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STRUCTURE OF NONSTANDARD NUMBER SYSTEMS

by

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ABSTRACT

The order structure of some nonstandard number systems is investigated, especially order-properties and connections between external subsets such as infinitesimals, infinite numbers and galaxies.

Standard functions are used to exhibit some structure of models of (full) arithmetic, introducing the concept of sky and constellation. Applications to intersection of models are given. This also leads us to characterizations of some well known ultrafilters on N.

DEDICATION

À mes parents et amis, Elizabeth et Raymond tout spécialement

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I first worked on this subject during the summer 81 at Laval University under the supervision of W. S. Hatcher. Part of this thesis constitutes what we have achieved together, and will be published during the current year in the Zeit. fur Logik und Grund. der Math. I would like to express my sincere gratitude to prof. Hatcher.

Special thanks are due to my present supervisor Alan Mekler, who accepted to move a little from his field of studies to provide me invaluable help in nonstandard theory. I would also like to acknowledge the intellectual and personal debt I owe to prof. Greg Cherlin.

Sincere thanks to all the people who helped me in the preparation of this thesis.

PREFACE

The rigorous treatment of infinitesimals has been developed by Abraham Robinson in the 60's. Since then the theory has been studied extensively and used in various branches of mathematics as well as in other scientific enterprise.

We shall however not be interested in nonstandard number systems as a tool but rather as an object of study; their structure from various point of view will be investigated. We believe it is interesting and may be useful to have a picture of the models one uses.

We have collected some results starting principally with Zakon's paper in 1967, as very little was done earlier. We have tried to give a good account of what has been achieved, however a selection has been made. In particular, topological properties of models have been completely omitted, although this is due more to the limitations of the author than to the lack of importance of the topic.

Chapter 1 will introduce nonstandard models of different number systems we shall be interested in, those are mainly the real numbers and the natural numbers.

In chapter 2, we investigate the order structure of models. We answer questions about cofinalities and coinitialities of certain external subsets, and try to relate them by order-isomorphism if possible or by their order saturation.

The last chapter deals only with full arithmetic. We will see that standard functions on N constitute an important tool toward the understanding of the structure of these models. Special attention is given to basic ultrapowers (ie ultrapowers on N) and ultrafilters on N.

Special notions will be introduced and defined when necessary. However, the usual model-theoretic and set-theoretic concepts are assumed to be known. In particular, we assume the reader to be familiar with the ultrapower construction, although we briefly recall the definable ultrapower construction.

Some good introductions to the subject have been written, in particular [LS] and [M1].

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I. CHAPTER 1

NUMBER SYSTEMS

We shall be interested in some different number systems;one is a superstructure based on the real numbers,containing "all of analysis", one is the natural numbers $N=\{0,1,2,...\}$ together with all functions and relations on N;similarly the rationals and the reals with all functions and relations. After introducing these number systems,we shall study their structure quite independently in subsequent chapters.

Superstructures

The theory of infinitesimals can be applied to certain mathematical theories such as real or complex analysis,topology,etc. In order to do this,we construct a superstructure that contains all mathematical objects under study in the given theory. We shall describe the process briefly.

We start with a set of individuals ("non-sets") A,which may contain the real numbers or the underlying point set of a topological space,etc. The superstructure $V(A)$ is the set:

$V(A) := \bigcup_{n=0}^{\infty} A(n)$

where $A(0)=A$ and $A(n+1)=P(\bigcup_{i=0}^{n} A(i))$ (where P is the power set operation), together with the notions of equality "=" and membership "$\in$" on elements of $V(A)$.

Elements of $A(n)$ are said to be of sort n. As noted above, elements of type 0 (those of A) are individuals, that is if $x \in A$ then $y \notin x$ for all y (although $y \notin x$ and $y \in x$ are always meaningful). For x and y in A, we define $x \cup y = x \cap y = x - y = \emptyset$, and $y \subseteq x$ holds always.

~~We now describe some easy properties of V(A). We refer to~~ the set elements of $V(A)$ as entities.

1) for each n, $A(n)$ is in $V(A)$ and included in $V(A)$

2) $V(A)$ is transitive; if y is an entity and x is in y, then x is in $V(A)$

3) if y is an entity and x is included in y, then x is in $V(A)$

4) if y is an entity, $P(y)$ is an entity

5) if x is a finite subset of $V(A)$, then x is an entity

6) if x is an entity, then $x := \{y \mid y \ x\}$ is also an entity

Briefly the set theory of entities is contained in the entities. If x and y are in $A(n)$, the ordered pair $\{\{x,y\},\{y\}\}$ is in $A(n+2)$; similarly for n-tuples. A set of such n-tuples ("n-ary relation") is in $V(A)$ if all its tuples are of bounded type (ie belong to some $A(N)$). In particular, a binary relation r is in $V(A)$ iff its domain $D(r) := \{x \mid (x,y) \text{ is in } r \text{ for some } y\}$ and its range $D'(r) = D(r^{-1})$ are in $V(A)$ ($r^{-1} = \{(y,x) \mid (x,y) \text{ is in } r\}$). We define $r \upharpoonright x = \{y \mid (z,y) \text{ is in } r \text{ for some } z \text{ in } x\}$ the "r-image of x",clearly r and x in V(A) implies $r \restriction x$ in V(A).

We view V(A) as a multisorted structure (see [R1]). $M \subseteq V(B)$ is called a nonstandard model of V(A) if M is an elementary extension of V(A) (as a multisorted structure). In this situation, then for C in V(A), we write $^*C$ for its interpretation in V(B).

Elements of the form $^*C$ for C in V(A) are called standard members of V(B),their elements are called internal elements of V(B); in particular $^*A(n)$ is standard and all its elements are internal,so are all elements of $^*V(A) := \bigcup_{n=0}^{\infty} {}^*A(n)$. Other elements of V(B) are called external. The distinction is important as we shall see because all properties of V(A) transfer only to internal elements of V(B).

For an entity x in V(A), we have:

$$P(x) \subseteq {}^*P(x) \subseteq P({}^*x)$$

and:

$$\begin{tikzcd} V(A) \arrow[rr] \arrow[dr, "*"'] & & V(B) \\ & {}^*V(A) \arrow[ur, hook] & \end{tikzcd}$$

This superstructure V(A) is really huge and we shall sometimes only need some part of it.

ARITHMETIC

We call $A \cup \bigcup_{n=0}^{\infty} P(A^n)$ the elementary part of V(A),it consists of A together with all functions and relations on A. By (full) arithmetic,we mean the elementary part of V(N). Note that arithmetic has always uncountably many functions and relations.

Similarly, we consider the elementary part of $V(Q)$ as the rational number system; and the elementary part of $V(R)$ as the real number system.

II. Chapter Two

Order structure of nonstandard number systems

Introduction

We shall work, unless specified, with models $V(^*R)$ of $V(R)$; but as the reader will note, most of the propositions carry to the elementary part of $V(R)$, or $V(Q)$. In fact, we usually only require that the base set is a field.

It is known, [R1], that the order type of the nonstandard natural numbers $^*N$ has the form $\omega+(\tilde{\omega}+\omega)\theta$, where ω is the order type of the natural numbers, $\tilde{\omega}$ is its reverse order type and θ is a dense order type. We shall investigate further properties of θ, and see that most of its order properties come directly from the structure of $^*R$.

So we shall be interested mainly in the underlying set $^*R$ of a model $V(^*R)$. For this reason and also because we will often deal with many models simultaneously, we shall use the notation $^*R\alpha$, or just $R\alpha$, α an ordinal, to denote either the underlying set of a model or the whole structure based on that set.

In this chapter, "$\cong$" means order-isomorphism.

We now give some set-theoretical definitions and notations:

Definitions and notations

1) Let A,B be subsets of an ordered set $(X,<)$. We write $A<B$ to mean that every element of A is less than every element of B.

2) A is coinitial (cofinal) in X if for all x in X, there is an a in A with $a<x$ ($a>x$). A and X are said to be coinitial (cofinal) (with each other) if A is coinitial (cofinal) in X and vice-versa.

3) Given a set X, X+ denotes its positive elements, and Xi its positive infinite elements (whenever it makes sense).

We now introduce two useful equivalence relations on *R.

4) For x,y in *R, we write:

$x \sim y$ iff $|x-y|<r$ for some r in R+

$x \simeq y$ iff $|x-y|<r$ for all r in R+

The relations $\sim$ and $\simeq$ are equivalence relations and classes are denoted by Gx (galaxy of x) and Mx (monad of x) respectively. Both Gx and Mx are convex subsets of *R. Go is the set of finite Hyperreals, and Mo is the set of infinitesimals.

We define a (total) order relation $\tilde{<}$ on *R/Go by:

$Gx \tilde{<} Gy$ iff $x<y$ and $y-x$ is positive infinite

It is easily seen that the order type of the positive part of *R/Go coincides with Θ.

Similarly we may define an order $\overset{=}{<}$ on *R/Mo by:

$Mx \overset{=}{<} My$ iff $x<y$ and $y-x$ is positive not infinitesimal

With these definitions in mind, we shall study in the next sections some set-theoretical similarities, connections and properties of the sets $^*R$, $^*Ri$, $^*Ni$, Θ and No.

§1 Cofinalities, coinitialities and order-isomorphisms

We write cof(X) (coin(X)) for cofinality of X (coinitiality of X). The first few results are folklore, and they already appear in [KS].

1.1 Proposition

1) $\mathrm{cof}(\Theta)=\mathrm{cof}(^*N)=\mathrm{cof}(^*R)=\mathrm{coin}(No+)$

2) $\mathrm{coin}(\Theta)=\mathrm{coin}(^*Ni)=\mathrm{coin}(^*Ri)=\mathrm{cof}(No)$

proof: 1) Recall that $^*N \cong \omega+(\tilde{\omega}+\omega)\Theta$, so that the first equality is clear. Since N and R are cofinal, this is true for $^*N$ and $^*R$ which implies the second. Finally the last one is trivial by the correspondence $x \leftrightarrow 1/x$.

2) Again the first equality holds just by examining the order type of $^*N$. For the second equality, define:

$f: {^*Ni} \to {^*Ri}$ as the identity mapping,

then the range of f is coinitial in $^*Ri$. Also, the last equality is clear by $x \leftrightarrow 1/x$. -|

1.2 Proposition Let A, B, C be open intervals with endpoints in $^*N$, Θ, $^*R$ respectively, then:

1) A and $\tilde{A}$ are order-isomorphic, A has a first and last element

2) B and $\tilde{B}$ are order-isomorphic, $\mathrm{cof}(B)=\mathrm{coin}(B)=\mathrm{coin}(\Theta)$

3) C and $\tilde{C}$ are order-isomorphic, $\mathrm{cof}(C)=\mathrm{coin}(C)=\mathrm{cof}(^*R)$

proof: 1) Let $A=(n,m)\subseteq {^*N}$, define:

$f: A \to \tilde{A}$ by $a \mapsto n+m-a$

then f is the required isomorphism.

2) Let $B=(Gn,Gm) \le \Theta$, where n,m are in $^*Ni$. Define:

$f': B \to \tilde{B}$ by $Gb \mapsto Gf(b)$

where f is as in 1), f maps (n,m) onto $\widetilde{(n,m)}$ of $^*Ni$; so that f' is the required isomorphism. The last assertion follows by translation.

3) Easily follows by transfer principle, since R is order-isomorphic to any nondegenerate open interval and $R \cong \tilde{R}$.

<u>1.3 definition</u>

1) A binary relation r in $V(R)$ is said to be <u>concurrent</u> if given any finite number of elements $t_1, t_2, \ldots, t_k$ of its domain $D(r)$, there is a y in $V(R)$ such that (t_i, y) is in r for $i=1,\ldots,k$.

2) A nonstandard model $V(^*R)$ of $V(R)$ is called an <u>enlargment</u> if, for any concurrent relation r in $V(R)$, there is a y in $V(^*R)$ with (t,y) in $^*r$ for all t in $D(r)$ (standard domain).

The existence of enlargments follows immediately from compactness. There also exist ultrapowers which are enlargments. It is clear that elementary extension preserves the enlargment property. Enlargments are frequently used in applications.

Before the next result, we disgress slightly and recall the definable ultrapower construction (reference is mainly [C]).

Generally, consider a structure A with a definable subset I A, and let D be an ultrafilter on the Boolean algebra of A-definable subsets of I; we let $Def(A^I)$ be the set of definable functions from I to A. We obtain the definable ultrapower

$Def(A^I)/D$ by factoring the equivalence relation:

$$f \equiv g \text{ iff } \{i \text{ in } I \mid f(i)=g(i)\} \text{ is in } D$$

Thus to any function f in $Def(A^I)$, there is associated its equivalence class f/D in the definable ultrapower. To each a in A, we assign the class $\vec{a}/D$ of the constant function $\vec{a}:I \to \{a\}$ and this induces the diagonal embedding $\Delta : A \to Def(A^I)/D$, which is elementary if A possesses definable Skolem functions.

Here is a well-known fact as a first application.

1.4 Proposition Any model R_0 has an elementary (proper) elongation R_1; we denote this by $R_0 \ll R_1$ (ie $\exists x \in R_1 \forall y \in R_0\ (x > y)$)

proof: Let $F := \{\, [n, \infty[\ \mid n \text{ is in } N_0 \}$ (N_0 = underlying set of natural numbers of R). Since F has the finite intersection property, we can extend F to an ultrafilter D on definable subsets of N_0.

Now form $R_1 := Def(R_0^{N_0})/D$. It is clear that id/D, the equivalence class of the identity mapping witnesses the condition. -|

I would like to remark that in the case we carry the whole superstructure, we can extend F to an internal ultrafilter D, and the construction above provides an end-extension N_1 of N_0.

1.5 Proposition Let a be an infinite regular cardinal. Then there is an enlargment $^*R$ with $cof(^*R)=a$.

proof: Pick an enlargment R_0, then build an a-chain of elementary elongations -|

1.6 Proposition Ler R_0 be any nonstandard model, then there is an elementary extension R_1 of R_0 satisfying:

1) $\forall n \in N_1 \exists m \in N_0\ (m > n)$ (hence R_0 and R_1 are cofinal)

2) $\exists n \in N_i\ \forall m \in N_i\ (n < m)$ (N_1 has new "small" infinite numbers)

proof: Let $F := \{ [n, n_0] \mid n \in N$ and $n_0 \in N_i\}$ F has the finite intersection property and thus can be extended to an ultrafilter D on definable subsets of N_0.

Form $R_1 := \mathrm{Def}(R^{N_0})/D$. clearly id/D satisfies 2). Further, given f in $\mathrm{Def}(N_0^{N_0})$ and $[n, n_0]$ as above, there is a m in N_0 such that $f[n, n_0] < m$ (by transfer); hence $f/D < \vec{m}/D$ so N_0 is cofinal in N_1, which proves 1). -|

Here is a slight improvement of a result in [KS].

1.7 Proposition Let a, b be infinite regular cardinals, then there is an enlargment *R such that cof(*R) = a and coin(*Ri) = b

proof: By corollary 1.5, choose R_0 with $\mathrm{cof}(R_0) = a$. We define by transfinite induction an elementary chain $\{R_\kappa \mid \kappa < b\}$ of nonstandard models and a coinitial subset $\{n_\kappa \mid \kappa < b\}$ of power b such that:

1) R_0 and R_κ are cofinal for each κ

2) $n_\kappa \in N_{i_\kappa}$ and $n_\kappa < \bigcup_{\beta<\kappa} N_{i_\beta}$

The construction is as follows. Choose any $n_0 \in N_{i_0}$, Suppose that $\{R_\kappa \mid \kappa<\beta\}$ and $\{n_\kappa \mid \kappa<\beta\}$ have been defined satisfying 1 & 2, where $\beta<b$. Let $R' = \bigcup_{\kappa<\beta} R_\kappa$, then $N' = \bigcup_{\kappa<\beta} N_\kappa$ and N' and N are cofinal. Now we use proposition 1.6 to get $R_\beta \succ R'$, cofinal with each other, and n_β in N_{i_β} satisfying 2.

Finally put $*R = \bigcup_{\kappa<b} R_\kappa$. Since *N and N_0 are cofinal, cof(*R) = cof(R_0) = a, and coin(*Ri) = coin(*Ni) = coin($\{n_\kappa \mid \kappa<b\}$) = b. -|

Proposition 1.7 answers several questions raised by Zakon [Z1]. In particular, we see that *R can have countable cofinality. Further, if we ask that coin(*Ri) is uncountable, then *R can obviously not be order-isomorphic with its monads Mx since cofinalities do not match.

Now it is clear that if *R is order-isomorphic with Mo, (or equivalently any Mx), then cof(*R)=coin(*Ri) (or equivalently cof(Θ)=coin(Θ) (by 1.1)). Kamo [KS] thought the converse to be true. Although he gave a proof which remains true in the countable situation, an error[1] has been found for the uncountable case. In fact we are able to provide a counterexample[2].

For any structure A, we denote the ultrapower of A modulo U by U-Prod(A).

First recall that $R_\alpha \ll R_\beta$ means that R_β is an elementary elongation of R_α. In this case, we write (R_α, R_β) to denote the gap (A,B) of R_β where $A=\{r \text{ in } R_\beta \mid \exists s \in R_\alpha\ (r \leq s)\}$, and $B=R_\beta \setminus A$.

We split the construction in two seperate parts, upward and downward.

1) <u>Upward</u> Start with any nonstandard model R_o

a) Successor stage:

If R_κ is constructed, $\kappa < (2^{\aleph_o})^+$; use a compactness argument (or whatever you want) to produce an elongation $R_{\kappa+1} \gg R_\kappa$ making also sure that you have filled all gaps (R_β, R_κ) for all $\beta<\kappa$. (We shall see later that we can fill gaps in a much nicer way)

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[1]I am indebted to Alan Mekler for pointing out the error

[2] Here again, I am very indebted to both Greg Cherlin and Alan Mekler

b) Limit stages:

If $R_\beta, \beta<\kappa \le (2^{\aleph_0})^+$ are constructed, let $R_\kappa = \bigcup_{\beta<\kappa} R_\beta$. We put $R' = \bigcup_{\beta<(2^{\aleph_0})^+} R_\beta$; then $\mathrm{cof}(R') = (2^{\aleph_0})^+$. Further note that for limit ordinals $\kappa < (2^{\aleph_0})^+$, every gap (R_κ, R') has character $(\mathrm{cof}(\kappa), (2^{\aleph_0})^+)$.

2) Downward Let $S_0 = R'$ as constucted above, and $I_0 = R'i$. Let U be any non-principal ultrafilter on N.

a) Successor steps:

If S_κ is constructed, $\kappa<(2^{\aleph_0})^+$, let $S_{\kappa+1} := U\text{-}\mathrm{Prod}(S_\kappa)$. Hence $I_{\kappa+1} := S_{\kappa+1}i$ contains some infinite number smaller than any element of I_κ (eg the class of the identity mapping)

b) Limit stages:

If $S_\beta, \beta<\kappa \le (2^{\aleph_0})^+$ are constructed, just put $S_\kappa = \bigcup_{\beta<\kappa} S_\beta$.

Finally just put $*R := \bigcup_{\kappa<(2^{\aleph_0})^+} S_\kappa$, we get $I = \bigcup_{\kappa<(2^{\aleph_0})^+} I_\kappa$ the infinite positive part of $*R$.

We remark that basic ultrapowers, as used in the downward construction, never fill gaps (A,B) of uncountable character (ie both cof(A) and coin(B) uncountable) since the index set N is countable. Hence for limit $\kappa < (2^{\aleph_0})^+$ of uncountable cofinality, (I, I_κ) has character $(b, \mathrm{cof}(\kappa))$ where $b \le 2^{\aleph_0}$. (in fact $b = \mathrm{cof}(U\text{-}\mathrm{Prod}(N))$) (The gap (I, I_κ) has similar meaning as above).

Note also that $\mathrm{cof}(*R) = \mathrm{coin}(*Ri) = (2^{\aleph_0})^+$ so that $*R$ is a candidate for a counterexample.

Since all R_κ and all S_κ are embedded in $*R$, we introduce a notation for clarity:

i) for $\kappa < (2^{\aleph_0})^+$, let A_κ = initial segment of $*R+$ determined by R_κ

ie $A = \{r \text{ in } *R+ \mid \exists s \in R_\kappa \ (r \le s)\}$

ii) for $\alpha < (2^{\aleph_0})^+$, let B_α = terminal segment of $^*R$ determined by I_α

ie $B_\alpha = \{r \text{ in } {}^*R \mid \exists s \in I_\alpha\, (s < r)\}$

Of course we have $^*R+ = \bigcup_\alpha A_\alpha$ and $I = \bigcup_\alpha B_\alpha$

Now let us assume $^*R \cong Mo$, or what is equivalent $^*R+ \cong Mo+$. By the correspondence $x \leftrightarrow 1/x$ between $Mo+$ and I, we may further assume that we have an inverse order-isomorphism:

$f: {}^*R+ \xrightarrow{1/\cong} I$

We shall derive a contradiction, but we need the following lemma:

__1.8__ __Lemma__ In this situation ie $f: {}^*R+ \xrightarrow{1/\cong} I$ an inverse order-isomorphism, then for all regular $a < (2^{\aleph_0})^+$ there is a v with $\mathrm{cof}(v) = a$ and such that:

$f\restriction A_v : A_v \xrightarrow{1/\cong} B_v$

If we assume the lemma with $a = \omega_1$, we find a v with $\mathrm{cof}(v) = \omega_1$, and an inverse order-isomorphism $f\restriction : A_v \xrightarrow{1/\cong} B_v$. However $\mathrm{char}(A_v, {}^*R) = (\mathrm{cof}(v), (2^{\aleph_0})^+) = (\omega_1, (2^{\aleph_0})^+)$ and on the other side $\mathrm{char}(I, B_v) = (b, \mathrm{cof}(v)) = (b, \omega_1)$ where $b \le 2^{\aleph_0}$. This is a contradiction.

It remains to prove the lemma:

__proof__ __of__ __lemma__ __1.8__ Pick your favourite ordinal $\kappa_0 < (2^{\aleph_0})^+$, and let $\beta_0 \geq \kappa_0$ be such that $f(A_{\kappa_0}) \subseteq B_{\beta_0}$.

Successor steps:

Given $A_{\kappa_\gamma}, B_{\beta_\gamma}$ with $f(A_{\kappa_\gamma}) \subseteq B_{\beta_\gamma}$. let $\kappa_{\gamma+1} \geq \beta_\gamma$ such that $A_{\kappa_{\gamma+1}} \supseteq f^{-1}(B_{\beta_\gamma})$ and $\beta_{\gamma+1} \geq \kappa_{\gamma+1}$ such that $f(A_{\kappa_{\gamma+1}}) \subseteq B_{\beta_{\gamma+1}}$

Limit stages:

Put $A_{\kappa_\gamma} = \bigcup_{\eta<\gamma} A_{\kappa_\eta}$, $B_{\beta_\gamma} = \bigcup_{\eta<\gamma} B_{\beta_\eta}$. Note that:

$f\restriction : A_{\kappa_\gamma} \xrightarrow{1/\cong} B_{\beta_\gamma}$ for all limit ordinals γ

Now let $A_{\kappa_a} = \bigcup_{\eta<a} A_{\kappa_\eta}$, $B_{\beta_a} = \bigcup_{\eta<a} B_{\beta_\eta}$. Let $v = \sup_\eta(\kappa_\eta) = \sup_\eta(\beta_\eta)$. Then clearly cof(v

)=a and $f\restriction : A_\nu \xrightarrow{\cong} B_\nu$ -|

This completes the counterexample.

<u>Remark</u>:For which cardinals a does there exist *R such that cof(*R)=coin(*Ri)=a but *R $\not\cong$ Mo? The obvious conjecture is for every uncountable regular cardinals. The proof above can be modified to show that if a is a regular uncountable cardinal and there is an ultrafilter U such that cof(U-Prod(N)) $\neq$ a, then there exists *R as above. It is consistent (see Proposition 1.11 below) that the conjecture is true.

~~Here is a way to fill gaps keeping track of what we are~~ doing. We use internal ultrapowers to fill gaps (R_1, R_2) in the situation of $R_1 \ll R_2$.

<u>1.9 Lemma</u> Assume $R_1 \ll R_2$, then there is an elementary extension R_3 of R_2 filling the gap (R_1, R_2), and such that R_2 and R_3 are cofinal.

<u>proof</u>:The proof goes exactly like 1.6, just replace F by $F' = \{[n_1, n_2] \mid n_1 \in N_1, n_2 \in N_2 \text{ and } n_2 > N_1\}$ -|

This allows to describe the possible characters of the gap (R_1, R_2).

<u>1.10</u> <u>Proposition</u> For every regular infinite cardinal a,b: there are models $R_1 \ll R_2$ with char(R_1, R_2)=(a,b)

<u>proof</u>:Start with R_1 of cofinality a (by 1.5), now form an elongation $R_1 \ll R_3$ (1.4); finally iterate 1.9 b times. -|

~~Concluding this section, we would like to say a word on the~~ ~~cofinality of U-Prod(N) for a npuf U on N.~~

We recall the definition of a scale on ${}^\omega\omega$, the set of all functions from N to N. We first define a partial ordering $\prec$ on ${}^\omega\omega$

by:

$$f \prec g \text{ iff } \exists n \forall m \geq n\ (f(m) < g(m))$$

then for all cardinals k ($\leq 2^{\aleph_0}$), a k-scale is a set $S = \{f_\alpha \mid \alpha < k\}$ such that:

1) $f_\alpha \prec f_\beta$ for all $\alpha < \beta < k$

2) for every $g: N \to N$, there exists f_α in S such that $g \prec f_\alpha$

It is clear that for any npuf U on N, a k-scale determines a cofinal sequence in U-Prod(N), so that its cofinality is cof(k) independently of U. It is consistent that for any fixed k, there is a k-scale (cf. [J]).

The question is whether it is consistent that the cofinality depends on U. We show that it is consistent (with ZFC) that for all regular uncountable cardinals $a, b \leq 2^{\aleph_0}$, there is an ultrafilter U such that cof(U-Prod(N)) $= b$ and the coinitiality of the nonstandard part is a. The proof given here comes from ideas by A. Mekler; a different proof appears in [CA], but the theorem should be attributed to folklore.

First we note that it is sufficient to show there is a npuf U_1 with cofinality (of U_1-Prod(N)) equal to b, and a npuf U_2 with coinitiality (of the nonstandard part) equal to a; since we can form $^*N = U_2$-Prod(U_1-Prod(N)), which is isomorphic to a basic ultrapower with the required cofinality and coinitiality.

Since both constructions are similar, we concentrate on the coinitiality requirement.

1.11 ***Proposition*** Let $M \models ZFC$, a uncountable and regular in M, and $a \leq k$. Consider $P = Fn(k \times \omega, \omega)$ and G P-generic over M. Then in

$M[G]$, there is a npuf U with coinitiality equal to a.

__proof__: Let $S=\{b<k \mid a\leq b<k\}$, G' the restriction of G to $S\times w$, Ga the restriction of G to $a\times w$. Then $M[G]=M[G'][Ga]$; hence we are reduced to $P'=Fn(a\times w, w)$, Ga P'-generic over $M[G']$, and $cof(a)$ uncountable. So it suffices to prove the proposition with $k=a$, regular uncountable. Ga gives the sequence $\{g_\kappa \mid \kappa<a\}$.

We work in $M[G]$, and construct U as follows:

Let U_0 be any nonprincipal ultrafilter in $P(w)^M$. U_0, and in fact for all κ, U_κ will not need be in $M[G_\kappa]$, but the important thing is that all their set-elements are.

Given U_κ, an ultrafilter in $P(w)^{M[G_\kappa]}$, define:

$X:=\{\{n\in N \mid g_{\kappa+1}(n)<f(n)\} \mid f:N\rightarrow N$ is in $M[G]$, and f/U_κ is nonstandard$\}$

$Y:=\{\{n\in N \mid g_{\kappa+1}(n)>m\} \mid$ for each $m\in N\}$

Genericity of $g_{\kappa+1}$ over $M[G_\kappa]$ shows that $F:=U_\kappa\cup X\cup Y$ has the finite intersection property. Hence extend F to an ultrafilter $U_{\kappa+1}$ in $P(w)^{M[G_{\kappa+1}]}$.

At limit stages, if U_β, $\beta<\kappa$ is constructed, let $U'_\kappa=\bigcup_{\beta<\kappa}U_\beta$ and U_κ an ultafilter on $P(w)^{M[G_\kappa]}$ extending U'_κ.

Finally, let $U=\bigcup_{\kappa<a}U_\kappa$. U_κ is an ultrafilter since a is regular and uncountable. So it suffices to show that $\{g_\kappa \mid \kappa<a\}$ is coinitial in U-Prod(N)i. If f/U is nonstandard, then f is in $M[G_\kappa]$ for some κ (because a is regular uncountable and P is ccc). But f/U_κ is also nonstandard, so $g_{\kappa+1}/U<f/U$. $\dashv$

The reader interested in further results on countable ultraproducts is refered to [CA].

§2 Particular models and η_α-sets

In this section, we investigate properties of some particular models mostly related to η_α-sets. The results are taken from [HL].

2.1 Definitions

1) Let α be an ordinal. An ordered set $(X,<)$ is said to be an η_α-set provided that whenever $A<B$ are subsets of X of power less than $\aleph_\alpha$, then there are x_1, x_2, x_3 in X such that:

$x_1 < A < x_2 < B < x_3$

By a back-and-forth argument, Hausdorff has shown that any two η_α-sets of power $\aleph_\alpha$ are order-isomorphic. Such sets are said to be of order type η_α. Assuming ZFC only, there are $\eta_{\alpha+1}$-sets of power $2^{\aleph_\alpha}$ for all α.

2) $V({}^*R)$ is said to be ***comprehensive*** if every function $f:A\to{}^*B$ has an internal extension $f':{}^*A\to{}^*B$, where A and B are any elements of $V(R)$ (of the same type).

3) $V({}^*R)$ is ***weakly comprehensive*** if comprehensiveness holds in any case $A\subseteq R^n$ for any n in N and $B\subseteq R$ (that is for the elementary part of $V(R)$)

4) $V({}^*R)$ is ***sequentially comprehensive*** if $V({}^*R)$ satisfies weak comprehensiveness for $A=N$.

2.2 Proposition If $V({}^*R)$ is sequentially comprehensive, then ${}^*R$ is an η_1-set.

proof: Consider two countable subsets $A<B$ of ${}^*R$. We distinguish

four cases concerning the order structure of A and B.

If A has a last element a and B has a first element b, then $A \leq a < (a+b)/2 < b \leq B$ so that we found a "between" element in that case.

Suppose A has no last element but B has a first element b. Since A has countable cofinality, we can extract an increasing sequence $f: N \to A$ whose range is cofinal in A. If we consider f as a function with codomain $^*R$, we can apply sequential comprehensiveness to obtain an internal extension $f': {}^*N \to {}^*R$.

We now define $S := \{n \text{ in } {}^*N \mid \forall k \in {}^*N (k < n \to f'(k) < f'(n) < b)\}$. S is internal (see internal definition principle in Stroyan[LS]). Also, clearly $S \supseteq N$. But N is an external subset of $^*N$, hence there exists an n_0 in $^*N$i with $n_0 \in S$; this implies that $f'(k) < f'(n_0) <$ b for all k in N. Since the range of f is cofinal in A, we have:

$A < f'(n_0) < b \leq B$

This is again a "between" element.

The case where B has no first element while A has a last element is treated similarly.

If A has no last element and B has no first, we define a cofinal increasing sequence $f: N \to A$ and a coinitial decreasing sequence $g: N \to B$ with respective internal extensions f' and g'. We define S as:

$S = \{n \text{ in } {}^*N \mid \forall k \in {}^*N (k < n \to f'(k) < f'(n) < g'(k))\}$

and the result follows similarly. This proves condition for "between" element.

It only remains to show that ~~cof(*R) is uncountable.~~ Consider an increasing sequence $f:N \to {}^*R$ with internal extension $f':{}^*N \to {}^*R$. By an argument similar to the above, we find a $n_0 \in {}^*N$ with $f'(k) < f'(n_0)$ for all k in N, proving the assertion. -|

Now since any densely ordered set without endpoints is $\aleph_1$-saturated iff it is an η_1-set, we deduce that the order structure of any sequentially comprehensive *R is $\aleph_1$-saturated. It is not hard to see that any ultrapower model is comprehensive, hence sequentially comprehensive; in fact it is well known that any ultrapower is $\aleph_1$-saturated.

It follows from Hausdorff's theorem that the order structure of the hyperreal line is determined up to isomorphism in any sequentially comprehensive model of analysis *R of power $\aleph_1$. Assuming CH, this will be the case for any basic ultrapower model *R=U-Prod(R) for any nonprincipal ultrafilter U on N. Further Erdös et al have shown that for any ordinal $\kappa > 0$, any two real closed fields whose order-structures are η_κ-sets of power $\aleph_\kappa$ are isomorphic as ordered fields. In our situation, the next proposition follows also from uniqueness of saturated structures. We thus have the following:

2.3 Proposition (CH) For any non-principal ultrafilter U on N, *R=U-Prod(R) is unique up to isomorphism of ordered field. -|

We have seen that *R, Θ, and Mo are ~~not in general isomorphic, however we have the following:~~

2.4 Proposition *R is an η_κ-set iff Mo (thus any Mx) is an η_κ-set proof: The proof is straightforward via $x \leftrightarrow 1/x - 1$

2.5 Proposition $^*R$ is an η_α-set iff θ is an η_α-set

proof: We take the positive part of $^*R/Go$ as a typical set of order type θ. It is sufficient to show that $^*R+$ is an η_α-set iff $^*R+/Go$ is an η_α-set.

It is not hard to see that $^*R+/Go$ is an η_α-set if $^*R+$ is; so we concentrate on the converse.

Assume $^*R+/Go$ is an η_α-set. Consider two subsets $A<B$ of $^*R+$ of power less than $\aleph_\alpha$. Because $cof(^*R+)=coin(^*R+)=cof(\theta) \geq \aleph_\alpha$ (see 1.1), we can find x_1 and x_2 in $^*R+$ such that

$x_1 < A < B < x_2$

It remains to find a "between" element, we consider two cases:

1) Suppose first that $B-A := \{b-a \mid b \in B \text{ and } a \in A\}$ contains no infinitesimals. Choose w in $^*Ri$ such that $w\bar{a}$ is infinite for some $\bar{a}$ in A, and define:

$A' := \{a \text{ in } A \mid a \geq \bar{a}\}$.

We now jump in $^*R+/Go$ by forming:

$G_{A'} := \{ \xi_{wa} \mid a \in A'\}$, $G_B := \{ \xi_{wb} \mid b \in B \}$

Clearly we have:

$|G_{A'}| \leq |A'| \leq |A| < \aleph_\alpha$, $|G_B| \leq |B| < \aleph_\alpha$

Further by our choice of $\bar{a}$ and w, $G_{A'} > 0$ (in $^*R+/Go$). Now given any a in A', b in B, we have:

$wb-wa = w(b-a)$ is infinite, since $b-a$ is not infinitesimal, this shows that $0 < G_{A'} < G_B$. But $^*R+/Go$ is an η_α-set, hence $G_{A'} < G_r < G_B$ for some r. We then have $A < r/w < B$.

2) Suppose $b'-a'$ is infinitesimal for some a' in A and b' in B. Pick a nonzero positive infinitesimal i and define:

$$A' := \{a-a'+i \mid a\in A \text{ and } a\geq a'\}$$

$$B' := \{b-a'+i \mid b\in B \text{ and } b\leq b'\}$$

Hence $A'\subseteq Mo+$ (this is why we added i), $B'\subseteq Mo+$ and also $B'-A'\subseteq Mo+$. Further $|A'|\leq|A|<\aleph_\alpha$, $|B'|\leq|B|<\aleph_\alpha$ and hence $|B'-A'|\leq|B'||A'|<\aleph_\alpha$. By proposition 1.1, coin$(Mo+)$=cof$(G)\geq\aleph_\alpha$ by hypothesis, so we can find j in $Mo+$ with $B'-A'>j$ (ie $b-a>j$ for all b in B' and a in A'). Now to use ${}^*R+/Go$ we define:

$$G_{A'} := \{G_{1/ja} \mid a\in A'\} \quad G_{B'} := \{G_{1/jb} \mid b\in B'\}$$

It is clear that $G_{B'}>0$ since $B'\subseteq Mo+$. Now consider $a\in A'$, $b\in B'$; we have:

$$1/ja - 1/jb = (b-a)/jba > j/jba = 1/ba \text{ is infinite}$$

hence $0<G_{B'}<G_{A'}$ in ${}^*R+/Go$. Also $|G_{A'}|\leq|A'|\leq|A|<\aleph_\alpha$, $|G_{B'}|<\aleph_\alpha$ and hence $0<G_{B'}<G_r<G_{A'}$ for some r. It follows that

$$A<1/jr + a' - i<B$$

and the proof is complete. -|

Since Q is dense in R, we get from 2.5 :

2.6 <u>Corollary</u> ${}^*Q$ is an η_α-set iff θ is an η_α-set. -|

III. CHAPTER 3

Nonstandard Models of Arithmetic

Introduction

Let F denote the set of all functions $f:N \to N$. We shall be interested in $\Phi(F)=\{{}^*f \mid f \text{ is in } F\}$, the set of all standard functions for a monomorphism Φ. Note that $\Phi(F)={}^*F$, the set of internal functions, properly includes $\Phi(F)$ if ${}^*N$ properly includes N. We first investigate the stucture of ${}^*N$ using the extensions of standard functions. Then some applications of this on intersections of submodels of a given model will be given in section 2. Finally, in section 3 we develop some combinatorial results about ultrafilters on N.

§1 Functions in nonstandard arithmetic

We denote by F the set of all functions $f:N \to N$, by $F0$ the subset of F consisting of finite-to-one(hence unbounded) functions, and by $F1$ the class of functions which are both finite-to-one and monotonic (equivalently monotonic unbounded). We begin with some definitions, recall that ${}^*Ni={}^*N\backslash N$.

1.1 definitions (Puritz [PC1])

1) We define the exact range of an infinite number a by:

$$er(a) := \{{}^*f(a) \mid f \text{ is in } F\} \cap {}^*Ni$$

2) For a, b in $^*Ni$, we write $a \nearrow b$ (b is accessible from a), if $^*f(a) \geq b$ for some f in F. If not $a \nearrow b$, we write $a \ll b$ (ie $^*f(a) < b$ for all f in F; $a°b$ denotes $a \nearrow b$ and $b \nearrow a$

For a in $^*Ni$, let $sk(a) := \{x \text{ in } {}^*Ni \mid x°a\}$, called the sky of a.

3) We define $a \dashrightarrow b$ if $^*f(a)=b$ for some f in F, and $a \leftarrow\rightarrow b$ if both $a \dashrightarrow b$ and $b \dashrightarrow a$, a and b are said to be linked.

Put $con(a) := \{b \text{ in } {}^*Ni \mid a \leftarrow\rightarrow b\}$, called the constellation of a.

In the sequel we write f for $^*f$; it will be clear from the context whether we mean f or $^*f$.

__1.2 lemma__: ° is an equivalence relation on $^*Ni$.

__proof__: Reflexivity and symmetry are obvious. Suppose $a°b$ and $b°c$, then $f(a) \geq b$ and $g(b) \geq c$ for some f, g in F. It is easy to construct f1 and g1 in F1 dominating f and g for all n. Hence we have: $g1 \cdot f1(a) \geq g1 \cdot f(a) \geq g1(b) \geq g(b) \geq c$ so that $a \nearrow c$. Similarly $c \nearrow a$ hence $a°c$. -|

__1.3 Proposition__:

1) $\dashrightarrow$ is reflexive and transitive

2) $a \dashrightarrow b$ iff $er(a) \supseteq er(b)$ (so $a \leftarrow\rightarrow b$ iff $er(a)=er(b)$)

3) $a \leftarrow\rightarrow b$ iff $f(a)=b$ for some 1-1 function f

4) $\leftarrow\rightarrow$ is an equivalence relation

__proof__: All is obvious except maybe first implication of 3). So suppose $a \leftrightarrow b$, ie $f(a)=b$ and $g(b)=a$ for some f, g. Let $S=\{n$ in $N \mid g \cdot f(n)=n\}$. Then a is in $*S$ and f is 1-1 on S. We may assume $S=S1 \cup S2$ are all infinite, $S1 \cap S2=\emptyset$ and a is in $*S1$. Just redefine f on $N \setminus S1$ so that f is 1-1 everywhere. -|

<u>1.4</u> <u>Lemma</u> $f(a)$ is infinite for all infinite number a iff f is in FO (finite-to-one)

<u>proof</u>:Let f be finite-to-one, a be infinite and n be in N; we want to show that $f(a)$ is infinite ie $f(a)>n$ or equivalently $a \in \{k$ in $*N \mid f(k)>n\}$. Since f is in FO, $\exists m(k>m$ implies $f(k)>n)$ is true in N, hence in $*N$. But since n is standard, m may be chosen standard. So $a \in \{k$ in $*N \mid k>m\} \subseteq \{k$ in $*N \mid f(k)>n\}$.

Conversely, suppose f is not finite-to-one, ie constant on some infinite set $S \subseteq N$;say $f(S)=n$. We have $\forall k(k \in S$ implies $f(k)=n)$. Since S is infinite, $*S$ contains an infinite number a and hence $f(a)=n$ is not infinite. -|

<u>1.5</u> <u>Lemma</u> $f(a)°a$ for all infinite a iff f is finite-to-one

<u>proof</u>:Assume f is finite-to-one, we need only show $f(a) \nearrow a$.

Define $f1$ as follows:

for all n N, $f1(n)$=largest m for which $f(m) \leq n$

$f1$ is well defined since $f \in FO$. Moreover $f1 \cdot f(n) \geq n$ for all n.

Hence $f1 \cdot f(a) \geq a$ and $f(a) \nearrow a$.

Conversely, if f is not finite-to-one, then by 1.4 pick a infinite with $f(a)$ finite, then clearly $f(a) \ll a$. -|

<u>1.6</u> <u>Proposition</u> (Puritz [PC1]) Let a in $*N$ be infinite, then:

1) $er^1(a):=\{f(a) \mid f$ is in $F1\}$ and $sk(a)$ are coinitial and cofinal

2) No countable set is either coinitial or cofinal in $sk(a)$

__proof__: 1) By lemma 1.5, $f1(a)^{\circ}a$ for all $f1$ in ~~F1~~, hence $er1(a) \subseteq sk(a)$. We have to show that given b in $sk(a)$, there are $f1$, $q1$ in $F1$ with $f1(a) \leq b \leq q1(a)$. Since $a \urcorner b$, $q(a) \gg b$ for some q in F. It is easy to construct $q1$ in $F1$ that dominates q for all n, hence $q1(a) \gg b$. On the other side, $b \urcorner a$, so let $f(b) \gg a$ and assume f is already chosen from $F1$. Define $f1$ in $F1$ as follows, a kind of inverse of f:

For all n in N, $f1(n) :=$ smallest m with $f(m) > n$

Then $f1 \in F1$, and $f1(a)$ is the smallest b' with $f(b') > a$; hence $f1(a) \leq b$.

2) Let S be a countable subset of $sk(a)$ and suppose S has a subset $S1 = \{a(n) \mid n \in N\}$ with $a(0) > a(1) > a(2) > \ldots$ (if no such set exists, S is obviously not coinitial in $sk(a)$) Let $q \in F1$ such that $q_n(a(n)) \gg a$ $n = 0, 1, 2, \ldots$ and define f in $F1$ as follows:

For all n, $f(n) := \max\{ q_k(l) \mid k, l \leq n\}$

then $(\forall m)(\forall n \geq m)$ $f(n) > q_m(n)$. Let $f1$ the "inverse" of f as defined in 1); then since $f(a(m)) \gg q_m(a(m)) \gg a$, we have $f1(a) < a(m)$ for all m. $f1(a)$ is in $sk(a)$ by 1.5, so S is not coinitial in $sk(a)$. The rest of the proof is straightforward. -|

__1.7__ __Lemma__ Let a be infinite and $A \subseteq sk(a)$ be a bounded subset of $sk(a)$ (ie $x < A < y$ for some x, y in $sk(a)$) Then there is a function f in $F1$ and $b^{\circ}a$ such that $f\{A\} = \{b\}$

__remark__: By proposition 1.6, any countable subset of any sky is bounded

__proof__: Since A is bounded in $sk(a)$, we can find f, q in $F1$ such that $q(a) < A < f(a)$. Hence $q(a)^{\circ}a^{\circ}f(a)$ and there is an r in $F1$, which we can assume strictly increasing such that $r\cdot q(a) \geq f(a)$. Now define $k(1)=0$, $k(2)=r(k(1))$, $k(3)=r(k(2))$, ... ;we have $k(1) < k(2) < k(3) < \dots$ Let m be the largest integer of $*N$ such that $k(m) \leq g(a)$, then $k(m+1) > q(a)$ and

$k(m+2)=r(k(m+1)) > r(q(a)) > f(a) > A > q(a) \geq k(m)$

so we define:

1) if m is odd $D(0)=[0,\ k(1)-1]$, $D(l)=[k(2m-1),\ k(2m+1)-1]$ for $l>0$

2) if m is even $D(l)=[k(2m),\ k(2m+2)-1]$

and in either case, $A \subseteq [k(m),\ k(m+2)-1] = D(l)$ for some l. Now we need only define $f(n)=p$ for all n in $D(p)$, then $f(A)=l$ a constant. Note that f is in F1 so that $f(a(n))\,{}^{\circ}a(n)$ by 1.5. This completes the proof-|

A negative instance of the above is with a model $*N$ equipped with a highest sky $sk(a)$ and $A=$ a, (, which is unbounded in $sk(a)$. If a standard function is constant on A, this constant has to be finite.

Let M be any nonstandard model of arithmetic. As arithmetic contains Skolem functions for all formulas, every submodel is an elementary submodel. Thus if a M. the set:

$N \cup er(a)=\{f(a) \mid f:N \to N\}$

is the universe of an elementary submodel of M. Such submodels, generated by a single element will be called principal. In fact, we shall see now that for a infinite in M, this principal model is isomorphpic to a basic ultrapower Ua-Prod(N) for a suitable non-principal ultrafilter (npuf from now on) Ua on N.

The following can be found in [PC1] and [CH].

1.8 Proposition Let M be any nonstandard model of arithmetic and a in M be infinite. Put $Ua=\{S \subseteq N \mid a \in {}^*S\}$. Then Ua is a npuf and

$$N \cup er(a) \cong Ua\text{-}Prod(N)$$

proof: It is straightforward to check that Ua is a npuf on N. For example Ua is non-principal, for let $S(n)=\{n+1, n+2, \dots\}$ then $a \in {}^*S(n)$ for all n, but $\bigcap_n S(n)=\emptyset$.

Now we define:

$$j: Ua\text{-}Prod(N) \dashrightarrow N \cup er(a)$$

$$: f/Ua \dashrightarrow {}^*f(a)$$

j is well-defined and 1-1 for:

$${}^*f(a)={}^*g(a) \text{ iff } a \in \{n \in {}^*N \mid {}^*f(n)={}^*g(n)\}$$

$$\text{iff } a \in {}^*\{n \in N \mid f(n)=g(n)\}$$

$$\text{iff } \{n \in N \mid f(n)=g(n)\} \in Ua$$

$$\text{iff } f/Ua=g/Ua$$

By its definition, j is onto. Further for any formula $\phi(x_1, \dots, x_k)$ and $f_1/Ua, \dots, f_k/Ua$ in $Ua\text{-}Prod(N)$,

$$Ua\text{-}Prod(N) \models \phi(f_1/Ua, \dots, f_k/Ua)$$

$$\text{iff } \{n \mid N \models \phi(f_1(n), \dots, f_k(n))\} \in Ua$$

$$\text{iff } a \in {}^*\{n \mid N \models \phi(f_1(n), \dots, f_k(n))\}$$

$$\text{iff } N \cup er(a) \models \phi(j(f_1/Ua), \dots, j(f_k/Ua)) \dashv$$

§2 Intersection of nonstandard models

By proposition 1.8, a model generated by a single element is just a basic ultrapower. Remember that assuming CH, all basic ultrapowers are isomorphic with $\theta=\aleph_1$.

A model $\prec M$ generated by a single element a of a given nonstandard model M will be denoted by M(a). Recall that:

M(a)={f(a) | f:N-->N} (in M)

From now on, we fix a nonstandard model of arithmetic M.

2.1 Proposition ~~Let a, b∈M; then~~

M(a) and M(b) are cofinal iff a°b

proof: Suppose that M(a) and M(b) are cofinal, then there is a c in M(a) with c≥b; but c=f(a) for some f in F and hence a $\nearrow$ b. Similarly b $\nearrow$ a so a°b.

Conversely, suppose a°b and consider c=f(a) in M(a). We search for d in M(b) with d>c. Construct f1 monotonic dominating f for all n. Since b $\nearrow$ a, g(b)>a for some g and hence

f1·g(b)≥f1(a)≥f(a)=c

but f1·g(b)∈M(b), hence M(b) is cofinal in M(a). Similarly, M(a) is cofinal in M(b) and the proof is complete. -|

With the basic ultrapower picture at hand, the last proposition is quite natural since for every principal model M(x), x corresponds to id/Ux of Ux-Prod(N) which is, as we have seen, in the highest sky.

Here is another characterization of cofinality:

2.2 Proposition (Blass [B1]) Let c∈M and f∈F, then:

$\mathbb{N}(c)$ and $\mathbb{N}(f(c))$ are cofinal iff there is a set $S \subseteq \mathbb{N}$ such that $c \in {}^*S$ and $f \restriction S$ is finite-to-one.

Further if these equivalent conditions hold, there exists $f' \in FO$ with $f(c)=f'(c)$.

proof: Suppose first that we have a set $S \subseteq \mathbb{N}$ with $f \restriction S$ finite-to-one and $c \in {}^*S$. Define f' in FO as follows:

$$f'(n)=f(n) \text{ if } n \in S, \text{ or } f'(n)=n \text{ if } n \notin S$$

Since $\forall n(n \in S \rightarrow f(n)=f'(n))$ is true in $\mathbb{N}$, it is true in M, so $f(c)=f'(c)$ and this proves the last assertion. But now $c^{o}f'(c)$ by 1.5, hence $\mathbb{N}(c)$ and $\mathbb{N}(f'(c))=\mathbb{N}(f(c))$ are cofinal by 2.1.

Conversely, suppose $\mathbb{N}(c)$ and $\mathbb{N}(f(c))$ are cofinal. Then $\mathbb{N}(f(c))$ has an element, say $h \cdot f(c)$ with $h \cdot f(c) \geq c$. Put $S=\{n \mid n \leq h \cdot f(n)\}$, then $c \in {}^*S$. Also for n in S and m in $\mathbb{N}$ arbitrary, $f(n)=m \rightarrow n \leq h(m)$ so that $f \restriction S$ takes the value m at most $h(m)+1$ times; which completes the proof. -|

2.3 Corollary Let $c \in M$ and $f \in F$, then:

$c^{o}f(c)$ iff there is a set S $\mathbb{N}$ such that $c \in {}^*S$ and $f \restriction S$ is finite-to-one. -|

2.4 Proposition Let $\{ {}^*\mathbb{N}_i \mid i \in I\}$ be submodels of a given model M. Suppose there are infinite $a^{o}b$ in M such that ${}^*\mathbb{N}_i \cap [a, b] \neq \emptyset$ for all i.

Then $\bigcap_i {}^*\mathbb{N}_i$ is nonstandard; in fact $\bigcap_i {}^*\mathbb{N}_i$ contains a principal model.

proof: Let $a_i \in {}^*\mathbb{N}_i \cap [a, b]$. By lemma 1.7, there is an f in F such that $f(a_i)=f(a_j)^{o}a$ for all $i,j \in I$. If we let $c=f(a_i)$, then:

$\mathbf{N}(c) \subseteq \bigcap_i {}^*\mathbf{N}_i$ is nonstandard. -|

The corollary appears in Blass [B1] , but we replace the countability of I. In fact , if I is countable, condition 1 implies condition 2.

<u>2.5</u> <u>Corollary</u> Let $\{{}^*\mathbf{N}_i \mid i \in I\}$ be pairwise cofinal submodels of $\mathbf{N}$ and suppose that:

1) at least one of ${}^*\mathbf{N}_i$ (hence all ${}^*\mathbf{N}_i$) has a highest sky, sk(c) say,

2) there are a°b in sk(c) such that ${}^*\mathbf{N}_i \cap [a, b] \neq \emptyset$ for all i,

then $\bigcap_i {}^*\mathbf{N}_i$ is cofinal with each ${}^*\mathbf{N}_i$; in fact $\bigcap_i {}^*\mathbf{N}_i$ contains a principal model cofinal with each ${}^*\mathbf{N}_i$.

Further if I is countable, we may drop condition 2).

<u>proof</u>:The proof is straightforward by 2.4. If I is countable, then by 1.6 condition 2) is always satisfied. -|

The next lemma is analogous to Corollary 2.3, the proof is left as an exercise.

<u>2.6</u> <u>Lemma</u> Let $c \in \mathbf{N}$ and $f \in \mathbf{F}$, then:

$f(c) \longleftrightarrow c$ iff there is a set $S \subseteq \mathbf{N}$ with $c \in {}^*S$ and $f \restriction S$ is one-one -|

Of course, $a \longleftrightarrow b$ iff $\mathbf{N}(a) = \mathbf{N}(b)$.

We conclude this section with a proposition on descending chains of principal models, due to Cherlin and Hirschfeld [CH] .

<u>2.7</u> <u>Proposition</u> Suppose $\mathbf{N}(a_0) \supsetneq \mathbf{N}(a_1) \supsetneq \mathbf{N}(a_2) \supsetneq \dots$ is a strictly descending chain where all a_i are infinite; then there is an infinite b with

$$\mathbf{N}(b) \subseteq \bigcap_{0}^{\infty} \mathbf{N}(a_i)$$

<u>proof</u>:We first note that we may assume $a_{i+1} < a_i$ for all i. Indeed,

since $f_i(a_i)=a_{i+1}$ for some f_i, put:

$k_i(x) := \min\{y \mid f_i(y)=x\}$, and 0 otherwise.

then $k_i(x) \longleftrightarrow x$ (use lemma 2.6 with $S=f_i(\mathbb{N})$) whenever $f_i(y)=x$ for some y; hence $\mathbb{N}(k_i(a_{i+1}))=\mathbb{N}(a_{i+1})$ and $k_i(a_{i+1})<a_i$; so just replace a_{i+1} by $k_i(a_{i+1})$.

Now let $f_n(a_n)=a_{n+1}$, $a_{n+1}<a_n$ and put

$q_n := f_n \cdot f_{n-1} \cdot f_{n-2} \cdot \ldots\ f_1$

so that $q_n(a_0)=a_n$, and define

$h(x) := \min\{q_n(x) \mid q_n(x) \geq n$ and $\forall i \leq n\ q_i(x) > q_{i+1}(x)\}$

(ie we look at the sequence $\{q_n(x)\}$ and take the last element up to where the sequence decreases or at the last that is still bigger than its index)

Then $h(a_0)$ is infinite, for $q_n(a_0) > n$ for all $n \in \mathbb{N}$ and $\{q_n(a_0)\}$ is decreasing by assumption. So let $h(a_0)=b$. We show that $b \in \bigcap_0^\infty \mathbb{N}(a_i)$

Fix j and define:

$q'_n(x) := x$ if $n \leq j$, $f_n \cdot f_{n-1} \cdot \ldots \cdot f_{j+1}(x)$ if $n > j$

and put:

$h'(x) := \min\{q'_n(x) \mid q'_n(x) \geq n$ and $\forall j < k \leq n\ q'_k(x) > q'_{k+1}(x)\}$

Since the following is true in $\mathbb{N}$:

$(y=q_j(x)$ and $q_j(x) > j$ and $x > q_1(x)$ and $\ldots$ and $q_j(x) > q_{j+1}(x)) \longrightarrow$

$(h(x)=h'(y))$

and the left side is true in $\mathbb{N}$ for a, we have:

$b=h(a_0)=h'(q_j(a_0))=h'(a_j)\ \mathbb{N}(a_j)$; this completes the proof.

$\dashv$

§3 Skies of basic Ultrapowers

We know that any basic ultrapower U-Prod(N) has a highest sky, the sky of the identity mapping id.

However, the number of skies may depend on U. Further we may ask about the structure of a constellation inside a sky. In fact, this gives us some characterization of ultrafilters. The results are from Puritz [PC1] and [PC2].

3.1 Definition

Let be any npuf on N, and a, $b \leq \aleph_1$ cardinal numbers (possibly finite). Let P be a partition of N into countably many disjoint sets D_m, $m \in N$, some of which may be empty.

U is said to be b-sparse with respect to P if there is a set S in U such that $|S \cap D_m| < b$ for all m. U is said to be ab-sparse if for every partition P satisfying $|D_m| < a$ for all m, either $D_m \in U$ for some m, or U is b-sparse with respect to P.

We denote by S(ab) the set of all ab-sparse npuf on N.

Definition

A npuf U on N is called δ-stable, or a P-point, if every function on N is finite-to-one or constant on some set in U.

3.2 Proposition Let U be a npuf on N, TFAE:

1) U is δ-stable

2) U is $\aleph_1\aleph_0$-sparse

3) *N=U-Prod(N) has only one sky

proof: 1) --> 2) Given a partition $P = \{D_m\}$ of N, consider the function $f(n) = m$ for all $n \in D_m$, $m = 0, 1, 2, \ldots$ Then f is constant on some set in U means that $D_m \in U$ for some m, and finite-to-one on some set S in U means that $|S \cap D_m| < \aleph_0$ for all m. Hence U is $\aleph_1\aleph_0$-sparse.

2) --> 3) $^*N$ has only one sky means that $a \nearrow id$ for every infinite a. For each $m \in N$, let $D_m := \{n \mid a_n = m\}$ and put $Pa = \{D_m\}$. Pa is a partition of N. If $D \in U$ for some m, then a is finite. Otherwise there must be a set S U such that $S \cap D_m$ is finite for all m. Define :

$f(m)$:=largest number in $S \cap D_m$ if $\neq \emptyset$, and 1 otherwise.

Since $S \subseteq \{n \mid f(a_n) \geq n\}$, $f(a) \geq id$ and $a \nearrow id$ so $^*N = N \cup sk(id)$.

3) --> 1) Let f be any function on N. If f/U is finite, then f is constant on some U-set. If f/U is infinite, then using 3), $g(f/U) \geq id$ for some g in F. Hence $S = \{n \mid g \cdot f(n) \geq n\} \in U$ and f is finite-to-one on S. -|

<u>Definition</u>

A npuf U on N is called rare, or a Q-point, if every finite-to-one function on N is one-to-one on some set in U.

<u>3.3</u> <u>Proposition</u> Let U be a npuf on N, TFAE:

1) U is rare

2) U is $\aleph_0 2$ -sparse

3) The highest sky of $^*N$=U-Prod(N) is a single constellation

<u>proof</u>:1) --> 2) Let $P = \{D_m\}$ be a partition of N with D finite for all m. We define:

$f(n) = m$ for all $n \in D_m$

then f is finite-to-one, hence one-to-one on some set S in U; hence $|S \cap D_m| < 2$ for all m in N.

2) ⟶ 3) Let $a \in sk(id)$ and consider again, for each $m \in N$, $D_m := \{n \mid a_n = m\}$. Since $f(a) \not> id$ for some f in F ie $S := \{n \mid f(a_n) \not> n\} \in U$ then $D'_m = S \cap D_m$ is finite for all m. If we partition $N \setminus S$ into singletons, we get a partition:

$$P' := \{D'_m\} \cup \{\text{singletons of } N \setminus S\}$$

which consists of finite sets only.

Assuming 2), there is a U-set S' which meets every set in P' in at most one point. Let $S'' := S' \cap S$ and define $f1$ in F by:

$f1(n) :=$ the number in $S'' \cap D'$ if nonempty, 1 otherwise

then for all n in S'' (and for U-almost all n) $f1(a_n) = n$, so that $f1(a) = id$. But $a \in er(id)$, so $a \leftrightarrow id$ for all $a \in sk(id)$.

3) ⟶ 1) Consider a finite-to-one function $a(n)$, and form $D_m := \{n \mid a(n) = m\}$. Define:

$f(m) := \max\{n \mid n \in D_m\}$ or 1 if empty

then $f(a) \not> id$ so $a°id$. By assumption $g(a) = id$ for some g. Hence $\{n \mid g \cdot a(n) = n\} \in U$, hence a is one-to-one on some U-set. -|

<u>Definition</u>

A npuf U on N is called selective, minimal, Ramsey or absolute, if every function on N is one-to-one or constant on some U-set.

<u>3.4</u> <u>Proposition</u> Let U be any npuf on N; TFAE:

1) U is selective

2) U is $\aleph_1$ 2-sparse

3) $*Ni = U\text{-Prod}(N)i$ is a single constellation

<u>proof</u>: Follows from 3.2 3.3 since $S(\aleph_1 2) = S(\aleph_1, \aleph_0)$ $S(\aleph_2)$ -|

<u>Definition</u>

A npuf $\tilde{U}$ on N is called rapid if for each function f on N, there is a U-set whose n^{th} element is $\geq f(n)$

<u>3.5 Proposition</u> Let U be any npuf on N, TFAE:

1) U is rapid

2) con(id) is coinitial in sk(id)

<u>proof</u>: 1) --> 2) Suppose U is rapid. Consider a/U in sk(id) and f such that $f(a) \geq id$, ie $S := \{n \mid f \cdot a(n) \geq n\} \in U$. By assumption there is a U-set $t = \{t_1, t_2, t_3, \ldots\}$ such that $t_n \geq f(n)$ for all n. Hence for all n in S:

$$t \cdot a(n) \geq f \cdot a(n) \geq n$$

Since t, as a functio on N, is 1-1 increasing, we may define:

$\bar{t}^{-1}(m) = n$ where n is determined by $t(n) \leq m < t(n+1)$

then $\forall n\ \bar{t}^{-1} \cdot t(n) = n$, and $\bar{t}^{-1}$ is monotonic. Hence $a(n) \geq \bar{t}^{-1}(n)$ for all n in S so that $a/U \geq \bar{t}^{-1}/U$.

But for all n in $t \cap S$, $n = t_k$ for some k and :

$t \cdot \bar{t}^{-1}(n) = t \cdot \bar{t}^{-1} \cdot t(k) = t(k) = n$ ie $t(\bar{t}^{-1}/U) = id$; hence $t \longleftrightarrow id$.

2) --> 1) Consider any f in F. It is easy to majorize f by a 1-1 strictly increasing function h. We show 1) holds for h, hence for f.

First define $\bar{h}^{-1}$ as follows:

$\bar{h}^{-1}(n) := m$ determined by $h(m) \leq n < h(m+1)$, 0 for $n < h(0)$

Clearly $\bar{h}^{-1} \circ id \circ h$. By hypothesis, there are functions q and k with $q < \bar{h}^{-1}$, ie $S := \{n \mid q(n) < \bar{h}^{-1}(n)\} \in U$; and $T := \{n \mid k \cdot q(n) = h(n)\} \in U$. Let $R := S \cap T$. $R \in U$, $R = \{r_1, r_2, r_3, \ldots\}$. A picture will help to see that $r_n \geq h(n)$ for all n. -|

The existence of the previous npuf is proved using CH. Further S. Shelah proved the existence of P-points is independent of ZFC.

I would like to summarize some possible structure of the set of skies for a basic ultrapower. Assuming CH, there are basic ultrapowers with n skies for every n in N, and we may prescribe which skies will constitute a single constellation. Irrespective of CH, there are basic ultrapowers with skies. The interested reader is referred to [PC1] for more details.

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