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ON COMPLEXITY OF COMPLETE FIRST-ORDER THEORIES

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Abstract

This thesis defines a partially ordered set $(\mathcal{P},\le)$ where each $\pi \in \mathcal{P}$ is a "property" which a complete first-order theory T may or may not possess and $\pi_0 \le \pi_1$ denotes that $\forall T$ (T possesses $\pi_1 \rightarrow$ T possesses π_0). In this way a "complexity" preorder $\blacktriangleleft$ on the class $\mathcal{T}$ of all such theories is obtained by letting $T_0 \blacktriangleleft T_1$ denote that $\forall \pi$ (T_0 possesses $\pi \rightarrow T_1$ possesses π). Some density results concerning $(\mathcal{P},\le)$ and $(\mathcal{T},\blacktriangleleft)$ are given after some basic properties are examined. In particular Keisler's finite cover property and Shelah's independence property are found quite useful.

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Introduction

Keisler (1967) employs ultrapowers to define an ordering $\lhd$ on the class $\mathcal{T}$ of all complete theories T which provides a measure of complexity for such theories. He defines the finite cover property and shows that if some formula $\varphi(x,\bar{y})$ of T admits the finite cover property in T then T is not $\lhd$-minimum and if T is not $\lhd$-minimum then T is not $\aleph_1$-categorical. He also defines the versatility property and shows that if some formula $\varphi(x,\bar{y})$ of T admits the versatility property in T then T is $\lhd$-maximum.

Shelah (1971) defines the order, strict order and independence properties and shows that T is unstable iff some formula $\varphi(\bar{x},\bar{y})$ of T admits the order property in T iff some formula $\psi(\bar{x},\bar{w})$ of T admits the strict order or independence property in T. Furthermore he shows that if some formula $\varphi(\bar{x},\bar{y})$ of T admits the order property in T then some formula $\psi(\bar{z},\bar{w})$ of T admits the finite cover property in T. He also shows that if some formula $\varphi(\bar{x},\bar{y})$ of T admits the finite cover, order, or independence property in T then some formula $\psi(z,\bar{w})$ of T admits the finite cover, order, or independence property (respectively) in T. This result makes it easier to decide whether any formula of T admits any one of these properties in T.

Shelah (1972) shows that if T is not $\lhd$-minimum then some formula $\varphi(\bar{x},\bar{y})$ of T admits the finite cover property in T.

Lachlan (1975) shows that if some formula $\varphi(\bar{x},\bar{y})$ of T admits the strict order property in T then some formula $\psi(z,\bar{w})$ of T admits the strict order property in T .

Evidently the above properties are useful in providing a measure of complexity for such theories.

This thesis constructs a poset of properties of complete theories $(\mathcal{P},\leq)$ which is used to define an ordering $\blacktriangleleft$ on $\mathcal{T}$ which provides another measure of complexity for such theories.

In §0 some preliminaries are covered.

In §1 some basic definitions are given. In 1.0 properties of formulas are defined which include the above properties in a natural way. In 1.1 $\mathcal{P}$ is obtained by identifying any pair of properties of formulas if they are admitted by the same complete theories. In 1.2 $\leq$ is obtained by identifying any property of complete theories with the class of all complete theories admitting it. Here $(\mathcal{P},\leq)$ is shown to be a lower semilattice. In 1.3 $\blacktriangleleft$ is obtained by identifying any complete theory with the set of all properties of complete theories which it admits. Here $(\mathcal{T},\blacktriangleleft)$ is shown to be an upper semilattice. In 1.4 an archetypal property of complete theories is defined to be any property of complete theories for which the class of all complete theories admitting it contains a smallest member. Such properties of complete theories are shown to be meet-irreducible. In 1.5 a prime property of complete theories is defined to be any property of complete theories for which the class of all complete theories admitting it contains any disjoint sum of complete theories iff it contains one of the complete theories being summed. Such properties of complete theories are shown to be join-irreducible (and vice versa). From this it follows easily that the meet of join-irreducible properties of complete theories is again join-irreducible.

In §2 some basic examples are provided which yield information about $(P,\leq)$ and $(T,\blacktriangleleft)$. In 2.0 the minimum and maximum properties of complete theories are defined and it is shown that the theory of atomless Boolean algebras does not admit the latter. From this it follows easily that $\triangleleft$ and $\blacktriangleleft$ are different orderings. Furthermore the maximum property of formulas is 1-dimensional in the sense that if some $\varphi(\overline{x},\overline{y})$ admits it in T then some $\psi(z,\overline{w})$ admits it in T. From this it follows easily that the maximum property of complete theories is prime. In 2.1 the finite cover and partition properties of complete theories are defined and it is shown that each may be used to provide an embedding of the poset of subsets of ω (modulo finite sets) into $(P,\leq)$ and to show that $(P,\leq)$ is not a lattice. Furthermore the partition property of formulas is 1-dimensional and from this it follows easily that the partition property of complete theories is prime. In 2.2 the order, strict order and independence properties of complete theories are defined and the latter are shown to be archetypal. In 2.3 the strong independence and versatility properties of complete theories are defined and the former is shown to be prime although whether the strong independence property of formulas is 1-dimensional is unknown. In 2.4 some remarks are given about the relative positions in $(P,\leq)$ of the above properties of complete theories. In 2.5 regular and Whitman theories are defined and are used to show that certain complete theories omit certain properties of complete theories. In 2.6 the partial order and line properties of complete theories are defined and the former is shown to be archetypal while the latter is shown to be quite weak. In 2.7 some remarks are given about the relative positions in $(\mathcal{P},\leq)$ of the above properties of complete theories. In 2.8 independent and countable properties of complete theories are defined and it is shown that if a countable complete theory admits every independent property of complete theories then it admits every property of complete theories.

In §3 some density results about $(\mathcal{P},\leq)$ are given which imply that much of $(\mathcal{P},\leq)$ is dense. However it is shown that prime archetypal properties of complete theories provide gaps in $(\mathcal{T},\blacktriangleleft)$ so it follows that $(\mathcal{T},\blacktriangleleft)$ is not dense.

In §4 some open questions are raised.

§0 Preliminaries

In this paper complete theories have infinite models. Standard notation is employed. If φ is a formula, T is a complete theory and $\mathcal{A},\mathcal{B}$ are structures of a language L then φ is a sentence if no variables occur free in φ, $\varphi(x_0,\ldots,x_{n-1})$ denotes that at most the variables $x_0,\ldots,x_{n-1}$ occur free in φ, φ^0 (φ^1) denotes φ $(\neg\varphi)$, $T \vdash \varphi$ denotes that φ is a theorem of T, $\mathcal{A} \models T$ denotes that $\mathcal{A}$ is a model of T, $T = \mathrm{Th}\mathcal{A}$ denotes that T is the theory of $\mathcal{A}$, $\mathcal{A} \subset \mathcal{B}$ denotes that $\mathcal{A}$ is a substructure of $\mathcal{B}$ and $\mathcal{A} \prec \mathcal{B}$ denotes that $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$. Let B_nT (O_nT) denote the Lindenbaum algebra of (open or quantifier-free) formulas $\varphi(x_0,\ldots,x_{n-1})$ of T and let $S_nT = SB_nT$ denote the corresponding Stone space of T. Obviously T is quantifier-eliminable iff $B_nT = O_nT$ $(n < \omega)$. A complete formula of T is any atom $\varphi(x_0,\ldots,x_{n-1})$ of B_nT. By Ryll-Nardzewski (1959) if T is countable then T is $\aleph_0$-categorical iff each formula $\varphi(x_0,\ldots,x_{n-1})$ of T is a finite disjunction of complete formulas $\varphi_i(x_0,\ldots,x_{n-1})$ of T. If $\bar{x} = (x_0,\ldots,x_{m-1})$ and $\bar{y} = (y_0,\ldots,y_{n-1})$ are sequences let $\ell(\bar{x}) = m$, $r(\bar{x}) = \{x_0,\ldots,x_{m-1}\}$, $\bar{x}(i) = x_i$ $(i < m)$, $\bar{x} \cap \bar{y} = (x_0,\ldots,x_{m-1},y_0,\ldots,y_{n-1})$ and let $\bar{z} = \bar{x} \cup \bar{y}$ denote that $r(\bar{z}) = r(\bar{x}) \cup r(\bar{y})$ and $r(\bar{x}) \cap r(\bar{y}) = \phi$. Furthermore let $\bar{x} \subset \bar{y}$ denote that $r(\bar{x}) \subset r(\bar{y})$. In 2.5 the distinction between $\bar{x}$ and $r(\bar{x})$ and between $\mathcal{A}$ and $|\mathcal{A}|$ is often ignored if no confusion results. Let $\varphi_{\mathcal{A}}(\bar{x},\bar{a}) = \{\bar{b} \in |\mathcal{A}|^{\ell(\bar{x})} \mid \mathcal{A} \models \varphi(\bar{b},\bar{a})\}$ denote the $\varphi(\bar{x},\bar{y})$-definable subset of $|\mathcal{A}|^{\ell(\bar{x})}$ defined by $\bar{a}$ in $\mathcal{A}$. If $\bar{a} \in |\mathcal{A}|^{\ell(\bar{a})}$ let

$t_A(\bar{a}) = \{\varphi(x) \mid A \models \varphi(\bar{a})\}$ denote the type realized by $\bar{a}$ in A and let $t^0_A(\bar{a}) = \{\varphi(\bar{x}) \mid \varphi(x)$ is open and $A \models \varphi(\bar{a})\}$ denote the open type realized by $\bar{a}$ in A. Each ordinal $\alpha = \{\beta \mid \beta < \alpha\}$ is equal to the set of ordinals smaller than it and each cardinal is an initial ordinal. If A,B are sets then $A \Delta B$ denotes $(A-B) \cup (B-A)$, $|A|$ denotes the cardinality of A, $P(A)$ denotes the power set of A, A^B denotes the set of functions from B into A (or $|A^B|$ if convenient) and whenever $A \subset B$ let A^0 (A^1) denote A $(B-A)$ and let the cocardinality of A in B be $|B-A|$. A preorder $(T,\leq)$ is any set (or class) T together with a reflexive transitive binary relation $\leq$ on T. Let $(T|_{\equiv},\leq|_{\equiv})$ denote the poset (partial order) obtained from $(T,\leq)$ by the congruence $\equiv$ on $(T,\leq)$ defined by $s \equiv t$ iff $s \leq t$ and $t \leq s$. Suppose $(T,\leq)$ is a poset. If $s,t \in T$ let $(s,t) = \{r \in T \mid s < r < t\}$ (the other intervals are defined similarly). Any $S \subset T$ is dense in $(T,\leq)$ if $s,t \in T$ and $s < t$ implies $(s,t) \cap S \neq \phi$. Let $\wedge$ $(\vee)$ denote meets (joins) in $(T,\leq)$. Any $t \in T$ is $\wedge$-irreducible ($\vee$-irreducible) in $(T,\leq)$ if $t = r \wedge s$ $(t = r \vee s)$ implies $t = r$ or $t = s$. A lattice is any poset $(T,\leq)$ where $s \wedge t$ and $s \vee t$ exist whenever $s, t \in T$.

Model theory can be found in Sacks (1972), Shoenfield (1973) or Chang and Keisler (1973).

Keisler's order can be found in Keisler (1967).

Stable theories can be found in Shelah (1971) or Shelah (1978).

The following well-known automorphism test for quantifier-elimination is useful. If T is a countable complete theory then

(1) T is quantifier-eliminable

(2) For every countable $\mathcal{A} \models T$ and $\bar{a}, \bar{b} \in |\mathcal{A}|^{\ell(\bar{a})}$ such that $t^0_{\mathcal{A}}(\bar{a}) = t^0_{\mathcal{A}}(\bar{b})$ there exists $\mathcal{B} \succ \mathcal{A}$ and $f \in \mathrm{Aut}(\mathcal{B})$ such that $f(\bar{a}) = \bar{b}$

are equivalent. To prove this assume that T is a countable complete theory. If (1) holds, $\mathcal{A} \models T$ is countable, $\bar{a}, \bar{b} \in |\mathcal{A}|^{\ell(\bar{a})}$ and $t^0_{\mathcal{A}}(\bar{a}) = t^0_{\mathcal{A}}(\bar{b})$ let $\mathcal{B} \succ \mathcal{A}$ be countable and $\aleph_0$-homogeneous. To construct $f \in \mathrm{Aut}(\mathcal{B})$ such that $f(\bar{a}) = \bar{b}$ use the $\aleph_0$-homogeneity of $\mathcal{B}$ in a back and forth argument after noting that $(\mathcal{B},\bar{a})$ and $(\mathcal{B},\bar{b})$ are elementarily equivalent (since T is quantifier-eliminable, $t^0_{\mathcal{A}}(\bar{a}) = t^0_{\mathcal{A}}(\bar{b})$ and $\mathcal{A} \prec \mathcal{B}$). Thus (2) holds. If (2) holds it follows easily that each ultrafilter on 0_nT extends uniquely to an ultrafilter on B_nT. By Makinson (1969) $0_nT = B_nT$. Thus (1) holds.

The following well-known partial isomorphism test is also useful. If T is a countable consistent theory with only infinite models then

(1) For every countable $\mathcal{A}, \mathcal{B} \models T$, $\bar{a} \in |\mathcal{A}|^{\ell(\bar{a})}$, $\bar{b} \in |\mathcal{B}|^{\ell(\bar{a})}$ such that $t^0_{\mathcal{A}}(\bar{a}) = t^0_{\mathcal{B}}(\bar{b})$ and $a \in |\mathcal{A}|$ there exists $b \in |\mathcal{B}|$ such that $t^0_{\mathcal{A}}(\bar{a} \frown a) = t^0_{\mathcal{B}}(\bar{b} \frown b)$

implies

(2) T is complete, $\aleph_0$-categorical and quantifier-eliminable.

To prove this assume that T is a countable consistent theory with only infinite models. If (1) holds then T is $\aleph_0$-categorical (use a back and forth argument), T is complete (use the Łos'-Vaught test) and T is quantifier-eliminable (use the automorphism test). Thus (2) holds.

The next result is used to show that certain complete theories omit the versatility property of complete theories. Let T be a complete quantifier-eliminable theory in a finite language without functions. Then there exists a polynomial f such that $|S_n T| \leq 2^{f(n)}$ $(n < \omega)$. To prove this assume that T is a complete quantifier-eliminable theory in a language consisting of constants $c_i (i < m)$ and predicates P_{ij} of arity $i+1$ $(i,j < m)$. A <u>basic formula</u> is any formula of the form $c_i = x_0$, $x_0 = x_1$ or $P_{ij}(x_0,\ldots,x_i)$. Thus there are $m^2 + m + 1$ basic formulas. An <u>n-arrangement</u> is any formula $\varphi(x_0,\ldots,x_{n-1})$ which states which $\bar{x} \in \{x_0,\ldots,x_{n-1}\}^{i+1}$ $(i < m)$ satisfy each basic formula. Thus there are

$$\underbrace{2^n\ldots 2^n}_{m}\; n^2\; \underbrace{2^n\ldots 2^n}_{m}\; \underbrace{2^{n^2}\ldots 2^{n^2}}_{m}\ldots\underbrace{2^{n^m}\ldots 2^{n^m}}_{m} \leq 2^{mn}2^{n+1}2^{mn}2^{mn^2}\ldots 2^{mn^m}$$

$$= 2^{1+(1+2m)n+mn^2+\ldots+mn^m} = 2^{f(n)}$$ n-arrangements where $f(x) = 1 + (1 + 2m)x + mx^2 + \ldots + mx^m$. But each ultrafilter in $S_n T$ is generated by an n-arrangement so $|S_n T| \leq 2^{f(n)}$.

The following result concerns the definability of complete theories within other complete theories and is used to compare the complexity between such theories. Although the following definition admits obvious generalizations it is sufficient for the purposes of this thesis. If A_0, A_1 are structures for languages (without functions) L_0, L_1 then A_0 is <u>definable</u> in A_1 if $|A_0| = |A_1|^n$ for some $n < \omega$ and if for every constant c of L_0 there exists a sequence of constants $\bar{c}$ (of length n) of L_1 such that $c_{A_0} = \bar{c}_{A_1}$ and if for every predicate P of L_0 there exists a formula $\bar{P}$ of L_1 such that $P_{A_0} = \bar{P}_{A_1}$. By changing the formulas $\bar{P}$ (if necessary) it may be assumed that infinitely many of the variables of L_1 do not occur in any of the formulas $\bar{P}$. For each variable x of L_0 let $\bar{x}$ be a distinct sequence (of length n) of distinct variables of L_1 which do not occur in any of the formulas $\bar{P}$ and for each formula φ of L_0 let $\bar{\varphi}$ be the formula of L_1 obtained from φ by the following rules: If φ is $s = t$ where s, t are terms of L_0 then $\bar{\varphi}$ is $\bar{s} = \bar{t}$. If φ is $P(s_0, \ldots, s_{m-1})$ where P is a predicate of L_0 and $s_0, \ldots, s_{m-1}$ are terms of L_0 then $\bar{\varphi}$ is $\bar{P}(\bar{s}_0, \ldots, \bar{s}_{m-1})$. If φ is $\neg\psi$, $\psi \vee \chi$ or $\exists x \psi$ then $\bar{\varphi}$ is $\neg\bar{\psi}$, $\bar{\psi} \vee \bar{\chi}$ or $\exists \bar{x} \bar{\psi}$ (respectively). It is easy to show that if φ is a sentence of L_0 then $A_0 \models \varphi$ iff $A_1 \models \bar{\varphi}$. Suppose T_0, T_1 are complete theories in L_0, L_1. If some model of T_0 is definable in some model of T_1 then T_0 is <u>definable</u> in T_1. If every finite reduct of T_0 is definable in T_1 then T_0 is <u>locally definable</u> in T_1. From the above it follows easily that if T_0 is locally definable in T_1 and $\varphi(\bar{x}, \bar{y})$ is a formula of T_0 there exists a formula $\psi(\bar{z}, \bar{w})$ of T_1 (namely $\bar{\varphi}(\bar{\bar{x}}, \bar{\bar{y}})$) such that if $\varphi_{A_0}(\bar{x}, \bar{a}_0), \ldots, \varphi_{A_0}(\bar{x}, \bar{a}_{\ell-1})$ are

$\varphi(\bar{x},\bar{y})$-definable subsets of $|A_0|^{\ell(\bar{x})}$ for some $A_0 \models T_0$ there exist $\psi(\bar{z},\bar{w})$-definable subsets $\psi_{A_1}(\bar{z},\bar{b}_0),\ldots,\psi_{A_1}(\bar{z},\bar{b}_{\ell-1})$ of $|A_1|^{\ell(\bar{z})}$ for some $A_1 \models T_1$ which have the same nonempty Boolean combinations in $|A_1|^{\ell(\bar{z})}$ as the corresponding $\varphi(\bar{x},\bar{y})$-definable subsets have in $|A_0|^{\ell(\bar{x})}$. Furthermore if T_0, T_1 are countable, T_0 is definable in T_1 and T_1 is $\aleph_0$-categorical then by Ryll-Nardzewski (1959) it follows easily that T_0 is $\aleph_0$-categorical.

The next result of this chapter concerns disjoint sums of theories and is used to characterize prime properties of complete theories. If $L_\alpha (\alpha < \beta)$ are languages without functions their disjoint sum is the language $\sum_{\alpha<\beta} L_\alpha$ obtained by adding unary predicates $P_\alpha (\alpha < \beta)$ to their disjoint union. If A_α is a structure for $L_\alpha (\alpha < \beta)$ their disjoint sum is the structure $\sum_{\alpha<\beta} A_\alpha$ for $\sum_{\alpha<\beta} L_\alpha$ obtained by interpreting P_α as $|A_\alpha|$ $(\alpha < \beta)$ in their disjoint union. If φ is a formula of L_α then φ^{P_α} is the formula of $\sum_{\alpha<\beta} L_\alpha$ obtained by replacing each subformula $\exists y\psi$ of φ with $\exists y(\psi \wedge P_\alpha(y))$. Note that if φ is a sentence of L_α then $A_\alpha \models \varphi$ iff $\sum_{\alpha<\beta} A_\alpha \models \varphi^{P_\alpha}$. If T is a theory in L_α then T^{P_α} is the theory in $\sum_{\alpha<\beta} L_\alpha$ whose axioms are the formulas $\exists x\, P_\alpha(x)$, $P_\alpha(c)$, $P(x_0,\ldots,x_{n-1}) \rightarrow \bigwedge_{i<n} P_\alpha(x_i)$ and φ^{P_α} where c is any constant of L_α, P is any predicate of L_α and φ is the

universal closure of any axiom of T. If T_α is a theory in $L_\alpha (\alpha < \beta)$ then $\sum_{\alpha<\beta} T_\alpha$ is the theory in $\sum_{\alpha<\beta} L_\alpha$ whose axioms are $T_\alpha^{P_\alpha}$ $(\alpha < \beta)$ together with the formulas $\neg\exists x(P_\alpha(x) \wedge P_{\alpha'}(x))(\alpha<\alpha'<\beta)$. Furthermore if $\beta < \omega$ then the formula $\forall x \bigvee_{\alpha<\beta} P_\alpha(x)$ is also an axiom of $\sum_{\alpha<\beta} T_\alpha$. Note that $A_\alpha \models T_\alpha$ $(\alpha < \beta)$ iff $\sum_{\alpha<\beta} A_\alpha \models \sum_{\alpha<\beta} T_\alpha$. For each $\alpha < \beta$ and $\bar{x}$ let $P_\alpha(\bar{x})$ denote the formula $\bigvee_{i<\ell(\bar{x})} P_\alpha(\bar{x}(i))$ and let $\neg P_\alpha(\bar{x})$ denote the formula $\bigwedge_{i<\ell(\bar{x})} \neg P_\alpha(\bar{x}(i))$. Let EQ denote the theory of equality on an infinite set. Then it is easy to prove that if $\varphi(\bar{x})$ is a formula of $\sum_{\alpha<\beta} T_\alpha$ there exist formulas $\varphi_{ij}(\bar{x}_{ij})$ of certain T_{α_j} $(i,j<n)$ and open formulas $\varphi_{in}(\bar{x}_{in})$ of EQ $(i < n)$ such that $\sum_{\alpha<\beta} T_\alpha \vdash \varphi(\bar{x}) \leftrightarrow \bigvee_{i<n} \bigwedge_{j\leq n} \varphi^*_{ij}(\bar{x}_{ij})$ where $\bar{x} = \bar{x}_{i0} \cup \ldots \cup \bar{x}_{in}$ $(i < n)$ and $\varphi^*_{ij}(\bar{x}_{ij})$ is

$\varphi_{ij}^{P_{\alpha_j}}(\bar{x}_{ij}) \wedge P_{\alpha_j}(\bar{x}_{ij})$ $(i,j<n)$ and $\varphi^*_{in}(\bar{x}_{in})$ is

$\varphi_{in}(\bar{x}_{in}) \wedge (\bigwedge_{j<n} \neg P_{\alpha_j}(\bar{x}_{in}))$ $(i<n)$. From this it follows easily that T_α is complete $(\alpha<\beta)$ iff $\sum_{\alpha<\beta} T_\alpha$ is complete. Furthermore $A_\alpha \prec B_\alpha$ $(\alpha<\beta)$ iff $\sum_{\alpha<\beta} A_\alpha \prec \sum_{\alpha<\beta} B_\alpha$. Using the automorphism test for quantifier-elimination it then follows easily that T_α is complete and quantifier-eliminable $(\alpha<\beta)$ iff $\sum_{\alpha<\beta} T_\alpha$ is complete and quantifier-eliminable.

The final results of this chapter concern generic structures of countable languages but the proofs of these results are omitted since they are similar to the proofs found in Woodrow (1977) which concerns generic structures of finite languages. Let L be a language consisting of finitely many predicates P_{ij} of each arity $i < \omega$. A structure A of L is good if $\mathrm{Th}A \vdash P_{ij}(x_0,\ldots,x_{i-1}) \rightarrow \bigwedge_{k<\ell<i} x_k \neq x_\ell$ for each P_{ij}. A class of good structures of L is good. Let Σ be a class of finite structures of L closed under isomorphism and let M be a countable structure of L. If

(HP) If $f : A \rightarrow B$ is an embedding of A into B and $B \in \Sigma$ then $A \in \Sigma$

holds then Σ admits the hereditary property.

If

(JEP) If $A, B \in \Sigma$ then there exist embeddings $f : A \rightarrow C$ and $g : B \rightarrow C$ for some $C \in \Sigma$

holds then Σ admits the joint embedding property.

If

(AP) If $f_i : A \rightarrow B_i$ $(i<2)$ are embeddings and $A, B_0, B_1 \in \Sigma$ then there exist embeddings $g_i : B_i \rightarrow C$ $(i<2)$ for some $C \in \Sigma$ such that $g_0 f_0 = g_1 f_1$

holds then Σ admits the amalgamation property.

If

(BP) If there exists a function $f : \omega \to \omega$ such that

if $A \subset B \in \Sigma$ then

$f(\|A\|) = \min\{\|C\| \mid A \subset C \in \Sigma\}$

holds then Σ admits the bounding property.

If $A \subset M$ and $\|A\| < \aleph_0$ implies that $A \subset B \subset M$ for some $B \in \Sigma$ then M is Σ-finite. If $A \in \Sigma$ implies that there exists an embedding $f : A \to M$ then M is Σ-universal. If $A,B \in \Sigma$, $A,B \subset M$ and $f : A \to B$ is an isomorphism implies that $f = g|_A$ for some isomorphism $g : M \to M$ then M is Σ-homogeneous. Finally if M is Σ-finite, Σ-universal and Σ-homogeneous then M is Σ-generic.

Let Σ be a class of finite structures of L closed under isomorphism. Then

(1) If M and N are Σ-generic then there exists and isomorphism $f : M \to N$

(2) If Σ is countable and admits JEP and AP then M is Σ-generic for some M

(3) If M is Σ-generic and Σ is good and admits BP then M is $\aleph_0$-categorical

(4) If M is Σ-generic and Σ is good and admits HP then M is quantifier-eliminable

(5) If Σ is good and admits HP, JEP and AP then M is Σ-generic, $\aleph_0$-categorical and quantifier-eliminable for some M

hold. In 2.3 (5) is used to show that certain complete theories admit the versatility property of complete theories but omit the partition property of complete theories.

§1 Basic Definitions

1.0 Properties of Formulas

A property of formulas is any sequence $\rho = (\rho(i) \mid i<\omega)$ of open formulas of BA (theory of Boolean algebra in the language $L = \{0,1,c,\cap,\cup\}$ interpreted in the usual sense). Note that each open formula of BA may be viewed as a finite disjunction of finite Venn diagrams. If $\varphi(\bar{x},\bar{y})$ is a formula of a complete theory T and $\psi(\bar{z})$ is an open formula of BA then $\varphi(\bar{x},\bar{y})$ admits $\psi(\bar{z})$ in T if

$$P(|A|^{\ell(\bar{x})}) \models \psi(\varphi_A(\bar{x},\bar{a}_0),\ldots,\varphi_A(\bar{x},\bar{a}_{\ell(\bar{z})-1}))$$

for some $A \models T$ and $\bar{a}_0,\ldots,\bar{a}_{\ell(\bar{z})-1} \in |A|^{\ell(\bar{y})}$ (where $P(|A|^{\ell(\bar{x})})$ is viewed as the power set Boolean algebra of $|A|^{\ell(\bar{x})}$). Otherwise $\varphi(\bar{x},\bar{y})$ omits $\psi(\bar{z})$ in T. Thus $\varphi(\bar{x},\bar{y})$ admits $\psi(\bar{z})$ in T iff some finite Venn diagram of $\psi(\bar{z})$ is admitted by some $\psi(\bar{x},\bar{y})$-definable subsets $\varphi_A(\bar{x},\bar{a}_0),\ldots,\varphi_A(\bar{x},\bar{a}_{\ell(z)-1})$ of $|A|^{\ell(\bar{x})}$ for some $A \models T$ (note that since T is complete any $A \models T$ may be chosen). If $\varphi(\bar{x},\bar{y})$ is a formula of a complete theory T and ρ is a property of formulas then $\varphi(\bar{x},\bar{y})$ admits ρ in T if there exists a strictly increasing sequence $\alpha \in \omega^\omega$ such that $\varphi(\bar{x},\bar{y})$ admits $\rho(\alpha(i))$ in T for every $i < \omega$. Otherwise $\varphi(\bar{x},\bar{y})$ omits ρ in T. If T is a complete theory and ρ is a property of formulas then T admits ρ if there exists a formula $\varphi(\bar{x},\bar{y})$ of T such that $\varphi(\bar{x},\bar{y})$ admits ρ in T. Otherwise T omits ρ. A principal property of formulas is any property of formulas ρ such that the following holds: If $\varphi(\bar{x},\bar{y})$ is a formula of a complete theory T and $\varphi(\bar{x},\bar{y})$ admits ρ in T then $\varphi(\bar{x},\bar{y})$ admits $\rho(i)$

in T for every $i < \omega$. A <u>1-dimensional</u> property of formulas is any property of formulas ρ such that the following holds: If T is a complete theory and T admits ρ then some formulas $\varphi(x,\overline{y})$ of T admits ρ in T . The <u>principal part</u> of any property of formulas ρ is the property of formulas $\overline{\rho} = (\bigwedge_{j<i} \rho(i) \mid i<\omega)$ where it may be assumed by changing variables (if necessary) that for each $i < \omega$ no variable occurs in more than one conjunct of $\bigwedge_{j<i} \rho(i)$. The <u>α-th part</u> of any property of formulas ρ (where $\alpha \in \omega^{\omega}$ is a strictly increasing sequence) is the property of formulas $\rho(\alpha) = (\rho(\alpha(i)) \mid i<\omega)$. The <u>intersection</u> of any properties of formulas ρ_0, ρ_1 is the property of formulas $\rho_0 \cap \rho_1 = (\rho_0(i) \vee \rho_1(i) \mid i<\omega)$. The <u>union</u> of any properties of formulas ρ_0, ρ_1 is the property of formulas $\rho_0 \cup \rho_1 = (\rho_0(i) \wedge \rho_1(i) \mid i<\omega)$ where it may be assumed by changing variables (if necessary) that for each $i < \omega$ no variable occurs in more than one conjunct of $\rho_0(i) \wedge \rho_1(i)$. Using the above definitions the following lemma may be easily proved.

<u>Lemma 1</u>

If $\varphi(\overline{x},\overline{y})$ is a formula of a complete theory T and ρ, ρ_0, ρ_1 are properties of formulas then the following hold:

(1) $\overline{\rho}$ is principal

(2) If ρ_0, ρ_1 are principal then $\rho_0 \cap \rho_1$, $\rho_0 \cup \rho_1$ are principal

(3) $\varphi(\overline{x},\overline{y})$ admits $\overline{\rho}$ in T

$\leftrightarrow$

$\varphi(\overline{x},\overline{y})$ admits $\rho(i)$ in T for every $i < \omega$

$\rightarrow$

$\varphi(\overline{x},\overline{y})$ admits ρ in T

$\leftrightarrow$

$\varphi(\overline{x},\overline{y})$ admits $\overline{\rho(\alpha)}$ in T for some strictly increasing sequence $\alpha \in \omega^{\omega}$

(4) $\varphi(\overline{x},\overline{y})$ admits $\rho_0 \cap \rho_1$ in T

$\leftrightarrow$

$\varphi(\overline{x},\overline{y})$ admits ρ_0 or ρ_1 in T

(5) $\varphi(\overline{x},\overline{y})$ admits $\rho_0 \cup \rho_1$ in T

$\rightarrow$

$\varphi(\overline{x},\overline{y})$ admits ρ_0 and ρ_1 in T (if ρ_0 or ρ_1 is principal the converse holds)

If $\varphi(\overline{x},\overline{y})$ is a formula of a complete theory T let $\rho(\varphi(\overline{x},\overline{y}),T)$ be some property of formulas enumerating those open formulas of BA which $\varphi(\overline{x},\overline{y})$ admits in T. If $\varphi_i(\overline{x}_i,\overline{y}_i)(i < n)$ are formulas of a complete theory T and $\ell(\overline{x}_i) = \ell(\overline{x}_j)(i,j<n)$ let $\rho(\varphi_i(x_i,y_i)(i < n), T)$ be the property of formulas $\rho(\varphi(\overline{x},\overline{y} \frown \overline{z} \frown \overline{w}),T)$ where $\varphi(\overline{x},\overline{y} \frown \overline{z} \frown \overline{w})$ is the formula $\bigvee_{i<n} (\varphi_i(\overline{x}_i,\overline{y}_i) \wedge z_i = w_i \wedge (\bigwedge_{j \neq i} z_j \neq w_j))$. Thus $\rho(\varphi(\overline{x},\overline{y} \frown \overline{z} \frown \overline{w}), T)$ enumerates those open formulas of BA which the <u>parameterized disjunction</u> $\bigvee_{i<n} (\varphi_i(\overline{x}_i,\overline{y}_i) \wedge z_i = w_i \wedge (\bigwedge_{j \neq i} z_j \neq w_j))$ of the formulas $\varphi_i(\overline{x}_i,\overline{y}_i)(i < n)$ admits in T.

1.1 Properties of Complete Theories

If ρ_0, ρ_1 are properties of formulas then ρ_0 is equivalent to ρ_1 (written $\rho_0 \sim \rho_1$) if ρ_0, ρ_1 are admitted by the same complete theories. Clearly $\sim$ is an equivalence relation. A property of complete theories is any equivalence class of $\sim$. If ρ is a property of formulas let $[\rho]$ be the equivalence class of $\sim$ containing ρ. If $\varphi(\overline{x},\overline{y})$ is a formula of a complete theory T and π is a property of complete theories then $\varphi(\overline{x},\overline{y})$ admits π in T if $\varphi(\overline{x},\overline{y})$ admits ρ in T for some $\rho \in \pi$. Otherwise $\varphi(\overline{x},\overline{y})$ omits π in T. If T is a complete theory and π is a property of complete theories then T admits π if T admits ρ for some $\rho \in \pi$. Otherwise T omits π. A principal property of complete theories is any property of complete theories containing a principal property of formulas. A 1-dimensional property of complete theories is any property of complete theories containing a 1-dimensional property of formulas. Using the above definitions the following lemma may be easily proved.

Lemma 2

If T is a complete theory and ρ, ρ_0, ρ_1 are properties of formulas then the following hold.

(1) $[\overline{\rho}]$ is principal

(2) T admits $[\overline{\rho}]$

$\leftrightarrow$

T admits $[\rho]$

$\leftrightarrow$

T admits $[\overline{\rho(\alpha)}]$ for some strictly increasing sequence $\alpha \in \omega^\omega$

(3) T admits $[\rho_0 \cap \rho_1]$

$\leftrightarrow$

T admits $[\rho_0]$ or $[\rho_1]$

(4) T admits $[\rho_0 \cup \rho_1]$

$\rightarrow$

T admits $[\rho_0]$ and $[\rho_1]$ (if ρ_0 or ρ_1 is principal the converse holds).

1.2 Ordering Properties of Complete Theories

Let P be the set of properties of complete theories and let PP be the subset of P consisting of principal properties of complete theories. If $\pi_0, \pi_1 \in P$ let $\pi_0' \leq \pi_1$ mean that every complete theory admitting π_1 admits π_0. Clearly $\pi_0 \leq \pi_1$ iff every complete theory (in the language of one binary predicate) admitting π_1 admits π_0. Obviously $(P,\leq)$ and $(PP,\leq)$ are posets.

Theorem 1

The poset $(P,\leq)$ is a lower semilattice which is distributive in the following sense: If $\pi, \pi_0, \pi_1 \in P$ and $\pi_0 \vee \pi_1$ exists then $(\pi \wedge \pi_0) \vee (\pi \wedge \pi_1)$ exists and $\pi \wedge (\pi_0 \vee \pi_1) = (\pi \wedge \pi_0) \vee (\pi \wedge \pi_1)$. If $\pi, \pi_0, \pi_1 \in P$ and $\pi \vee \pi_0, \pi \vee \pi_1$ exists then $\pi \vee (\pi_0 \wedge \pi_1)$ exists and $\pi \vee (\pi_0 \wedge \pi_1) = (\pi \vee \pi_0) \wedge (\pi \vee \pi_1)$. The poset $(PP,\leq)$ is a distributive sublattice of $(P,\leq)$.

Proof

Suppose $\pi_0 = [\rho_0] \in P$ and $\pi_1 = [\rho_1] \in P$. If $\pi = [\rho_0 \cap \rho_1]$ then Lemma 2 shows that

(1) If T is a complete theory then T admits π iff T admits π_0 or π_1

and from (1) it follows that for every $\pi \in P$

(2) $\pi = \pi_0 \wedge \pi_1$ iff π satisfies (1)

so (2) characterizes meets in $(P,\leq)$. Thus $(P,\leq)$ is a lower semilattice. By Lemma 1 $\pi_0 \wedge \pi_1 \in PP$ if $\pi_0, \pi_1 \in PP$ so $(PP,\leq)$

is a lower semilattice. If ρ_0 is principal or ρ_1 is principal and $\pi = [\rho_0 \cup \rho_1]$ then Lemma 2 shows that

(3) If T is a complete theory then T admits π iff T admits π_0 and π_1

and from (3) it may be proved that for every $\pi \in P$

(4) $\pi = \pi_0 \vee \pi_1$ iff π satisfies (3)

so (4) characterizes joins in $(P, \leq)$. Obviously if π satisfies (3) then $\pi = \pi_0 \vee \pi_1$. Thus suppose $\pi = \pi_0 \vee \pi_1$. If $\pi_0 \in PP$ or $\pi_1 \in PP$ then Lemma 2 shows that π satisfies (3) since it may be assumed that ρ_0 is principal or ρ_1 is principal. Thus suppose $\pi_0 \notin PP$ and $\pi_1 \notin PP$. Then by Lemma 2 $[\overline{\rho_0(\alpha)}] > \pi_0$ and $[\overline{\rho_1(\beta)}] > \pi_1$ whenever $\alpha, \beta \in \omega^\omega$ are strictly increasing sequences (hence $[\overline{\rho_0(\alpha)} \cup \overline{\rho_1(\beta)}] \geq \pi_0 \vee \pi_1 = \pi$ since $[\overline{\rho_0(\alpha)} \cup \overline{\rho_1(\beta)}] \geq [\overline{\rho_0(\alpha)}], [\overline{\rho_1(\beta)}]$ by Lemma 2). But if T is a complete theory which admits π_0 and π_1 then Lemma 2 shows that T admits $[\overline{\rho_0(\alpha)}]$ and $[\overline{\rho_1(\beta)}]$ for some strictly increasing sequences $\alpha, \beta \in \omega^\omega$ so by Lemma 2 T admits $[\overline{\rho_0(\alpha)} \cup \overline{\rho_1(\beta)}]$ so T admits π. Hence π satisfies (3). From the above it follows that if $\pi_0 \in PP$ or $\pi_1 \in PP$ then $\pi_0 \vee \pi_1$ exists. By Lemma 1 $\pi_0 \vee \pi_1 \in PP$ if $\pi_0 \in PP$ and $\pi_1 \in PP$ so $(PP, \leq)$ is an upper semilattice so it is a lattice. The distributivity of $(P, \leq)$ and $(PP, \leq)$ follow from (2) and (4).

If T is a complete theory let $\mathcal{I}(T) = \{\pi \in \mathcal{PP} \mid T$ admits $\pi\}$. From (2) and (4) in the above proof it follows that $\mathcal{I}(T)$ is a prime ideal of $(\mathcal{PP}, \leq)$.

<u>Theorem 2</u>

If $T_\alpha (\alpha < \beta)$ are complete theories (without functions) then $\mathcal{I}(\sum_{\alpha<\beta} T_\alpha) = \sum_{\alpha<\beta} \mathcal{I}(T_\alpha)$.

<u>Proof</u>

Suppose $T_\alpha (\alpha < \beta)$ are complete theories (without functions) in the languages $L_\alpha (\alpha < \beta)$ and suppose $T = \sum_{\alpha<\beta} T_\alpha$ is their disjoint sum in the language $\sum_{\alpha<\beta} L_\alpha$ obtained by adding unary predicates $P_\alpha (\alpha < \beta)$ to the disjoint union of the ~~languages~~ $L_\alpha (\alpha < \beta)$. It suffices to show that $\sum_{\alpha<\beta} \mathcal{I}(T_\alpha) \subset \mathcal{I}(T)$ and $\mathcal{I}(T) \subset \sum_{\alpha<\beta} \mathcal{I}(T_\alpha)$. To show that $\sum_{\alpha<\beta} \mathcal{I}(T_\alpha) \subset \mathcal{I}(T)$ it suffices to show that $\mathcal{I}(T_\alpha) \subset \mathcal{I}(T)$ for every $\alpha < \beta$. Suppose $\pi \in \mathcal{I}(T_\alpha)$ for some $\alpha < \beta$. Then $\varphi(\bar{x},\bar{y})$ admits π in T_α for some formula $\varphi(\bar{x},\bar{y})$ of T_α. It suffices to show that some formula $\varphi(\bar{x},\bar{y} \frown \bar{z})$ of T admits π in T. But letting $\varphi(\bar{x},\bar{y} \frown \bar{z})$ be the formula

$$(\varphi^{P_\alpha}(\bar{x},\bar{y}) \wedge P_\alpha(\bar{x}) \wedge P_\alpha(\bar{y}) \wedge z_0 = z_1) \vee ((\bigvee_{i<\ell(\bar{x})} \neg P_\alpha (\bar{x}(i))) \wedge z_2 = z_3)$$

of T it is easy to show that $\psi(\bar{x},\bar{y} \frown \bar{z})$ admits in T any open formula of BA which $\varphi(\bar{x},\bar{y})$ admits in T_α. Thus $\psi(\bar{x},\bar{y} \frown \bar{z})$ admits π in T. To show that $\mathcal{I}(T) \subset \sum_{\alpha<\beta} \mathcal{I}(T_\alpha)$ it suffices to show that for every $\pi \in I(T)$ there exist $\pi_{ij} \in I(T_{\alpha_j})(i,j < n)$ such that $\pi \leq \bigvee_{j<n} \bigvee_{i<n} \pi_{ij}$. Suppose $\pi \in I(T)$. Then $\varphi(\bar{x},\bar{y})$ admits π in T for some formula $\varphi(\bar{x},\bar{y})$ of T. It is easy to prove that there exist formulas $\varphi_{ij}(\bar{x}_{ij},\bar{y}_{ij})$ of T_{α_j} $(i,j<n)$ and open formulas $\varphi_{in}(\bar{x}_{in},\bar{y}_{in})$ of EQ $(i < n)$ such that

$$T \vdash \varphi(\bar{x},\bar{y}) \leftrightarrow \bigvee_{i<n} \bigwedge_{j\leq n} \varphi^*_{ij}(\bar{x}_{ij},\bar{y}_{ij}) \quad \text{where}$$

$\bar{x} = \bar{x}_{i0} \cup \cdots \cup \bar{x}_{in}$ $(i < n)$, $\bar{y} = \bar{y}_{i0} \cup \cdots \cup \bar{y}_{in}$ $(i < n)$,

$\varphi^*_{ij}(\bar{x}_{ij},\bar{y}_{ij})$ is $\varphi_{ij}^{P_{\alpha_j}}(\bar{x}_{ij},\bar{y}_{ij}) \wedge P_{\alpha_j}(\bar{x}_{ij}) \wedge P_{\alpha_j}(\bar{y}_{ij})(i,j<n)$ and

$\varphi^*_{in}(\bar{x}_{in},\bar{y}_{in})$ is $\varphi_{in}(\bar{x}_{in},\bar{y}_{in}) \wedge (\bigwedge_{j<n} \neg P_{\alpha_j}(\bar{x}_{in})) \wedge (\bigwedge_{j<n} \neg P_{\alpha_j}(\bar{y}_{in}))(i < n)$.

For notational convenience assume that $r(\bar{x}_{i_0 j}) \neq r(\bar{x}_{i_1 j})$ whenever $i_0 < i_1 < n$ and $j < n$. Let $\bar{x} = (x_0,\ldots,x_{\ell(\bar{x})-1})$. For each $i,j<n$ let $\pi_{ij} = [\bar{\rho}_{ij}]$ where $\rho_{ij} = \rho(\varphi_{ij}(\bar{x}_{ij},\bar{y}_{ij}), T_{\alpha_j})$. For each $i < n$ let $\pi_{in} = [\bar{\rho}_{in}]$ where $\rho_{in} = \rho(\varphi_{in}(\bar{x}_{in},\bar{y}_{in}), EQ)$.

Obviously $\pi_{ij} \in I(T_{\alpha_{ij}})$ $(i,j<n)$. It suffices to show that $\pi \leq \bigvee_{j<n} \bigvee_{i<n} \pi_{ij}$. Thus suppose T' is a complete theory which admits $\bigvee_{j<n} \bigvee_{i<n} \pi_{ij}$. It suffices to show that T' admits π. Since T' admits $\bigvee_{j<n} \bigvee_{i<n} \pi_{ij}$ there exist formulas $\psi_{ij}(\bar{z}_{ij},\bar{w}_{ij})$ of T' $(i,j<n)$ such that $\psi_{ij}(\bar{z}_{ij},\bar{w}_{ij})$ admits ρ_{ij} in $T'(i,j<n)$. Obviously T'

admits $\pi_{in} (i < n)$ so there exist formulas $\psi_{in}(\bar{z}_{in},\bar{w}_{in})$ of T' $(i < n)$ (namely the formulas $\varphi_{in}(\bar{x}_{in},\bar{y}_{in})$ $(i < n)$) such that $\psi_{in}(\bar{z}_{in},\bar{w}_{in})$ admits ρ_{in} in T' $(i < n)$. By changing or adding variables (if necessary) it may be assumed that $\bar{z}_j = \bar{z}_{ij}$ $(i<n, j\leq n)$ for sequences of variables $\bar{z}_j$ $(j \leq n)$ such that $r(\bar{z}_{j_0}) \cap r(\bar{z}_{j_1}) = \phi$ $(j_0 < j_1 \leq n)$. For each $k < \ell(\bar{x})$ let $\bar{x}_k$ be a distinct sequence of variables of length n and let $\bar{\bar{x}} = \bar{x}_0 \cap \ldots \cap \bar{x}_{\ell(\bar{x})-1}$. Also let $R_j(x_0,\ldots,x_{n-1})$ $(j < n)$ be formulas of T' such that $T' \vdash \exists x_0 \ldots \exists x_{n-1} R_j(x_0,\ldots,x_{n-1})$ $(j < n)$, $T' \vdash \neg\exists x_0 \ldots \exists x_{n-1}(R_{j_0}(x_0,\ldots,x_{n-1}) \wedge R_{j_1}(x_0,\ldots,x_{n-1}))$ $(j_0 < j_1 < n)$ and $T' \vdash \exists x_0 \ldots \exists x_{n-1} \bigwedge_{j<n} \neg R_j(x_0,\ldots,x_{n-1})$. Furthermore let $\psi^*_{ij}(\bar{\bar{x}} \cap \bar{z}_j,\bar{w}_{ij})$ be $\psi_{ij}(\bar{z}_j,\bar{w}_{ij}) \wedge (\bigwedge_{x_k \in r(\bar{x}_{ij})} R_j(\bar{x}_k))$ $(i,j<n)$ and let $\psi^*_{in}(\bar{\bar{x}} \cap \bar{z}_n,\bar{w}_{in})$ be $\psi_{in}(\bar{z}_n,\bar{w}_{in}) \bigwedge (\bigwedge_{j<n} \bigwedge_{x_k} \neg R_j(\bar{x}_k))$ $(i < n)$. Then letting $\psi^*(\bar{\bar{x}} \cap \bar{z},\bar{w})$ be $\bigvee_{i<n} \bigwedge_{j\leq n} \psi^*_{ij}(\bar{\bar{x}} \cap \bar{z}_j,\bar{w}_{ij})$ it is not difficult to show that $\psi^*(\bar{\bar{x}} \cap \bar{z},\bar{w})$ admits in T' any open formula of BA which $\varphi(\bar{x},\bar{y})$ admits in T. Hence $\psi^*(\bar{\bar{x}} \cap \bar{z},\bar{w})$ admits π in T' so T' admits π.

It may be proved that Theorem 2 fails for direct products of theories. In fact Wierzejewski (1976) provides a structure $\mathcal{A}$ such that $T = Th\mathcal{A}$ admits the order property of complete theories yet $T \times T = Th(\mathcal{A} \times \mathcal{A})$ does not. Hence $I(T \times T) \neq I(T) = I(T) + I(T)$.

1.3 Ordering Complete Theories

Let $\mathcal{T}$ be the class of all complete theories. If $T_0, T_1 \in \mathcal{T}$ let $T_0 \blacktriangleleft T_1$ mean that every property of complete theories admitted by T_0 is admitted by T_1. Clearly $T_0 \blacktriangleleft T_1$ iff $I(T_0) \subset I(T_1)$. Hence $(\mathcal{T}, \blacktriangleleft)$ is a preorder. If $T_0, T_1 \in \mathcal{T}$ let $T_0 \equiv T_1$ mean that $T_0 \blacktriangleleft T_1$ and $T_1 \blacktriangleleft T_0$. Clearly $\equiv$ is a congruence on $(\mathcal{T}, \blacktriangleleft)$. Hence $(\mathcal{T}|_{\equiv}, \blacktriangleleft|_{\equiv})$ is a poset. Note that if $T_0 \in \mathcal{T}$ then $T_0 \equiv T_1$ for some T_1 (without functions) $\in \mathcal{T}$ such that $|T_1| \le \min\{|T_0|, 2^{\aleph_0}\}$. In fact suppose $T_0 \in \mathcal{T}$. Let $J(T_0) = \{\pi \in I(T_0) | \pi = \overline{[\rho(\varphi(\bar{x},\bar{y}), T_0)]}$ for some formula $\varphi(\bar{x},\bar{y})$ of $T_0\}$ and for each $\pi \in J(T_0)$ let T_π be the complete theory in the language of one predicate $P_\pi(\bar{x},\bar{y})$ obtained by interpreting $P_\pi(\bar{x},\bar{y})$ as $\varphi(\bar{x},\bar{y})$ in T_0. Letting $T_1 = \sum_{\pi \in J(T_0)} T_\pi$ it follows from Theorem 2 that $I(T_1) = I(T_0)$. Thus $\mathcal{T}$ may be viewed as the set of all complete theories in a language consisting of $2^{\aleph_0}$ predicates of each arity (including 0). Note that if $T_\alpha (\alpha < \beta)$ are complete theories without functions then it follows from Theorem 2 that $(\sum_{\alpha<\beta} T_\alpha)|_{\equiv} = \bigvee_{\alpha<\beta} (T_\alpha|_{\equiv})$. In fact Theorem 2 shows that $T \mapsto I(T)$ induces a join-preserving embedding $(\mathcal{T}|_{\equiv}, \blacktriangleleft|_{\equiv}) \to (\text{Ideals}\,(\mathcal{PP}, \le), \subset)$. Hence $(\mathcal{T}|_{\equiv}, \blacktriangleleft|_{\equiv})$ is an upper semilattice. Finally note that if T_0 is locally definable in T_1 then $T_0 \blacktriangleleft T_1$. Thus $T \times T \blacktriangleleft T$ (so $I(T \times T) \subset I(T) = I(T) + I(T)$) for every $T \in \mathcal{T}$. Also $EQ \blacktriangleleft T$ for every $T \in \mathcal{T}$. Hence $(\mathcal{T}|_{\equiv}, \blacktriangleleft|_{\equiv})$ is an upper semilattice with a smallest element (namely $EQ|_{\equiv}$).

1.4 Archetypal Properties of Complete Theories

If $T \in \mathcal{T}$ and $\pi \in \mathcal{P}$ then T is archetypal for π (and π is archetypal) if the following holds: T admits π' iff $\pi' \leq \pi$. Note that if $\pi \in \mathcal{P}$ and π is archetypal then $\pi \in \mathcal{PP}$ and π is $\wedge$-irreducible in $(\mathcal{P}, \leq)$. Hence T is archetypal for π iff $\pi \in \mathcal{PP}$ and $\mathcal{I}(T) = \{\pi' \in \mathcal{PP} \mid \pi' \leq \pi\}$. From Theorem 2 it follows that if T_i is archetypal for π_i $(i < 2)$ then $T_0 + T_1$ is archetypal for $\pi_0 \vee \pi_1$. Hence if π_i is archetypal $(i < 2)$ then $\pi_0 \vee \pi_1$ is archetypal. Finally note that if T_0 is archetypal for π_0 and T_1 admits π_0 then $T_0 + T_1 \equiv T_1$ since by Theorem 2 $\mathcal{I}(T_0 + T_1) = \mathcal{I}(T_0) + \mathcal{I}(T_1) \subset \mathcal{I}(T_1) + \mathcal{I}(T_1) = \mathcal{I}(T_1) \subset \mathcal{I}(T_0) + \mathcal{I}(T_1) = \mathcal{I}(T_0 + T_1)$.

1.5 Prime Properties of Complete Theories

If $\pi \in P$ then π is <u>prime</u> if the following holds: $T_0 + T_1$ admits π iff T_0 admits π or T_1 admits π.

Theorem 3

If $\pi \in P$ then π is prime iff π is $\vee$-irreducible in $(P,\le)$.

Proof

Suppose $\pi \in P$, π is prime and π is not $\vee$-irreducible in $(P,\le)$. Then $\pi = \pi_0 \vee \pi_1$ for some $\pi_0, \pi_1 \in P$ such that $\pi_0, \pi_1 < \pi$. Hence there exist $T_0, T_1 \in T$ such that T_0 admits π_0 but not π and T_1 admits π_1 but not π. But $T_0 \blacktriangleleft T_0 + T_1$ so $T_0 + T_1$ admits π_0. Similarly $T_0 + T_1$ admits π_1. Hence $T_0 + T_1$ admits $\pi_0 \vee \pi_1 = \pi$. Since π is prime it follows that T_0 admits π or T_1 admits π and this is a contradiction. Suppose $\pi \in P$ and π is $\vee$-irreducible in $(P,\le)$. It suffices to show that π is prime. Thus suppose $T_0 + T_1$ admits π. It suffices to show that T_0 admits π or T_1 admits π. Let $\pi = [\rho]$. Then $T_0 + T_1$ admits $[\overline{\rho(\alpha)}]$ for some strictly increasing sequence $\alpha \in \omega^\omega$. Hence $[\overline{\rho(\alpha)}] \in I(T_0 + T_1) = I(T_0) + I(T_1)$ so $[\overline{\rho(\alpha)}] = \pi_0 \vee \pi_1$ for some $\pi_0 \in I(T_0)$ and $\pi_1 \in I(T_1)$. But $\pi \le [\overline{\rho(\alpha)}] = \pi_0 \vee \pi_1$ so $\pi = (\pi \wedge \pi_0) \vee (\pi \wedge \pi_1)$. Since π is $\vee$-irreducible in $(P,\le)$ it follows that $\pi = \pi \wedge \pi_0$ or $\pi = \pi \wedge \pi_1$. Hence T_0 admits π or T_1 admits π.

Corollary 1

If $\pi_0, \pi_1 \in \mathcal{P}$ are $\vee$-irreducible in $(\mathcal{P},\leq)$ then $\pi_0 \wedge \pi_1$ is $\vee$-irreducible in $(\mathcal{P},\leq)$.

Proof

Suppose $\pi_0, \pi_1 \in \mathcal{P}$ are $\vee$-irreducible in $(\mathcal{P},\leq)$. By Theorem 3 π_0, π_1 are prime. It follows easily that $\pi_0 \wedge \pi_1$ is prime. By Theorem 3 $\pi_0 \wedge \pi_1$ is $\vee$-irreducible in $(\mathcal{P},\leq)$.

§2 Basic Examples

2.0 The Minimum and Maximum Properties of Complete Theories

Let O be a property of formulas which enumerates the open formulas of BA. Since $EQ \blacktriangleleft T$ for every $T \in \mathcal{T}$ it follows that EQ is archetypal for $[O]$. Hence $[O]$ is archetypal so $[O] \in \mathcal{PP}$ and $[O]$ is $\wedge$-irreducible in $(\mathcal{P}, \leq)$. Furthermore $[O] \leq \pi$ for every $\pi \in \mathcal{P}$. Note that if $T \in \mathcal{T}$ then T is archetypal for $[O]$ iff $T \blacktriangleleft EQ$.

Example 1

Let $0 < n < \omega$. If $T_n = Th\mathcal{A} = Th(|\mathcal{A}|, P_{\mathcal{A}}, E_{\mathcal{A}})$ where $\|\mathcal{A}\| \geq \aleph_0$, $P_{\mathcal{A}} \subset |\mathcal{A}|$ and $E_{\mathcal{A}} \subset P_{\mathcal{A}} \times (|\mathcal{A}| - P_{\mathcal{A}})^n$ is the graph of some bijection $P_{\mathcal{A}} \to (|\mathcal{A}| - P_{\mathcal{A}})^n$ it is easy to show that T_n is definable in EQ. Hence $T_n \blacktriangleleft EQ$ so T_n is archetypal for $[O]$.

Example 2

Let $T = Th\mathcal{A} = Th(|\mathcal{A}|, P_i^{\mathcal{A}})_{i<\omega}$ where $|\mathcal{A}| \geq \omega$ and the $P_i^{\mathcal{A}} \subset |\mathcal{A}|$ $(i<\omega)$ are independent (each finite Boolean combination of the $P_i^{\mathcal{A}}$ $(i<\omega)$ is nonempty). It is easy to show that T is locally definable in EQ. Hence $T \blacktriangleleft EQ$ so T is archetypal for $[O]$. Note that T is superstable but not $\aleph_0$-stable.

Example 3

Let $T = Th\mathcal{A} = Th(|\mathcal{A}|, E_i^{\mathcal{A}})_{i<\omega}$ where $|\mathcal{A}| \geq \aleph_0$ and the $E_i^{\mathcal{A}} \subset |\mathcal{A}| \times |\mathcal{A}|$ $(i<\omega)$ are equivalence relations such that for every $i < \omega$ each equivalence class of $E_i^{\mathcal{A}}$ is the union of infinitely many equivalence classes of $E_{i+1}^{\mathcal{A}}$. It is easy to show that T is locally definable in EQ. Hence $T \blacktriangleleft EQ$ so T is archetypal for $[O]$. Note that T is stable but not superstable.

Let $1 = \overline{0}$. Obviously $[1] \in PP$ and $\pi \leq [1]$ for every $\pi \in P$. Note that if $\varphi(\overline{x},\overline{y})$ is a formula of a complete theory T then $\varphi(\overline{x},\overline{y})$ admits 1 in T iff for arbitrarily large $n < \omega$ there exists $A \models T$ and a partition of $|A|^{\ell(\overline{x})}$ into n $\varphi(\overline{x},\overline{y})$-definable subsets such that the union of any $m \leq n$ of them is also $\varphi(\overline{x},\overline{y})$-definable. For each $0 < n < \omega$ let 1_n be a property of formulas such that if $\varphi(\overline{x},\overline{y})$ is a formula of a complete theory T then $\varphi(\overline{x},\overline{y})$ admits 1_n in T iff for arbitrarily large $n \leq m < \omega$ there exists $A \models T$ and m $\varphi(\overline{x},\overline{y})$-definable subsets of $|A|^{\ell(\overline{x})}$ such that the union of any $\ell \leq m$ of them is also $\varphi(\overline{x},\overline{y})$-definable but the intersection of any $\ell \leq m$ of them is nonempty iff $\ell \leq n$.

<u>Lemma 3</u>

If T is a complete theory and $0 < n < \omega$ then the following hold:

(1) If some formula $\varphi(\overline{x},\overline{y})$ of T admits 1_n in T then some formula $\psi(\overline{x},\overline{z})$ of T admits 1_1 in T

(2) If some formula $\psi(\overline{x},\overline{z})$ of T admits 1_1 in T then some formula $\chi(\overline{x},\overline{w})$ of T admits 1 in T .

In particular $[1] = [1_n]$ $(0 < n < \omega)$.

<u>Proof</u>

Suppose the premise of (1) holds. Letting $\psi(\overline{x},\overline{z})$ be $\varphi(\overline{x},\overline{y}) \wedge (\bigwedge_{i<n-1} \varphi(\overline{x},\overline{y}_i))$ where $\overline{z} = \overline{y} ^\frown \overline{y}_0 ^\frown \ldots ^\frown \overline{y}_{n-2}$ it follows easily that $\psi(\overline{x},\overline{z})$ admits 1_1 in T . Suppose the premise of (2) holds. Letting $\chi(\overline{x},\overline{w})$ be $(\psi(\overline{x},\overline{z}) \wedge w_0 = w_1) \vee (\psi(\overline{x},\overline{z}) \wedge w_2 = w_3)$ where $\overline{w} = \overline{z} ^\frown w_0 ^\frown w_1 ^\frown w_2 ^\frown w_3$ it follows easily that $\chi(\overline{x},\overline{w})$ admits 1 in T .

Theorem 4

1 is 1-dimensional.

Proof

Suppose $\varphi(\overline{x},\overline{y})$ is a formula of a complete theory T which admits 1 in T. It suffices to show that some formula $\varphi(z,\overline{w})$ of T admits 1 in T. For notational convenience assume that $\overline{x} = x_0 {}^\frown x_1$. Thus $\varphi(x_0 {}^\frown x_1, \overline{y})$ admits 1 in T. By the compactness theorem there exists $A \models T$ and nonempty, disjoint $\varphi(x_0 {}^\frown x_1, \overline{y})$-definable subsets A_i of $|A| \times |A|$ $(i < \omega)$ such that $\bigcup_{i \in S} A_i$ is also $\varphi(x_0 {}^\frown x_1, \overline{y})$-definable whenever $S \subset \omega$ and $|S| < \aleph_0$. For each $i < 2$ let $f_i : |A| \times |A| \to |A|$ be the i-th projection of $|A| \times |A|$ onto $|A|$ (thus $f_i(a_0,a_1) = a_i$ whenever $a_0,a_1 \in |A|$). By the compactness and Ramsey theorems it may be assumed that $\bigcap_{i \in S_0} f_0(A_i) \neq \phi$ iff $\bigcap_{i \in S_1} f_0(A_i) \neq \phi$ whenever $S_0, S_1 \subset \omega$ and $|S_0| = |S_1| < \aleph_0$. Thus either (1) there exists $0 < n < \omega$ such that $\bigcap_{i \in S} f_0(A_i) \neq \phi$ iff $|S| \leq n$ whenever $|S| < \omega$ or (2) $\bigcap_{i \in S} f_0(A_i) \neq \phi$ whenever $|S| < \omega$. If (1) holds let $\chi(z,\overline{w})$ be $\exists x_1 \varphi(x_0 {}^\frown x_1, \overline{y})$ where $z = x_0$ and $\overline{w} = \overline{y}$. Since f_0 preserves unions it follows easily that $\chi(z,\overline{w})$ admits 1_n in T. By Lemma 3 some formula $\psi(z,\overline{w})$ admits 1 in T. If (2) holds let $\chi(z,\overline{w})$ be $\varphi(x_0 {}^\frown x_1, \overline{y})$ where $z = x_1$ and $\overline{w} = x_0 {}^\frown \overline{y}$. Since $f_1|_{f_0^{-1}(a_0)}$ preserves disjointness whenever $a_0 \in |A|$ it follows easily that $\psi(z,\overline{w})$ admits 1_1 in T . By Lemma 3 some formula $\psi(z,\overline{w})$ of T admits 1 in T .

Corollary 2

[1] is prime.

Proof

Suppose $T_i (i<2)$ are complete theories and $\sum_{i<2} T_i$ admits [1]. It suffices to show that T_i admits [1] for some $i < 2$. Since $\sum_{i<2} T_i$ admits [1] Theorem 4 shows that some formula $\varphi(x,\overline{y})$ of $\sum_{i<2} T_i$ admits 1 in $\sum_{i<2} T_i$. Let $\sum_{i<2} A_i \models \sum_{i<2} T_i$. It is easy to prove that there exist formulas $\varphi_i(x,\overline{y}_i)$ of T_i $(i < 2)$ such that every $\varphi(x,\overline{y})$-definable subset of $|\sum_{i<2} A_i|$ is the union of a $\varphi_0(x,\overline{y}_0)$-definable subset of $|A_0|$ and a $\varphi_1(x,\overline{y}_1)$-definable subset of $|A_1|$. From this it follows easily that $\varphi_i(x,\overline{y}_i)$ admits 1_1 in T_i for some $i < 2$. By Lemma 3 T_i admits [1].

Example 4

Let $T = \mathrm{Th}A = \mathrm{Th}(|A|, P_A, E_A)$ where $\|A\| \geq \aleph_0$, $P_A \subset |A|$ and $E_A \subset P_A \times (|A| - P_A)$ is the graph of some bijection $P_A \to P(|A| - P_A)$. Letting $\varphi(x,y)$ be $E(y,x)$ it is clear that $\varphi(x,y)$ admits 1_1 in T . Hence T admits [1].

Example 5

Let $T = \mathrm{Th}A = \mathrm{Th}(\omega, 0, 1, +, \cdot)$ where A is the standard model of Peano arithmetic. Letting $\varphi(x,y)$ be a formula of T which asserts that

x is a prime divisor of y it is clear that $\varphi(x,y)$ admits 1_1 in T. Hence T admits [1].

Example 6

Let $T = \mathrm{Th}\mathcal{A}$ where $\mathcal{A}$ is an infinite Boolean algebra containing infinitely many atoms. Letting $\varphi(x,y)$ be a formula of T which asserts that x is an atom contained in y it is clear that $\varphi(x,y)$ admits 1_1 in T. Hence T admits [1].

Example 7

Let $T = \mathrm{Th}\mathcal{A}$ where $\mathcal{A}$ is an infinite Boolean algebra containing no atoms. Then T omits [1]. Otherwise some formula $\varphi(x,\overline{y})$ of T admits 1_1 in T. By the compactness theorem it may be assumed that there exist nonempty, disjoint, $\varphi(x,\overline{y})$-definable subsets A_i of $|\mathcal{A}|$ $(i<\omega)$ such that the union of any finite number of them is also $\varphi(x,\overline{y})$-definable. From the well-known result that T is quantifier-eliminable and $\aleph_0$-categorical it follows easily that for some $n < \omega$ every $\varphi(x,\overline{y})$-definable subset of $|\mathcal{A}|$ is the union of at most n n-basic subsets of $|\mathcal{A}|$ where an n-basic subset of $|\mathcal{A}|$ is any subset of $|\mathcal{A}|$ of the form

$$[\overline{a},\overline{b}] = \{a \in |\mathcal{A}| \,|\, a_0 \subset a \subset a_1 \wedge (\bigwedge_{i<n} (b_i \neq 0 \rightarrow b_i \cap a \neq 0 \wedge b_i - a \neq 0))\}$$

where $\overline{a} = a_0 \cap a_1$, $\overline{b} = b_0 \cap \ldots \cap b_{n-1}$, $a_0 \subset a_1$ and A_{ij} of $b_i \subset a_1 - a_0$ $(i<n)$. Thus there exist n-basic subsets of $|\mathcal{A}|$ $(i<\omega,\ j<n)$ such that $A_i = \bigcup_{j<n} A_{ij}$ for every $i<\omega$. Note that if $A = [\overline{a},\overline{b}]$ and $A' = [\overline{a}',\overline{b}']$ are n-basic subsets of $|\mathcal{A}|$ then

$A = A'$ iff $a_0 = a_0'$, $a_1 = a_1'$ and $\{b_0,\ldots,b_{n-1}\} = \{b_0',\ldots,b_{n-1}'\}$. If $A = [\bar{a},\bar{b}]$ and $A' = [\bar{a}',\bar{b}']$ are n-basic subsets of $|A|$ let $A \leq A'$ mean that $a_0' \subset a_0 \wedge a_1 \subset a_1'$ and let $A \equiv A'$ mean that $A \leq A'$ and $A' \leq A$. Using Ramsey's theorem it may be assumed that $A_{i_0j_0} \leq A_{i_1j_1}$ iff $A_{i_2j_0} \leq A_{i_3j_1}$ whenever $i_0,i_1,i_2,i_3,j_0,j_1 < \omega$ and i_1-i_0, i_3-i_2 have the same sign. If A, A' are n-basic subsets of $|A|$ then A' <u>covers</u> A if $A \leq A' \wedge A \cap A' \neq \phi$. Obviously A' covers A if $A' \equiv A$. It is easy to prove that if $A = [\bar{a},\bar{b}]$ and $A' = [\bar{a}',\bar{b}']$ are n-basic subsets of $|A|$ and $A \leq A'$ then A' does not cover A iff $0 \neq b_i' \subset a_0 - a_0'$ or $0 \neq b_i' \subset a_1' - a_1$ for some $i<n$. From this it follows easily that if $A \leq A' \leq A''$ are n-basic subsets of $|A|$ and A'' covers A then A'' covers A'. It is also easy to prove that if A, $A_0, \ldots, A_{m-1}$ are n-basic subsets of $|A|$ and $A \subset A_0 \cup \ldots \cup A_{m-1}$ then A_i covers A for some $i < m$. From this it may be proved that for each $j_0 < n$ there exists some $f(j_0) < n$ such that $A_{0j_0} \leq A_{1f(j_0)}$ and $A_{1f(j_0)}$ is $\leq$-maximal among the $A_{1j'}$, $(j'<n)$ or $A_{1j_0} \leq A_{0f(j_0)}$ and $A_{0f(j_0)}$ is $\leq$-maximal among the $A_{0j'}$, $(j'<n)$. Indeed let $j_0 < n$. Since $\bigcup_{i\leq n} A_i$ is $\varphi(x,\bar{y})$-definable there exist n-basic subsets B_k of $|A|$ $(k < n)$ such that $\bigcup_{i\leq n} A_i = \bigcup_{k<n} B_k$. In particular $A_{ij_0} \subset \bigcup_{k<n} B_k$ for $i \leq n$ so there exist $i_0 < i_1 \leq n$ and $k_0 < n$ such that B_{k_0} covers $A_{i_0j_0}$ and $A_{i_1j_0}$ (thus

$A_{i_0 j_0} \leq B_{k_0}$ and $A_{i_1 j_0} \leq B_{k_0}$). Since $B_{k_0} \subset \bigcup_{i \leq n} A_{ij}$ there exist $i_2 \leq n$ and $j_1 < n$ such that $A_{i_2 j_1}$ covers B_{k_0} (thus $B_{k_0} \leq A_{i_2 j_1}$). Choose $j_2 < n$ such that $A_{i_2 j_1} \leq A_{i_2 j_2}$ and $A_{i_2 j_2}$ is $\leq$-maximal among the $A_{i_2 j'}$, $(j' < n)$. Obviously $A_{i_0 j_0} \leq A_{i_2 j_2}$ and $A_{i_1 j_0} \leq A_{i_2 j_2}$. If $i_2 \geq i_1$ it follows easily that $A_{0 j_0} \leq A_{1 j_2}$ and $A_{1 j_2}$ is $\leq$-maximal among the $A_{1 j'}$, $(j' < n)$. Similarly if $i_2 < i_1$ it follows easily that $A_{1 j_0} \leq A_{0 j_2}$ and $A_{0 j_2}$ is $\leq$-maximal among the $A_{0 j'}$, $(j' < n)$. Letting $f(j_0) = j_2$ concludes the argument. From this it follows that for some $j_0 < n$ either $A_{0 j_0} \leq A_{m j_0}$ or $A_{m j_0} \leq A_{0 j_0}$ for some $m \leq n$. Indeed choose $j_0 < n$ such that $O_f(j_0)$ is cyclic where $O_f(j_0) = \{f^i(j_0) \mid i<m\}$ for some $m \leq n$, is the orbit of j_0 under f. It suffices to prove that if $A_{0 j_0} \leq A_{1 f(j_0)}$ then $A_{0 j_0} \leq A_{m j_0}$ but if $A_{1 j_0} \leq A_{0 f(j_0)}$ then $A_{m j_0} \leq A_{0 j_0}$. Suppose $A_{0 j_0} \leq A_{1 f(j_0)}$. Then $A_{i f^i(j_0)} \leq A_{1+1 f^{i+1}(j_0)}$ for every $i < m$. Otherwise $A_{i-1 f^{i-1}(j_0)} \leq A_{i f^i(j_0)}$ yet $A_{i+1 f^i(j_0)} \leq A_{i f^{i+1}(j_0)}$ for some $i < m$. But then $A_{i f^i(j_0)} \leq A_{i-1 f^{i+1}(j_0)}$ so $A_{i-1 f^{i-1}(j_0)} \leq A_{i f^i(j_0)} \leq A_{i-1 f^{i+1}(j_0)}$ so $A_{i-1 f^{i-1}(j_0)} \equiv A_{i f^i(j_0)} \equiv A_{i-1 f^{i+1}(j_0)}$ (since $A_{i-1 f^{i-1}(j_0)}$ is

$\leq$-maximal among the $A_{i-1j'}$ $(j' < n)$) so $A_{i-1f^{i-1}(j_0)} \cap A_{if^i(j_0)} \neq \phi$ so $A_{i-1} \cap A_i \neq \phi$ and this is a contradiction. Hence

$A_{0j_0} = A_{0f^0(j_0)} \leq A_{mf^m(j_0)} = A_{mj_0}$. Similarly if $A_{1j_0} \leq A_{0f(j_0)}$ then $A_{m-if^i(j_0)} \leq A_{m-i-1\ f^{i+1}(j_0)}$ for every $i < m$ so

$A_{mj_0} = A_{mf^0(j_0)} \leq A_{0f^m(j_0)} = A_{0j_0}$. From this it follows that for some $j_0 < n$ either $A_{ij_0} \leq A_{i+1j_0}$ $(i \leq 2n)$ or $A_{i+1j_0} \leq A_{ij_0}$ $(i \leq 2n)$.

Since $\bigcup_{i\leq n} A_{2i}$ is $\varphi(x,\overline{y})$-definable there exist n-basic subsets B_k of $|A|$ $(k < n)$ such that $\bigcup_{i\leq n} A_{2i} = \bigcup_{k<n} B_k$. In particular

$A_{2ij_0} \subset \bigcup_{k<n} B_k$ for $i \leq n$ so there exist $i_0 < i_1 \leq n$ and $k_0 < n$ such that B_{k_0} covers $A_{2i_0j_0}$ and $A_{2i_1j_0}$. But

$A_{2i_0j_0} \leq A_{2i_0+1j_0} \leq B_{k_0}$ or $A_{2i_1j_0} \leq A_{2i_1-1j_0} \leq B_{k_0}$ so B_{k_0} covers $A_{2i_0+1j_0}$ or $A_{2i_1-1j_0}$. In either case $B_{k_0} \cap (\bigcup_{i\leq n} A_{2i+1}) \neq \phi$ so $(\bigcup_{k<n} B_k) \cap (\bigcup_{i\leq n} A_{2i+1}) \neq \phi$ so $(\bigcup_{i\leq n} A_{2i}) \cap (\bigcup_{i\leq n} A_{2i+1}) \neq \phi$ and this is a contradiction since the A_i $(i < 2n+1)$ are disjoint.

Examples 6 and 7 show that some theories of infinite Boolean algebras are $\triangleleft$-maximum while others are not. In fact if A,B are infinite Boolean algebras then $\mathrm{Th}A \equiv \mathrm{Th}B$ iff either both A,B contain only finitely many atoms or both A,B contain infinitely many atoms since it may be shown that if A contains only finitely many atoms and $\mathcal{B}$ contains no atoms then $\mathcal{A}$ is essentially definable in $\mathcal{B}$ (and vice versa). But all theories of infinite Boolean algebras are $\lhd$-maximum since it may be shown that they admit the versatility property. Thus $T_0 \lhd T_1$ does not imply that $T_0 \blacktriangleleft T_1$.

<u>Example 8</u>

Let $T = Th\mathcal{A} = Th(|\mathcal{B}| \cup S\mathcal{B}, 0, 1, \cap, \cup, c, E)$ where $S\mathcal{B}$ is the Stone space of an infinite atomless Boolean algebra $\mathcal{B} = (|\mathcal{B}|, 0, 1, \cap, \cup, c)$ and $E \subset |\mathcal{B}| \times S\mathcal{B}$ is defined by $E(b,c)$ iff $b \in c$ $(b \in |\mathcal{B}|, c \in S\mathcal{B})$. Letting $\varphi(x,y)$ be $E(y,x)$ it is clear that $\varphi(x,y)$ admits 1_1 in T. Hence T admits [1]. Note that T is $\aleph_0$-categorical (use a back and forth argument).

Example 8 shows that some countable complete $\aleph_0$-categorical theories may be $\blacktriangleleft$-maximum even though countable complete non-$\aleph_0$-categorical theories cannot be definable in them. Thus $T_0 \blacktriangleleft T_1$ does not imply that T_0 is definable in T_1 .

2.1 The Finite Cover and Partition Properties of Complete Theories

Let fcp be a property of formulas such that if $\varphi(\overline{x},\overline{y})$ is a formula of a complete theory T then $\varphi(\overline{x},\overline{y})$ admits fcp in T iff for arbitrarily large $n < \omega$ there exists $A \models T$ and n $\varphi(\overline{x},\overline{y})$-definable subsets of $|A|^{\ell(\overline{x})}$ such that the union of them is $|A|^{\ell(\overline{x})}$ but the union of any $n - 1$ of them is not $|A|^{\ell(\overline{x})}$. In fact let fcp(n) be $(x_0 \cup \ldots \cup x_{n-1} = 1) \wedge (\bigwedge_{i<n} x_0 \cup \ldots \cup x_{i-1} \cup x_{i+1} \cup \ldots \cup x_{n-1} \neq 1)$ for every $n < \omega$. Shelah (1971) proved that fcp is 1-dimensional. Using this result it is easy to prove that [fcp] is prime (see the proof of Corollary 2). Keisler (1967) proved that if T is a countable complete theory which admits [fcp] then T is not $\aleph_1$-categorical.

Example 9

Let L be a language consisting of a binary predicate $\sim$ and let EQV be the theory in L whose axioms are

$$x \sim x$$

$$x \sim y \rightarrow y \sim x$$

$$x \sim y \wedge y \sim z \rightarrow x \sim z$$

$$\exists x_0 \ldots \exists x_{n-1} \bigwedge_{i<j<n} x_i \not\sim x_j \qquad (n < \omega)$$

$$\exists^{!n} y(y \sim x_0) \wedge \exists^{!n} y(y \sim x_1) \rightarrow x_0 \sim x_1 \qquad (n < \omega)$$

If $S, T \subset \omega$ let EQV(S,T) be the theory in L whose axioms are

EQV

$\exists x \exists^{!n} y(y \sim x)$ $(n \in S)$

$\neg\exists x \exists^{!n} y(y \sim x)$ $(n \in T)$

If $S \subset \omega$ let EQV(S) be EQV(S,ω-S). Note that EQV(S) is complete whenever $S \subset \omega$. Furthermore if $|S| < \aleph_0$ then EQV(S) < EQ (so EQV(S) omits [fcp]). But if $|S| = \aleph_0$ and $\varphi(x,y)$ is $x = y \vee x \nsim y$ then $\varphi(x,y)$ admits fcp in EQV(S) (so EQV(S) admits [fcp]).

If $S, T \subset \omega$ and $|S|, |T| < \aleph_0$ let (S,T) be the sentence

$(\bigwedge_{n \in S} \exists x \exists^{!n} y(y \sim x)) \wedge (\bigwedge_{n \in T} \neg\exists x \exists^{!n} y(y \sim x))$ of EQV.

<u>Lemma 4</u>

If φ is a sentence of EQV there exist sentences (S_i,T_i) of EQV $(i < \omega)$ such that EQV $\vdash \varphi \leftrightarrow \bigvee_{i<n} (S_i,T_i)$.

<u>Proof</u>.

Suppose φ is a sentence of EQV. Let Φ be the set of all Boolean combinations of the sentences $\exists x \exists^{!n} y(y \sim x)$ $(n < \omega)$. Clearly $\mathrm{Th}\mathcal{A} = \mathrm{Th}\mathcal{B}$ whenever $\mathcal{A},\mathcal{B} \models$ EQV and $\mathrm{Th}\mathcal{A} \cap \Phi = \mathrm{Th}\mathcal{B} \cap \Phi$. Let $\Psi = \{\psi \in \Phi \mid \text{EQV} \vdash \varphi \rightarrow \psi\}$. It suffices to show that EQV $\cup\, \Psi \vdash \varphi$. Suppose not. Then there exists $\mathcal{A} \models$ EQV $\cup\, \Psi$ such that $\mathcal{A} \models \neg\varphi$. Let $\mathbf{X} = \{\chi \in \Phi \mid \mathcal{A} \models \chi\}$. Note that if $\mathcal{B} \models$ EQV $\cup\, \mathbf{X}$ then $\mathcal{B} \models \neg\varphi$ (since $\mathrm{Th}\mathcal{A} = \mathrm{Th}\mathcal{B}$). Hence EQV $\cup\, \mathbf{X} \vdash \neg\varphi$ so EQV $\vdash \chi \rightarrow \neg\varphi$ for some $\chi \in \mathbf{X}$. But then EQV $\vdash \varphi \rightarrow \neg\chi$ so $\neg\chi \in \Psi \subset \mathbf{X}$. Hence both $\chi, \neg\chi \in \mathfrak{X}$ and this is a contradiction.

If $S \subset \omega$ and $|S| = \aleph_0$ then S is <u>thin</u> for fcp if the following holds: If $\alpha, \beta \subset S$ and $|\alpha| = |\beta| = \aleph_0$ then EQV(α) admits $\overline{\text{fcp}(\beta)}$ iff $|\beta - \alpha| < \aleph_0$. Note that infinite subsets of thin sets for fcp are thin for fcp.

<u>Lemma 5</u>

Thin sets for fcp exist.

<u>Proof</u>

For each formula $\varphi(\bar{x},\bar{y})$ of EQV and $n < \omega$ let $(\varphi(\bar{x},\bar{y}), n)$ be a sentence of EQV which asserts that $\varphi(\bar{x},\bar{y})$ admits fcp(n). By Lemma 4 EQV $\vdash (\varphi(\bar{x},\bar{y}),n) \leftrightarrow \bigvee_{i<m} (S_i,T_i)$ for some $S_i,T_i \subset \omega$ $(i < m)$. If $S_i = \phi$ for some $i < m$ then $\varphi(\bar{x},\bar{y})$ <u>admits</u> n. Otherwise $\varphi(\bar{x},\bar{y})$ omits n. Obviously $\varphi(\bar{x},\bar{y})$ omits n for sufficiently large $n < \omega$ (otherwise $\varphi(\bar{x},\bar{y})$ admits fcp in EQV(ϕ)). For each $n < \omega$ such that $\varphi(\bar{x},\bar{y})$ omits n let $S(\varphi(\bar{x},\bar{y}),n) \subset \omega$ be defined by choosing some $\ell_i \in S_i$ for each $i < m$ such that $S_i \cap T_i = \phi$. Suppose $S \subset \omega$ and $|S| < \aleph_0$. Then for sufficiently large $n < \omega$ $S(\varphi(\bar{x},\bar{y}),n)$ may be defined so that $S(\varphi(\bar{x},\bar{y}),n) \cap S = \phi$. To show this note that for every $T \subset S$ $\varphi(\bar{x},\bar{y})$ omits fcp in EQV(T) (since $|T| < \aleph_0$). Thus for sufficiently large $n < \omega$ it follows that for every $T \subset S$ EQV(T) $\vdash \neg(\varphi(\bar{x},\bar{y}),n)$ (since $|S| < \aleph_0$). But for such $n < \omega$ it follows that $S_i \not\subset S$ for each $i < m$ such that $S_i \cap T_i = \phi$ so define $S(\varphi(\bar{x},\bar{y}),n)$ by choosing some $\ell_i \in S_i - S$ for each $i < m$

such that $S_i \cap T_i = \phi$. Let $\varphi_i(\overline{x}_i,\overline{y}_i)(i < \omega)$ be the formulas of EQV and let $\psi_i(\overline{z}_i,\overline{w}_i)$ $(i < \omega)$ be the formulas of EQV obtained by letting $\psi_i(\overline{z}_i,\overline{w}_i)$ be the parametrized disjunction of the formulas $\varphi_j(\overline{x}_j,\overline{y}_j)$ $(j \leq i)$ for every $i < \omega$. Let $S = \{n_i | i < \omega\}$ where $n_0 < n_1 < n_2 < \ldots < \omega$ are chosen as follows: Choose $n_0 < \omega$ so that $\psi_0(\overline{z}_0,\overline{w}_0)$ omits n_0 . If $n_0 < \ldots < n_{i-1} < \omega$ have been chosen so that $\psi_j(\overline{z}_j,\overline{w}_j)$ omits n_j $(j < i)$ and so that $S(\psi_j(\overline{z}_j,\overline{w}_j),n_j) \cap \{n_0,\ldots,n_{j-1},n_{j+1},\ldots,n_{i-1}\} = \phi$ $(j < i)$ choose $n_i > n_{i-1}$ so that $\psi_i(\overline{z}_i,\overline{w}_i)$ omits n_i and so that $n_i \notin \bigcup_{j<i} S(\psi_j(\overline{z}_j,\overline{w}_j),n_j)$ and so that $S(\psi_i(\overline{z}_i,\overline{w}_i),n_i) \cap \{n_0,\ldots,n_{i-1}\} = \phi$. It follows easily that if $i < \omega$, $\alpha \subset S$ and $EQV(\alpha) \vdash (\psi_i(\overline{z}_i,\overline{w}_i),n_i)$ then $n_i \in \alpha$ (since $S(\psi_i(\overline{z}_i,\overline{w}_i),n_i) \cap S \subset \{n_i\}$). From this it follows that if α, $\beta \subset S$ and $|\alpha| = |\beta| = \aleph_0$ then $EQV(\alpha)$ admits $\overline{fcp(\beta)}$ iff $|\beta - \alpha| < \aleph_0$ (since $EQV(\alpha)$ admits $\overline{fcp(\beta)}$ iff $\psi_i(\overline{z}_i,\overline{w}_i)$ admits $\overline{fcp(\beta)}$ in $EQV(\alpha)$ for sufficiently large $i < \omega$). Hence S is a thin set for fcp .

<u>Theorem 5</u>

The poset of subsets of ω(modulo finite sets) may be embedded into $(PP,\leq)$ in such a way that finite joins are preserved.

<u>Proof</u>

Let S be a thin set for fcp and let $\mathcal{S}$ be the poset of subsets of S (modulo finite sets). It suffices to show that $\mathcal{S}$ may be embedded into $(PP,\leq)$ in such a way that finite joins are preserved. For each $\alpha \subset S$ let $f(\alpha) = \overline{[fcp(\alpha)]}$ (where $f(\alpha) = [0]$ if $|\alpha| < \aleph_0$) and note that $f(\beta) \leq f(\alpha)$ iff $|\beta - \alpha| < \aleph_0$ ($\alpha, \beta \subset S$) since S is thin for fcp. Furthermore $f(\alpha) \vee f(\beta) = f(\alpha \cup \beta)$ ($\alpha, \beta \subset S$). Thus f induces an embedding $f|_{\equiv} : \mathcal{S} \to (\mathcal{PP}, \leq)$ which preserves finite joins.

Corollary 3

$|\mathcal{PP}| = 2^{\aleph_0}$.

Lemma 6

If $\alpha \subset \omega$ and $|\alpha| = \aleph_0$ then $[fcp(\alpha)] \notin \mathcal{PP}$.

Proof

Suppose $\alpha \subset \omega$, $|\alpha| = \aleph_0$ and $[fcp(\alpha)] \in \mathcal{PP}$. Then $[fcp(\alpha)] = [\rho]$ where ρ is some principal property of formulas. Let $\varphi_i(\overline{x}_i, \overline{y}_i)$ ($i < \omega$) be the formulas of EQV. For each $i < \omega$ and $n_0 < \ldots < n_{j-1} < \omega$ there exists $n_j > n_{j-1}$ such that $\varphi(\overline{x}_i, \overline{y}_i)$ omits ρ in EQV (β) whenever $\beta \cap (n_{j-1}, n_j) = \phi$. To prove this note that $\varphi_i(\overline{x}_i, \overline{y}_i)$ omits ρ in EQV ($\{n_0, \ldots, n_{j-1}\}$) since EQV ($\{n_0, \ldots, n_{j-1}\}$) omits [fcp] and $[fcp] \leq [fcp(\alpha)] = [\rho]$. Hence $\varphi_i(\overline{x}_i, \overline{y}_i)$ omits $\rho(k)$ in EQV ($\{n_0, \ldots, n_{j-1}\}$) for some $k < \omega$. By the compactness theorem there exists $n_j > n_{j-1}$ such that $\varphi_i(\overline{x}_i, \overline{y}_i)$ omits $\rho(k)$ in EQV(β) whenever $\beta \cap (n_{j-1}, n_j) = \phi$. But then $\varphi_i(\overline{x}_i, \overline{y}_i)$ omits ρ in EQV(β) whenever $\beta \cap (n_{j-1}, n_j) = \phi$ since ρ is principal. From this it follows easily that $n_0 < n_1 < \ldots < n_{i-1} < n_i < \ldots < \omega$ in α may be chosen so that for each $i < \omega$ $\varphi_i(\overline{x}_i, \overline{y}_i)$ omits ρ in EQV(β) whenever $\beta \cap (n_{i-1}, n_i) = \phi$. Let $\beta = \{n_i \mid i < \omega\} \subset \alpha$. Then for each $i < \omega$ $\varphi_i(\overline{x}_i, \overline{y}_i)$ omits ρ in EQV(β) so EQV(β) omits $[\rho]$. But EQV(β) admits $[fcp(\alpha)] = [\rho]$ and this is a contradiction.

Theorem 6

The poset of subsets of ω (modulo finite sets) may be embedded into $(\mathcal{P} - \mathcal{PP}, \leq)$.

Proof

Let S be a thin set for fcp and let $\mathcal{S}$ be the poset of subsets of S (modulo finite sets). It suffices to show that $\mathcal{S}$ may be embedded into $(\mathcal{P} - \mathcal{PP}, \leq)$. For each $\alpha \subset S$ let $f(\alpha) = [fcp\ (\omega - \alpha)]$ (where $f(\alpha) = [fcp]$ if $|\omega - \alpha| < \aleph_0$) and note that $f(\beta) \leq f(\alpha)$ iff $|\beta - \alpha| < \aleph_0$ $(\alpha, \beta \subset S)$. By Lemma 6 $f(\alpha) \in \mathcal{P} - \mathcal{PP}$ $(\alpha \subset S)$. Thus f induces an embedding $f|_{\equiv}: \mathcal{S} \to (\mathcal{P}-\mathcal{PP}, \leq)$.

Corollary 4

$|\mathcal{P} - \mathcal{PP}| = 2^{\aleph_0}$.

Lemma 7

If $\varphi(\overline{x})$ is a formula of EQV there exist formulas $\alpha_i(\overline{x})$, $\beta_i(\overline{x})$, $\gamma_i(\overline{x})$ of EQV $(i < n)$ and sentences δ_i of EQV $(i < n)$ such that

$$\text{EQV} \vdash \varphi(\overline{x}) \leftrightarrow \bigvee_{i<n} (\alpha_i(\overline{x}) \wedge \beta_i(\overline{x}) \wedge \gamma_i(\overline{x}) \wedge \delta_i)$$ and each

$\alpha_i(\overline{x})$ states which variables occurring in $\overline{x}$ are equal, each

$\beta_i(\overline{x})$ states which variables occurring in $\overline{x}$ are equivalent, each

$\gamma_i(\overline{x})$ states that each variable occurring in $\overline{x}$ is either contained in an equivalence class of some given finite cardinality or is not contained in an equivalence class of any cardinality among a given finite number of finite cardinalities and each δ_i is (S_i, T_i) for some $S_i, T_i \subset \omega$.

<u>Proof</u>

Similar to the proof of Lemma 4.

If $\varphi(\overline{x},\overline{y})$ is a formula of EQV let $\rho(\varphi(\overline{x},\overline{y})) < \omega$ be the smallest cardinality greater than all the cardinalities occurring in the formula $\bigvee_{i<n} (\alpha_i(\overline{x},\overline{y}) \wedge \beta_i(\overline{x},\overline{y}) \wedge \gamma_i(\overline{x},\overline{y}) \wedge \delta_i)$ of EQV given by Lemma 7. If $A, B \models$ EQV and $m < \omega$ let $A \subseteq_m B$ denote that B may be obtained from A by adding equivalence classes of cardinalities $\geq m$ to A.

<u>Lemma 8</u>

If $\varphi(\overline{x},\overline{y})$ is a formula of EQV and $A \subset_{\rho(\varphi(\overline{x},\overline{y}))} B$ are models of EQV then $\rho(\varphi(\overline{x},\overline{y}), \text{Th}A) \subset \rho(\varphi(\overline{x},\overline{y}), \text{Th}B)$.

<u>Proof</u>

Suppose $\varphi(\overline{x},\overline{y})$ is a formula of EQV and $A \subset_{\rho(\varphi(\overline{x},\overline{y}))} B$ are models of EQV. Let the formula $\bigvee_{i<n} (\alpha_i(\overline{x},\overline{y}) \wedge \beta_i(\overline{x},\overline{y}) \wedge \gamma_i(\overline{x},\overline{y}) \wedge \delta_i)$ of EQV be given by Lemma 7. Thus

$\text{EQV} \vdash \varphi(\overline{x},\overline{y}) \leftrightarrow \bigvee_{i<n} (\alpha_i(\overline{x},\overline{y}) \wedge \beta_i(\overline{x},\overline{y}) \wedge \gamma_i(\overline{x},\overline{y}) \wedge \delta_i)$. Let $\overline{a}_0, \ldots, \overline{a}_{m-1} \in |A|^{\ell(\overline{y})}$. If $\overline{b} \in |A|^{\ell(\overline{x})}$ it follows easily that

$A \models \varphi(\bar{b},\bar{a}_i)$ iff $B \models \varphi(\bar{b},\bar{a}_i)$ $(i < m)$. Similarly if $\bar{b} \in |B|^{\ell(\bar{x})}$ and $\bar{c} \in |A|^{\ell(\bar{x})}$ is obtained from $\bar{b}$ by replacing distinct, equivalent or inequivalent constants occurring in $\bar{b}$ which are not contained in $|A|$ with distinct, equivalent or inequivalent constants (respectively) contained in equivalence classes of A of cardinality $\geq p(\varphi(\bar{x},\bar{y}))$ which contain no constant occurring in $\bar{a}_0,\ldots,\bar{a}_{m-1}$ it follows easily that $A \models \varphi(\bar{c},\bar{a}_i)$ iff $B \models \varphi(\bar{b},\bar{a}_i)$ $(i < m)$. Hence the $\varphi(\bar{x},\bar{y})$-definable subsets $\varphi_A(\bar{x},\bar{a}_i)$ $(i < m)$ of $|A|^{\ell(\bar{x})}$ have the same nonempty Boolean combinations in $|A|^{\ell(\bar{x})}$ as the corresponding $\varphi(\bar{x},\bar{y})$-definable subsets $\varphi_B(\bar{x},\bar{a}_i)$ $(i < m)$ of $|B|^{\ell(\bar{x})}$ have in $|B|^{\ell(\bar{x})}$. From this it follows that $\rho(\varphi(\bar{x},\bar{y}), \text{Th}A) \subset \rho(\varphi(\bar{x},\bar{y}), \text{Th}B)$.

<u>Lemma 9</u>

If $S,T \subset \omega$ and $|S - T| < \aleph_0$ then $EQV(S) \blacktriangleleft EQV(T)$.

<u>Proof</u>

Suppose $S,T \subset \omega$ and $|S - T| < \aleph_0$. Let $\varphi(\bar{x},\bar{y})$ be a formula of EQV. It suffices to prove that $\rho(\varphi(\bar{x},\bar{y}), EQV(S)) \subset \rho(\psi(\bar{z},\bar{w}), EQV(T))$ for some formula $\psi(\bar{z},\bar{w})$ of EQV. By Lemma 8 $\rho(\varphi(\bar{x},\bar{y}), EQV(S)) \subset \rho(\varphi(\bar{x},\bar{y}), EQV(S \cup (T - p(\varphi(\bar{x},\bar{y})))))$. But $EQV(S \cup (T-p(\varphi(\bar{x},\bar{y})))) \equiv EQV(T)$ since $|(S \cup (T-p(\varphi(\bar{x},\bar{y}))) \,\Delta\, T| < \aleph_0$ so $\chi(\bar{z},\bar{w})$ admits $\rho(\varphi(\bar{x},\bar{y}), EQV(S \cup (T-p(\varphi(\bar{x},\bar{y})))))$ in EQV(T) for some formula $\psi(\bar{z},\bar{w})$ of EQV. Hence $\rho(\varphi(\bar{x},\bar{y}), EQV(S)) \subset \rho(\psi(\bar{z},\bar{w}), EQV(T))$.

<u>Theorem 7</u>

The poset of subsets of ω (modulo finite sets) may be embedded into $(T|_{\equiv}, \blacktriangleleft|_{\equiv})$.

Proof

Let S be a thin set for fcp and let $\mathcal{S}$ be the poset of subsets of S (modulo finite sets). It suffices to show that $\mathcal{S}$ may be embedded into $(\mathcal{T}|_{\equiv}, \blacktriangleleft|_{\equiv})$. For each $\alpha \subset S$ let $f(\alpha) = EQV(\alpha)$ and note that $f(\beta) \blacktriangleleft f(\alpha)$ iff $|\beta - \alpha| < \aleph_0$ $(\alpha, \beta \subset S)$ by Lemma 9 and the fact that S is thin for fcp . Thus f induces an embedding $f|_{\equiv} : \mathcal{S} \to (\mathcal{T}|_{\equiv}, <|_{\equiv})$.

Corollary 5

$|(\mathcal{T}|_{\equiv}, \blacktriangleleft|_{\equiv})| = 2^{\aleph_0}$.

Lemma 10

If $\varphi(\overline{x},\overline{y})$ is a formula of EQV, ψ is an open formula of BA and $S \subset p(\varphi(\overline{x},\overline{y}))$ there exist finite $S_i \subset \omega - p(\varphi(\overline{x},\overline{y}))$ $(i < \omega)$ such that the following holds: If $T \cap p(\varphi(\overline{x},\overline{y})) = S$ then $\varphi(\overline{x},\overline{y})$ admits ψ in EQV(T) iff $\exists i(S_i \subset T)$.

Proof

Suppose $\varphi(\overline{x},\overline{y})$ is a formula of EQV, ψ is an open formula of BA and $S \subset p(\varphi(\overline{x},\overline{y}))$. Let $(\varphi(\overline{x},\overline{y}),\psi)$ be a sentence of EQV which asserts that $\varphi(\overline{x},\overline{y})$ admits ψ . By Lemma 4 there exist sentences (S_i,T_i) of EQV $(i < n)$ such that

$EQV(S, p(\varphi(\overline{x},\overline{y}))-S) \vdash (\varphi(\overline{x},\overline{y}),\psi) \leftrightarrow \bigvee_{i<n} (S_i,T_i)$. Clearly it may be assumed that $(\bigcup_{i<n} (S_i \cup T_i)) \cap p(\varphi(\overline{x},\overline{y})) = \emptyset$. By Lemma 8 it may also be assumed that $T_i = \emptyset$ $(i < \omega)$. Thus if T ~~$\subset \omega$~~ and $T \cap p(\varphi(\overline{x},\overline{y})) = S$ then $\varphi(\overline{x},\overline{y})$ admits ψ in EQV(T) iff $\exists i(S_i \subset T)$.

Lemma 11

Suppose $S_0, S_1 \subset \omega$, $|S_0| = |S_1| = \aleph_0$ and $\pi \in P$. If (1) EQV(S) admits π whenever $S \subset S_0 \cup S_1$ and $|S \cap S_0| = |S \cap S_1| = \aleph_0$ then (2) either EQV(S_0) admits π or EQV(S_1) admits π.

Proof

Suppose $S_0, S_1 \subset \omega$, $|S_0| = |S_1| = \aleph_0$, $\pi \in P$, (1) holds and (2) fails. Let $\pi = [\rho]$ and let $\varphi_i(\bar{x}_i, \bar{y}_i)$ $(i < \omega)$ be the formulas of EQV. To obtain a contradiction it suffices to prove that each $\varphi_i(\bar{x}_i, \bar{y}_i)$ omits ρ in EQV(S) for some $S \subset S_0 \cup S_1$ such that $|S \cap S_0| = |S \cap S_1| = \aleph_0$. Note that if $p(\varphi_0(\bar{x}_0, \bar{y}_0)) < n$ there exist infinite $S_2 \subset S_0$, $S_3 \subset S_1$ such that $(S_2 \cup S_3) \cap n = (S_0 \cup S_1) \cap n$ and such that $\varphi_0(\bar{x}_0, \bar{y}_0)$ omits ρ in EQV(T) whenever $T \subset S_2 \cup S_3$ and $T \cap n = (S_0 \cup S_1) \cap n$. To prove this first note by Lemma 10 that for each i there exist finite $S_{ij} \subset \omega - p(\varphi_0(\bar{x}_0, \bar{y}_0))$ $(j < n_i)$ such that the following holds: If $T \subset \omega$ and $T \cap p(\varphi_0(\bar{x}_0, \bar{y}_0)) = (S_0 \cup S_1) \cap p(\varphi_0(\bar{x}_0, \bar{y}_0))$ then $\varphi_0(\bar{x}_0, \bar{y}_0)$ admits $\rho(i)$ in EQV(T) iff $\exists j(S_{ij} \subset T)$. If $\varphi_0(\bar{x}_0, \bar{y}_0)$ omits ρ in EQV($S_0 \cup S_1$) simply let $S_2 = S_0$, $S_3 = S_1$ (use Lemma 9). Hence assume that $\varphi_0(\bar{x}_0, \bar{y}_0)$ admits ρ in EQV($S_0 \cup S_1$). For notational convenience assume that $\varphi_0(\bar{x}_0, \bar{y}_0)$ admits each $\rho(i)$ in EQV($S_0 \cup S_1$). Note that $S_{ij} \subset S_0 \cup S_1$ implies $S_{ij} \cap S_0 \neq \phi$ and

$S_{ij} \cap S_i \neq \phi \ (i < n_i)$ for sufficiently large $i < \omega$ because otherwise it follows easily that (2) holds. For such $i < \omega$ let

$$f_0(i) = \min_{\substack{j < n_i \\ S_{ij} \subset S_0 \cup S_1}} (\max(S_{ij} \cap S_0))$$

$$f_0(i) = \min_{\substack{j < n_i \\ S_{ij} \subset S_0 \cup S_1}} (\max(S_{ij} \cap S_1))$$

and note that $\lim_{i \to \omega} f_0(i) = \lim_{i \to \omega} f_1(i) = \omega$ because otherwise it follows easily that (2) holds. If $m_0 < m_1 < \ldots < \omega$ let

$$T_{2\ell} = \{\max(S_{ij} \cap S_0) \mid m_{2\ell-1} \leq i < m_{2\ell}, j < n_i, \ S_{ij} \subset S_0 \cup S_1, \ S_{ij} \cap S_0 \not\subset n\}$$

$$T_{2\ell+1} = \{\max(S_{ij} \cap S_1) \mid m_{2\ell} \leq i < m_{2\ell+1}, j < n_i, \ S_{ij} \subset S_0 \cup S_1, \ S_{ij} \cap S_1 \not\subset n\}$$

$$S_2 = S_0 - \bigcup_{\ell<\omega} T_{2\ell}$$

$$S_3 = S_1 - \bigcup_{\ell<\omega} T_{2\ell+1}$$

and note that $(S_2 \cup S_3) \cap n = (S_0 \cup S_1) \cap n$ and $\varphi(\overline{x}_0,\overline{y}_0)$ omits ρ in EQV(T) whenever $T \subset S_2 \cup S_3$ and $T \cap n = (S_0 \cup S_1) \cap n$. Thus choose $m_0 < m_1 < \ldots < \omega$ so that $|S_2| = |S_3| = \aleph_0$ (this is possible since $\lim_{i<\omega} f_0(i) = \lim_{i<\omega} f_1(i) = \omega$). Note that EQV(T) admits π whenever $T \subset S_2 \cup S_3$ and $|T \cap S_2| = |T \cap S_3| = \aleph_0$ but neither $EQV(S_2)$ admits π nor $EQV(S_3)$ admits π (use Lemma 9). Hence the above argument may be repeated with S_0, S_1 replaced by S_2, S_3 (respectively) and $\varphi_0(\overline{x}_0,\overline{y}_0)$ replaced by $\varphi_1(\overline{x}_1,\overline{y}_1)$. Continuing this way ω times yields infinite $\omega \supset S_0 \supset S_2 \supset \ldots$ and infinite $\omega \supset S_1 \supset S_3 \ldots$ and $n(\varphi_0(\overline{x}_0,\overline{y}_0)) < n(\varphi_1(\overline{x}_1,\overline{y}_1)) < \ldots < \omega$ such that the following holds: For each $i < \omega$ $S_{2i} \cap (n(\varphi_{i-1}(\overline{x}_{i-1},\overline{y}_{i-1})), n(\varphi_i(\overline{x}_i,\overline{y}_i))) \neq \phi$, $S_{2i+1} \cap (n(\varphi_{i-1}(\overline{x}_{i-1},\overline{y}_{i-1})), n(\varphi_i(\overline{x}_i,\overline{y}_i))) \neq \phi$, $(S_{2i+2} \cup S_{2i+3}) \cap n(\varphi_i(\overline{x}_i,\overline{y}_i)) = (S_{2i} \cup S_{2i+1}) \cap n(\varphi_i(\overline{x}_i,\overline{y}_i))$ and $\varphi_i(\overline{x}_i,\overline{y}_i)$ omits ρ in EQV(T) whenever $T \subset S_{2i+2} \cup S_{2i+3}$ and $T \cap n(\varphi_i(\overline{x}_i,\overline{y}_i)) = (S_{2i} \cup S_{2i+1}) \cap n(\varphi_i(\overline{x}_i,\overline{y}_i))$. Letting $S = (\bigcap_{i<\omega} S_{2i}) \cup (\bigcap_{i<\omega} S_{2i+1}) \subset S_0 \cup S_1$ it follows easily that $|S \cap S_0| = |S \cap S_1| = \omega$ and each $\varphi_i(\overline{x}_i,\overline{y}_i)$ omits ρ in EQV(S).

<u>Theorem 8</u>

$(P,\leq)$ is not a lattice.

Proof

Let S be a thin set for fcp and let $|\alpha - \beta| = |\beta - \alpha| = \aleph_0$ for some $\alpha, \beta \subset S$. It suffices to show that $[\mathrm{fcp}(\alpha)] \vee [\mathrm{fcp}(\beta)]$ does not exist. Suppose not. Then $[\mathrm{fcp}(\alpha)] \vee [\mathrm{fcp}(\beta)] = \pi$ for some $\pi \in \mathcal{P}$. Clearly $\mathrm{EQV}(\gamma)$ admits $\pi = [\mathrm{fcp}(\alpha)] \vee [\mathrm{fcp}(\beta)]$ whenever $\gamma \subset \alpha \cup \beta$ and $|\gamma \cap \alpha| = |\gamma \cap \beta| = \aleph_0$ (since $\mathrm{EQV}(\gamma)$ admits both $[\mathrm{fcp}(\alpha)]$ and $[\mathrm{fcp}(\beta)]$) yet both $\mathrm{EQV}(\alpha)$ and $\mathrm{EQV}(\beta)$ omit π (since $\mathrm{EQV}(\alpha)$ omits $[\mathrm{fcp}(\beta)]$ and $\mathrm{EQV}(\beta)$ omits $[\mathrm{fcp}(\alpha)]$). By Lemma 11 this is a contradiction.

Let pp be a property of formulas such that if $\varphi(\overline{x},\overline{y})$ is a formula of a complete theory T then $\varphi(\overline{x},\overline{y})$ admits pp in T iff for arbitrarily large $n < \omega$ there exists $A \models T$ and n $\varphi(\overline{x},\overline{y})$-definable subsets of $|A|^{\ell(\overline{x})}$ which partition $|A|^{\ell(\overline{x})}$. In fact let pp(n) be $(x_0 \cup \ldots \cup x_{n-1} = 1) \wedge (\bigwedge_{i<n} x_i \neq 0) \wedge (\bigwedge_{i<j<n} x_i \cap x_j = 0)$ for every $n < \omega$. Note that Example 9 and the results which follow it remain true if fcp is replaced by pp. In particular $[\mathrm{pp}] \notin \mathcal{PP}$.

Theorem 9

pp is 1-dimensional.

Proof

Suppose $\varphi(\overline{x},\overline{y})$ is a formula of a complete theory T which admits pp in T. It suffices to show that some formula $\psi(\overline{z},\overline{w})$ of T admits pp in T. For notational convenience assume that $\overline{x} = x_0 {}^\frown x_1$. Thus $\varphi(x_0 {}^\frown x_1, \overline{y})$ admits pp in T. By the compactness theorem there exists $A \models T$ such that for arbitrarily large

$n < \omega$ there exist n $\varphi(x_0{}^\frown x_1,\bar{y})$-definable subsets A_{ni} $(i < n)$ of $|A| \times |A|$ which partition $|A| \times |A|$. For such $n < \omega$ let $g(n) = \max\{|S| \mid \bigcap_{i\in S} f_0(A_{ni}) \neq \phi\}$ where $f_i : |A| \times |A| \to |A|$ is the i-th projection of $|A| \times |A|$ onto $|A|$ $(i < 2)$. Thus either (1) $\sup\{g(n) \mid n < \omega\} = \omega$ or (2) $\sup\{g(n) \mid n < \omega\} = m < \omega$ for some $m < \omega$. If (1) holds let $\psi(z,\bar{w})$ be $\varphi(x_0{}^\frown x_1,\bar{y})$ where $z = x_1$ and $\bar{w} = x_0{}^\frown \bar{y}$. Since $f_1|_{f_0^{-1}(a_0)}$ preserves disjointness and unions whenever $a_0 \in |A|$ it follows easily that $\psi(z,\bar{w})$ admits pp in T. If (2) holds let $\psi(z,\bar{w})$ be $(\bigwedge_{i<m} \exists x_1 \varphi(x_0{}^\frown x_1,\bar{y}_1)) \wedge (\forall x_1 \bigvee_{i<m} \varphi(x_0{}^\frown x_1,\bar{y}_1))$ where $z = x_0$ and $\bar{w} = \bar{y}_0{}^\frown \ldots {}^\frown \bar{y}_{m-1}$. It suffices to prove that $\psi(z,\bar{w})$ admits pp in T. For each $n < \omega$ such that $g(n)$ is defined let $\underset{n}{\sim}$ be the equivalence relation on $|A|$ defined as follows:

$a_0 \underset{n}{\sim} a_0'$ iff $\forall i(a_0 \in f_0(A_{ni}) \leftrightarrow a_0' \in f_0(A_{ni}))$ $(a_0, a_0' \in |A|)$. For such $n < \omega$ let $h(n) < \omega$ be the number of equivalence classes of $\underset{n}{\sim}$. Evidently $n \leq m\,h(n)$ for every $n < \omega$ such that $h(n)$ is defined. Thus $\sup\{h(n) \mid n<\omega\} = \omega$. But for each $n < \omega$ such that $h(n)$ is defined the equivalence classes of $\underset{n}{\sim}$ are $\psi(z,\bar{w})$-definable. Hence $\psi(z,\bar{w})$ admits pp in T.

Corollary 6

[pp] is prime.

Proof

Similar to the proof of Corollary 2.

Example 10

Let L be a language consisting of a unary predicate P and a binary predicate E and let IND be the theory in L whose axioms are

$$E(x,y) \to P(x) \wedge \neg P(y)$$

$$\exists x_0 \exists x_1 (x_0 \neq x_1 \wedge P(x_0) \wedge P(x_1))$$

$$(\bigwedge_{i<2n} P(x_i)) \wedge (\bigwedge_{i,j<n} x_i \neq x_{n+j}) \to \exists y \bigwedge_{i<n} (E(x_i,y) \wedge \neg E(x_{n+i},y))\ (n < \omega)$$

$$(\bigwedge_{i<2n} \neg P(y_i)) \wedge (\bigwedge_{i,j<n} y_i \neq y_{n+i}) \to \exists x \bigwedge_{i<n} (E(x,y_i) \wedge \neg E(x,y_{n+i}))\ (n < \omega)$$

It may be proved that IND is complete, $\aleph_0$-categorical and quantifier-eliminable by using the partial isomorphism test. It may be also proved that IND omits [pp]. Suppose not. Then by Theorem 9 some formula $\varphi(x,\bar{y})$ of IND admits pp in IND. Let $A \models$ IND. Then for arbitrarily large $n < \omega$ there exist n $\varphi(x,\bar{y})$-definable subsets of $|A|$ which partition $|A|$. Since IND is $\aleph_0$-categorical it follows easily by Ryll-Nardzewski (1959) that

(1) For every $n < \omega$ there exist n infinite, disjoint, $\varphi(x,\bar{y})$-definable subsets of $|A|$

holds. But there exist complete formulas $\varphi_i(x,\bar{y})$ $(i < n)$ of IND such that IND $\vdash \varphi(x,\bar{y}) \leftrightarrow \bigwedge_{i<n} \varphi_i(x,\bar{y})$ since IND is $\aleph_0$-categorical. Hence $\varphi_i(x,\bar{y})$ satisfies (1) for some $i < n$. Assume that

IND $\vdash \varphi_i(x,\bar{y}) \to P(x)$ (a similar argument holds if IND $\vdash \varphi_i(x,y) \to \neg P(x)$). Since IND is quantifier-eliminable every $\varphi_i(x,\bar{y})$-definable subset of $|A|$ is $\psi(z,\bar{w})$-definable where $\psi(z,\bar{w})$ is the formula

$$\bigwedge_{i<m} (E(z,w_i) \wedge \neg E(z,w_{m+i}) \wedge z \neq w_{2m+i} \wedge \neg P(w_i) \wedge \neg P(w_{m+i}) \wedge P(w_{2m+i}))$$

of IND for some $m < \omega$. Hence $\psi(z,\bar{w})$ satisfies (1). By the compactness theorem it may be assumed that there exist $\bar{a}_i \in |A|^{3m}$ $(i < \omega)$ such that the $\psi(z,\bar{w})$-definable subsets $\psi_A(z,\bar{a}_i)$ of $|A|$ $(i < \omega)$ are infinite and disjoint. Since IND is $\aleph_0$-categorical it may be assumed by Ramsey's theorem that $t_A(\bar{a}_{i_0} \cap \bar{a}_{i_1}) = t_A(\bar{a}_{i_2} \cap \bar{a}_{i_3})$ whenever $i_0 < i_1 < \omega$ and $i_2 < i_3 < \omega$. Since $\psi_A(z,\bar{a}_0)$ and $\psi_A(z,\bar{a}_1)$ are disjoint it follows easily that there exist $k < m \leq \ell < 2m-1$ such that either $\bar{a}_0(k) = \bar{a}_1(\ell)$ or $\bar{a}_1(k) = \bar{a}_0(\ell)$. Assume that $\bar{a}_0(k) = \bar{a}_1(\ell)$ (a similar argument holds if $\bar{a}_1(k) = \bar{a}_0(\ell)$). Then $\bar{a}_0(k) = \bar{a}_2(\ell)$ and $\bar{a}_1(k) = \bar{a}_2(\ell)$. Hence $\bar{a}_1(k) = \bar{a}_1(\ell)$ so $\psi_A(z,\bar{a}_1)$ is empty and this is a contradiction.

Note that the proof in Example 10 shows that for each formula $\varphi(x,\bar{y})$ of IND there exists $p(\varphi(x,\bar{y})) < \omega$ such that if $A \models$ IND then (1) if A is a finite $\varphi(x,\bar{y})$-definable subset of $|A|$ then $|A| < p(\varphi(x,\bar{y}))$ and (2) if $A_0,\ldots,A_{n-1}$ are infinite disjoint $\varphi(x,\bar{y})$-definable subsets of $|A|$ then $n < p(\varphi(x,\bar{y}))$.

The following set-theoretical result may be used to show that certain complete theories omit [pp] (see Example 13). Let X be a set, F a set of subsets of X and $\bar{F}$ the Boolean closure of F in

$\mathcal{P}(X)$. The complexity of each $A \in \overline{F}$ is the smallest number of members of F needed to generate A. If there exists a partition $F_0, \ldots, F_{n-1}$ of F such that

(S) If $A \in F_i$, $B \in F_j$ and $A \subsetneq B$ then $i > j$

(W) If $\mathcal{A}, \mathcal{B} \subset F$ are finite and $\phi \neq \cap \mathcal{A} \subset \cap \mathcal{B}$ then $A \subset B$ for some $A \in \mathcal{A}$, $B \in \mathcal{B}$

hold then F admits the stratified-Whitman property. If there exists $n < \omega$ such that for arbitrarily large $m < \omega$ there exists a partition of X into m members of $\overline{F}$ of complexity $< n$ then F admits the partition property.

Theorem 10

If F admits the stratified-Whitman property then F does not admit the partition property.

Proof

Suppose F admits the stratified-Whitman property. Then there exists a partition $F_0, \ldots, F_{n-1}$ of F such that (S) and (W) hold. By the following results it will follow that F does not admit the partition property. A basic set is any nonempty set of the form $\cap\mathcal{A} - \cup\mathcal{B}$ where $\mathcal{A}, \mathcal{B} \subset F$ are finite. Evidently

(1) A basic set has a unique irredundant form

holds. Indeed let $\cap\mathcal{A}_0 - \cup\mathcal{B}_0 = \cap\mathcal{A}_1 - \cup\mathcal{B}_1$ be irredundant forms of a basic set. It suffices to prove that $\mathcal{A}_0 = \mathcal{A}_1$ and $\mathcal{B}_0 = \mathcal{B}_1$. To prove that $\mathcal{A}_0 = \mathcal{A}_1$ it suffices to prove that $\mathcal{A}_0 \subset \mathcal{A}_1$ since the other case admits a similar argument. Let $A_0 \in \mathcal{A}_0$. It suffices to prove that $A_0 \in \mathcal{A}_1$. Since $\phi \neq \cap\mathcal{A}_1 \subset \cup(\mathcal{B}_1 \cup \{A_0\})$ it follows easily by (W) that $A_1 \subset A_0$ for some $A_1 \in \mathcal{A}_1$. Similarly $A_0' \subset A_1$ for some $A_0' \in \mathcal{A}_0$. By irredundancy $A_0' = A_0$ so $A_1 = A_0$. Hence $A_0 \in \mathcal{A}_1$. To prove that $\mathcal{B}_0 = \mathcal{B}_1$ it suffices to prove that $\mathcal{B}_0 \subset \mathcal{B}_1$ since the other case admits a similar argument. Let $B_0 \in \mathcal{B}_0$. It suffices to prove that $B_0 \in \mathcal{B}_1$. Since $\mathcal{A}_0 = \mathcal{A}_1$ it follows easily by irredundancy that $\phi \neq \cap(\mathcal{A}_0 \cup \{B_0\}) = \cap(\mathcal{A}_1 \cup \{B_0\}) \subset \cup\mathcal{B}_1$. It follows easily by (W) that $B_0 \subset B_1$ for some $B_1 \in \mathcal{B}_1$. Similarly $B_1 \subset B_0'$ for some $B_0' \in \mathcal{B}_0$. By irredundancy $B_0 = B_0'$ so $B_0 = B_1$. Hence $B_0 \in \mathcal{B}_1$. Using (1) the rank of a basic set of the irredundant form $\cap\mathcal{A} - \cup\mathcal{B}$ may be unambiguously defined as the finite sequence

$$(\underbrace{n-1,\ldots,n-1}_{i_{n-1}},\ \underbrace{n-2,\ldots,n-2}_{i_{n-2}},\ \ldots,\ \underbrace{0,\ldots,0}_{i_0}) \text{ where } i_j = |\mathcal{A} \cap F_j| \ (j < n).$$

By ordering these ranks lexicographically it follows easily that any set of ranks contains a least member. In what follows basic sets are always of the irredundant form. Next

(2) A basic set cannot be covered by finitely many basic sets of greater rank

holds. In fact suppose $\cap\mathcal{A} - \cup\mathcal{B} \subset \bigcup_{i<m} (\cap\mathcal{A}_i - \cup\mathcal{B}_i)$ are basic sets and rank $(\cap\mathcal{A} - \cup\mathcal{B}) <$ rank $(\cap\mathcal{A}_i - \cup\mathcal{B}_i)$ $(i < \omega)$. Using the definition of rank ordering, (S) and irredundancy choose $A_i \in \mathcal{A}_i$ for each $i < m$

so that $A \notin A_i$ $(A \in \mathcal{A})$. Since $\phi \neq \cap\mathcal{A} \subset \cup(\mathcal{B} \cup \{A_i \mid i < m\})$ it follows easily by (W) that $A \subset A_i$ for some $A \in \mathcal{A}$ and $i < m$. But this is a contradiction. It may be proved that

(3) If $\cap\mathcal{A}_0 - \cup\mathcal{B}_0 \subset \cap\mathcal{A}_1 - \cup\mathcal{B}_1$ are basic sets with equal rank then $\mathcal{A}_0 = \mathcal{A}_1$

(4) If $\cap\mathcal{A}_0 - \cup\mathcal{B}_0$, $\cap\mathcal{A}_1 - \cup\mathcal{B}_1$ are basic sets and $\mathcal{A}_0 = \mathcal{A}_1$ then $(\cap\mathcal{A}_0 - \cup\mathcal{B}_0) \cap (\cap\mathcal{A}_1 - \cup\mathcal{B}_1) \neq \phi$

hold. To prove (3) assume that $\mathcal{A}_0 \neq \mathcal{A}_1$. Since rank $(\cap\mathcal{A}_0 - \cup\mathcal{B}_0)$ = rank $(\cap\mathcal{A}_1 - \cup\mathcal{B}_1)$ there exists $i < n$ such that $\mathcal{A}_0 \cap F_j = \mathcal{A}_1 \cap F_j$ $(j > i)$ yet $\mathcal{A}_0 \cap F_i \neq \mathcal{A}_1 \cap F_i$. Choose $A_1 \in (\mathcal{A}_1 \cap F_i) - \mathcal{A}_0$. Since $\phi \neq \mathcal{A}_0 \subset \cup(\mathcal{B}_0 \cup \{A_1\})$ it follows easily by (W) that $A \subset A_1$ for some $A \in \mathcal{A}_0$. By (S) $A \in \mathcal{A}_0 \cap F_j$ for some $j > i$. Hence $A \in \mathcal{A}_1 \cap F_j$ but by irredundancy this is a contradiction. To prove (4) assume that $(\cap\mathcal{A}_0 - \cup\mathcal{B}_0) \cap (\cap\mathcal{A}_1 - \cup\mathcal{B}_1) = \phi$. Then $\phi \neq \cap\mathcal{A}_0 \subset \cup(\mathcal{B}_0 \cup \mathcal{B}_1)$ so by (W) it follows easily that $A \subset B$ for some $A \in \mathcal{A}_0$, $B \in \mathcal{B}_i$ and $i < 2$. But then $\cap\mathcal{A}_i - \cup\mathcal{B}_i = \phi$ (since $\mathcal{A}_0 = \mathcal{A}_1$) and this is a contradiction. Also

(5) The rank of a basic set partitioned into finitely many basic sets in equal to the rank of one member of the partition and smaller than the rank of the other members of the partition

holds. To prove (5) suppose that $\bigcap\mathcal{A} - \bigcup\mathcal{B}$ is a basic set partitioned into the basic sets $\bigcap\mathcal{A}_i - \bigcup\mathcal{B}_i$ $(i < m)$. From (2) it follows that rank $(\bigcap\mathcal{A} - \bigcup\mathcal{B}) \leq$ rank $(\bigcap\mathcal{A}_i - \bigcup\mathcal{B}_i)$ for every $i < m$ since $\bigcap\mathcal{A}_i - \bigcup\mathcal{B}_i \subset \bigcap\mathcal{A} - \bigcup\mathcal{B}$. From this and (2) it follows that rank $(\bigcap\mathcal{A} - \bigcup\mathcal{B}) =$ rank $(\bigcap\mathcal{A}_i - \bigcup\mathcal{B}_i)$ for at least one $i < m$ since $\bigcap\mathcal{A} - \bigcup\mathcal{B} \subset \bigcup_{i<m} (\bigcap\mathcal{A}_i - \bigcup\mathcal{B}_i)$. From (3) it follows that $\mathcal{A}_i = \mathcal{A}$ for such $i < m$ since $\bigcap\mathcal{A}_i - \bigcup\mathcal{B}_i \subset \bigcap\mathcal{A} - \bigcup\mathcal{B}$. But from this and (4) it follows that rank $(\bigcap\mathcal{A} - \bigcup\mathcal{B}) =$ rank $(\bigcap\mathcal{A}_i - \bigcup\mathcal{B}_i)$ for at most one $i < m$ since $(\bigcap\mathcal{A}_i - \bigcup\mathcal{B}_i) \cap (\bigcap\mathcal{A}_j - \bigcup\mathcal{B}_j) = \phi$ $(i < j < m)$.
Finally

(6) If $\bigcap\mathcal{A} - \bigcup\mathcal{B}$ is a basic set then complexity $(\bigcap\mathcal{A} - \bigcup\mathcal{B}) \geq |\mathcal{A}|$

holds. To prove (6) assume that complexity $(\bigcap\mathcal{A} - \bigcup\mathcal{B}) = m < |\mathcal{A}|$. Then $\bigcap\mathcal{A} - \bigcup\mathcal{B}$ is a Boolean combination of m members of $\mathcal{F}$ and so $\bigcap\mathcal{A} - \bigcup\mathcal{B}$ may be partitioned into finitely many basic sets each of which is a Boolean combination of $\leq m$ members of $\mathcal{F}$. In particular the rank of each such basic set is unequal to rank $(\bigcap\mathcal{A} - \bigcup\mathcal{B})$. But by (5) this is a contradiction. Now suppose $\mathcal{F}$ admits the partition property. Then it follows easily that there exists $n < \omega$ such that for arbitrarily large $m < \omega$ there exists a partition of X into m basic sets A_{mi} $(i < m)$ of complexity $< n$. From (6) it follows easily that $|\{\text{rank } (A_{mi}) \mid i < m < \omega\}| < \omega$. To obtain a contradiction it suffices to prove that $|\{\text{rank } (A_{mi}) \mid i < m < \omega\}| = \omega$. Since complexity $(A_{mi}) < n$ $(i < m < \omega)$ it follows easily that for each

$j < \omega$ there exists $f(j) < \omega$ such that if A_{mi} $(i < m)$ is one of the above partitions and $j < i$ then $X - \bigcup_{i<j} A_{mi}$ may be partitioned using at most $f(j)$ basic sets. It may be assumed that f is strictly increasing. Let $g(j) = 1$ if $j = 0$ and $g(j) = f(\sum_{i<j} g(i))$ if $j > 0$. Let $\ell < \omega$. Choose $m > \sum_{i<\ell} g(i)$ so that the partition A_{mi} $(i < m)$ is defined. It suffices to prove that $|\{\text{rank } (A_{mi}) \mid i < m\}| > \ell$. By (5) exactly one of the A_{mi} $(i < m)$ has smallest rank α_0. For notational convenience assume that rank $(A_{m0}) = \alpha_0$. Hence rank $(A_{mi}) > \alpha_0$ $(0 < i < m)$. But $X - A_{m0}$ may be partitioned using at most $f(1)$ basic sets B_{0j} $(j \in J_0)$ and if α_1 is the smallest rank of these basic sets then $\alpha_1 > \alpha_0$ by (5). Furthermore since $\bigcup_{j \in J_0} B_{0j} = \bigcup_{0<i<m} A_{mi}$ it follows easily by (5) that for each B_{0j} there exists exactly one of the A_{mi} such that rank $(A_{mi} \cap B_{0j})$ = rank (B_{0j}) and vice versa. From this it follows easily that there exist at most $f(1)$ basic sets A_{mi} $(i \in I_0)$ such that rank $(A_{mi}) = \alpha_1$ (and by (5) rank $(A_{mi}) > \alpha_1$ for the remaining A_{mi}). By replacing X with $X - (A_{m0} \cup (\bigcup_{i \in I_0} A_{mi}))$ this argument may be repeated ℓ times. Hence $|\{\text{rank } (A_{mi}) \mid i < m\}| > \ell$.

2.2 The Order, Strict Order and Independence Properties of Complete Theories

Let op be a property of formulas such that if $\varphi(\bar{x},\bar{y})$ is a formula of a complete theory T then $\varphi(\bar{x},\bar{y})$ admits op in T iff for arbitrarily large $n < \omega$ there exists $A \models T$ and $\varphi(\bar{x},\bar{y})$-definable subsets A_i of $|A|^{\ell(\bar{x})}$ $(i < n)$ such that $(\bigcap_{i<m} A_i^c) \cap (\bigcap_{m \le i < n} A_i) \neq \phi$ for every $m < n$. Let sop be a property of formulas such that if $\varphi(\bar{x},\bar{y})$ is a formula of a complete theory T then $\varphi(\bar{x},\bar{y})$ admits sop in T iff for arbitrarily large $n < \omega$ there exists $A \models T$ and $\varphi(\bar{x},\bar{y})$-definable subsets A_i of $|A|^{\ell(\bar{x})}$ $(i < n)$ such that $A_i \subsetneq A_{i+1}$ for every $i < n - 1$. Finally let ip be a property of formulas such that if $\varphi(\bar{x},\bar{y})$ is a formula of a complete theory T then $\varphi(\bar{x},\bar{y})$ admits ip in T iff for arbitrarily large $n < \omega$ there exists $A \models T$ and $\varphi(\bar{x},\bar{y})$-definable subsets A_i of $|A|^{\ell(\bar{x})}$ $(i < n)$ such that $\bigcap_{i<n} A_i^{\alpha(i)} \neq \phi$ for every $\alpha \in 2^n$. Obviously [op], [sop], [ip] $\in PP$. Shelah (1971) proved that op and ip are 1-dimensional. Lachlan (1975) proved that sop is 1-dimensional. Using these results it is easy to prove that [op], [sop] and [ip] are prime (see the proof of Corollary 2).

Example 11

Let L be a language consisting of a binary predicate $<$. Let PO be the theory in L whose axioms are

$$x \nless x$$

$$x < y < z \rightarrow x < z$$

and let DLO be the theory in L whose axioms are

$$PO$$

$$x < y \vee x = y \vee y < x$$

$$x < y \rightarrow \exists z\ (x < z < y)$$

$$\exists y \exists z\ (y < x < z).$$

It is well-known that DLO is complete, $\aleph_0$-categorical and quantifier-eliminable. Letting $\varphi(x,y)$ be $x < y$ it is clear that $\varphi(x,y)$ admits sop in DLO. Hence DLO admits [sop].

Lemma 12

If $\mathcal{A} \models$ DLO, $\mathcal{B} \models$ PO and $\mathcal{A} \subset \mathcal{B}$ then $Th\mathcal{A} \blacktriangleleft Th\mathcal{B}$.

Proof

Suppose $\mathcal{A} \models$ DLO, $\mathcal{B} \models$ PO and $\mathcal{A} \subseteq \mathcal{B}$. Let $\varphi(\overline{x},\overline{y})$ be a formula of $Th\mathcal{A}$. It suffices to prove that $\rho(\varphi(\overline{x},\overline{y}), Th\mathcal{A}) \subset \rho(\psi(\overline{x},\overline{y} \cap \overline{z}), Th\mathcal{B})$ for some formula $\psi(\overline{x},\overline{y} \cap \overline{z})$ of $Th\mathcal{B}$. Since $Th\mathcal{A}$ is quantifier-eliminable there exist atomic formulas (or their negations) $\varphi_{ij}(\overline{x},\overline{y})$ of $Th\mathcal{A}$ (i, j < n) such that $Th\mathcal{A} \vdash \varphi(\overline{x},\overline{y}) \leftrightarrow \bigvee_{i<n} \bigwedge_{j<n} \varphi_{ij}(\overline{x},\overline{y})$. If x occurs in $\overline{x}$ and y occurs in $\overline{y}$ it may be assumed that no $\varphi_{ij}(\overline{x},\overline{y})$ is of the form $y < x$ or $y \nless x$ (since $Th\mathcal{A} \vdash x < y \vee x = y \vee y < x$). Let $\chi(\overline{x})$ be a formula of $Th\mathcal{A}$ which asserts that some permutation of the variables occurring in $\overline{x}$ is (not necessarily strictly) increasing. Obviously $\chi_{\mathcal{A}} = |\mathcal{A}|^{\ell(\overline{x})} \subset \chi_{\mathcal{B}} \subset |\mathcal{B}|^{\ell(\overline{x})}$ since $\mathcal{A} \models$ DLO and $\mathcal{A} \subset \mathcal{B}$. Let $\psi(\overline{x},\overline{y} \cap \overline{z})$ be the formula $((\bigvee_{i<n} \bigwedge_{j<n} \varphi_{ij}(\overline{x},\overline{y}) \wedge \chi(\overline{x})) \vee (\neg\chi(\overline{x}) \wedge z_0 = z_1)$.

of $Th\mathcal{B}$. It suffices to prove that $\rho(\varphi(\bar{x},\bar{y}), Th\mathcal{A}) \subset \rho(\psi(\bar{x},\bar{y}^\frown\bar{z}), Th\mathcal{B})$.

Let $\bar{a}_k \in |\mathcal{A}|^{\ell(\bar{y})}$ $(k < m)$ and $\alpha \in 2^m$. It suffices to prove that

$$\mathcal{A} \models \exists\bar{x} \bigwedge_{k<m} (\bigvee_{i<n} \bigwedge_{j<n} \varphi_{ij}(\bar{x},\bar{a}_k))^{\alpha(k)}$$

$$\leftrightarrow$$

$$\mathcal{B} \models \exists\bar{x} ((\bigwedge_{k<m} (\bigvee_{i<n} \bigwedge_{j<n} \varphi_{ij}(\bar{x},\bar{a}_k))^{\alpha(k)}) \wedge \chi(\bar{x})) .$$

Suppose $\mathcal{B} \models \exists\bar{x} ((\bigwedge_{k<m} (\bigvee_{i<n} \bigwedge_{j<n} \varphi_{ij}(\bar{x},\bar{a}_k))^{\alpha(k)}) \wedge \chi(\bar{x}))$. It suffices to prove that $\mathcal{A} \models \exists\bar{x} \bigwedge_{k<m} (\bigvee_{i<n} \bigwedge_{j<n} \varphi_{ij}(\bar{x},\bar{a}_k))^{\alpha(k)}$ since the other implication is obvious. Choose $\bar{b} = (b_0,\ldots,b_{\ell(\bar{x})-1}) \in \chi_{\mathcal{B}}$ so that $\mathcal{B} \models \bigwedge_{k<m} (\bigvee_{i<n} \bigwedge_{j<n} \varphi_{ij}(\bar{b},\bar{a}_k))^{\alpha(k)}$. Let $\bar{a} = \bar{a}_0{}^\frown \ldots {}^\frown \bar{a}_{m-1} = (a_0,\ldots,a_{\ell-1})$. Since $\mathcal{A} \models DLO$ and $\mathcal{B} \models \chi(\bar{b})$ it follows that there exists $\bar{c} = (c_0,\ldots,c_{\ell(\bar{x})-1}) \in |\mathcal{A}|^{\ell(\bar{x})}$ such that

$$b_i = b_j \leftrightarrow c_i = c_j$$

$$b_i < b_j \leftrightarrow c_i < c_j$$

$$b_i = a_k \leftrightarrow c_i = a_k$$

$$b_i < a_k \leftrightarrow c_i < a_k$$

for every $i, j < \ell(\bar{x})$ and $k < \ell$. But since no $\varphi_{ij}(\bar{x},\bar{y})$ is of the form $y < x$ or $y \nless x$ it follows that $\mathcal{A} \models \bigwedge_{k<m} (\bigvee_{i<n} \bigwedge_{j<n} \varphi_{ij}(\bar{c},\bar{a}_k))^{\alpha(k)}$.

Hence $\mathcal{A} \models \exists \bar{x} \bigwedge_{k<m} (\bigvee_{i<n} \bigwedge_{j<n} \varphi_{ij}(\bar{x},\bar{a}_k))^{\alpha(k)}$.

Theorem 11

DLO is archetypal for [sop].

Proof

By the compactness theorem it is easy to prove that a complete theory T admits [sop] iff there exists $\mathcal{A} \models$ DLO, $\mathcal{B} \models$ PO and $\mathcal{C} \models T$ such that $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B}$ is definable in $\mathcal{C}$. From this and Lemma 12 it follows easily that DLO is archetypal for [sop].

Example 12

Let L be a language consisting of a unary predicate P and a binary predicate E and for each $n < \omega$ let IND(n) be the theory in L whose axioms are

$$E(x,y) \rightarrow P(x) \wedge P(y)$$

$$\exists x_0 \ldots \exists x_{m-1} \bigwedge_{i<j<m} (x_i \neq x_j \wedge P(x_i)) \qquad (m < \omega)$$

$$\exists y_0 \ldots \exists y_{m-1} \bigwedge_{i<j<m} (y_i \neq y_j \wedge \neg P(y_i)) \qquad (m < \omega)$$

$$(\bigwedge_{i<2m} P(x_i) \wedge (\bigwedge_{i,j<m} x_i \neq x_{m+j}) \rightarrow \exists y \bigwedge_{i<m} (E(x_i,y) \wedge \neg E(x_{m+i},y)) \qquad (m < n)$$

$$(\bigwedge_{i<2m} \neg P(y_i)) \wedge (\bigwedge_{i,j<m} y_i \neq y_{m+j}) \rightarrow \exists x \bigwedge_{i<m} (E(x,y_i) \wedge \neg E(x,y_{m+i})) \qquad (m < n)$$

Note that IND(ω) is IND so IND(ω) is complete, $\aleph_0$-categorical and quantifier-eliminable. Letting $\varphi(x,y)$ be $E(x,y)$ it is clear that $\varphi(x,y)$ admits ip in $IND(\omega)$. Hence $IND(\omega)$ admits [ip].

Lemma 13

If $\mathcal{A} \models IND(\omega)$, $\mathcal{B} \models IND(0)$ and $\mathcal{A} \subset \mathcal{B}$ then $Th\mathcal{A} \blacktriangleleft Th\mathcal{B}$.

Proof

Suppose $\mathcal{A} \models IND(\omega)$, $\mathcal{B} \models IND(0)$ and $\mathcal{A} \subset \mathcal{B}$. Let $\varphi(\bar{x},\bar{y})$ be a formula of $Th\mathcal{A}$. It suffices to prove that $\rho(\varphi(\bar{x},\bar{y}), Th\mathcal{A}) \subset \rho(\psi(\bar{x},\bar{y}), Th\mathcal{B})$ for some formula $\psi(\bar{x},\bar{y})$ of $Th\mathcal{B}$. Since $Th\mathcal{A}$ is quantifier-eliminable there exists an open formula $\psi(\bar{x},\bar{y})$ of $Th\mathcal{A}$ such that $Th\mathcal{A} \vdash \varphi(\bar{x},\bar{y}) \leftrightarrow \psi(\bar{x},\bar{y})$. It suffices to prove that $\rho(\psi(\bar{x},\bar{y}), Th\mathcal{A}) \subset \rho(\psi(\bar{x},\bar{y}), Th\mathcal{B})$ since $\rho(\varphi(\bar{x},\bar{y}), Th\mathcal{A}) = \rho(\psi(\bar{x},\bar{y}), Th\mathcal{A})$. Let $\bar{a}_i \in |\mathcal{A}|^{\ell(\bar{y})}$ $(i < n)$ and $\alpha \in 2^n$. It suffices to prove that

$$\mathcal{A} \models \exists\bar{x} \bigwedge_{i<n} \psi(\bar{x},\bar{a}_i)^{\alpha(i)}$$

$$\leftrightarrow$$

$$\mathcal{B} \models \exists\bar{x} \bigwedge_{i<n} \psi(\bar{x},\bar{a}_i)^{\alpha(i)} .$$

Suppose $\mathcal{B} \models \exists\bar{x} \bigwedge_{i<n} \psi(\bar{x},\bar{a}_i)^{\alpha(i)}$. It suffices to prove that $\mathcal{A} \models \exists\bar{x} \bigwedge_{i<n} \psi(\bar{x},\bar{a}_i)^{\alpha(i)}$ since the other implication is obvious. Choose $\bar{b} \in |\mathcal{B}|^{\ell(\bar{x})}$ so that $\mathcal{B} \models \bigwedge_{i<n} \psi(\bar{b},\bar{a}_i)^{\alpha(i)}$. Let $\bar{a} = \bar{a}_0 {}^\frown \cdots {}^\frown a_{\ell(\bar{y})-1}$. Since $\mathcal{A} \models IND(\omega)$ it follows easily that there exists $\bar{c} \in |\mathcal{A}|^{\ell(\bar{x})}$ such

that $t^0_{\mathcal{A}}(\bar{c} \frown \bar{a}) = t^0_{\mathcal{B}}(\bar{b} \frown \bar{a})$. In particular $\mathcal{A} \models \bigwedge_{i<n} \psi(\bar{c},\bar{a}_i)^{\alpha(i)}$.

Hence $\mathcal{A} \models \exists x \bigwedge_{i<n} \psi(\bar{x},\bar{a}_i)^{\alpha(i)}$.

<u>Theorem 12</u>

IND(ω) is archetypal for [ip].

<u>Proof</u>

By the compactness theorem it is easy to prove that a complete theory T admits [ip] iff there exists $\mathcal{A} \models$ IND(ω), $\mathcal{B} \models$ IND(0) and $\mathcal{C} \models$ T such that $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B}$ is definable in $\mathcal{C}$. From this and Lemma 13 it follows easily that IND(ω) is archetypal for [ip].

Shelah (1971) proved that [op] = [sop] ∧ [ip]. But both [sop] and [ip] are ∧-irreducible (since both are archetypal) so it follows easily that Shelah's result is optimal in the sense that if [op] $= \pi_0 \wedge \pi_1$ and [op] $\neq \pi_0, \pi_1$ then [op] $< \pi_i \leq$ [sop] and [op] $< \pi_{1-i} \leq$ [ip] for some $i < 2$.

2.3 The Strong Independence and Versatility Properties of Complete Theories

Let sip be a property of formulas such that if $\varphi(\bar{x},\bar{y})$ is a formula of a complete theory T then $\varphi(\bar{x},\bar{y})$ admits sip in T iff for arbitrarily large $n < \omega$ there exist $\mathcal{A} \models T$ and $\varphi(\bar{x},\bar{y})$-definable subsets A_{ij} of $|\mathcal{A}|^{\ell(\bar{x})}$ $(i,j<n)$ such that

$$A_{i_0 j} \cap A_{i_1 j} = \phi$$

for every $i_0 < i_n < n$ and $j < n$ and such that

$$\bigcap_{j<n} A_{\alpha(j)j} \neq \phi$$

for every $\alpha \in n^n$. Obviously [sip] $\in PP$.

Theorem 13

If T is a complete theory then the following hold:

(1) If $\varphi_k(\bar{x},\bar{y})$ $(k \leq n)$ are formulas of T and $\bigvee_{k<n} \varphi_k(\bar{x},\bar{y})$ admits sip in T then $\varphi_k(\bar{x},\bar{y})$ admits sip in T for some $k < n$.

(2) If $\varphi_k(\bar{x}_k,\bar{y})$ $(k < n)$ are formulas of T, $r(\bar{x}_k) \cap r(\bar{x}_{k'}) = \phi$ $(k < k' \leq n)$ and $\bigwedge_{k\ n} \varphi_k(\bar{x}_k,\bar{y})$ admits sip in T then $\varphi_k(\bar{x}_k,\bar{y})$ admits sip in T for some $k < n$.

Proof

Suppose the premise of (1) holds. For notational convenience assume that $n = 2$ and $\ell(\bar{x}) = \ell(\bar{y}) = 1$. Thus $\varphi_0(x,y) \vee \varphi_1(x,y)$ admits sip in T . By the compactness theorem there exist $\mathcal{A} \models T$ and $\varphi_0(x,y) \vee \varphi_1(x,y)$-definable subsets A_{ij} of $|\mathcal{A}|$ $(i,j<\omega)$ such that $A_{i_0 j} \cap A_{i_1 j} = \phi$ for every $i_0 < i_1 < \omega$ and $j < \omega$ and such that $\bigcap_{j<\omega} A_{\alpha(j)j} \neq \phi$ for every $\alpha \in \omega^\omega$. For each $i, j < \omega$ let $A_{ij} = A^0_{ij} \cup A^1_{ij}$ where A^k_{ij} is $\varphi_k(x,y)$-definable $(k < 2)$. By using the compactness and Ramsey theorems it may be assumed that for some $k < 2$ $\bigcap_{j<\omega} A^k_{\alpha(j)j} \neq \phi$ for every $\alpha \in \omega^\omega$. But then $\varphi_k(x,y)$ admits sip in T .

Suppose the premise of (2) holds. For notational convenience assume that $n = 2$ and $\ell(\bar{x}_0) = \ell(\bar{x}_1) = \ell(\bar{y}) = 1$. Thus $\varphi_0(x_0,y) \wedge \varphi_1(x_1,y)$ admits sip in T . By the compactness theorem there exists $\mathcal{A} \models T$ and $\varphi_0(x_0,y) \wedge \varphi_1(x_1,y)$-definable subsets A_{ij} of $|\mathcal{A}| \times |\mathcal{A}|$ $(i, j < \omega)$ such that $A_{i_0 j} \cap A_{i_1 j} = \phi$ for every $i_0 < i_1 < \omega$ and $j < \omega$ and such that $\bigcap_{j<\omega} A_{\alpha(j)j} \neq \phi$ for every $\alpha \in \omega^\omega$. For each $i, j < \omega$ let $A_{ij} = A^0_{ij} \times A^1_{ij}$ where A^k_{ij} is $\varphi_k(x_k,y)$-definable $(k < 2)$. By using the compactness and Ramsey theorems it may be assumed that some $k < 2$ $A^k_{i_0 j} \cap A^k_{i_1 j} = \phi$ for every $i_0 < i_1 < \omega$ and $j < \omega$. But then $\varphi_k(x_k,y)$ admits sip in T .

<u>Corollary 7</u>

[sip] is prime.

Proof

Suppose $T_j (j < 2)$ are complete theories and $\sum_{j<2} T_j$ admits [sip]. It suffices to prove that T_j admits [sip] for some $j < 2$. Since $\sum_{j<2} T_j$ admits [sip] some formula $\varphi(\overline{x},\overline{y})$ of $\sum_{j<2} T_j$ admits sip in $\sum_{j<2} T_j$. It is easy to prove that there exist formulas $\varphi_{ij}(\overline{x}_{ij},\overline{y}_{ij})$ of T_j $(i < n,\ j < 2)$ such that

$\sum_{j<2} T_j \vdash \varphi(\overline{x},\overline{y}) \leftrightarrow \bigvee_{i<n} \bigwedge_{j<2} \varphi^*_{ij}(\overline{x}_{ij},\overline{y}_{ij})$ where $\overline{x} = \overline{x}_{i0} \cup \overline{x}_{i1}$ $(i < n)$, $\overline{y} = \overline{y}_{i0} \cup \overline{y}_{i1}$ $(i < n)$ and $\varphi^*_{ij}(\overline{x}_{ij},\overline{y}_{ij})$ is

$\varphi_{ij}^{P_j}(\overline{x}_{ij},\overline{y}_{ij}) \wedge P_j(\overline{x}_{ij}) \wedge P_j(\overline{y}_{ij})$ $(i < n,\ j < 2)$. By Theorem 13 $\varphi^*_{ij}(\overline{x}_{ij},\overline{y}_{ij})$ admits sip in $\sum_{j<2} T_j$ for some $i < n$, $j < 2$. But then $\varphi_{ij}(\overline{x}_{ij},\overline{y}_{ij})$ admits sip in T_j. Hence T_j admits [sip].

Example 13

Let L be a language consisting of a unary predicate P, binary predicates $E, \sim$ and a ternary predicate D and let SIND be the theory in L whose axioms are

$$E(x,y) \to P(x) \wedge \neg P(y)$$

$$y_0 \sim y_1 \to \neg P(y_0) \wedge \neg P(y_1)$$

$$D(x_0,x_1,y_0) \leftrightarrow \exists y_1 (y_0 \sim y_1 \wedge E(x_0,y_1) \wedge E(x_1,y_1))$$

$$\neg P(y_0) \rightarrow y_0 \sim y_0$$

$$y_0 \sim y_1 \rightarrow y_1 \sim y_0$$

$$y_0 \sim y_1 \wedge y_1 \sim y_2 \rightarrow y_0 \sim y_2$$

$$\neg P(y) \rightarrow \exists y_0 \ldots \exists y_{n-1} \bigwedge_{i<j<n} (y \sim y_i \wedge y_i \not\equiv y_j) \qquad (n < \omega)$$

$$\exists y_0 \ldots \exists y_{n-1} \bigwedge_{i<j<n} (y_i \not\sim y_j \wedge \neg P(y_i)) \qquad (n < \omega)$$

$$y_0 \sim y_1 \wedge y_0 \not\equiv y_1 \rightarrow \neg \exists x (E(x, y_0) \wedge E(x, y_1))$$

$$\bigwedge_{i<j<n} (y_i \not\sim y_j \wedge \neg P(y_i)) \rightarrow \exists x \bigwedge_{i<n} E(x, y_i) \qquad (n < \omega)$$

$$P(x) \wedge \neg P(y_0) \rightarrow \exists y_1 (y_0 \sim y_1 \wedge E(x, y_1))$$

$$\bigwedge_{i<j<m} (x_i \not\equiv x_j \wedge P(x_i))$$

$$\rightarrow$$

$$\exists y_0 \ldots \exists y_{n-1} \bigwedge_{k<\ell<n} (y_k \not\sim y_\ell \wedge \neg P(y_k) \wedge (\bigwedge_{\substack{i,j<m \\ i \not\equiv j}} D(x_i, x_j, y_k)) \wedge (\bigwedge_{\substack{i,j<m \\ i \not\equiv j}} \neg D(x_i, x_j, y_k)))$$

$(m, n < \omega$

$\equiv$ is an equivalence relation on $m)$

To prove that SIND is consistent it suffices to build a model for it. For each $n < \omega$ and $f : n \rightarrow \omega$ let p_f be a distinct prime number and let $\overline{f} : \omega \rightarrow \omega$ be defined by

$$\overline{f}(i) = f(i) \quad \text{if} \quad i < n$$
$$= \text{the multiplicity of } p_f \text{ in the prime factorization of } i \text{ if } i \geq n .$$

Let $F = \{\overline{f} \mid f : n \to \omega\}$ and let $A = (|A|, P_A, E_A, \sim_A, D_A)$ be defined by

$$|A| = F \cup (\omega \times \omega)$$

$$P_A(a) \leftrightarrow a \in F$$

$$E_A(a,b) \leftrightarrow b \in a \in F$$

$$b_0 \underset{A}{\sim} b_1 \leftrightarrow \exists i \exists j_0 \exists j_1 (i, j_0, j_1 < \omega \wedge b_0 = (i, j_0) \wedge b_1 = (i, j_1))$$

$$D_A(a_0, a_1, b) \leftrightarrow a_0, a_1 \in F \wedge \exists i \exists j (i, j < \omega \wedge b = (i,j) \wedge a_0(i) = a_1(i)) .$$

Then $A \models$ SIND. It may be proved that SIND is complete, $\aleph_0$-categorical and quantifier-eliminable by using the partial isomorphism test. Letting $\varphi(x,y)$ be $E(x,y)$ it is clear that $\varphi(x,y)$ admits sip in SIND. Hence SIND admits [sip]. It may be proved that SIND omits [pp]. Suppose not. Then by Theorem 9 some formula $\varphi(x,\overline{y})$ of SIND admits pp in SIND. Let $A \models$ SIND. Then for arbitrarily large $n < \omega$ there exist n $\varphi(x,\overline{y})$-definable subsets of $|A|$ which partition $|A|$. But then either

(1) For arbitrarily large $n < \omega$ there exist n $\varphi(x,\overline{y}) \wedge P(x)$-definable subsets of P_A which partition P_A

or

(2) For arbitrarily large $n < \omega$ there exist n $\varphi(x,\bar{y})$-definable subsets of $|A| - P_A$ which partition $|A| - P_A$

holds. Suppose (1) holds. Since SIND is quantifier-eliminable there exist $m < \omega$ such that every $\varphi(x,\bar{y}) \wedge P(x)$-definable subset of P_A is a Boolean combination of at most m $P(x)$-definable, $x = y$-definable or $E(x,y)$-definable subsets of P_A. But the set of such subsets has the stratified-Whitman property so by Theorem 10 it does not admit the partition property and this is a contradiction. Suppose (2) holds. Since SIND is quantifier-eliminable it may be assumed that there exists $m < \omega$ such that every

$\varphi(x,\bar{y}) \wedge \neg P(x)$-definable subset of $|A| - P_A$ is $x = y$-definable, $\bigwedge_{i<m} (x \neq y_i \wedge x \sim y_i)$-definable or $\psi_{J,\equiv}(x,\bar{y} \frown \bar{z})$-definable (for some $J \subset m$ and equivalence relation $\equiv$ on m) where $\psi_{J,\equiv}(x,\bar{y} \frown \bar{z})$ is

$$\bigwedge_{i<m} (\neg P(x) \wedge x \nsim y_i \wedge (\bigwedge_{\substack{j<m \\ j\in J}} E(z_j,x)) \wedge (\bigwedge_{\substack{j<m \\ j\notin J}} \neg E(z_j,x)) \wedge (\bigwedge_{\substack{k,\ell<m \\ k\equiv\ell}} D(z_k,z_\ell,x)) \wedge (\bigwedge_{\substack{k,\ell<m \\ k\not\equiv\ell}} \neg D(z_k,z_\ell,x)))$$

whenever $J \subset m$ and $\equiv$ is an equivalence relation on m. Let $\mathcal{J} = \{(J,\equiv) \mid J \subset m$ and $\equiv$ is an equivalence relation on $m\}$. Clearly each $x = y$-definable subset of $|A| - P_A$ is a subset of cardinality 1 of some equivalence class and each $\bigwedge_{i<m} (x \neq y_i \wedge x \sim y_i)$-definable subset of $|A| - P_A$ is a subset of cocardinality $\leq m$ of some equivalence class. It is easy to prove

that each $\psi_{J,\equiv}(x,\bar{y} \cap \bar{z})$-definable subset of $|A|-P_A$ either intersects with cardinality 1 infinitely many equivalence classes (and does not intersect the remaining infinitely many other equivalence classes) or intersects with cocardinality $\leq m$ infinitely many equivalence classes (and does not intersect the remaining infinitely many other equivalence classes). Furthermore it is easy to prove that each subset of $|A|-P_A$ of the form $\bigwedge_{i<\ell} \psi^A_{J_i,\equiv_i}(x,\bar{a}_i \cap \bar{b}_i)$ either contains every equivalence class not intersecting $\bigcup_{i<\ell} (r(\bar{a}_i) \cup r(\bar{b}_i))$ or does not contain infinitely many equivalence classes not intersecting $\bigwedge_{i<\ell} (r(\bar{a}_i) \cup r(\bar{b}_i))$.

For each $(J,\equiv) \in \mathcal{J}$ it follows easily by Ramsey's theorem that there exist at most $\ell(J,\equiv) < \omega$ nonempty, disjoint, $\psi_{J,\equiv}(x,\bar{y} \cap \bar{z})$-definable subsets of $|A|-P_A$. Hence there exist at most $\ell = \sum_{(J,\equiv)\in\mathcal{J}} \ell(J,\equiv)$ nonempty, disjoint subsets of $|A|-P_A$ each of which is $\psi_{J,\equiv}(x,\bar{y} \cap \bar{z})$-definable for some $(J,\equiv) \in \mathcal{J}$. From this it follows that for some $k < \omega$ there exist at most k $\bigwedge_{i<m} (x = y_i \wedge x \sim y_i)$-definable subsets of $|A|-P_A$ in any $\varphi(x,\bar{y}) \wedge \neg P(x)$-definable partition of $|A|-P_A$ since such subsets must be contained in different equivalence classes. Similar reasoning shows that for sufficiently large $n < \omega$ every $\varphi(x,\bar{y}) \wedge \neg P(x)$-definable partition of $|A|-P_A$ into n sets must contain $> m$ $x = y$-definable subsets of $|A|-P_A$ which belong to the same equivalence class. But this is a contradiction since the remaining part of such equivalence classes cannot be partitioned using only finitely many $\bigwedge_{i<m} (x \neq y_i \wedge x \sim y_i)$-definable or $\psi_{J,\equiv}(x,\overline{y} \frown \overline{z})$-definable sets.

A <u>weak</u> <u>ideal</u> of a set J is a set $\mathcal{I}$ of subsets of J such that $\phi \neq I' \subset I \in \mathcal{I}$ implies $I' \in \mathcal{I}$. Let vp be a property of formulas such that if $\varphi(\overline{x},\overline{y})$ is a formula of a complete theory T then $\varphi(\overline{x},\overline{y})$ admits vp in T iff for arbitrarily large $n < \omega$ and every weak ideal $\mathcal{I}$ of n there exists $\mathcal{A} \models T$ and $\varphi(\overline{x},\overline{y})$-definable subsets A_i of $|\mathcal{A}|^{\ell(\overline{x})} (i < n)$ such that

$$(\bigwedge_{I\in\mathcal{I}} \bigcap_{i\in I} A_i \neq \phi) \wedge (\bigwedge_{I\in P(n)-\mathcal{I}} \bigcap_{i\in I} A_i = \phi).$$ Obviously $[vp] \in PP$.

<u>Example 14</u>

Let $T = Th\mathcal{A}$ where $\mathcal{A}$ is an infinite Boolean algebra. If $\mathcal{A}$ contains an atomless element and $\varphi(x,y)$ is $0 \neq x \subset y$ it is clear that $\varphi(x,y)$ admits vp in T. If $\mathcal{A}$ contains no atomless element then $\mathcal{A}$ contains infinitely many atoms so T admits [1] (see Example 6). In either case T admits [vp].

<u>Example 15</u>

Let L be a language consisting of a unary predicate P, binary predicate Q and n-ary predicates R_n $(1 < n < \omega)$ and let T be the theory of L whose axioms are

$$Q(x,y) \rightarrow P(x) \wedge \neg P(y)$$

$$R_n(y_0,\ldots,y_{n-1}) \to \bigwedge_{i<j<n} (\neg P(y_i) \wedge y_i \neq y_j) \quad (1 < n < \omega)$$

$$R_n(y_0,\ldots,y_{i-1},y_i,\ldots,y_{n-1}) \to R(y_0,\ldots,y_i,y_{i-1},\ldots,y_{n-1}) \; (1 \leq i < n < \omega)$$

$$R_n(y_0,\ldots,y_{n-2},y_{n-1}) \to R_{n-1}(y_0,\ldots,y_{n-2})$$

$$\bigwedge_{i<j<n} (Q(x,y_i) \wedge y_i \neq y_j) \to R_n(y_0,\ldots,y_{n-1}) \quad (1 < n < \omega)$$

Letting Σ be the class of finite models of T it is clear that Σ is good and admits HP, JEP and AP . Hence by §0 M is Σ-generic, $\aleph_0$-categorical and quantifier-eliminable for some M . Let VP = ThM. Since M is Σ-generic it follows easily that

(1) If $b_0,\ldots,b_{n-1} \in |M|-P_M$ then $\bigcap_{i<n} Q_M(x,b_i) \neq \phi$

iff $R_n^M(b_0,\ldots,b_{n-1})$

holds. Since M is Σ-generic it follows easily from (1) that

(2) If $\mathcal{I}$ is a weak ideal of n there exist

$b_0,\ldots,b_{n-1} \in |M|-P_M$ such that $\bigcap_{i\in I} Q_M(x,b_i) \neq \phi$

iff $I \in \mathcal{I}$

holds. Letting $\varphi(x,y)$ be $Q(x,y)$ it is clear from (2) that $\varphi(x,y)$ admits vp in VP . Hence VP admits [vp] . It may be proved that

VP omits [pp]. Suppose not. Then by Theorem 9 some formula $\varphi(x,\overline{y})$ of VP admits pp in VP. Then for arbitrarily large $n < \omega$ there exist n $\varphi(x,\overline{y})$-definable subsets of $|M|$ which partition $|M|$. But then either

(3) For arbitrarily large $n < \omega$ there exist n $\varphi(x,\overline{y}) \wedge P(x)$-definable subsets of P_M which partition P_M

or

(4) For arbitrarily large $n < \omega$ there exist n $\varphi(x,\overline{y}) \wedge \neg P(x)$-definable subsets of $|M|-P_M$ which partition $|M|-P_M$.

holds. Suppose (3) holds. Since VP is quantifier-eliminable there exists $m < \omega$ such that every $\varphi(x,\overline{y}) \wedge P(x)$-definable subset of P_M is a Boolean combination of at most m $P(x)$-definable, $Q(x,y)$-definable or $x = y$-definable subsets of P_M. Since M is Σ-generic it follows easily that the set of such subsets admits the stratified-Whitman property so by Theorem 10 it does not admit the partition property and this is a contradiction. Suppose (4) holds. Since VP is quantifier-eliminable there exist $\ell,m<\omega$ such that every $\varphi(x,\overline{y}) \wedge \neg P(x)$-definable subset of $|M|-P_M$ is a Boolean combination of at most m $P(x)$-definable, $R_2(x,y_0)$-definable, ..., $R_\ell(x,y_0,\ldots,y_{\ell-2})$-definable, $Q(y,x)$-definable or $x = y$-definable subsets of $|M|-P_M$. Since M is Σ-generic it follows easily that the set of such subsets admits the stratified-Whitman property so by

Theorem 10 it does not admit the partition property and this is a contradiction.

The following result may be used to show that certain complete theories omit [vp].

Theorem 14

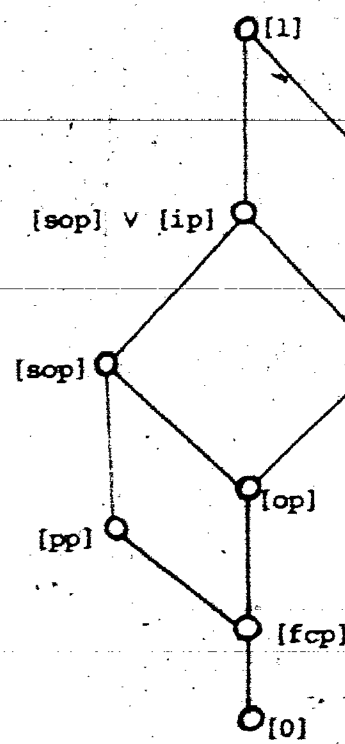
If T is a complete quantifier-eliminable theory in a finite language without functions then T omits [vp].

Proof

Suppose T is a complete, quantifier-eliminable theory in a finite language without functions. Suppose that T admits [vp]. Then some formula $\varphi(\overline{x},\overline{y})$ of T admits vp in T. Note that for every $n < \omega$ there exist at least $2^{\binom{2n}{n}}$ weak ideals of 2n. In fact if I is a set of subsets (of cardinality n) of 2n let $\overline{I}$ be the weak ideal of 2n generated by I and observe that $I \to \overline{I}$ is one-one. Since $\varphi(\overline{x},\overline{y})$ admits vp in T it follows easily $2^{\binom{2n}{n}} \leq |S_{2n\ell(\overline{y})}T|$ $(n < \omega)$. But by §0 there exists a polynomial f such that $|S_{2n\ell(\overline{y})}T| \leq 2^{f(2n\ell(\overline{y}))}$ $(n < \omega)$. Hence $\binom{2n}{n} \leq f(2n\ell(\overline{y}))$ $(n < \omega)$ yet this is clearly impossible.

2.4 Remark

The above examples of properties of complete theories are ordered in $(\mathcal{P},\leq)$ in the following manner:



By Keisler (1967) $[0] \nleq [\mathrm{fcp}]$. By Shelah (1971) $[\mathrm{fcp}] < [\mathrm{op}] < [\mathrm{sop}]$, $[\mathrm{ip}]$. To show that $[\mathrm{fcp}] \leq [\mathrm{pp}]$ note that if $\varphi(\bar{x},\bar{y})$ admits pp in T then $\neg\varphi(\bar{x},\bar{y})$ admits fcp in T. To show that $[\mathrm{fcp}] \nleq [\mathrm{pp}]$ note that IND admits [fcp] but omits [pp]. To show that $[\mathrm{pp}] \leq [\mathrm{sop}]$ note that if $\varphi(\bar{x},\bar{y})$ admits sop in T and $\psi(\bar{x},\bar{y}_0 {}^\frown \bar{y}_1 {}^\frown \bar{y}_2 {}^\frown \bar{y}_3 {}^\frown \bar{z})$ is $(\varphi(\bar{x},\bar{y}_0) \wedge z_0=z_1) \wedge (\neg\varphi(\bar{x},\bar{y}_1) \wedge \varphi(\bar{x},\bar{y}_2) \wedge z_2=z_3) \vee (\neg\varphi(\bar{x},\bar{y}_3) \wedge z_4=z_5)$ then $\psi(\bar{x}, \bar{y}_0 {}^\frown \bar{y}_1 {}^\frown \bar{y}_2 {}^\frown \bar{y}_3 {}^\frown \bar{z})$ admits pp in T. To show that $[\mathrm{pp}] \nleq [\mathrm{sop}]$ note that $[\mathrm{pp}] \notin \mathcal{PP}$ yet $[\mathrm{sop}] \in \mathcal{PP}$. By Shelah (1971)

[sop] $\not\leq$ [ip] and [ip] $\not\leq$ [sop] so [sop], [ip] $\neq$ [sop] ∨ [ip]. To show that [sop] ∨ [ip] $\neq$ [1] note that [1] is ∨-irreducible (since it is prime). To show that [ip] ≤ [sip] note that if $\varphi(\overline{x},\overline{y})$ admits sip in T then $\varphi(\overline{x},\overline{y})$ admits ip to T. To show that [ip] $\neq$ [sip] note that IND admits [ip] yet by Theorem 13 it is easy to prove that IND omits [sip]. To show that [sip] ≤ [vp] note that if $\varphi(\overline{x},\overline{y})$ admits vp in T then $\varphi(\overline{x},\overline{y})$ admits sip in T. To show that [sip] $\neq$ [vp] note that SIND admits [sip] yet by Theorem 14 SIND omits [vp]. To show that [vp] $\neq$ [1] note that the theory of infinite atomless Boolean algebras admits [vp] but omits [1]. To show that [vp] $\not\leq$ [pp] note that VP admits [vp] but omits [pp].

2.5 Regular and Whitman Theories

Let $\mathcal{A}$ be a structure of a language L. If $A \subset |\mathcal{A}|^n$ and $B \subset |\mathcal{A}|$ then A is B-definable if $A = \varphi_{\mathcal{A}}(\overline{x},B)$ for some formula $\varphi(\overline{x},\overline{y})$ of L. If $A \subset |\mathcal{A}|^n$ is B-definable for some $B \subset |\mathcal{A}|$ then A is definable. If for every definable $A \subset |\mathcal{A}|^n$ there exists a unique minimal $B \subset |\mathcal{A}|$ such that A is B-definable then $\mathcal{A}$ is a n-regular. Thus $\mathcal{A}$ is n-regular iff for every definable $A \subset |\mathcal{A}|^n$ the following holds: If A is B-definable and C-definable then A is $B \cap C$-definable. If for every $n < \omega$ $\mathcal{A}$ is n-regular then $\mathcal{A}$ is regular. A complete theory T is n-regular if every model of T is n-regular.

Example 16

Let $T = Th\mathcal{A} = Th(\omega,<)$ where $\mathcal{A}$ is the standard model of infinite discrete linear orders with a least element. Then $\mathcal{A}$ is regular (since each $a \in |\mathcal{A}|$ is ϕ-definable) but it is easy to prove that every other model of T is irregular. Hence elementary equivalence does not preserve regularity.

Theorem 15

Suppose T is a countable, complete, $\aleph_0$-categorical theory. If T is 1-regular then T is n-regular for every $n < \omega$.

Proof

Suppose T is a countable, complete, $\aleph_0$-categorical, 1-regular and n-regular theory. It suffices to prove that T is (n+1)-regular. Suppose $\mathcal{A} \models T$ and $A \subset |\mathcal{A}|^{n+1}$ is B-definable and C-definable. It suffices to prove that A is $B \cap C$-definable. For each $\overline{a} \in |A|^n$ let $A(\overline{a}) = \{b \in |A| \mid (\overline{a},b) \in A\} \subset |A|$ and note that $A(\overline{a})$ is $\overline{a} \cup B$-definable and $\overline{a} \cup C$-definable. Since T is 1-regular it follows easily that for each $\overline{a} \in |A|^n$ there exists a unique minimal $S(\overline{a}) \subset |A|$ disjoint from $\overline{a}$ such that $A(\overline{a})$ is $\overline{a} \cup S(\overline{a})$-definable and note that $S(\overline{a}) \subseteq B \cap C$ since $A(\overline{a})$ is $\overline{a} \cup B$-definable and $\overline{a} \cup C$-definable. Since T is $\aleph_0$-categorical it follows by Ryll-Nardzewski (1959) that for each $\overline{a} \in |A|^n$ there exists a unique finite set $F(\overline{a})$ of complete formulas $\varphi(\overline{x},y)$ of T such that $A(\overline{a}) = \bigcup_{\varphi(\overline{x},y) \in F(\overline{a})} \varphi_A(x,\overline{a} \cup S(\overline{a}))$. Write $\overline{a} \sim \overline{b}$ if $S(\overline{a}) = S(\overline{b})$ and $F(\overline{a}) = F(\overline{b})$ and note that since T is $\aleph_0$-categorical $\sim$ is an equivalence relation on $|A|^n$ with only finitely many equivalence classes $D_i \subset |A|^n$ $(i < m)$. Obviously each D_i is B-definable and C-definable. Since T is n-regular it follows that for each D_i there exists a unique minimal $E_i \subset |A|$ such that D_i is E_i-definable. In particular each $E_i \subset B \cap C$ since each D_i is B-definable and C-definable. But since each $E_i \subset B \cap C$ it follows easily that A is $B \cap C$-definable.

If T is a countable, complete, $\aleph_0$-categorical, 1-regular theory then T is <u>regular</u>. A complete theory T admits the <u>exchange property</u> if

(EP) If $A \cup \{a\} \subset A \models T$ and $a \notin A$ then

$$t_A(A \cup \{a\}) = t_A(A \cup \{a'\}) \text{ for some } a' \neq a$$

holds. A complete theory T admits the splitting property if

(SP) If $A \cup \{a\} \cup \{a'\} \subset \mathcal{A} \models T$, $t_{\mathcal{A}}(A \cup \{a\}) = t_{\mathcal{A}}(A \cup \{a'\}$, $a \neq a'$, $\varphi(x,A)$ is a complete formula of $Th(\mathcal{A},A)$ and $B \subset \varphi_{\mathcal{A}}(x,A)$ is $A \cup \{a\}$-definable and $A \cup \{a'\}$-definable then $B = \phi$ or $B = \varphi_{\mathcal{A}}(x,A)$

holds.

Theorem 16

Suppose T is a countable, complete, $\aleph_0$-categorical theory. If T admits EP and SP then T is regular.

Proof

Suppose T is a countable, complete, $\aleph_0$-categorical theory which admits EP and SP. It suffices to prove that T is 1-regular. Suppose $\mathcal{A} \models T$ and $A \subset |\mathcal{A}|$ is B-definable and C-definable. It suffices to prove that A is $B \cap C$-definable. Suppose not. Then there exists some $c \in C-(B \cap C)$ and it may be assumed that A is not D-definable where $D = C-\{c\}$. By EP $t_{\mathcal{A}}(B \cup D \cup \{c\} = t_{\mathcal{A}}(B \cup D \cup \{c'\}$ for some $c' \neq c$. Let $A = \varphi_{\mathcal{A}}(x,B) = \psi_{\mathcal{A}}(x,C) = \psi_{\mathcal{A}}(x,D \cup \{c\})$ for some formulas $\varphi(x,\bar{y})$ and $\psi(x,\bar{z})$ of T. Then $A = \varphi_{\mathcal{A}}(x,B) = \psi_{\mathcal{A}}(x,D \cup \{c'\})$ since $t_{\mathcal{A}}(B \cup D \cup \{c\}) = t_{\mathcal{A}}(B \cup D \cup \{c'\})$. In particular

$A = \psi_{\mathcal{A}}(x,D \cup \{c\}) = \psi_{\mathcal{A}}(x,D \cup \{c'\})$, $t_{\mathcal{A}}(D \cup \{c\}) = t_{\mathcal{A}}(D \cup \{c'\}$ and $c \neq c'$.

Since T is $\aleph_0$-categorical and A is not D-definable it follows by Ryll-Nardzewski (1959) that $\phi \neq \chi_{\mathcal{A}}(x,D) \cap A \neq \chi_{\mathcal{A}}(x,D)$ for some complete formula $\chi(x,D)$ of $Th(\mathcal{A},D)$. But $\chi_{\mathcal{A}}(x,D) \cap A \subset \chi_{\mathcal{A}}(x,D)$ is

$D \cup \{c\}$-definable and $D \cup \{c'\}$-definable and by SP this is a contradiction.

Example 17

Let L be a language consisting of a binary predicate $<$ and let PO be the theory in L whose axioms are

$$x \nless x$$

$$x < y < z \rightarrow x < z .$$

Letting Σ be the class of finite models of PO it is clear that Σ is good and admits HP, JEP and AP. Hence by §0 M is Σ-generic, $\aleph_0$-categorical and quantifier-eliminable for some M. Let GPO = ThM. It is easy to prove that GPO admits EP and SP. Hence GPO is regular.

It is easy to prove that DLO and IND admit EP and SP so by Theorem 16 it follows that DLO and IND are regular.

If T is a countable, complete, $\aleph_0$-categorical, quantifier-eliminable theory such that

(A) If $A, B \subset \mathcal{A} \models T$, $\varphi(x,\bar{y})$ and $\psi(x,\bar{z})$ are atomic formulas of T and $\phi \neq \varphi_{\mathcal{A}}(x,A) = \psi_{\mathcal{A}}(x,B)$ then $A = B$

(W) If $A_i, B_j \subset \mathcal{A} \models T$ $(i < m,\ j < n)$, $\varphi_i(x,\bar{y}_i)$ $(i < m)$ and $\psi_j(x,\bar{y}_j)$ $(j < n)$ are atomic formulas of T and $\phi \neq \bigcap_{i<m} \varphi_i^{\mathcal{A}}(x,A_i) \subset \bigcup_{j<n} \psi_j^{\mathcal{A}}(x,B_j)$ then $\varphi_i^{\mathcal{A}}(x,A_i) \subset \psi_j^{\mathcal{A}}(x,B_j)$ for some $i < m$, $j < n$

hold then T is an <u>atomic-Whitman</u> theory.

Theorem 17

If T is an atomic-Whitman theory then T is regular.

Proof

Suppose T is an atomic-Whitman theory. It suffices to prove that T is 1-regular. Suppose $\mathcal{M} \models T$. It suffices to prove that $\mathcal{M}$ is 1-regular. Any nonempty set of the form $\bigcap_{i<m} \varphi_i^{\mathcal{M}}(x,A_i) - \bigcup_{j<n} \psi_j^{\mathcal{M}}(x,B_j)$ where $A_i, B_j \subset \mathcal{M}$ $(i < m, j < n)$ and $\varphi_i(x,\overline{y}_i)$ $(i < m)$ and $\psi_j(x,\overline{z}_j)$ $(j < n)$ are atomic formulas of T is basic and the representation $\bigcap_{i<m} \varphi_i^{\mathcal{M}}(x,A_i) - \bigcup_{j<n} \psi_j^{\mathcal{M}}(x,B_j)$ is irredundant if

$\varphi_{i_0}^{\mathcal{M}}(x,A_{i_0}) \subset \varphi_{i_1}^{\mathcal{M}}(x,A_{i_1})$ implies $i_0 = i_1$. If $\bigcup_{i<m} (\bigcap \mathcal{A}_i - \bigcup \mathcal{B}_i)$ is a finite union of basic sets the representation $\bigcup_{i<m} (\bigcap \mathcal{A}_i - \bigcup \mathcal{B}_i)$ is irredundant if each representation $\bigcap \mathcal{A}_i - \bigcup \mathcal{B}_i$ is irredundant. It may be proved that

(1) If $\bigcup_{i<m} (\bigcap \mathcal{A}_i - \bigcup \mathcal{B}_i) = \bigcup_{j<n} (\bigcap \mathcal{C}_j - \bigcup \mathcal{D}_j)$ are irredundant representations of the same set then $\mathcal{A}_i = \mathcal{C}_j$ for some $i < m, j < n$

holds. To prove (1) let $\mathcal{A}_i \le \mathcal{C}_j$ denote that $\forall C(C \in \mathcal{C}_j \to \exists A(A \in \mathcal{A}_i \wedge A \subset C))$ and note that $\le$ is transitive. Furthermore by irredundancy $\mathcal{A}_i \le \mathcal{C}_j \le \mathcal{A}_i$ implies $\mathcal{A}_i = \mathcal{C}_j$. Hence it suffices to prove that $\forall i \exists j(\mathcal{A}_i \le \mathcal{C}_j)$ and $\forall j \exists i(\mathcal{C}_j \le \mathcal{A}_i)$. Suppose $\neg\forall i \exists j(\mathcal{A}_i \le \mathcal{C}_j)$ (the other case admits a similar argument). Then

$\exists i \forall j \exists c_j (c_j \in C_j \wedge \forall A(A \in A_i \to A \notin c_j))$. But

$\phi \neq \cap A_i \subset (\cup B_i) \cup c_0 \cup \ldots \cup c_{n-1}$ so by (W) $\exists A \exists j(A \in A_i \wedge A \subset c_j)$

and this is a contradiction. Suppose M is not 1-regular. Then for some definable $S \subset A$ there exist distinct minimal $A, B \subset A$ such that S is a A-definable and B-definable. In particular S is not $A \cap B$-definable. Since T is $\aleph_0$-categorical it may be assumed by Ryll-Nardzewski (1959) that this counterexample is A-minimal in the sense that if $S' \underset{\neq}{\subset} S$ is A-definable then there exists a unique minimal $C \subset A$ such that S' is C-definable. Let $S = \varphi_A(x,A) = \psi_A(x,B)$ for some formulas $\varphi(x,\bar{y})$ and $\psi(x,\bar{z})$ of T. Since T is quantifier-eliminable $\varphi_A(x,A) = \bigcup_{i<m} (\cap A_i - \cup B_i)$ for some basic sets $\cap A_i - \cup B_i$ defined by formulas of the form $\chi(x,A')$ where $\chi(x,\bar{z})$ is an atomic formula of T and $A' \subset A$ (and the representation $\bigcup_{i<m} (\cap A_i - \cup B_i)$ may be assumed irredundant). Similarly $\psi_A(x,B) = \bigcup_{j<n} (\cap C_j - \cup D_j)$ for some basic sets $\cap C_j - \cup D_j$ defined by formulas of the form $\chi(x,B')$ where $\chi(x,\bar{z})$ is an atomic formula of T and $B' \subset B$ (and the representation $\bigcup_{j<n} (\cap C_j - \cup D_j)$ may be assumed irredundant). In particular

$\phi \neq S = \bigcup_{i<m} (\cap A_i - \cup B_i) = \bigcup_{j<n} (\cap C_j - \cup D_j) \neq M$. Since the counter-example is A-minimal it follows that $P \cup Q = \{\chi_M(x,C) \mid \psi(x,\bar{z})$ atomic and $C \subset A \cap B\}$ where $P = \{\chi_M(x,C) \mid \chi(x,\bar{z})$ atomic, $C \subset A \cap B$ and

and $S \underset{\neq}{\subset} \chi_M(x,C)\}$ and $\mathcal{Q} = \{\chi_M(x,C) \mid \chi(x,\overline{z})$ atomic, $C \subset A \cap B$ and $\chi_M(x,C) \underset{\neq}{\subset} M - S\}$. In particular $\forall_i(\phi \neq \cap A_i - \cup B_i \subset S \underset{\neq}{\subset} \cap \mathcal{P} - \cup \mathcal{Q})$.

From this it may be proved that

(2) $\forall i(\cap A_i \subset \cap \mathcal{P})$

holds. To prove (2) it suffices to prove that $\forall i(A_i \leq \mathcal{P})$. Suppose $\neg\forall i(A_i \leq \mathcal{P})$. Then $\exists i \exists P(P \in \mathcal{P} \wedge \forall A(A \in A_i \rightarrow A \not\subset P))$. But $\phi \neq \cap A_i \subset (\cup B_i) \cup P$ so by (W) $\exists A(A \in A_i \wedge A \subset P)$ and this is a contradiction. From (2) it may be proved that

(3) $\exists i(\cap A_i = \cap \mathcal{P})$

holds. To prove (3) suppose $\neg\exists_i(\cap A_i = \cap \mathcal{P})$. Then by (2) it follows that $\forall i(\cap A_i \underset{\neq}{\subset} \cap \mathcal{P})$. Since $\bigcup_{i<m}(\cap A_i - \cup B_i) = \bigcup_{j<n}(\cap C_j - \cup D_j)$ are irredundant representations of the same set it follows by (1) that $A_i = C_j$ for some $i < m$, $j < n$. Since A_i contains sets of the form $\chi_M(x,A')$ where $\chi(x,\overline{z})$ is an atomic formula of T and $A' \subset A$ and C_j contains sets of the form $\chi_M(x,B')$ where $\chi(x,\overline{z})$ is an atomic formula of T and $B' \subset B$ it follows by (A) that $A_i = C_j$ contains sets of the form $\chi_M(x,C)$ where $\chi(x,\overline{z})$ is an atomic formula of T and $C \subset A \cap B$. In particular $A_i \subset \mathcal{P} \cup \mathcal{Q}$. But since

$\phi \neq \cap A_i - \cup B_i \underset{\neq}{\subset} S$ it follows that $A_i \subset P$ so $\cap P \subset \cap A_i$ and this is a contradiction. For notational convenience assume that $\cap A_0 = \cap P$. Since $\phi \neq \bigcup_{i<m} (\cap A_i - \cup B_i) = \bigcup_{j<n} (\cap C_j - \cup D_j) \underset{\neq}{\subset} \cap P - \cup Q$ it follows that $\phi \neq (\cap P - \cup Q) - \bigcup_{i<m} (\cap A_i - \cup B_i) = (\cap P - \cup Q) - \bigcup_{j<n} (\cap C_j - \cup D_j) \underset{\neq}{\subset} \cap P - \cup Q$.

For each $f : m \to \bigcup_{i<m} (A_i \cup B_i)$ such that $f(i) \in A_i \cup B_i$ $(i < m)$ let

$T_f = \{f(i) | f(i) \in B_i\}$ and $U_f = \{f(i) | f(i) \in A_i\}$ and let

$I = \{f | \cap(T_f \cup P) - \cup(U_f \cup Q) \neq \phi\}$. Similarly for each

$g : n \to \bigcup_{j<n} (C_j \cup D_j)$ such that $g(j) \in C_j \cup D_j$ $(j < n)$ let

$V_g = \{g(j) | g(j) \in D_j\}$ and $W_g = \{g(j) | g(j) \in C_j\}$ and let

$J = \{g | \cap(V_g \cup P) - \cup(W_g \cup Q) \neq \phi\}$. Since $A_i \cap B_i = \phi$ $(i < m)$ and

$C_j \cap D_j = \phi$ $(i < n)$ it follows that

$(\cap P - \cup Q) - \bigcup_{i<m} (\cap A_i - \cup B_i) = \bigcup_{f\in I} (\cap(T_f \cup P) - \cup(U_f \cup Q))$ and

$(\cap P - \cup Q) - \bigcup_{j<n} (\cap C_j - \cup D_j) = \bigcup_{g\in J} (\cap(V_g \cup P) - \cup(W_g \cup Q))$. In particular

$\bigcup_{f\in I} (\cap(T_f \cup P) - \cup(U_f \cup Q)) = \bigcup_{g\in J} (\cap(V_g \cup P) - \cup(W_g \cup Q))$. For each $f \in I$ choose a minimal $T'_f \subset T_f \cup P$ such that $T'_f \leq T_f \cup P$ and for each $g \in J$ choose a minimal $V'_g \subset V_g \cup P$ such that $V'_g \leq V_g \cup P$. Then

$\bigcup_{f\in I} (\bigcap T'_f - \bigcup(\mathcal{U}_f \cup \mathcal{Q})) = \bigcup_{g\in J} (\bigcap V'_g - \bigcup(\mathcal{W}_g \cup \mathcal{Q}))$ are irredundant representations of the same set

$\bigcup_{f\in I} (\bigcap(T_f \cup P) - \bigcup(\mathcal{U}_f \cup \mathcal{Q})) = \bigcup_{g\in J} (\bigcap(V_g \cup P) - \bigcup(\mathcal{W}_g \cup \mathcal{Q}))$ so it follows by (1) that $T'_f = V'_g$ for some $f \in I$, $g \in J$. Since T'_f contains sets of the form $\chi_{\mathcal{M}}(x,A')$ where $\chi(x,\overline{z})$ is an atomic formula of T and $A' \subset A$ and V'_g contains sets of the form $\chi_{\mathcal{M}}(x,B')$ where $\chi(x,\overline{z})$ is an atomic formula of T and $B' \subset B$ it follows by (A) that $T'_f = V'_g$ contains sets of the form $\chi_{\mathcal{M}}(x,C)$ where $\chi(x,\overline{z})$ is an atomic formula of T and $C \subset A \cap B$. In particular $T'_f \subset \mathcal{P} \cup \mathcal{Q}$. But since $\bigcap T'_f - \bigcup(\mathcal{U}_f \cup \mathcal{Q}) \neq \phi$ it follows that $T'_f \subset P$ so $P \leq T'_f$ so $P \leq T_f \cup P$ so $P \leq T_f$. But then $T_f \neq \phi$ since

$\forall B(B \in T_f \rightarrow \exists i(B \in \mathcal{B}_i))$ and $\forall i(\phi \neq \bigcap A_i - \bigcup B_i \underset{\neq}{\subset} \bigcap P - \bigcup \mathcal{Q})$. But if $T_f = \phi$ then $f(0) \in \mathcal{U}_f$ so $\bigcap P \subset \mathcal{U}_f \cup \mathcal{Q}$ since $\bigcap A_0 = \bigcap P$. But then $\bigcap(T_f \cup P) - \bigcup(\mathcal{U}_f \cup \mathcal{Q}) = \phi$ since

$\bigcap(T_f \cup P) - \bigcup(\mathcal{U}_f \cup \mathcal{Q}) = \bigcap P - \bigcup(\mathcal{U}_f \cup \mathcal{Q}) \subset \bigcap P - \bigcap P = \phi$ and this is a contradiction.

It is easy to prove that VP is an atomic-Whitman theory so by Theorem 17 VP is regular.

2.6 The Partial Order and Line Properties of Complete Theories

Let pop be a property of formulas such that if $\varphi(\overline{x},\overline{y})$ is a formula of a complete theory T then $\varphi(\overline{x},\overline{y})$ admits pop in T iff for every finite partial order $(A,<)$ there exists $A \models T$ and $\overline{a}_i \in |A|^{\ell(\overline{y})} (i < |A|)$ such that $(A,<)$ is isomorphic to $(\{\varphi_A(\overline{x},\overline{a}_i) \mid i < |A|\},\subset)$. Obviously $[\text{pop}] \in PP$.

Lemma 14

If $A \models GPO$, $B \models PO$ and $A \subset B$ then $\text{Th}A \blacktriangleleft \text{Th}B$.

Proof

Similar to the proof of Lemma 13.

Theorem 18

GPO is archetypal for [pop].

Proof

Similar to the proof of Theorem 12.

Let $ACF(0) = \text{Th}A = \text{Th}(\mathbb{C},0,1,+,\cdot)$ where A is the standard model of algebraically closed fields of characteristic 0 and let $RCF = \text{Th}B = \text{Th}(\mathbb{R},0,1,+,\cdot,<)$ where B is the standard model of real closed fields. Let $\ell p = \rho(\varphi(\overline{x},\overline{y}), ACF(0))$ where $\varphi(\overline{x},\overline{y})$ is the formula $y_0x_0 + y_1x_1 = y_2$ of $ACF(0)$. Note that $\rho(\varphi(\overline{x},\overline{y}), RCF) \subset \rho(\varphi(\overline{x},\overline{y}), ACF(0))$. For $1 < m < n < \omega$ let $S(m,n)$ denote a finite Venn diagram corresponding to the set of lines of $\mathbb{R} \times \mathbb{R}$ incident to a pair of distinct points of $n \times n$ less than distance m apart (see the remarks preceeding Lemma 15 in §3). Each

$L \subset S(m,n)$ corresponding to a line (horizontal line, vertical line) of $\mathbb{R} \times \mathbb{R}$ is a line (horizontal line, vertical line) of $S(m,n)$. Each $\alpha \in S(m,n)$ corresponding to an intersection of a pair of distinct lines of $S(m,n)$ is a vertex of $S(m,n)$. Each $\alpha \in S(m,n)$ corresponding to an intersection of a pair of horizontal and vertical lines of $S(m,n)$ is a proper vertex of $S(m,n)$. For $\ell < \omega$ $S(m,n)$ is ℓ-incident if each proper vertex of $S(m,n)$ corresponds to the intersection of at least ℓ lines of $S(m,n)$. It may be proved that $S(m,n)$ is m-incident and rank $S(m,n) \leq 12m^2n$. Note that if a formula $\varphi(\overline{x},\overline{y})$ of a complete theory T admits ℓp in T then $\varphi(\overline{x},\overline{y})$ admits each $S(m,n)$ in T.

Theorem 19

If T is a regular theory then T omits $[\ell p]$.

Proof

Suppose $\varphi(\overline{x},\overline{y})$ $(\ell(\overline{x}) = \ell(\overline{y}) = p)$ is a formula of a regular theory T. It suffices to prove that $\varphi(\overline{x},\overline{y})$ omits $S(m,n)$ in T for some $1 < m < n < \omega$. Let $A \models T$. Since T is p-regular it follows easily that

(1) If $\overline{a}_i \in |A|^p$ $(i < 2p)$ and $\varphi_A(\overline{x},\overline{a}_i) \cap \varphi_A(\overline{x},\overline{a}_j) = \varphi_A(\overline{x},\overline{a}_k) \cap \varphi_A(\overline{x},\overline{a}_\ell)$ $(i < j < 2^{2p}, k < \ell < 2^{2p})$ then $\varphi_A(\overline{x},\overline{a}_0) \cap \varphi_A(\overline{x},\overline{a}_1)$ is $\overline{a}_i$-definable for some $i < 2^{2p}$

holds. Since T is $\aleph_0$-categorical it follows by Ryll-Nardzewski (1959) that

(2) $|S_{2p}T| = q$ for some $q < \omega$

(3) If $\bar{a} \in |A|^p$, $\psi_i(\bar{x},\bar{y})$ $(i < k)$ are formulas of T,

$\psi_i^A(\bar{x},\bar{a}) \neq \phi$ $(i < k)$ and

$\psi_i^A(\bar{x},\bar{a}) \cap \psi_j^A(\bar{x},\bar{a}) = \phi$ $(i < j < k)$ then $k \leq q$

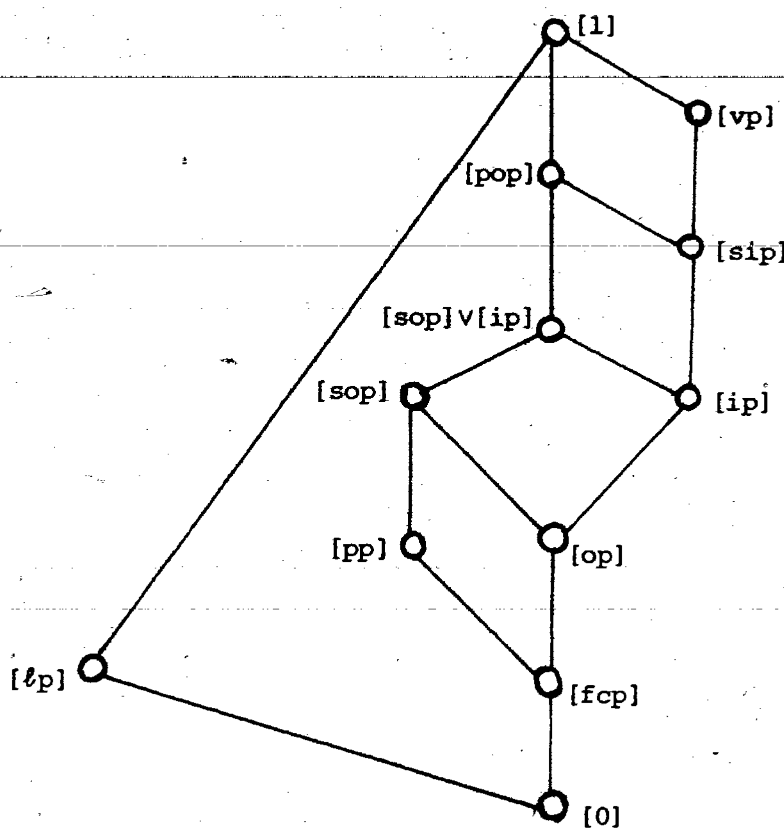
hold. Suppose $\varphi(\bar{x},\bar{y})$ admits $S(2^{2p},n)$ in T for some $n > 12(2^{2p})^3 q$. Then there exists a set S of $\varphi(\bar{x},\bar{y})$-definable subsets $\varphi_A(\bar{x},\bar{a}_i)$ $(i < \text{rank } S(2^{2p},n))$ of $|A|^p$ which admits $S(2^{2p},n)$ in A. Obviously the sets $\varphi_A(\bar{x},\bar{a}_i)$ $(i < \text{rank } S(2^{2p},n))$ are the lines of S. Let α_i $(i < n^2)$ be the proper vertices of S. For each $i < \text{rank } S(2^{2p},n)$ let $f(i)$ denote the number of α_j which are $\bar{a}_i$-definable. By (3) $f(i) \leq q$ $(i < \text{rank } S(2^{2p},n))$. For each $j < n^2$ let $g(j)$ denote the number of i such that α_j is $\bar{a}_i$-definable. By (1) $g(j) \geq 1$ $(j < n^2)$. But $\sum_{i < \text{rank } S(2^{2p},n)} f(i) = \sum_{j < n^2} g(j)$ so

$$12(2^{2p})^3 nq \geq \text{rank } S(2^{2p},n)\, q \geq \sum_{i < \text{rank } S(2^{2p},n)} f(i) = \sum_{j<n^2} g(j) \geq n^2$$

so $12(2^{2p})^3 q \geq n$ and this is a contradiction.

2.7 <u>Remark</u>

The above examples of properties of complete theories are ordered in $(P,\leq)$ in the following manner:



To show that [sop] $\leq$ [pop] note that $x < y$ admits sop in GPO and GPO is archetypal for [pop]. To show that [sip] $\leq$ [pop], note that $x < y_0 \wedge x \nleq y_1 \wedge x \nleq y_1$ admits sip in GPO and GPO is archetypal for [pop]. Hence [sop] $\vee$ [ip] $\leq$ [pop]. To show that [sop] $\vee$ [ip] $\nleq$ [pop] note that [sop] $\vee$ [ip] $\nleq$ [sip]. To show that [pop] $\nleq$ [sip] note that [vp] $\nleq$ [pp]. To show that [pop] $\nleq$ [vp] note that [vp] $\nleq$ [pp]. To show that [pop] $\nleq$ [vp]

note that GPO admits [pop] but by Theorem 14 GPO omits [vp]. To show that [pop] $\nRightarrow$ [1] note that [pop] $\nRightarrow$ [vp]. To show that [ℓp] $\nRightarrow$ [fcp] note that ACF(0) admits [ℓp] but by Keisler (1967) ACF(0) omits [fcp]. Hence [ℓp] $\nRightarrow$ [1]. To show that [ℓp] $\nRightarrow$ [pop] note that GPO admits [pop] but by Theorem 19 GPO omits [ℓp] since GPO is regular. In particular [ℓp] $\nRightarrow$ [0]. To show that [ℓp] $\nRightarrow$ [vp] note that VP admits [vp] but by Theorem 19 VP omits [ℓp] since VP is regular.

It may be proved that if $A \models$ GPO then A admits no definable infinite linear order. Suppose not. Then there exist formulas $\varphi(\overline{x},\overline{y})$ and $\psi(\overline{x},\overline{y})$ ($\ell(\overline{x}) = \ell(\overline{y}) = n$) of GPO such that $\varphi_A(\overline{x},\overline{y})$ is an equivalence relation on $|A|^n$ with infinitely many equivalence class and $\psi_A(\overline{x},\overline{y})$ is a preorder on $|A|^n$ which induces a linear order on the equivalence classes of $\varphi_A(\overline{x},\overline{y})$. Since $\varphi_A(\overline{x},\overline{y})$ has infinitely many equivalence classes and GPO is $\aleph_0$-categorical it follows by Ryll-Nardzewski (1959) that $\neg\varphi_A(\overline{a},\overline{b})$ for some $\overline{a}, \overline{b} \in |A|^n$ such that $t_A(\overline{a}) = t_A(\overline{b})$. Choose $\overline{c} \in |A|^n$ such that $t_A(\overline{c} \frown \overline{a}) = t_A(\overline{a} \frown \overline{c}) = t_A(\overline{c} \frown \overline{b}) = t_A(\overline{b} \frown \overline{c})$. Then $\psi_A(\overline{c},\overline{a})$ iff $\psi_A(\overline{a},\overline{c})$ and $\psi_A(\overline{c},\overline{b})$ iff $\psi_A(\overline{b},\overline{c})$. Hence $\varphi_A(\overline{c},\overline{a})$ and $\varphi_A(\overline{c},\overline{b})$ since $\psi_A(\overline{x},\overline{y})$ induces a linear order on the equivalence classes of $\varphi_A(\overline{x},\overline{y})$. But then $\varphi_A(\overline{a},\overline{b})$ and this is a contradiction. Thus GPO admits no definable infinite linear order. From this it follows that the property of admitting a definable infinite linear order cannot be viewed as a property of complete theories since DLO admits a definable infinite linear order and DLO ◄ GPO (since DLO is archetypal for [sop] and GPO admits [sop]).

Schmerl (1979) proved that if $A \models PO$ is countable and ThA is quantifier-eliminable then A is an anti-chain, countably many copies of the rationals with the usual order, countably many copies of the rationals with the weak order, or a generic partial order. In particular it follows easily that ThA is definable in EQ, DLO or GPO. Since EQ is archetypal for [0], DLO is archetypal for [sop] and GPO is archetypal for [pop] it follows that ThA ≡ EQ, ThA ≡ DLO or ThA ≡ GPO. But EQ ◄ DLO ◄ GPO so it follows that the class of
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quantifier-eliminable theories of partial orders with the order ◄ is a linear order with three elements.

2.8 Independent and Countable Properties of Complete Theories

A sequence of finite Venn diagrams $S_i (i < \omega)$ is <u>independent</u> if rank $(S_i) \geq i$ $(i < \omega)$ and IND $(S_i) = i$ $(i < \omega)$ (see the remarks preceeding Lemma 15 in §3). If $\pi \in P$ then π is <u>independent</u> if there exists an independent sequence of finite Venn diagrams S_i $(i < \omega)$ such that the following holds: If $\varphi(\overline{x},\overline{y})$ is a formula of a complete theory T then $\varphi(\overline{x},\overline{y})$ admits π in T iff $\varphi(\overline{x},\overline{Y})$ admits each S_i in T. Obviously [ip] is independent and if π is independent then $\pi \geq$ [ip] and $\pi \in PP$.

<u>Theorem 20</u>

If π is independent then $\pi \not\geq$ [pp] $\wedge$ [sip].

<u>Proof</u>

Suppose π is independent. It suffices to prove that some complete theory T admits π but omits [pp] $\wedge$ [sip]. Let S_i $(i < \omega)$ be an independent sequence of finite Venn diagrams associated with π. For each $i < \omega$ let $S_i = \{\alpha_{ij} | j < |S_i|\}$. Let L be a language consisting of constants a_{ij} $(i < \omega, j < |S_i|)$ and b_{ij} $(i < \omega,\ j < \text{rank}\ (S_i))$, unary predicate P and binary predicates Q and R and let T be the theory in L whose axioms are

$$Q(x,y) \rightarrow P(x) \wedge \neg P(y) \wedge R(x,y)$$

$$R(x,x)$$

$$R(x,y) \rightarrow R(y,x)$$

$$R(x,y) \wedge R(y,z) \rightarrow R(x,z)$$

$$\bigwedge_{j<k<|S_i|} (a_{ij} \neq a_{ik} \wedge R(a_{ij},a_{ik}) \wedge P(a_{ij})) \quad (i < \omega)$$

$$\bigwedge_{j<k<\mathrm{rank}(S_i)} (b_{ij} \neq b_{ik} \wedge R(b_{ij},b_{ik}) \wedge \neg P(b_{ij})) \quad (i < \omega)$$

$$R(a_{i0},b_{i0}) \quad (i < \omega)$$

$$\neg R(a_{i0},a_{j0}) \quad (i < j < \omega)$$

$$P(x) \rightarrow \exists y_0 \ldots \exists y_{n-1} \bigwedge_{i<j<n} (y_i \neq y_j \wedge (\bigwedge_{i<n} Q(x,y_i))) \quad (n < \omega)$$

$$\neg P(y) \rightarrow \exists x_0 \ldots \exists x_{n-1} \bigwedge_{i<j<n} (x_i \neq x_j \wedge (\bigwedge_{i<n} Q(x_i,y))) \quad (n < \omega)$$

$$\bigwedge_{k<\mathrm{rank}(S_i)} Q(a_{ij},b_{ik})^{\alpha_{ij}(k)} \quad (i < \omega,\ j < |S_i|)$$

$$\neg\exists x(P(x) \wedge R(x,a_{i0}) \wedge (\bigwedge_{k<\mathrm{rank}(S_i)} Q(x,b_{ik})^{\alpha(k)}))$$

$$(i < \omega,\ \alpha \in 2^{\mathrm{rank}(S_i)} - S_i)$$

$$\bigwedge_{k<\ell<n} (x_k \neq x_\ell \wedge P(x_k) \wedge R(x_k,x_\ell))$$

$$\rightarrow$$

$$\exists y(\neg P(y) \wedge R(y,x_0) \wedge (\bigwedge_{k<n} Q(x_k,y)^{\beta(k)}))$$

$$(n < \omega,\ \beta \in 2^n)$$

$$\bigwedge_{\substack{k<\ell<n \\ j<\mathrm{rank}(S_i)}} (y_k \neq y_\ell \wedge y_k \neq b_{ij} \wedge P(y_k) \wedge R(y_k,b_{ij})$$

$$\rightarrow$$

$$\exists x(P(x) \wedge R(x,y_0) \wedge (\bigwedge_{j<\mathrm{rank}(S_i)} Q(x,b_{ij})^{\alpha(j)}) \wedge (\bigwedge_{k<n} Q(x,y_k)^{\beta(k)}))$$

$$(i < \omega,\ \alpha \in S_i,\ n < \omega,\ \beta \in 2^n)$$

$$\bigwedge_{k<\ell<n} (y_k \neq y_\ell \wedge \neg P(y_k) \wedge R(y_k,y_\ell) \wedge (\bigwedge_{i<n} \neg R(y_0,b_{i0}))$$

$$\rightarrow$$

$$\exists x(P(x) \wedge R(x,y_0) \wedge (\bigwedge_{k<n} Q(x,y_k)^{\beta(k)}))$$

$$(n < \omega,\ \beta \in 2^n)$$

Since $\mathrm{IND}(S_i) < \mathrm{IND}(S_{i+1})$ $(i < \omega)$ it follows easily that T is consistent. To prove that T is complete and quantifier-eliminable note that each finite reduct of T is complete and quantifier-eliminable by the partial isomorphism test. Letting $\varphi(x,\overline{y})$ be $(Q(x,y) \wedge y_0 = y_1) \vee (\neg P(x) \wedge y_2 = y_3)$ it follows easily that $\varphi(x,\overline{y})$

admits each S_i in T. Hence $\varphi(x,\overline{y})$ admits π in T so T admits π. To prove that T omits $[pp] \wedge [sip]$ it suffices to prove that

(1) T omits $[pp]$

(2) T omits $[sip]$

hold. To prove (1) suppose T admits $[pp]$. Then by Theorem 9 some formula $\varphi(x,\overline{y})$ of T admits pp in T. Since T is quantifier-eliminable there exists an open formula $\psi(x,\overline{y})$ of T such that $T \vdash \varphi(x,\overline{y}) \leftrightarrow \psi(x,\overline{y})$. In particular $\psi(x,\overline{y})$ admits pp in T. By replacing constants with variables (if necessary) it may be assumed that $\psi(x,\overline{y})$ contains no constants. Let $A \models T$ and let $T' = \mathrm{Th}A'$ where A' is the reduct of A to L minus the constants. Then $\psi(x,\overline{y})$ admits pp in T' so

(3) For arbitrarily large $n < \omega$ there exist n $\psi(x,\overline{y})$-definable subsets of $|A'|$ which partition $|A'|$

holds. For each $i < \omega$ let $T_i = \mathrm{Th}A'_i$ where $A'_i \models \mathrm{IND}(i)$ is obtained by restricting A' to the R-equivalence class of A' containing the constants $a_{ij} (j < |S_i|)$ and $b_{ij} (j < \mathrm{rank}\ (S_i))$. For each $\overline{a} \in |A'|^{\ell(\overline{y})}$ let $I(\overline{a}) = \{i < \omega \mid r(\overline{a}) \cap |A'_i| = \phi\}$. Since $\psi(x,\overline{y})$ is open and contains no constants it follows easily that

(4) $A' \models (P(b_0) \leftrightarrow P(b_1)) \rightarrow (\psi(b_0,\overline{a}) \leftrightarrow \psi(b_1,\overline{a}))$

$$(\overline{a} \in |A'|^{\ell(\overline{y})},\ i_0, i_1 \in I(\overline{a}),\ b_0 \in |A'_{i_0}|,\ b_1 \in |A'_{i_1}|)$$

holds. From (3) and (4) it follows easily that for some open formula $\chi(x,\bar{y})$ of T which contains no constants

(5) For arbitrarily large $n < \omega$ there exist $f(n) < \omega$ and n $\chi(x,\bar{y})$-definable subsets A_{ni} $(i < n)$ of $|A'_{f(n)}|$ which partition $|A'_{f(n)}|$.

holds. If $\sup \{f(n) \mid n < \omega\} < \omega$ then clearly $\chi(x,\bar{y})$ admits pp in T_n for some $n < \omega$. But for every $n < \omega$ it may be proved that T_n omits [pp] (see Example 10) and this is a contradiction. Thus suppose $\sup \{f(n) \mid n < \omega\} = \omega$. For each $n < \omega$ such that $f(n)$ is defined let $g(n) = |\{i < n \mid |A_{ni}| > \ell(\bar{y})\}|$. If $\sup \{g(n) \mid n < \omega\} = m < \omega$ for some $m < \omega$ let $\chi^m(x,\bar{z})$ be the formula $\neg \bigvee_{i<m} \chi(x,\bar{y}_i)$ where $\bar{z} = \bar{y}_0 \cap \ldots \cap \bar{y}_{m-1}$. Choose $n > m + m\ell(\bar{y})$ such that $f(n)$ is defined. For notational convenience assume that $|A_{ni}| > \ell(\bar{y})$ iff $i < g(n)$ $(i < n)$. Since $g(n) \leq m$ it follows that $|A_{ni}| \leq \ell(\bar{y})$ $(m \leq i < n)$ so $n - m \leq |\bigcup_{m \leq i < n} A_{ni}| < \aleph_0$ so $m\ell(\bar{y}) < |\bigcup_{m \leq i < n} A_{ni}| < \aleph_0$. Note that $\bigcup_{m \leq i < n} A_{ni}$ is a $\chi^m(x,\bar{z})$-definable subset of $|A'_{f(n)}|$ and $|\bigcup_{m \leq i < n} A_{ni}| > \ell(\bar{z})$. But since $\chi^m(x,\bar{z})$ is an open formula of T which contains no constants it is easy to prove that if A is a $\chi^m(x,\bar{z})$-definable subset of $|A'_{f(n)}|$ and $|A| < \ell(\bar{z})$ then $|A| \geq \aleph_0$ and this is a contradiction. Thus suppose $\sup \{g(n) \mid n < \omega\} = \omega$. By the compactness theorem it follows that there exist $A'_\omega \models IND(\omega)$ and infinitely many disjoint $\chi(x,\overline{y})$-definable subsets $A_i (i < \omega)$ of $|A'_\omega|$ such that $|A_i| > \ell(\overline{y}) (i < \omega)$. Since $\chi(x,\overline{y})$ is an open formula of T which contains no constants it is easy to prove that $|A_i| \geq \aleph_0$ $(i < \omega)$. But by remark (2) following Example 10 this is a contradiction. To prove (2) suppose T admits [sip]. Then some formula $\varphi(\overline{x},\overline{y})$ of T admits sip in T. Since T is quantifier-eliminable there exist atomic formulas (or their negations) $\varphi_{ij}(\overline{x},\overline{y}) (i,j<n)$ of T such that $T \vdash \varphi(\overline{x},\overline{y}) \leftrightarrow \bigvee_{i<n} \bigwedge_{j<n} \varphi_{ij}(\overline{x},\overline{y})$. By Theorem 13 (1) it may be assumed that $\bigwedge_{j<n} \varphi_{0j}(\overline{x},\overline{y})$ admits sip in T. Let $x = (x_0,\ldots,x_{\ell-1})$ and $\overline{y} = (y_0,\ldots,y_{m-1})$. By replacing constants with variables (if necessary) it may be assumed that $\bigwedge_{j<n} \varphi_{0j}(\overline{x},\overline{y})$ contains no constants. Let $A \models T$ and let $T' = ThA'$ where A' is the reduct of A to L minus the constants. Then $\bigwedge_{j<n} \varphi_{0j}(\overline{x},\overline{y})$ admits sip in T'. Let $\psi(\overline{x},\overline{y})$ be $\psi_0(x_0,\overline{y}) \wedge \ldots \wedge \psi_{\ell-1}(x_{\ell-1},\overline{y}) \wedge \psi_\ell(\overline{x}) \wedge \psi_{\ell+1}(\overline{y})$ where each $\psi_i(x_i,\overline{y})$ is the conjunction of those $\varphi_{0j}(\overline{x},\overline{y})$ containing one occurrence of x_i and no occurrences of the other variables occurring in $\overline{x}$, $\psi_\ell(\overline{x})$ is the conjunction of those $\varphi_{0j}(\overline{x},\overline{y})$ containing two occurrences of the variables occurring in $\overline{x}$ and $\psi_{\ell+1}(\overline{y})$ is the conjunction of those $\varphi_{0j}(\overline{x},\overline{y})$ containing no occurrences of the variables

occurring in $\bar{x}$. Thus, $\psi(\bar{x},\bar{y})$ admits sip in T' . Since $|O_m T'| < \aleph_0$ it follows easily from the definition of sip that it may be assumed that $\psi_{\ell+1}(\bar{y})$ is an atom of $O_m T'$. From this and Theorem 13 (1) it follows easily that it may then be assumed that $\psi(\bar{x},\bar{y})$ is an atom of $O_{\ell+m}T'$ since $|O_{\ell+m}T'| < \aleph_0$. By Theorem 13 (2) it may then be assumed that

$$(6)\quad \forall i \forall j (T' \vdash \psi(\bar{x},\bar{y}) \rightarrow \neg R(x_i,y_j))$$

or

$$(7)\quad \forall i \forall j (T' \vdash \psi(\bar{x},\bar{y}) \rightarrow R(x_i,y_j))$$

holds. If (6) holds then a pair of $\psi(\bar{x},\bar{y})$-definable subsets of $|A'|$ cannot be disjoint so $\psi(\bar{x},\bar{y})$ omits sip in T' and this is a contradiction. Suppose (7) holds. By Theorem 13 (2) it may be assumed that

$$(8)\quad \forall i \forall j (T' \vdash \psi(\bar{x},\bar{y}) \rightarrow R(x_i,y_i))$$

$$(9)\quad \forall i \forall j (T' \vdash \psi(\bar{x},\bar{y}) \rightarrow x_i \neq y_j)$$

hold. From (8) and (9) it follows easily that if a pair of $\psi(\bar{x},\bar{y})$-definable subsets of $|A'|$ is disjoint then the corresponding pair of $\psi_0(\overline{x}_0,\overline{y}) \wedge \ldots \wedge \psi_{\ell-1}(x_{\ell-1},\overline{y})$-definable subsets of $|A'|$ is disjoint. But then it follows easily from the definition of sip that $\psi_0(x_0,\overline{y}) \wedge \ldots \wedge \psi_{\ell-1}(x_{\ell-1},\overline{y})$ admits sip in T'. From Theorem 13 (2) it may be assumed that $\psi_0(x_0,\overline{y})$ admits sip in T'. It may also be assumed that $\psi_0(x_0,\overline{y})$ is an atom of $O_{l+m}T'$ since $\psi(\overline{x},\overline{y})$ is an atom of $O_{\ell+m}T'$. From this and the definition of sip it follows easily that $\psi_0(x_0,\overline{y})$ admits sip in T'' where $T'' = \mathrm{Th}A''$ for some substructure A'' of A' obtained by restricting A' to some R-equivalence class of A'. But it may be proved that there do not exist infinitely many, disjoint, infinite, $\psi_0(x_0,\overline{y})$-definable subsets of $|A''|$ (see Example 10) so it follows that $\psi_0(x_0,\overline{y})$ omits sip in T'' and this is a contradiction.

If $\pi \in P$ then π is <u>countable</u> if the following holds: If T is a countable complete theory which admits every $\pi' < \pi$ then T admits π .

<u>Theorem 21</u>

If $\pi \geq [pp]$ then π is countable.

<u>Proof</u>

Suppose $\pi \geq [pp]$ and T is a countable complete theory which omits π . It suffices to prove that T omits some $\pi' < \pi$. Let $\varphi_i(\overline{x}_i,\overline{y}_i)$ $(i < \omega)$ be the formulas of T . Since T omits π each $\varphi_i(\overline{x}_i,\overline{y}_i)$ omits π in T . In particular each $\varphi_i(\overline{x}_i,\overline{y}_i)$ omits [1] in T . By Lemma 15 it follows that each $\varphi_i(\overline{x}_i,\overline{y}_i)$ omits S_i in T

for some finite Venn diagram S_i such that rank $(S_i) \geq i$ and IND $(S_i) = i$. Let π'' be the independent property of complete theories associated with the independent sequence of finite Venn diagrams S_i $(i < \omega)$. Then T omits π'' since each $\varphi_i(\bar{x}_i, \bar{y}_i)$ omits π'' in T. Hence T omits π' where $\pi' = \pi \wedge \pi''$. Obviously $\pi' \leq \pi$. But $\pi' \nleq \pi$ since $\pi \geq [pp]$ yet $\pi'' \ngeq [pp]$ by Theorem 20. Hence $\pi' < \pi$.

Corollary 8

$|\{\pi \in P | \pi \text{ is independent}\}| > \aleph_0$.

Proof

Let $T = \sum_{\pi \text{ independent}} T_\pi$ where each T_π is a countable complete theory which admits π but omits $[pp] \wedge [sip]$. In particular each T_π omits [1]. Since [1] is prime it follows easily that T omits [1]. Since $[1] \geq [pp]$ it follows by Theorem 21 that [1] is countable. But then $|T| > \aleph_0$ since T admits every $\pi \in P$ such that π is independent so it follows that $|\{\pi \in P | \pi \text{ is independent}\}| > \aleph_0$.

§3 Density Results

The following result shows that $\mathcal{PP}$ is not a dense subset of $\mathcal{P}$.

Theorem 22

If $\pi \in \mathcal{PP}$ then $(\pi, \pi \vee [pp]] \cap \mathcal{PP} = \phi$.

Proof

Suppose $\pi \in \mathcal{PP}$. If $(\pi, \pi \vee [\mathcal{PP}]] \cap \mathcal{PP} \neq \phi$ then $\pi < \pi' \leq \pi \vee [pp]$ for some $\pi' \in \mathcal{PP}$. In particular $\pi' = [\rho']$ where ρ' is some principal property of formulas. Let T be a countable complete theory which admits π but omits π' and let $\varphi_i(\overline{x}_i, \overline{y}_i)$ $(i < \omega)$ be the formulas of $T + EQV$. Since ρ' is principal and $T{+}EQV(\alpha) < T$ whenever $|\alpha| < \aleph_0$ it follows by the compactness theorem that for each $i < \omega$ and $n_0 < \ldots < n_{j-1} < \omega$ there exists $n_j > n_{j-1}$ such that $\varphi_i(\overline{x}_i, \overline{y}_i)$ omits ρ' in $T{+}EQV(\alpha)$ whenever $\alpha \cap (n_{j-1}, n_j) = \phi$ (see the proof of Lemma 6). From this it follows easily that $n_0 < n_1 < \ldots < n_{i-1} < n_i < \ldots < \omega$ may be chosen so that for each $i < \omega$ $\varphi_i(\overline{x}_i, \overline{y}_i)$ omits ρ' in $T{+}EQV(\alpha)$ whenever $\alpha \cap (n_{i-1}, n_i) = \phi$. Let $\alpha = \{n_i \mid i < \omega\}$. Then for each $i < \omega$ $\varphi_i(\overline{x}_i, \overline{y}_i)$ omits ρ' in $T{+}EQV(\alpha)$ so $T{+}EQV(\alpha)$ omits $[\rho'] = \pi'$. But $T{+}EQV(\alpha)$ admits $\pi \vee [pp] \geq \pi'$ since T admits π and $EQV(\alpha)$ admits $[pp]$ and this is a contradiction.

By letting $\pi = [0]$ in Theorem 22 it follows that $([0], [pp]] \cap PP = \phi$. In particular $([fcp], [pp]) \cap PP = \phi$ although $([fcp], [pp]) \neq \phi$ since $[fcp] < [pp] \wedge [sip] < [pp]$ (note that IND admits [fcp] but omits both [pp] and [sip] while SIND admits [sip] but omits [pp]). Theorem 22 also shows that if $[pp] \nleq \pi \in PP$ then $[[pp], \pi \vee [pp]] \cap PP = \phi$. In fact if $[pp] \leq \pi' \leq \pi \vee [pp]$ for some $\pi' \in PP$ then $\pi \vee [pp] \in PP$ since $\pi \vee [pp] = \pi \vee \pi'$ and $\pi, \pi' \in PP$. But $\pi \vee [pp] \in (\pi, \pi \vee [pp]]$ since $[pp] \nleq \pi$ and this is a contradiction. From this remark it follows that $[[pp], [ip] \vee [pp]] \cap PP = \phi$ since $[pp] \nleq [ip] \in PP$ (note that IND admits [ip] but omits [pp]).

The next result shows that PP is a fairly dense subset of P.

<u>Theorem 23</u>

If $\pi_0 < \pi_1$, $\pi_0 \notin PP$ and $\pi_1 \in PP$ then $(\pi_0, \pi_1) \cap PP \neq \phi$.

<u>Proof</u>

Suppose $\pi_0 < \pi_1$, $\pi_0 \notin PP$ and $\pi_1 \in PP$. Let $\pi_0 = [\rho_0]$ where ρ_0 is some property of formulas. Since $\pi_0 < \pi_1$ it follows that $\pi_1 \nleq [\overline{\rho_0(\alpha_0)}]$ for some strictly increasing sequence $\alpha_0 \in \omega^\omega$. Let $\pi = \pi_1 \wedge [\overline{\rho_0(\alpha_0)}]$ so $\pi_0 \leq \pi \leq \pi_1$. Obviously $\pi \in PP$ since $\pi_1, [\overline{\rho_0(\alpha_0)}] \in PP$. But $\pi_0 \neq \pi$ since $\pi_0 \notin PP$ and $\pi \neq \pi_1$ since $\pi_1 \nleq [\overline{\rho_0(\alpha_0)}]$. Hence $\pi_0 < \pi < \pi_1$ so $(\pi_0, \pi_1) \cap PP \neq \phi$

Theorem 23 shows that if $[0] < \pi_1 \in PP$ then $([0], \pi_1) \cap PP \neq \phi$. In fact let $\pi_0 = \pi_1 \wedge [pp]$ so $[0] \leq \pi_0 \leq \pi_1$. Since $[0]$ is archetypal it is $\wedge$-irreducible so $[0] \neq \pi_0$. Hence $\pi_0 \in ([0], [pp]]$ so $\pi_0 \notin PP$ and $\pi_0 \neq \pi_1$. Thus $(\pi_0, \pi_1) \cap PP \neq \phi$ so $([0], \pi_1) \cap PP \neq \phi$. Theorem 23 also shows that if $\pi < [1]$ and $\pi \notin PP$ then $(\pi, [1]) \cap PP \neq \phi$.

The following result shows that $([[0], [\overline{pp}]] \cap PP, \leq)$ is dense.

<u>Theorem 24</u>

If $\pi_0 < \pi_1$, $\pi_0 \in PP$, $\pi \in PP$ and $\pi_0 \not\geq \pi_1 \wedge [\overline{pp}]$ then $(\pi_0, \pi_1) \cap PP \neq \phi$.

<u>Proof</u>

Suppose $\pi_0 < \pi_1$, $\pi_0 \in PP$, $\pi_1 \in PP$ and $\pi_0 \not\geq \pi_1 \wedge [\overline{pp}]$. Obviously $\pi_0 \leq \pi_1 \wedge (\pi_0 \vee [\overline{pp(\alpha)}]) \leq \pi_1 \wedge (\pi_0 \vee [\overline{pp}]) \leq \pi_1$ and $\pi_1 \wedge (\pi_0 \vee [\overline{pp(\alpha)}]) \in PP$ for every strictly increasing sequence $\alpha \in \omega^\omega$. Hence it suffices to prove that $\pi_0 < \pi_1 \wedge (\pi_0 \vee [\overline{pp(\alpha)}]) < \pi_1 \wedge (\pi_0 \vee [\overline{pp}])$ for some strictly increasing sequence $\alpha \in \omega^\omega$. Let $\varphi_i(\overline{x}_i, \overline{y}_i)$ $(i < \omega)$ be the formulas of a countable complete theory which admits π_0 but omits $\pi_1 \wedge [\overline{pp}]$. For each $i < \omega$ let $S_i = \{j < \omega \mid$ for some $A \models T$ there exists a partition of $|A|^{\ell(\overline{x}_i)}$ into j $\varphi_i(\overline{x}_i, \overline{y}_i)$-definable sets$\}$. Since T omits $[\overline{pp}]$ it follows easily that $|\omega - S_i| = \aleph_0$ $(i < \omega)$. Let $\psi_i(\overline{z}_i, \overline{w}_i)$ $(i < \omega)$ be the formulas of T+EQV and let $\pi_1 = [\rho_1]$ where ρ_1 is some principal property of formulas. Since $\rho_1 \cap \overline{pp}$ is principal and $T{+}EQV(\alpha) \blacktriangleleft T$ whenever $|\alpha| < \aleph_0$ it follows by the compactness theorem that for each $i < \omega$ and $n_0 < \ldots < n_{j-1} < \omega$ there exists $n_j > n_{j-1}$ such that $\psi_i(\overline{z}_i, \overline{w}_i)$ omits $\rho_1 \cap \overline{pp}$ in $T{+}EQV(\alpha)$ whenever $\alpha \cap (n_{j-1}, n_j) = \phi$ (see the proof of Lemma 6). From this it follows easily that $n_0 < n_1 < \ldots < n_{i-1} < n_i < \ldots < \omega$ may be chosen so that $n_i \in \omega - S_i$ $(i < \omega)$ and so that for each $i < \omega$ $\psi_i(\overline{z}_i, \overline{w}_i)$ omits $\rho_1 \cap \overline{pp}$ in $T{+}EQV(\alpha)$ whenever $\alpha \cap (n_{i-1}, n_i) = \phi$. Let $\alpha = \{n_i | i < \omega\}$. Then for each $i < \omega$ $\psi_i(\overline{z}_i, \overline{w}_i)$ omits $\rho_1 \cap \overline{pp}$ in $T{+}EQV(\alpha)$ so $T{+}EQV(\alpha)$ omits $\pi_1 \wedge (\pi_0 \vee \overline{[pp]})$. However $T{+}EQV(\alpha)$ admits $\pi_1 \wedge (\pi_0 \vee \overline{[pp(\alpha)]})$ since T admits π_0 and $EQV(\alpha)$ admits $[pp(\alpha)]$. Hence $\pi_1 \wedge (\pi_0 \vee \overline{[pp(\alpha)]}) < \pi_1 \wedge (\pi_0 \vee \overline{[pp]})$. On the other hand for each $i < \omega$ $\varphi_i(\overline{x}_i, \overline{y}_i)$ omits $\overline{pp(\alpha)}$ in T (since $n_i \in \omega - S_i$) so T omits $\overline{[pp(\alpha)]}$. Since T omits π_1 it follows that T omits $\pi_1 \wedge (\pi_0 \vee \overline{[pp(\alpha)]})$. However T admits π_0 so $\pi_0 < \pi_1 \wedge (\pi_0 \vee \overline{[pp(\alpha)]})$.

The next result shows that $([[ip], [1]] \cap PP, \leq)$ is dense. However some preliminary remarks are required.

For convenience a finite Venn diagram will be viewed as a set $\phi \neq S \subset 2^{<\omega}$ such that $S \subset 2^n$ for some $0 < n < \omega$ (view S as the set of nonempty Boolean cominations of a finite Venn diagram). Let n be the <u>rank</u> of S. If $0 \le m \le n$ then S is m-<u>independent</u> if for each $i_0 < \dots < i_{m-1} < n$ and $\alpha \in 2^{\{i_0,\dots,i_{m-1}\}}$ there exists $\beta \in 2^n$ such that $\alpha \subset \beta \in S$. Thus S is m-independent iff every Boolean combination of m members of S is nonempty. Let $\mathrm{ind}(S) = \max\{m \le n \mid S \text{ is m-independent}\}$ be the <u>independence</u> of S. For each $0 < m < \omega$ let 1^m be a property of formulas such that if $\varphi(\bar{x},\bar{y})$ is a formula of a complete theory T then $\varphi(\bar{x},\bar{y})$ admits 1^m in T iff for every $m \le n < \omega$ and m-independent $\phi \neq S \subset 2^n$ there exist $A \models T$ and n $\varphi(\bar{x},\bar{y})$-definable subsets of $|A|^{\ell(\bar{x})}$ which admit S in A.

<u>Lemma 15</u>

$[1^m] = [1] \quad (0 < m < \omega)$.

<u>Proof</u>

Suppose $0 < m < \omega$ and T is a complete theory which admits $[1^m]$. It suffices to prove that T admits $[1]$. Since T admits $[1^m]$ some formula $\varphi(\bar{x},\bar{y})$ of T admits 1^m in T. Letting $\psi(\bar{x},\bar{y}_0 \cap \dots \cap \bar{y}_m \cap \bar{z})$ be

$$((\bigwedge_{i<m} \varphi(\bar{x},\bar{y}_i)) \wedge \varphi(\bar{x},\bar{y}_m) \wedge z_0 = z_1) \vee ((\neg \bigwedge_{i<m} \varphi(\bar{x},\bar{y}_i)) \wedge z_2 = z_3)$$

it is easy to prove that $\psi(\bar{x},\bar{y}_0 \cap \dots \cap \bar{y}_m \cap \bar{z})$ admits 1 in T. Hence T admits $[1]$.

Let L be a language consisting of constants $c_{ij}(i, j < \omega)$, a unary predicate P and binary predicates $E, \sim$ and let T be the theory in L whose axioms are

$x \sim x$

$x \sim y \to y \sim x$

$x \sim y \wedge y \sim z \to x \sim z$

$E(x,y) \to x \sim y \wedge P(x) \wedge \neg P(y)$

$P(c_{ij}) \quad (i,j < \omega)$

$c_{ij} \sim c_{ik} \wedge c_{ij} \neq c_{ik} \quad (i < \omega,\ j < k < \omega)$

$c_{i0} \nsim c_{j0} \quad (i < j < \omega)$

$P(x) \to \exists y_0 \ldots \exists y_{n-1} \bigwedge_{i<j<n} (y_i \neq y_j \wedge y_i \sim x \wedge \neg P(y_i)) \quad (n < \omega)$

$\neg P(y) \to \exists x_0 \ldots \exists x_{n-1} \bigwedge_{i<j<n} (x_i \neq x_j \wedge x_i \sim y \wedge P(x_i)) \quad (n < \omega)$.

If T' is a theory in L and $\alpha \in (\omega + 1)^\omega$ then T' is α-<u>independent</u> if

$$T' \vdash \bigwedge_{i<j<m} (x_i \neq x_j \wedge x_i \sim x_j \wedge P(x_i)) \to (\bigvee_{i<n} x_0 \sim c_{i0}) \vee \exists y \bigwedge_{i<m} E(x_i,y)^{\beta(i)}$$

$$T' \vdash \bigwedge_{i<j<m} (y_i \neq y_j \wedge y_i \sim y_j \wedge \neg P(y_i)) \to (\bigvee_{i<n} y_0 \sim c_{i0}) \vee \exists x \bigwedge_{i<m} E(x,y_i)^{\beta(i)}$$

hold for every $n < \omega$, finite $m \leq \alpha(n)$ and $\beta \in 2^m$. If S_n $(n < \omega)$ are finite Venn diagrams let $T(S_n | n < \omega)$ be the theory in L whose axioms are

T

$$\bigwedge_{i<j<m} (x_i \neq x_j \wedge x_i \sim x_j \wedge P(x_i)) \rightarrow \exists y \bigwedge_{i<m} E(x_i,y)^{\alpha(i)}$$

$$\bigwedge_{\substack{i<j<m \\ k<\mathrm{rank}(S_n)}} (y_i \neq y_j \wedge y_i \sim y_j \wedge \neg P(y_i) \wedge y_0 \sim c_{n0} \wedge y_i \neq c_{nk}) \rightarrow$$

$$\exists x((\bigwedge_{i<m} E(x,y_i)^{\alpha(i)}) \wedge (\bigwedge_{i<\mathrm{rank}(S_n)} E(x,c_{ni})^{\beta(i)}))$$

$$\neg\exists x \bigwedge_{i<\mathrm{rank}(S_n)} (E(x,c_{ni})^{\gamma(i)} \wedge P(x))$$

$$\bigwedge_{i<j<\mathrm{ind}(S_n)} (y_i \neq y_j \wedge y_i \sim y_j \wedge \neg P(y_i)) \rightarrow$$

$$(\bigvee_{i<n} y_0 \sim c_{i0}) \vee \exists x \bigwedge_{i<\mathrm{ind}(S_n)} E(x,y_i)^{\delta(i)}$$

where $m, n < \omega$, $\alpha \in 2^m$, $\beta \in S_n$, $\gamma \in 2^{\mathrm{rank}(S_n)} - S_n$ and $\delta \in 2^{\mathrm{ind}(S_n)}$.

If $S_0,\ldots,S_{m-1}$ are finite Venn diagrams let $T(S_n|n<m)$ be $T(S_n|n<\omega)$ where $S_n = 2^1$ $(n \geq m)$. Finally let T_ω be $T(S_n|n<\omega)$ where $S_n = 2^1 (n < \omega)$.

<u>Lemma 16</u>

If $S_n (n < \omega)$ are finite Venn diagrams then

(1) $T(S_n|n<\omega)$ is $(\mathrm{ind}(S_n)|n<\omega)$-independent

(2) $T(S_n|n<\omega)$ is consistent iff $(\mathrm{ind}(S_n)|n<\omega)$ is increasing

(3) $T(S_n|n<\omega)$ is complete if $(\mathrm{ind}(S_n)|n<\omega)$ is increasing and

$$\lim_{n\to\omega} (\mathrm{ind}(S_n)) = \omega$$

(4) If $T(S_n|n<\omega)$ is consistent and $m<\omega$ then $T(S_n|n<\omega)$ is $(\text{ind}(S_0),\ldots,\text{ind}(S_{m-1}),\omega,\omega,\ldots)$-independent iff $\text{ind}(S_n) = \text{rank}(S_n)$ $(n \geq m)$

(5) If $m<\omega$ and $T(S_0,\ldots,S_{m-1})$ is consistent then $T(S_0,\ldots,S_{m-1})$ is archetypal for $[ip]$

hold.

Proof

It is easy to prove (1), (2) and (4) using the definitions. To prove (3) note that if $\lim_{n\to\omega} (\text{ind}(S_n)) = \omega$ then every pair of countable models of $T(S_n|n<\omega)$ have countable, isomorphic, elementary extensions. To prove (5) note that $T(S_0,\ldots,S_{m-1}) \blacktriangleleft T_\omega$ (since $T(S_0,\ldots,S_{m-1})$ is definable in T_ω) and T_ω is archetypal for $[ip]$ (see the proof of Lemma 13).

Theorem 25

If $\pi_0 < \pi_1$, $\pi_0, \pi_1 \in PP$ and $\pi_0 \vee [ip] \not\geq \pi_1$ then $(\pi_0, \pi_1) \cap PP \neq \phi$.

Proof

Suppose $\pi_0 < \pi_1$, $\pi_0, \pi_1 \in PP$ and $\pi_0 \vee [ip] \not\geq \pi_1$. Let T be a countable complete theory which admits both π_0 and $[ip]$ but omits π_1. Let $\varphi_i(\overline{x}_i,\overline{y}_i)$ $(i<\omega)$ be the formulas of T and let $\psi_i(\overline{z}_i,\overline{w}_i)$ $(i<\omega)$ be the formulas of $T + T_\omega$. Also let $\pi_1 = [\rho_1]$ where ρ_1 is some principal property of formulas. By induction on $n < \omega$ it may be proved that there exist finite Venn diagrams

S_n $(n < \omega)$ such that for every $n < \omega$

(1) $\mathrm{ind}(S_{n-1}) < \mathrm{ind}(S_n)$

(2) $\psi_{n-1}(\bar{z}_{n-1}, \bar{w}_{n-1})$ omits ρ_1 in $T+T(S_0,\ldots,S_{n-1},S'_n,S'_{n+1},\ldots)$ whenever $T+T(S_0,\ldots,S_{n-1},S'_n,S'_{n+1},\ldots)$ is complete and $\mathrm{ind}(S'_n) \geq \mathrm{ind}(S_n)$

(3) $\varphi_n(\bar{x}_n, \bar{y}_n)$ omits S_n in T

hold. To prove this assume that $S_0,\ldots,S_n$ satisfy (1), (2) and (3). Then $T+T(S_0,\ldots,S_n) \blacktriangleleft T$ since T admits [ip] and $T(S_0,\ldots,S_n)$ is archetypal for [ip]. Hence $T+T(S_0,\ldots,S_n)$ omits π_1 since T omits π_1. In particular $\psi_n(\bar{z}_n,\bar{w}_n)$ omits ρ_1 in $T+T(S_0,\ldots,S_n)$. Since ρ_1 is principal it follows easily by the compactness theorem that there exists $m > \mathrm{IND}(S_n)$ such that $\psi_n(\bar{z}_n,\bar{w}_n)$ omits ρ_1 in $T+T(S_0,\ldots,S_n,S'_{n+1},S'_{n+2},\ldots)$ whenever $T+T(S_0,\ldots,S_n,S'_{n+1},S'_{n+2},\ldots)$ is complete and $\mathrm{ind}(S'_{n+1}) \geq m$ (see the proof of Lemma 6). Since T omits [1] and $[1^m] = [1]$ by Lemma 15 it follows that $\varphi_{n+1}(\bar{x}_{n+1},\bar{y}_{n+1})$ omits 1^m in T. In particular $\varphi_{n+1}(\bar{x}_{n+1},\bar{y}_{n+1})$ omits S in T for some finite Venn diagram S such that $\mathrm{ind}(S) = m$. Letting $S_{n+1} = S$ completes the induction. Let $\pi = [\bar{p}]$ where $\rho = (S_n | n<\omega)$. Obviously $\pi_0 \leq \pi_1 \wedge (\pi_0 \vee [ip] \vee \pi) \leq \pi_1$ and $\pi_0 \vee [ip] \vee \pi \in PP$. But $\pi_0 \not\geq \pi_1 \wedge (\pi_0 \vee [ip] \vee \pi)$ since T admits π_0 but omits both π_1 and π. Furthermore $\pi_1 \wedge (\pi_0 \vee [ip] \vee \pi) \not\geq \pi_1$

since $T+T(S_n | n<\omega)$ admits $\pi_0 \vee [ip] \vee \pi$ but omits π_1. Hence $(\pi_0,\pi_1) \cap PP \neq \phi$.

Theorem 25 shows that if $[op] < \pi < [sop]$ and $\pi \in PP$ then $([op],\pi) \cap PP \neq \phi$ and $(\pi,[sop]) \cap PP \neq \phi$. In fact suppose $[op] < \pi < [sop]$ and $\pi \in PP$. Then $[op] \vee [ip] \nleq \pi$ (since $[op] = [sop] \wedge [ip]$) and $\pi \vee [ip] \nleq [sop]$ (since $[sop]$ is $\vee$-irreducible). Hence $([op],\pi) \cap PP \neq \phi$ and $(\pi,[sop]) \cap PP \neq \phi$. It should be noted that $([op], [sop]) \cap PP \neq \phi$. First note that $[op] < [op] \vee [pp] < [sop]$ and $[op] \vee [pp] \notin PP$ (Theorem 22 shows that $[op] \vee [pp] \notin PP$ since IND shows that $[op] \nleq [pp]$). By Theorem 23 $([op] \vee [pp], [sop]) \cap PP \neq \phi$ so $([op], [sop]) \cap PP \neq \phi$. It may also be noted that $([op], [ip]) \cap PP \neq \phi$. Obviously $([sop] \vee [\ell p]) \wedge [ip] \in PP$ and $[op] \leq ([sop] \vee [\ell p]) \wedge [ip] \leq [ip]$. But DLO admits $[op]$ but omits both $[\ell p]$ and $[ip]$ (since $[\ell p] \nleq [pop]$ and $[ip] \nleq [sop]$) and DLO + ACF(0) admits $([sop] \vee [\ell p]) \wedge [ip]$ but omits $[ip]$ (since $[ip]$ is prime). Hence $[op] < ([sop] \vee [\ell p]) \wedge [ip] < [ip]$. Theorem 25 also shows that if $\pi < [1]$ and $\pi \in PP$ then $(\pi,[1]) \cap PP \neq \phi$ since $\pi \leq \pi \vee [ip] < [1]$ and $(\pi \vee [ip], [1]) \cap PP \neq \phi$ (note that $\pi \vee [ip] \neq [1]$ since $[1]$ is $\vee$-irreducible.

The following result is useful.

Theorem 26

If $\pi < \pi_n$ $(n < \omega)$, $\pi \in PP$ and $\pi_n \in P$ $(n < \omega)$ then the following holds: $\pi < \pi_\omega \leq \pi_n$ $(n < \omega)$ for some $\pi_\omega \in P$ iff some complete theory T admits π but omits each π_n.

Proof

Suppose $\pi < \pi_n$ $(n < \omega)$, $\pi \in PP$ and $\pi_n \in P$ $(n < \omega)$. If $\pi < \pi_\omega \leq \pi_n$ $(n < \omega)$ for some $\pi_\omega \in P$ let T be some complete theory which admits π but omits π_ω. Then T omits each π_n $(n < \omega)$. If some complete theory T admits π but omits each π_n let $\pi_\omega \in P$ be defined as follows: Since it may be assumed that T is countable (if necessary replace T with some finite reduct of T admitting π) let $\varphi_i(\overline{x}_i,\overline{y}_i)$ $(i < \omega)$ be the formulas of T. For each $n < \omega$ let $\pi_n = [\rho_n]$ where ρ_n is some property of formulas and let $\sigma_n = \{\rho_n(i) \,|\, i<\omega\}$. Furthermore let $\pi^\omega = [\rho^\omega]$ where ρ^ω is a property of formulas enumerating the set $\sigma^\omega = \{\rho_n(i) \,|\, \forall_j (j \leq n \rightarrow \varphi_j(\overline{x}_j,\overline{y}_j)$ omits $\rho_n(i)$ in $T)\}$. For each $n < \omega$ it follows easily that $|\sigma_n| = \aleph_0$ (since $\pi_n \neq [0]$) and $|\sigma_n - \sigma^\omega| < \aleph_0$ (since T omits π_n) so $|\sigma_n \cap \sigma^\omega| = \aleph_0$ and $\pi^\omega \leq \pi_n$. It is easy to prove that T omits π^ω so let $\pi_\omega = \pi \wedge \pi^\omega$. Then $\pi < \pi_\omega \leq \pi_n$ $(n < \omega)$.

Corollary 8

If $\pi < \pi_n$ $(n < \omega)$, $\pi \in \mathcal{PP}$, $\pi_n \in \mathcal{P}$ $(n < \omega)$ and π is archetypal then

$\pi < \pi_\omega \leq \pi_n$ $(n < \omega)$ for some $\pi_\omega \in \mathcal{P}$.

Proof

Immediate.

Since $[0]$ is archetypal Corollary 8 shows that every countable set of nonzero properties of complete theories has a nonzero lower bound. Thus every countable set of complete theories admitting nonzero properties of complete theories admits a common nonzero property of complete theories.

It may also be noted that if $\pi \in \mathcal{P}$ is countable then there does not exist $[0] < \pi_0 < \pi$ such that $([0],\pi) = ([0],\pi_0]$ (otherwise some countable complete theory admits every $\pi' < \pi$ but omits π). Since every $\pi \geq [pp]$ is countable by Theorem 21 it follows that each such property of complete theories has no greatest property of complete theories below it.

This chapter concludes by showing that the ordering $\blacktriangleleft$ on $\mathcal{T}$ is not dense.

Theorem 27

$(\mathcal{T}|_{\equiv}, \blacktriangleleft|_{\equiv})$ is not dense.

Proof

Let T_0 be a complete theory which is archetypal for some nonzero, prime, archetypal property of complete theories π_0 and let $T_1 = \sum_{\pi \nleq \pi_0} T_\pi$ where each T_π is a complete theory admitting π but omitting π_0 . Clearly $T_1 \blacktriangleleft T_0 + T_1$. Furthermore T_1 omits π_0 (since π_0 is prime and each T_π omits π_0) but $T_0 + T_1$ admits π_0 (since T_0 admits π_0) . Hence $T_1 \not\equiv T_0 + T_1$. It suffices to prove that if $T_1 \blacktriangleleft T$ and $T_1 \not\equiv T$ then $T_0 + T_1 \blacktriangleleft T$. Suppose $T_1 \blacktriangleleft T$ and $T_1 \not\equiv T$. Then T admits π_0 so $T_0 \blacktriangleleft T$ (since T_0 is archetypal for π_0) so $T_0 + T_1 \blacktriangleleft T$. Thus $(T_1|_{\equiv}, (T_0+T_1)|_{\equiv}) = \phi$.

§4 Open Questions

This thesis shows that $(PP,\leq)$ is an infinite distributive lattice with no minimum element above [0] and no maximum element below [1]. Is

(1) If $\pi_0 < \pi_1$ and $\pi_0, \pi_1 \in PP$ then $(\pi_0, \pi_0) \cap PP \neq \phi$

true? This thesis also shows that $(P,\leq)$ is an infinite lower semilattice with no maximum element below [1]. Is

(2) If $[0] < \pi \in P$ then $([0], \pi) \neq \phi$

true? More generally is

(3) If $\pi_0 < \pi_1$ and $\pi_0, \pi_1 \in P$ then $(\pi_0, \pi_1) \neq \phi$

true?

Shelah (1975) proved that if a complete stable theory T admits [fcp] then T admits the E-property (that is, there exists a formula $\varphi(\overline{x},\overline{y},\overline{z})$ of T such that $\ell(\overline{x}) = \ell(\overline{y})$ and such that for arbitrarily large $n < \omega$ there exists $A \models T$ and $\overline{c} \in |A|^{\ell(\overline{z})}$ such that $\varphi_A(\overline{x},\overline{y},\overline{c}) = \{(\overline{a},\overline{b}) \in |A|^{\ell(\overline{x})} \times |A|^{\ell(\overline{y})} \mid A \models \varphi(\overline{a},\overline{b},\overline{c})\}$ is an equivalence relation on $|A|^{\ell(x)}$ with exactly n equivalence classes). It is easy to prove that if a complete theory T admits the E-property then T admits [pp]. Hence Shelah's result implies that [fcp] = [pp] $\wedge$ [ip] since unstable complete theories admit [op] = [sop] $\wedge$ [ip]. Note that the E-property cannot be viewed as a property of complete theories since there exist countable, complete, $\aleph_0$-categorical theories which admit every property of complete theories yet by Ryll-Nardzewski (1959) it is easy to prove that such theories omit the E-property. From this it follows that if T is a complete, $\aleph_0$-categorical theory which admits [fcp] then T admits [op]. Is

(4) If T is a complete, $\aleph_0$-categorical theory which admits [fcp] then T admits [sop]

true?

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