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NEUTRINO MASSES WITH TRIPLET LEPTONS IN THE

GLASHOW-WEINBERG-SALAM ELECTROWEAK THEORY

by

Ho Fan Jang

B.Sc.(Hons.), Simon Fraser University, 1980

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

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ABSTRACT

A study of neutrino masses and mixings in the Glashow-Weinberg-Salam (GWS) SU(2)xU(1) electroweak theory with the addition of heavy lepton triplet fields is made. The general framework of the extended model, which consists of F-families of new triplets and N-families of doublets, is presented. Contrary to other extended GWS models, this model will retain the standard relation between the masses of the charged and neutral gauge bosons, and also give lepton-number-violating processes. Significant phenomena, which cannot occur in the minimal GWS model, are considered in detail: neutrino oscillations, neutrinoless double beta decays, radiative decays of massive neutrinos and charged leptons, and their anomalous magnetic moments.

Numerical results are provided in various cases with reasonable assumptions that the mass of the light neutrino is 100eV and that of the new lepton is 20GeV. It is concluded that no known experimental limit is violated with this newly-added triplet; and the new massive leptons are unstable and decay with a lifetime between that of the muon and tau.

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I. Chapter 1 Introduction

The Neutrino, an elementary particle produced in beta decay, has one fundamental and important property which has not been understood: namely, its mass. Not only are we unable to determine the neutrino mass exactly, we cannot even conclude whether it has a mass or not. All the experimental evidence indicates that the rest mass of the neutrino should be vanishingly small even if it is nonzero. In calculating classical β-decay, it is justifiable to neglect its mass. But apart from the fundamental question, the mass of the neutrino has many important implications in modern theories of different fields[1,2].

Experimentally, some indications of a non-zero mass have been found by Lyubimov[3] (1980) though not yet confirmed. Theoretically, Salam, Lee and Yang, and Landau (1957) considered the masslessness of the neutrino as a consequence of global γ_5 invariance. Recently, a greater understanding in the field of elementary particle physics has been made with the recognition of "Gauge theories"[4]. Local gauge invariance is now believed to be the underlying principle for describing strong, weak and electromagnetic interactions. Unlike the case of the photon, for which both the masslessness of the photon and charge conservation are consequence of local gauge invariance of Maxwell's equations, there is no corresponding gauge symmetry to ensure the masslessness of the neutrino. Similarly, the conservation of lepton and baryon numbers is not supported by any local group symmetry.

It has been experimentally demonstrated that there are different neutrinos associated with different charged leptons in weak decays. Naturally, we would like to find out what properties and characteristics differentiate these neutral leptons. Could it be their masses?

A non-vanishing mass for the neutrino may lead to the possibility of a Cabibbo-like mixing for neutrino decays or oscillations. Neutrino oscillations, proposed long ago by Pontecorvo[5] (1967) and Maki[6] (1962), if they exist, may provide a natural explanation for the solar neutrino "puzzle"[1,7].

The importance of the neutrino mass problem is not restricted to particle physics. If the neutrino is found to have a mass of order 10eV , this may have very significant implications in both cosmology and astrophysics[2] ; for instance (i) the mean energy density of the universe, (ii) the constitution of galactic halos, and (iii) the formation of galaxies.

In 1960's Weinberg, Salam and Glashow[8] (GWS) successfully constructed a renormalizable theory which unifies electromagnetic and weak interactions. The theory was based on the invariance under the gauge group SU(2)xU(1). In their model, all left-handed fermions transform according to doublet representations, while right-handed charged fermions are singlets. In the minimal model, the existence of the right-handed neutrino is not assumed because the right-handed current has not been observed yet; and only Higgs doublets are assumed. Consequently, only the charged leptons can acquire masses after spontaneous symmetry breakdown[4] (SSB) but not the neutrino. Also lepton number is conserved. Essentially experimental results[9-13] in the low energy domain are known to be in approximate agreement with the minimal GWS model, but they do not rule out the possibilities that the neutrino has a non-zero mass or that lepton-number-violating processes do exist.

Recently, some theories which unify strong, weak and electromagnetic interactions have shown that baryon and lepton numbers are generally not conserved, and the neutrino is likely to have a non-zero mass[14,15]. For instance, the grand-unified-theory based on SO(10) constructed by H. Georgi (1974) can allow the neutrino to acquire a mass in the range $10^{0\pm2}$ eV[16].

A lot of interest in the massive neutrino has been stimulated by grand-unified-theory considerations. We believe that it is still very useful to investigate the possibility of having massive neutrinos in the SU(2)xU(1) model. Since all grand-unified models contain the electroweak theory, it will be easier to study extensions of the minimal SU(2)xU(1) model directly.

Obviously, the extensions would mean the addition of Higgs scalar and/or fermion fields to the theory. In the past few years, some work has been done in generating a mass for the neutrino in the GWS electroweak theory[17-20]. For instance, right-handed neutrinos in singlet representation, a Higgs triplet or both are added in order to generate a neutrino mass. The extra right-handed components can allow the neutrino to acquire Dirac and Majorana masses, but since there are no right-handed charged currents in the minimal model, the lepton number is still conserved. On the other hand, the extra Higgs triplet can only allow the neutrino to acquire a Majorana mass and lepton-number-violating processes are possible. However, the predicted relation $\left(\rho = \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = 1\right)^{\ddagger}$ between the masses of the charges boson W and the neutral boson Z, which has already been experimentally verified, must be altered (see Chapter 3).

From purely theortical considerations, we try to construct an alternative theory which will retain $\rho = 1$ and also give lepton-number-violating processes. We find that if real lepton triplets are added to the minimal SU(2)xU(1) theory, these requirements will be satisfied. In this thesis, we present the structure of this modified model and study the significant phenomena which may arise from the theory.

In Chapter 2, we review the difficulties with weak interaction phenomenology in its early period. We then briefly introduce the gauge theory and the Higgs mechanism which are the two main ingredients in constructing a successful electroweak theory. The SU(2)xU(1) electroweak theory for leptons

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$\ddagger$ The parameter $\sin^2\theta_W$ describes the amount of mixing between the SU(2) and U(1) gauge boson (see Chapter2) constructed by GWS in 1967 is reviewed in some detailed. Finally, the present experimental data for the weak interactions are discussed briefly.

In Chapter 3 we discuss neutrino masses of the Majorana type, the Dirac type, and the mixed-type . We also review some modified GWS models which will allow neutrinos to acquire masses of various types. The generalized non-diagonal mass matrices for N-families of leptons are considered. The generalized Cabibbo-type mixing angles and CP violating phases are discussed as well.

In Chapter 4 we present our modified model. We restate our motivation to construct such a model with more detail. The general frame-work, which consists of F-families of triplets and N-families of doublets, will be constructed. For simplicity, we restrict our calculations only to one triplet and the three well-known families of doublets : electron, muon and tau.

The subsequent Chapter 5-9 are devoted to various phenomena which arise if the new triplet is added.

Being massive particles, neutrinos of different families will mix through Cabibbo rotations. In Chapter 5 we study the most interesting phenomenon : neutrino oscillations. A beam of neutrinos produced through weak interactions can oscillate in vacuum into neutrinos of a different family. We also show that the Kurie plot for the β-decay of tritium will also depend on the mixing angles and masses of all the neutrino mass eigenstates which couple to the electron. In Chapter 6 we investigate the possibility of neutrinoless double β-decay in our model. In Chapter 7 we calculate the decay rates of the new leptons in the lowest order. In Chapter 8 we calculate the radiative decays of heavy neutrinos; we also consider the magnetic moments of the Majorana neutrinos. In Chapter 9 we calculate the radiative decays of charged leptons and we also consider the contributions of the new heavy leptons to the anomalous magnetic moments of the light charged leptons. All the calculations in Chapters 8 and 9 are based on the existing formulations provided by Lee and Shrock (1977) and the summary of the relevant results can be found in appendix F.

Numerical results will be provided for various phenomena. Since these triplet fields have not yet been observed experimentally, we conclude that the masses of such leptons must be heavy and are likely heavier than 20GeV [46]. Therefore, all the numerical values calculated are based on the assumption that the masses of such heavy leptons are 20GeV.

Finally, Chapter 10 contains our conclusions.

II. Chapter 2 Unification of Electromagnetic and Weak Interactions

2.1 The Weak Interaction‡

The weak interaction, known from the process of nuclear beta decay, was observed during the early period of nuclear physics. However, little progress on its understanding was made until 1933. Pauli then proposed a new neutral particle called the neutrino, with spin-1/2 and very low or even zero mass. In 1934, Fermi, based on the hypothetical existence of the neutrino, postulated an effective four-fermion point-like interaction with effective coupling constant G_F at low energy.

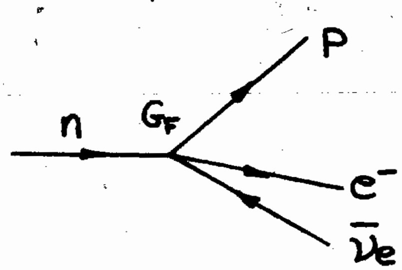

Fig.2.1 Four-fermion interaction for neutron β-decay

But the point-like interaction faced a great difficulty because the theory violated the unitarity bound for processes

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‡The material in this section can be found with details in Aitchison and Hey[2](1982)

such as $\nu_e e^- \rightarrow \nu_e e^-$.

Later, the massive Intermediate Vector Boson (IVB) $W^{\pm}$ was proposed in the hope that the unitarity disease would be cured. The exchange of an IVB in weak interaction just imitate the exchange of a photon in Quantum Electrodynamics (QED).

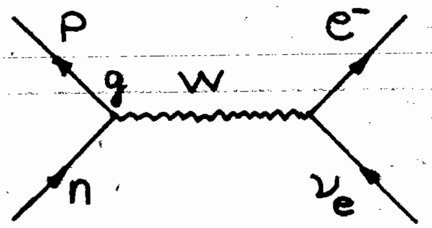

Fig.2.2 IVB exchange in weak interaction

But different unitarity-violating processes appeared, such as those processes involving external W particles. (This violation is understood to be due to the longitudinal polarization states.) IVB theory simply puts the problems into higher energy processes.

The four-fermion interaction and IVB models have a related disease: nonrenormalizability. There is a simple criterion that if the coupling constant has a dimension of $[mass]^n$ where $n<0$, the theory is nonrenormalizable; whereas, if $n>0$, the theory has fewer divergence than QED; and if the coupling constant is dimensionless, further detailed investigation is necessary. The Fermi coupling constant G_F has the dimension of $[M]^{-2}$; therefore, the theory is nonrenormalizable. But how about the IVB model which has a dimensionless coupling constant as in QED? It turns out that 'dimensionless' alone is not enough. The difficulty again comes from the longitudinal polarization states which produce a nonrenormalizable ultraviolet divergence as in unitarity-violating process.

Searching for a clue to cure the problems, we further look at QED which has no such disease. The longitudinal states of polarization for real photons can always be transformed away, and there exist cancellation mechanisms within the theory so that the contributions of the longitudinal states of virtual photons do not cause bad high-energy behaviour. In fact, these are the properties of the assumption that QED is an abelian gauge invariant theory. Gauge invariance seems to play an important role in ensuring renormalizability.

The problem now is to construct a gauge invariant weak interaction theory in the hope that both unitarity violations and nonrenormalizability may be cured. In 1954, Yang and Mills constructed a mathematical framework to generalize an abelian gauge theory to a non-abelian one. For the case of SU(2), there exist three massless vector gauge bosons in the theory; while, there is only one gauge boson in QED, the photon. At that time, it was not known whether any of the interactions observed in nature could be described by a non-abelian gauge theory. For instance, the weak interaction is mediated by the exchange of massive vector bosons. But if we want to retain the property of gauge invariance, the vector gauge bosons must remain massless.

The dilemma was finally resolved after 10 years by introducing a new mechanism into the theory: spontaneous symmetry breakdown -- the Higgs mechanism. The main feature is that the theory is still gauge invariant. The invariance is only hidden when the intermediate vector bosons acquire mass through the spontaneous breakdown of gauge symmetry.

We have tried to explain both electromagnetic and weak interactions by the same kind of theory. It seems natural to try to construct a unified theory for these two forces.

In 1967, Glashow, Salam and Weinberg (GSW) successfully constructed a simple model which unifies electromagnetic and weak interactions. The theory was based on invariance under the gauge group SU(2)xU(1). In 1971-72, it was proved by 't Hooft[48] that theories of this type were renormalizable. In this model, besides the massless photon and two massive charged bosons $W^{\pm}$, there exists a neutral intermediate boson Z. This implies neutral currents, which were discovered at CERN in 1973. More recently, the $W^{\pm}$, Z have been experimentally confirmed in their predicted mass range.

The discovery of unified renormalizable theories of electroweak interactions is one of the triumphs of modern particle physics. The understanding of gauge theories may be the key to understanding the interactions in nature. Here, a brief review to gauge theories, the Higg mechanism and the electroweak model by GWS will be given.

2.2 Gauge Invariance[4]

Gauge symmetry is an internal symmetry which differs from space-time symmetry. A gauge transformation of the first kind (also called a global gauge transformation) for the abelian group U(1) is the transformation:

$$\psi(x) \longrightarrow \psi'(x) = e^{i\alpha}\psi(x) = U(\alpha)\psi(x) \qquad (2.1)$$

where α is a real constant.

As an example, consider the electron field $\psi(x)$. The free Lagrangian $\mathcal{L}_o$ (see appendix A) of this field is

$$\mathcal{L}_o = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \qquad (2.2)$$

where m is the mass of the electron and $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$. Clearly (2.2) is invariant with respect to the transformation (2.1).

Noether's theorem assures the existence of conserved quantities whenever a continuous transformation of the coordinates and the fields leaves the Lagrangian invariant. Gauge invariance (2.1) gives rise to the conservation of a "charge".

It is clear that if α is a space-time dependent function, the Lagrangian is not invariant under local gauge transformations or gauge transformations of the second kind:

$$\psi(x) \longrightarrow \psi'(x) = U(\alpha(x))\psi(x) \qquad (2.3)$$

and

$$\partial_\mu \Psi(x) \longrightarrow \partial_\mu \Psi'(x) = U(\alpha(x))(\partial_\mu + i\,\partial_\mu \alpha(x))\Psi(x). \qquad (2.4)$$

Clearly, $\Psi'(x)$ and $\partial_\mu \Psi'(x)$ transform differently.

Local gauge invariance can be satisfied if a new field, A_μ, which is called a gauge field, is introduced.

First, let us consider the quantity $(\partial_\mu - ieA_\mu)\Psi$, where e is any constant, (here e is the electric charge of an electron). We have

$$(\partial_\mu - ieA_\mu)\Psi(x) = U^{-1}(\alpha(x))(\partial_\mu - ieA'_\mu(x))\Psi'(x) \qquad (2.5)$$

where

$$A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x) . \qquad (2.6)$$

Hence, $(\partial_\mu - ieA_\mu)$ transforms like $\Psi(x)$ if $A_\mu(x)$ transforms as in (2.6).

Therefore, if we change ∂_μ into the covariant derivative D_μ:

$$\partial_\mu \longrightarrow D_\mu = (\partial_\mu - ieA_\mu) . \qquad (2.7)$$

then, the Lagrangian $\mathcal{L}_0$ is invariant with respect to the gauge transformations (2.3) and (2.6).

We should add a kinetic energy term $\mathcal{L}_{K.E.}$ for A_μ which is also gauge invariant, (A_μ here should be identified as the electromagnetic vector field.)

$$\mathcal{L}_{K.E.} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad (2.8)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad . \tag{2.9}$$

Consequently, the complete gauge invariant Lagrangian density $\mathcal{L}$ for electrons and photons takes the form

$$\mathcal{L} = \overline{\Psi}(i\gamma^\mu(\partial_\mu - ieA_\mu) - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad . \tag{2.10}$$

As it is clear that the photon mass term $-\frac{1}{2}m^2A_\mu A^\mu$ will violate local gauge invariance, the requirement of local gauge invariance implies the masslessness of photons.

To construct a gauge invariant theory for the interactions different from electromagnetic interaction, we have to generalize to gauge invariance of a non-abelian type. Let us take the special case of SU(2)‡ because this group is important in weak interactions.

Consider two fermion fields grouped in an SU(2) doublet

$$\Psi = \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix} . \tag{2.11}$$

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‡SU(2) is the group of 2x2 unitary matrices with determinant one. It has three generators T^a (a=1,2,3) which are referred to as the weak isospin generators. These generators T^a have commutation relations $[T^a, T^b] = i\epsilon_{abc}T^c$, where ϵ_{abc} are called structure constants; they are antisymmetric in all three indices with ϵ_{123} = +1. The fundamental 2x2 representation is $L^a = \frac{\tau^a}{2}$,where τ^a are Pauli matrices:

$$\tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

The Lagrangian $\mathcal{L}_0$ of the field Ψ is written

$$\mathcal{L}_0 = \overline{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \tag{2.12}$$

where m is the common mass for Ψ_1 , Ψ_2 .

Again, $\mathcal{L}_0$ is invariant under the global SU(2) transformations:

$$\Psi(x) \longrightarrow \Psi'(x) = \exp.\left\{i\frac{1}{2}\tau\cdot\lambda\right\}\Psi(x) = U(\lambda)\Psi(x) \tag{2.13}$$

where τ_a , a=1,2,3 are three Pauli matrices and λ_a are real constants associated with each τ_a . When λ_a are real functions of space and time, the transformation of $\partial_\mu\Psi$ is different from Ψ . The requirement of gauge invariance leads to the introduction of three fields $A_\mu^a(x)$ (a=1,2,3), which are called "Yang-Mills fields, such that the derivative of the fields becomes

$$\partial_\mu \longrightarrow D_\mu\Psi(x) = \left(\partial_\mu - ig\frac{\tau^a A_\mu^a(x)}{2}\right)\Psi(x) . \tag{2.14}$$

The interaction constant g is introduced. (Summation convention $\tau^a A_\mu^a = \sum_{a=1}^{3}\tau^a A_\mu^a = \tau\cdot A_\mu$ is used.) It can be shown that

$$\left(\partial_\mu - ig\frac{\tau\cdot A_\mu}{2}\right)\Psi(x) = U^{-1}(x)\left(\partial_\mu - ig\frac{\tau\cdot A'_\mu}{2}\right)\Psi'(x) \tag{2.15}$$

with

$$\frac{\tau\cdot A_\mu}{2} \longrightarrow \frac{\tau\cdot A'_\mu}{2} = \frac{U\tau\cdot A_\mu U^{-1}}{2} - \frac{i}{g}[\partial^\mu U(x)]U^{-1}(x). \tag{2.16}$$

Then

$$\mathcal{L}_0 = \overline{\Psi}(i\gamma^\mu D_\mu - m)\Psi \tag{2.17}$$

is invariant under the gauge transformations and (2.16).

In analogy with A_μ in U(1) theory, we add the K.E. terms for $A^a_\mu(x)$,

$$\mathcal{L}_0' = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} \tag{2.18}$$

but with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, C_{abc} A^b_\mu A^c_\nu \tag{2.19}$$

where C_{abc} are structure constants of the group; for SU(2), they are ϵ_{abc}.

The total Lagrangian $\mathcal{L}$ has the form

$$\mathcal{L} = \overline{\Psi}(i\gamma^\mu D_\mu + m)\Psi - \frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} . \tag{2.20}$$

Again, no mass term for A^a_μ is possible without violating the gauge invariance.

In conclusion, the requirement of gauge invariance is only satisfied if new interacting fields are introduced through the substitution of the covariant derivative 'D_μ' for the partial derivative '∂_μ'.

2.3 Symmetry Breaking And The Higgs Mechanism

A symmetry of a system is said to be "spontaneously broken" if the lowest state (vacuum) of the system is not invariant under the operations of that symmetry. Let us consider a Lagrangian density for a complex scalar field which is invariant under U(1) symmetry:

$$\mathcal{L} = \frac{1}{2}(D_\mu\phi)^* D^\mu\phi - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (2.21)$$

where $V(\phi)$ is the potential and has the form

$$V(\phi) = \mu^2\phi^*\phi + \lambda(\phi^*\phi)^2 \, . \qquad (2.21)$$

Obviously, $V(\phi)$ is invariant under local gauge transformation $\phi = e^{i\omega(x)}\phi$.

The parameters λ , μ^2 can be any real constant. If we choose λ >0 and μ^2 >0, then the potential is

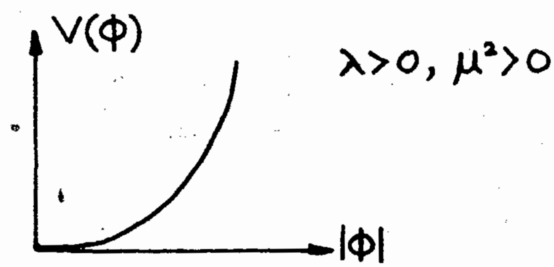

Fig.2.3 Non-symmetry-breaking scalar potential

The vacuum expectation value of ϕ would be zero. Therefore, there is no spontaneous symmetry breaking . However, if we choose μ^2 <0 and λ >0, the potential $V(\phi)$ is

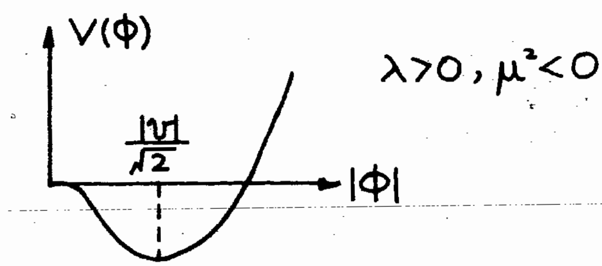

Fig.2.4 A symmetry-breaking scalar potential

The vacuum expectation of ϕ would be nonzero,

$$\langle\phi\rangle_0 = \frac{|v|}{\sqrt{2}} . \tag{2.23}$$

By minimizing $V(\phi)$, we have

$$|v|^2 = \frac{-\mu^2}{\lambda} . \tag{2.24}$$

We could choose the vacuum in some region to be

$$\phi_{vac.} = \frac{v}{\sqrt{2}} , \qquad v \in \mathbb{R} . \tag{2.25}$$

The vacuum is not invariant under the U(1) symmetry of the Lagrangian density. Therefore, the symmetry has been spontaneously broken (SSB).

The perturbation theory should be developed in terms of small departures from the vacuum state. Consider the parametrization of ϕ with new real fields ξ(x) and ϵ(x) such that

$$\phi = \frac{(v+\epsilon(x))}{\sqrt{2}} \exp\left[i\,\xi(x)/v \right] . \tag{2.26}$$

Now, $\mathcal{L}$ is written

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial^\mu\epsilon\,\partial_\mu\epsilon + \frac{1}{2}\partial_\mu\xi\,\partial^\mu\xi \\ &+ \frac{1}{2}e^2v^2A_\mu A^\mu - \sqrt{2}\,ev A_\mu\partial^\mu\xi + \mu^2\epsilon(x) \\ &+ \text{Cubic and higher order terms} .\end{aligned} \tag{2.27}$$

The ϵ(x) field has mass $-2\mu^2$, but the fields A_μ and ξ have mixed together. Without the term $-\sqrt{2}evA_\mu\partial^\mu\xi$ in (2.27), we would have concluded that the vector field has mass $\mu^2=e^2v^2$ and that the ξ field is massless. To straighten this up, let us consider a local gauge transformation of the following type, in what is called the unitary gauge,

$$\phi \longrightarrow \phi' = e^{i\xi(x)/v}\phi = \frac{v+\epsilon(x)}{\sqrt{2}} \qquad (2.28)$$

and

$$A_\mu \longrightarrow A'_\mu = A_\mu - \frac{1}{ev}\partial_\mu \xi \quad . \qquad (2.29)$$

Since the Lagrangian $\mathcal{L}$ is invariant under this transformation,

$$\mathcal{L} = -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + D'_\mu\phi' D'^\mu\phi' - V(\phi') \qquad (2.30)$$

which can be expanded as follows:

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}\partial_\mu\epsilon\partial^\mu\epsilon + \frac{1}{2}e^2v^2A'_\mu A'^\mu \\ & + \frac{1}{2}e^2A'^2_\mu\epsilon(2v+\epsilon) - \frac{1}{2}\epsilon^2(3\lambda v^2+\mu^2) \\ & - \lambda v\epsilon^2 - \frac{1}{4}\lambda\epsilon^4 \quad . \end{aligned} \qquad (2.31)$$

In this gauge there are no terms coupling different particles, so that the masses can be simply read off the quadratic terms.

We notice from (2.31) that

(i) A_μ has acquired a mass $M=|e|v$.

(ii) the scalar field has acquired a mass $(3\lambda^2v^2+\mu^2)^{1/2}$.

(iii) ξ-field has disappeared.

We started from a system describing a charged scalar field with two states and a massless gauge field with two polarization states. After SSB, we have one real scalar field and one massive vector field with three polarization states. The degrees of freedom have been conserved, and the ξ-field has been transmuted into the longitudinal polarization state of the vector field. This mechanism which gives mass to the gauge field is called the "Higgs Mechanism". The massive scalar field ϵ is called the Higgs particle.

The previous mechanism can be extended to a non-abelian gauge theory. For instance, consider a Lagrangian density invariant under local gauge transformation of some group G of dimension N. There are n-scalar fields which transform under an n-dimensional representation and there are N gauge vector bosons A_μ^a. Suppose the symmetry breaking leaves the vacuum invariant under an M dimensional subgroup H of G. Then, there are M generators satisfying $L^a \Phi_{vac.}=0$. Goldstone's theorem states that in the absence of the gauge fields, there exist (N-M) massless (Goldstone) bosons and (n-(N-M)) massive scalars. Now under the Higgs mechanism, these (N-M) Goldstone Bosons will become the longitudinal modes and give mass to (N-M) vector bosons. The remaining M vector bosons will remain massless .

With the Higgs mechanism understood, we can now discuss the GWS Model.

2.4 The GWS SU(2)xU(1) Electroweak Theory For Leptons[4,22]

It is well-known that it is the left-handed states which participate in the charged-current weak interaction. The GWS model is a chiral model in which parity violation is incorporated by assigning left- and right-handed fermions to different representations of SU(2). All left-handed fermions transform according to doublet representations, while right-handed fermions are singlets. L and R fields transform nontrivially under U(1). The weak hypercharge (the U(1) charge Y) is chosen so that the electric charge Q is

$$Q = T^3 + \frac{Y}{2} \tag{2.32}$$

where T^3 is the third component of the weak isospin (SU(2) charge).

The minimal GWS model involves one complex doublet of scalar fields

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{2.33}$$

(from now on, ϕ is denoted for the Higgs doublet.)
where Y is chosen to be one, the ϕ^+ and ϕ^0 have charge +1 and 0.

Let us first consider the left-handed electron e_L and its associated neutrinos ν_L are grouped in an SU(2) doublet L(D):

$$L(D) = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \tag{2.34}$$

where $e_L = \frac{1}{2}(1-\gamma_5)e$ (see appendix B) and similarly ν_L, while e_R is a SU(2) singlet: $e_R = \frac{1}{2}(1+\gamma_5)e$.

We assign to the doublet Y=-1 and to the singlet, e_R, Y=-2, so that $Q = T_L^3 + \frac{1}{2}Y$ holds for all particles. Since all members of each irreducible multiplet of SU(2) have the same hypercharge, we have

$$[T_i^a, Y] = 0 \tag{2.35}$$

for all a=1,2,3.

The group generated by T^a and Y is SU(2)xU(1) which is the symmetry group of the model. Under the local gauge transformation:

$$L \longrightarrow L' = \exp.\{i(\vec{\Lambda}(x)\cdot\frac{\vec{\tau}}{2} - \frac{1}{2}\theta(x)\}L, \tag{2.36a}$$

$$\phi \longrightarrow \phi' = \exp.\{i(\vec{\Lambda}(x)\cdot\frac{\vec{\tau}}{2} + \frac{1}{2}\theta(x)\}\phi, \tag{2.36b}$$

$$e_R \longrightarrow e_R' = \exp.\{-i\theta(x)\}e_R \,. \tag{2.36c}$$

As discussed earlier, in order to make a gauge invariant Lagrangian, we need to introduce three vector gauge fields $A_\mu^a(x)$ (a=1,2,3) associated with the three generators of SU(2), and $B_\mu(x)$ associated with the generator of U(1).

The covariant derivatives applied to the fields are the following,

$$D_\mu L(x) = (\partial_\mu + ig\frac{A_\mu^a\tau^a}{2} - i\frac{g'}{2}B_\mu(x))L(x)\,, \tag{2.37a}$$

$$D_\mu \phi(x) = (\partial_\mu + ig\frac{A_\mu^a\tau^a}{2} + i\frac{g'}{2}B_\mu(x))\phi(x)\,, \tag{2.37b}$$

$$D_\mu e_R(x) = (\partial_\mu - ig'B_\mu(x))e_R \tag{2.37c}$$

where g and g' are the gauge coupling constants of SU(2) and U(1).

The total Lagrangian $\mathcal{L}_T$ is

$$\mathcal{L}_T = \mathcal{L}_0' + \mathcal{L}_\phi + \mathcal{L}_0 \tag{2.38}$$

where

$$\mathcal{L}_0' = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \ , \tag{2.39a}$$

$$\mathcal{L}_\phi = (D^\mu\phi)^\dagger D_\mu\phi - V(\phi) \ , \tag{2.39b}$$

$$\mathcal{L}_0 = \overline{L}(D)i\gamma^\mu D_\mu L(D) + \overline{e}_R i\gamma^\mu D_\mu e_R \ , \tag{2.39c}$$

and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon_{abc}A^b_\mu A^c_\nu \ , \tag{2.40a}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \ , \tag{2.40b}$$

$$V(\phi) = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \ , \ \lambda > 0, \ \mu^2 < 0. \tag{2.40c}$$

Yukawa Couplings

We cannot put in by hand mass terms for the electrons because the gauge invariance would be broken. The SU(2)xU(1) invariant Yukawa couplings are introduced such that fermions acquire mass through SSB. In searching for these coupling terms, we examine the tramsformation properties of the lepton fields and scalar field under SU(2)xU(1):

$$e_R \sim (\underline{1}, -2) \ ; \tag{2.42a}$$

$$L(D) = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \sim (\underline{2}, -1) \ , \tag{2.42b}$$

and

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \sim (\underline{2}, 1) \tag{2.42C}$$

where the second entries in the parentheses are the U(1) hypercharges.

A bilinear term in fermion fields is

$$\bar{L}(D) e_R \sim (\underline{2}, 1) \otimes (\underline{1}, -2) = (\underline{2}, -1) . \tag{2.43}$$

Therefore, with $\phi \sim (\underline{2}.1)$, the only Yukawa couplings allowed is of the form

$$\mathcal{L}_{yuk.} = -G(\bar{L}(D)\phi e_R) + H.C. \tag{2.44}$$

where G is a constant.

We will see soon how the electron acquires a mass while the neutrino remains massless.

Spontaneous Symmetry Breaking And The Higgs Mechanism in SU(2)xU(1)

Let the vacuum expectation $|\langle\phi\rangle| = \frac{v}{\sqrt{2}}$, $|v^2| = -\frac{\mu^2}{\lambda}$. Again, we would choose the vacuum in some region to be

$$\phi_{vac.} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} , \qquad v \in \mathbb{R} . \tag{2.45}$$

It is easy to check that

$$\frac{1}{2}\tau^a \phi_{vac} \neq 0 \qquad (2.46a)$$

and

$$\frac{Y\phi_{vac}}{2} = \frac{1}{2}v \neq 0 \quad , \qquad (2.46b)$$

but

$$\left(\frac{\tau^3}{2} + \frac{Y}{2}\right)\phi_{vac} = 0 \quad . \qquad (2.46c)$$

Therefore, after SSB, the symmetries associated with the generators T^1, T^2 and $T^3 - \frac{Y}{2}$ are broken. However, the subgroup $U^{EM}(1)$ generated by the electric charge $Q = T^3 + \frac{Y}{2}$ is unbroken. Hence, SU(2)xU(1) is broken down to $U^{EM}(1)$. From section (2.3), we expect that three gauge bosons acquire masses while one gauge boson remains massless.

The complex Higgs doublets can be written in terms of four real fields which are three $\xi^i(x)$ associated with the generators of SU(2), and $\eta(x)$:

$$\phi = \frac{1}{\sqrt{2}} \exp.\left(i\sum \xi^a L^a\right)\begin{pmatrix} 0 \\ v+\eta(x)\end{pmatrix} . \qquad (2.47)$$

In the unitary gauge, we have

$$\phi' = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+\eta(x)\end{pmatrix} . \qquad (2.48)$$

There is only one physical Higgs particle left. The other three fields have been gauged away to give masses to three of the gauge bosons. (From now on, it is understood that A_μ, and ϕ are in the unitary gauge.)

The Lagrangian $\mathcal{L}_\phi$ in (2.39b) can be written as,

$$\mathcal{L}_\phi = M_W^2 W^{+\mu} W_\mu^- + \frac{M_Z^2}{2} Z^\mu Z_\mu - \left(-\frac{\mu^4}{4\lambda} - \mu^2\eta^2 + \lambda v \eta^3 + \frac{\lambda}{4}\eta^4\right) \tag{2.49}$$

where

$$W_\mu^\pm = \frac{A_\mu^1 \mp i A_\mu^2}{\sqrt{2}} \tag{2.50a}$$

and

$$Z_\mu = \frac{g' B_\mu - g A_\mu^3}{(g^2 + g'^2)^{1/2}} \tag{2.50b}$$

are charged and neutral massive gauge fields with masses

$$M_{W^\pm}^2 = \frac{g^2 v^2}{4} , \tag{2.51a}$$

$$M_Z^2 = (g^2 + g'^2)\frac{v^2}{4} = \frac{M_W^2}{\cos^2\theta_W} . \tag{2.51b}$$

The weak (Weinberg) angle is defined by

$$\tan\theta_W = \frac{g'}{g} . \tag{2.52}$$

The mass of the Higgs boson is given by

$$M_\eta^2 = -2\mu^2 > 0 . \tag{2.53}$$

The massless gauge boson is

$$A_\mu = \frac{g B_\mu + g' A_\mu^3}{(g^2 + g'^2)^{1/2}} \quad . \tag{2.54}$$

In the unitary gauge, the Yukawa interaction becomes

$$\mathcal{L}_{yuk.} = -G(\bar{\nu}_L, \bar{e}_L)\begin{pmatrix} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{pmatrix} e_R + H.C.$$

$$= -\left(\frac{G v \bar{e}_L e_R}{\sqrt{2}} + \frac{G \bar{e}_L \eta(x) e_R}{\sqrt{2}}\right) + H.C. \quad . \tag{2.55}$$

Therefore, the mass of the electron is

$$m_e = \frac{G v}{\sqrt{2}} \tag{2.56}$$

and the associated neutrino remains massless.

Interactions

The free Lagrangian $\mathcal{L}_\ell^0$ for the lepton fields is

$$\mathcal{L}_\ell^0 = \bar{\nu}_L i \gamma^\mu \partial_\mu \nu_L + \bar{e} i \gamma^\mu \partial_\mu e - \left(\frac{G v}{\sqrt{2}} \bar{e}_L e_R + H.C.\right). \tag{2.57}$$

The interacting Lagrangian $\mathcal{L}_{int}$ for the electromagnetic and weak interactions can be written in terms of $W^\pm$, Z and A:

$$\mathcal{L}_{int.} = -\frac{g}{\sqrt{2}}\left(J_W^\mu W_\mu^- + J_W^{\mu\dagger} W_\mu^+\right)$$

$$+ \frac{1}{2}(g^2 + g'^2)^{1/2} J_Z^\mu Z_\mu + e J_{EM}^\mu A_\mu \tag{2.58}$$

where

$$e = \frac{gg'}{\sqrt{g'^2+g^2}} = g\sin\theta_W \quad . \tag{2.59}$$

The electromagnetic current is

$$J^\mu_{EM} = \bar{e}\gamma^\mu e \quad . \tag{2.60}$$

The charged weak currents are

$$J^\mu_W = \bar{e}\gamma^\mu \frac{(1-\gamma^5)}{2}\nu \tag{2.61a}$$

and

$$J^{\mu^\dagger}_W = \bar{\nu}\gamma^\mu \frac{(1-\gamma^5)}{2} e \quad . \tag{2.61b}$$

The neutral current is

$$J^\mu_Z = \bar{\nu}\gamma^\mu \frac{(1-\gamma^5)}{2}\nu + \frac{1}{2}\bar{e}\gamma^\mu(g_V - g_A\gamma^5)e \tag{2.62}$$

where

$$g_V = -1 + 4\sin^2\theta_W \quad , \quad g_A = -1 \quad . \tag{2.63}$$

For momentum transfers small compared to M_W ‡ , $\mathcal{L}_{int}$ leads to an effective four-Fermi charged current interaction

$$\mathcal{L}_c^{eff.} = \frac{4G_F}{\sqrt{2}} J^\mu_W J^\dagger_{W\mu} \tag{2.64}$$

where the Fermi constant is given by

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} = \frac{1}{2v^2} \quad . \tag{2.65}$$

Similarly, at small momentum transfers, an effective four-Fermi

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‡The IVB propagator $\dfrac{-g_{\mu\nu}+k_\mu k_\nu/M_W^2}{k^2-M_W^2} \cong \dfrac{-g_{\mu\nu}}{M_W^2}$ for $|k|^2 \ll M_W^2$.

neutral current interaction is

$$\mathcal{L}_Z^{eff.} = \frac{G_F}{\sqrt{2}} \rho J_Z^{\mu} J_{\mu Z} \tag{2.66}$$

where

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2\theta_W} \tag{2.67}$$

In the minimal GWS model, there is only one complex doublet of Higgs scalar fields. The relation between M_W and M_Z is given in (2.51), and ρ is equal to one . If a new Higgs scalar is added; for instance, a Higgs triplet which has a nonzero vacuum expectation value, the relation between M_W and M_Z will be altered (see Chapter 3).

Generalized To N-Family Leptons

It is easy to generalize the single lepton family of GWS to an N-family case, let

$$L_n(D) = \begin{pmatrix} \nu_{n_L} \\ \ell_{n_L} \end{pmatrix} \tag{2.68a}$$

and

$$\ell_{n_R} \tag{2.68b}$$

where n=1,2,3,.....,N. Then, the generalized Lagrangian density $\mathcal{L}_{0}L(D)$ would be

$$\mathcal{L}_{OL(D)} = \sum_{n=1}^{N} \bar{L}_n(D) i\gamma^{\mu} D_{\mu} L_n(D) + \bar{\ell}_{R_n} i\gamma^{\mu} D_{\mu} \ell_{R_n} \qquad (2.69)$$

and the generalized Yukawa coupling would be

$$\mathcal{L}_{Yuk.} = -\sum_{n,m=1}^{N} (G_{mn} \bar{L}_m(D) \phi \ell_{R_n} + H.C.) \qquad (2.70)$$

where G_{mn} are some constants for m,n=1,2,3,.....,N.

In general $G_{mn}(\neq 0)$ for $m\neq n$ is the mixing term between different families. Since the masses for the neutrinos are degenerate, namely all are massless, we can always redefine the neutrino states as we wish. More details will be discussed in Chapter 3. Now, for simiplicity, we assume $G_{mn} = \delta_{mn} G_n$ and then the masses for the leptons will be

$$m_n = \frac{v G_n}{\sqrt{2}} , \qquad (2.71)$$

and the interactions for the electroweak currents will just be the sum of the interactions of each lepton field with the gauge vector fields.

2.5 The Success Of GWS Model

The GWS model is not only the electroweak model for leptons. It was shown that hadrons could be in corporated in the model by a mechanism due to Glashow, Iliopoulas and Maiane (GIM). The GWS model with the GIM mechanism successfully predicted both the existence of flavour conserving neutral currents and the existence of the charmed quark, both of which were discovered‡.

One of the most important predictions of the GWS model is the existence of neutral currents. In 1973 at CERN, both the processes of neutrino deep inelastic scattering on nucleons $\nu_\mu(\bar{\nu}_\mu)+N\rightarrow\nu_\mu(\bar{\nu}_\mu)+N$, and the pure leptonic process $\nu_\mu(\bar{\nu}_\mu)+e\rightarrow\nu_\mu(\bar{\nu}_\mu)+e$, which are not induced by charged currents, were first observed. The observation of these processes has marked the discovery of neutral currents.

The processes $\nu_\mu(\bar{\nu}_\mu)+N\rightarrow\nu_\mu(\bar{\nu}_\mu)+N$ have been studied in a wide interval of neutrino energy up to $\sim$ 200GeV. They are the best investigated neutrino processes induced by neutral currents. Experimental data which determines the strength of the left-handed and right-handed couplings of the neutral quark current are well described by the standard GWS model. The best experimental data of such processes is obtained by the CDHS (1979) and CHARM (1980).

Parity violating asymmetries are predicted from the interference between the weak (the Z-boson exchange) and electromagnetic (the photon exchange) amplitudes. At the present accessible energy, the expected asymmetry is quite small because of the dominantly parity symmetric electromagnetic interactions. However, such effects have been observed and the most precise result is obtained from a SLAC experiment (1978,1979) in the inelastic scattering of longitudinally polarized electrons from

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‡For theoretical and experimental review of the weak neutral current, see review articles by J.E. Kim, P. Langacker, etc., (1981)[9] and S.M. Bilenky and J. Hosek[10] (1982)

an unpolarized deuteruim target,

$$e(\text{polarized}) + D(\text{unpolarized}) \rightarrow e' + X \, . \qquad (2.72)$$

The experimental results are consistent with the expected value of $\sin^2\theta_W$ in the GWS model.

It has been theoretically shown that the process $e^+ + e^- \rightarrow \mu^+ + \mu^-$ is forward-backward asymmetric due to the nonzero axial-currents. Experimental measurements (at PETRA, 1980,1981) of such effects are still preliminary, but the present data are in agreement with the GWS theory. The value of the contribution of neutral currents to the cross sections of the processes $e^+ + e^- \rightarrow \ell^+ + \ell^- \, (\ell = e, \mu, \tau)$ has been measured (at PETRA, 1980,1981). These data, which have enabled us to determine the value of the parameter $\sin^2\theta_W$, agree with all other experimental data.

Conclusively, the parameters of the model $\sin^2\theta_W$ and ρ, have been intensively investigated by measuring various neutral current processes, are found approximately to have the same experimental values:[11]

$$\sin^2\theta_W^{exp.} = 0.224 \pm 0.019 \qquad (2.73a)$$

$$\rho^{exp.} = 0.992 \pm 0.020 \qquad (2.73b)$$

Where $\rho^{exp.}$ is in agreement with the theoretical value of 1.

Using the above experimental value for $\sin^2\theta_W$, we can predict the masses M_W and M_Z by using (2.51):

$$M_W = \frac{gv}{2} = \left(\frac{\pi\alpha}{\sqrt{2}G_F}\right)^{1/2}\frac{1}{\sin\theta_W} \cong 78\,GeV/c^2 \,, \qquad (2.74a)$$

$$M_Z = \frac{M_W}{\cos\theta_W} = \frac{74.6\,GeV/c^2}{\sin 2\theta_W} \cong 89\,GeV/c^2 . \qquad (2.74b)$$

(Without radiative corrections).

Recently at CERN, the W and Z bosons have been successfully created by the Proton-Antiproton Collider. The W boson then decays to a charged lepton and a neutrino, and the Z boson decays to a pair of charged leptons. The masses M_W[12] and M_Z[13] of the intermediate bosons have been measured in this experiment:

$$M_W^{exp} = (81 \pm 5)\,\frac{GeV}{c^2} \,, \qquad (2.75a)$$

$$M_Z^{exp} = (95.2 \pm 2.5)\,\frac{GeV}{c^2} \qquad (2.75b)$$

which are in agreement with the theoretical values.

Finally, one more important mass has to be determined: the mass of the neutrino. Although the GWS model assumes the masslessness of the neutrinos, the experimental limits on the neutrino masses are not very stringent. The limits[11] are

$$m_{\nu_e} < 46\,eV \,, \qquad (2.76a)$$

$$m_{\nu_\mu} < 0.52\,MeV \,, \qquad (2.76b)$$

$$m_{\nu_\tau} < 250\,MeV \qquad (2.76c)$$

which are still in agreement with the GWS model. However, there is one positive result which has been reported for the mass of the electron neutrino by Lyubimov et, al[3] (1980):

$$14\ \mathrm{eV} < m_{\nu_e} < 46\ \mathrm{eV} \tag{2.77}$$

which is still unconfirmed by other experiments.

Essentially experimental results are known to be in agreement with the GWS model which is now taken as the standard model for electroweak theories. Even if the neutrino has nonzero mass, the basic structures of the model for the known leptons and quarks probably will not be altered. It leads to the possibility of extending the minimal model when we consider the neutrino mass.

III. Chapter 3 Neutrino Mass Terms In SU(2)xU(1) Model

3.1 Definition Of A Mass Term

A fermion mass term is any proper Lorentz invariant term which is formed by bilinear fermion fields in the Lagrangian. With this definition in mind, let us examine what possible types of mass terms for a neutrino.

3.2 Dirac Mass Terms

Both left-handed neutrino fields ν_L and right-handed neutrino fields ν_R are needed to construct a mass term of the Dirac type. For instance, a neutrino field ν has

$$-\mathcal{L}_D^{\nu} = m(\bar{\nu}_L \nu_R + \bar{\nu}_R \nu_L) = m\bar{\nu}\nu \qquad (3.1)$$

where m is called the mass of the neutrino ν .

It is easy to see that $\mathcal{L}_D^{\nu}$ is invariant under the global transformation $\nu \to e^{i\theta}\nu$. This implies that $\mathcal{L}_D^{\nu}$ conserves an additive quantum number which is referred to as a lepton charge L or a fermion number F. A massive lepton of the Dirac type can be distinguished from its anti-particle by the value of the fermion number F.

Due to the absence of a right-handed neutrino, in the minimal SU(2)xU(1) model, the neutrino can not acquire Dirac mass. But if the right-handed neutrino ν_R[18,19] is introduced as an SU(2) singlet, the Dirac neutrino mass term arises naturally. Now, let us consider

$$\nu_R \sim (\underline{1}, 0) \tag{3.2}$$

and $L(D)=\begin{pmatrix}\nu_L\\ e_L\end{pmatrix}\sim(\underline{2},-1)$ in (24.2a), then a new bilinear term can be formed and has the following transformation property:

$$\overline{L}(D)\nu_R \sim (\underline{2},1)\otimes(\underline{1},0)=(\underline{2},1) \quad . \tag{3.3}$$

The charge-conjugate of the standard Higgs doublet $\phi=\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}$ is

$$\phi^c = i\tau_2\phi^* = \begin{pmatrix}\phi^0\\ -\phi^-\end{pmatrix} \sim (\underline{2},-1) \ . \tag{3.4}$$

Therefore, the Yukawa couplings which are invariant under SU(2)xU(1) can be formed as

$$\mathcal{L}^{\nu}_{yuk.} = G_\nu(\overline{L}(D)\phi^c\nu_R) + \text{H.C.} \tag{3.5}$$

After SSB, the couplings give mass to the neutrino $m = G_\nu\langle\phi^0\rangle_0$, where $\langle\phi^0\rangle_0$ is a nonzero vacuum expectation value.

For the generalized case of N-families $L_n(D)\sim(\underline{2},-1)$, $\nu_{nR}\sim(\underline{1},0)$ (n=1,2,3,...,N), the Lagrangian $\mathcal{L}^{\nu}_{yuk.}$ for the Yukawa couplings‡ is

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‡Since we are only interested in how to generate masses for neutrinos, we neglect other Yukawa couplings here.

$$\mathcal{L}^{\nu}_{yuk.} = \sum_{m,n=1}^{N} (\bar{L}_m(D) G_{\nu mn} \Phi^c \nu_{n_R}) + H.C. \tag{3.6}$$

where $G_{\nu mn}$ is an arbitrary complex constant matrix. After SSB, the Lagrangian $\mathcal{L}^{\nu}_{D}$ for the mass terms is

$$\mathcal{L}^{\nu}_{D} = \sum_{m,n=1}^{N} \bar{\nu}_{m_L} M_{mn} \nu_{n_R} + H.C. \tag{3.7}$$

where

$$M_{mn} = G_{\nu mn} \langle \Phi^0 \rangle_0 \tag{3.8}$$

is in general complex and non-Hermitian and may be diagonalized by means of a transformation:[22]

$$A^{\nu +}_{L} M A^{\nu}_{R} = M_D = \mathrm{Diag.}[m_1, \ldots, m_N], \tag{3.9}$$

M_D is real, and m_n (n=1,2,3,...,N) are the mass eigenvalues of M. A^{ν}_{L} and A^{ν}_{R} are NxN unitary matrices and can be determined almost uniquely by observing that MM^+ and M^+M are Hermitian nxn matrices such that

$$\begin{aligned} A^{\nu +}_{L} M M^{+} A^{\nu}_{L} &= A^{\nu +}_{R} M^{+} M A^{\nu}_{R} \\ &= M_D^2 = \mathrm{Diag.}[m_1^2, \ldots, m_N^2]. \end{aligned} \tag{3.10}$$

Clearly, A^{ν}_{L} and A^{ν}_{R} can only be determined up to N arbitrary phases. That is, if A^{ν}_{L} and A^{ν}_{R} satisfy (3.10), so do

$$A'_L = A_L P_L(\varphi) \tag{3.11a}$$

and

$$A'_R = A_R P_R(\varphi) \tag{3.11b}$$

where

$$P_{L,R}(\varphi)=\begin{pmatrix} \exp.(i\varphi_{1L,R}) & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & \ddots & \\ 0 & & & \exp.(i\varphi_{NL,R}) \end{pmatrix} . \qquad (3.11c)$$

The relative phases $(\varphi_{n_L}-\varphi_{n_R})$ are determined from (3.9) such that M_D is real and positive, but the absolute phases are arbitrary. That means if $P_L(\varphi) = P_R(\varphi)$, the form of the mass terms is left invariant.

Let the fields $(\tilde{\nu}_n)_{L,R}$ (n=1,2,3,....,N) in a column vector form,

$$\tilde{\nu}_{L,R} = \begin{pmatrix} \tilde{\nu}_1 \\ \tilde{\nu}_2 \\ \vdots \\ \tilde{\nu}_N \end{pmatrix}_{L,R} \qquad (3.12)$$

be the mass eigenstates which are the physical observable states. Then, $\tilde{\nu}_L$, and $\tilde{\nu}_R$ are obtained by the transformations as follows:

$$\tilde{\nu}_L = A_L^{\nu\dagger} \nu_L , \qquad (3.13a)$$

$$\tilde{\nu}_R = A_R^{\nu\dagger} \nu_R . \qquad (3.13b)$$

Similar unitary matrices A_L^{ℓ}, A_R^{ℓ} exist for the charged leptons $\ell=\begin{pmatrix} \ell_1 \\ \vdots \\ \ell_N \end{pmatrix}$ such that

$$\tilde{\ell}_L = A_L^{\ell\dagger}\ell_L \ , \tag{3.14a}$$

$$\tilde{\ell}_R = A_R^{\ell\dagger}\ell_R \ . \tag{3.14b}$$

Let us consider the effect of these matrices A_L, A_R in charged weak currents

$$\begin{aligned} J_W^\mu &= \overline{\ell}_L \gamma^\mu \nu_L \\ &= \overline{\tilde{\ell}}_L \gamma^\mu A_L^{\ell\dagger} A_L^\nu \tilde{\nu}_L \\ &= \overline{\tilde{\ell}}_L \gamma^\mu A_c \tilde{\nu}_L \end{aligned} \tag{3.15}$$

where

$$A_c = A_L^{\ell\dagger} A_L^\nu \tag{3.16}$$

is the generalized cabbibo matrix.‡

The presence of the generalized cabibbo matrix A_c is due to the mismatch between the weak and the mass eigenstates. For N-families, A_c is an NxN unitary matrix which can be expressed in N^2 real parameters. (2N-1) parameters are not observable because they correspond to the relative phases of the fermion fields. These phases, which corresponds to the undetermined matrices $P_L^\ell(\varphi)$ and $P_L^\nu(\varphi)$, can be eliminated from A_c by

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‡notice that if the neutrinos are massless, they cannot be distinguished from each other except through their weak interactions. We can therefore simply define $\tilde{\nu}_{m_L} = (A_L^{\ell\dagger})_{mn}\nu_{Ln}$ as the neutrino associated with ℓ_{m_L}. Therefore, there is no mixing between different families.

redefining the phases of the fermion fields‡ . Therefore, the unitary matrix A_c can be expressed in terms of $N^2 - (2N-1) = (N-1)^2$ observable parameters.

3.3 Majorana Mass Terms For Neutrinos

Besides the Dirac mass terms, there exists a different type of Lorentz invariant bilinear form for neutrino:

$$-\mathcal{L}_M^{\nu} = \frac{m}{2}\left(\overline{(\nu_L)^c}\,\nu_L\right) + \text{H.C.} \tag{3.17}$$

where m is a real constant and $(\nu_L)^c = (\nu^c)_R$ (see appendix C) is a right-handed charge conjugate field and transforms as ν_L under the proper Lorentz transformation. It is obvious that $\mathcal{L}_M^{\nu}$ is not invariant under the transformation in (2.1).

Let us define

$$\chi = \nu_L + (\nu^c)_R \tag{3.18}$$

then χ is a self-conjugate field

$$\chi^c = \chi \tag{3.19}$$

Now, the Lagrangian $\mathcal{L}_M^{\nu}$ can be written in terms of the field χ :

$$-\mathcal{L}_M^{\nu} = -\mathcal{L}_M^{\chi} = \frac{1}{2} m \bar{\chi} \chi \tag{3.20}$$

Hence, m is the mass of the field χ which satisfies (3.18) and

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‡In a theory with more interactions, some of these phases may be observable.

is called a Majorana field.

The Majorana mass term does not need a right-handed field in the theory. The main difference in the descriptions of the massive Dirac field and the Majorana field is that the former has four independent components and has a well-defined fermion or lepton charge number; whereas, the latter has only two independent components, no charge carried by ν_L is conserved, and the transition of a neutrino into an antineutrino at one space-time point becomes possible. Obviously, the Majorana term is only allowed for the neutrino because it has no charge.

It is impossible to generate a Majorana mass for a neutrino in the minimal GWS model, since the bilinear field is

$$\overline{(L(D))^c}L(D) \sim (\underline{2},-1)\otimes(\underline{2},-1) = (\underline{1},-2)\oplus(\underline{3},-2) \quad (3.21)$$

which cannot form a invariant term with the usual Higgs doublet. However, it becomes possible if we introduce some new scalars which can couple to this bilinear field to form an SU(2)xU(1) invariant terms:

(i) Triplet[17,18]: $H = \begin{bmatrix} H^+ & \sqrt{2}H^{++} \\ \sqrt{2}H^0 & -H^+ \end{bmatrix} \sim (\underline{3},2)$, (3.22)

(ii) Single-charged singlet‡ : $h^+ \sim (\underline{1},2)$ (3.23)

First, let us consider only a triplet of scalars. Then the Yukawa couplings would be

$$-\mathcal{L}^{\prime\nu}_{yuk} = \frac{G'_\nu}{2}\left(\overline{L^c}(D)\,H\,L(D)\right) + H.C. \quad (3.24)$$

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‡this case has been considered by Zee(1980)[20].

If the triplet H has a nonzero vacuum expectation value $\langle H^0\rangle_0 \neq 0$, then after SSB, it leads to (3.17) with

$$m = G'_\nu \sqrt{2} \langle H^0\rangle_0 \; . \tag{3.25}$$

The introduction of the coupling of H to the gauge fields changes the relation between W and Z masses. Now we have

$$\frac{M_Z^2}{M_W^2} = \frac{1}{\cos^2\theta_W}\left(\frac{4\langle H^0\rangle_0^2 + \langle \Phi^0\rangle_0^2}{2\langle H^0\rangle_0^2 + \langle \Phi^0\rangle_0^2}\right) \tag{3.26}$$

where $\langle \Phi^0\rangle_0$ is the vacuum value of the usual Higgs doublet. With the assumption $\frac{\langle H^0\rangle_0^2}{\langle \Phi^0\rangle_0^2}$ is small, we obtan

$$\frac{M_Z^2}{M_W^2} \cong \frac{1}{\cos^2\theta_W}\left(1 + \frac{2\langle H^0\rangle_0^2}{\langle \Phi^0\rangle_0^2}\right) . \tag{3.27}$$

Therefore, $\rho = \frac{M_W^2}{M_Z^2\cos^2\theta_W}$ is less than one. Because there is no compelling reason for $\frac{\langle H^0\rangle_0^2}{\langle \Phi^0\rangle_0^2} \ll 1$, this addition is unnatural.

For the generalized case of N-families of lepton doublets $L(D) = \begin{pmatrix}\nu_n\\ \ell_n\end{pmatrix}_L$ (n=1,2,3,......,N), the Lagrangian for the Yukawa coupling terms is

$$-\mathcal{L}'^{\nu}_{yuk} = \sum_{n,m=1}^{N} G'_{\nu\, mn}\, \overline{L_m^c}(D)\, H\, L_n(D) + \text{H.C.} \quad . \tag{3.28}$$

Then, the Lagrangian for the neutrino mass terms can be written as:

$$-\mathcal{L}'^{\nu}_{M} = \sum_{m,n=1}^{N} \overline{(\nu_{m_L})^c}\, M_{mn}\, \nu_{n_L} + \text{H.C.} \tag{3.29}$$

where $M_{mn} = G'_{\nu mn}\sqrt{2}\langle H^0\rangle_0$ is the mass matrix which in general is a complex symmetric matrix‡ which can be diagonalized with real non-negative elements by a transformation:[17,19]

$$U^T M U = M_D = \mathrm{Diag.}(m_1, \ldots\ldots, m_n) \tag{3.30}$$

where M_D is diagonal mass matrix and U is NxN unitary matrix $U^{\dagger}U$ =1. The physical neutrino fields χ_m with masses (m_m) can be found:

$$\chi_m = \sum_{n=1}^{N} U^{\dagger}_{mn} \nu_{n_L} + (U^{\dagger}_{mn})^{*} (\nu_{n_L})^{c} \tag{3.31}$$

where '*' means complex conjugate. It is clear from (3.31) that the χ_m satisfy the Majorana condition

$$\chi^c_m = \chi_n \quad , \quad \text{for all } m = 1, \ldots\ldots, N \, . \tag{3.32}$$

Then the Lagrangian $\mathcal{L}^{\prime\nu}_M$ can be written: in terms of these Majorana fields as follows:

$$-\mathcal{L}^{\prime\nu}_M = -\mathcal{L}^{\chi}_M = \sum_{n=1}^{N} \frac{m_n \bar{\chi}_n \chi_n}{2} \, . \tag{3.34}$$

Again, we will get the mixing matrix A_c (3.16) as in the Dirac case.

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‡since $\overline{(\nu_{m_L})^c}\,\nu_{n_L} = \overline{(\nu_{n_L})^c}\,\nu_{m_L}$, see appendix C.

3.4 The CP Invariant Mass Matrix

The interesting case is the one in which the matrix M is a real symmetric matrix which can be diagonalized by an orthogonal matrix O[23]:

$$\sum_{l,m=1}^{N} O^T_{kl} M_{lm} O_{mn} = \eta_k m_k \delta_{kn} \tag{3.35}$$

where m_k is a positive real number and $\eta_k = \pm 1$.

The Majorana mass eigenstates would be defined as

$$\chi_m = \sum_{n=1}^{N} O^T_{mn} \nu_{n_L} + \eta_m O^T_{mn} (\nu_{n_L})^c \quad . \tag{3.36}$$

Obviously, we have

$$\chi^c_m = \eta_m \chi_m \quad , \quad m = 1, \ldots\ldots, N \tag{3.37}$$

and note that

$$(CP)\chi_m (CP)^{-1} = (\chi_m)^c \quad , \tag{3.38}$$

hence, χ_m is an eigenstate of CP with the eigenvalue $\eta_m = \pm 1$.

The Lagrangian for the mass term can be written in terms of χ_m as in (3.34). m_m is the mass of the Majorana field χ_m which is always positive because of the factor η_m.

If we use a unitary matrix U to diagonalize the real symmetric mass matrix M with real non-negative masses, this corresponds to the orthogonal matrix O as follows:

$$\text{(i)} \quad \eta_m = 1 \;, \qquad U_{mn} = O_{mn} \qquad (3.39a)$$

$$\text{(ii)} \quad \eta_m = -1 \;, \qquad U_{mn} = i\, O_{mn} \qquad (3.39b)$$

With U, $\chi_m^c = \chi_m$ in (3.32) for all m=1,2,3,.....,N, therefore, the Majorana fields are all chosen to have positive CP. However, those for which η_m=-1 now have pure imaginary couplings for the interactions. Thus people have discussed these as if they were CP-violating interactions. However, redefining CP as in (3.37), this CP violation can simply be removed. As noticed recently by Wolfenstein[23] (1981), the product of the η-factors of two neutrinos is significant. We will discuss the significance in later chapters with more details.

3.5 Dirac And Majorana Mass Terms

If the theory contains N two-component left-handed neutrinos ν_L belonging to SU(2) doublets and F two-component‡ right-handed neutrinos which are singlets under SU(2)xU(1) the most general case for the neutrino mass Lagrangian $\mathcal{L}^{\nu}_{mass}$ may include both $\mathcal{L}^{\nu}$ (Dirac mass) and $\mathcal{L}'^{\nu}$ (Majorana mass) and other Majorana mass terms formed by ν_R.

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‡It is not necessary to identify the singlets as the right-handed neutrinos. Instead, V. Barger, P. Langacker and J.P. Leveille and S. Pakvasa (1980)[24] have proposed that there exist left-handed fields which are singlets in SU(2)xU(1).

The generalized Lagrangian $\mathcal{L}^{\nu}_{mass}$ is written:

$$-\mathcal{L}^{\nu}_{mass} = \frac{1}{2}(\overline{\nu_L^c})M\nu_L + \text{H.C.} \tag{3.40}$$

where

$$M = \begin{bmatrix} M_1 & D_1 \\ D_2 & M_2 \end{bmatrix} \tag{3.41}$$

is a (N+F)x(N+F) mass matrix, and

$$\nu_L = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_N \\ \nu_1^c \\ \vdots \\ \nu_F^c \end{pmatrix}_L \tag{3.42}$$

is a 1x(N+F) column vector.

M_1 is the NxN Majorana mass matrix as in (3.29) and D_1 is the FxN Dirac mass matrix as in (3.7). M_2 is another FxF Majorana mass matrix which is already SU(2)xU(1) invariant without the need for any extra Higgs fields because ν_{m_R} are singlets under the group. Since $\overline{\nu^c_{m_R}}\nu^c_{n_L} = \overline{\nu_{n_R}}\nu_{m_L}$, we have $D_2 = D_1^T$.

Again $M=M^T$ is a complex symmetric matrix which can be diagonalized with real non-negative elements from the transformation $U^T M U = M_D$, where U is (N+F)x(N+F) unitary matrix. Physical neutrino fields χ_m (m=1,2,3,....,N+F) are defined as in (3.31). For N=F, there exists 2N Majorana fields. If N=F and $M_1 = M_2 = 0$, we could get back N-Dirac neutrinos. Thus the most general mass term for a four-component fermion field actually decribes two Majorana particles with distinct masses.

IV. Chapter 4 The Modified GWS Model

4.1 Motivations

As we have discussed in the last chapter, neutrinos are massless in the minimal GWS SU(2)xU(1) model. In order for a neutrino to acquire mass, whether it is a Dirac type or a Majorana type, something must be added to the theory. If additional new components ν_R are introduced, neutrinos can acquire Dirac masses and also can acquire Majorana masses for the new components ν_R . Since there is no right-handed charged current, lepton number will remain conserved. Therefore, lepton-number violating processes such as neutrinoless double β -decays can not arise from such a theory. If a Higgs triplet is added, left-handed neutrinos can acquire Majorana masses, and lepton number will not remain conserved. Processes involving lepton-number violation now become possible, but we can no longer predict the value for ρ , which has already been experimental verified.

Searching for a massive neutrino theory which will retain ρ =1 and also give lepton-number-violating processes, we introduce real fermion triplets $L_m(T)$ (m=1,2,.....,F) in SU(2). These triplets transform as

$$L_m(T) \sim (\underline{3}, 0) \tag{4.1}$$

in the SU(2)xU(1) model and have the following form:

$$L_m(T) = \begin{bmatrix} N_m & -\sqrt{2}E_m^c \\ \sqrt{2}E_m & -N_m \end{bmatrix}_L . \tag{4.2}$$

The subscript 'L' denotes left-handed fields. E_m is a negatively charged lepton and N_m is its associated neutral lepton (neutrino). E_m^c is the charge conjugate of E_m.

Because of the presence of a lepton E_m and its anti-lepton E_m^c in the same triplet, we are no longer able to assign the lepton number for $L_m(T)$. This signifies that lepton-number-violating processes are possible. It follows from (4.1) that

$$Q_m = T_3 \, , \qquad \text{for all} \quad m=1,\ldots\ldots,F \tag{4.3}$$

and E_m has unit negative charge.

The inclusion of these triplets does not create any axial anomaly problem because the anomalous term in SU(2)xU(1) is weighted by the quantity[45]

$$\begin{aligned} A_{SU(2)\times U(1)} &= Tr\,(T_3^2 Q_m) \\ &= 0 \, . \end{aligned} \tag{4.4}$$

At this point, the introduction of these new triplets into the theory is purely speculative. Nevertheless, it is theoretically interesting to investigate the effects of such new added triplets.

4.2 The F-Family Triplets

We consider F-families of lepton triplets ($m=1,2,\ldots,F$) as described in (4.1). Under the local gauge transformation SU(2)xU(1), each triplet $L_m(T)$ transforms as follow:

$$L_m(T) \longrightarrow L'_m(T) = U(\Lambda_m(x)) L_m(T) U^{\dagger}(\Lambda_m(x)) \qquad (4.5)$$

where $U(\Lambda_m(x)) = \exp i(\vec{\Lambda}_m(x)\cdot\frac{\vec{\tau}}{2})$,

$\vec{\Lambda}_m(x)$ is a function of space-time and τ^a are Pauli matrices.

It can be shown that the covariant derivative for such a triplet has an expression

$$D_\mu L_m(T) = \partial_\mu L_m(T) + ig[A_\mu, L_m(T)] \qquad (4.6)$$

with

$$A_\mu = \frac{1}{2}\sum_{a=1}^{3} A_\mu^a \tau^a \qquad (4.7)$$

We may now write the gauge invariant Lagrangian density $\mathcal{L}_{oL(T)}$ for the triplets $L_m(T)$ (excluding mass terms):

$$\mathcal{L}_{oL(T)} = \frac{1}{2}\sum_{m=1}^{F} \mathrm{Trace}\,(\bar{L}_m(T)\, i\gamma^\mu D_\mu L_m(T)) \qquad (4.8)$$

$\mathcal{L}_{oL(T)}$ can be expanded in terms of $W^{\pm}$ in (2.5a), A_3, E_L^c, E_L and N_L as

$$\mathcal{L}_{0L(T)} = \sum_m \{ \overline{N}_{m_L} i\gamma^\mu \partial_\mu N_{m_L} + \overline{E}_{m_L} i\gamma^\mu \partial_\mu E_{m_L} + \overline{E}^c_{m_L} i\gamma^\mu \partial_\mu E^c_{m_L}$$

$$- g(\overline{N}_{m_L}\gamma^\mu E_{m_L} + \overline{E}^c_{m_L}\gamma^\mu N_{m_L}) W^+_\mu$$

$$- g(\overline{N}_{m_L}\gamma^\mu E^c_{m_L} + \overline{E}_{m_L}\gamma^\mu N_{m_L}) W^-_\mu$$

$$- g(\overline{E}^c_m \gamma^\mu E^c_{m_L} - \overline{E}_{m_L}\gamma^\mu E_{m_L}) A_{3\mu} \} \tag{4.9}$$

Furthermore, it may be simplified by using the identities (see appendix C) for the anticommuting fermions E_m :

$$\text{(i)} \quad (\overline{E^c_m})_L \gamma^\mu \partial_\mu (E^c_m)_L = \overline{E}_{m_R} \gamma^\mu \partial_\mu E_{m_R} , \tag{4.10}$$

$$\text{(ii)} \quad (\overline{E^c_m})_L \gamma^\mu (E^c_m)_L = -\overline{E}_{m_R} \gamma^\mu E_{m_R} . \tag{4.11}$$

We also use

$$A_{3\mu} = \frac{g' A_\mu - g Z_\mu}{\sqrt{g'^2 + g^2}} , \quad e = \frac{g g'}{\sqrt{g'^2 + g^2}} , \quad \cos\theta_W = \frac{g}{\sqrt{g'^2 + g^2}} .$$

Finally, the Lagrangian $\mathcal{L}_{0L(T)}$ is

$$\mathcal{L}_{0L(T)} = \sum_{m=1}^{F} \{ (\overline{N}_{m_L} i\gamma^\mu \partial_\mu N_{m_L} + \overline{E}_m i\gamma^\mu \partial_\mu E_m)$$

$$- g(\overline{N}_{m_L}\gamma^\mu E_{m_L} + (\overline{E^c_m})_L \gamma^\mu N_{m_L}) W^+_\mu$$

$$- g(\overline{N}_{m_L}\gamma^\mu (E^c_m)_L + \overline{E}_{m_L}\gamma^\mu N_{m_L}) W^-_\mu$$

$$+ e\,\overline{E}_m \gamma^\mu E_m A_\mu - (g^2 + g'^2)^{1/2} \cos^2\theta_W\, \overline{E}_m \gamma^\mu E_m Z_\mu \} . \tag{4.12}$$

4.3 The N-Family Doublets And F-Family Triplets

Add the usual N-families of doublets $L_n(D) = \begin{pmatrix} \nu_{n_L} \\ \ell_{n_L} \end{pmatrix}$ and the associated right-handed charged lepton ℓ_{n_R} which are singlets for each family, n=1,2,....,N. The total Lagrangian $\mathcal{L}_0$ we have is

$$\mathcal{L}_0 = \mathcal{L}_{0\,L(T)} + \mathcal{L}_{0\,L(D)} \tag{4.13}$$

where $\mathcal{L}_{0\,L(D)}$ is described in (2.69).

The Mass Terms

In the standard model, only the Yukawa couplings $\overline{L}_n(D)\phi\,\ell_{n_R}$ are present as in (2.70). With the additional triplets introduced, we now search for the possible new invariant Yukawa couplings or gauge invariant mass terms. First, let us construct all possible independent bilinear fermion fields with the triplets $L_m(T)$:

$$\text{(i)}\quad \overline{L^c_m}(D)L_n(T) \sim (\underline{2},-1)\otimes(\underline{3},0) = (\underline{4},1)\oplus(\underline{2},-1), \tag{4.14a}$$

$$\text{(ii)}\quad \overline{L^c_m}(T)L_n(T) \sim (\underline{3},0)\otimes(\underline{3},0) = (\underline{5},0)\oplus(\underline{3},0)\oplus(\underline{1},0), \tag{4.14b}$$

$$\text{(iii)}\quad \overline{L^c_m}(T)\,\ell_{n_R} \sim (\underline{3},0)\otimes(\underline{1},-2) = (\underline{3},-2). \tag{4.14c}$$

Obviously, with only the Higgs doublets $\phi \sim (\underline{2},1)$ in the theory, only the bilinear term (i) is possible to join with ϕ to form

Yukawa couplings. Therefore, we obtain‡

$$-\mathcal{L}_{yuk.} = \sum_{m=1}^{N}\sum_{n=1}^{F} G'_{mn}(\overline{L^c_m}(D) L_n(T)\Phi) + H.C. \qquad (4.15)$$

where G'_{mn} are the Yukawa coupling constants.

Also, there exist gauge invariant mass terms formed by the bilinear term $\overline{L^c_m}(T) L_n(T)$:

$$-\mathcal{L}_{mass\,L(T)} = \sum_{m,n=1}^{F} G''_{mn}\, Tr\,(\overline{L^c_m}(T) L_n(T)) + H.C. \qquad (4.16)$$

Adding (4.15), (4.16) and (2.70) together, we have

$$\begin{aligned} &-(\mathcal{L}_{total\,yuk.} + \mathcal{L}_{mass\,L(T)}) \\ &= \sum_{m,n=1}^{N}(G_{mn}\overline{L_m}(D)\phi\, l_{R_n}) + \sum_{m=1}^{N}\sum_{n=1}^{F}(G'_{mn}\overline{L^c_m}(D) L_n(T)\Phi) \\ &+ \sum_{m,n=1}^{F}(G''_{mn}\, Tr\,(\overline{L^c_m}(T) L_n(T)) + H.C. \qquad (4.17) \end{aligned}$$

Then, the Lagrangian $\mathcal{L}_{mass}$ for the mass terms would be

$$\begin{aligned} -\mathcal{L}_{mass} = \Big\{ &\sum_{m,n=1}^{N} m_{mn}\,\overline{l}_{m_L} l_{n_R} \\ &+ \sum_{m=1}^{N}\sum_{n=1}^{F}(D^0_{mn}\,\overline{l}_{m_L} E_{n_R} - \frac{D^0_{mn}}{\sqrt{2}}\,\overline{\nu^c_{m_L}} N_{n_L}) \\ &+ \sum_{m,n=1}^{F}(-M_{mn}\overline{(N_{m_L})^c} N_{n_L} + M_{mn}\overline{E}_{m_R} E_{n_L})\Big\} + H.C. \end{aligned} \qquad (4.18)$$

---

‡ $\phi^{+}\overline{L^c_m}(T) L_n(D) + H.C.$ is also an invariant term but it is equivalent to (4.15).

where we have taken

$$\mathcal{M}_{mn} = G_{mn} \frac{v}{\sqrt{2}} , \quad (4.19a)$$

$$D^{\circ}_{mn} = -G'_{mn} v, \quad (4.19b)$$

$$M_{mn} = -2 G''_{mn} . \quad (4.19c)$$

Now, let us introduce column vectors for leptons and neutrinos as follows:

$$\ell^{\circ} = \begin{pmatrix} \ell_1^{\circ} \\ \vdots \\ \ell_N^{\circ} \\ E_1^{\circ} \\ \vdots \\ E_F^{\circ} \end{pmatrix} , \quad \nu^{\circ} = \begin{pmatrix} \nu_1^{\circ} \\ \vdots \\ \nu_N^{\circ} \\ N_1^{\circ} \\ \vdots \\ N_F^{\circ} \end{pmatrix} \quad (4.20a)$$

where the superscript zero is introduced to denote that these fields are not mass eigenstates.‡ Then $-\mathcal{L}_{mass}$ in (4.18) can be written as

$$-\mathcal{L}_{mass} = \overline{\ell_L^{\circ}} M_\ell \ell_R^{\circ} + \frac{1}{2} \overline{(\nu_L^{\circ})^c} M_\nu \nu_L^{\circ} + H.C. \quad (4.20b)$$

with

$$M_\ell = \begin{bmatrix} \mathcal{M} & D^{\circ} \\ 0 & M \end{bmatrix} , \quad (4.21a)$$

$$M_\nu = \begin{bmatrix} 0 & \frac{-D^{\circ}}{\sqrt{2}} \\ -\frac{D^{\circ T}}{\sqrt{2}} & -M \end{bmatrix} . \quad (4.21b)$$

M_ℓ, M_ν are (N+F)x(N+F) mixing mass matrices for leptons and

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‡Notice that we use a new set of notations which are different from chapter 3 to distinguish the fields whether they are the mass eigenstates or not.

their associated neutrinos. $\mathcal{M}$ is a NxN matrix, and D° is a FxN matrix, and M is a FxF matix.

Clearly M in (4.21b) is the Majorana mass matrix for N°_{m}. But what is D° which is the mass matrix for the bilinear terms $\overline{\nu^{\circ}_{m_L}}(N^{\circ}_{n_L})^{c} = \overline{\nu^{\circ}_{m_L}}(N^{c}_{n})_{R}$? In fact, we can always identify $(N^{\circ c}_{m})_{R}$ as the right-handed components of the usual neutrinos in doublets, we then can interpret the matrix D° as the Dirac mass matrix for ν°_{m}. Hence, the neutrino mass matrix can be thought of as Dirac and Majorana types.

Lagrangian Density

Now, the free Lagrangian density $\mathcal{L}^{\circ}_{\ell}$ for lepton fields can be written

$$\mathcal{L}^{\circ}_{\ell} = \overline{\nu^{\circ}_{L}} i\gamma^{\mu}\partial_{\mu}\nu^{\circ}_{L} + \overline{\ell^{\circ}} i\gamma^{\mu}\partial_{\mu}\ell^{\circ}$$

$$- (\overline{\ell^{\circ}_{L}} M_{\ell}\ell^{\circ}_{R} + \frac{1}{2}\overline{(\nu^{\circ}_{L})^{c}} M_{\nu}\nu^{\circ}_{L} + \mathrm{H.C.}) . \qquad (4.22)$$

And the Lagrangian for the electromagnetic and weak interactions can be written

$$\mathcal{L}_{int.} = -g J^{\mu}_{W} W^{-}_{\mu} - g J^{\mu +}_{W} W^{+}_{\mu}$$

$$+ (g^{2}+g'^{2})^{1/2} J^{\mu}_{Z} Z_{\mu} + e J^{\mu}_{EM} A_{\mu} \qquad (4.23)$$

where

$$J_W^\mu = \overline{\ell_L^0}\gamma^\mu T_1 \nu_L^0 + \overline{\nu_L^0}\gamma^\mu T_2 (\ell^{0c})_L , \tag{4.24a}$$

$$J_W^{\mu\dagger} = \overline{\nu_L^0}\gamma^\mu T_1 \ell_L^0 + \overline{(\ell^{0c})}_L \gamma^\mu T_2 \nu_L^0 , \tag{4.24b}$$

$$J_Z^\mu = \overline{\ell_L^0}\gamma^\mu T_1^Z \ell_L^0 + \overline{\nu_L^0}\gamma^\mu T_2^Z \nu_L^0 + \overline{\ell_R^0}\gamma^\mu T_3^Z \ell_R^0 \tag{4.24c}$$

$$J_{EM}^\mu = \overline{\ell^0}\gamma^\mu \ell^0 . \tag{4.24d}$$

T_1, T_2, T_1^Z, T_2^Z and T_3^Z are (N+F)x(N+F) diagonal matrices:

$$T_1 = \text{diag.}\left[\frac{1}{\sqrt{2}}, \cdots\cdots, \frac{1}{\sqrt{2}}, 1, \cdots\cdots, 1\right], \tag{4.45a}$$

$$T_2 = \text{diag.}\left[\, 0, \cdots\cdots, 0, 1, \cdots\cdots, 1\right], \tag{4.45b}$$

$$T_1^Z = \text{diag.}\left[-\frac{1}{2}+\sin^2\theta_W, \cdots\cdots, -\frac{1}{2}+\sin^2\theta_W, -\cos^2\theta_W, \cdots, -\cos^2\theta_W\right], \tag{4.45c}$$

$$T_2^Z = \text{diag.}\left[\frac{1}{2}, \cdots\cdots, \frac{1}{2}, 0, \cdots\cdots, 0\right], \tag{4.45d}$$

$$T_3^Z = \text{diag.}\left[\sin^2\theta_W, \cdots\cdots, \sin^2\theta_W, -\cos^2\theta_W, \cdots\cdots, -\cos^2\theta_W\right]. \tag{4.45e}$$

Mass Eigenstates

Since M_ℓ, M_ν are in general not diagonal, ℓ^0, ν^0 are not mass eigenstates and are not physical observable states. As discussed in Chapter 3, M_ℓ and M_ν can always be diagonalized by means of (N+F)x(N+F) unitary matrices A_L, A_R and U such that

$$A_L^\dagger M_\ell A_R = \mathrm{Diag.}(m_{\ell_1}, \ldots\ldots, m_{\ell_{n+F}}) = M_{\ell_D} \qquad (4.26)$$

and

$$U^T M_\nu U = \mathrm{Diag.}(m_{\nu_1}, \ldots\ldots, m_{\nu_{n+F}}) = M_{\nu_D} \qquad (4.27)$$

where M_{ℓ_D} and M_{ν_D} are real diagonal matrices, m_{ℓ_m} and m_{ν_m} are the mass eigenvalues for the charged leptons and the neutrinos.

The mass matrix in (4.21b) has an NxN zero matrix in the upper left corner. It is easy to verify that an arbitrary matrix of this type has an (N-F) dimensional null space[17]. Since the rank of a matrix is preserved under the transformation (4.27), we conclude that the diagonal matrix for neutrinos M_{ν_D} has (N-F) zeros. This implies that in order for N-families of neutrinos in the usual doublets to acquire nonzero masses, at least N-families of triplets are needed. We also notice that it is possible to make a U(N-F) transformation without affecting the Lagrangian; therefore, in general less parameters are needed to parametrize the unitary matrix U in (4.27).

Lagrangian Density In Mass Eigenstates

Let us take ℓ, ν to be the column vectors of mass eigenstates for leptons and neutrinos:

$$\ell_L^0 = A_L \ell_L \,, \tag{4.28a}$$

$$\ell_R^0 = A_R \ell_R \,, \tag{4.28b}$$

$$\nu_L = U \nu_L \,. \tag{4.28c}$$

Then, $\mathcal{L}_\ell^0$ is written in terms of the mass eigenstates ℓ, ν :

$$\begin{aligned}\mathcal{L}_\ell^0 = {} & \overline{\nu}_L i \gamma^\mu \partial_\mu \nu_L + \overline{\ell} i \gamma^\mu \partial_\mu \ell \\ & -\left(\overline{\ell}_L M_{\ell_D} \ell_R + \frac{1}{2} \overline{\nu_L^c} M_{\nu_D} \nu_L + \mathrm{H.C.} \right) .\end{aligned} \tag{4.29}$$

And the currents in (4.24) are written as follows:

$$J_W^\mu = \overline{\ell}_L \gamma^\mu (A_L^\dagger T_1 U) \nu_L + \overline{\nu}_L \gamma^\mu (U^\dagger T_2 A_R) \ell_L^c \,, \tag{4.30a}$$

$$J_W^{\mu\dagger} = \overline{\nu}_L \gamma^\mu (U^\dagger T_1 A_L) \ell_L + \overline{(\ell^c)}_L \gamma^\mu (A_R^\dagger T_2 U) \nu_L \,, \tag{4.30b}$$

$$\begin{aligned}J_Z^\mu = {} & \overline{\ell}_L \gamma^\mu (A_L^\dagger T_1^Z A_L) \ell_L + \overline{\nu}_L \gamma^\mu (U^\dagger T_2^Z U) \nu_L \\ & + \overline{\ell}_R \gamma^\mu (A_R^\dagger T_3^Z A_R) \ell_R \,,\end{aligned} \tag{4.30c}$$

$$J_{EM}^\mu = \overline{\ell} \gamma^\mu \ell \,. \tag{4.30d}$$

4.4 Three-Families Of Lepton Doublets And One Triplet

Instead of working with the most general case, for simiplicity, we restrict the calculations to the three known families of leptons in SU(2) doublets and one family of leptons in an SU(2) triplets; that is N=3 and F=1. Although this restricted model will have two massless neutrinos, it does retain all the main features of the more generalized case. Again for simiplicity, we assume CP invariance; i.e. M_ν and M_ℓ are real matrices. Since the mass of these new triplet fields probably will be a lot heavier than the known leptons, we will retain only those nonzero leading terms of order $\frac{m_\ell}{M}$ ($m_\ell(M)$ is the mass for a light (heavy) lepton) in our calculations.

The three-family doublets in SU(2) are

$$\begin{pmatrix} \nu_e^0 \\ e^0 \end{pmatrix}_L , \quad \begin{pmatrix} \nu_\mu^0 \\ \mu^0 \end{pmatrix}_L , \quad \begin{pmatrix} \nu_\tau^0 \\ \tau^0 \end{pmatrix}_L ; \; e_R^0, \mu_R^0, \tau_R^0$$

and the triplet is

$$\begin{bmatrix} N^0 & -\sqrt{2}E^{0c} \\ \sqrt{2}E^0 & -N^0 \end{bmatrix}_L$$

The column vectors in (4.20a) now become

$$\ell^\circ = \begin{pmatrix} e^\circ \\ \mu^\circ \\ \tau^\circ \\ E^\circ \end{pmatrix} \quad , \quad \nu^\circ = \begin{pmatrix} \nu_e^\circ \\ \nu_\mu^\circ \\ \nu_\tau^\circ \\ N^\circ \end{pmatrix} \quad . \tag{4.31}$$

Now, the matrix $\mathcal{M}$ in (4.21a) is a 3x3 matrix, M in (4.21) is just a scalar, D° is 1x3 matrix and will denoted by the column vector $D^\circ = \begin{pmatrix} d_1^\circ \\ d_2^\circ \\ d_3^\circ \end{pmatrix}$,and $D^{\circ T}$ is the row vector $(d_1^\circ, d_2^\circ, d_3^\circ)$. We shall assume that M is much larger than any other elements in the mass matrices M_e and M_ν , ie. $M \gg d_i, m_{ij}$.

As in section (3.4), M_ν can be diagonalized by an orthogonal matrix O . If only the leading contribution will be retained, we have the eigenvalues for the mass matrix M_ν as follows:

$$O^T M_\nu O = M_{\nu_D} = \mathrm{Diag}\left[0, 0, \frac{D^{\circ T} D^\circ}{2M}, -M\right] . \tag{4.32}$$

To find the 4x4 orthogonal matrix O with aproximation, we use the following ansatz:

$$O = \left[\begin{array}{c|c} \mathcal{U}^\circ & \dfrac{D^\circ}{\sqrt{2}M} \\ \hline -\dfrac{D^{\circ T}\mathcal{U}^\circ}{\sqrt{2}M} & 1 \end{array} \right] \tag{4.33}$$

where $\mathcal{U}^\circ$ is 3x3 orthogonal matrix. It is easy to show that the matrix O is orthogonal up to terms of order $\frac{m}{M}$ (m is just any matrix elements other than M): $O^T O \cong 1$. We also obtain

$$O^T M_\nu O = M_{\nu_D} = \left[\begin{array}{c|c} \dfrac{\mathcal{U}^{oT} D^o D^{oT} \mathcal{U}^o}{2M} & 0 \\ \hline 0 & -M \end{array}\right] \tag{4.34}$$

Hence, in order to satisfy (4.32), the following relation

$$\mathcal{U}^{oT} D^o = \begin{pmatrix} 0 \\ 0 \\ \pm\sqrt{D^{oT} D^o} \end{pmatrix} \tag{4.35}$$

must be satisfied. Clearly, instead of the usual three parameters, only two parameters are needed to parametrize the 3x3 orthogonal matrix $\mathcal{U}^o$. It is because there are two degenerate masses for the neutrinos.

The two mass-eigenstate neutrinos ν_1, ν_2 with zero-mass eigenvalues will still remain as left-handed two-component massless fields:

$$\nu_{m_L} = \sum_{n=1}^{4} O^T_{mn} \nu^o_{n_L} \quad , \qquad \text{for } m = 1, 2 \tag{4.36}$$

The other two massive-eigenstate neutrinos χ_3, χ_4 of the Majorana type as in (3.36) with mass eigenvalues $m_{\chi_3} = \dfrac{D^{oT} D^o}{2M}$ and $m_{\chi_4} = M$ will be

$$\begin{aligned} \chi_m &= \sum_{n=1}^{4} O^T_{mn} \nu^o_{n_L} + \eta_m O^T_{mn} (\nu^o_{n_L})^c \\ &= \nu_{m_L} + \eta_m (\nu_{m_L})^c , \qquad \text{for } m = 3, 4 \end{aligned} \tag{4.37}$$

where $\nu_m = \sum_{n=1}^{4} O^T_{mn} \nu^o_{n_L}$.

The CP eigenvalues of χ_3 and χ_4 are $\eta_3=1$ and $\eta_4=-1$ (see the discussion in section 3.4).

The mass matrix M_ℓ is also diagonalized by means of a transformation as in (3.9). The mass eigenvalues for the charged leptons are:

$$A_L^{+} M_\ell A_R = M_{\ell_D} = \mathrm{Diag}\,[m_e, m_\mu, m_\tau, M] \,. \qquad (4.38)$$

Again, to find the 4x4 matrices A_L, A_R, we use the following ansätze:

$$A_L = \left[\begin{array}{c|c} V_L & \dfrac{D^{\circ}}{M} \\ \hline -\dfrac{D^{\circ T} V_L}{M} & 1 \end{array}\right] \qquad (4.39a)$$

and

$$A_R = \left[\begin{array}{c|c} V_R & \dfrac{\mathcal{M}^T D^{\circ}}{M^2} \\ \hline -\dfrac{D^{\circ T} \mathcal{M} V_R}{M^2} & 1 \end{array}\right] \qquad (4.39b)$$

where V_L and V_R are 3x3 orthogonal matrices. It is easy to show that the A_L and A_R‡ are orthogonal up to terms of order $\frac{\mathcal{M}}{M}$: $A_L^T A_L \cong 1$, $A_R^T A_R \cong 1$.. We also find

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‡(i)Since all matrices are real, $V_L^{+} = V_L^T$, $V_R^{+} = V_L^T$, and $\mathcal{M}^{+} = \mathcal{M}^T$.
(ii)The matrix O (4.33), A_L and A_R (4.39) are also applicable for a general n-families of doublets and one triplet model.

$$A_L^T M_L A_R = \begin{bmatrix} V_L^T \mathcal{M} V_R & 0 \\ 0 & M \end{bmatrix} . \tag{4.40}$$

Hence, from (4.38), we obtain

$$V_L^T \mathcal{M} V_R = \mathcal{M}_D = \mathrm{diag.}[m_e, m_\mu, m_\tau] . \tag{4.41}$$

Now, let us define the column vectors for the mass eigenstates of neutrinos‡ and leptons:

$$\ell = \begin{pmatrix} e \\ \mu \\ \tau \\ E \end{pmatrix} \quad , \quad \nu = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \chi_3 \\ \chi_4 \end{pmatrix} . \tag{4.42}$$

Finally, the weak currents in (4.30) are

$$J_W^\mu = \bar{\ell}_L \gamma_\mu \tau_1 \nu_L + \overline{\nu_L} \gamma_\mu \tau_2 (\ell^c)_L \quad , \tag{4.43a}$$

$$J_W^{\mu +} = \bar{\nu}_L \gamma_\mu \tau_1^T \ell_L + \overline{(\ell^c)}_L \gamma_\mu (\tau_2)^T \nu_L \quad , \tag{4.43b}$$

$$J_Z^\mu = \bar{\ell}_L \gamma_\mu \tau_1^Z \ell_L + \bar{\nu}_L \gamma_\mu \tau_2^Z \nu_L + \bar{\ell}_R \gamma_\mu \tau_3^Z \ell_R . \tag{4.43c}$$

with the definitions

$$D = V_L^T D^0 \tag{4.44a}$$

---

‡ Notice that we have

$$\nu_L = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \chi_3 \\ \chi_4 \end{pmatrix}_L = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \end{pmatrix}_L \quad , \quad (\nu_L)^c = \begin{pmatrix} \nu_{1L}^c \\ \nu_{2L}^c \\ \eta_3 \chi_{3R} \\ \eta_4 \chi_{4R} \end{pmatrix} .$$

and

$$\mathcal{U} = V_L^T \mathcal{U}^\circ \tag{4.44b}$$

We have

$$\tau_1 = A_L^T T_1 O = \begin{bmatrix} \frac{1}{\sqrt{2}}\mathcal{U} & -\frac{D}{2M} \\ 0 & 1 \end{bmatrix}, \tag{4.45a}$$

$$\tau_3 = O^T T_2 A_R = \begin{bmatrix} \frac{\mathcal{U}^T D D^T \mathcal{M}_D}{\sqrt{2} M^3} & \frac{\mathcal{U}^T D}{\sqrt{2} M} \\ \frac{-D^T \mathcal{M}_D}{M^2} & 1 \end{bmatrix}, \tag{4.45b}$$

$$\tau_1^z = A_L^T T_1^z A_L = \begin{bmatrix} (-\frac{1}{2} + \sin^2\theta_W) I & \frac{D}{2M} \\ \frac{D^T}{2M} & -\cos^2\theta_W \end{bmatrix}, \tag{4.45c}$$

$$\tau_2^z = O^T T_2^z O = \begin{bmatrix} \frac{1}{2} I & \frac{\mathcal{U}^T D}{2\sqrt{2} M} \\ \frac{D^T \mathcal{U}}{2\sqrt{2} M} & \frac{D^T D}{4M^2} \end{bmatrix}, \tag{4.45d}$$

$$\tau_3^z = A_R^T T_3^z A_R = \begin{bmatrix} \sin^2\theta_W I & \frac{\mathcal{M}_D D}{M^2} \\ \frac{D^T \mathcal{M}_D}{M^2} & -\cos^2\theta_W \end{bmatrix}. \tag{4.45e}$$

where I is 3x3 identity matrix and $\mathcal{M}_D$ is in (4.41). Notice that the condition of the GIM mechanism in the neutral currents for the usual three families of leptons is still satisfied in this order of approximaton. That is, a lepton in one family will not change to another one up to order $\frac{m}{M}$ in a neutral current process. However, there exist family violating neutral interactions between light leptons and the new heavy leptons with order $\frac{d_i}{M}$ suppressed. We also notice from the matrix τ_2 that the new lepton-number-violating interactions, which are possible in our model, are naturally much weaker than the standard weak interactions for the known families of leptons.

It follows from (4.35) and (4.44) that

$$\mathcal{U}^T D = \mathcal{U}^{0T} D^0 = \begin{pmatrix} 0 \\ 0 \\ \sqrt{D^T D} \end{pmatrix}^{\ddagger} \tag{4.46}$$

where $\mathcal{U}$ can be paramatrized with two parameters θ_1 and θ_2 :

$$\mathcal{U} = \begin{bmatrix} C_1 & 0 & -S_1 \\ -S_1 S_2 & C_2 & -C_1 S_2 \\ S_1 C_2 & S_2 & C_1 C_2 \end{bmatrix} \tag{4.47}$$

with $C_i = \cos\theta_i$, $S_i = \sin\theta_i$. θ_1 and θ_2 are related to d_1, d_2, d_3 and $d = \sqrt{D^T D}$ as follows:

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‡ we take the positive sign.

$$\frac{d_1}{d} = -S_1 \tag{4.48a}$$

$$\frac{d_2}{d} = -C_1 S_2 \tag{4.48b}$$

$$\frac{d_3}{d} = C_1 C_2 \tag{4.48c}$$

With θ_1, θ_2, $m_{\chi_3} = \frac{d^2}{2M}$ and M, we have introduced four new undetermined parameters into the theory.

V. Chapter 5 Neutrino Oscillations

The concept of oscillations among different families of neutrinos was first postulated by Pontecorvo[5] (1967) and Maki[6] (1962). The first phenomenological theory of neutrino oscillations was constructed by Gribov and Pontecorvo[25] (1968). Neutrino oscillations are possible provided

(i) the neutrinos have non-degenerate masses, and

(ii) the neutrino mass eigenstates ν_n with mass m_{ν_n} differ from the weak charged current eigenstates ν_n^0 .

5.1 The Formulation Of Neutrino Oscillations

Suppose that there exist N physical neutrinos (mass eigenstates) $|\nu_n\rangle$ with mass m_{ν_n} . The weak eigenstates $|\nu_n^0\rangle$ are linear superpositions of the mass eigenstates

$$|\nu_m^0\rangle = \sum_{n=1}^{N} U_{mn} |\nu_n\rangle \quad , \quad m, n = 1, \cdots\cdots, N \qquad (5.1)$$

Now, let us discuss the behaviour of a beam of neutrinos produced in some weak processes. The neutrino starts out with definite family index m at time t=0, x=0: $|\nu(0,0)\rangle = |\nu_m^0\rangle$. Its wave function will evolve in space-time as follows:

$$|\nu(x,t)\rangle = \sum_{n=1}^{N} U_{mn} e^{i(P_n x - E_n t)} |\nu_n\rangle \, . \qquad (5.2)$$

We assume each neutrino has momentum P_n and energy E_n. These should be understood here as the average values for a wave-packet‡. As we shall show in Chapter 8, the radiative decay rates of the heavier neutrinos into the light neutrinos are very small§. Hence we can treat neutrinos as stable particles. By using $|\nu_m\rangle = \sum_n U^{+}_{mn} |\nu^{o}_n\rangle$, we have

$$|\nu(x,t)\rangle = \sum_{n,\ell=1}^{N} U_{m\ell} U^{*}_{n\ell} e^{i(P_\ell x - E_\ell t)} |\nu^{o}_n\rangle \tag{5.3}$$

which is a superposition of weak eigenstates $|\nu^{o}_n\rangle$.

Hence, if we initially have ν^{o}_m, the transition amplitude A for observing ν^{o}_n is

$$A(\nu^{o}_m \longrightarrow \nu^{o}_n) = \sum_{\ell=1}^{N} U_{m\ell} U^{*}_{n\ell} e^{i(P_\ell x - E_\ell t)} . \tag{5.4}$$

The transition probability P is

$$\begin{aligned} P(\nu^{o}_m \longrightarrow \nu^{o}_n) &= |A(\nu^{o}_m \rightarrow \nu^{o}_n)|^2 \\ &= \sum_{k=1}^{N} \sum_{\ell=1}^{N} U_{m\ell} U_{nk} U^{*}_{mk} U^{*}_{n\ell} e^{i\phi_{\ell k}} \end{aligned} \tag{5.5}$$

where the phase

$$\begin{aligned} \varphi_{\ell k}(x,t) &= (P_\ell - P_k) x - (E_\ell - E_k) t \\ &= (P_\ell - P_k)\left[x - \frac{P_\ell + P_k}{E_\ell + E_k} t \right] - \frac{(m_\ell^2 - m_k^2) t}{E_\ell + E_k} . \end{aligned} \tag{5.6}$$

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‡Boris Kayser[26] has shown that the general analysis for the case of the wave-packet with small momentum and energy spreads will lead to the same results as the analysis here.

§except the decay rate of the heavy neutrino χ_4, but no oscillation for χ_4 is considered.

The velocity of the combined wave packets for the neutrinos ν_ℓ and ν_k is

$$v_{\ell k} = \frac{P_\ell + P_k}{E_\ell + E_k} \tag{5.7}$$

The interference between the two neutrinos ν_ℓ and ν_k can only be observed when

$$x = v_{\ell k}\, t \tag{5.8}$$

where x is the distance from the source. Therefore, the first term in (5.6) vanishes, and the phase is

$$\varphi_{\ell k} = \frac{-(m_\ell^2 - m_k^2)t}{P_\ell + P_k} . \tag{5.9}$$

For $m_\ell, m_k \ll P_\ell, P_k$, $v_{\ell k} \cong 1$ for all ℓ, k, and $P_\ell \simeq P_k = P$,[27] we obtain

$$\varphi_{\ell k} \cong \frac{-(m_\ell^2 - m_k^2)\,x}{2P} . \tag{5.10}$$

Defining the oscillation length $L_{\ell k}$

$$L_{\ell k} = \frac{4\pi P}{m_\ell^2 - m_k^2} \tag{5.11}$$

and substituting the numerical values of $\hbar$ & c, we get the convenient formula

$$L_{\ell k} = \frac{2.5\, P/\mathrm{MeV}}{|m_\ell^2 - m_k^2|/(\mathrm{eV})^2} \text{ meters} . \tag{5.12}$$

If we assume CP invariance, then U is a real matrix. With the initial neutrino ν_m^0, the probability of finding a neutrino ν_n^0 at a distance x is

$$P(\nu_m^0 \longrightarrow \nu_n^0) = \delta_{mn} - \sum_{\ell<k} 4\, U_{m\ell} U_{n\ell} U_{mk} U_{nk} \sin^2\left(\frac{2\pi x}{L_{\ell k}}\right) . \tag{5.13a}$$

Since CP is conserved, we have

$$P(\overline{\nu}_m^0 \longrightarrow \overline{\nu}_n^0) = P(\nu_m^0 \longrightarrow \nu_n^0) . \tag{5.13b}$$

It is clear that the oscillating terms in $P(\nu_m^0 \rightarrow \nu_n^0)$ come from the interference between the different mass eigenstates in the neutrino wave function. It is purely a quantum mechanical effect. The oscillation length in (5.12) depends on the momentum as well as the mass difference among neutrinos while the amplitude of the oscillations depends on the mixing matrix U. Needless to say, neutrino oscillations are of great importance in finding the neutrino mass scales and mixing angles.

Let us apply (5.13) to our model which consists of four weak eigenstates $\nu_n^0 = (\nu_e, \nu_\mu, \nu_\tau, N)$ and four mass eigenstates $\nu_n = (\nu_1, \nu_2, \chi_3, \chi_4)$. We will neglect the existence of the neutrino χ_4 and the oscillations for N because χ_4 is too massive and unstable (see Chapter 7 & 8), and the couplings of the N with (ν_1, ν_2, χ_3) are of order $\sqrt{\frac{m_{\chi_3}}{M}}$. Now we can identify $U = \mathcal{U}$ in (4.47). We obtain

$$P(\nu_{e_L} \longrightarrow \nu_{e_L}) = 1 - 4 C_1^2 S_1^2 L(x) , \tag{5.14a}$$

$$P(\nu_{\mu_L} \longrightarrow \nu_{\mu_L}) = 1 - 4(s_1^2 s_2^2 + \cancel{c_2^2}) c_1^2 s_2^2 L(x), \qquad (5.14b)$$

$$P(\nu_{\tau_L} \longrightarrow \nu_{\tau_L}) = 1 - 4(s_1^2 c_2^2 + s_2^2) c_1^2 c_2^2 L(x), \qquad (5.14c)$$

$$P(\nu_{e_L} \longrightarrow \nu_{\mu_L}) = 4\,(s_1 c_1 s_2)^2 L(x)\,, \qquad (5.14d)$$

$$P(\nu_{e_L} \longrightarrow \nu_{\tau_L}) = 4\,(c_1 s_1 c_2)^2 L(x)\,, \qquad (5.14e)$$

$$P(\nu_{\mu_L} \longrightarrow \nu_{\tau_L}) = 4\,(c_1^2 s_2 c_2)^2 L(x)\,. \qquad (5.14f)$$

where

$$L(x) = \sin^2\left(\frac{2\pi x}{L}\right) \qquad (5.15)$$

and the oscillation length L,

$$L = 2.5\,\frac{P/\mathrm{MeV}}{m_{\chi_3}^2/\mathrm{eV}^2} \qquad (5.16)$$

is the same in all cases. This is due to the fact that only the neutrino χ_3 has a nonzero mass m_{χ_3}. Theoretically the mass can be determined by an oscillation experiment.

Oscillation experiments can be done with various sources. The oscillation effect can in principle be observed provided that the mixing angles are large, the neutrino source is localized within a region much smaller than the oscillation length, and the coherence of neutrino beams is not absent[1,2]. The following table is listed with the typical observation lengths which can be achieved, and the typical mass to which they are sensitive.

Table 5.1 - Sensitivity For Various Neutrino Sources[28]

Source	Mean energy	L(m)	Lower limit on Δm(eV)
Solar ν_e	100 keV	10^{11}	10^{-6}
Atmospheric ($\Phi_{\nu_\mu} \sim 2\Phi_{\nu_e}$)	0.5 GeV	10^{7}	10^{-3}
Reactors $\bar{\nu}_e$	3 MeV	10	10^{-1}
Meson Factories (ν_μ, ν_e)	30 MeV	10-100	10^{-1}
Accelerators (ν_μ)	1-30 GeV	10^{2}-10^{4}	10^{-1}

If only the average value of the probability is observed at a sufficiently large distance, then $\langle L(x)\rangle$ =1/2, we obtain

$$\langle P(\nu_{e_L} \longrightarrow \nu_{\mu_L})\rangle = 2(S_1 C_1 S_2)^2 , \qquad (5.17a)$$

$$\langle P(\nu_{e_L} \longrightarrow \nu_{\tau_L})\rangle = 2(C_1 S_1 C_2)^2 , \qquad (5.17b)$$

$$\langle P(\nu_{\mu_L} \longrightarrow \nu_{\tau_L})\rangle = 2(C_1^2 S_2 C_2)^2 . \qquad (5.17c)$$

If the maximum mixings are assumed $\theta_1 = \theta_2 = \frac{\pi}{4}$, then the oscillations between $\nu_\mu \leftrightarrow \nu_\tau$ are suppressed by factor (1/2) relative to $\nu_e \leftrightarrow \nu_\mu$ and $\nu_e \leftrightarrow \nu_\tau$ oscillations. If the mixing angle θ_2 is small, then the oscillatins between $\nu_e \leftrightarrow \nu_\mu$ and $\nu_\mu \leftrightarrow \nu_\tau$ are suppressed with respect to $\nu_e \leftrightarrow \nu_\tau$ oscillations.

No firm evidence for neutrino oscillations is found although several anomalous experimental results have been reported.[2,28]

5.2 Neutrino-Antineutrino Oscillations

Besides the neutrino oscillations of the type $\nu_m^0 \to \nu_n^0$, there exists a different kind of neutrino oscillation: neutrino-antineutino oscillation. As we know, different weak-eigenstate neutrinos are different linear superpositions of mass eigenstates. Different kinds of weak-eigenstate neutrinos can only be identified by their associated leptons. The neutrino oscillation experiments are being done by identifying the kind of antilepton ℓ_m^+ (associated with ν_m^0) created at t=0 in a weak decay; and then what kind of leptons will be created by a neutrino beam ν_m^0 in a weak process at a later time. The amplitude of finding ℓ_n^- is proportional to $A(\nu_m^0 \to \nu_n^0)$ in (5.14). In our model, there exists a massive Majorana neutrino χ_3; therefore, the same neutrino beam which contains χ_3 can create a charged antilepton ℓ_n^+. The amplitude for such a process is proportional to[30]

$$A(\nu_m^0 \longrightarrow (\nu_n^0)_R^c) = \sum_{\ell=1}^{3} \frac{m_{\chi_\ell}}{E} U_{m\ell} U_{n\ell}\, e^{i(P_\ell x - E_\ell t)} \qquad (5.18)$$

This is the amplitude for the neutrino-antineutrino oscillation. With the assumption that CP is conserved, $U^* = U$, this implies

$P(\nu_m^0 \to (\nu_n^{0c})_R) = P((\nu_m^{0c})_R \to \nu_{n_L})$ and

$$P(\nu_{e_L} \longrightarrow \nu_{e_R}^c) = \left(\frac{m_{x_3}}{E} s_1^2\right)^2 , \qquad (5.19a)$$

$$P(\nu_{e_L} \longrightarrow \nu_{\mu_R}^c) = \left(\frac{m_{x_3}}{E} s_1 c_1 s_2\right)^2 , \qquad (5.19b)$$

$$P(\nu_{e_L} \longrightarrow \nu_{\tau_R}^c) = \left(\frac{m_{x_3}}{E} s_1 c_1 c_2\right)^2 , \qquad (5.19c)$$

$$P(\nu_{\mu_L} \longrightarrow \nu_{\mu_R}^c) = \left(\frac{m_{x_3}}{E} c_1^2 s_2^2\right)^2 , \qquad (5.19d)$$

$$P(\nu_{\mu_L} \longrightarrow \nu_{\tau_R}^c) = \left(\frac{m_{x_3}}{E} c_1^2 s_2 c_2\right)^2 , \qquad (5.19e)$$

$$P(\nu_{\mu_L} \longrightarrow \nu_{\tau_R}^c) = \left(\frac{m_{x_3}}{E} c_1^2 c_2^2\right)^2 . \qquad (5.19f)$$

The lepton-number violating processes are suppressed in intensity by $\left(\frac{m_{x_3}}{E}\right)^2$ as compared to the usual neutrino oscillations.

The process is proportional to $\frac{m_{x_n}}{E}$, the heavy neutrino with mass M is still neglected because it needs energy E >> M to create such a heavy neutrino. For present facilities, there is not enough energy to produce them; moreover, this heavy neutrino is very unstable.

The charged antilepton $\ell_{n_R}^+$ can also be created from the weak lepton-number-violating currents. The amplitude for such a process is of order $\sqrt{\frac{m_{x_3}}{M}} \cdot \frac{m_\ell}{M}$ which is even weaker than the previous process. (see (4.45a))

5.3 Neutrino Masses From Beta Decay ‡

The most sensitive way to determine the ν_e mass is to observe the deviations from a straight line Kurie plot near the end-point of tritium β-decay. The Kurie plot, in the presence of neutrino mixing, depends on the mixing angles and masses of all neutrino mass eigenstates which couple to the electron.

Let us consider a neutrino ν_n of mass m_{ν_n} emitted in β-decay. The Kurie function K takes the form

$$K_n^2 = F^2 \Delta \left(\Delta^2 - m_{\nu_n}^2\right)^{1/2} \theta\left(\Delta - m_{\nu_n}\right) \tag{5.20}$$

where $\Delta = E_0 - E_\beta$. Here E_0 is the maximum allowed electron kinetic energy and E_β is the kinetic energy of the electron. F is the nuclear Coulomb factor.

When there is neutrino mixing, the weak-eigenstate neutrino ν_e which couples to the electron is a linear combination of mass eigenstates ν_n. The Kurie function then becomes

$$K^2 = \sum_{n=1}^{N} P_n K_n^2 = \sum_{n=1}^{N} P_n \Delta\left(\Delta^2 - m_{\nu_n}^2\right)^{1/2} \theta\left(\Delta - m_{\nu_n}\right) F^2 \tag{5.21}$$

where P_n is the probability that the neutrino ν_n is emitted in β-decay. Since χ_4 is heavy, $m_{\chi_4} \gg E_0$, only the three neutrinos (ν_1, ν_2, χ_3) are present as in the neutrino oscillations. From (4.43a) and (4.47) we obtain

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‡This section follows closely the analysis of J. W. F. Valle and M. Singer.[31]

$$\frac{K^2}{F^2} = S_1^{\,2}\Delta(\Delta^2 - m_{\chi_3}^2)^{1/2} + C_1^{\,2}\Delta^2 \qquad (5.22)$$

As we see the deviation from a straight line near the end point of the kurie plot is determined by the masses of the neutrinos; the end-point of the Kurie spectrum is determined by the lightest neutrinos. We also notice that (5.22) does not depend on θ_2.

VI. Chapter 6 Neutrinoless Double Beta Decay

6.1 Possibilities Of Neutrinoless Double Beta Decay

Nuclear double beta decay is a second-order semileptonic processes accompanying a transition from a nucleus Z to Z+2. Theoretically the process can proceed in two ways:

(i) from standard second-order beta decay

$$Z \longrightarrow (Z+2) + \ell_1^- + \nu_1 + \ell_2^- + \nu_2 , \qquad (6.1)$$

(ii) by the no neutrino process

$$Z \longrightarrow (Z+2) + \ell_1^- + \ell_2^- \qquad (6.2)$$

known as neutrinoless double β-decay.

This second process violates lepton-number conservation. If observed, it would signify two possibilities or both:

(i) Neutrino has a finite Majoranà mass.

(ii) Lepton-number-violating currents exist.

These two cases correspond to two different mechanisms as shown in fig.6.1.

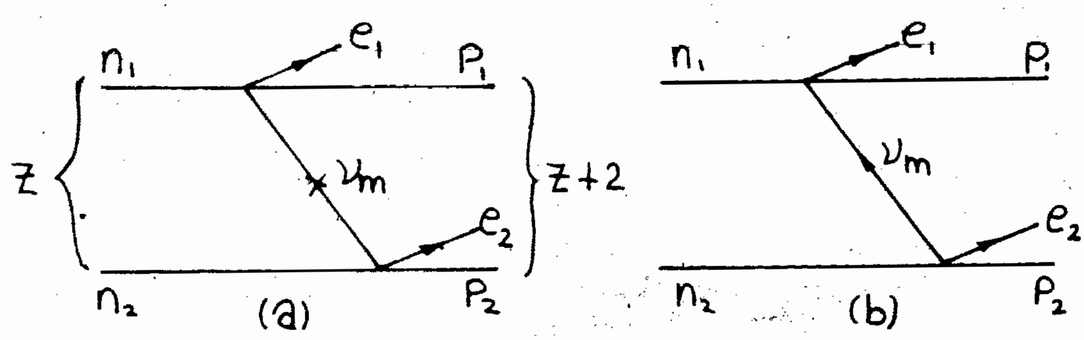


Fig.6.1 (a) neutrinoless double beta decay from a massive Majorana neutrino, (b) from a lepton-number-violating currents.

Both cases do exist in our model; here we will calculate the relative contribution to the decay amplitude of the neutrinoless double beta decay from each case.

6.2 Double Beta Decay From a Massive Majorana Neutrino

The decay amplitude A of the neutrinoless double-β decay is expressed as

$$A[(A,Z) \longrightarrow (A,Z+2)+e^-+e^-] \propto \sum_j \eta_j m_j (\tau_1)^2_{ej} \qquad (6.3)$$

where m_j is the mass of a Majorana neutrino χ_j, η_j is the eigenvalues for CP and τ_1 is the matrix given in (4.45a).

In our model, only two neutrinos χ_3, χ_4 have nonzero masses m_{χ_3} and M. It is easy to see from (6.3) that the contributions from these two neutrinos to the decay amplitude will tend to cancel each other because they have opposite CP eigenvalues; namely, $\eta_3 = 1$, $\eta_4 = -1$. We find that in this model, the cancellation is complete and the double beta decay cannot arise from such process. It can be understood by noting, using (4.34), that $(m_\nu)_{i\ell}$ in (4.21b) can be written as

$$
\begin{aligned}
(m_\nu)_{i\ell} &= \sum_{j,k=1}^{4} O_{ij}\,\eta_j\, m_{\nu_j}\,\delta_{jk}\,O^T_{k\ell} \\
&= \sum_{j=1}^{4} \eta_j\, m_{\nu_j}\, O_{ij}\, O_{\ell j} \quad . \qquad (6.4)
\end{aligned}
$$

Using (4.45a), we then see that the decay amplitude is

$$
\begin{aligned}
A &= \sum_{j=1}^{4} \eta_j\, m_{\nu_j}\,(\tau_1)^2_{ej} \\
&= \sum_{j=1}^{4}\sum_{i,\ell=1}^{3} \eta_j\, m_{\nu_j}\left(\frac{(V_L^T)_{ei}}{\sqrt{2}}\,O_{ij}\right)\left(\frac{(V_L^T)_{e\ell}}{\sqrt{2}}\,O_{\ell j}\right) \\
&= \frac{1}{2}\sum_{\ell,i=1}^{3}\left\{(V_L^T)_{ei}(V_L^T)_{e\ell}\sum_{j=1}^{4}\eta_j\, m_{\nu_j}\, O_{ij}\, O_{\ell j}\right\} \\
&= \frac{1}{2}\sum_{i,\ell=1}^{3}(V_L^T)_{ei}(V_L^T)_{e\ell}\,(m_\nu)_{i\ell} \qquad (6.5)
\end{aligned}
$$

where V_L is defined in (4.39a). Thus the decay amplitude is directly proportional to the linear superposition of $(m_\nu)_{i\ell}$ (i, ℓ =1,2,3). However, these elements are zero in our model, and therefore the amplitude for this process vanishes identically. A similar feature was obtained in a model constructed by Zee.

6.3 Double Beta Decay From A Lepton-Number-Violating Current

The decay amplitude A from this process is

$$A[(A,Z)\longrightarrow(A,Z+2)+e^-+e^-]$$

$$\propto\sum_j(\tau_2^T)_{ej}(\tau_1^T)_{je} = \frac{2m_{\chi_3}m_e\sin^2\theta_1}{M^2}\,. \qquad (6.6)$$

Therefore, the neutrinoless double beta decay arises naturally by lepton-number-violating currents. Again, we notice that (6.6) does not depend on the mixing angle θ_2.

Numerical Results

The decay amplitude A in (6.6) will vanish if θ_1 or m_{χ_3} vanishes.

If we put the maximum values for $\sin\theta_1=1$ and $m_{\chi_3}=100\text{eV}$ and $M=20\text{GeV}$, we have

$$\frac{2m_{\chi_3}m_e}{M^2}\cong 2.5\times10^{-13}\,. \qquad \cancel{(6.7)}$$

The experimental upper limit for such parameter is 2×10^{-5}.[33] Hence, our result is well within the upper limit.

VII. Chapter 7 The Decay Of Heavy Leptons

7.1 The General Formulation For The Decay Of A Lepton

In this chapter, we consider the decay rates of the new heavy lepton E and χ_4 to the known leptons and quarks in their lowest-order diagrams. Since we only calculate the approximate decay rates in the low energy domain, we use the effective Lagrangian density $\mathcal{L}^{eff}$ as in (2.64) or (2.66) for a four-fermion pointlike interactions.

It is shown in appendix D that if the Lagrangian for the interactions has the following form

$$\mathcal{L}^{eff.}_{int.} = \frac{4G_F}{\sqrt{2}}\bar{\psi}_2\gamma^\mu\left(\frac{g_L(1-\gamma_5)}{2}+\frac{g_R(1+\gamma_5)}{2}\right)\psi_1\cdot\bar{\psi}_3\gamma_\mu\left(\frac{\tilde{g}_L(1-\gamma_5)}{2}+\frac{\tilde{g}_R(1+\gamma_5)}{2}\right)\psi_4 \quad (7.1)$$

and if the masses of the fermions ψ_2, ψ_3 and ψ_4, which are small compared to the mass M of the lepton ψ_1, are neglected; then the approximate decay rate Γ (D.24) for the lepton ψ_1 decaying into two different fermions ψ_2, ψ_3 and antifermion $\bar{\psi}_4$ is

$$\Gamma_{\psi_1\neq\psi_3}(\psi_1\longrightarrow\psi_2\psi_3\bar{\psi}_4) = \frac{G^2M^5}{192\pi^3}(g_L^2+g_R^2)(\tilde{g}_L^2+\tilde{g}_R^2)$$

$$= \Gamma(\mu\longrightarrow e\nu_\mu\bar{\nu}_e)\frac{M^5}{m_\mu^5}(g_L^2+g_R^2)(\tilde{g}_L^2+\tilde{g}_R^2) \quad (7.2a)$$

where

$$\Gamma(\mu \to e \nu_\mu \bar{\nu}_e) = \Gamma_\mu = \frac{G^2 m_\mu^5}{192\pi^3} \quad . \tag{7.3}$$

In our approximation, this decay-rate formula is also applicable to the case if the ψ_3 and ψ_4 are identical neutrinos of the Majorana type (for example, see the final paragraph of appendix D).

If the fermions ψ_2 and ψ_3 are identical, then the decay rate Γ (D.33) is

$$\Gamma_{\psi_2 \neq \psi_3} = \frac{G^2 M^5}{192\pi^3}\left\{ (g_L^2 + g_R^2)(\tilde{g}_L^2 + \tilde{g}_R^2) + (g_L^2 \tilde{g}_L^2 + g_R^2 \tilde{g}_R^2) \right\}. \tag{7.2b}$$

If the ψ_1 and ψ_2 are Majorana neutrinos, then the decay rate Γ (D.36) is

$$\Gamma = \frac{G^2 M^5}{192\pi^2}\left\{ (g_L - \eta_1\eta_2 g_R)^2 + (g_R - \eta_1\eta_2 g_L)^2 \right\}(\tilde{g}_L^2 + \tilde{g}_R^2) \tag{7.2c}$$

where $\eta_1 = \pm 1, \eta_2 = \pm 1$ are the CP parity for the ψ_1 and ψ_2 .

For the case of one heavy Majorana neutrino ψ_1 decaying into three identical light Majorana neutrinos ψ_2, ψ_3, ψ_4, then the decay rate Γ (D.37) is

$$\Gamma = \frac{G^2 M^5}{192\pi^3}\Big\{ \left((g_L - \eta_1\eta_2 g_R)^2 + (g_R - \eta_1\eta_2 g_L)^2 \right)(\tilde{g}_L^2 + \tilde{g}_R^2) + \left((g_L - \eta_1\eta_2 g_R)^2 \tilde{g}_L^2 + (g_R - \eta_1\eta_2 g_L)^2 \tilde{g}_R^2 \right) \Big\} \quad . \tag{7.2d}$$

The generalized effective Lagrangian which is

$$\mathcal{L}^{eff} = \frac{4 G_F}{\sqrt{2}}\left(2 J_W^{\ell\mu} J_{W\mu}^{\ell\,+} + J_Z^\mu J_{Z\mu} \right) \quad . \tag{7.4}$$

where $J_W^{\ell\mu}$, $J_W^{\ell\mu +}$ and $J_Z^{\ell\mu}$ ǂ are given in (4.43). Then the ψ_1, ψ_2, ψ_3 and ψ_4 are now column vectors of fermion fields as in (4.42), and $g_L, g_R, \tilde{g}_L$ and $\tilde{g}_R$ are now matrices, a lepton $(\psi_1)_k$ (k=1,..,k) can possibly decay into L fermions $(\psi_2)_\ell$ (ℓ =1,..,L), M fermions $(\psi_3)_m$ (m=1,..,M) and N antifermions $(\psi_4^c)_n$ (n=1,..,N). For the case ψ_2 and ψ_3 are different fermion fields, the total decay rate in (7.2a) can be generalized as follow:

$$\Gamma_{\psi_2\neq\psi_3}\left((\psi_1)_k \longrightarrow \sum_\ell (\psi_2)_\ell + \sum_m (\psi_3)_m + \sum_n (\psi_4^c)_n \right)$$
$$= \frac{G^2M^5}{192\pi^3}\sum_\ell \left((g_L)^2_{\ell k} + (g_R)^2_{\ell k}\right) \sum_m \sum_n \left((\tilde{g}_L)^2_{mn} + (\tilde{g}_R)^2_{mn}\right) \qquad (7.5a)$$

which is just the summation § of all the decay rates in different decay processes of the lepton $(\psi_1)_k$.

For the case where the fermion field vectors ψ_2 and ψ_3 are identical, the total decay rate in (7.2b) can be generalized to

$$\Gamma_{\psi_2=\psi_3} = \frac{G^2M^5}{192\pi^3}\left\{ \sum_\ell \left((g_L)^2_{\ell k} + (g_R)^2_{\ell k}\right) \sum_m \sum_n \left((\tilde{g}_L)^2_{mn} + (\tilde{g}_R)^2_{mn}\right) \right.$$
$$\left. + \sum_\ell \sum_n \left((g_L)^2_{\ell k} (\tilde{g}_L)^2_{\ell n} + (g_R)^2_{\ell k} (\tilde{g}_R)^2_{\ell n}\right) \right\} . \qquad (7.5b)$$

ǂ the superscribe 'ℓ' is put here in order to distinguish the weak currents for leptons from the weak currents for quarks.

§ notice that the summation over the indices l,m,n does not necessarily start from one and end with L,M,N; it depends on the processes which we want to calculate.

7.2 The Decay Of Heavy Leptons To Charged Leptons And Neutrinos

First, let us consider the decay rate of the charged lepton E (which corresponds to k=4 in our representation). It is clear from the left-handed charged currents (4.43a) and the matrix $(g_L)_{\ell k}=\sqrt{2}\,(\tau_1^T)_{\ell k}$ in (4.45a) that the lepton E cannot decay through the lepton-number-conserving charged currents because the couplings $(\tau_1^T)_{\ell 4}$ (ℓ =1,2,3), which couple to the neutrinos ν_1,ν_2,χ_3 are zero and there exists no right-handed charged current. However, the charged lepton E can decay through the lepton-number-violating charged currents‡ and neutral currents. With only the leading contribution retained, the decay rates for the lepton E through different processes as shown in fig.7.1 have been calculated.

---

‡ We have not considered the decay rate for a heavy lepton ψ_1 into antileptons ψ_2^c , $\bar{\psi}_4$ and a lepton ψ_3 . In fact, with the same assumptions: $m_1=m_3=m_4=0$, and the similar procedures as in appendix D, it can be shown that if we have the effective Lagrangian as follows:

$$\mathcal{L}=\frac{4G_F}{\sqrt{2}}\,\overline{\psi_1^c}\gamma^\mu\left(\frac{g_L(1-\gamma_5)}{2}+\frac{g_R(1+\gamma_5)}{2}\right)\psi_2\,\bar{\psi}_3\gamma_\mu\left(\frac{\tilde{g}_L(1-\gamma_5)}{2}+\frac{\tilde{g}_R(1+\gamma_5)}{2}\right)\psi_4\,,$$

then for a process with two different outgoing antileptons, the decay rate for the ψ_1 is the same as in (7.2a); while for a process with two identical outgoing antileptons, the decay rate is the same as in (7.2b).

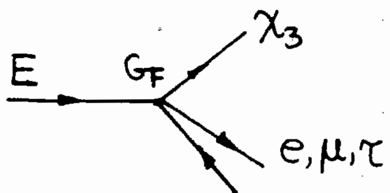
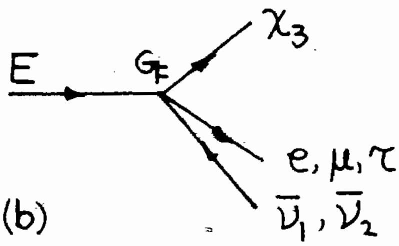
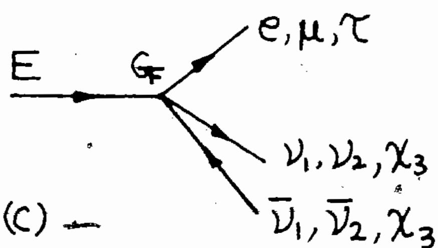
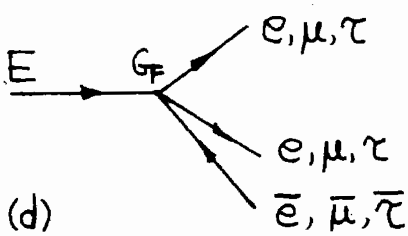
FEYNMAN DIAGRAMS	COUPLING STRENGTH
 (a)	$g_L = \frac{d}{M}$, $g_R = 0$ $\tilde{g}_L = \sqrt{2}\,\tau_1$, $\tilde{g}_R = 0$
 (b)	$g_L = \frac{d}{M}$, $g_R = 0$ $\tilde{g}_L = \sqrt{2}\,\tau_1$, $\tilde{g}_R = 0$
 (c)	$g_L = \sqrt{2}\,\tau_1^z$, $g_R = \sqrt{2}\,\tau_3^z$ $\tilde{g}_L = \sqrt{2}\,\tau_2^z$, $\tilde{g}_R = 0$
 (d)	$g_L = \tilde{g}_L = \sqrt{2}\,\tau_1^z$ $g_R = \tilde{g}_R = \sqrt{2}\,\tau_3^z$

Fig.7.1 Four-fermion point interactions for the decays of lepton E

The decay rates for the corresponding diagrams are

$$\Gamma(7.1(a)) = \frac{G^2 M^5}{192\pi^3}\left(\frac{4 m_{\chi_3}}{M}\right) , \qquad (7.6)$$

$$\Gamma(7.1(b)) = \frac{G^2M^5}{192\pi^3}\left(\frac{4\,m_{\chi_3}}{M}\right) , \qquad (7.6b)$$

$$\Gamma(7.1(c)) = \frac{G^2M^5}{192\pi^3}\left(\frac{3\,m_{\chi_3}}{2\,M}\right) , \qquad (7.6c)$$

$$\Gamma(7.1(d)) = \frac{G^2M^5}{192\pi^3}\left\{6\,\frac{m_{\chi_3}}{M}\left(2\left(-\frac{1}{2}+\sin^2\theta_w\right)^2+\sin^4\theta_w\right)\right\} \qquad (7.6d)$$

where (7.5b) is used to calculate the rates for the diagrams in fig.7.1(a) and fig.7.1(d) because they have two identical outgoing particles while (7.5a) is used for the diagrams in fig.7.5(b) and fig.7.5(c). The specific decay rates of E into electrons, muons and taus depends on specific values of the mixing parameters in (4.47).

Notice that we do not consider the decay of the lepton E to the lepton χ_4. This is because the mass for the lepton E is about the same as the lepton χ_4 even if the radiative corrections for the mass of the lepton E are taken. The rate of such a decay process will be small compared to other decay processes.

Comparing (7.6a), (7.6b) with (7.6c) and (7.6d), we can see that the lepton E will mostly decay through the lepton-number-violating charged currents rather than neutral currents.

Now, let us calculate the decay rates of the neutral lepton (the Majorana neutrino) χ_4 to different leptons. The diagrams of the decay processes are shown in the following figure:

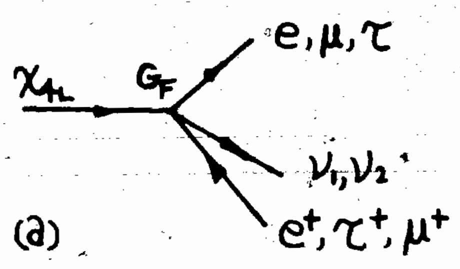
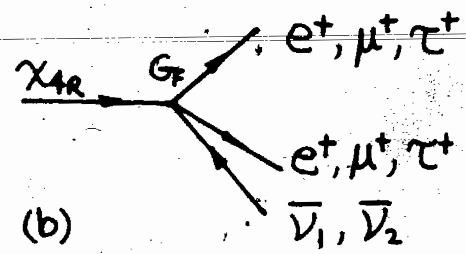
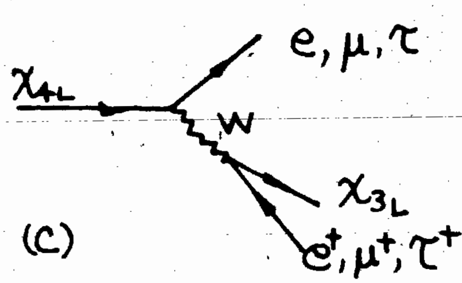
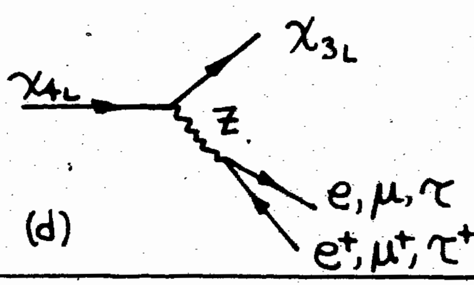
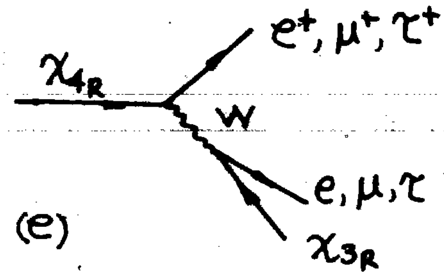
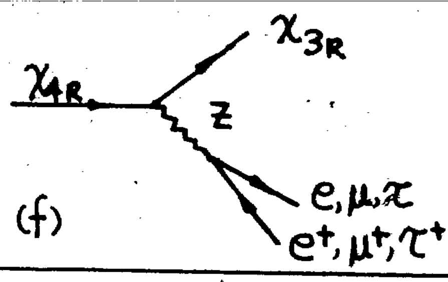
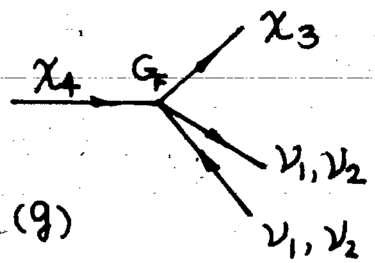
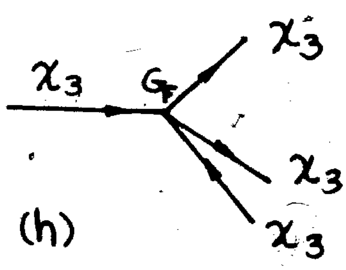
FEYNMAN DIAGRAMS	COUPLING STRENGTH
 (a)	$g_L=\sqrt{2}\,\tau_1$, $g_R=0$ $\tilde{g}_L=\sqrt{2}\,\tau_1^T$, $\tilde{g}_R=0$
 (b)	$g_L=0$, $g_R=\sqrt{2}\,\tau_1$ $\tilde{g}_L=\sqrt{2}\,\tau_1$, $\tilde{g}_R=0$
 (c)	$g_L=\sqrt{2}(\tau_1)_{i4}$, $g_R=0$ $\tilde{g}_L=\sqrt{2}(\tau_1^T)_{3j}$, $\tilde{g}_R=0$
 (d)	$g_L=\sqrt{2}(\tau_2^Z)_{34}$, $g_R=0$ $\tilde{g}_L=\sqrt{2}\,\tau_1^Z$, $\tilde{g}_R=\tau_3^Z$

Fig.7.2 Four-fermion point interactions for the decays of the χ_4

Continue Fig.7.2 Four-fermion point interactions for the decays of the χ_4

FEYNMAN DIAGRAMS	COUPLING STRENGTH
	$g_L = 0$, $g_R = \sqrt{2}(\tau_1)_{i4}$ $\tilde{g}_L = \sqrt{2}(\tau_1)_{j3}$, $\tilde{g}_R = 0$
	$g_L = 0$, $g_R = \sqrt{2}(\tau_2^Z)_{34}$ $\tilde{g}_L = \sqrt{2}\tau_1^Z$, $\tilde{g}_R = \sqrt{2}\tau_3^Z$
	$g_L = \sqrt{2}(\tau_2^Z)_{34}$, $g_R = 0$ $\tilde{g}_L = \sqrt{2}\tau_2^Z$, $\tilde{g}_R = 0$
	$g_L = \sqrt{2}(\tau_2^Z)_{34}$, $g_R = 0$ $\tilde{g}_L = \sqrt{2}(\tau_2^Z)_{33}$, $\tilde{g}_R = 0$

First, the process in fig.7.2(b) is possible because the χ_4 is the Majorana neutrino. Clearly, the rate for this process is the same as the one in fig.7.1(a). Notice that fig.7.2(c) and fig.7.2(d) correspond to the decay of the χ_4 through the charged current and the neutral current for the same process. With the help of a Fiertz transformation, the total Lagrangian interaction for these two diagrams in low energy can be written as

$$\mathcal{L} = \frac{4G_F}{\sqrt{2}} \bar{\chi}_3 \gamma^\mu \frac{(1-\gamma_5)}{2} \chi_4 \Big[\sum_i \sum_j \bar{\ell}_i \gamma_\mu \Big\{ \big(2(\tau_1)_{i4} (\tau_1^T)_{3j} + 2(\tau_2^z)_{34} (\tau_1^z)_{ij} \big) \cdot \frac{(1-\gamma_5)}{2} + 2(\tau_2^z)_{34} (\tau_3^z)_{ij} \frac{(1+\gamma_5)}{2} \Big\} \ell_j \Big] .$$

Therefore, we can identity the couplings for the process as follows:

$$g_L = 1 \quad , \qquad g_R = 0 \quad ,$$

$$(\tilde{g}_L)_{ij} = 2(\tau_1)_{i4} (\tau_1^T)_{3j} + 2(\tau_2^z)_{34} (\tau_1^z)_{ij} \, ,$$

$$(\tilde{g}_R)_{ij} = 2(\tau_2^z)_{34} (\tau_3^z)_{ij} \, .$$

Clearly, the process which corresponds to fig.7.2(e) and fig.7.2(f) has the same decay rate as the process corresponding to fig.7.2(c) and fig.7.2(d).

We use (7.5a) to calculate the decay rates for the processes in fig.7.2(a) to fig.7.2(f),

$$\Gamma(7.2(a)) + \Gamma(7.2(b)) = 2\Gamma(7.2(a)) = \frac{G^2 M^5}{192\pi^3} \left(4 \frac{m_{\chi_3}}{M} \right) , \qquad (7.7a)$$

$$\Gamma(7.2(c)+7.2(d))+\Gamma(7.2(e)+7.2(f))$$

$$=\frac{G^2M^5}{192\pi^3}\left\{\frac{2m_{x_3}}{M}+\frac{4m_{x_3}}{M}\left(-\frac{1}{2}+\sin^2\theta_W\right)+\frac{6m_{x_3}}{M}\left((-\frac{1}{2}+\sin^2\theta_W)^2+(\sin^2\theta_W)^2\right)\right\} \tag{7.7b}$$

Applying (7.2c) and (7.2d) to calculate the decay rates for the processes in fig.7.2(g) and fig.7.2(h) respectively, we obtain

$$\Gamma(7.2(g)) = \frac{G^2M^5}{192\pi^3}\left(\frac{m_{x_3}}{M}\right), \tag{7.7c}$$

$$\Gamma(7.2(h)) = \frac{G^2M^5}{192\pi^3}\left(\frac{3}{4}\frac{m_{x_3}}{M}\right). \tag{7.7d}$$

Notice that we do not consider the decay of the χ_4 through the lepton-number-violating charged currents because the couplings $(\tau_2^T)_{\ell 4}$ (ℓ =1,2,3), which couple χ_4 to the charged antileptons, are m_W/M weaker than the processes shown in fig.7.2. Again as in E decays, we see that the charge current decay modes of the χ_4 will predominate.

7.3 The Decay Of Heavy Leptons To Hadrons

It is also possible for a heavy lepton E or χ_4 to decay into a lepton and hadrons. Before we calculate the decay rate for such a process, let us briefly describe the weak interactions for quarks which are the constituents of hadrons. Similar to the case of lepton fields, there exist three families of quarks (up and down quarks(u,d), charm and strange(c,s), top and bottom(t,b)) which correspond to three families of leptons ; the left-handed quark fields are grouped into three SU(2) doublets and the right-handed quark fields are in SU(2) singlets:

$$\begin{pmatrix} u^\alpha \\ d^\alpha \end{pmatrix}_L , \begin{pmatrix} c^\alpha \\ s^\alpha \end{pmatrix}_L , \begin{pmatrix} t^\alpha \\ b^\alpha \end{pmatrix} , u_R^\alpha , d_R^\alpha , c_R^\alpha , s_R^\alpha , t_R^\alpha , b_R^\alpha \quad (7.8)$$

where each quark is assumed to exist in three states which differ among themselves only by a new quantum number called color, α=1,2,3. Each u,c,t (d,s,b) quark is assumed to have electric charge 2/3 (-1/3) of the unit charge.

Since the masses of the t and b quarks are heavy, we shall not consider the decays of the heavy leptons to such quarks. Hence, let us consider only the weak current interactions for u,s,d,c quarks. Similar to the case of leptons in the weak interaction, we can write down the charged currents J_W^μ for quarks:

$$J_W^{q\mu} = \sum_{\alpha=1}^{3} \overline{q}_{1L}^{\alpha} \gamma_\mu \frac{A_c}{\sqrt{2}} q_{2L}^{\alpha} \quad , \tag{7.9a}$$

$$J_W^{q\mu +} = \sum_{\alpha=1}^{3} \overline{q}_{2L}^{\alpha} \gamma_\mu \frac{A_c^{+}}{\sqrt{2}} q_{2L}^{\alpha} \tag{7.9b}$$

where the column vectors q_1^α and q_2^α for quarks are

$$q_1^\alpha = \begin{pmatrix} d^\alpha \\ s^\alpha \end{pmatrix} \quad , \quad q_2^\alpha = \begin{pmatrix} u^\alpha \\ c^\alpha \end{pmatrix} \tag{7.10}$$

and the Cabibbo matrix A_c is

$$A_c = \begin{bmatrix} \cos\theta_c & -\sin\theta_c \\ \sin\theta_c & \cos\theta_c \end{bmatrix} \tag{7.11}$$

where θ_c is the Cabibbo angle which arises from the mismatch between the weak eigenstates and the mass eigenstates of quarks (see Chapter 3). Similarly, the neutral currents for quarks $J_Z^{q\mu}$ are

$$J_Z^{q\mu} = \sum_{\alpha=1}^{3} \Big\{ \left(-\tfrac{1}{2} + \tfrac{1}{3}\sin^2\theta_W\right)\left(\overline{q}_{1L}^{\alpha}\gamma^\mu q_{1L}^{\alpha}\right) + \tfrac{1}{3}\sin^2\theta_W\left(\overline{q}_{1R}^{\alpha}\gamma^\mu q_{1R}^{\alpha}\right)$$

$$+ \left(\tfrac{1}{2} - \tfrac{2}{3}\sin^2\theta_W\right)\left(\overline{q}_{2L}^{\alpha}\gamma^\mu q_{2L}^{\alpha}\right) + \left(-\tfrac{2}{3}\sin^2\theta_W\right)\left(\overline{q}_{2R}^{\alpha}\gamma^\mu q_{2R}^{\alpha}\right)\Big\} \; . \tag{7.9c}$$

Now, the total charged and neutral currents for leptons and quarks are:

$$J_W^{\mu} = J_W^{\ell\mu} + J_W^{q\mu} \quad , \tag{7.12a}$$

$$J_Z^{\mu} = J_Z^{\ell\mu} + J_Z^{q\mu} \; . \tag{7.12b}$$

The total effective Lagrangian $\mathcal{L}_T^{eff}$ is

$$
\begin{aligned}
\mathcal{L}_T^{eff} &= \frac{4G_F}{\sqrt{2}}\left(2J_W^{\mu}J_{W\mu} + J_Z^{\mu}J_{Z\mu}\right) \\
&\simeq \frac{4G_F}{\sqrt{2}}\Big\{2\left(J_W^{\ell\mu}J_{W\mu}^{\ell\dagger} + J_W^{\ell\mu}J_{W\mu}^{q\dagger} + J_W^{q\mu}J_{W\mu}^{\ell\dagger} + J_W^{q\mu}J_{W\mu}^{q\dagger}\right) \\
&\qquad + \left(J_Z^{\ell\mu}J_{Z\mu}^{\ell} + 2J_Z^{\ell\mu}J_{Z\mu}^{q} + J_Z^{q\mu}J_{Z\mu}^{q}\right)\Big\} . \qquad (7.13)
\end{aligned}
$$

With the Lagrangian for the lepton-quark interactions, we can now calculate the decays of the heavy leptons to light leptons and hadrons (quarks).

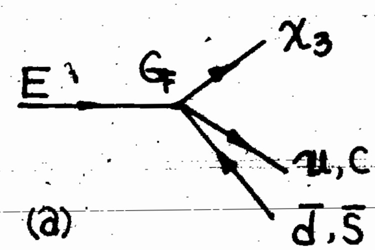
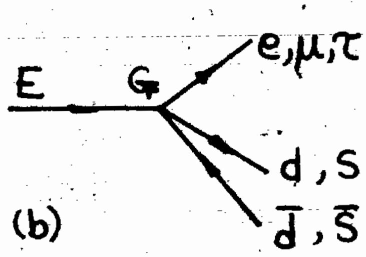
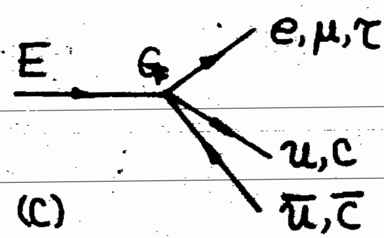
FEYNMAN DIAGRAMS	COUPLING STRENGTH
	$g_L = \frac{d}{M}$, $g_R = 0$ $\tilde{g}_L = A_c$, $\tilde{g}_R = 0$
	$g_L = \sqrt{2}\,\tau_1^Z$, $g_R = \sqrt{2}\,\tau_3^Z$ $\tilde{g}_L = \sqrt{2}\left(-\frac{1}{2} + \frac{1}{3}\sin^2\theta_W\right)I$, $\tilde{g}_R = \sqrt{2}\left(\frac{1}{3}\sin^2\theta_W\right)I$.
	$g_L = \sqrt{2}\,\tau_1^Z$, $g_R = \sqrt{2}\,\tau_3^Z$ $\tilde{g}_L = \sqrt{2}\left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_W\right)I$, $\tilde{g}_R = \sqrt{2}\left(-\frac{2}{3}\sin^2\theta_W\right)I$.

Fig.7.3 The decays of the lepton E to leptons and quarks

Using (7.5a), we obtain the decay rates for the above diagrams:

$$\Gamma(7.3(a)) = \frac{G^2M^5}{192\pi^3}\left(\frac{12m_{x_3}}{M}\right), \tag{7.14a}$$

$$\Gamma(7.3(b)) = \frac{G^2M^5}{192\pi^3}\left\{\frac{12m_{x_3}}{M}\left(\left(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_w\right)^2+\left(\frac{1}{3}\sin^2\theta_w\right)^2\right)\right\}, \tag{7.14b}$$

$$\Gamma(7.3(c)) = \frac{G^2M^5}{192\pi^3}\left\{\frac{12m_{x_3}}{M}\left(\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)^2+\left(\frac{2}{3}\sin^2\theta_w\right)^2\right)\right\} \tag{7.14c}$$

where the factor '3' is multiplied to the decay rates of the corresponding diagrams because each diagram consists of three different processes (which correspond to three different color states of the quarks) with the same decay rate.

Finally, let us consider the decay of the lepton χ_4 to leptons and quarks.

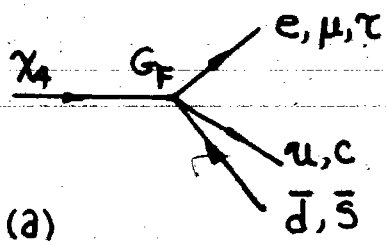
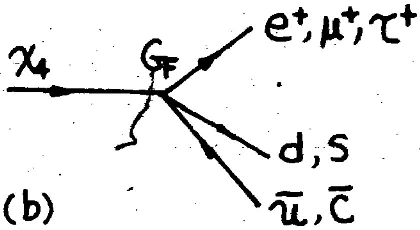
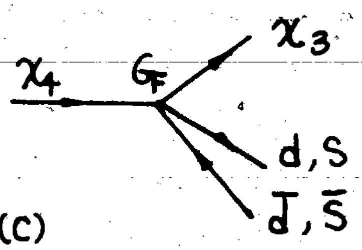
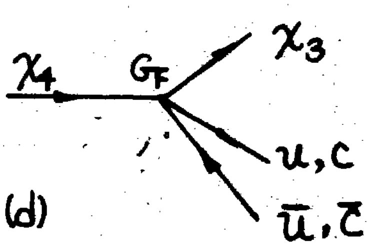
FEYNMAN DIAGRAMS	COUPLING STRENGTH
 (a)	$g_L = \sqrt{2}\,\tau_1$, $g_R = 0$ $\tilde{g}_L = A_c$, $\tilde{g}_L = 0$
 (b)	$g_L = 0$, $g_R = \sqrt{2}\,\tau_1$ $\tilde{g}_L = A_c^T$, $\tilde{g}_R = 0$
 (c)	$g_L = \sqrt{2}(\tau_2^z)_{34}$, $g_R = 0$ $\tilde{g}_L = \sqrt{2}(-\frac{1}{2} + \frac{1}{3}\sin^2\theta_W)I$, $\tilde{g}_R = \sqrt{2}(\frac{1}{3}\sin^2\theta_W)I$
 (d)	$g_L = \sqrt{2}(\tau_2^z)_{34}$, $g_R = 0$ $\tilde{g}_L = \sqrt{2}(\frac{1}{2} - \frac{2}{3}\sin^2\theta_W)I$, $\tilde{g}_R = \sqrt{2}(-\frac{2}{3}\sin^2\theta_W)I$

Fig.7.4 The decays of the lepton χ_4 to leptons and quarks

Using (7.5a) and (7.2c), we obtain the decay rates for the above diagrams:

$$\Gamma(7.4(a)) + \Gamma(7.4(b)) = 2\,\Gamma(7.4(a))$$

$$= \frac{G^2M^5}{192\pi^3}\left(\frac{12m_{x_3}}{M}\right), \qquad (7.15a)$$

$$\Gamma(7.4(c)) = \frac{G^2M^5}{192\pi^3}\left\{\frac{12m_{x_3}}{M}\left[\left(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_W\right)^2+\left(\frac{1}{3}\sin^2\theta_W\right)^2\right]\right\}, \qquad (7.15b)$$

$$\Gamma(7.4(d)) = \frac{G^2M^5}{192\pi^3}\left\{\frac{12m_{x_3}}{M}\left[\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_W\right)^2+\left(\frac{2}{3}\sin^2\theta_W\right)^2\right]\right\}. \qquad (7.15c)$$

Numerical Results

The total decay rate for the lepton E would be the addition of the decay rates of the processes in fig.7.1 and fig.7.3. It is

$$\Gamma(E) = \frac{G^2M^5}{192\pi^3}\frac{m_{x_3}}{M}\left(\frac{61}{2} - 24\sin^2\theta_W + \frac{94}{3}\sin^4\theta_W\right) \qquad (7.16a)$$

$$= \Gamma(\mu\to e\bar{\nu}_e\nu_\mu)\frac{m_{x_3}M^4}{m_\mu^5}\left(\frac{61}{2} - 24\sin^2\theta_W + \frac{94}{3}\sin^4\theta_W\right).$$

Similarly, the total decay rate for the lepton χ_4 would be the addition of the decay rates of the processes in fig.7.2 and fig.7.4. It is

$$\Gamma(\chi_4) = \frac{G^2M^5}{192\pi^3}\frac{m_{x_3}}{M}\left(\frac{109}{4} - 18\sin^2\theta_W + \frac{76}{3}\sin^4\theta_W\right) \qquad (7.16b)$$

$$= \Gamma(\mu\to e\bar{\nu}_e\nu_\mu)\frac{m_{x_3}M^4}{m_\mu^5}\left(\frac{101}{4} - 14\sin^2\theta_W + \frac{76}{3}\sin^4\theta_W\right).$$

If we assume M=20GeV, m_{χ_3}=100eV, and take $\sin^2\theta_W$=0.224, m_μ =105.6MeV and the lifetime for the muon decay $\tau_\mu = \frac{1}{\Gamma_\mu}$ =2.2x10^{-6} sec, then the lifetime for the heavy lepton decays are

$$\tau_E = 6.8 \times 10^{-11} \text{ sec.}, \tag{7.17a}$$

$$\tau_{\chi_4} = 7.8 \times 10^{-11} \text{ sec.} \tag{7.17b}$$

which are unstable compared to muon but more stable than tau ($\tau_\tau \sim 10^{-12}$ sec).

The lepton E would naively be expected to be stable because it seems that it can only decay to the lepton N through the charged currents. However, since the lepton E is in the triplet representation, the lepton-number-violating charged currents exist. Also the GIM mechanism in the neutral currents has been destroyed; there exist nonzero couplings which couple the lepton E to the leptons e, μ, τ. Although the strength of the couplings is proportional to $\sqrt{\frac{m_{\chi_3}}{M}}$ which is weak, the lepton E is unstable compared to the muon because it has large mass.

VIII. Chapter 8 Radiative Decays Of Massive Neutrinos And Magnetic Moments Of Neutrinos

8.1 The Possibility Of Radiative Decays Of Massive Neutrinos

If neutrinos are massive and if the mass eigenstates are not degenerate, then it is possible to have a radiative decay from a heavy neutrino ν_1 to a lighter one ν_2 of the form $\nu_1 \rightarrow \nu_2 + \gamma$[34]. In this chapter, we will calculate the decay rates of such processes in one-loop diagrams. We use the existing formulation due to Lee and Shrock[43] which is valid for a general SU(2)xU(1) gauge model. Relevant results have been summarized in appendix F. In order to use their formulation, we shall assume that the masses of the heavy leptons (E and χ_4) are lighter than the mass of the intermediate bosons M_W here and the subsequent chapter, i.e. $\left(\frac{M}{M_W}\right)^2 << 1$.

Because of gauge invariance, the most general form for the decay amplitude $\nu_1 \rightarrow \nu_2 + \gamma$ is

$$i\mathcal{M}(\nu_1(P_1) \longrightarrow \nu_2(P_2) + \gamma(q))$$

$$= \bar{u}_2(P_2)\, \frac{i\,\sigma_{\mu\nu} q^{\nu} \epsilon^{\mu}}{(m_1+m_2)} \left(F_{21}^{V} + F_{21}^{A} \gamma_5 \right) u_1(P_1) \qquad (8.1)$$

and the decay rate $\nu_1 \rightarrow \nu_2 + \gamma$ is

$$\Gamma(\nu_1 \longrightarrow \nu_2+\gamma)$$
$$= \frac{m_1}{8\pi}\left(1-\frac{m_2}{m_1}\right)^2\left(1-\frac{m_2^2}{m_1^2}\right)\left\{|F_{21}^{V}|^2+|F_{21}^{A}|^2\right\} \qquad (8.2)$$

where F_{21}^{V}, F_{21}^{A} are the transition magnetic moment and the transition electric dipole moment for ν_1 to ν_2, and they can be decomposed into two parts

$$F^{V,A} = F^{V,A}_{LL,RR} + F^{V,A}_{LR,RL} \qquad (8.3)$$

as indicated in the appendix F ‡. $F^{V,A}_{LL,RR}$ comes from those processes without chirality being changed; whereas $F^{V,A}_{LR,RL}$ comes from those pocesses where chirality is changed.

8.2 Radiative Decays Of Majorana Neutrinos

Let us first consider the case of the decays of a heavy Majorana neutrino χ_4. Obviously, only one-loop diagrams § which involve charged currents will contribute to radiative decays. From (4.43), the charged currents are

$$J_W^{\mu} = \overline{\ell}_L \gamma_\mu \tau_1 \nu_L + \overline{\nu}_L \gamma_\mu \tau_2 (\ell^c)_L, \qquad (8.4a)$$
$$J_W^{\mu +} = \overline{\nu}_L \gamma_\mu \tau_1^T \ell_L + \overline{(\ell^c)}_L \gamma_\mu \tau_2^T \nu_L \qquad (8.4b)$$

which can also be written as

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‡(8.1), (8.2) and (8.3) correspond (F.14), (F.15) and (F.6) in appendix F.

§see fig.F.1(a) and fig.F.1(b) in appendix F

$$J_W^{\mu} = -\overline{(\nu^c)}_R \gamma_\mu \tau_1^T (l^c)_R - \overline{l}_R \gamma_\mu \tau_2^T (\nu^c)_R \,, \tag{8.4c}$$

$$J_W^{\mu +} = -\overline{(l^c)}_R \gamma_\mu \tau_1 (\nu^c)_R - \overline{(\nu^c)}_R \gamma_\mu \tau_2\, l_R \,. \tag{8.4d}$$

There are eight possible mechanisms by which a massive neutrino or antineutrino will decay as illustrated in fig.8.1:

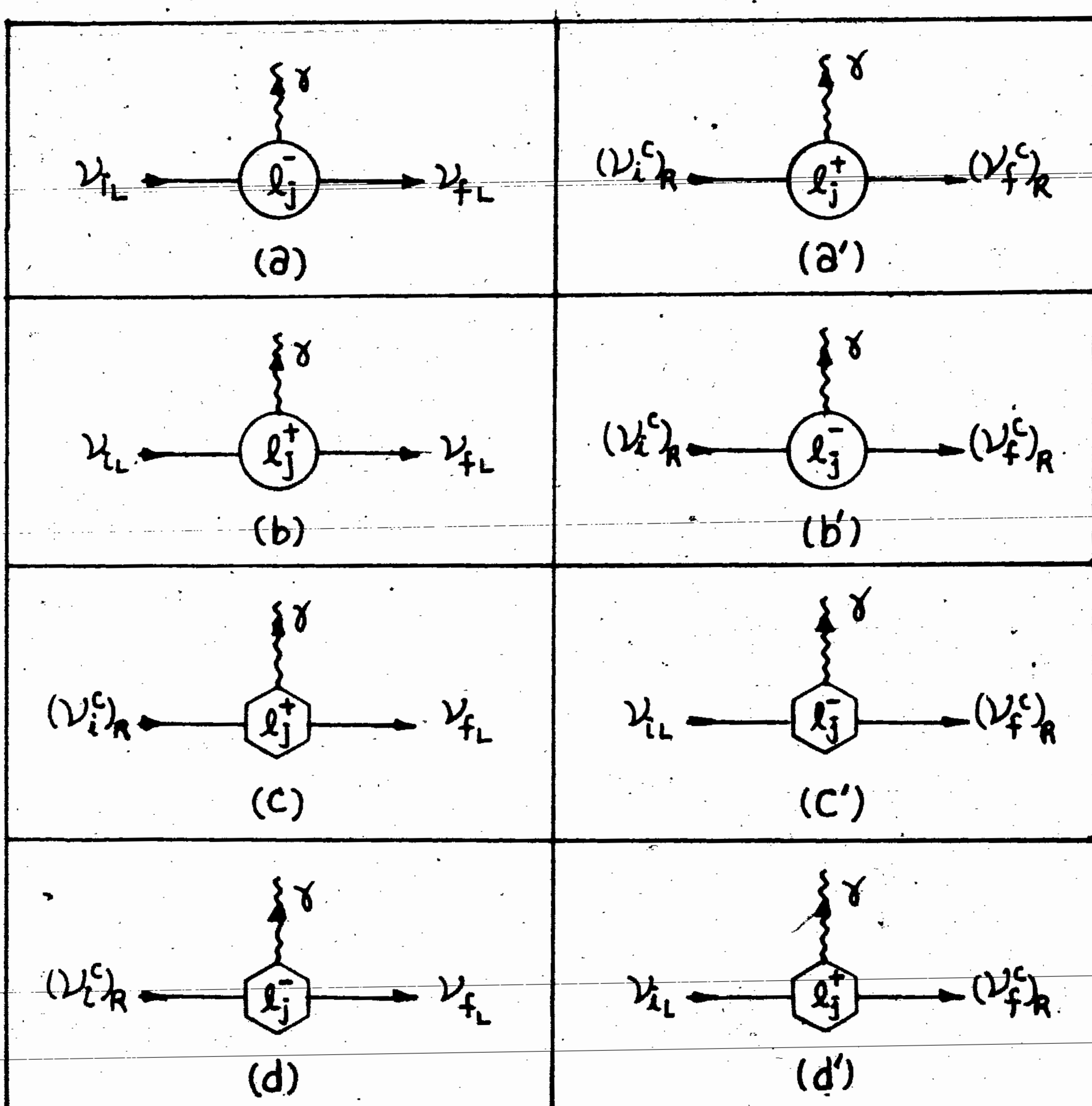
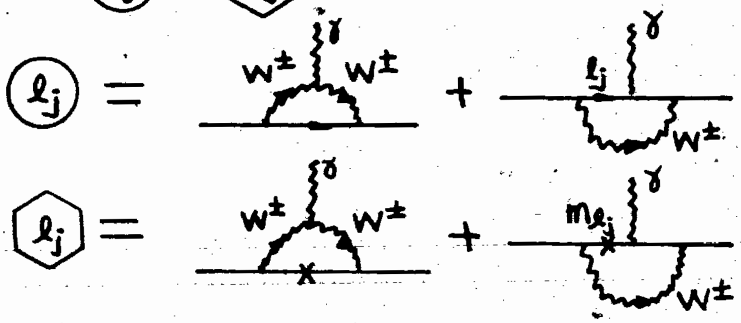


Fig.8.1 Diagrams contributing to the processes $\nu_i, \nu_i^c \to \nu_f, \nu_f^c + \gamma$ where ν_i and ν_f are the initial and the final neutrinos. l_j denotes

any charged leptons which can couple in these graphs, and the symbols ℓ_j and ℓ_j representing one-loop diagrams are as follows:



For instance, the first term in (8.4b) will give the decay $(\nu_i)_L \longrightarrow (\nu_f)_L + \gamma$ (see fig.8.1(a)), while the first term in (8.4c) give the decay $(\nu_i^c)_R \longrightarrow (\nu_f^c)_R + \gamma$ (see fig.8.1(a')). Clearly, if neutrinos are of the Dirac type, we can divide those diagrams in fig.8.1 into several different kinds of processes. If neutrinos are of the Majorana type such as in this case, all these processes correspond to the same decay process because neutrinos are self-conjugate as defined in previous chapters (see appendix E). Using the formulation in appendix F, the transition magnetic moment F^V_{fi} and the transition electric dipole moment F^A_{fi} for the process $\chi_i \longrightarrow \chi_f + \gamma$ will be

$$(F^V_{LL,RR})_{fi} = \sum_j (m_i + m_f)^2 \left[\frac{eG_F}{4\sqrt{2}\pi^2}\right] \Big[(\tau_1^T)_{fj}(\tau_1)_{ji} - \eta_f \eta_i (\tau_1^T)_{fj}(\tau_1)_{ji} - (\tau_2)_{fj}(\tau_2^T)_{ji} + \eta_f \eta_i (\tau_2)_{fj}(\tau_2^T)_{ji}\Big] \cdot C_j^{LL,RR} , \qquad (8.5a)$$

$$(F^A_{LL,RR})_{fi} = \sum_j (m_i^2 - m_f^2) \left[\frac{eG_F}{4\sqrt{2}\pi^2}\right] \Big[(\tau_1^T)_{fj}(\tau_1)_{ji} + \eta_f \eta_i (\tau_1^T)_{fj}(\tau_1)_{ji} - (\tau_2)_{fj}(\tau_2^T)_{ji} - \eta_f \eta_i (\tau_2)_{fj}(\tau_2^T)_{ji}\Big] \cdot C_j^{LL,RR} \qquad (8.5b)$$

and

$$(F^V_{LR,RL})_{fi} = \sum_j (m_i+m_f)\left[\frac{eG_F}{4\sqrt{2}\pi^2}\right]\Big[\eta_i(\tau_2)_{fj}(\tau_1)_{ji} - \eta_i(\tau_1^T)_{fj}(\tau_2^T)_{ji}$$

$$- \eta_f(\tau_2)_{fj}(\tau_1)_{ji} + \eta_f(\tau_1^T)_{fj}(\tau_2^T)_{ji}\Big]\cdot m_{\ell_j} C_j^{LR,RL}, \quad (8.5c)$$

$$(F^A_{LR,RL})_{fi} = \sum (m_i-m_f)\left[\frac{eG_F}{4\sqrt{2}\pi^2}\right]\Big[\eta_i(\tau_2)_{fj}(\tau_1)_{ji} - \eta_i(\tau_1^T)_{fj}(\tau_2^T)_{ji}$$

$$+ \eta_f(\tau_2)_{fj}(\tau_1)_{ji} - \eta_f(\tau_1^T)_{fj}(\tau_2^T)_{ji}\Big]\cdot m_{\ell_j} C_j^{LR,RL} \quad (8.5d)$$

where

$$C_j^{LL,RR} = \frac{3}{2} - \frac{3}{4}\epsilon_i \,, \qquad (8.6a)$$

$$C_j^{LR,RL} = \left(-4+\frac{3}{2}\epsilon_i\right) - \epsilon_i\left(-4\ln\frac{1}{\epsilon_i}+6\right) \qquad (8.6b)$$

with

$$\epsilon_i = \frac{m^2_{\ell_i}}{M_W^2} \,. \qquad (8.7)$$

m_{ℓ_i} is the mass of i^{th} virtual charged lepton and $\eta_{i,f}$ are CP eigenvalues of the initial and the final Majorana neutrinos. The contributions to the F^V and F^A in (8.5a,b) arise from the diagrams in fig.8.1(a),(b),(a'),(b'); while the F^V and F^A in (8.5c,d) arise from the diagrams in fig.8.1(c),(d),(c'),(d').

Notice that equations (8.5) lead to two possibilities in general:

(i) $\eta_f\eta_i = 1$, this implies

$$F^V_{LL,RR} = F^V_{LR,RL} = 0 \qquad (8.8a)$$

hence, $F^V = 0$. $\qquad (8.8b)$

(ii) $\eta_f\eta_i=-1$, this implies

$$F^A_{LL,RR} = F^A_{LR,RL} = 0 \tag{8.9a}$$

hence $F^A=0$. (8.9b)

These show that if the initial and the final neutrinos have the same CP parity (eigenvalue) $\eta_f=\eta_i$, then there is no transition magnetic moment; whereas, if they have opposite CP parity $\eta_f=-\eta_i$, then there is no transition electric dipole moment. Although we have derived these results by calculating the lowest-order diagrams in a particular model, this in fact be true in general for a CP-invariant theory.[35-37]

Now, in our model, we have $\eta_3=1, \eta_4=-1$; therefore, no transition electric dipole moment exists. The contributions to the transition magnetic moment F^V from the diagrams without "Prime" in fig.8.1 will be the same as the contributions from the diagrams with "Prime". For, $m_4=M$, $m_3=m_{\chi_3}$, $M\gg m_{\chi_3}$, we have

$$(F^V_{LL,RR})_{34} \cong 2M^4\left[\frac{eG_F}{4\sqrt{2}\pi^2}\right]\left[\sum_j\left((\tau_1^T)_{3j}(\tau_1)_{j4}-(\tau_2)_{3j}(\tau_2^T)_{j4}\right)\right] \cdot C_j^{LL,RR} \tag{8.10a}$$

$$(F^V_{LR,RL})_{34} \cong 2M^4\left[\frac{eG_F}{4\sqrt{2}\pi^2}\right]\left[\sum_j\left((\tau_1^T)_{3j}(\tau_2^T)_{j4}-(\tau_2)_{3j}(\tau_1)_{j4}\right)\right] \cdot \frac{m_{\ell j}}{M} C_j^{LR,RL}. \tag{8.10b}$$

The leading contribution for each type of one-loop diagrams has been calculated as follows:

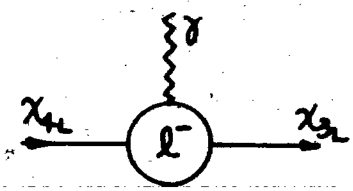
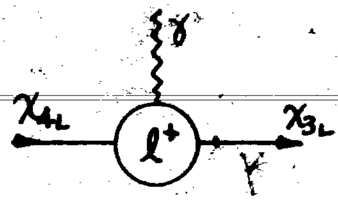
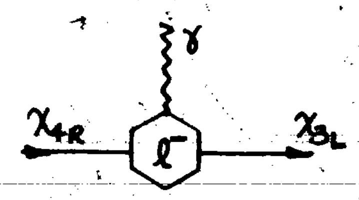
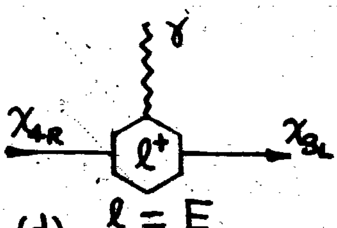
 (a) $\ell = e, \mu, \tau$	$\sum_j (\tau_1^T)_{3j}(\tau_1)_{j4} C_j^{LL,RR} = -\frac{3d}{4\sqrt{2}M}$
 (b) $\ell = E$	$\sum_j (\tau_2)_{3j}(\tau_2^T)_{j4} C_j^{LL,RR} = -\frac{3d}{2\sqrt{2}M}$
 (c) $\ell = e, \mu, \tau$	$\sum_j (\tau_1^T)_{3j}(\tau_2^T)_{j4} C_j^{LR,RL} \frac{m_{\ell j}}{M} = \frac{4}{\sqrt{2}}\left(\frac{s_1^2 m_e^2 + c_1^2 s_2^2 m_\mu^2 + c_1^2 c_2^2 m_\tau^2}{M^2}\right)\frac{d}{M}$
 (d) $\ell = E$	$\sum_j (\tau_2)_{3j}(\tau_1)_{j4} C_j^{LR,RL} \frac{m_{\ell j}}{M} = \frac{4d}{\sqrt{2}M}$

Fig.8.2 Diagrams contributing to the process $\chi_4 \longrightarrow \chi_3 + \gamma$.

Since the contribution of fig.8.2(c) is $\left(\frac{m_\ell}{M}\right)^2$ smaller than the contributions from other diagrams, it will be neglected. Retaining only the leading contributions from the diagrams in fig.8.2(a),(b),(d), we have

$$F_{34}^{V} = (F_{LL,RR}^{V})_{34} + (F_{LR,RL}^{V})_{34}$$

$$= -\frac{13}{2\sqrt{2}}\left(\frac{eG_F}{4\sqrt{2}\pi^2}\right) M d \qquad (8.11)$$

Using (8.2), we obtain the rate

$$\Gamma(\chi_4 \to \chi_3 + \gamma) = \frac{6\alpha}{\pi}\left(\frac{13}{2\sqrt{2}}\right)^2 \left(\frac{G_F^2 M^5}{192\pi^3}\right)\frac{m_{\chi_3}}{M} \qquad (8.12)$$

where $\alpha = \frac{e^2}{4\pi}$ is the electromagnetic coupling constant.

8.3 Radiative Decays Of Majorana Neutrinos To Massless Neutrinos

It is also possible for a massive Majorana neutrino decay radiatively to a massless one. But now, $\chi_i \to \nu_f + \gamma$ and $\chi_i \to (\nu_f^c)_R + \gamma$ are two distinct processes because the final neutrino is not a Majorana type. Hence, the processes with "Prime" in fig.8.1 will be different from the processes without "Prime". For the process $\chi_i \to \nu_f + \gamma$, we have F^V and F^A as follows:

$$(F_{LL,RR}^{A})_{fi} = (F_{LL,RR}^{V})_{fi}$$

$$= m_i^2 \left[\frac{eG_F}{4\sqrt{2}\pi^2}\right] \sum_j \left[(\tau_1^T)_{fj}(\tau_1)_{ji} - (\tau_2)_{fj}(\tau_2^T)_{ji}\right] C_j^{LL,RR}, \qquad (8.13a)$$

$$(F_{LR,RL}^{A})_{fi} = (F_{LR,RL}^{V})_{fi}$$

$$= m_i \left[\frac{eG_F}{4\sqrt{2}\pi^2}\right] \sum_j \left[\eta_i(\tau_2)_{fj}(\tau_1)_{ji} - \eta_i(\tau_1^T)_{fj}(\tau_2^T)_{ji}\right] m_{\ell_j} C_j^{LR,RL}. \qquad (8.13b)$$

Notice that both the transition magnetic moment F^V and the transition electric dipole moment F^A are nonzero in general.

The leading contribution for each type of one-loop diagrams has been calculated as follows:

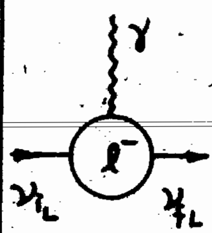
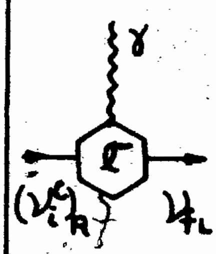
	PROCESSES	$\sum_{j=1}^{4}(\tau_1^T)_{fj}(\tau_1)_{ji}C_j^{LL,RR}$
	$\chi_3 \to \nu_1+\gamma$	$-\frac{3}{8}\left(\frac{-c_1s_1m_e^2+c_1s_1s_2^2m_\mu^2+c_1c_2^2s_1m_\tau^2}{M_W^2}\right)$
	$\chi_4 \to \nu_1+\gamma$	$\frac{3}{8\sqrt{2}}\left(\frac{-c_1s_1m_e^2+c_1s_1s_2^2m_\mu^2+c_1c_2^2s_1m_\tau^2}{M_W^2}\right)\frac{d}{M}$
$\ell=e,\mu,\tau$	$\chi_3 \to \nu_2+\gamma$	$-\frac{3}{8}\left(\frac{c_1c_2s_2(m_\tau^2-m_\mu^2)}{M_W^2}\right)$
(a)	$\chi_4 \to \nu_2+\gamma$	$\frac{3}{8\sqrt{2}}\left(\frac{c_1c_2s_2(m_\tau^2-m_\mu^2)}{M_W^2}\right)\frac{d}{M}$
		$\sum_{j=1}^{4}(\tau_1^T)_{fj}(\tau_2^T)_{ji}C_j^{LR,RL}\frac{m_{\ell j}}{m_\tau}$
	$\chi_3 \to \nu_1+\gamma$	$-\frac{8}{\sqrt{2}}\left(\frac{-c_1s_1m_e^2+c_1s_1s_2^2m_\mu^2+c_1c_2^2s_1m_\tau^2}{M^2}\right)$
	$\chi_4 \to \nu_1+\gamma$	$\frac{4}{\sqrt{2}}\left(\frac{-c_1s_1m_e^2+c_1s_1s_2^2m_\mu^2+c_1c_2^2s_1m_\tau^2}{M^2}\right)\frac{d}{M}$
$\ell=e,\mu,\tau$	$\chi_3 \to \nu_2+\gamma$	$-\frac{8}{\sqrt{2}}\left(\frac{c_1c_2s_2(m_\tau^2-m_\mu^2)}{M^2}\right)$
(b)	$\chi_4 \to \nu_2+\gamma$	$\frac{4}{\sqrt{2}}\left(\frac{c_1c_2s_2(m_\tau^2-m_\mu^2)}{M^2}\right)\frac{d}{M}$

Fig.8.3 Diagrams contributing to the process $\chi_f \to \nu_i+\gamma$.

The other two types of one-loop diagrams fig.8.1(b) and fig.8.1(c) have no contribution for the given matrices τ_1 and τ_2. Since we have assumed $\left(\frac{M}{M_W}\right)^2 \ll 1$, the contributions from fig.8.3(a) will be small compared to the contributions from fig.8.3(b), and hence will be neglected. Notice that the coupling strength for the processes in fig.8.3(a) is stronger than those in fig.8.3(b). However, their leading contributions are cancelled by a "leptonic G.I.M. mechanism". We have

$$F_{13}^{V} = F_{13}^{A} = (m_{\chi_3})^2 \left[\frac{eG_F}{4\sqrt{2}\pi^2}\right]\left[\frac{8}{\sqrt{2}}\left(\frac{-c_1 s_1 m_e^2 + c_1 s_1 s_2^2 m_\mu^2 + c_1 c_2^2 s_1 m_\tau^2}{M^2}\right)\right], \quad (8.15a)$$

$$F_{14}^{V} = F_{13}^{A} = M^2 \left[\frac{eG_F}{4\sqrt{2}\pi^2}\right]\left[\frac{4}{\sqrt{2}}\left(\frac{-c_1 s_1 m_e^2 + c_1 s_1 s_2^2 m_\mu^2 + c_1 c_2^2 s_1 m_\tau^2}{M^2}\right)\frac{d}{M}\right], \quad (8.15b)$$

$$F_{23}^{V} = F_{23}^{A} = (m_{\chi_3})^2 \left[\frac{eG_F}{4\sqrt{2}\pi^2}\right]\left[\frac{8}{\sqrt{2}}\left(\frac{c_1 c_2 s_2 (m_\tau^2 - m_\mu^2)}{M^2}\right)\right], \quad (8.15c)$$

$$F_{24}^{V} = F_{24}^{A} = M^2 \left[\frac{eG_F}{4\sqrt{2}\pi^2}\right]\left[\frac{4}{\sqrt{2}}\left(\frac{c_1 c_2 s_2 (m_\tau^2 - m_\mu^2)}{M^2}\right)\right]. \quad (8.15d)$$

Since $m_e, m_\mu \ll m_\tau$, we finally obtain the rates

$$\Gamma(\chi_3 \to \nu_1 + \gamma) = \frac{192\alpha}{\pi}\frac{G_F^2 m_{\chi_3}^5}{192\pi^3}\left(\frac{c_1 c_2^2 s_1 m_\tau^2}{M^2}\right)^2, \quad (8.16a)$$

$$\Gamma(\chi_4 \to \nu_1 + \gamma) = \frac{96\alpha}{\pi}\frac{G_F^2 M^5}{192\pi^3}\left(\frac{c_1 c_2^2 s_1 m_\tau^2}{M^2}\right)^2 \frac{m_{\chi_3}}{M}, \quad (8.16b)$$

$$\Gamma(\chi_3 \to \nu_2 + \gamma) = \frac{192\alpha}{\pi}\frac{G_F^2 m_{\chi_3}^5}{192\pi^3}\left(\frac{c_1 s_2 c_2 m_\tau^2}{M^2}\right)^2, \quad (8.16c)$$

$$\Gamma(\chi_4 \to \nu_2 + \gamma) = \frac{96\alpha}{\pi}\frac{G_F^2 M^5}{192\pi^3}\left(\frac{c_1 c_2 s_2 m_\tau^2}{M^2}\right)^2 \frac{m_{\chi_3}}{M}. \quad (8.16d)$$

Numerical Results

Comparing (8.12) with (8.16), we notice that the decay rate for $\chi_4 \rightarrow \nu_{1,2} + \gamma$ will be $\left(\frac{m_\tau}{M}\right)^4$ smaller than the decay rate $\chi_4 \rightarrow \chi_3 + \gamma$. Hence, the total decay rate Γ_{χ_4} for χ_4 will be dominated by the latter decay mode. As for the decay rate of χ_3, there exists a second decay mode $\chi_3 \rightarrow \nu_{1,2}^c + \gamma$ with the same rate as $\chi_3 \rightarrow \nu_{1,2} + \gamma$. Therefore, the total decay rate Γ_{χ_3} for χ_3 would be twice the rates of (8.16a) and (8.16c). The total decay rates for χ_3 and χ_4 would be

$$\Gamma_{\chi_3} = 2\left(\frac{192\alpha}{\pi}\right)\left(\frac{m_{\chi_3}^5}{m_\mu^5}\right)\left(\left(\frac{c_1 s_2^2 s_1 m_\tau^2}{M^2}\right)^2 + \left(\frac{c_1 c_2 s_2 m_\tau^2}{M^2}\right)^2\right)\Gamma_\mu, \quad (8.17a)$$

$$\Gamma_{\chi_4} = \frac{6\alpha}{\pi}\left(\frac{13}{2\sqrt{2}}\right)^2\left(\frac{M^5}{m_\mu^5}\right)\left(\frac{m_{\chi_3}}{M}\right)\Gamma_\mu \quad (8.17b)$$

where $\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3}$.

It is interesting to notice that the radiative decay rate of χ_4 in (8.17b) differs the decay rate of χ_4 in (7.16b) only by a factor proportional to the electromagnetic coupling constant 'α'.

We assume the maximum mixing angles $\theta_1 = \theta_2 = \frac{\pi}{4}$, $m_{\chi_3} = 100$eV and the heavy neutrino has mass $m_{\chi_4} = 20$GeV, then the rates would be

$$\Gamma_3 \cong 8.6 \times 10^{-3}\,\Gamma_\mu, \quad (8.18a)$$

$$\Gamma_4 \cong 3.7 \times 10^{2}\,\Gamma_\mu. \quad (8.18b)$$

Putting the lifetime $\tau_\mu \cong 2.2 \times 10^{-6}$ sec for the muon decay, we obtain the lifetime τ_{χ_3} and τ_{χ_4} for the neutrinos χ_3 and χ_4:

$$\tau_{\chi_3} \cong 8.0 \times 10^{21} \text{ years} , \qquad (8.19a)$$

$$\tau_{\chi_4} \cong 6.0 \times 10^{-9} \text{ sec} . \qquad (8.19b)$$

8.4 Magnetic Moments Of Neutrinos

As it is well known, a massless neutrino cannot have a magnetic moment. In fact, this is also true for a Majorana neutrino in contrast to a massive Dirac neutrino. If the theory is CP invariant, the zero magnetic moment for the Majorana neutrino immediately follows from (8.8b). If the theory is CPT invariant rather than CP invariant, the antiparticle must be defined through the CPT operation. It is well known that CPT invariance implies that particle and antiparticle have opposite magnetic moments. Since the particle and the antiparticle are the same for a Majorana neutrino, their magnetic moment must be zero if CPT invariance holds.

IX. Chapter 9 The Radiative Decays Of Charged Leptons And Their Anomalous Magnetic Moments

9.1 The Radiative Decays Of Light Charged Leptons

In the first part of this chapter, we consider the rare weak processes of the type $l_i \to l_j + \gamma$ where l_i and l_j are charged leptons of different families. The occurrence of such processes would signify that the lepton numbers defined in different families are not conserved. Processes of this kind are forbidden in the minimal SU(2)xU(1) model, but become possible if there exist neutrino mixings. In this chapter, the rate of the process $\mu \to e\gamma$ is calculated for one-loop diagrams within the modified model. Before we present our calculation, we report on some early work.

(i) One possibility is that ν_e and ν_μ are linear superpositions of two neutrinos ν_1 , ν_2 with finite masses m_1, m_2 ; i.e.

$$\nu_e = \nu_1 \cos\theta + \nu_2 \sin\theta \quad , \tag{9.1a}$$

$$\nu_\mu = -\nu_1 \sin\theta + \nu_2 \cos\theta \tag{9.1b}$$

where θ is a mixing angle.

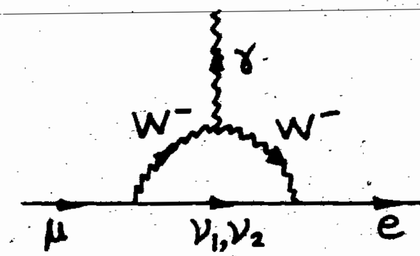


Fig.9.1 Diagrams of the process $\mu \to e\gamma$ with virtual neutrinos

The ratio R_μ of the $\mu \to e\gamma$ rate to the $\mu \to e\bar{\nu}_e\nu_\mu$ rate has been calculated in GWS model and is given[1]

$$R_\mu = \frac{\Gamma(\mu \to e\gamma)}{\Gamma(\mu \to e\bar{\nu}_e\nu_\mu)} = \frac{3}{32}\frac{\alpha}{\pi}\left(\frac{|m_1^2 - m_2^2|}{M_W^2}\right)^2 \sin^2\theta\cos^2\theta \qquad (9.2)$$

where α =1/137 and M_W is the mass of the intermediate charged boson. It is found that R_μ in this case turns out to be smaller by many orders of magnitude than the experimental upper limit[11], which is

$$R_\mu^{exp.} < 1.9 \times 10^{-10} \quad . \qquad (9.3)$$

(ii) Heavy neutral leptons

The situation might change radically[1,36] if there exist heavy leptons. Let us assume that besides the left-handed doublets in the GWS model, there are right-handed doublets:

$$\begin{pmatrix} N_e \\ e \end{pmatrix}_R , \begin{pmatrix} N_\mu \\ \mu \end{pmatrix}_R \qquad (9.4)$$

where

$$N_e = N_1\cos\theta' + N_2\sin\theta' \quad , \qquad (9.5a)$$

$$N_\mu = -N_1\sin\theta' + N_2\cos\theta' \quad . \qquad (9.5b)$$

N_1 and N_2 are mass eigenstates with masses M_1 and M_2 ($M_1, M_2 > M_K$, M_K being the kaon mass), and θ' is the mixing angle.

In this model the charge current has an additional right-handed current J'_μ:

$$J_W'^{\mu} = \bar{N}_{e_R}\gamma^\mu e_R + \bar{N}_{\mu_R}\gamma^\mu \mu_R \quad . \tag{9.6}$$

Hence, there exist extra one-loop diagrams as follows:

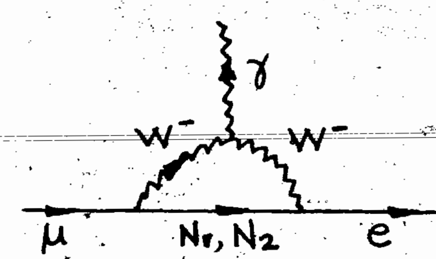


Fig.9.2 Diagrams of the process $\mu \rightarrow e\gamma$ with virtual neutral heavy leptons N_1 and N_2.

Neglecting the small contribution from fig.9.1, we find the ratio

$$R_\mu = \frac{3}{32}\frac{\alpha}{\pi}\left(\frac{M_1^2 - M_2^2}{M_W^2}\right)\sin^2\theta' \cos^2\theta' \quad . \tag{9.7}$$

We now can assume that the mass difference $|M_1 - M_2|$ is an order of GeV and the mixing is maximum $\theta' = \frac{\pi}{4}$. Thus, the $\mu \rightarrow e\gamma$ decay probability would turn out closer to its upper experimental limit.

As in other models, the rate of the processes $f_1 \rightarrow f_2 + \gamma$ in our model can also be calculated in one-loop diagrams. As mentioned before, we use the existing formulation due to Lee and Shrock. Although their results are based on the the assumption that massive neutrinos are of the Dirac type, their results are still applicable to our calculations here because the propagator for a Majorana field χ is just the same as the usual Dirac case (see appendix E). Now, let us consider the process $\mu \to e\gamma$. It has been found that the leading contribution to the decay amplitude will come from the following one-loop diagrams:

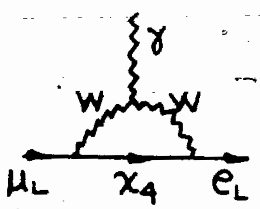
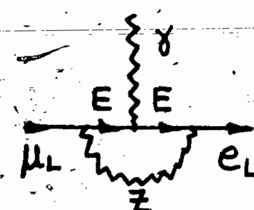
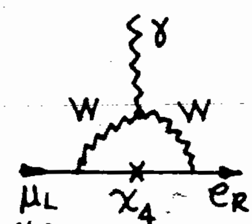
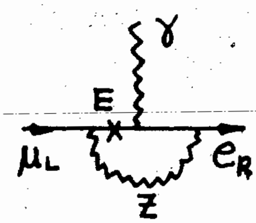
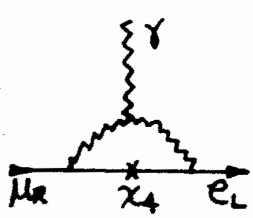
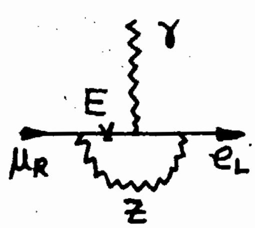
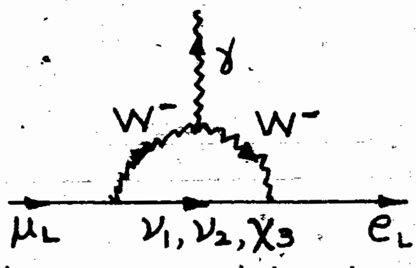
ONE-LOOP DIAGRAMS	RELATIVE AMPLITUDE	ONE-LOOP DIAGRAMS	RELATIVE AMPLITUDE
 (a)	$-\frac{5}{12} S_1 C_1 S_2 \frac{m_{\chi_3}}{M}$	 (d)	$\frac{1}{3} S_1 C_1 S_2 \frac{m_{\chi_3}}{M}$
 (b)	$2 S_1 C_1 S_2 \times \frac{m_e}{m_e+m_\mu} \frac{m_{\chi_3}}{M}$	 (e)	$-2 S_1 C_1 S_2 \times \frac{m_e}{m_e+m_\mu} \frac{m_{\chi_3}}{M}$
 (c)	$2 S_1 C_1 S_2 \times \frac{m_\mu}{m_e+m_\mu} \frac{m_{\chi_3}}{M}$	 (f)	$-2 S_1 C_1 S_2 \times \frac{m_\mu}{m_e+m_\mu} \frac{m_{\chi_3}}{M}$

Fig.9.3 Diagrams of the process $\mu \to e\gamma$ in our model and their relative contributions to the amplitude

First, we notice that all leading contributions come from diagrams in which the internal virtual fermions are the charged heavy leptons E or the neutral heavy leptons χ_4. Notice that the diagrams as follows:



$$\sim \left(\frac{m_{\chi_3}}{M_W}\right)^2$$

where the leading term vanishes by a leptonic G.I.M. mechanism. The contribution of these diagrams is usually important in some other models but is negligible compared to the other contributions here.

The loops mediated by the neutral currents rather than by the weak charged currents are possible because the neutral current matrix is not diagonal, and these contributions to the amplitude are important. Fig.9.3(b) and fig.9.3(c) are possible because of the existence of Majorana neutrinos χ_4 and the lepton-number-violating currents within the model.

The amplitudes of the diagrams in fig.9.3(b),(c),(e),(f) would be expected to be $\frac{M}{m_e}$ times larger than those in fig.9.3(a),(b) because they are proportional to the masses (M) of the heavy virtual leptons. However, the amplitudes of the diagrams in fig.9.3 are in the same order of magnitude because the right-handed couplings are $\frac{m_e}{M}$ times weaker than the left-handed one.

Since $m_\mu \gg m_e$ and the second term in $F^V_{LL,RR}$ and $F^A_{LL,RR}$ is unimportant (see appendix F, eq(F.7)), hence

$$F^V_{LL,RR} = F^A_{LL,RR}$$
$$= \frac{eG_F}{4\sqrt{2}\pi^2}\left[-\frac{5}{12}S_1C_1S_2\frac{m_{x_3}}{M}+\frac{1}{3}S_1C_1S_2\frac{m_{x_3}}{M}\right] \quad (9.8)$$

which are the contributions from fig.9.3(a) and (d).

Since the contribution of fig.9.3(b) is cancelled completely by the contribution of fig.9.3(e), and the same for fig.9.3(c) and (f), we thus see that

$$F^{V,A}_{LR,RL} = 0 \quad . \quad (9.9)$$

Finally, adding (9.8) and (9.9) together, we obtain

$$F^V = F^A$$
$$= \frac{-eG_F m_\mu^2}{48\sqrt{2}\pi^2}\left[S_1C_1S_2\frac{m_{x_3}}{M}\right] \quad . \quad (9.10)$$

Using (8.2), we obtain the rate

$$\Gamma(\mu \to e\gamma) \cong \frac{m_\mu^5}{2^{11}3^2\pi^5}e^2G_F^2(S_1C_1S_2)^2\left(\frac{m_{x_3}}{M}\right)^2 . \quad (9.11)$$

With $\alpha = \frac{e^2}{4\pi}$, $\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3}$, the rate can be written

$$\Gamma(\mu \to e\gamma) \cong \left(\frac{\alpha}{24\pi}\right)\left(\frac{m_{x_3}}{M}\right)^2(S_1C_1S_2)^2\cdot\Gamma_\mu \quad (9.12a)$$

We find that the same diagrams as in fig.9.3 will be involved in the leading contribution for the amplitude of the decays $\tau \to e\gamma$ and $\tau \to \mu\gamma$. Similarly, we obtain

$$\Gamma(\tau \to e\gamma) \cong \left(\frac{\alpha}{24\pi}\right)\left(\frac{m_{x_3}}{M}\right)^2\left(\frac{m_\tau}{m_\mu}\right)^5 (C_1S_1C_2)^2 \cdot \Gamma_\mu , \quad (9.12b)$$

and

$$\Gamma(\tau \to \mu\gamma) \cong \left(\frac{\alpha}{24\pi}\right)\left(\frac{m_{x_3}}{M}\right)^2\left(\frac{m_\tau}{m_\mu}\right)^5 (C_1^2S_2C_2)^2 \cdot \Gamma_\mu . \quad (9.12C)$$

Comparing (5.17) and (9.12), we find that the neutrino oscillations and the radiative decays of the known charged leptons are dependent on the same mixing parameters. That is,

$$P(\nu_e \to \nu_\mu) \propto \Gamma(\mu \to e\gamma) , \quad (9.13a)$$

$$P(\nu_e \to \nu_\tau) \propto \Gamma(\tau \to e\gamma) , \quad (9.13b)$$

$$P(\nu_\mu \to \nu_\tau) \propto \Gamma(\tau \to \mu\gamma) . \quad (9.13C)$$

hence, the existence of oscillations between the electron and muon neutrinos implies the existence of radiative decays for muon to electron.

9.2 The Radiative Decays Of Heavy Charged Leptons

Now, let us consider the radiative decays of the heavy charged lepton. It has been found that the leading contribution to the decay amplitude $E \to e\gamma$ will come from the following one-loop diagrams:

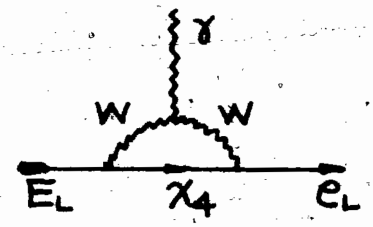
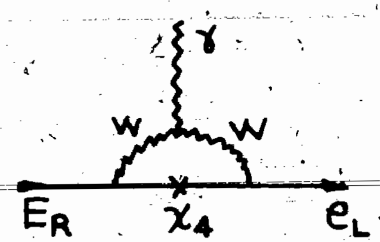
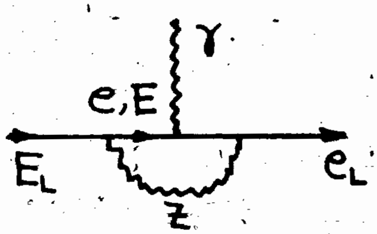
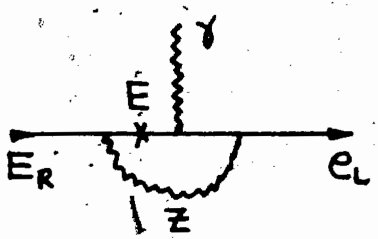
ONE-LOOP DIAGRAMS	RELATIVE AMPLITUDE
(a) 	$\frac{5}{2}\,\frac{d_1}{M}$
(b) 	$-\frac{m_{\chi_4}}{m_e+m_E}\,\frac{d_1}{M}$
(c) 	$\left\{\frac{1}{3}\left(-\frac{1}{2}+\sin^2\theta_W\right)-\frac{\cos^2\theta_W}{3}\right\}\frac{d_1}{M}$
(d) 	$\cos^2\theta_W\,\frac{m_E}{m_E+m_e}\,\frac{d_1}{M}$

Fig.9.4 Diagrams of the process $E \to e\gamma$ and their relative contributions

With $M_E = M \gg m_e, m_\mu, m_\tau$, we obtain

$$F^V_{LL,RR} = F^A_{LL,RR} \simeq \frac{cG_F M^2}{4\sqrt{2}\pi^2}\left\{\frac{5}{12}+\frac{1}{3}\left(-\frac{1}{2}+\sin^2\theta_W-\cos^2\theta_W\right)\right\}\frac{d_1}{M}, \tag{9.14a}$$

$$F^V_{LR,RL} = F^A_{LR,RL} \cong \frac{eG_F M^2}{4\sqrt{2}\,\pi^2}\left(\cos^2\theta_W - 1\right)\frac{d_1}{M} \quad . \qquad (9.14b)$$

Finally, we have

$$F^V = F^A \cong \frac{eG_F}{4\sqrt{2}\,\pi^2} M\left(-\frac{5}{12} + \frac{1}{3}\cos^2\theta_W\right)d_1 \quad . \qquad (9.15)$$

Using (8.2), we obtain the decay rate

$$\Gamma(E \to e\gamma) \cong \frac{M^5}{3\cdot 2^7 \pi^5} e^2 G_F^2 \left(-\frac{5}{4} + \cos^2\theta_W\right)^2 \sin^2\theta_1 \frac{m_{\chi_3}}{M} \qquad (9.16)$$

or

$$\Gamma(E \to e\gamma) \cong K \sin^2\theta_1 \Gamma_\mu \qquad (9.17a)$$

where

$$K = \left(\frac{2\alpha}{\pi}\right)\left(\frac{m_{\chi_3}}{M}\right)\left(\frac{M}{m_\mu}\right)^5\left[-\frac{5}{4} + \cos^2\theta_W\right]^2 .$$

Similarly, we find

$$\Gamma(E \to \mu\gamma) \cong K(\cos\theta_1 \sin\theta_2)^2 \Gamma_\mu \qquad (9.17b)$$

and

$$\Gamma(E \to \tau\gamma) \cong K(\cos\theta_1 \cos\theta_2)^2 \Gamma_\mu \qquad (9.17c)$$

hence

$$\Gamma(E \to (e,\mu,\tau)) = K\Gamma_\mu \quad . \qquad (9.18)$$

Similar to χ_4, the radiative decay rate of E differs from the decay rate of E in (7.16a) only by a factor proportional to the electromagnetic coupling constant 'α'. This character of E is extremely different from the known leptons as discussed in the early sections.

9.3 The Anomalous Magnetic Moments Of The Muon, Electron And Tau

We consider here the implications of the heavy leptons to the anomalous magnetic moments of the electron, muon and tau. The anomalous magnetic moment a of a lepton is defined as (see appendix F)

$$a = \frac{(g-2)}{2} = \frac{F^V}{Q} \tag{9.19}$$

where g is the gyromagnetic ratio and Q is the charge of the lepton.

The calculation on the contribution of the minimal GWS model to the anomalous magnetic moment of the leptons has been done. Here, we just calculate the contribution which may arise from the internal virtual heavy leptons in the one-loop diagrams. It is found that the leading contribution to the anomalous magnetic of e, μ and τ will be the diagrams in fig.9.5.

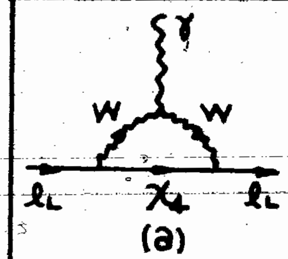
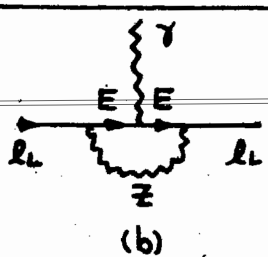
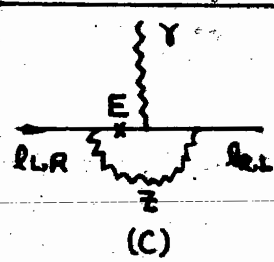
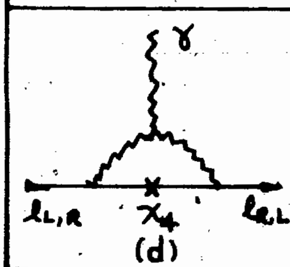
ONE-LOOP DIAGRAMS	RELATIVE AMPLITUDE		
	ELECTRON	MUON	TAU
	$-\frac{5}{3}\frac{d_1^2}{M^2}$	$-\frac{5}{3}\frac{d_2^2}{M^2}$	$-\frac{5}{3}\frac{d_3^2}{M^2}$
	$\frac{4}{3}\frac{d_1^2}{M^2}$	$\frac{4}{3}\frac{d_2^2}{M^2}$	$\frac{4}{3}\frac{d_3^2}{M^2}$
	$-\frac{8d_1^2}{M^2}$	$-\frac{8d_2^2}{M^2}$	$-\frac{8d_3^2}{M^2}$
	$\frac{8d_1^2}{M^2}$	$\frac{8d_2^2}{M^2}$	$\frac{8d_3^2}{M^2}$

Fig.9.5 Diagrams contribute to the anomalous magnetic moments of the electron, muon and tau

The contributions of the heavy leptons to the anomalous magnetic moments a' of electron, muon and tau are

$$a'_e = \frac{G_F m_e^2}{12\sqrt{2}\pi^2} S_1^2 \frac{m_{x_3}}{M} , \tag{9.20a}$$

$$a'_\mu = \frac{G_F m_\mu^2}{12\sqrt{2}\pi^2} (C_1 S_2)^2 \frac{m_{x_3}}{M} , \tag{9.20b}$$

$$a'_\tau = \frac{G_F m_\tau^2}{12\sqrt{2}\pi^2} (C_1 C_2)^2 \frac{m_{x_3}}{M} . \tag{9.20c}$$

Notice that the above contributions are all $\frac{m_{x_3}}{M}$ times smaller that the corresponding weak contributions from the minimal GWS model.

Numerical Results

Let us first estimate the numerical value for the ratio of the R_μ rate to the $\mu \to e\gamma$ rate. We expect that the decay rate (9.12a) in our model would be many order larger than the rate in (9.2) because $M < M_W$, $\left(\frac{m_{x_3}}{M}\right)^2 \gg \left(\frac{m_1^2 - m_2^2}{M_W^2}\right)$.

If we assume the maximum mixings $\theta_1 = \theta_2 = \frac{\pi}{4}$, m_{x_3}=100eV and M=20GeV, we obtain the ratio

$$R_\mu \cong 3.0 \times 10^{-22} . \tag{9.21}$$

Although it is still many order smaller than the experimental upper limit, it is greatly improved over the previous model (i).

We are also interested in the value for the anomalous magnetic moment a'_μ of muon because it has been found with great accuracy experimentally. With the above assumptions, we find

$$a'_\mu \cong 9.5\times 10^{-19} \tag{9.22}$$

which is negligibly small compared to the experimental value for a_μ,[47]

$$a_\mu^{\text{exp.}} = (1165924 \pm 8.5)\cdot 10^{-9} \quad . \tag{9.23}$$

X. Chapter 10 Conclusions

We have extended the GWS electroweak theory by the addition of heavy triplet fields. It is found that the neutrino in this model can acquire a nonzero mass without the need for any extra Higgs scalar fields, and lepton-number-violating processes are possible. As discussed in chapter 4, these new lepton number violating interactions are naturally much weaker than the standard weak interactions for the three known families of leptons.

As discussed in chapter 3, this triplet does not create any anomaly problem. Also it is found in chapter 9 that the contributions of the heavy leptons to the anomalous magnetic moments of the electron, muon and tau are insignificant, and thus consistent with the present experimental data.

These new heavy leptons are very unstable compared to the decay of the muon but stable compared to the decay of the tau (chapter 7). It is interesting to notice that the decay rates of heavy leptons in four-fermions pointlike interactions and the radiative decays are only different by the electromagnetic coupling constant. However, the decay rates for the known leptons in the latter decay processes are $\sim 10^{-16}$ smaller than the former one.

It is found that the radiative decays of the known charged leptons and the neutrino oscillations are dependent on the same mixing parameters. This is also an interesting feature of our model.

Neutrinoless double beta decays are also possible, their existence is a direct consequence of the lepton-number-violating currents and the lepton mixings.

As illustrated in this thesis, the existence of these heavy triplet fields will not alter the basic structure of the known lepton and quark interactions; nevertheless, they provide the possibilities for massive neutrinos and some new phenomena. Numerical results have shown that no known experimental limit is violated with the assumptions that the mass of light neutrino is 100eV and that of the new lepton is 20GeV . Finally, we would like to postulate that these new leptons have masses in the range between 20GeV to 30GeV because if they exist, they should be detected soon with our present experimental facilities.

APPENDIX A THE DIRAC EQUATION

The Dirac equation plays a fundamental role in relativistic quantum theory because it naturally describes the spin-1/2 particles such as the electron. The derivations of it have been given in many standard relativistic quantum theory texts; for example, Relativistic Quantum Mechanics by J. D. Bjorken and S. D. Drell[39] (1964). Here, we just want to provide some basic results and the notation used in this text.

The Dirac equation for a particle of spin-1/2 and mass m is

$$\frac{i\partial\psi}{\partial t} = (i\,\alpha\cdot\nabla + \beta m)\psi = H\psi \tag{A.1}$$

where the wave function ψ contains four components; α_i, β are 4x4 matrices which satisfy the anticommutation relations:

$$\begin{aligned} \{\alpha_i, \alpha_k\} &= 0 \quad , \quad \text{for } i \neq k,\ i,k = 1,2,3 \\ \{\alpha_i, \beta\} &= 0 \quad , \\ \alpha_i^2 = \beta^2 &= I \quad . \end{aligned} \tag{A.2}$$

One can introduce the notation $\gamma^\mu = (\gamma^0, \vec{\gamma})$:

$$\begin{aligned} \gamma^0 &= \beta \quad , \\ \gamma^i &= \beta\alpha^i \quad , \qquad (i = 1,2,3) \\ \{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu} \qquad (\mu = 0,1,2,3) \end{aligned} \tag{A.3}$$

(Latin letters for 1,2,3; Greek letters for 0,1,2,3)

and the Feynman "slash"

$$
\begin{aligned}
\not{a} &= \gamma^\mu a_\mu \qquad \text{(summation convention used)} \\
&= g_{\mu\nu}\gamma^\mu a^\nu \\
&= \gamma^0 a^0 - \vec{\gamma}\cdot\vec{a}
\end{aligned} \tag{A.4}
$$

is introduced. Then the Dirac equation in covariant form is

$$(i\gamma^\mu\partial_\mu - m)\psi \equiv (i\not{\partial} - m)\psi = 0 \tag{A.5}$$

with $\partial_\mu = (\partial_t, \nabla)$, $\nabla = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$ where x^1, x^2, x^3 are space coordinates.

In the Dirac-Pauli representation:

$$
\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix},
$$

$$
\gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}, \quad \alpha^i = \begin{bmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{bmatrix} \tag{A.6}
$$

where I and σ^i are the 2x2 unit matrix and the Pauli σ^i matrices

$$
\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{A.7}
$$

The positive- and negative-energy solutions, $\psi_\pm$, of the covariant free-particle Dirac equation are given by

$$(i\not{\partial} - m)\psi_\pm = 0 \quad . \tag{A.8}$$

For the positive-energy solution with momentum P,

$$\psi_+ = u(p)e^{iP\cdot x}e^{-iEt} \quad , \tag{A.9a}$$

while for the negative-energy solution with energy $-E\,(E=\sqrt{p^2+m^2})$ and momentum -P

$$\psi_-(x) = v(p)e^{ip\cdot x}e^{iEt}\,. \tag{A.9b}$$

Substituting (A.9a) and (A.9b) into (A.8), we have,

$$(\not{p}-m)u(p) = 0\,, \tag{A.10a}$$

$$(\not{p}+m)v(p) = 0\,. \tag{A.10b}$$

There are two linearly independent solutions for u and v with the normalizations‡ $\bar{u}u=2m,\ \bar{v}v=-2m$:

$$u^\lambda(p) = \frac{\not{p}+m}{\sqrt{(E+m)}}\begin{pmatrix}\varphi^\lambda(\hat{p})\\ 0\end{pmatrix}\,, \tag{A.11a}$$

$$v^\lambda(p) = \frac{-\not{p}+m}{\sqrt{(E+m)}}\begin{pmatrix}0\\ \chi^\lambda(\hat{p})=e^{-i\phi}\varphi^\lambda(-\hat{p})\end{pmatrix} \tag{A.11b}$$

where $\varphi^\lambda(\hat{p})$ are the eigenstates of the helicity operator $\lambda=S\cdot\hat{P}$. For a spin-1/2 particle $S=\frac{1}{2}\sigma$, σ^i are Pauli matrices and $\varphi^\lambda(\hat{p})$ satisfies

$$\frac{1}{2}\sigma\cdot\hat{p}\,\varphi^\lambda(\hat{p}) = \lambda\varphi^\lambda(\hat{p})\,. \tag{A.12}$$

The eigenvalues $\lambda=\pm\frac{1}{2}$ are for the corresponding helicity eigenstates. We now redefine $\sigma=\begin{bmatrix}\sigma & 0\\ 0 & \sigma\end{bmatrix}$; now σ is the four-component Dirac spin matrix . Since $[\sigma\cdot\hat{p},\not{p}]=0$, $u^\lambda(p)$ and $v^\lambda(p)$ satisify

---

‡Our normalizations of u and v differ by $\left(\frac{1}{2m}\right)^{1/2}$ from those defined in Bjorken and Drell.

$$\frac{1}{2}\sigma\cdot\hat{p}\, u^\lambda(p) = \lambda u^\lambda(p)\,, \tag{A.13a}$$

$$-\frac{1}{2}\sigma\cdot\hat{p}\, v^\lambda(p) = \lambda v^\lambda(p)\,. \tag{A.13b}$$

Some useful matrices and their relations:

The anticommutation relations of γ matrices:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,. \tag{A.14}$$

The chirality operator:

$$\begin{aligned}\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 &= -i\gamma_0\gamma_1\gamma_2\gamma_3 \\ &= \gamma_5^\dagger = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\,,\end{aligned} \tag{A.15a}$$

$$\gamma_5^2 = 1\,, \tag{A.15b}$$

$$\{\gamma_5, \gamma^\mu\} = 0\,. \tag{A.15c}$$

Commutation relations of γ matrices:

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]\,, \tag{A.16a}$$

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\sigma^{\mu\nu}\,, \tag{A.16b}$$

$$[\gamma^5, \sigma^{\mu\nu}] = 0\,. \tag{A.16c}$$

Hermitian conjugates:

$$\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}\,, \tag{A.17a}$$

$$\gamma^0\gamma_5\gamma^0 = -\gamma_5^\dagger = -\gamma_5\,, \tag{A.17b}$$

$$\gamma^0(\gamma_5\gamma^\mu)\gamma^0 = (\gamma_5\gamma^\mu)^\dagger , \tag{A.17c}$$

$$\gamma^0 \sigma^{\mu\nu}\gamma^0 = (\sigma^{\mu\nu})^\dagger . \tag{A.17d}$$

The projection operators:

$$\sum_S u(P,S)\bar{u}(P,S) = \not{P} + m , \tag{A.18a}$$

$$\sum_S v(P,S)\bar{v}(P,S) = \not{P} - m . \tag{A.18b}$$

Trace theorems and γ Identities:

$$\not{a}\not{b} = a\cdot b - i\,\sigma_{\mu\nu}a^\nu b^\nu . \tag{A.19}$$

Trace of odd number γ_μ's vanishes.

$$\mathrm{Tr}\,\gamma_5 = 0 , \tag{A.20a}$$

$$\mathrm{Tr}\,1 = 4 , \tag{A.20b}$$

$$\mathrm{Tr}\,\not{a}\not{b} = 4a\cdot b , \tag{A.20c}$$

$$\mathrm{Tr}\,\not{a}_1\not{a}_2\not{a}_3\not{a}_4 = 4[a_1\cdot a_2\, a_3\cdot a_4 - a_1\cdot a_3\, a_2\cdot a_4 + a_1\cdot a_4\, a_2\cdot a_3] , \tag{A.20d}$$

$$\mathrm{Tr}\,\gamma_5\not{a}\not{b} = 0 , \tag{A.20e}$$

$$\mathrm{Tr}\,\gamma_5\not{a}\not{b}\not{c}\not{d} = 4i\,\epsilon_{\alpha\beta\gamma\delta}a^\alpha b^\beta c^\gamma d^\delta , \tag{A.20f}$$

$$\gamma_\mu\not{a}\gamma^\mu = -2\not{a} , \tag{A.20g}$$

$$\gamma_\mu\not{a}\not{b}\gamma^\mu = 4a\cdot b , \tag{A.20h}$$

$$\gamma_\mu\not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a} . \tag{A.20i}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is the Levi-Civita pseudotensor ($\epsilon_{0123}=1$).

APPENDIX B THE TWO-COMPONENT THEORY OF MASSLESS SPIN-1/2 PARTICLES

The representation-independence (Pauli-Good) theorem states that all representations of γ-matrices are equivalent up to a similarity transfomation U:

$$\gamma'^{\mu} = U\gamma^{\mu}U^{+} . \tag{B.1}$$

Let us consider $U = \frac{1}{\sqrt{2}}(1-\gamma_5\gamma_0) = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}$,

then we find a new set of γ matrices:

$$\gamma^0 = \beta = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}, \quad \alpha^i = \begin{bmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{bmatrix}, \quad i = 1,2,3$$

$$\gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}, \quad \gamma^5 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad k = 1,2,3 \tag{B2}$$

This representatin was first introduced by H. Weyl in 1929.

With the wave function Ψ written as:

$$\Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \tag{B3}$$

in which ψ_R and ψ_L are two-component spinors, the Dirac equation can be written as two coupled equations:

$$i\frac{\partial\psi_R}{\partial t} + i\sigma\cdot\nabla\psi_R = -m\psi_L , \tag{B.3a}$$

$$i\frac{\partial\psi_L}{\partial t} - i\sigma\cdot\nabla\psi_L = -m\psi_R . \tag{B3b}$$

Clearly, if m=0, the two coupled equations will be decoupled.

As in (A.8) for positive-energy solutions are

$$\psi_{R,L} = u_{R,L}(P)\, e^{iP\cdot x}\, e^{-iEt} \qquad \text{(B4a)}$$

Substituting (b.4a) into (b.3a), we have

$$(E \mp \sigma\cdot P)\, u_{R,L} = 0 \quad . \qquad \text{(B.5)}$$

Let $u_{R,L}$ be the eigenstates of the helicity operator:

$$\frac{1}{2}\sigma\cdot\hat{P}\, u_{R,L} = \lambda u_{R,L} \quad , \quad \lambda = \pm\frac{1}{2} \, . \qquad \text{(B6)}$$

Since $E=|P|$, clearly only the λ=1/2 state survives for u_R, whereas only the λ=-1/2 state survives for u_L.

Similarly, for negative-energy solution $-E=-|P|$ and momentum $-P$

$$\psi^-_{R,L} = v_{R,L}(P)\, e^{-iP\cdot x}\, e^{iEt} \, . \qquad \text{(B.4b)}$$

One finds that only the λ=-1/2 state survives for v_R, whereas only the λ =1/2 state survives for v_L.

The chirality operator γ^5 in this representation is diagonized. One defines an operator $a=\frac{1}{2}(1-\gamma_5)$ which projects out the left-handed spinor, whereas $\bar{a}=\frac{1}{2}(1+\gamma_5)$ which projects out the right-handed spinor:

$$\frac{1}{2}(1-\gamma_5)\Psi = \begin{pmatrix} 0 \\ \psi_L \end{pmatrix} = \Psi_L \, , \qquad \text{(B.7a)}$$

$$\frac{1}{2}(1+\gamma_5)\Psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} = \Psi_R \, . \qquad \text{(B.7b)}$$

One important point about the set of equations (b.3) is that they are not invariant under spatial reflection ($\psi_{L,R} \rightarrow \psi_{R,L}$). Due to this reason, they had been rejected for a long time until the parity violation experiments in weak interactions were found in 1957. It was Lee and Yang[49] who first pointed out that there was no evidence for conservation of parity in weak interactions. Now, experiments have agreed with the assumption that only ψ_L take part in charged weak interactions.

APPENDIX C DISCRETE SYMMETRIES

Parity:

A parity transfomation means $\mathbf{x} \to \mathbf{x}' = -\mathbf{x}$ and $t \to t' = t$. Under parity, ψ transforms

$$\psi(t,\mathbf{x}) \xrightarrow[P]{} \eta_p \gamma_0 \psi(t,-\mathbf{x}) \quad , \quad |\eta_p| = 1 , \tag{C.1}$$

For the quantum Dirac field, we need a unitary operator $\mathcal{P}$ satisifying

$$\mathcal{P}\psi(x)\mathcal{P}^\dagger = \eta_p \gamma^0 \psi(\tilde{x}) \quad , \tag{C.2}$$

where $\tilde{x} = (t, -\mathbf{x})$. It is easy to show that

$$\psi_{L,R}(t,\mathbf{x}) \xrightarrow[P]{} \gamma_0 \psi_{R,L}(t,-\mathbf{x}) \quad , \tag{C.3a}$$

$$\bar{\psi}_{L,R}(t,\mathbf{x}) \xrightarrow[P]{} \bar{\psi}_{R,L}(t,-\mathbf{x})\gamma_0 \quad . \tag{C.3b}$$

Charge Conjugation:

The charge conjugation operation converts particle to antiparticle:

$$\psi \xrightarrow[C]{} \psi^c = \eta_c C \bar{\psi}^T \quad , \quad |\eta_c| = 1 \tag{C.4a}$$

$$\bar{\psi} \xrightarrow[C]{} \bar{\psi}^c = -\eta_c^\dagger \psi^T C^{-1} \quad . \tag{C.4b}$$

C is a Dirac matrix defined by

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T \tag{C.5a}$$

and has the following properties:

$$C\gamma_5 C^{-1} = \gamma_5^T \tag{C.5b}$$

$$C\sigma_{\mu\nu} C^{-1} = -\sigma_{\mu\nu}^T \tag{C.5c}$$

$$C(\gamma_5\gamma_\mu)C^{-1} = (\gamma_5\gamma_\mu)^T \tag{C.5d}$$

$$C^T = C^\dagger = -C \tag{C.5e}$$

$$CC^\dagger = C^\dagger C = I \ , \ C^2 = -1 \ . \tag{C.5f}$$

For the quantum Dirac field, we need a unitary operator $\mathcal{C}$ satisifying

$$\mathcal{C}\psi(x)\mathcal{C}^\dagger = \eta_c C\bar{\psi}^T \ . \tag{C.6}$$

It is easy to show that

$$\psi^c_{L,R} \xrightarrow[C]{} (\psi^c)_{L,R} = C\bar{\psi}^T_{R,L} \ , \tag{C.7a}$$

$$\bar{\psi}_{L,R} \xrightarrow[C]{} (\bar{\psi}^c)_{L,R} = -\psi^T_{R,L} C^{-1} \tag{C.7b}$$

where $(\psi^c)_L((\psi^c)_R)$ is the field which annihilates a left(right)-handed antiparticle or creates a right(left)-handed particle.

Let us now investigate the transformation properties of bilinear forms: $\bar{\Psi}O\psi$ where O is a Dirac matrix and $\bar{\Psi}$, ψ are the fermion field operators. However, in field theory, such a form $\bar{\Psi}O\psi$ will lead to difficulties (see Bjorken and Drell) unless we antisymmetrize (or, equivalently, normal-order) the fermion field operators which is

$$\bar{\Psi}O\psi \longrightarrow \frac{1}{2}[\bar{\Psi}, O\psi] \quad . \tag{C.8}$$

Hence, under charge conjugation, the bilinear form transforms as

$$\begin{aligned}
\mathcal{C}[\bar{\Psi}, O\psi]\mathcal{C}^{\dagger} &= O_{\alpha\beta}[\mathcal{C}\bar{\Psi}_{\alpha}\mathcal{C}^{\dagger}, \mathcal{C}\psi_{\beta}\mathcal{C}^{\dagger}] \\
&= O_{\alpha\beta}[(-\psi^{T}C^{-1})_{\alpha}, (C\bar{\Psi}^{T})_{\beta}] \\
&= O_{\alpha\beta}[-\psi_{\nu}C^{-1}_{\nu\alpha}, C_{\beta\mu}\bar{\Psi}_{\mu}] \\
&= O_{\alpha\beta}[C_{\beta\mu}\bar{\Psi}_{\mu}, C^{-1}_{\nu\alpha}\psi_{\nu}] \\
&= C^{-1}_{\nu\alpha}O_{\alpha\beta}C_{\beta\mu}[\bar{\Psi}_{\mu}, \psi_{\nu}] \\
&= (C^{-1}OC)_{\nu\mu}[\bar{\Psi}_{\mu}, \psi_{\nu}] \\
&= [\bar{\Psi}, O'\psi]
\end{aligned} \tag{C.9}$$

where

$$O' = (C^{-1}OC)^{T} \quad .$$

Finally, we list the transformation properties of bilinear forms in the Dirac field under Parity and charge conjugation:[40]

Table C.1

The Transformation Properties Of Bilinear Forms Under Discrete Symmetries

	$S(x)$	$V^{\mu}(x)$	$T^{\mu\nu}(x)$	$A^{\mu}(x)$	$P(x)$
$\mathcal{P}$	$S(\tilde{x})$	$V_{\mu}(\tilde{x})$	$T_{\mu\nu}(\tilde{x})$	$-A_{\mu}(\tilde{x})$	$-P(\tilde{x})$
$\mathcal{C}$	$S(x)$	$-V^{\mu}(x)$	$-T^{\mu\nu}(x)$	$A^{\mu}(x)$	$P(x)$

where $\tilde{x} = (t, -x)$

$$S(x) = :\bar{\Psi}(x)\Psi(x): \quad ,$$
$$V^{\mu}(x) = :\bar{\Psi}(x)\gamma^{\mu}\Psi(x): \quad ,$$
$$T^{\mu\nu}(x) = :\bar{\Psi}(x)\sigma^{\mu\nu}\Psi(x): \quad ,$$
$$A^{\mu}(x) = :\bar{\Psi}(x)\gamma^{5}\gamma^{\mu}\Psi(x): \quad ,$$
$$P(x) = i:\bar{\Psi}(x)\gamma^{5}\Psi(x): \quad .$$

The double-dot symbol denotes for "normal-ordered".

APPENDIX D THE DECAY OF A LEPTON IN FOUR-FERIMON POINTLIKE WEAK INTERACTIONS

The lowest order calculations for the weak decay rate of a heavy leptons ψ_1,

$$\psi_1 \longrightarrow \psi_2 + \psi_3 + \overline{\psi}_4 \quad , \qquad (D.1)$$

to fermions ψ_2, ψ_3 and $\overline{\psi}_4$ is shown in this appendix for four different cases

(i) the ψ_2 , ψ_3 are two different fermions,

(ii) the ψ_2 , ψ_3 are identical fermions,

(iii) the ψ_1 , ψ_2 are Majorana neutrinos,

(iv) the ψ_1 , ψ_2 , ψ_3 are all identical Majorana neutrinos.

In the low energy domain, the weak interaction can be approximated as a four-fermion pointlike interaction; the effective Vector and Axial-vector (V-A) current-current interaction Lagrangian density $\mathcal{L}^{\text{eff.}}$ can be written

$$\mathcal{L}^{\text{eff}} = \frac{G}{\sqrt{2}} J^{\dagger}_{21\mu} J^{\mu}_{34}$$

$$= \frac{G}{\sqrt{2}} \overline{\psi}_2 \gamma_\mu (g_V + g_A \gamma_5) \psi_1 \overline{\psi}_3 \gamma^\mu (\tilde{g}_V + \tilde{g}_A \gamma_5) \psi_4 \qquad (D.2)$$

where

$$J_{21\mu} = \overline{\psi}_2 \gamma_\mu (g_V + g_A \gamma_5) \psi_1 \quad , \qquad (D.3)$$

$$J^{\mu}_{34} = \overline{\psi}_3 \gamma^\mu (\tilde{g}_V + \tilde{g}_A \gamma_5) \psi_4 \qquad (D.4)$$

and $(g_V + g_A\gamma_5) = g_L(1-\gamma_5) + g_R(1+\gamma_5)$, $(\tilde{g}_V + \tilde{g}_A\gamma_5) = \tilde{g}_L(1-\gamma_5) + \tilde{g}_R(1+\gamma_5)$; $g_V, g_A, \tilde{g}_V, \tilde{g}_A$ are some real constants.

Case (i)

In the lowest order calculations, only one Feynman tree diagram for such a decay process is possible.

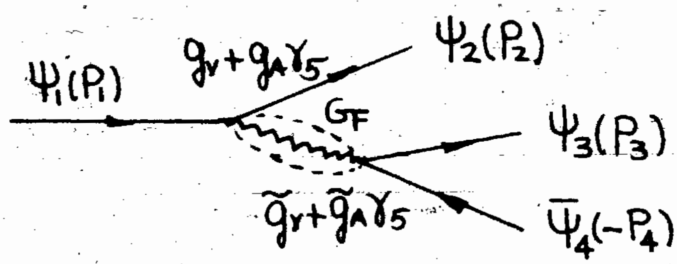


Fig.D.1 Feynman diagram for four-fermion interation for heavy lepton Ψ_1 decay.

The Lorentz invariant amplitude $\mathcal{M}$ for the diagram in Fig.D.1 is

$$\mathcal{M} = \left[\bar{u}_2 \gamma^\mu (g_v + g_A\gamma_5) u_1\right] \frac{G_F}{\sqrt{2}} \left[\bar{u}_3 \gamma^\mu (\tilde{g}_v + \tilde{g}_A \gamma_5) v_4\right] \qquad \text{(D.5)}$$

where $u_m = u_m(P_m, S_m)$ is a Dirac Spinor for the fermion Ψ_m of physical momentum P_m and polarization S_m, while v_4 is for the antifermion $\overline{\Psi}_4$.

The differential decay rate for an unpolarized fermion Ψ_1 is

$$d\Gamma = \frac{1}{2}\left(\frac{1}{2\pi}\right)^5 \left(\frac{1}{2E_1}\right) \frac{d^3P_2\, d^3P_3\, d^3P_4}{2E_2\; 2E_3\; 2E_4}\, \delta^4(P_1 - P_2 - P_3 - P_4) \underbrace{\sum |M|^2}_{\text{sum over all initial and final spins}}, \qquad \text{(D.6)}$$

which we obtain from Bjorken and Drell. The factor 1/2 in front of (D.6) is due to the fact that we average over all possible spins of the initial fermion Ψ_1.

Let us first evaluate $\sum|\mathcal{M}|^2$, with $\mathcal{M}$ given in (D.5), we have

$$\sum|\mathcal{M}|^2 = \frac{G^2}{2}\sum_{spins}(\bar{u}_2\gamma^\mu g u_1 \bar{u}_3\gamma_\mu \tilde{g} v_4)(\bar{u}_2\gamma^\nu g u_1 \bar{u}_3\gamma_\nu \tilde{g} v_4)^*$$

$$= \frac{G^2}{2}\sum_{spins}(\bar{u}_2\gamma^\mu g u_1 \bar{u}_1\gamma^\nu g u_2)(\bar{u}_3\gamma_\mu \tilde{g} v_4 \bar{v}_4\gamma_\nu \tilde{g} u_3)$$

$$= \frac{G^2}{2} T_1^{\mu\nu} T_{2\mu\nu} \qquad (D.7)$$

where we denote $g=(g_A+g_V\gamma_5)$ and $\tilde{g}=(\tilde{g}_A+\tilde{g}_V\gamma_5)$, and

$$T_1^{\mu\nu} = \sum_{spins} \bar{u}_2\gamma^\mu g u_1 \bar{u}_1\gamma^\nu g u_2 \quad , \qquad (D.8)$$

$$T_2^{\mu\nu} = \sum_{spins} \bar{u}_3\gamma_\mu \tilde{g} v_4 \bar{v}_4\gamma_\nu \tilde{g} u_3 \; . \qquad (D.9)$$

To evaluate $T_1^{\mu\nu}$, we first write it explicitly with indices:

$$T_1^{\mu\nu} = \sum_{spins} (\bar{u}_2)_k \gamma^\mu_{kl} g_{lm} (u_1\bar{u}_1)_{mn} \gamma^\nu_{no} g_{op} u_{2p}$$

$$= \sum_{spins} (u_2\bar{u}_2)_{pk} \gamma^\mu_{kl} g_{lm} (u_1\bar{u}_1)_{mn} \gamma^\nu_{no} g_{op} \; . \qquad (D.10)$$

With the projection operator $\sum_S u_\alpha(p.s)\bar{u}_\alpha(p.s) = \not{p}_\alpha + m_\alpha$ in (A.18), we obtain

$$T_1^{\mu\nu} = Tr[(\not{p}_2+m_2)\gamma^\mu g(\not{p}_1+m_1)\gamma^\nu g] \; . \qquad (D.11a)$$

Similarly, we have

$$T_{2\mu\nu} = Tr[(\not{p}_3+m_3)\gamma_\mu \tilde{g}(\not{p}_4-m_4)\gamma_\nu \tilde{g}] \; . \qquad (D.11b)$$

With the assumption that the masses of ψ_1 , ψ_2 and ψ_4 are much lighter compared to the mass M of ψ_1 , and only the leading contributions to the decay rate of ψ_1 are of interest, the masses of these fermions can be neglected, hence

$$T_1^{\mu\nu} = \mathrm{Tr}\left[\not{P}_2 \gamma^\mu g (\not{P}_1 + M) \gamma^\nu g\right] \quad , \tag{D.12a}$$

$$T_{2\mu\nu} = \mathrm{Tr}\left[\not{P}_3 \gamma_\mu \tilde{g} \not{P}_4 \gamma_\nu \tilde{g}\right] \quad . \tag{D.12b}$$

Using the properties Tr(odd number of γ^μ's)=0 (A.20) and $\{\gamma^5, \gamma^\mu\}=0$ (A.15), we have

$$\begin{aligned} T_1^{\mu\nu} &= \mathrm{Tr}\,(g^2 \not{P}_2 \gamma^\mu \not{P}_1 \gamma^\nu) \\ &= 2\mathrm{Tr}\left[(g_L^2 + g_R^2 + (g_R^2 - g_L^2)\gamma_5)\not{P}_2 \gamma^\mu \not{P}_1 \gamma^\nu\right] \quad . \end{aligned} \tag{D.13}$$

Applying the Trace therom in (A.20),

$\mathrm{Tr}(\gamma^5 \not{a} \not{b} \not{c} \not{d}) = 4i\epsilon_{\alpha\beta\gamma\delta} a^\alpha b^\beta c^\gamma d^\delta$ and

$\mathrm{Tr}(\not{a} \not{b} \not{c} \not{d}) = 4(a\cdot b\, c\cdot d - a\cdot c\, b\cdot d + a\cdot d\, b\cdot c)$, we have

$$T_1^{\mu\nu} = 8\left\{(g_L^2 + g_R^2)(P_2^\mu P_1^\nu - P_2\cdot P_1 g^{\mu\nu} + P_2^\nu P_1^\mu) + i(g_R^2 - g_L^2)\epsilon^{\alpha\mu\beta\nu} P_{2\alpha} P_{1\beta}\right\} . \tag{D.14a}$$

Similarly, we have

$$T_2^{\mu\nu} = 8\left\{(\tilde{g}_L^2 + \tilde{g}_R^2)(P_{3\mu} P_{4\nu} - P_3\cdot P_4 g_{\mu\nu} + P_{3\nu} P_{4\mu}) + i(\tilde{g}_R^2 - \tilde{g}_L^2)\epsilon_{\xi\mu\eta\nu} P_{3\xi} P_{4\eta}\right. . \tag{D.14b}$$

Now, the decay amplitude, $\sum|M|^2$ is

$$\begin{aligned} \sum_s |M|^2 &= \frac{G^2}{2} T_1^{\mu\nu} T_{2\mu\nu} \\ &= \frac{G^2}{2} 64\left\{(g_L^2 + g_R^2)(\tilde{g}_L^2 + \tilde{g}_R^2)(P_2^\mu P_1^\nu + P_2^\nu P_1^\mu - P_1\cdot P_2 g^{\mu\nu})(P_{3\mu} P_{4\nu} + P_{3\nu} P_{4\mu} - P_3\cdot P_4 g_{\mu\nu})\right. \\ &\quad \left. - (g_R^2 - g_L^2)(\tilde{g}_R^2 - \tilde{g}_L^2)\epsilon^{\alpha\mu\beta\nu}\epsilon_{\xi\mu\eta\nu} P_{2\alpha} P_{1\beta} P_3^\xi P_4^\eta\right\} \end{aligned} \tag{D.15}$$

$$\begin{aligned} &= 32G^2\left\{(g_L^2 + g_R^2)(\tilde{g}_L^2 + \tilde{g}_R^2)(2P_1\cdot P_4\, P_2\cdot P_3 + 2P_1\cdot P_3\, P_2\cdot P_4)\right. \\ &\quad \left. + 2(g_L^2 - g_R^2)(\tilde{g}_L^2 - \tilde{g}_R^2)(P_1\cdot P_4\, P_2\cdot P_3 - P_2\cdot P_4\, P_1\cdot P_3)\right\} \end{aligned} \tag{D.16}$$

where we have used

$$\epsilon^{\alpha\mu\beta\nu}\epsilon_{\xi\mu\eta\nu} = -2\left[g^{\alpha}_{\xi}g^{\beta}_{\eta} - g^{\alpha}_{\eta}g^{\beta}_{\xi}\right] . \qquad (D.17)$$

Finally, after a few steps of algebra, it is easy to show

$$\sum_{\text{spins}} |M|^2 = 128G^2(A\,P_1\cdot P_4\,P_2\cdot P_3 + B\,P_1\cdot P_3\,P_2\cdot P_4) \qquad (D.18)$$

where we put $A=(g_L^2\tilde{g}_L^2+g_R^2\tilde{g}_R^2)$, $B=(g_L^2\tilde{g}_R^2+g_R^2\tilde{g}_L^2)$.

To proceed further, we integrate over all possible momentum P_2, P_3 of the fermions ψ_2 and ψ_3, the decay rate in (D.6) can be written as

$$d\Gamma = \frac{1}{2}\left(\frac{1}{2\pi}\right)^5\left(\frac{1}{2E_1}\right)\frac{d^3P_4}{2E_4}\iint\frac{d^3P_2\,d^3P_3}{2E_2\;2E_3}\delta^4(P_1-P_2-P_3-P_4)(128G^2)$$

$$\times(A\,P_1\cdot P_4\,P_2\cdot P_3 + B\,P_1\cdot P_3\,P_2\cdot P_4)$$

$$= \frac{G^2}{\pi^5E_1}\frac{d^3P_4}{2E_4}(AP_1\cdot P_4\,I^0 + B\,P_{1\mu}P_{4\nu}I^{\mu\nu}) \qquad (D.19)$$

where

$$I^0 = \iint\frac{d^3P_2\,d^3P_3}{2E_2\;2E_3}\delta^4(Q-P_2-P_3)P_2\cdot P_3 = \frac{\pi}{4}Q^2, \qquad (D.20a)$$

$$I^{\mu\nu} = \iint\frac{d^3P_2\,d^3P_3}{2E_2\;2E_3}\delta^4(Q-P_2-P_3)P_3^{\mu}P_2^{\nu} = \frac{\pi}{24}(g^{\mu\nu}Q^2+2Q^{\mu}Q^{\nu}), \qquad (D.20b)$$

and

$$Q = P_1 - P_4 . \qquad (D.20c)$$

The results of the integrals I^0 and $I^{\mu\nu}$ can be found in Bjorken and Drell.

Now, let us choose the initial frame in which the heavy lepton ψ_1 is at rest, hence,

$$P_1^{\mu} = (M, 0, 0, 0) . \tag{D.21a}$$

With the mass of ψ_4 is neglected, P_4^{μ} is

$$P_4^{\mu} = (E_4 = |\vec{P_4}| , \vec{P_4}) \tag{D.21b}$$

Therefore, we have

$$Q = (P_1 - P_4)^2 = P_1^2 - 2P_1 \cdot P_4 + P_4^2 = M^2 - 2ME_4 \tag{D.22}$$

where $P_1 \cdot P_4 = ME_4$ and $P_4^2 = 0$.

Finally, $d\Gamma$ becomes

$$d\Gamma = \frac{G^2}{\pi^5 M} \frac{d^3P_4}{2E_4} \left[\frac{A\pi M^3}{4} \left(E_4 - \frac{2E_4^2}{M}\right) + \frac{B\pi}{24} M^3 \left(3E_4 - \frac{4E_4^2}{M}\right) \right] . \tag{D.23}$$

Integrating over the lepton ψ_4 angles and all possible energies of the ψ_4, $0 < E_4 \leq \frac{1}{2}M$,‡ we finally obtain the total decay rate Γ for the ψ_1 :

$$\begin{aligned} \Gamma &= \frac{G^2M^2}{12\pi^3} \int_0^{\frac{M}{2}} \left[6A\left(E_4 - \frac{2E_4^2}{M}\right) + B\left(3E_4 - \frac{4E_4^2}{M}\right) \right] dE_4 \\ &= \frac{G^2M^2}{12\pi^3} \frac{M^3}{16} (A + B) \\ &= \frac{G^2M^5}{192\pi^3} (g_L^2 \tilde{g}_L^2 + g_R^2 \tilde{g}_R^2 + g_L^2 \tilde{g}_R^2 + g_R^2 \tilde{g}_L^2) \\ &= \frac{G^2M^5}{192\pi^3} (g_L^2 + g_R^2)(\tilde{g}_L^2 + \tilde{g}_R^2) . \end{aligned} \tag{D.24}$$

‡The maximum possible energy for the lepton ψ_4, $E_{4max} = \frac{1}{2}M$, is taken when the other two leptons ψ_2, ψ_3 are emitted in same direction and the lepton ψ_4 in the opposite direction.

Case (ii)

If we have the identical fermions ψ_2 and ψ_3 in the final states, we must antisymmetrize the product wave functions for ψ_2 and ψ_3; the amplitude must be antisymmetric under the exchange of the fermions ψ_2 and ψ_3. In terms of Feynman diagrams, we must have two diagrams as follows:

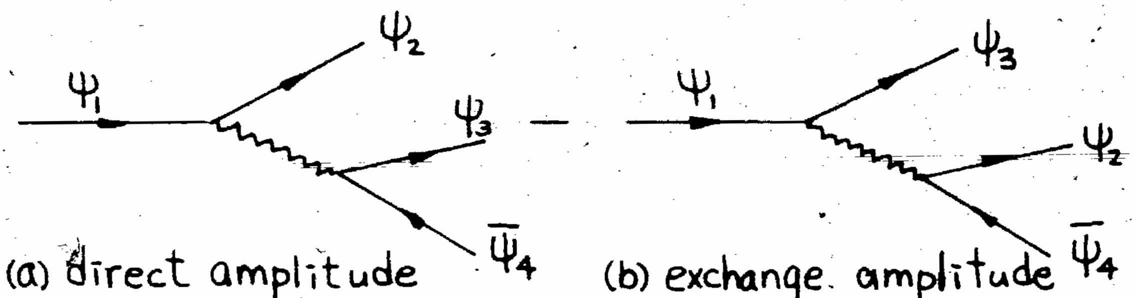


Fig.D.2 The Feynman diagrams for the heavy lepton ψ_1 decay into two identical fermions ψ_2 and ψ_3.

The resultant amplitude $\mathcal{M}_T$ would be the addition of the amplitude of the diagram in Fig.D.2(a) and the amplitude of the diagram in Fig.D.2(b). Since it involves the interchange of two fermions, the amplitudes for these two diagrams should be opposite in sign, hence

$$\mathcal{M}_T = \mathcal{M}_1 - \mathcal{M}_2 \tag{D.25}$$

where $\mathcal{M}_1$ is given in (D.3) and $\mathcal{M}_2$ is the same as $\mathcal{M}_1$ with subscripts 2 and 3 interchanged. Again, let us first evaluate $\sum_S |\mathcal{M}_T|^2$, we have

$$\sum_S |\mathcal{M}_T|^2 = \sum_S |\mathcal{M}_1|^2 - \sum_S |\mathcal{M}_1\mathcal{M}_2^\dagger + \mathcal{M}_2\mathcal{M}_1^\dagger| + \sum_S |\mathcal{M}_2|^2. \tag{D.26}$$

Clearly, with the low energy approximation, the contribution of the third term in (D.26) to the total decay rate is the same as the contribution of the first term which is found previously (D.24). Hence, we only need to evaluate the contribution from the second term: the interference terms. We have

$$\sum |\mathcal{M}_1 \mathcal{M}_2^+ + \mathcal{M}_2 \mathcal{M}_1^+| = \mathcal{M}_{int}^2$$

$$= \frac{G^2}{2} \sum_S \left\{ [(\bar{u}_2 \gamma_\mu g u_1)(\bar{u}_3 \gamma^\mu \tilde{g} v_4)][(\bar{u}_3 \gamma_\mu g u_1)(\bar{u}_2 \gamma^\nu \tilde{g} v_4)]^* + (2 \longleftrightarrow 3) \right\}$$

$$= \frac{G^2}{2} \sum_S \left\{ [(\bar{u}_2 \gamma_\mu g u_1)(\bar{u}_1 \gamma_\nu g u_3)(\bar{u}_3 \gamma^\mu \tilde{g} v_4)(\bar{v}_4 \gamma^\nu \tilde{g} u_2) + (2 \longleftrightarrow 3) \right\} . \qquad (D.27)$$

Using the same tricks as before and neglecting the masses for the leptons ψ_2, ψ_3 and ψ_4, we obtain

$$\mathcal{M}_{int.} = \frac{G^2}{2} \mathrm{Tr}[\not{P}_2 \gamma_\mu g (\not{P}_1 + m) \gamma_\nu g \not{P}_3 \gamma^\mu \tilde{g} \not{P}_4 \gamma^\nu \tilde{g}] + (2 \longleftrightarrow 3)$$

$$= \frac{G^2}{2} \mathrm{Tr}[g^2 \tilde{g}^2 (\not{P}_2 \gamma_\mu \not{P}_1 \gamma_\nu \not{P}_3 \gamma^\mu \not{P}_4 \gamma^\nu) + (2 \longleftrightarrow 3)\} . \qquad (D.28)$$

Using $\gamma_\mu \not{a} \not{b} \gamma^\mu = 4 a \cdot b$, $\gamma_\mu \not{a} \not{b} \not{c} \gamma^\mu = -2 \not{c} \not{b} \not{a}$ in (A.20), we have

$$\not{P}_2 \gamma_\mu \not{P}_1 \gamma_\nu \not{P}_3 \gamma^\mu \not{P}_4 \gamma^\nu = -2 \not{P}_2 \not{P}_3 \gamma_\nu \not{P}_1 \not{P}_4 \gamma^\nu$$

$$= -8 \not{P}_2 \not{P}_3 P_1 \cdot P_4 , \qquad (D.29)$$

therefore,

$$\mathcal{M}_{int}^2 = -32 G^2 \mathrm{Tr}[(g_L^2 \tilde{g}_L^2 + g_R^2 \tilde{g}_R^2) + (g_R^2 \tilde{g}_R^2 - g_L^2 \tilde{g}_L^2) \gamma_5]$$

$$\times (\not{P}_2 \not{P}_3 P_1 \cdot P_4) + (2 \longleftrightarrow 3) . \qquad (D.30)$$

Since $\mathrm{Tr}\, \gamma_5 \not{a} \not{b} = 0$, $\mathrm{Tr}\, \not{a} \not{b} = 4 a \cdot b$ in (A.20), we have

$$\mathcal{M}_{int}^2 = -32G^2(g_L^2\tilde{g}_L^2 + g_R^2\tilde{g}_R^2)\mathrm{Tr}[\not{P}_2\not{P}_3 P_1\cdot P_4] + (2\leftrightarrow 3)$$

$$= -128G^2(g_L^2\tilde{g}_L^2 + g_R^2\tilde{g}_R^2)[P_1\cdot P_4\, P_2\cdot P_3] + (2\leftrightarrow 3)\,. \quad (D.31)$$

Finally, the amplitude in (D.26) can be written

$$\sum_S |\mathcal{M}_T|^2 = 128G^2\{2(g_L^2\tilde{g}_L^2 + g_R^2\tilde{g}_R^2)P_1\cdot P_4\, P_2\cdot P_3$$
$$+(g_L^2\tilde{g}_R^2 + g_R^2\tilde{g}_R^2)P_1\cdot P_3\, P_2\cdot P_4\} + (3\leftrightarrow 4)\,. \quad (D.32)$$

Comparing the first two terms with (D.18), they have the same form except for the extra factor '2' in the first term. Clearly, the other two terms which are the same as the first two terms with subscripts 2 and 3 interchanged, will have the same contribution to the total decay rate as the first two terms (see (D.20)). Using the previous result, we obtain the total decay rate

$$\Gamma = \frac{G^2M^5}{192\pi^3}\left\{(g_L^2+g_R^2)(\tilde{g}_L^2+\tilde{g}_R^2) + (g_L^2\tilde{g}_L^2 + g_R\tilde{g}_R^2)\right\} \qquad (D.33)$$

where the factor 1/2 is multiplied to the final decay rate because we have not antisymmetrized the product wave functions for the ψ_2 and ψ_3. Comparing the total decay rates of (D.33) and (D.24), (D.24) has extra terms which arise from the interference of two amplitudes; they can only be nonzero whenever both currents are left-handed or right-handed in our approximation.

Case(iii)

Let us consider the case when the leptons ψ_1 and ψ_2 are Majorana neutrinos($\psi_1 = \psi_{1L} + \eta_1(\psi_{1L})^c, \psi_2 = \psi_{2L} + \eta_2(\psi_{2L})^c,\ \eta_1 = \pm 1,\ \eta_2 = \pm 1$)and the leptons ψ_3 and ψ_4 are different kinds of fermions. Clearly, the decay of a neutrino to another neutrino must go through the neutral current processes. The neutral currents J_Z^μ for the neutrinos ψ_1 and ψ_2 can be written be

$$J_Z^\mu = \overline{\psi}_2 \gamma_\mu (g_V + g_A\gamma_5)\psi_1 + \overline{\psi}_1 \gamma_\mu (g_V + g_A\gamma_5)\psi_2 \quad . \qquad (D.34)$$

Since $\psi_1^c = \eta_1\psi_1$, $\psi_2^c = \eta_2\psi_2$, we have

$$\begin{aligned} J_Z^\mu &= \overline{\psi}_2 \gamma_\mu (g_V + g_A\gamma_5)\psi_1 + \eta_1\eta_2 \overline{\psi}_1^c \gamma_\mu (g_V + g_A\gamma_5)\psi_2^c \\ &= \overline{\psi}_2 \gamma_\mu (g_V + g_A\gamma_5)\psi_1 + \eta_1\eta_2 \overline{\psi}_2 \gamma_\mu (-g_V + g_A\gamma_5)\psi_1 \\ &= \overline{\psi}_2 \gamma_\mu \{(1-\eta_1\eta_2)g_V + (1+\eta_1\eta_2)g_A\gamma_5\}\psi_1 \end{aligned} \qquad (D.35a)$$

where we use $\overline{\psi}_1^c \gamma_\mu \psi_2^c = -\overline{\psi}_2\gamma_\mu\psi_1$ and $\overline{\psi}_1^c \gamma_\mu\gamma_5 \psi_2^c = \overline{\psi}_2\gamma_\mu\gamma_5\psi_1$ (see appendix C).

Clearly, if $\eta_1\eta_2 = 1$, there is no vector current; whereas, if $\eta_1\eta_2 = -1$, there is no axial-vector current.

Putting $g_V = g_L + g_R$, $g_A = g_R - g_L$, we finally have

$$J_Z^\mu = \overline{\psi}_2 \gamma_\mu \{(g_L - \eta_1\eta_2 g_R)(1-\gamma_5) + (g_R - \eta_1\eta_2 g_L)(1+\gamma_5)\}\psi_1 \qquad (D.35b)$$

which has the same form of currents in (D.3)with g_L replaced by $(g_L - \eta_1\eta_2 g_R)$ and g_R replaced by $(g_R - \eta_1\eta_2 g_L)$. Hence, the decay rate Γ for ψ_1 in this process would be the same as in Case(i) with g_L and g_R replaced. Using (D.24), we obtain the decay rate

$$\Gamma = \frac{G^2 M^5}{192\pi^3}\left\{(g_L - \eta_1\eta_2 g_R)^2 + (g_R - \eta_1\eta_2 g_L)^2\right\}(\tilde{g}_L^2 + \tilde{g}_R^2) \quad . \qquad (D.36)$$

Case (iv)

For the case where the letpon ψ_2, ψ_3 and ψ_4 are all identical Majorana neutrinos, we must antisymmetrize the product wave functions for ψ_2, ψ_3 and ψ_4 because the particle and the antipacticle are the same for the Majorana field. The ψ_4 can no longer be treated as a different particle from the ψ_2 and ψ_3.

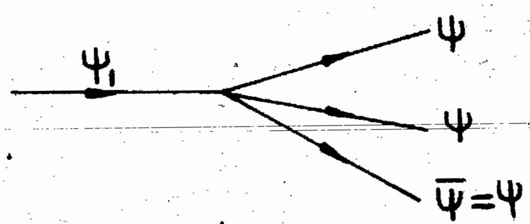


Fig.D.3 The decay of ψ_1 into three identical Majorana neutrinos

There are six Feynman diagrams which correspond to the permutation of the neutrinos ψ_2, ψ_3 and ψ_4. Because in our approximation, we neglect the mass of the light neutrino, the right-handed components $(\nu^c)_R$ of χ_4 which participates in the interaction can be treated as an independent two-component spinor from the left-handed two component spinor ν_L of the ψ_2 and ψ_3. Hence, we get back approximately the two identical particle processes whose decay rates were found previously in Case (ii). Using (D.33) with g_L and g_R replaced, we obtain the decay rate Γ for this process:

$$\Gamma = \frac{G^2M^5}{192\pi^3}\Big\{\big((g_L-\eta_1\eta_2 g_R)^2+(g_R-\eta_1\eta_2 g_L)^2\big)(\tilde{g}_L^2+\tilde{g}_R^2) + \big((g_L-\eta_1\eta_2 g_R)^2\tilde{g}_L^2+(g_R-\eta_1\eta_2 g_L)^2\tilde{g}_R^2\big)\Big\}. \qquad (D.37)$$

APPENDIX E THE MAJORANA FIELD

Let us consider the Lagrangian for the left-handed field with a Majorana mass term:

$$\mathcal{L} = \overline{\psi}_L i \gamma^\mu \partial_\mu \psi_L - \frac{m}{2}(\overline{\psi_L^c}\psi_L + \overline{\psi}_L \psi_L^c) \quad \text{(E.1)}$$

Let us define

$$\chi = (\psi_L)^c + \psi_L \quad , \quad \text{(E.2)}$$

then we have

$$\overline{\chi}\chi = \overline{(\psi_L)^c}\psi_L + \overline{\psi}_L(\psi_L)^c \quad \text{(E.3)}$$

and

$$\begin{aligned}\overline{\chi} i\gamma^\mu \partial_\mu \chi &= (\overline{(\psi_L)^c} + \overline{\psi}_L)\, i\gamma^\mu \partial_\mu ((\psi_L)^c + \psi_L) \\ &= \overline{(\psi_L)^c}\, i\gamma^\mu \partial_\mu (\psi_L)^c + \overline{\psi}_L i\gamma^\mu \partial_\mu \psi_L \\ &= 2\overline{\psi}_L \gamma^\mu \partial_\mu \psi_L \end{aligned} \quad \text{(E.4)}$$

where $\overline{\psi_L^c} i\gamma^\mu \partial_\mu \psi_L^c = \overline{\psi}_L i\gamma^\mu \partial_\mu \psi_L$ is used. Hence , the Lagrangian in (E.1) can be written as

$$\mathcal{L} = \frac{1}{2}\overline{\chi} i\gamma^\mu \partial_\mu \chi - \frac{m}{2}\overline{\chi}\chi \quad \text{(E.5)}$$

This shows that propagator for the χ field is just the usual Dirac case:

$$\langle 0|T(\overline{\chi}(x)\chi(0)|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot x}}{\not{k}-m+i\epsilon} \quad . \quad \text{(E.6)}$$

The left-handed Dirac field is just the left handed projection of χ,

$$\psi_L = \frac{1}{2}(1-\gamma_5)\chi \quad . \tag{E.7}$$

Therefore, the weak charged currents in terms of χ can be written:

$$J_\mu = \bar{e}\gamma_\mu \frac{1}{2}(1-\gamma_5)\nu_e = \bar{e}\gamma_\mu \frac{1}{2}(1-\gamma_5)\chi_e , \tag{E.8a}$$

$$J_\mu^+ = \bar{\nu}_e \gamma_\mu \frac{1}{2}(1-\gamma_5) e = \bar{\chi}_e \gamma_\mu \frac{1}{2}(1-\gamma_5) e . \tag{E.8b}$$

The current J_μ^+ can be written in terms of charge-conjugate fields as

$$\begin{aligned} J_\mu^+ &= -\overline{e^c}\gamma_\mu \frac{1}{2}(1-\gamma_5)\nu_e^c \\ &= -\overline{e^c}\gamma_\mu \frac{1}{2}(1+\gamma_5)\chi_e^c \\ &= -\overline{e^c}\gamma_\mu \frac{1}{2}(1+\gamma_5)\chi_e \quad . \end{aligned} \tag{E.9}$$

Therefore, χ_e can produce e^- or e^+ but with different chirality.

In the zero-mass limit, chirality is the same as the helicity. Processes which involve different chiralities will not interfere with each other and the Majorana field is equivalent to the Dirac field.

APPENDIX F THE GENERAL FERMION ELECTROMAGNETIC VERTEX TO ONE-LOOP ORDER‡

The general fermion electromagnetic vertex to one-loop order for fermions within an SU(2)xU(1) framework has been calculated by using ξ-limiting procedure as formulated for spontaneously broken non Abelian gauge theories by Fujikawa[42]. In this formulation, there is no interaction term of the type $em_W[A_\mu W^{-\mu}\phi^+ + A_\mu W^{+\mu}\phi^-]$, where $\phi^\pm$ are unphysical scalar fields, in constrast to the regular R_ξ gauge, in which this term is present. The advantages of using this procedure are that there are no diagrams involving $\phi^\mp W^\pm A$ vertices, also the physical quantities which we calculate at the one-loop level, diagrams such as those of Fig. F.1 (a) and (b), but with $W^\pm$ replaced by $\phi^\pm$, both vanish in the limit $\xi \to 0$.

The gauge invariant amplitude for $f_1 \to f_2 + \gamma$ has the general Lorzentz and Dirac with $P_1 = P_2 + q$:

$$\begin{aligned} \mathcal{M}_\mu(f_1(P_1) &\longrightarrow f_1(P_2) + \gamma(q)) \\ = -i\,\bar{u}_2(P_2)\Big[&\gamma_\mu(F_1^V(q^2) + F_1^A(q^2)\gamma_5) \\ &+ \frac{i\,\sigma_{\mu\nu}q^\nu}{(m_1+m_2)}(F_2^V(q^2) + F_2^A(q^2)\gamma_5) \\ &+ q_\mu(F_3^V(q^2) + F_3^A(q^2)\gamma_5)\Big]u_1(P_1) . \end{aligned} \qquad (F.1)$$

‡This appendix is just a straight summary part of the results which have been presented by B.W.Lee and R.E. Shrock[43].

$\mathcal{U}_1(P_1)$ is to be regarded as a tensor product of a Dirac four-spinor and an n-dimensional vector, where n denotes the number of leptonic flavors in mass eigenstates. The form factors $F^{V,A}$ are n x n matrices in the space of physical lepton fields. The $F^{V,A}$ matrices have been normalized so that the diagonal elements are equal to the anomalous magnetic moment(times the charge) and electric dipole moment of the corresponding fermions.

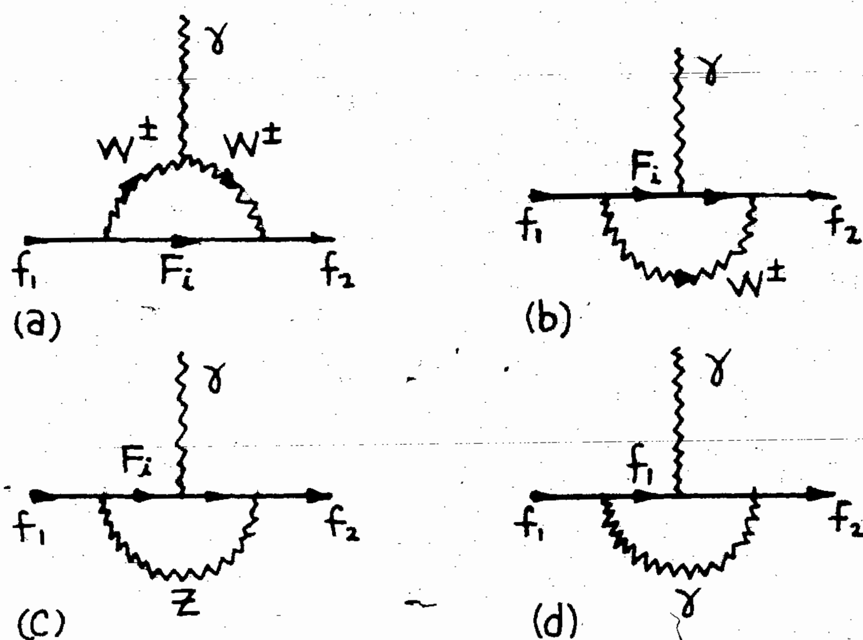


Fig.F.1 Diagrams contributing in a general SU(2)xU(1) gauge model to the process $f_1 \to f_2 + \gamma$, where $f_{1,2}$ are external fermions. The symbol F_i denotes any fermion which can contribute in these graphs.

Let us consider the processed with real photon: $q^2=0$. Electromagnetic current conservation requires $q^\mu m_\mu = 0$ which implies $F_1^V(0) = Q_{f_1}\delta_{f_1 f_2}$ and $F_1^A(0)=0$, where f_1 and f_2 label the initial and final fermions. For the decays $\mu \to e\gamma$ or $\nu_2 \to \nu_1 \gamma$, $F_1^V(0) = 0$. Furthermore, for a real photon, the full amlplitude is

$$\mathcal{M} = \epsilon^{\mu}(q)\,\mathcal{M}_{\mu} \tag{F.2}$$

and $\epsilon \cdot q = 0$ so that $F_3^{V,A}$ terms make no contribution to $f_1 \to f_2 + \gamma(q^2=0)$. Thus only $F_2^{V,A}(0)$ is needed to determine the rate $\Gamma(f_1 \to f_2 + \gamma(q^2=0))$.

For convenience, let us define

$$F^{V,A} = F_2^{V,A}(0) \tag{F.3}$$

Before presenting the results for calculating $F^{V,A}$, we introduce some notations. Let ψ_L and ψ_R be the mass eigenstates of the left- and right-handed lepton fields with a diagonal mass matrix M_0. The Lagrangian density for the charged currents and the neutral currents is written

$$\mathcal{L}_{int.} = (g J_{\mu}^{+} W_{\mu}^{+} + H.C.) + (g^2 + g'^2)^{1/2} J_Z^{\mu} Z_{\mu} \tag{F.4}$$

where

$$J_W^{\mu} = \bar{\psi}_L \gamma^{\mu} \tau_+^L \psi_L + \bar{\psi}_R \gamma^{\mu} \tau_+^R \psi_R \tag{F.5a}$$

$$J_Z^{\mu} = \bar{\psi}_L \gamma^{\mu} \tau_Z^L \psi_L + \bar{\psi}_R \gamma^{\mu} \tau_Z^R \psi_R \tag{F.5b}$$

where ψ_L and ψ_R are column vectors and $\tau_+^L, \tau_+^R, \tau_Z^L, \tau_Z^R$ are in general non-diagonal nxn square matrices.

We now present the results for evaluating $F^{V,A}$. First we separate the form factos into LL, RR and LR, RL parts which correspond to the processes $f_{1L} \to f_{2L} + \gamma$, $f_{1R} \to f_{2L} + \gamma$ and $f_{1L} \to f_{2R} + \gamma$,

$f_{iL} \to f_{jL} + \gamma$. Hence, we have

$$F^{V,A} = F^{V,A}_{LL,RR} + F^{V,A}_{LR,RL} \quad . \tag{F.6}$$

The general structure of LL, RR and LR, RL parts of the form factors as list below. The sum over the indices $a(\bar{a}) = +(-), -(+), z(z)$ is understood corresponding to the contributions of W^-, W^+ and z graphs, respectively:

$$F^V_{LL,RR} = (m_1+m_2)^2 \left[\tau^L_a C_a^{LL,RR} \tau^L_{\bar{a}} + \tau^R_a C_a^{LL,RR} \tau^R_{\bar{a}} \right] , \tag{F.7a}$$

$$F^A_{LL,RR} = (m_1^2 - m_2^2) \left[\tau^L_a C_a^{LL,RR} \tau^L_{\bar{a}} - \tau^R_a C_a^{LL,RR} \tau^R_{\bar{a}} \right] , \tag{F.7b}$$

$$F^V_{LR,RL} = (m_1+m_2) \left[\tau^L_a C_a^{LR,RL} M_D \tau^R_{\bar{a}} + \tau^R_a C_a^{LR,RL} M_D \tau^L_{\bar{a}} \right] , \tag{F.7c}$$

$$F^A_{LR,RL} = (m_1+m_2) \left[\tau^L_a C_a^{LR,RL} M_D \tau^R_{\bar{a}} - \tau^R_a C_a^{LR,RL} M_D \tau^L_{\bar{a}} \right] . \tag{F.7d}$$

The $C_a^{LL,RR}$ and $C_a^{LR,RL}$ are real n x n diagonal matrices. The values for C have been calculated with the approximation: all external masses and all internal masses are much smaller than the W-boson mass.

Let us denote

$$C_{ij} = \frac{e G_F}{4\sqrt{2}\pi^2} c_i \delta_{ij} \tag{F.8}$$

The diagrams of fig.1(a) and fig.1(b) yield‡

$$\left(C_{\pm}^{(a)+(b)}\right)_i^{LL,RR} = (Q_1 - Q_{F_i})\left(\frac{5}{6} - \frac{1}{4}\epsilon_i\right) + Q_{F_i}\left(-\frac{2}{3} + \frac{1}{2}\epsilon_i\right), \quad (F.9a)$$

$$\left(C_{\pm}^{(a)+(b)}\right)_i^{LR,RL} = (Q_1 - Q_{F_i})\left(-2 + \frac{3}{2}\epsilon_i\right) + Q_{F_i}\left(2 + \epsilon_i\left(-4\ln\frac{1}{\epsilon_i} + 6\right)\right). \quad (F.9b)$$

The diagram of fig.1(c) yields

$$\left(C_Z^{(c)}\right)_i^{LL,RR} = Q_{F_i}\left(-\frac{2}{3} + \frac{1}{2}\delta_i\right), \quad (F.10a)$$

$$\left(C_Z^{(c)}\right)_i^{LR,RL} = Q_{F_i}\left(2 + \delta_i\left(-4\ln\frac{1}{\delta_i} + 6\right)\right) \quad (F.10b)$$

where Q_1 and Q_{F_i} are the charges of the initial and i^{th} virtual fermion, and

$$\epsilon_i = \frac{m_{F_i}^2}{m_W^2}, \quad (F.11a)$$

$$\delta_i = \frac{m_{F_i}^2}{m_Z^2}, \quad (F.11b)$$

m_{F_i} is the mass of the i^{th} virtual fermion.

Let us separate the Dirac and weak-gauge-group matrix structures, defining

$$F_{ab}^{V,A} = \langle f_a | F^{V,A} | f_b \rangle \quad (F.12)$$

then the invariant matrix element for the radiative decay $f_1 \to f_2 + \gamma$ is given by

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‡More exact formula for $f_1 \to f_2 + \gamma$ type processes for $\left(C_{\pm}^{(a)+(b)}\right)^{LL,RR}$ was computed by Ernest Ma and P. Pramudita[44].

$$i\mathcal{M}(f_1(P_1) \longrightarrow f_2(P_2) + \gamma(q=0))$$

$$= \bar{u}_2(P_2)\,\frac{i\sigma_{\mu\nu}q^\nu\epsilon^\mu}{(m_2+m_1)}\,(F^V_{21} + F^A_{21}\gamma_5)u_1(P_1)\ . \qquad (F.14)$$

The rate is then

$$\Gamma(f_1 \longrightarrow f_2 + \gamma)$$

$$= \frac{m_1}{8\pi}\left(1-\frac{m_2}{m_1}\right)^2\left(1-\frac{m_2^2}{m_1^2}\right)\left[|F^V_{21}|^2 + |F^A_{21}|^2\right]\ . \qquad (F.15)$$

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