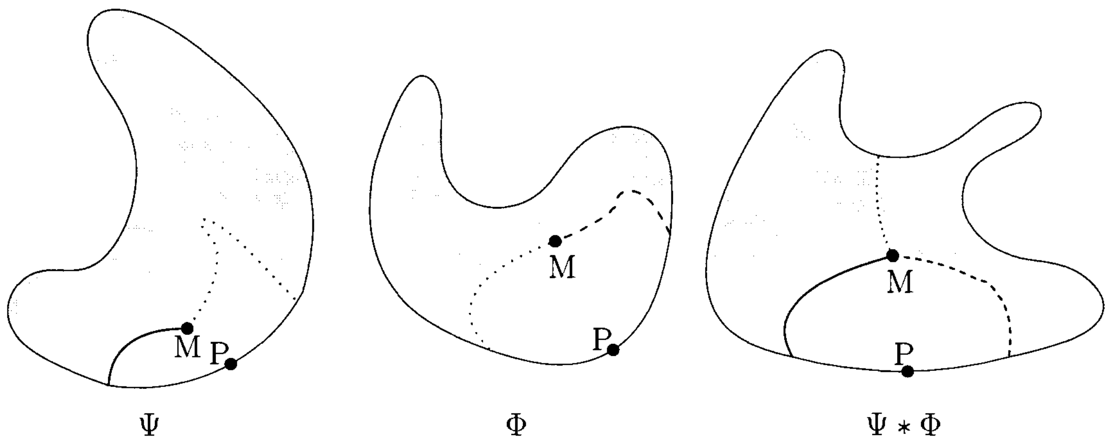

Figure 5.2: star product of surface states

To take the star product of two states, we 'glue' the left-string dependence of one to the right-string dependence of the other, as in eqn.(1.16) or eqn.(5.12). This is shown in fig.5.2. The two states Ψ and Φ are shown with their string boundaries emphasized in heavier lines. The solid line is the left-string boundary of Ψ and the dashed line is the right-string boundary of Φ. The dotted lines are to be glued together and this results in the surface representing $\Psi * \Phi$. Conformal mapping of the surfaces to a new coordinate system may be necessary in order to geometrically glue them together. A new local patch has been attached; in order to compare surface states, they must of course be specified with respect to the same local patch geometry. Ψ, Φ and $\Psi * \Phi$ in the figure could be directly compared by remapping each of them so that the local patch consists of the canonical half-disc.

5.6 Wedge states

Here we discuss the set of 'wedge states', which forms a sub-algebra of the surface-state algebra. The function

$$h(\xi) = \frac{1+i\xi}{1-i\xi} \tag{5.67}$$

maps the upper unit half-disc to a unit half-disc 'pointing' to the right (shaped like a 'D'). This function is such that between the two half-discs the straight and curved sides are interchanged in the mapping. Now, the function

$$\omega_n(\xi) = h^{2/n} = \left(\frac{1+i\xi}{1-i\xi}\right)^{2/n} = \exp\left(\frac{4i}{n}\tan^{-1}\xi\right) \tag{5.68}$$

maps the canonical unit half-disc to a wedge shape with unit radius and opening angle $2\pi/n$ as shown in fig.5.3. Let us consider the state defined by taking this wedge as the

Figure 5.3: mapping for wedge state

local patch and taking the rest of the disc as the state surface. Referring back to the general case, fig.5.1, this corresponds to assigning Λ to the wedge shape in fig.5.3, D to the unit disc, and $\Upsilon = D \backslash \Lambda$. The mapping function ω is ω_n given in eqn.(5.68). We refer to this state as a wedge state, $|\omega_n\rangle$ [127]. We now represent $|\omega_n\rangle$ by a surface with a local patch of uniform geometry by using the mapping

$$\zeta = \frac{1}{2i} \log h = \frac{i}{2} \log \frac{1 - i\xi}{1 + i\xi} \tag{5.69}$$

which maps the wedge state surface to a rectangular region in the ζ plane which we call C_n. As shown in fig.5.4 C_n is the cylinder $\mathfrak{Im}\ \zeta > 0$, $-\pi/4 < \mathfrak{Re}\ \zeta < (2n-1)\pi/4$ where

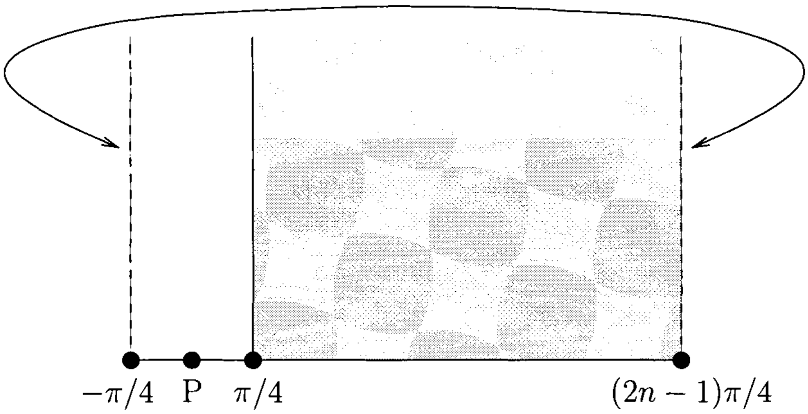

Figure 5.4: the surface C_n

the left and right sides at $\mathfrak{Re}\ \zeta = -\pi/4$ and $\mathfrak{Re}\ \zeta = (2n-1)\pi/4$ are identified. The local patch is the portion of C_n satisfying $\mathfrak{Re}\ \zeta < \pi/4$ and the puncture P is at the origin $\xi = 0$. The left and right string boundaries border the local patch in the left and right; they meet at the midpoint which is at $\zeta = \infty$.

The ξ coordinate system allows us to see clearly and intuitively that the star-product of two wedge states is again a wedge state. Two wedge states $|\omega_n\rangle$ and $|\omega_m\rangle$ are represented in the ξ plane by surfaces C_n and C_m. Taking the star-product of the two involves removing the local patch from each, gluing the remaining surfaces together using the left half of one string boundary and the right half of the other, and attaching a new local patch. This results in a wedge state defined by the surface C_{m+n-1}. We thus have

$$|\omega_n\rangle * |\omega_m\rangle = |\omega_{n+m-1}\rangle \tag{5.70}$$

(up to normalisation). We first notice that the wedge state with $n = 1$ is special; it star-squares to itself. This is in fact the identity state $|I\rangle = |\omega_1\rangle$, as shown on the left of fig.5.5. The figure is meant to illustrate that the local patch takes up the entire unit disc, with the two string boundaries coinciding. We see that in the ω_n coordinate system, the two halves of the string are identified; the identification represents the δ function of eqn.(1.17) or eqn.(5.13). Now, it also turns out that the limiting wedge state

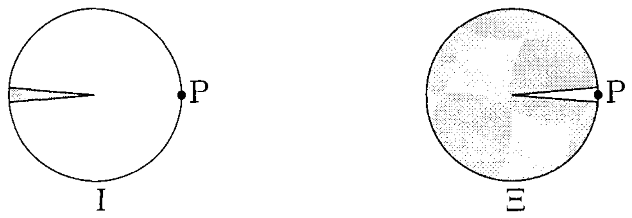

Figure 5.5: identity and sliver wedge states

$n \to \infty$, which we label $|\Xi\rangle = |\omega_\infty\rangle$ exists, and star-squares to itself (notwithstanding $\lim_{n\to\infty} 2n - 1 \neq n$). This state is called the sliver state [128, 129, 130, 131] and is exhibited in the ω_∞ coordinate plane on the right of fig.5.5. The local patch is reduced to an infinitesimal sliver, while the state surface Υ takes up the whole unit disc.

The sliver state may be used to represent a D-brane [132, 131], and in chapter 6 we investigate tachyon fluctuations about this state to derive brane tension. We mention that the sliver state also has a supersymmetric generalisation [133].

5.7 Butterfly

The geometrical formulation of the star product shown in fig.5.2 provides an intuitive way to think about the projection equation $\Psi = \Psi \star \Psi$. We only need imagine a surface which, when joined to itself in the appropriate way, retains the same shape. In this way we can intuitively see how wedge states star-multiply and that they form a closed sub-algebra. The surface in C_n (not the local patch) in fig.5.4 also has this property, at least heuristically; an infinitely large rectangle glued to another infinitely large rectangle produces what might be expected to be a similar infinite rectangle.

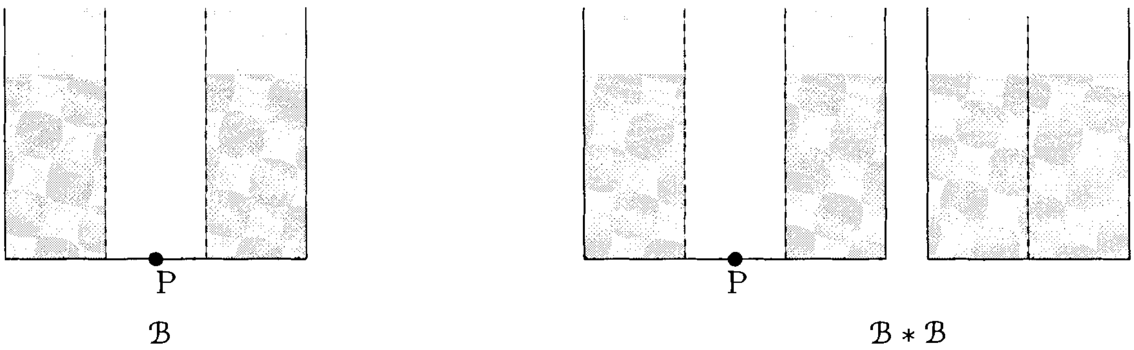

Figure 5.6: surface states $\mathcal{B}$ and $\mathcal{B} * \mathcal{B}$

Along these intuitive lines, we may present another state which obviously star-squares to itself. The butterfly state $\mathcal{B}$, shown on the left in fig.5.6, exhibits this property in a trivial way. Some papers dealing with the butterfly state are [122, 124, 134, 135, 136, 137, 138]. The wavefunction factorises into left and right string functionals. When the star product is taken, two of these 'wings' form a new butterfly, and the other two contribute to normalisation; the second surface in $\mathcal{B} * \mathcal{B}$ has no string boundary or local patch.

The butterfly state is localized in the same way as is the sliver, that is to say it has gaussian dependence on the zero modes. If it is to be interpreted as a brane-type object, possibly a D-brane, it must exhibit tachyon fluctuations and tension similar to the sliver state. This will be the subject of chapter 7.

Chapter 6

Sliver Brane Tension

6.1 Introduction

The problem of tachyon fluctuations and the resulting D25-brane tension in the context of VSFT was first considered by Hata and Kawano [139], and by Rastelli, Sen and Zwiebach [140]. In the former paper the D25-brane tension did not seem to be compatible with the standard result, and in the latter doubt was expressed as to the validity of the linearised equations of motion for the tachyon perturbation, when the BPZ-product with the perturbation state (which is slightly outside the Fock-space) is taken. The analysis in [141] clarified the limiting procedure involved when taking star- and BPZ-products, and showed that the linearised equation of motion is indeed valid in the 'strong' sense, obtaining a result for the D25-brane tension compatible with the assertion that the sliver solution represents a single D25-brane.

The single tachyon-vertex insertion of [139] was thought for some time to be the proper representation of the tachyon state in VSFT. Although it is possible to perform calculations using this state, such as our tension calculation of the present chapter, the situation was later clarified by Okawa [142], who introduced the line-integrated vertex operator prescription which we use in chapter 7 to analyse the butterfly state. The main difference is that surface states defined using the formulation of Okawa, in which the vertex operator providing the field degrees of freedom is integrated along the boundary of the string surface, have proper conformal transformation properties, whereas defining states using the single insertion method of Hata and Kawano and others is not a conformally-invariant state-definition method and thus not strictly correct.

Here we consider tachyon fluctuations about a Dp-brane solution for arbitrary p and in this way evaluate the tension of the D-brane. We make use of boundary condition-changing twist operators, as suggested in [126], in the 'geometrical' conformal field theory approach. We find the correct ratios of tensions between branes of differing dimension. Our results may be compared to those found in the paper [143] which were obtained using the algebraic (oscillator) [144, 26, 27] approach. We also investigate the effect of a B-field and again obtain the expected ratios of tensions.

This chapter is organised as follows. In section 6.2 we discuss the construction of perturbative states around a D-brane solution, in particular focusing our attention on the tachyon field. Next, in section 6.3 we construct this perturbative tachyon state around a Dp-brane configuration and evaluate the resulting tension. Section 6.4 contains an analysis of the effect of a B-field background [145, 146, 147], and we again obtain the standard expression for the tension of a non-commutative D-brane. We conclude with some comments in section 6.5.

6.2 Dp-brane

In Fock-space notation, as used in the geometrical approach of Rastelli, Sen and Zwiebach [43], the action is given by

$$S[\Psi] = -\kappa \left[\frac{1}{2} \langle \Psi | \mathcal{Q} \Psi \rangle + \frac{1}{3} \langle \Psi | \Psi * \Psi \rangle \right] , \tag{6.1}$$

and the equation of motion is written

$$\mathcal{Q} \, |\Psi\rangle = |\Psi * \Psi\rangle \, . \tag{6.2}$$

As discussed in the previous chapter, in VSFT the operator $\mathcal{Q}$ affects only the ghost sector of the theory; the equation of motion thus factorises and the matter part becomes

$$|\Psi_m\rangle = |\Psi_m * \Psi_m\rangle \, . \tag{6.3}$$

It is conjectured that the ghost part of the solution is universal [36].

We have already seen one solution Ξ to the matter equation of motion, the sliver state, and it is believed to correspond to a D25-brane. We showed how this state was defined by

$$\langle \Xi | \phi \rangle = \lim_{n \to \infty} \mathcal{N} \langle f \circ \phi(0) \rangle_{C_n} \tag{6.4}$$

where on the RHS the brackets denote a correlation function in the matter boundary CFT on the semi-infinite cylinder C_n, shown in fig.5.4. ϕ is an operator representing an arbitrary state in the Fock space and $|\phi\rangle$ is the corresponding state obtained by inserting ϕ at $z = 0$ in the 'local patch' $-\pi/4 < \mathfrak{Re}\ z < \pi/4$.

In order to extend this to the case of a Dp-brane, we need to introduce boundary condition-changing twist operators [126, 144]. These are σ^+ and σ^- and are inserted at $z = \pm\pi/4 \pm \epsilon$. This effectively imposes Neumann boundary conditions on $-\pi/4 < z < \pi/4$ and Dirichlet conditions on the rest of the boundary, $\pi/4 < z < (2n-1)\pi/4$. The correlator in eqn.(6.4) is taken over the 26 independent conformal fields X^μ. In this case, we define

$$\langle \mathrm{D}p | \phi \rangle = \lim_{n\to\infty} \mathcal{N} \langle f \circ \phi(0) \rangle_{C_n}^{\sigma\perp}, \tag{6.5}$$

where the superscript indicates the presence of the σ operators in the $25-p$ Dirichlet directions, while the $p+1$ Neumann directions are unchanged. The state $|\mathrm{D}p\rangle$ so defined represents a Dp-brane [126] and it satisfies the equation of motion (6.2), provided that it is renormalised as follows. When we take the star-product of the sliver with itself, we will obtain a short-distance singularity from the proximity of σ^+ from one sliver, and σ^- from the other. As noted in [126], the leading term of the operator expansion will have no operator content, and so this singularity will only contribute $(1/2\epsilon)^h$ to the product, where h depends on the conformal dimensions of the σ-operators. This divergent factor may simply be absorbed into the definition of the sliver state.

6.3 VSFT Perturbations and Dp-brane Tension

We wish to start with a background string-field solution $\Phi_0 = \Phi_{\mathrm{ghost}} \otimes \Phi$ and consider perturbations parametrised by fields. We follow here the procedure used in [141], generalised to a the case of a Dp-brane. As in [140], we use the perturbative expansion

$$\big|\Psi\big\rangle = \big|\Phi_g\big\rangle \otimes \left\{ \big|\Phi\big\rangle + \big|T\big\rangle + \cdots \right\}, \tag{6.6}$$

where $|T\rangle$ is a tachyon excitation and terms corresponding to vector and higher excitations follow. The tachyon perturbation is

$$\big|T\big\rangle = \int \mathrm{d}k n^{-k_{\parallel}^2} T(k) \big|\chi(k)\big\rangle, \tag{6.7}$$

with $T(k)$ the momentum-space tachyon field. We use $k_{\parallel}$ to refer to directions longitudinal to the D-brane, and $k_{\perp}$ for transverse directions.

We may insert this expansion for $|\Psi\rangle$ into the action (6.1) to obtain

$$S[T] = S[\Phi_g \otimes \Phi] - \big\langle \Phi_g \big| \mathcal{Q} \Phi_g \big\rangle \left[\frac{1}{2} \big\langle T \big| T \big\rangle - \big\langle \Phi \big| T * T \big\rangle + \frac{1}{3} \big\langle T \big| T * T \big\rangle \right] \tag{6.8}$$

whence the linearised equation of motion for $|T\rangle$ can be obtained;

$$\big|\chi(k)\big\rangle = \big|\chi(k) * \Phi\big\rangle + \big|\Phi * \chi(k)\big\rangle. \tag{6.9}$$

In this section we take as the background solution the Dp-brane state $|\Phi\rangle = |\mathrm{D}p\rangle$. It was shown in [141] that the linearised equation of motion for a perturbation $\chi(k)$ about the sliver state holds even when the BPZ-product with another solution $\chi(k')$ is taken, that is

$$\big\langle \chi(k) \big| \chi(k') \big\rangle = \big\langle \chi(k) * \Xi + \Xi * \chi(k) \big| \chi(k') \big\rangle. \tag{6.10}$$

Equation (6.9) has been referred to as the 'weak' equation of motion, with eqn.(6.10) being a stronger version, since Ξ and thus χ are not quite Fock space states.

The linearised equation of motion (6.9) for a tachyon perturbation $\chi(k)$ about the sliver state Ξ was solved in [139] and this solution was expressed in the CFT language in [140]. In the present case of a Dp-brane, the solution may be written

$$\langle\chi(k)|\phi\rangle = \lim_{n\to\infty} \mathcal{N} n^{2k_\parallel^2}\langle f\circ\phi(0)e^{ik\cdot X}(n\pi/4)\rangle_{C_n}^{\sigma\perp}, \tag{6.11}$$

so that the momentum degrees of freedom are carried by a tachyon vertex operator inserted diametrically opposite the ϕ-insertion puncture. The factor $n^{2k_\parallel^2}$ is inserted to compensate for a factor $n^{-2k_\parallel^2}$ which will come from the correlator.

Let us now show that the state in eqn.(6.11) satisfies the equation of motion (6.10). Computing first the LHS, we find

$$\langle\chi(k)|\chi(k')\rangle = \mathcal{N}^2 \lim_{n,m\to\infty} n^{2k_\parallel^2} m^{2k'^2_\parallel}\left\langle e^{ik\cdot X}(0)e^{ik'\cdot X}\left((n+m-2)\frac{\pi}{4}\right)\right\rangle_{C_{m+n-2}}^{\sigma\perp}. \tag{6.12}$$

This is calculated in the appendix (section 6.6) and is given by eqn.(6.60),

$$\langle\chi(k)|\chi(k')\rangle = \mathcal{K} \lim_{n\to\infty} n^{2k_\parallel^2} 2^{2k_\parallel^2}(2\pi)^{26}\delta(k_\parallel + k'_\parallel)e^{ix_0\cdot(k_\perp+k'_\perp)}. \tag{6.13}$$

Shifting our attention to the RHS of eqn.(6.10), we may express either of the two terms as

$$\langle\Xi*\chi(k)|\chi(k')\rangle = \mathcal{N}^2 \lim_{n_1,n_2,n_3\to\infty} n_2^{2k_\parallel^2} n_3^{2k'^2_\parallel}\left\langle e^{ik\cdot X}(0)e^{ik'\cdot X}\left((n_2+n_3-2)\frac{\pi}{4}\right)\right\rangle_{C_{n_1+n_2+n_3-3}}^{\sigma\perp}. \tag{6.14}$$

Evaluating this correlator in the same way we find

$$\langle\Xi*\chi(k)|\chi(k')\rangle = \mathcal{K} \lim_{n\to\infty} n^{2k_\parallel^2} 2^{k_\parallel^2}(2\pi)^{26}\delta(k_\parallel + k'_\parallel)e^{ix_0\cdot(k_\perp+k'_\perp)}. \tag{6.15}$$

Substituting eqn.(6.13) and eqn.(6.15) into the equation of motion (6.10), it reduces to

$$k_\parallel^2 = 1. \tag{6.16}$$

We see that the tachyon living on the Dp-brane satisfies the strong equation of motion, provided that it is on-shell. This is to be compared with the case of a D25-brane [141]. Off-shell we have

$$2^{1-k_{\parallel}^2}\langle\chi(k)|\chi(k')\rangle = \langle\chi(k) * \Xi + \Xi * \chi(k)|\chi(k')\rangle. \tag{6.17}$$

To calculate the Dp-brane tension, we first examine the quadratic term $S^{(2)}$ of the action (6.8) to fix the normalisation of the tachyon field $T(k)$. Using eqn.(6.17) we may write

$$S^{(2)} = -\frac{1}{2}\langle\Psi_g|\mathcal{Q}\Psi_g\rangle \int \mathrm{d}k\mathrm{d}k' T(k)T(k') \left(1 - 2^{1-k_{\parallel}^2}\right) \langle\chi(k)|\chi(k')\rangle. \tag{6.18}$$

After substituting the product from eqn.(6.13), we have

$$\begin{aligned} S^{(2)} &= -\frac{1}{2}\langle\Psi_g|\mathcal{Q}\Psi_g\rangle\mathcal{K} \int \mathrm{d}^{p+1}k_{\parallel}\mathrm{d}^{p+1}k'_{\parallel} \left(1 - 2^{1-k_{\parallel}^2}\right) 2^{k_{\parallel}^2}(2\pi)^{26}\delta(k_{\parallel} + k'_{\parallel}) \\ &\quad \left\{\int \mathrm{d}^{25-p}k_{\perp}\mathrm{d}^{25-p}k'_{\perp}e^{ix_0\cdot(k_{\perp}+k'_{\perp})}T(k)T(k')\right\} \end{aligned} \tag{6.19}$$

Taking $T(k)$ near on-shell, $k_{\parallel}^2 \approx 1$, we may write the $k_{\parallel}$-dependent factor as $(k_{\parallel}^2-1)4\log 2$. We see that a re-definition of the tachyon field

$$T_{\mathrm{D}p}(k_{\parallel}) \equiv \sqrt{\mathcal{K}(2\pi)^{25-p}\langle\Psi_g|\mathcal{Q}\Psi_g\rangle 4\log 2} \int \mathrm{d}^{25-p}k_{\perp}e^{ix_0\cdot k_{\perp}}T(k) \tag{6.20}$$

would cast $S^{(2)}$ into the canonical form

$$S^{(2)} = -\frac{1}{2}(2\pi)^{p+1} \int \mathrm{d}^{p+1}k_{\parallel}({k_{\parallel}}^2 - 1)T_{\mathrm{D}p}(k_{\parallel})T_{\mathrm{D}p}(-k_{\parallel}). \tag{6.21}$$

The cubic term in the action involves

$$\begin{aligned} &\langle\chi(k_1)|\chi(k_2) * \chi(k_3)\rangle \\ &= \left\langle e^{ik_1\cdot X}(0)e^{ik_2\cdot X}\big((n_1+n_2-2)\frac{\pi}{4}\big)e^{ik_3\cdot X}\big(-(n_1+n_3-2)\frac{\pi}{4}\big)\right\rangle^{\sigma\perp}_{C_{n_1+n_2+n_3-3}}. \end{aligned} \tag{6.22}$$

Once again mapping to the unit disc, we may use Wick's theorem and the tachyon correlator from eqn.(6.58) to find

$$\langle \chi(k_1) | \chi(k_2) * \chi(k_3) \rangle = \lim_{n\to\infty} 2n^{\sum k_{\|}^2} (2\pi)^{26} e^{ix_0\cdot(\sum k_\perp)} \delta\left(\sum k_{\|}\right) \tag{6.23}$$

Substituting this result into the action, the cubic term $S^{(3)}$ is

$$\begin{aligned} S^{(3)} &= -\frac{2}{3}\langle \Psi_g | \Omega \Psi_g \rangle \mathcal{K} \int \mathrm{d}k_{1,2,3\|} (2\pi)^{26} \delta\left(\sum k_{\|}\right) \\ & \left\{ \int \mathrm{d}k_{1,2,3\perp} T(k_1) T(k_2) T(k_3) e^{ix_0\cdot(\sum k_\perp)} \right\}, \end{aligned} \tag{6.24}$$

which we can write in terms of the D-brane tachyon field (6.20) as

$$\begin{aligned} S^{(3)} &= -\frac{1}{3} \frac{2(2\pi)^{p+1}}{(4\log 2)^{3/2} \sqrt{\mathcal{K}(2\pi)^{p-25} \langle \Psi_g | \Omega \Psi_g \rangle}} \\ & \int \mathrm{d}k_{1,2,3\|} T_{\mathrm{D}p}(k_{1\|}) T_{\mathrm{D}p}(k_{2\|}) T_{\mathrm{D}p}(k_{3\|}) \delta\left(\sum k_{\|}\right). \end{aligned} \tag{6.25}$$

The three-tachyon coupling can be read off;

$$g_T = \frac{2}{(4\log 2)^{3/2} \sqrt{\mathcal{K}(2\pi)^{p-25} \langle \Psi_g | \Omega \Psi_g \rangle}}. \tag{6.26}$$

As explained by Polchinski in [148], the Dp-brane tension is inversely proportional to the closed-string coupling. The open string coupling is identified with the tachyon coupling in the present case, and following [149], we have

$$\mathcal{T} = \frac{1}{2\pi^2 g_T^2}, \tag{6.27}$$

so that using eqn.(6.26) we can recover the standard formula for the ratio of D-brane tensions,

$$\frac{\mathcal{T}_{p+1}}{\mathcal{T}_p} = \frac{1}{2\pi\sqrt{\alpha'}}, \tag{6.28}$$

where we have restored α'.

6.4 Background B-field

Here we investigate the effect of a constant background B-field, and compute the tension as in the previous section. The worldsheet CFT is given by the action [150]

$$S = \frac{1}{2} \int \mathrm{d}^2 z \big(g_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu h^{\alpha\beta} - B_{ij} \partial_\alpha X^i \partial_\beta X^j \epsilon^{\alpha\beta} \big), \tag{6.29}$$

where μ and ν run from 0 to 25, while for simplicity i and j will take only the values 24 and 25. That is, the B-field has non-zero components only in these two directions. This action leads us to the usual Neumann boundary conditions for the first 24 directions, while for the last two, we have

$$\big(\delta_{ij} \partial_n + B_{ij} \partial_t \big) X^j(z) \Big|_{\partial\Sigma} = 0. \tag{6.30}$$

One can see from this expression that the B-field effectively interpolates between Neumann and Dirichlet boundary conditions. In the following, we concentrate on these two directions and often omit explicit indices. The Green function on the unit disc is given by

$$\big\langle X(z) X(w) \big\rangle_{\text{Disc}} = -\log|z-w| - \frac{1}{2} \left(\frac{1+B}{1-B} \right) \log(z - \frac{1}{\bar{w}}) - \frac{1}{2} \left(\frac{1-B}{1+B} \right) \log(\bar{z} - \frac{1}{w}). \tag{6.31}$$

This can be written [150] in terms of the open-string metric G and the non-commutativity parameter θ. These are related to the B-field by

$$\theta = -\frac{1}{1+B} B \frac{1}{1-B} \qquad \text{and} \qquad G = \frac{1}{1+B} 1 \frac{1}{1-B}. \tag{6.32}$$

On the boundary, $|z| = 1$, the correlator (6.31) becomes

$$\big\langle X^i(e^{i\phi}) X^j(e^{i\phi'}) \big\rangle_{\partial \mathrm{D}} = -G^{ij} \log \left(2 \sin \frac{\phi - \phi'}{2} \right)^2 + \frac{i}{2} \theta^{ij} \epsilon(\phi - \phi'). \tag{6.33}$$

6.4.1 (B,g)-Parametrisation

In section 6.3, to define the D-brane state $|\mathrm{D}p\rangle$ in eqn.(6.5) and the tachyon perturbation $|\chi(k)\rangle$ in eqn.(6.11), we inserted operators $\sigma^{\pm}$ to change the boundary condition to Dirichlet on part of the cylinder C_n. We can accommodate the new boundary condition (6.30) by instead inserting new operators $\beta^{\pm}$ at the same positions. Thus on the cylinder C_n, we have effectively a Neumann boundary condition on the region $-\pi/4 < z < \pi/4$, and eqn.(6.30) will apply to the rest of the boundary. We will again absorb the various singular factors due to coincident β^+ and β^- operators into the states.

Analogous to eqn.(6.5) we define the sliver state in the presence of a B-field as

$$\langle \Xi_B | \phi \rangle = \lim_{n\to\infty} \mathcal{N} \langle f \circ \phi(0) \rangle^{\beta}_{C_n}, \tag{6.34}$$

where now the superscript indicates that we have inserted the operators $\beta^{\pm}$ in the 24- and 25-directions.

We now define a tachyon perturbation in the presence of the B-field as

$$|T_B\rangle = \int \mathrm{d}k n^{-k_\sharp^2 - k_\flat \frac{1}{1-B^2} k_\flat} T_B(k) |\chi_B(k)\rangle, \tag{6.35}$$

where $k_\sharp$ refers to the first 24 directions, and $k_\flat$ contains the last two components of k. $T_B(k)$ is the tachyon field, and we have used the notation

$$k_\flat \frac{1}{1-B^2} k_\flat \equiv k_\flat^\mu \left(\frac{1}{1-B^2} \right)_{\mu\nu} k_\flat^\nu. \tag{6.36}$$

As in the previous section, we can write down the solution to the linearised equation of motion for the perturbation $\chi_B(k)$;

$$\langle \chi_B(k) | \phi \rangle = \lim_{n\to\infty} \mathcal{N} n^{2k_\sharp + 2k_\flat \frac{1}{1-B^2} k_\flat} \langle f \circ \phi(0) e^{ik\cdot X}(n\pi/4) \rangle^{\beta\perp}_{C_n}, \tag{6.37}$$

where this time we have inserted the operators $\beta^{\pm}$ to impose the B-field boundary condition. Exactly analogously to the case of the Dp-brane, this state can be shown to satisfy the following,

$$\begin{aligned}\langle \chi_B(k) | \chi_B(k') \rangle = \mathcal{K}\sqrt{\det(1+B)} \lim_{n\to\infty} n^{2k_\sharp^2 + 2k_\flat \frac{1}{1-B^2} k_\flat} 2^{2k_\sharp^2 + 2k_\flat \frac{1}{1-B^2} k_\flat} \\ (2\pi)^{26} 4\delta(k_\sharp + k'_\sharp) \det \frac{1}{1-B^2} \delta\left((k_\flat + k'_\flat) \det \frac{1}{1-B^2} \right),\end{aligned} \tag{6.38}$$

$$\begin{aligned}\langle \Xi_B * \chi_B(k) | \chi_B(k') \rangle = \mathcal{K}\sqrt{\det(1+B)} \lim_{n\to\infty} n^{2k_\sharp^2 + 2k_\flat \frac{1}{1-B^2} k_\flat} 2^{k_\sharp^2 + k_\flat \frac{1}{1-B^2} k_\flat} \\ (2\pi)^{26} 4\delta(k_\sharp + k'_\sharp) \det \frac{1}{1-B^2} \delta\left((k_\flat + k'_\flat) \det \frac{1}{1-B^2} \right),\end{aligned} \tag{6.39}$$

so that off-shell

$$\langle \chi_B(k) | \chi_B(k') \rangle = 2^{k_\sharp^2 + k_\flat \frac{1}{1-B^2} k_\flat - 1} \langle \chi_B(k) * \Xi_B + \Xi_B * \chi_B(k) | \chi_B(k') \rangle. \tag{6.40}$$

We see that the on-shell condition is now

$$k_\sharp^2 + k_\flat \frac{1}{1-B^2} k_\flat = 1, \tag{6.41}$$

to be compared with eqn.(6.16). In the limit of a large B-field, $1/(1-B^2) \to 0$, and we recover the case of a D23-brane with on-shell tachyon condition $k_\sharp^2 = k_{\|(23)}^2 = 1$. In the $B \to 0$ limit, $1/(1-B^2) \to 1$ and a D25-brane obtains, with $k_\sharp^2 + k_\flat^2 = k_{\|(25)}^2 = 1$. Substituting eqn.(6.38) into the quadratic part of the action as before, and considering the large-B limit we are led to define a 24-dimensional tachyon field by

$$\widetilde{T_B}(k_\sharp) \equiv \sqrt{\mathcal{K}(2\pi)^2 \langle \Psi_g | \mathcal{Q}\Psi_g \rangle 4 \log 2 \sqrt{\det(1+B)}} \int \mathrm{d}^{25-p} k_\flat T_B(k). \tag{6.42}$$

The tension of the resulting Dp-brane (now taking k_b to represent $25-p$ dimensions rather than two) is thus given by

$$\frac{\mathcal{T}_p^B}{\mathcal{T}_{25}} = \sqrt{\det(1+B)}(2\pi)^{(25-p)}\alpha'^{(25-p)/2}. \tag{6.43}$$

6.4.2 (G,θ)-Parametrisation

Defining the tachyon, in the presence of the open-string metric G, we write

$$\big|T_{NC}\big\rangle = \int \mathrm{d}k n^{k^2} T_{NC}(k)\big|\chi_{NC}(k)\big\rangle, \tag{6.44}$$

where since G is the effective metric, it is understood that $k^2 \equiv k^\mu G_{\mu\nu} k^\nu$. Looking at eqn.(6.32), we see that this normalisation is the same as that used in eqn.(6.35).

The perturbation is now of the form

$$\big\langle \chi_{NC}(k)\big|\phi\big\rangle = \lim_{n\to\infty} \mathcal{N} n^{k^2} \big\langle f\circ\phi(0)e^{ik\cdot X}(n\pi/4)\big\rangle_{C_n}^{\beta\perp}, \tag{6.45}$$

and the correlators are found to be

$$\big\langle \chi_{NC}(k)\big|\chi_{NC}(k')\big\rangle = \mathcal{K}\sqrt{\det G}\lim_{n\to\infty} n^{k^2}2^{k^2}(2\pi)^{26}4\delta(k+k') \tag{6.46}$$

$$\big\langle \Xi_{NC} * \chi_{NC}(k)\big|\chi_{NC}(k')\big\rangle = \mathcal{K}\sqrt{\det G}\lim_{n\to\infty} n^{2k^2}2^{k^2}(2\pi)^{26}4\delta(k+k'), \tag{6.47}$$

with off-shell equation of motion

$$\big\langle \chi_{NC}(k)\big|\chi_{NC}(k')\big\rangle = 2^{k^2-1}\big\langle \chi_{NC}(k) * \Xi_{NC} + \chi_{NC}(k) * \Xi_{NC}\big|\chi_{NC}(k')\big\rangle. \tag{6.48}$$

We now have a 'non-commutative' on-shell condition

$$k^\mu G_{\mu\nu} k^\nu = 1, \tag{6.49}$$

where we have written G explicitly. We substitute eqn.(6.46) into the quadratic part of the action and re-define the tachyon field T_{NC};

$$\widetilde{T_{NC}} \equiv \sqrt{\mathcal{K}\langle\Psi_g|\mathcal{Q}\Psi_g\rangle\sqrt{\det G}4\log 2}T_{NC}. \tag{6.50}$$

With this, the tachyon action becomes

$$\begin{aligned} S &= S^{(2)} + S^{(3)} \\ &= -\frac{1}{2}(2\pi)^{26}\int \mathrm{d}k(k^2-1)\widetilde{T_{NC}}(k)\widetilde{T_{NC}}(-k) \\ &\quad -\frac{1}{3}\frac{2(2\pi)^{26}}{(4\log 2)^{3/2}\sqrt{\mathcal{K}\langle\Psi_g|\mathcal{Q}\Psi_g\rangle\sqrt{\det G}}} \\ &\qquad \int \mathrm{d}k_{1,2,3}\widetilde{T_{NC}}(k_1) * \widetilde{T_{NC}}(k_2) * \widetilde{T_{NC}}(k_3)\delta\left(\sum k\right) \end{aligned} \tag{6.51}$$

where $*$ represents the Moyal product, defined in position-space by

$$(f * g)(x) \equiv e^{\frac{i}{2}\theta^{jk}\frac{\partial}{\partial\xi^j}\frac{\partial}{\partial\zeta^k}} f(x+\xi)g(x+\zeta)\big|_{\xi=0,\zeta=0}. \tag{6.52}$$

We identify the cubic tachyon coupling to find the tension of this non-commutative D25-brane

$$\mathcal{T}_{25}^{NC} = \sqrt{\det G}\mathcal{T}_{25}. \tag{6.53}$$

Finally, we identify the ratio

$$\frac{\mathcal{T}_{23}^{B}}{\mathcal{T}_{25}^{NC}} = (2\pi)^2\alpha'\sqrt{\frac{\det(1+B)}{\det G}} = \frac{(2\pi)^2\alpha'}{(\det G)^{1/4}}, \tag{6.54}$$

as demonstrated in the papers [151] and [143].

6.5 Discussion

In [126] it was proposed that in Vacuum String Field Theory a D-brane of arbitrary dimension may be represented by inserting boundary condition-changing twist operators into the CFT description of the sliver state in the directions transverse to the D-brane. We have calculated the ratio of tensions of D-branes using this approach and obtain eqn.(6.28), in agreement with standard string theory. Similar calculations in VSFT have been done in [151] and [143], using the algebraic oscillator approach.

Additionally, in section 6.4 turning on a constant B-field in some directions can be represented in similar fashion, by the insertion of operators $\beta^{\pm}$ which suitably modify the boundary condition. In this way, we have shown that a non-commutative D-brane solution can also be constructed from the sliver state, for generic constant B-field.

Of course, for the case of vanishing B-field, we recover the D25-brane solution, while in the large-B limit, the boundary conditions become Dirichlet, and we find agreement with our tension calculations for the lower-dimensional D-brane in section 6.3. Although for finite B there is no clear way to re-define the tachyon field, in the D25-brane case we calculated the tension of a non-commutative D25-brane.

There was some discussion in [139], [140] and [141] as to whether the sliver state represents a single or multiple D-brane state. Although the quantities calculated here do not shed light on this question since they are ratios of tensions, they do provide further evidence that we are in fact dealing with a D-brane state, and that the proposals of [126] describe a method of representing D-branes in VSFT. As we mentioned at the beginning of this chapter, it was later realised [142] that a more correct, meaning conformally invariant, way of representing field-parametrised fluctuations about a string-field solution is to integrate the corresponding vertex operator around the boundary of the string state. We make use of this method in the next chapter, for the case of the butterfly state.

6.6 Appendix

Here we show the method of calculation of the correlators used in the main text; specifically, we make an example of eqn.(6.13). The other correlators used in the main text may be calculated in a similar way, and the details of the limiting procedure may be found in [141].

Starting with eqn.(6.12), we map C_{m+n-2} to the unit disc using $z \longrightarrow e^{\frac{4iz}{m+n-2}}$. We note that since the zero mode of the expansion for $X(z)$ contributes to the conformal dimension of the tachyon vertex operator, and this is absent in the Dirichlet case, the conformal dimension of $e^{ik\cdot X}$ (which is of course normal-ordered) is $k_{\|}^2/2$. On the disc, then,

$$\begin{aligned} \langle \chi(k) | \chi(k') \rangle = \mathcal{N}^2 \lim_{n,m\to\infty} n^{2k_{\|}^2} m^{k'^2_{\|}} \left(\frac{4i}{m+n-2} \right)^{k_{\|}^2} \left(\frac{-4i}{m+n-2} \right)^{k'^2_{\|}} \\ \times \langle e^{ik\cdot X}(1) e^{ik'\cdot X}(-1) \rangle^{\sigma\perp}_{\text{Disc}}. \end{aligned} \tag{6.55}$$

The zero mode we mentioned above also produces a δ-function in the correlator, [3] giving

$$\langle e^{ik\cdot X}(z) e^{ik'\cdot X}(w) \rangle^{\sigma\perp}_{\text{D}} = e^{ik\cdot k'\langle X(z)X(w)\rangle} \delta(k_{\|} + k'_{\|}) e^{ix_0\cdot(k_\perp + k'_\perp)} \tag{6.56}$$

for the tachyon propagator. The X-propagator is given by

$$\left\langle X(z)X(w)\right\rangle_D = -\log|z-w| \pm \log\left|z-\frac{1}{\bar{w}}\right| \tag{6.57}$$

where for Dirichlet and Neumann directions the + and − sign is used, respectively. The tachyon correlator on the boundary is thus

$$\left\langle e^{ik\cdot X}(e^{i\theta})e^{ik'\cdot X}(e^{i\theta'})\right\rangle^{\sigma\perp}_{\text{Disc}} = \frac{1}{(2\epsilon)^{2h}}\left(2\sin\frac{\theta-\theta'}{2}\right)^{2k_{\|}\cdot k'_{\|}}\delta(k_{\|}+k'_{\|})e^{ix_0\cdot(k_\perp+k'_\perp)} \tag{6.58}$$

where x_0 is the Dirichlet boundary condition. Substituting this into eqn.(6.55) we have (We absorb factors of $\frac{1}{(2\epsilon)^{2h}}$ into the state $\chi(k)$.)

$$\begin{aligned}\left\langle\chi(k)\middle|\chi(k')\right\rangle &= \mathcal{N}^2\lim_{n,m\to\infty} n^{2k_{\|}^2}m^{2k'^2_{\|}}2^{-2k_{\|}^2}\left(\frac{4i}{m+n-2}\right)^{k_{\|}^2}\left(\frac{-4i}{m+n-2}\right)^{k'^2_{\|}}\\ &\quad \delta(k_{\|}+k'_{\|})e^{ix_0\cdot(k_\perp+k'_\perp)}.\end{aligned} \tag{6.59}$$

Taking the $m\to\infty$ limit, we end up with the result

$$\left\langle\chi(k)\middle|\chi(k')\right\rangle = \mathcal{K}\lim_{n\to\infty} n^{2k_{\|}^2}2^{2k_{\|}^2}(2\pi)^{26}\delta(k_{\|}+k'_{\|})e^{ix_0\cdot(k_\perp+k'_\perp)}. \tag{6.60}$$

Chapter 7

Butterfly Tachyons

7.1 Introduction

There are several known solutions [40, 42, 152, 126] to the equations of motion of Vacuum String Field Theory (VSFT) [43]. These include the two which we introduced in chapter 5: the sliver state [40, 42, 37] and the butterfly state [152]. The sliver state was conjectured to represent a D25-brane, and subsequent calculations of its tension based on this assumption yielded the correct brane tension. The sliver state construction was used also to build solutions corresponding to Dp-branes of arbitrary dimension [126], and ratios of tensions were found [143, 7] which were in agreement with the known results from string theory. These calculations were based on a field expansion of fluctuations about the classical solution, using a tachyon field to find the brane tension. One requirement for the consistency of the interpretation of the sliver brane as a D-brane is that the equation of motion for the tachyon must be a consequence of the string field equation of motion; an on-shell string field must correspond to an on-shell tachyon. This means that the quadratic term in the resulting tachyon action must vanish, as is the case for the sliver state.[142] While it has already been assumed in the literature that the butterfly can be interpreted as a D-brane, this may have been premature, as the above properties had not been verified. Were the tachyon field not to satisfy these requirements, it would mean that the butterfly state, although known to be a brane (*i.e.* localized) solution and a rank-one projector, could not be viewed as a D-brane. We show in the present chapter that the butterfly does indeed support a tachyon field with vanishing on-shell quadratic term.

We now turn our attention to the brane tension. The tension of the brane may be obtained from the cubic term in the tachyon action [143, 142, 7], and this has been completed for the case of the sliver by Okawa [142]. In that calculation, the evaluation of the cubic terms turned out to be very tedious and lengthy, and in the present case of the butterfly we find that it is much more so. Thus rather than attempting a calculation of the entire cubic term, we content ourselves with motivating the procedure and conjecture how the result should obtain. In this way, we will see how the correct brane tension should arise.

This chapter is structured as follows. In section 7.2 we review the geometrical construction of the butterfly state as a surface state [43], along with the regularisation required for any concrete calculations.[152] We then in section 7.3 turn to the expansion of deformations of a VSFT solution [142]. In section 7.4 we investigate the quadratic term in the tachyon action. This surface-state calculation involves the construction of conformal mappings in order to perform star multiplication. In section 7.5 we begin the calculation of the ratios of tensions of different butterflies, suggest how this could be completed, and comment on the preliminary results. We conclude in section 7.6 with a discussion of surface states and the role of regularisation and conformal invariance with respect to deformations of string fields and definition of fields.

7.2 The Butterfly State

We use the geometrical, surface representation of string fields which we introduced in chapter 5; thus we specify states using a BPZ product with an arbitrary state $|\phi\rangle$.

The butterfly state $|\mathfrak{B}\rangle$ is a factorisable state, so that it may be decomposed into the product of a left-string functional and a right-string functional. The surface Σ, defined by $-\pi/2 < \mathfrak{Re}\ z < \pi/2$ and $\mathfrak{Im}\ z > 0$, used to define the butterfly is shown in fig.7.1. The unshaded region is the local patch, $-\pi/4 < \mathfrak{Re}\ z < \pi/4$, $\mathfrak{Im}\ z > 0$. The dashed lines

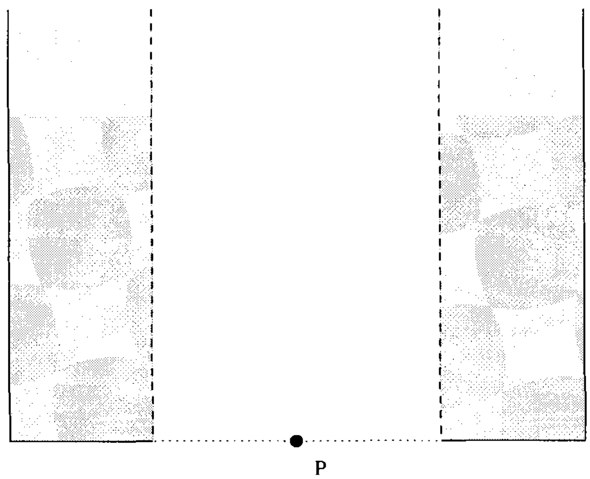

Figure 7.1: the butterfly, defined on the surface Σ

which border the local patch are the left- and right-string boundaries, and the solid line is the boundary of the surface on which we impose the standard open string boundary condition. In the centre of the local patch is the puncture P where we insert the operator ϕ, transformed to this coordinate system from the canonical half-disc via the mapping f. Thus the BPZ product of the butterfly with the arbitrary state represented by the operator ϕ is

$$\langle \mathcal{B} | \phi \rangle = \langle f \circ \phi(P) \rangle_\Sigma, \tag{7.1}$$

where the combination of the mapping f and the surface Σ really only need be defined up to conformal equivalence. This is not strictly true of the states we will define in subsequent sections, due to regularisation of operator short-distance singularities.

We will also have need of the regularised butterfly $|\mathcal{B}_h\rangle$, and we borrow the formulation from [152]. The surface Σ_h, shown in fig.7.2, is obtained from Σ by identifying the

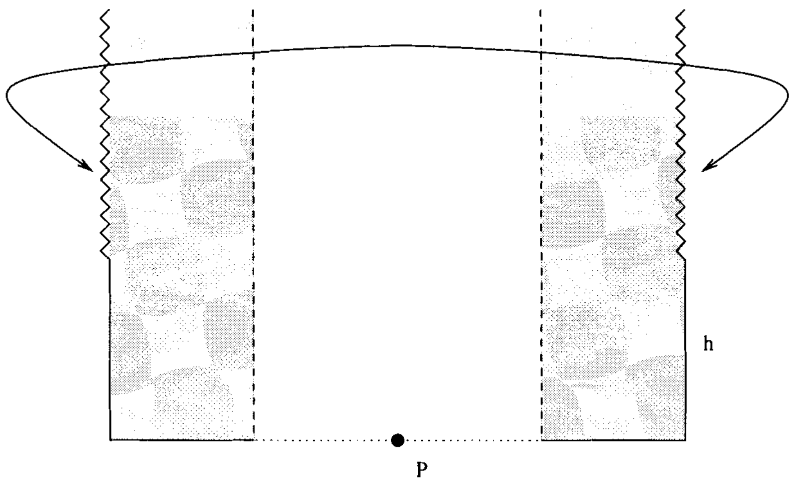

Figure 7.2: the regularised butterfly, defined on Σ_h

left and right edges, above some height h. This is the regularisation parameter, and the limit $h \longrightarrow \infty$ will be taken to obtain the surface Σ. The state is now defined by

$$\langle \mathcal{B}_h | \phi \rangle = \langle f_h \circ \phi(P) \rangle_{\Sigma_h}. \tag{7.2}$$

The regularised state $|\mathcal{B}_h\rangle$ does not satisfy the string-field equation of motion for finite h.

7.3 Tachyon Fluctuations

In [142] an elegant proposition was made regarding both the parametrisation of string-field fluctuations by fields and the construction of states representing the coefficients of these fields. We here present this briefly, referring the reader to that paper for details.

The string-field action is given by

$$S = -\frac{1}{2}\langle \Psi | \mathcal{Q} | \Psi \rangle - \frac{1}{3}\langle \Psi | \Psi * \Psi \rangle. \tag{7.3}$$

As we discussed in chapter 5, since the VSFT operator $\mathcal{Q}$ is purely ghost, there are factorisable solutions $\Psi = \Psi_m \otimes \Psi_g$ satisfying $\mathcal{Q}\Psi_g + \Psi_g * \Psi_g = 0$ and $\Psi_m * \Psi_m = \Psi_m$. As the ghost part of the solution is thought to be in some sense universal [149], attention has mainly been given to the matter part of the solution. From [142], a finite deformation of the matter solution parametrised by fields $\{\varphi_i\}$ is given by

$$\langle \Psi_{\{\varphi_i\}} | \phi \rangle = \mathcal{N} \left\langle \exp\left[-\int_{\tilde{\partial}\Sigma} \mathrm{d}z \int \mathrm{d}k \sum_i \varphi_i(k) \mathcal{O}_{\varphi_i(k)}(z) \right] f \circ \phi(P) \right\rangle_{\Sigma}, \tag{7.4}$$

where $\tilde{\partial}\Sigma$ refers to the portion of the boundary of Σ belonging to the state $|\Psi_{\{\varphi_i\}}\rangle$, as opposed to the reference state $|\phi\rangle$. $\{\varphi_i\}$ are fields which parametrise the deformation, while $\mathcal{O}_{\varphi_i(k)}$ are the corresponding vertex operators. The integral of such a vertex operator, which is of conformal dimension one for on-shell physical states, is thus conformally invariant.

In the case of a tachyon deformation, we have

$$\langle e^{-T}|\phi\rangle = \mathcal{N}\left\langle \exp\left[-\int_{\partial\Sigma} \mathrm{d}z \int \mathrm{d}k T(k) e^{-ik\cdot X}(z)\right] f\circ\phi(P)\right\rangle_\Sigma . \tag{7.5}$$

Expanding in powers of the tachyon field T, we have

$$\langle e^{-T}|\phi\rangle = \sum_j \langle T_j|\phi\rangle, \tag{7.6}$$

where

$$\langle T_j|\phi\rangle = \left\langle \frac{1}{j!}\left(-\int \mathrm{d}z \int \mathrm{d}k T(k) e^{-ik\cdot X}(z)\right)^j f\circ\phi(P)\right\rangle_\Sigma . \tag{7.7}$$

One must take care to regularise these states when taking BPZ products, as short-distance singularities will obtain.

$|T_0\rangle$ is nothing but the classical solution $|\Psi_m\rangle$. The next term in the series (7.6) is

$$|T_1\rangle = -\int \mathrm{d}k T(k)|\chi_T(k)\rangle, \tag{7.8}$$

where the tachyon state $|\chi_T(k)\rangle$ is given by the integral of the tachyon vertex operator along the boundary of the surface. The linearised equation of motion for the tachyon state is then

$$|\chi_T(k)\rangle = |\chi_T(k) * \Psi_m\rangle + |\Psi_m * \chi_T(k)\rangle. \tag{7.9}$$

It is shown in [142] that in the case of the sliver state $\Psi_m = \Xi_m$ this equation is satisfied on-shell not only for the tachyon $\chi_T(k)$ but for all physical string states $|\chi_{\varphi_i}(k)\rangle$. In the case of the butterfly, it is easy to see that the linearised equation of motion eqn.(7.9) is satisfied.

Inserting the expansion (7.6) into the action (7.3), the term quadratic in the tachyon field is immediately given by

$$S^{(2)} = -\langle\Psi_g|\mathcal{Q}\Psi_g\rangle\left(\frac{1}{2}\langle T_1|T_1\rangle + \langle T_2|T_0\rangle - \langle T_1|T_1 * T_0\rangle - \langle T_2|T_0 * T_0\rangle\right) \tag{7.10}$$

$$= -\frac{\mathcal{K}}{2}(2\pi)^{26}\int \mathrm{d}k K(k^2)T(k)T(-k), \tag{7.11}$$

where $K(k^2)$ consists of four pieces coming from the above four contributions;

$$\frac{1}{2}K(k^2) = \frac{1}{2}K_{11} + K_{20} - K_{110} - K_{200}. \tag{7.12}$$

In [142], $K(k^2)$ was found to be identically zero in the case of the sliver state. In the following section we perform an analogous calculation for the case of the butterfly.

Since we wish to deal with tension in section 7.5, we will also need the part of the action cubic in the tachyon, given by

$$S^{(3)} = -\langle\Psi_g|\mathcal{Q}\Psi_g\rangle\Big(\langle T_3|T_0\rangle - \langle T_3|T_0 * T_0\rangle + \langle T_2|T_1\rangle - \langle T_2|T_1 * T_0\rangle - \langle T_2|T_0 * T_1\rangle$$

$$-\frac{1}{3}\langle T_1|T_1 * T_1\rangle\Big) \tag{7.13}$$

$$= -\frac{\mathcal{K}}{3}(2\pi)^{26}\int \mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3\delta(k_1+k_2+k_3)V(k_1,k_2,k_3)T(k_1)T(k_2)T(k_3), \tag{7.14}$$

where V is a function containing contributions from each of the six terms in the action. We consider the on-shell case, and write this quantity as

$$-\frac{1}{3}V = V_{30} + V_{21} - V_{300} - V_{210} - V_{201} - \frac{1}{3}V_{111}. \tag{7.15}$$

For the case of the sliver, Okawa [142] showed that the first five terms together cancel;

the cubic action is given solely by the sixth term V_{111} and thus so is the brane tension. We will discuss the case of the butterfly in section 7.5.

7.4 Quadratic Tachyon Action

Here we calculate the quadratic term in the tachyon action, and show that it vanishes on-shell. From eqn.(7.10) we have four pieces.

For the term K_{11}, we use the mapping shown in fig.7.3 to map the surface obtained

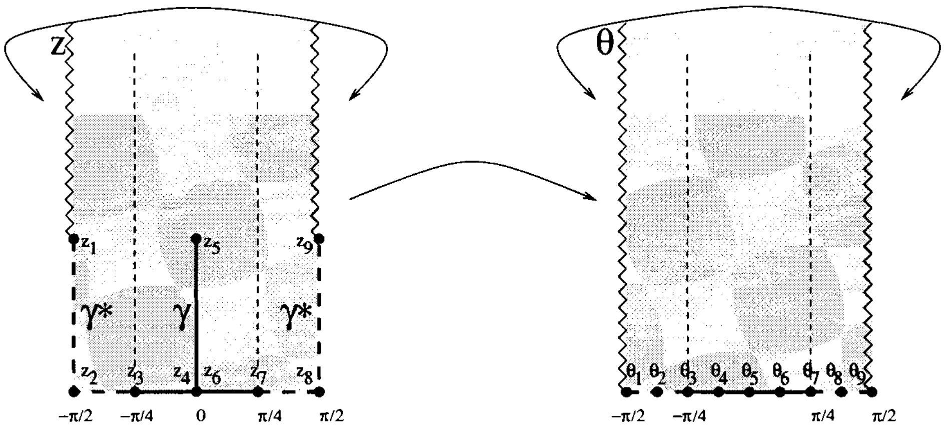

Figure 7.3: mapping for two-state BPZ product

by gluing together two copies of Σ_h onto a cone of circumference π. The boundary of the first and second copies of Σ_h will be denoted γ and $\gamma*$. They are shown in the figure as heavy dashed and solid lines. From the appendix, section 7.7, the mapping is given by

$$\cos 2\theta = \frac{1}{\eta} \cos 2z, \tag{7.16}$$

where $\eta = \cosh 2h$, so that

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} \equiv s(\theta) = \frac{\varepsilon(\theta)\sin 2\theta}{\sqrt{\frac{1}{\eta^2} - \cos^2 2\theta}}, \tag{7.17}$$

where $\varepsilon(\theta)$ denotes the sign of $\mathfrak{Re}\ \theta$. From the previous section, we have

$$K_{11} \sim \left\langle \int_\gamma \mathrm{d}z \int_{\gamma*} \mathrm{d}z' e^{ik\cdot X}(z) e^{ik'\cdot X}(z') \right\rangle, \tag{7.18}$$

where by $\sim$ we indicate that there this requires regularisation; we see there will be short-distance singularities at $z = \pm\pi/4$. We thus regularise the state by leaving a gap of $2\delta_z$ at each of these points. The limit $\delta_z \to 0$ will be taken at the end of the calculation. Transforming to the θ coordinate, we have

$$K_{11} = \int_{\theta_3+\delta_\theta}^{\theta_7-\delta_\theta} \mathrm{d}\theta \left\{ \int_{\theta_7+\delta_\theta}^{\theta_9} \mathrm{d}\theta' + \int_{\theta_1}^{\theta_3-\delta_\theta} \mathrm{d}\theta' \right\} \left\langle e^{ik\cdot X}(\theta) e^{ik'\cdot X}(\theta') \right\rangle_{C_\pi}, \tag{7.19}$$

where C_π is a cone with opening angle π, and δ_θ is our regulator δ_z, transformed to the theta system;

$$\delta_\theta = \left.\frac{1}{s(\theta)}\right|_{\theta=\pm\pi/4} \delta_z = \frac{1}{\eta}\delta_z. \tag{7.20}$$

The propagator on the boundary of the cone is given by

$$\left\langle e^{ik\cdot X}(\theta) e^{ik'\cdot X}(\theta') \right\rangle_{C_{n\pi}} = \delta(k+k') \left| n \sin\frac{\theta - \theta'}{n} \right|^{2kk'}, \tag{7.21}$$

and considering the on-shell case, $k^2 = 1$, we have

$$K_{11} = \int_{\theta_3+\delta_\theta}^{\theta_7-\delta_\theta} \mathrm{d}\theta \left\{ \int_{\theta_7+\delta_\theta}^{\theta_9} \mathrm{d}\theta' + \int_{\theta_1}^{\theta_3-\delta_\theta} \mathrm{d}\theta' \right\} \csc^2(\theta - \theta'). \tag{7.22}$$

Performing the integrals, we obtain

$$K_{11} = -2\log\sin 2\delta_\theta \tag{7.23}$$

$$\to -2\log 2 - 2\log\delta_z + 2\log\eta, \tag{7.24}$$

where we have made use of the limit $\delta_z \to 0$. We see that there is a finite and a divergent part to K_{11} for finite η, and also that there is a divergence as $\eta \to \infty$. We expect the part divergent in δ_z to cancel between K_{11} and K_{110} in eqn.(7.12), since products of $|T_1\rangle$ states must be regularised at the endpoints of vertex operator integration, while the $|T_2\rangle$ state contains a different, independently regularised divergence.

Turning our attention now to K_{110}, we use the mapping shown in fig.7.4. We may

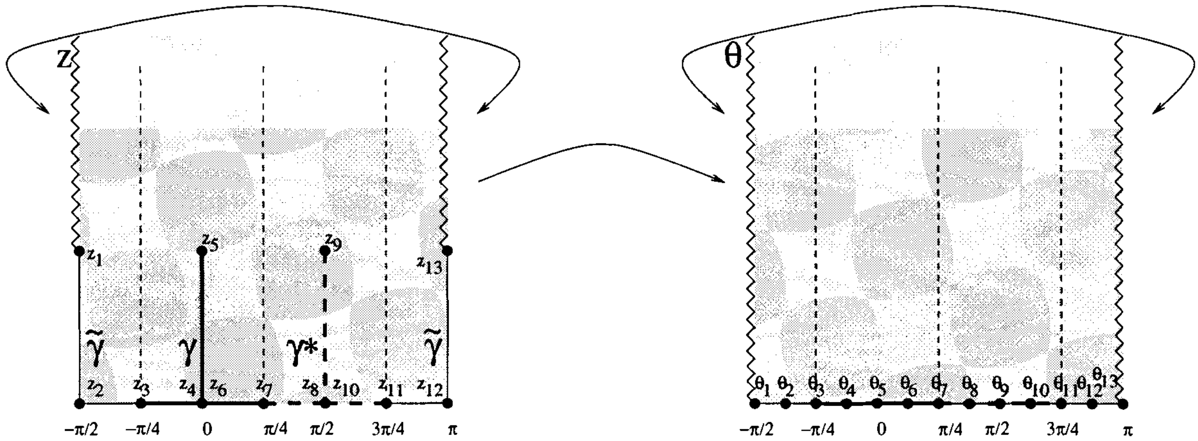

Figure 7.4: mapping for three-state BPZ product

again write

$$K_{110} \sim \left\langle \int_{\gamma} \mathrm{d}z \int_{\gamma*} \mathrm{d}z' e^{ik\cdot X}(z) e^{ik'\cdot X}(z') \right\rangle . \tag{7.25}$$

Recalling the mapping in section 7.7, the relation (7.16) and the derivative (7.17) remain unchanged. The mapping is now to a cone of angle $3\pi/2$, so substituting $n = 3/2$ into the two-point function (7.21), we may write

$$K_{110} = \frac{4}{9} \int_{\theta_3+\delta_\theta}^{\theta_7-\delta_\theta} \mathrm{d}\theta \int_{\theta_7+\delta_\theta}^{\theta_{11}-\delta_\theta} \mathrm{d}\theta' \csc^2 \frac{2}{3}(\theta - \theta'). \tag{7.26}$$

Again, performing the integrals, we have

$$K_{110} = \frac{3}{2}\log 3 - 3\log 2 + -\log\delta_z + \log\eta, \tag{7.27}$$

so that the combination

$$\frac{1}{2}K_{11} - K_{110} = 2\log 2 - \frac{3}{2}\log 3 \tag{7.28}$$

is finite, but non-zero.

In order to calculate the K_{20} and K_{200} contributions, greater care must be taken with the regularisation procedure. The $|T_2\rangle$ state from eqn.(7.7) involves the a double integral since

$$\left\langle\left(-\int \mathrm{d}z e^{-ik\cdot X}(z)\right)^2\right\rangle = \int \mathrm{d}z\mathrm{d}z' \left\langle e^{-ik\cdot X}(z)e^{-ik\cdot X}(z')\right\rangle, \tag{7.29}$$

but this will be divergent when $z \sim z'$.

Looking at both fig.7.3 and fig.7.4, let us calculate K_{20} and K_{200} simultaneously. We follow Okawa [142] and regularise the double integral (7.29) as

$$\int_{\gamma;z_3+\epsilon_z}^{z_7} \mathrm{d}z \int_{\gamma;z_3}^{z-\epsilon_z} \mathrm{d}z' \left\langle e^{-ik\cdot X}(z)e^{-ik\cdot X}(z')\right\rangle, \tag{7.30}$$

where the z integral is to be taken along the contour γ from 'just after' the beginning until the end at z_7, and the z' integral is taken along the γ from the beginning at z_3 until 'just before' the point z. That is, the small quantity ϵ_z is equal to ϵ, $i\epsilon$, $-i\epsilon$ and ϵ for each segment of γ respectively, where real ϵ is then the regularisation parameter. Using this formulation in the θ-system, we write

$$K_{20(0)} = \int_{\theta_3+\epsilon_\theta}^{\theta_7} \mathrm{d}\theta \int_{\theta_3}^{\theta-\epsilon_\theta} \mathrm{d}\theta'\omega^2 \csc^2\omega(\theta-\theta'), \tag{7.31}$$

where

$$\epsilon_\theta = \frac{1}{s(\theta)}\epsilon_z \tag{7.32}$$

so that in the inner integral, the upper limit of integration depends on θ. In order to do both calculations at once, we have introduced the constant ω; for the cases of K_{20} and K_{200}, $\omega = 1$ and $\omega = 2/3$, respectively. Let us first perform the inner integral, giving

$$K_{20(0)} = \int_{\theta_3+\epsilon_\theta}^{\theta_7} \mathrm{d}\theta \left(\frac{1}{\epsilon_\theta} - \omega \cot \omega(\theta - \theta_3)\right) \tag{7.33}$$

$$\rightarrow \left[\frac{1}{\epsilon_z}\int \mathrm{d}\theta s(\theta) - \log \sin \omega\theta\right]_{\theta_3+\epsilon_\theta}^{\theta_7} . \tag{7.34}$$

Substituting $s(\theta)$ from eqn.(7.17), we have

$$\int \mathrm{d}\theta s(\theta) = -\frac{\varepsilon(\theta)}{2} \tan^{-1} \frac{\sin 2\theta}{\sqrt{\frac{1}{\eta^2} - \cos^2 2\theta}}. \tag{7.35}$$

We may now evaluate eqn.(7.33), obtaining

$$K_{20(0)} = -\frac{1}{\epsilon} \tan^{-1} \frac{\sin 2\theta_4}{\sqrt{\frac{1}{\eta^2} - \cos^2 2\theta_4}} - \log \sin \frac{\omega\pi}{2} + \log \frac{\omega\epsilon}{2}. \tag{7.36}$$

We thus find the contribution

$$K_{20} - K_{200} = -2\log 2 + \frac{3}{2}\log 3 \tag{7.37}$$

to (7.12) to be again finite and non-zero, precisely cancelling the contribution (7.28) to the quadratic term (7.12).

7.5 Cubic Tachyon Action and Brane Tension

We have verified in section 7.4 that the butterfly does indeed support a tachyon field. In order to further motivate its interpretation as a D-brane state, the brane tension should also be calculated. This was properly done for the sliver in [142], and we consider the same procedure here for the butterfly.

We recall that the tachyon coupling constant can be identified by first canonically normalising the tachyon field using the quadratic term, and then extracting the coefficient of the cubic term as we did in the previous chapter. This coupling will be related to the tension, and the tension may be expressed in terms of the energy density.

The cubic term in the action contains the quantity from eqn.(7.15),

$$-\frac{1}{3}V = V_{30} + V_{21} - V_{300} - V_{210} - V_{201} - \frac{1}{3}V_{111}. \tag{7.38}$$

These may be calculated as follows. The three-point function on the boundary of the cone $C_{n\pi}$ is given by

$$\begin{aligned} P_n(w_1, w_2, w_3)\delta(k_1 + k_2 + k_3) &\equiv \langle e^{ik_1 \cdot X}(w_1)e^{ik_2 \cdot X}(w_2)e^{ik_3 \cdot X}(w_3)\rangle_{C_{n\pi}} \\ &= \frac{1}{n^3}\left|\csc\frac{|w_1 - w_2|}{n}\csc\frac{|w_2 - w_3|}{n}\csc\frac{|w_3 - w_1|}{n}\right|\delta(k_1 + k_2 + k_3). \end{aligned} \tag{7.39}$$

In the following, as before, we imply the use of fig.7.3 and fig.7.4 for two- and three-state products, respectively. We use the same regularisation parameters $\delta_\theta = \frac{1}{\eta}\delta_z$ and $\epsilon_\phi \equiv \frac{1}{s(\phi)}\epsilon_z$ as in the preceding section.

We begin with V_{111} and write

$$V_{111} \sim \left\langle \int_\gamma \mathrm{d}w_1 \int_{\gamma*} \mathrm{d}w_2 \int_{\tilde{\gamma}} \mathrm{d}w_3 e^{ik\cdot X}(w_1)e^{ik\cdot X}(w_2)e^{ik\cdot X}(w_3)\right\rangle. \tag{7.40}$$

Regularisation may be carried out as before, so that

$$\begin{aligned} V_{111} = \int_{\theta_3+\delta_\phi}^{\theta_7-\delta_\phi} \mathrm{d}\phi_1 \int_{\theta_7+\delta_\phi}^{\theta_{11}-\delta_\phi} \mathrm{d}\phi_2 \left\{\int_{\theta_{11}+\delta_\phi}^{\theta_{13}} \mathrm{d}\phi_3 + \int_{\theta_1}^{\theta_3-\delta_\phi} \mathrm{d}\phi_3\right\} \\ \left(\frac{2}{3}\right)^3 \csc\frac{2}{3}(\phi_1 - \phi_2)\csc\frac{2}{3}(\phi_2 - \phi_3)\csc\frac{2}{3}(\phi_3 - \phi_1). \end{aligned} \tag{7.41}$$

The three integrals in this case are independent, (*i.e.* the integration bounds do not depend on the integration variables) and we find that this is identical to V_{111} in the case of the sliver, calculated in [142], that is

$$V_{111} = \frac{\pi^2}{3}. \tag{7.42}$$

Moving on to V_{21} and V_{210} we may write the unregularised quantity as

$$V_{21(0)} \sim \int_{\gamma} \mathrm{d}w_1 \mathrm{d}w_2 \int_{\gamma *} \mathrm{d}w_3 P_{1/\omega}(w_1, w_2, w_3), \tag{7.43}$$

where we have written both states simultaneously using ω, as we did in section 7.4. We regularise as we did with $K_{20(0)}$;

$$V_{21(0)} = 2 \int_{\gamma; z_3+\epsilon_{z_3}}^{z_7} \mathrm{d}w_1 \int_{\gamma; z_3}^{w_1-\epsilon_{w_1}} \mathrm{d}w_2 \int_{\gamma *} \mathrm{d}w_3 P_{1/\omega}(w_1, w_2, w_3). \tag{7.44}$$

In the θ-system, this becomes

$$\begin{aligned} V_{21} = 2 \int_{\theta_3+\epsilon_{\theta_3}}^{\theta_7} \mathrm{d}\phi_1 \int_{\theta_3}^{\phi_1-\epsilon_{\phi_1}} \mathrm{d}\phi_2 &\left\{ \int_{\theta_7+\delta_\theta}^{\theta_9} \mathrm{d}\phi_3 + \int_{\theta_1}^{\theta_3-\delta_\theta} \mathrm{d}\phi_3 \right\} \\ &\times \csc(\phi_1 - \phi_2) \csc(\phi_2 - \phi_3) \csc(\phi_3 - \phi_1). \end{aligned} \tag{7.45}$$

Here we are using the regulator ϵ just as we did in the previous section.

$V_{210} = V_{201}$ may similarly be expressed as

$$\begin{aligned} V_{210} = 2 \int_{\theta_3+\epsilon_{\theta_3}}^{\theta_7} \mathrm{d}\phi_1 \int_{\theta_3}^{\phi_1-\epsilon_{\phi_1}} \mathrm{d}\phi_2 \int_{\theta_7+\delta_\theta}^{\theta_{11}-\delta_\theta} \mathrm{d}\phi_3 \left(\frac{2}{3}\right)^3 & \\ \times \csc \frac{2}{3}(\phi_1 - \phi_2) \csc \frac{2}{3}(\phi_2 - \phi_3) \csc \frac{2}{3}(\phi_3 - \phi_1).& \end{aligned} \tag{7.46}$$

Finally, V_{30} and V_{300} may be written

$$V_{30(0)} \sim \int_{\gamma} \mathrm{d}w_1 \mathrm{d}w_2 \mathrm{d}w_3 P_{1/\omega}(w_1, w_2, w_3), \tag{7.47}$$

which may be regularised and written as

$$V_{30(0)} = 4 \int_{\theta_3+2\tau_{\theta_3}}^{\theta_7} \mathrm{d}\phi_1 \int_{\theta_3+\tau_{\theta_3}}^{\phi_1-\tau_{\phi_1}} \mathrm{d}\phi_2 \int_{\theta_3}^{\phi_2-\tau_{\phi_2}} \mathrm{d}\phi_3 \omega^3$$
$$\times \csc\omega(\phi_1-\phi_2)\csc\omega(\phi_2-\phi_3)\csc\omega(\phi_3-\phi_1). \qquad (7.48)$$

Here, τ is a new regulator, which is used exactly as is ϵ in previous expressions. This means that $\tau_\phi = \frac{1}{s(\phi)}\tau_z$, and τ_z 'follows the contour' in the z-system, as explained for ϵ_z just before equation (7.31).

Explicit evaluation of the integrals for $V_{21(0)}$ and $V_{30(0)}$ in equations (7.45), (7.46) and (7.48) is not impossible, but very tedious. Primarily this is due to the complicated dependence of the upper limits of the inner integrals on the variables of the outer integrals. We leave this evaluation for future investigation and here make some comments. We first mention that the calculation must be done for fixed, finite η, taking the limit $\eta \to \infty$ only at the very end. The $\delta \to 0$, $\epsilon \to 0$ and $\tau \to 0$ limits may be considered while performing the calculation, but care must be taken to keep all divergent terms in these regulators. In addition, in individual expressions, these three limits do not always commute, and it is only at the end where the divergent parts should be seen to cancel. Given that these two regulators are independent, we again expect that the combinations $V_{30} - V_{300}$ and $V_{21} - V_{210} - V_{201}$ will not be divergent. We also expect that combined, they contribute zero to the expression (7.13) for the cubic tachyon coupling, through eqn.(7.15), leaving the tension dependent only on the term V_{111}. Finally, to calculate the tension one must calculate the overall normalisation of the quadratic term, so that it may be written canonically and the cubic term normalised appropriately. This calculation also is tedious for the case of the butterfly; we do not undertake it here. In fact, although the author did try to undertake it, only limited progress was made due to the complexity and length of the calculation. Although the quadratic normalisation and the cubic term evaluation are in principle accessible using the methods here, a full calculation of the tension does not seem practical. One way to simplify the problem is to use the conformal symmetry and seek coordinate systems in which the surface geometries lead to more tractable integrals.

7.6 Discussion

We have calculated the on-shell quadratic term in the tachyon action, and found that it vanishes. The structure of this calculation is the same as that for the sliver [142]. This is compatible with the assertion that the butterfly represents a D-brane state. The tension could be calculated by following the procedure outlined in the last section, and we expect that this will also agree with the canonical value of unity for a D-brane, matching the case of the sliver. Since it is not clear how these two D-brane formulations could differ physically, we are led to conjecture that there exists some gauge relationship [153, 154] between the butterfly and sliver states.

Finally, it is interesting to note that when defining surface states with operators on the boundary, such as the deformation states in equations (7.4)-(7.7), the definition of the state depends on the geometry used [134]. That is to say, although such specifications are formally conformally invariant, the necessary regularisation procedure will break this symmetry. When defined using operators requiring regularisation, conformally equivalent surfaces can correspond to different states. This information is contained in the function which maps the regulator from one coordinate system to another; here this was the function $s(\theta)$ which maps the regulator ϵ_z in the z-system to ϵ_θ in the θ-system.

7.7 Appendix: Conformal Mapping to the Cone

Let Ω be the region of the complex plane given by $-\pi/4 < \mathfrak{Re}\ z < \pi/4$, $\mathfrak{Im}\ z > 0$. First let us construct a map from Ω to itself, which transforms the boundary $\partial\Omega$ as shown in fig.7.5. That is, the contour given by line segments from $z = -\pi/4 + ih$ down

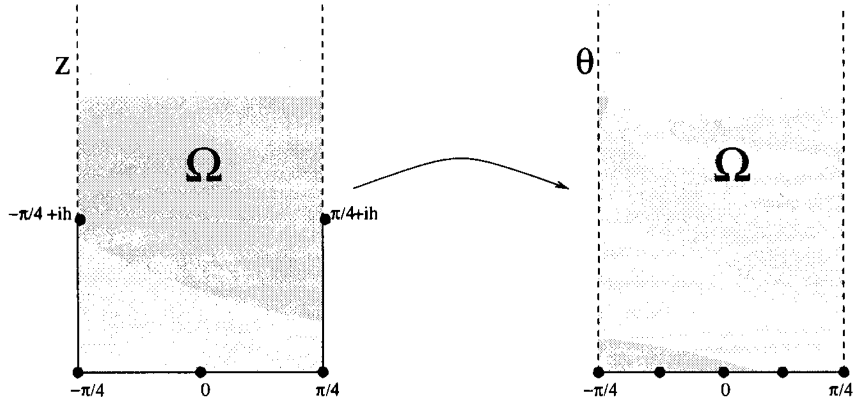

Figure 7.5: $\eta \cos 2\theta = \cos 2z$

to $z = -\pi/4$, across to $z = +\pi/4$ and up to $z = +\pi/4 + ih$ should be mapped to the segment of the real line $-\pi/4 < z < \pi/4$, with the rest of the boundary mapped accordingly. This map is given implicitly by the relation

$$\cos 2\theta = \frac{1}{\eta} \cos 2z, \tag{7.49}$$

where $\eta = \cosh 2h$. Now, due to the periodicity of the functions in (7.49), this map in fact extends to arbitrarily many copies of the surfaces, as shown in fig.7.6. Each 'bucket'

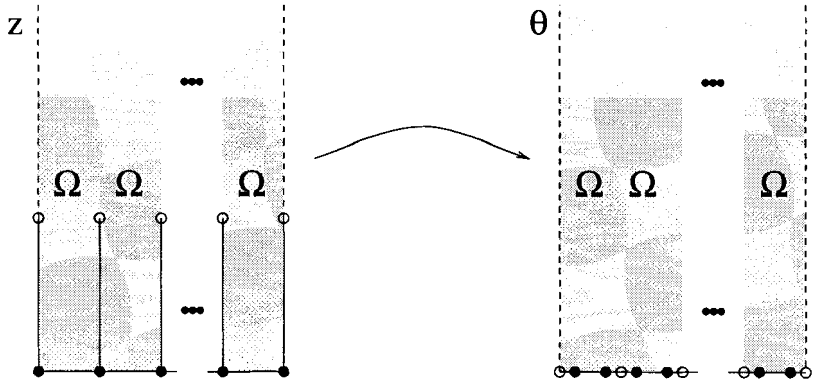

Figure 7.6: $\eta \cos 2\theta = \cos 2z$

of width $\pi/2$ is folded down onto the real line. We will use the map for two copies of Ω for surfaces corresponding to K_{11} and K_{20}, and the three-copy map for K_{110} and K_{200}. Explicitly, the map (7.49) can of course be written as

$$\theta = \frac{1}{2} \cos^{-1} \frac{\cos 2z}{\eta}, \tag{7.50}$$

although we must be careful to note that the inverse cosine function is not single-valued.

Chapter 8

The Future

In this chapter we conclude by mentioning some more recent directions and developments, and also some speculations, related to the research presented in his thesis.

8.1 BMN Correspondence

The BMN sector of Super Yang Mills theory consists of the set of large R-charge operators with impurities. On the string side one considers states with a large angular momentum around a compact direction, in correspondence with the R-charge, and with oscillator excitations in correspondence with the impurities. The underlying AdS/CFT duality is a holographic duality, in that the Yang-Mills theory can be thought of as living on the boundary of anti-de Sitter space and describing fully the string theory within. One might expect that an analogous structure might be preserved by the limits involved in the BMN correspondence. It has been noted that the boundary in the pp-wave case is a one-dimensional light-like boundary. It would be interesting to understand what becomes of the holography in the BMN limit. Presumably, one would think of looking for some kind of gauge theory on this boundary. Being only one-dimensional, this field theory would really just be some kind of quantum mechanics. Some progress has already been made along these lines, and we refer the reader to [155, 156].

Another area which requires attention is that of open strings [157, 158, 159, 160, 161]. Investigation of the BMN correspondence has mainly been focused on closed string theory. Dealing with open strings implies that one must include D-branes; this is a big step to make, since the existence of D-branes on the string side, being non-perturbative, may be difficult to map to the gauge theory. The BMN correspondence as explained in the present work is a mapping identifies between a specific sector of Yang mills operators and perturbative string states on the pp-wave. As in flat-space string theory, D-branes are connected with compactification and T-dual transformations. In a non-flat background, compactification generically makes sense along Killing vector directions, and this means that typically the group of T-dual transformations is 'smaller' (the T-dual group is usually discrete) in a less symmetric background; the T-dual transformations are essentially matched with the isometries. Although D-branes are dynamical objects, and in principle can have any geometry, the D-brane backgrounds obtainable from T-duality are thus limited in the pp-wave. In addition, the T-dual transformations are not guaranteed to preserve the supersymmetry; the supersymmetry algebra may suffer the loss (or gain) of some number of supercharges. Some studies on T-duality and D-branes in the plane-wave limit may be found in [162, 163, 164, 165]. Clearly, there are many questions to be addressed in this area.

The correspondence between anomalous dimensions and light-cone energies now has been attacked from several angles. The BMN quantum mechanics route which we have taken in the present work in chapter 3 is intuitive in that with it one deals directly with the dilatation operator, and thus understanding what is happening from the viewpoint of conformal field theory is clear. The dilatation operator must still be constructed in the first place, and this has been done by primarily diagrammatic means. As we have shown, the method of BNM quantum mechanics, although we have successfully made use of it for four impurities, is already rather tedious. We commented at the end of section 3.7 that the extension past the 'natural' four distinct scalar impurities would involve considerable complication in terms of counting. In [67, 81], for example, using perturbative methods rather than the dilatation operator, the authors were able to consider an arbitrary number of impurities. This would suggest that the use of three-point functions, which of course are to some extent more tractable in a conformal field theory, is a more efficient method, at least for larger numbers of impurities. One might speculate that the counting techniques used in these papers to deal with arbitrary-impurity states could be used to investigate non-perturbative string-field states such as the D-brane states investigated in chapters 6 and 7. At the least, this would require an infinite number of impurities and one might question whether these operators are well defined. The problem, of course, is that although the sliver and butterfly solutions have oscillator representations, no such solutions are known for the full string field theory. To the author's knowledge, no attempt has been made to make use of vacuum string field theory in a modified BMN correspondence, or in AdS/CFT at all. This would be a very interesting direction of research, and would shed light on some non-perturbative aspects of the duality.

One idea has been to find the representation of the dilatation operator on the space of gauge-invariant operators in the BMN gauge theory, and attempt to diagonalise it completely. It has been argued that this would enable one to solve the full gauge theory, at least in such a BMN sector, through the identification of the dilatation operator with the Hamiltonian under so-called radial quantisation [87, 86, 77]. As an example of such a matrix, considering h impurity states with a certain permutation symmetry (one can, say, [anti]symmetrise the ordering of the impurities, or the powers of Z between them) one can write down the explicit dilatation matrix elements between such states. This was done for two-impurity states in [54] where the resulting tri-diagonal matrices are easily diagonalised. Similar matrices may easily be written down [166] for the cases of three and four impurities (and arbitrary J), but although they were of somewhat similar form (one might describe them as tridiagonal in blocks of tridiagonal matrices) we were unable to achieve diagonalisation for arbitrarily many fields (*i.e.* large J). It did seem that similar methods to those used to diagonalise the matrices in the two-impurity case could have been used to diagonalise these, one set of blocks at a time. However, this method seemed more tedious than the BMN quantum mechanics used in chapter 3, which amounts to a continuum formulation of these matrices in which the tri-diagonal elements are expressed as second derivatives. One can neglect 'boundary terms' in such matrices, corresponding to (comparatively rare) elements where impurities are exchanged, but this does not result in significant simplification anyway when dealing with, for example, symmetrised-impurity states.

Another area with room for development is the string-bit model [84, 85] in which the string is discretised and considered as a large number of 'bits' each with some mass. The action for this model is constrained by supersymmetry, and can be constructed so that the spectrum matches the standard string-theory free spectrum. This method has been useful in the BMN correspondence and yields the prediction that the genus counting parameter g_2 appears only through the quantity $\lambda' g_2^2$ [52].

The semi-classical analysis which we described briefly in section 2.2.2 has yet to be developed as well as the BMN formulation. The BMN analysis focuses on operators with large R-charge, which corresponds to large angular momentum on the string side. Considering instead large Lorentz spin, S, on the string side, as in the 'propeller' geometry we mentioned in section 2.2.2, an analogous correspondence could be built up. Identifying operators in the gauge theory with large spin is not difficult and presumably such states, populated with some kind of impurities, could be related to fluctuations of string states, quantised around classical spinning solutions [68].

Finally, we mention the unresolved discrepancy in the correspondence found at cubic order in λ'. It has been possible on both the string and gauge sides of the BMN correspondence to compute 'finite-radius' corrections, which appear as powers of $1/J$. On the string side, this corresponds to moving away from the strict pp-wave limit and considering large but finite angular momentum J. On the gauge theory side, the BMN operators have a large number of fields in the trace, but corrections in $1/J$ appear since this number of fields is kept finite. This has been referred to as the 'near-BMN limit'. It has been possible to compute anomalous dimensions of some two-impurity BMN operators to three-loop order in this near-BMN limit. On the string side, introducing a finite radius of curvature produces an interacting worldsheet theory, which using the Green-Schwarz formalism has been tractable perturbatively. This, too, has been solved up to three-loop order and the $1/J$ corrections do not match those found in the gauge-theory computation [54, 167]. This is an important discrepancy, the resolution of which will be important for a deeper understanding of the BMN correspondence and for AdS/CFT.

8.2 Little String Theory

In chapter 4 we solved superstring theory in the Nappi-Witten background and by the holographic duality with six-dimensional Little String Theory found the supersymmetric spectrum of the high energy sector of that theory. Speaking more generally, lower-dimensional string theories have been studied for some time [168]. Since they do not live in the usual $d = 10$ background, they have been called non-critical strings [168, 169]. Such theories can be consistent; the standard critical dimension, as is well-known, arises from the use of worldsheet conformal invariance to achieve a free worldsheet theory. In the non-critical case the issue is somewhat more subtle; in order to obtain conformal invariance one typically has to allow the string to propagate in an extra dimension containing a tachyon and dilaton, which produces a Liouville interaction on the worldsheet [170]. One problem has been to understand the geometrical interpretation of such theories, which due to the extra dimension is not as clear as in the case of critical string theory. In [170] theories for $d \leq 6$ have been investigated and progress made in this geometrical understanding. Little String Theory in particular has attracted attention because of its holographic relation to string theory involving NS5-branes [171, 172], the thermal instability indicated by its Hagedorn spectrum [173], and its relation to gauge theory [174].

8.3 Vacuum String Field Theory

Vacuum String Field Theory represents a significant step forward in the understanding of string field theory. As we mentioned in chapter 1, progress in string field theory had been limited by the lack of any classical solutions. Very interesting early work [175, 176, 177, 178, 179, 180, 181, 182, 183] had provided hints of a background-independent formulation and the dynamical appearance of geometry, along with links between closed- and open-string theories, but in a computational sense 'traditional' string field theory had been very limited. Closed strings have been studied more recently in [122, 184, 145]. It has been fairly well established that the various sliver and butterfly solutions do represent D-branes, but how these objects may be related by gauge transformations [183, 185, 153] is a difficult and interesting problem. The gauge structure of the theory remains to be well-understood, and especially important is its relation to duality relationships, such as T-duality [186]. In VSFT one can write down solutions for various D-branes; it should be possible to identify a T-duality operator group on the space of string fields which maps these into one another.

There is also the problem of the actual identification of the vacuum. As we discussed in chapter 5, Sen's conjecture [36] allowed the investigation of this putative vacuum, along with the construction and analysis of D-brane states [40]. It has not, however, been possible to derive it from the cubic string field theory. For some recent comments on and review of the search for vacua in string theory, the reader is referred to [187]. Some progress has also been made with regard to dynamical solutions to VSFT. In particular, investigation of the 'rolling tachyon' [188, 189, 190] has led to some understanding of time-evolution in the theory.

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